AN EXPLICIT VERSION OF CHEBOTAREV'S DENSITY THEOREM

SOURABHASHIS DAS
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SOURABHASHIS DAS

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Dr. Habiba Kadiri Thesis Supervisor	Associate Professor	Ph.D.
Dr. Nathan Ng Thesis Supervisor	Professor	Ph.D.
Dr. Joy Morris Thesis Examination Committee Member	Professor	Ph.D.
Dr. Andrew Fiori Thesis Examination Committee Member	Assistant Professor	Ph.D.
Dr. John Sheriff Chair, Thesis Examination Committee	Assistant Professor	Ph.D.

Abstract

Chebotarev's density theorem generalizes the prime number theorem and Dirichlet's theorem for primes in arithmetic progressions to the setting of number fields. In particular, it asserts that prime ideals are equi-distributed over the conjugacy classes of the Galois group of any given normal extensions of number fields. The first part of the thesis investigates the works by Lagarias and Odlyzko together with the work of Winckler which provides an explicit error term for the prime counting function in Chebotarev's density theorem. We rework their argument and improve their bounds. The second part improves on the results from the first part by investigating more modern tools. The second part improves further by investigating more modern tools. Some of the main ideas are deriving an explicit formula for a smooth version of a certain prime counting function, and estimating associated sums over the zeros of Hecke L-functions.

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Chapter 1

Introduction

1.1 The prime number theorem

The main focus of this thesis is to study of distribution of primes in number fields of degree greater than 1. In the base number field, $\mathbb{Q}$, the distribution of primes is studied in the form of the Prime Number Theorem (denoted PNT) and Dirichlet's Theorem for Primes in Arithmetic Progressions. For $x > 1$, let us define

$$\pi(x) = \#\{p \leq x \mid p \text{ prime }\}$$

and the logarithmic integral

$$\text{Li}(x) = \int_2^x \frac{dt}{\log t}. \tag{1.1}$$

The famous prime number theorem (PNT) states that

$$\pi(x) \sim \text{Li}(x),$$

where $f(x) \sim g(x)$, for positive functions f, g, means $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$. PNT was proven independently by Hadamard and de la Vallée Poussin in 1896. More precisely, they proved

$$\pi(x) - \text{Li}(x) \ll x\exp(-c\sqrt{\log x}),$$

where c is a positive effective constant (effective means the constant c can be computed). The notation $f(x) \ll g(x)$ means there is a positive constant C such that for all sufficiently large x, $|f(x)| \leq Cg(x)$. For details of this proof, see Davenport's book [6, Pages 115 - 124].

Let us introduce the classical Riemann ζ-function defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

for $\Re(s) > 1$. Here the product is over all primes p. This function is holomorphic in the complex plane except at $s = 1$ where it has a simple pole. All its zeros outside the so-called critical strip $0 < \Re s < 1$ are known and it has been conjectured by Riemann in 1859 that the zeros inside that strip all lie in the line $\Re s = 1/2$. This conjecture, famously known as the Riemann Hypothesis (denoted RH), remains open to this day.

The PNT is equivalent to

$$\lim_{x\to\infty} \frac{\psi(x)}{x} = 1 \quad \text{where} \quad \psi(x) = \sum_{p^k \le x, k\ge 1} \log p$$

is a weighted prime counting function. As Riemann showed in his 1859 memoir, the approach of directly using $\psi(x)$ allows us to directly connect to the complex valued Riemann ζ-function and to make use of powerful analytic tools to extract information about prime numbers.

We denote the normalized error term in PNT by:

