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Groups with elements of order 8 do not have the DCI property

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Abstract

Let k be odd, and n an odd multiple of 3. Although this can also be deduced from known results, we provide a new proof that $C_k \rtimes C_8$ and $(C_n \times C_3) \rtimes C_8$ do not have the Directed Cayley Isomorphism (DCI) property. When k is prime, $C_k \rtimes C_8$ had previously been proved to have the Cayley Isomorphism (CI) property. To the best of our knowledge, the groups $C_p \rtimes C_8$ (where p is an odd prime) are only the second known infinite family of groups that have the CI property but do not have the DCI property. This also provides a new proof of the result (which follows from known results but was not explicitly published) that no group with an element of order 8 has the DCI property.

One piece of our proof is a new result that may prove to be of independent interest: we show that if a permutation group has a regular subgroup of index 2 then it must be 2-closed.

Keywords: CI property, DCI property, Cayley graphs, Cayley digraphs, 2-closed groups, 2-closure.

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1 Introduction

Let $\Gamma_1 = \mathrm{Cay}(G, S)$ be a Cayley digraph on G. We say that the digraph Γ_1 has the Directed Cayley Isomorphism (DCI) property if whenever $\Gamma_2 = \mathrm{Cay}(G, T) \cong \Gamma_1$, there is some $\alpha \in \mathrm{Aut}(G)$ such that $S^\alpha = T$. Analogously, we say that the graph Γ has the Cayley Isomorphism (CI) property if whenever $\Gamma_2 = \mathrm{Cay}(G, T) \cong \Gamma_1$, there is some $\alpha \in \mathrm{Aut}(G)$ such that $S^\alpha = T$. For graphs, we must have $S = S^{-1}$ and $T = T^{-1}$.

We say that the group G has the DCI property if every Cayley digraph on G has the DCI property, and that G has the CI property if every Cayley graph on G has the CI property. Observe that for a group, having the DCI property is stronger than having the CI property.

In the 1970s, Babai [1] proved a criterion involving only permutation groups that allows us to determine whether or not a group G has the DCI property: this happens if every 2-closed group has at most one conjugacy class of regular subgroups isomorphic to G.

Throughout this paper, A will denote a group that is isomorphic to either C_k or $C_k \times C_3$, where k is odd. Our main result is the following:

Theorem 1.1. Let A be a group that is isomorphic to either C_k or $C_k \times C_3$, where k is odd. Let $R = A \rtimes \langle r \rangle$, where $|r| = 8$ and $r^{-1}ar = a^{-1}$ for every $a \in A$.

Then R does not have the DCI property.

This result is not new, but has never been published explicitly. There are two reasonably straightforward ways of deducing this result from the existing literature. Both methods rely on key results from an early paper [2] of Babai and Frankl on CI groups. Lemma 3.2 of [2] shows that every subgroup of a CI group is CI. Although the proof is written for graphs rather than digraphs, their techniques apply to either situation. Since C_8 is not a DCI group (this was shown by Elspas and Turner in [5]), this means that no group with an element of order 8 can have the DCI property either.

We can also deduce the fact that the groups $A \rtimes C_8$ do not have the DCI property from the result of [2] showing that a quotient of a CI group by a characteristic subgroup is CI. Although again the proof is written for graphs, the techniques apply more generally. Since C_8 is not a DCI group and A is the unique maximal subgroup of odd order and therefore characteristic in $A \rtimes C_8$, this means that $A \rtimes C_8$ is not DCI.

We remark that both techniques in the preceding two paragraphs involve wreath products of graphs and digraphs, and hence the examples they produce of (di)graphs that do not have the DCI property, have large automorphism groups.

However, C_8 is a CI group, so neither of these techniques allows us to determine the CI-status of these groups. When k is square-free and not prime, the CI-status of $A \rtimes C_8$ remains open.

When k is prime (and A is cyclic), the groups $A \rtimes C_8$ have been proved to have the CI property by Li, Lu, and Pálfy [7]. As far as we are aware, this makes $C_p \rtimes C_8$ (where p is an odd prime) only the second known infinite family of groups that have the CI property but do not have the DCI property. The other family is $C_p \rtimes C_3$, where p is prime and $3 \mid p-1$. These groups were shown by the first author to be CI [3, Theorem 21].[1] That they are not DCI groups follows from work by Li showing that they are not k-DCI for most values of k [6, Theorem 1.3].