$$E(x) = \left| \frac{\psi(x) - x}{x} \right|.$$

The conjectured size of $E(x)$ is $x^{-\frac{1}{2}+\epsilon}$ and this is equivalent to the infamous Riemann Hypothesis (RH). Unconditional (without RH) results exist (valid for sufficiently large x) and we refer the interested readers to [6, Chapters 17 and 18]. Next, we provide a brief history about the explicit treatment of this error term (so valid for known values of x). This was initiated by works of Rosser and Schoenfeld ([23],[24],[25]) : Rosser in [23, Theorem 22] proved $E(x) \le \sqrt{\log x}e^{-\sqrt{\frac{\log x}{19}}}$ for $x \ge e^{4000}$ and in [23, Theorem 21] proved $E(x) \le 0.0119$ for $x \ge e^{50}$. This was later refined by Dusart [7] in his PhD thesis and successive articles. He proved that $E(x) \le 9.05 \times 10^{-8}$ for $x \ge e^{50}$. Later, Faber (Chinook 2010) and Kadiri [8, Theorem 1.1] proved $E(x) \le 9.47 \times 10^{-10}$ for $x \ge e^{50}$. They introduced a family of smooth weight functions to prove their theorem. This is one of the key tools used in this thesis. This idea was later used by Büthe [5, Theorem 1] (with a different weight). He established bounds on $E(x)$ for smaller values of x and improved bounds on $E(x)$ for previously established values of x, for instance, $E(x) \leq 1.12 \times 10^{-10}$ for $x \geq e^{50}$. Finally, Platt and Trudgian proved in 2020:

Theorem 1.1. *([20, Theorem 1]). Let* $R = 5.573412$*. For each row* X, A, B, C, ϵ_0 *from [20, Table 1], we have that, for all* $\log x \geq X$*,*

$$E(x) \leq A\left(\frac{\log x}{R}\right)^B \exp\left(-C\sqrt{\frac{\log x}{R}}\right),$$

and

$$|\psi(x) - x| \leq \epsilon_0 x.$$

For instance, they obtain $(X, A, B, C, \epsilon_0) = (6000, 611.6, 1.51, 1.94, 4.23 \times 10^{-21})$ is valid. New work of Fiori, Kadiri and Swindisky (USRA 2020) [9, Theorem 1.2] claims improvements to $(X, A, B, C, \epsilon_0) = (6000, 135.7, 1.5, 2, 1.349 \times 10^{-22})$.

Now let a and q be fixed co-prime numbers with $q \geq 3$. Let ϕ denote the Euler phi function defined by $\phi(q) = \#\{k \in \mathbb{Z} \mid 1 \leq k \leq q \text{ and } \gcd(q,k) = 1\}$ and let

$$\pi(x; a, q) = \#\{p \leq x \mid p \equiv a \bmod q\}.$$

Dirichlet's prime number theorem for arithmetic progressions states that, for any fixed modulus q,

$$\pi(x; a, q) \sim \frac{1}{\phi(q)} \operatorname{Li}(x), \quad \text{as} \quad x \to \infty.$$

This is equivalent to

$$\lim_{x\to\infty} \frac{\psi(x;a,q)}{\left(\frac{x}{\phi(q)}\right)} = 1, \quad \text{where} \quad \psi(x;a,q) = \sum_{\substack{p^k \leq x, k \geq 1, \\ p^k \equiv a \bmod q}} \log p.$$

Explicit results regarding the primes in arithmetic progression are well-known. Let

$$E(x; a, q) = \left| \frac{\psi(x;a,q) - \frac{x}{\phi(q)}}{\frac{x}{\phi(q)}} \right|.$$

In 1984, McCurley generalized the approach of Rosser and Schoenfeld to primes in arithmetic progressions and obtained the first explicit results regarding primes in arithmetic progressions.

In [17, Theorem 1.2], he obtained bounds for sufficiently large moduli. For instance he proved $E(x;a,q) \leq 11.50$ for $x \geq e^{(\log q)^2}$ and $q \geq 10^{13}$. In [22, Theorem 1], Ramaré and Rumely proved that $E(x;a,q) \leq 0.002$ for $x \geq 10^{100}$ and $q = 13$, and in [14, Theorem 1.1], Kadiri and Lumley using the technique of smooth weights proved that $E(x;a,q) \leq 4.992 \cdot 10^{-6}$ for $x \geq 10^{100}$ and $q = 13$. Both results are actually proven for a finite family of "small" moduli (including $q = 13$) for which the Generalized Riemann Hypothesis (GRH) was partially verified.