It has been unfortunately common for work on the CI problem and the DCI problem to be imprecise in terminology, so that some papers report that a group has the CI property

[1] A subsequent, self-contained proof of this also appears in [7].

when the proof actually shows the DCI property, or more importantly that a group is not CI when in fact the proof only shows that it is not a DCI group. Some of the confusion that has arisen through this lack of consistency in the terminology has been partially addressed in some recent papers [8, 9]. However, those papers do not touch at all on $A \rtimes C_8$.

There is a description that first appeared in the work of Li, Lu, and Pálfy [7] and has since been updated via [4], of groups that may have the CI property. For many of these, their status remains open. No equivalent description has been published for the DCI property. Groups not matching the description cannot be DCI since they are not CI, but groups that match the description need not be DCI even if they are CI. In the Li-Lu-Pálfy description (as updated), $A \rtimes C_8$ is the only family of groups that have elements of order 8, whose CI status is unknown. Since the groups $A \rtimes C_8$ are not in fact DCI groups, there are 2-closed groups that contain more than one conjugacy class of regular subgroups isomorphic to $A \rtimes C_8$. This has a significant impact on the proof techniques that may be required to prove that the groups $A \rtimes C_8$ are CI (if, in fact, they are). So it seemed to us important to write this note clarifying what we currently know about these groups.

We were able to produce a short and direct proof of Theorem 1.1. Additionally, the examples we provide in this paper have relatively small automorphism groups, compared to the large automorphism groups of the currently known examples. Our proof includes a new lemma that may be useful in other contexts, showing that a permutation group that has a regular subgroup of index 2 is always 2-closed.

2 The proof

We will define two regular permutation groups R_1 and R_2 that are isomorphic to $A \rtimes C_8$, and show that R_1 and R_2 are not conjugate in $G^{(2)}$ (the 2-closure of G), where $G = \langle R_1, R_2 \rangle$. Using Babai's criterion [1], this implies that $C_k \rtimes C_8$ does not have the DCI property.

Let the underlying set Ω of cardinality $8|A|$ be written as $\mathbb{Z}_8 \times \mathbb{Z}_k \times \mathbb{Z}_r$, where $r = 1$ if $A \cong C_k$, and $r = 3$ if $A \cong C_3 \times C_k$. When adding or subtracting in a coordinate, the operation is performed in $\mathbb{Z}_k$ or $\mathbb{Z}_r$ (as appropriate).

- Define $\tau_1 = ((0\,1\,2\,3\,4\,5\,6\,7), (1), (1))$;
- Define $\tau_2 = ((0\,1\,6\,7\,4\,5\,2\,3), (1), (1))$;
- Define ρ_1 by
$$(i, j, \ell)^{\rho_1} = \begin{cases} (i, j+1, \ell) & i \text{ even} \\ (i, j-1, \ell) & i \text{ odd.} \end{cases}$$
- Define ρ_2 by
$$(i, j, \ell)^{\rho_1} = \begin{cases} (i, j, \ell+1) & i \text{ even} \\ (i, j, \ell-1) & i \text{ odd.} \end{cases}$$
- Take $R_1 = \langle \tau_1, \rho_1, \rho_2 \rangle$ and $R_2 = \langle \tau_2, \rho_1, \rho_2 \rangle$.

It is straightforward to calculate that the group $H = \langle \tau_1, \tau_2 \rangle$ has order 16 and any of its elements can be written as $\tau_2^\ell h^\epsilon$, where $\ell \in \mathbb{Z}_8$, $h = ((1\,5)(3\,7), (1), (1))$, and $\epsilon \in \{0, 1\}$. It is also straightforward to see that $R_1 = \langle \tau_1, \rho_1, \rho_2 \rangle \cong A \rtimes C_8 \cong R_2 = \langle \tau_2, \rho_1, \rho_2 \rangle$. We need to show two things:

1. there is no $g \in G$ such that $R_2^g = R_1$; and
2. G is 2-closed.

We show these in the subsequent results.

Lemma 2.1. *For R_1, R_2 as defined above and $G = \langle R_1, R_2 \rangle$, there is no $g \in G$ such that $R_2^g = R_1$.*

Proof. Since τ_1 and τ_2 normalise $\langle \rho_1, \rho_2 \rangle$, any element of G can be written as $g = \tau \rho_1^i \rho_2^j$ for some $\tau \in H = \langle \tau_1, \tau_2 \rangle$ and some $i \in \mathbb{Z}_k$, $j \in \mathbb{Z}_r$. Additionally, as discussed above, we may write $\tau = \tau_2^\ell h^\epsilon$, where $h = ((1\,5)(3\,7), (1), (1))$, $\ell \in \mathbb{Z}_8$, and $\epsilon \in \mathbb{Z}_2$. So $g = \tau_2^\ell h^\epsilon \rho_1^i \rho_2^j$.

Now if $R_2^g = R_1$ we must have $\tau_2^g \in R_1$. Since τ_2^g has order 8, this implies $\tau_2^g \in \langle \tau_1 \rangle$ (the Sylow 2-subgroup of R_1).

We have

$$\tau_2^g = \tau_2^{\tau_2^\ell h^\epsilon \rho_1^i \rho_2^j} = \tau_2^{h^\epsilon \rho_1^i \rho_2^j}.$$

Now we calculate $\tau_2^h = \tau_2^5 \in R_2$; since $\rho_1, \rho_2 \in R_2$ we must have τ_2^g being an element of order 8 in the Sylow 2-subgroup of R_2. But none of these elements is in R_1. Therefore for every $g \in G$, $R_2^g \neq R_1$. □

We prove a more general lemma that implies the final piece we need.

Lemma 2.2. *Let G be a group with a regular subgroup R of index 2. Then G is 2-closed.*

Proof. Since R is acting regularly, we can label the points of the underlying set with the elements of R. Let $z \in G^{(2)}$, with $z \neq 1$. We will show that $z \in G$. Since R is regular, we may multiply z by an element of R if necessary to assume without loss of generality that $1^z = 1$.

Since R has index 2 in G, there is some nonidentity element $g \in G$ of order 2 that acts as an automorphism of R (so fixes 1), such that $G = R\langle g \rangle$.

Suppose that $a \in R$ with $a^z \neq a$. By the definition of 2-closure, there is some $h \in G$ such that $(1^h, a^h) = (1, a)^h = (1, a)^z = (1, a^z)$. Since g is the only nontrivial element of G that fixes 1, we must have $h = g$. Thus $a^z = a^g$.

Now suppose that $b \in R$ with $b^z = b$. Note that there is a unique element of G that fixes b and is not the identity, namely $b^{-1}gb$. By the definition of 2-closure, there is some $h \in G$ such that $(b, a)^h = (b, a)^z = (b, a^g) \neq (b, a)$, so h fixes b but does not fix a. The only nontrivial element of G that fixes b is $b^{-1}gb$, so $h = b^{-1}gb$.

Now

$$(b, a^g) = (b, a)^z = (b, a)^{b^{-1}gb} = (b, a^{b^{-1}gb}) = (b, (ab^{-1})^g b).$$

Since g is an automorphism of R, this is $(b, a^g (b^g)^{-1} b)$. Therefore $(b^g)^{-1} b = 1$, implying $b^g = b$. So $b^z = b = b^g$.

We have now shown that for any $c \in R$, whether or not $c^z = c$ we can conclude that $c^z = c^g$. Thus $z = g \in G$. Since z was an arbitrary element of $G^{(2)}$, G is 2-closed. □

Putting these together produces our main result.

Proof of Theorem 1.1. Let τ_1, τ_2, ρ_1, ρ_2, R_1, R_2, and G be as defined above. Then G contains two regular subgroups (R_1 and R_2) isomorphic to R. By Lemma 2.2, G is 2-closed. By Lemma 2.1, R_1 and R_2 are not conjugate in $G = G^{(2)}$. Therefore by Babai's criterion [1], R does not have the DCI property. □

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