In order to state precisely GRH, we first introduce Dirichlet characters. A Dirichlet character, $\chi \pmod q$ is a completely multiplicative function of period q on integers $n \in \mathbb{Z}$ which takes complex roots of unity as values when $\gcd(n,q) = 1$ and is 0 otherwise. For a Dirichlet character $\chi \pmod q$, the Dirichlet L-function associated to it is given by

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1},$$

for $\Re(s) > 1$. Here the product is over all primes. χ_0 is called the principal character if $\chi_0(n) = 1$ for all n with $\gcd(n,q) = 1$ and is 0 otherwise. If we take $q = 1$, then χ_0 is identically 1 and the corresponding L-function is the Riemann ζ-function, $\zeta(s)$. It is conjectured that each Dirichlet L-function satisfy the so-called Generalized Riemann Hypothesis, that is that all its zeros in the critical strip align on $\Re s = 1/2$. It can be proven that they do not vanish on a slim region to the left of $\Re s = 1$, namely that there exists some absolute positive constant R_1 (depending on q) such that $L(s,\chi)$ modulo q has at most one zero in the region $\Re s > 1 - \frac{1}{R_1 \log q}$ and $|\Im s| < 1$. This zero if it exists is real, and is called exceptional (also referred to as Siegel zero). It is another open famous conjecture in analytic number theory that this zero does not exist.

Similar to Theorem 1.1, Bennett et al. proved :

Theorem 1.2. *([4, Lemma 6.10]). Let $R_1 = 5.645908801$. For $q \geq 10^5$ and $x \geq e^{4R_1(\log q)^2}$,*

$$E(x;a,q) \leq 1.012 x^{\beta_0 - 1} + 1.4579\ \phi(q)\sqrt{\frac{\log x}{R_1}} \exp\left(-\sqrt{\frac{\log x}{R_1}}\right),$$

where the term in β_0 is present only if one of the Dirichlet L-function $\pmod q$ possesses an exceptional zero β_0.

The classical tools to prove above theorems are:

1. An explicit formula relating the prime counting functions $\psi(x)$ and $\psi(x; a, q)$ to some sum over the zeros of Riemann ζ-function or Dirichlet L-functions respectively.

2. An estimate of the count for the number of zeros in a specific region of the complex plane.

3. A zero-free region for the corresponding function.

4. Numerical verification of RH / GRH for the first zero. See the latest work of Platt-Trudgian [21] and Platt [19] for current records (for instance if the imaginary part is under 3×10^{12}, then the zeros of $\zeta(s)$ are on the $\Re(s) = 1/2$ line).

All these tools have been adapted to the general number field case recently except the fourth one which is yet to be done.

1.2 Primes in Chebotarev's density theorem

Our next step is to explore prime ideals in number fields. In order to do so, we require some notation and definitions.

Let L/K denote a normal extension of number fields with Galois group, $\mathrm{Gal}(L/K) = G$. Let $\mathcal{O}_K$ denote the ring of integers of the field K. Let N denote the absolute norm of an ideal I in $\mathcal{O}_K$ (i.e., $\mathrm{N}(I) = [\mathcal{O}_K : I]$). A prime ideal $\mathfrak{p}$ in $\mathcal{O}_K$ is said to be unramified in L if the ideal $\mathfrak{p}\mathcal{O}_L$ has a unique decomposition into a product of distinct prime ideals in $\mathcal{O}_L$ with multiplicity 1. For every unramified prime ideal $\mathfrak{p}$ in $\mathcal{O}_K$, let $\sigma_\mathfrak{p}$ denote the Artin symbol at $\mathfrak{p}$. Then Chebotarev's density theorem essentially tells us that the Artin symbols are equidistributed in the set of conjugacy classes of G. More precisely, Chebotarev proved in 1922 that :

Theorem 1.3. *(Chebotarev's density theorem [29]). Let $C \subset G$ be a fixed conjugacy class and denote*

$$\pi_C(x) = \#\{\mathfrak{p} \subset \mathcal{O}_K \mid \mathfrak{p} \text{ is unramified with } \mathrm{N}\,\mathfrak{p} \leq x \text{ and } \sigma_\mathfrak{p} = C\}.$$

Then

$$\pi_C(x) \sim \frac{|C|}{|G|}\,\mathrm{Li}(x).$$

This theorem tells us that for a random prime ideal $\mathfrak{p}$, the probability that $\sigma_\mathfrak{p}$ equals C is $\frac{|C|}{|G|}$. Note that if $K = L = \mathbb{Q}$, Chebotarev's density theorem reduces to the Prime Number Theorem ($C = G = \{1\}$ and $\pi_C(x) = \pi(x)$) and if $K = \mathbb{Q}$ and $L = \mathbb{Q}(\zeta_q)$ where ζ_q is a primitive q-th root of unity, then Chebotarev's density theorem reduces to Dirichlet's theorem for primes in a fixed arithmetic progression (modulo q) ($C = \sigma_a$ and $\pi_C(x) = \pi(x; a, q)$).

Let us introduce $\zeta_L(s)$, the Dedekind ζ-function corresponding to the field L, given by the Dirichlet series

$$\zeta_L(s) = \sum_{I \subseteq \mathcal{O}_L} \frac{1}{(\mathrm{N}(I))^s}, \quad \text{for } \Re(s) > 1,$$

where I runs through the non-zero ideals of the ring of integers, $\mathcal{O}_L$. As the Riemann ζ-function, $\zeta_L(s)$ is expected to satisfy the GRH for Dedekind ζ-function : for every complex number s with $\Re(s) > 0$ and $\zeta_L(s) = 0$, $\Re(s) = 1/2$. In 1974, Stark proved the first zero-free region for $\zeta_L(s)$:

Theorem 1.4. *([27, lemma 4]). If $n_L > 1$, then $\zeta_L(s)$ has at most one zero $\rho = \beta + i\gamma$ in the region*

$$|\gamma| \leq (4 \log d_L)^{-1} \text{ and } \beta \geq 1 - (4 \log d_L)^{-1}, \tag{1.2}$$

and if this zero exists, then it has to be real and simple and is denoted as β_0.

Effective versions of Chebotarev's density theorem provide explicit error terms which depend on field constants, namely on n_L, the degree of extension of L over $\mathbb{Q}$, and on d_L, the absolute value of the discriminant of L. These versions are important since many number theoretic applications depend on the size of error terms involved. In this regard, Serre provided explicit bounds which were conditional on the Generalized Riemann Hypothesis (GRH):

Theorem 1.5. *(Serre [26, Theorem 4]). Let L/K be a normal extension of number fields with $G = \mathrm{Gal}(L/K)$. Let $C \subset G$ be a conjugacy class. Assume the GRH for the Dedekind zeta function, $\zeta_L(s)$. Then there exists an absolute constant $c_1 > 0$ such that*

$$\left| \pi_C(x) - \frac{|C|}{|G|} \mathrm{Li}(x) \right| \leq c_1 \frac{|C|}{|G|} x^{\frac{1}{2}} (\log d_L + n_L \log(x))$$

for all $x \geq 2$.

In 1977, Lagarias and Odlyzko proved the first unconditional version of Chebotarev Density Theorem:

Theorem 1.6. *(Lagarias and Odlyzko [15]). Let L/K be a normal extension of number fields with $G = \mathrm{Gal}(L/K)$. Let $C \subset G$ be a conjugacy class. There exists absolute effectively com-*

putable constants c_2 and c_3 such that if $x \geq \exp(10n_L(\log d_L)^2)$, then

$$\left|\pi_C(X) - \frac{|C|}{|G|}\operatorname{Li}(x)\right| \leq \frac{|C|}{|G|}\operatorname{Li}(x^{\beta_0}) + c_2 x \exp\left(-c_3\sqrt{\frac{\log x}{n_L}}\right),$$

where β_0 term is present only if Dedekind ζ-function, ζ_L has an exceptional zero β_0.

Chebotarev's density theorem is equivalent to

$$\lim_{x\to\infty}\frac{\psi_C(x)}{\left(\frac{|C|}{|G|}x\right)} = 1 \quad \text{where} \quad \psi_C(x) = \sum_{\substack{\mathrm{N}\,\mathfrak{p}^m \leq x \\ \mathfrak{p}\ \text{unramified} \\ \sigma_{\mathfrak{p}}^m = C}} \log(\mathrm{N}\,\mathfrak{p}).$$

Let

$$E_\psi(x) = \left|\frac{\psi_C(x) - \frac{|C|}{|G|}x}{\frac{|C|}{|G|}x}\right|. \tag{1.3}$$

Lagarias and Odlyzko proved that:

Theorem 1.7. *([15, Theorem 9.2]) There exists $c_4, c_5 > 0$ constants such that if $x \geq \exp(4n_L(\log d_L)^2)$, then*

$$E_\psi(x) \leq \frac{x^{\beta_0 - 1}}{\beta_0} + c_4 \exp\left(-c_5\sqrt{\frac{\log x}{n_L}}\right),$$

where β_0 term is present only if the Dedekind ζ-function ζ_L has an exceptional zero β_0.

In his Ph.D (2013), Winckler made every result in the article of Lagarias and Odlyzko [15] explicit, and he proved:

Theorem 1.8. *([30, Theorem 1.2]). Let L/K be a normal extension of number fields with $G = \operatorname{Gal}(L/K)$. Let $C \subset G$ be a conjugacy class. If $x \geq \exp(8n_L(\log(150\ 867 d_L^{44/5}))^2)$, then*

$$\left|\pi_C(X) - \frac{|C|}{|G|}\operatorname{Li}(x)\right| \leq \frac{|C|}{|G|}\operatorname{Li}(x^{\beta_0}) + c_6 x \exp\left(-\frac{1}{99}\sqrt{\frac{\log x}{n_L}}\right),$$

where $c_6 = 783\,846\,699\,796\,966 < 7.84 \times 10^{14}$ and the β_0 term is present only if the Dedekind ζ-function, ζ_L has an exceptional zero β_0.

This follows by partial summation and integration by parts from:

Theorem 1.9. *([30, Theorem 8.2]). Let L/K be a normal extension of number fields with $G = \mathrm{Gal}(L/K)$. Let $C \subset G$ be a conjugacy class. If $x \geq \exp(4n_L(\log(150\ 867 d_L^{44/5}))^2)$,*

$$E_\psi(x) \leq \frac{x^{\beta_0 - 1}}{\beta_0} + c_7 \exp\left(-\frac{7-4\sqrt{3}}{5}\sqrt{\frac{\log x}{n_L}}\right),$$

where $c_7 < 1.51 \times 10^{12}$ and the β_0 term is present only if the Dedekind ζ-function, ζ_L has an exceptional zero β_0. Note that $\frac{7-4\sqrt{3}}{5} = 0.01436\ldots$.

In this thesis, we will only look at the unconditional versions of Chebotarev's density theorem. In Chapter 2 of the thesis, we study Winckler's [30] closely. We were able on one hand to correct some errors and on the other hand to improve some of his results which leads to the improvement of the constants c_6 ad c_7. For instance, we modify Theorem 1.8 and Theorem 1.9 into the following three theorems:

Theorem 1.10. *Let L/K be a normal extension of number fields with $G = \mathrm{Gal}(L/K)$. Let $C \subset G$ be a conjugacy class. If $n_L \geq 2$ and $x \geq \exp\left(8n_L(\log(1\,114\,759\, d_L^{\frac{44}{5}}))^2\right)$, then*

$$\left|\pi_C(x) - \frac{|C|}{|G|}\,\mathrm{Li}(x)\right| < \frac{|C|}{|G|}\,\mathrm{Li}(x^{\beta_0}) + \frac{|C|}{|G|}c_6' x \exp\left(-\frac{1}{99}\sqrt{\frac{\log x}{n_L}}\right), \tag{1.4}$$

where $c_6' = 0.4958$ and the β_0 term is present only if the Dedekind ζ-function, ζ_L has an exceptional zero β_0.

Theorem 1.11. *Let L/K be a normal extension of number fields with $G = \mathrm{Gal}(L/K)$. Let $C \subset G$ be a conjugacy class. If $n_L \geq 2$ and $x \geq \exp\left(8n_L(\log(10\,478\,733\, d_L^{\frac{44}{5}}))^2\right)$, then*

$$\left|\pi_C(x) - \frac{|C|}{|G|}\,\mathrm{Li}(x)\right| < \frac{|C|}{|G|}\,\mathrm{Li}(x^{\beta_0}) + c_6''\frac{|C|}{|G|} x \exp\left(-\frac{1}{99}\sqrt{\frac{\log x}{n_L}}\right),$$

where $c_6'' = 0.4$ and the β_0 term is present only if the Dedekind ζ-function, ζ_L has an exceptional zero β_0.

Theorem 1.12. *Let L/K be a normal extension of number fields with $G = \mathrm{Gal}(L/K)$. Let*

$C \subset G$ be a conjugacy class. If $n_L \geq 2$ and $x \geq \exp\left(4n_L(\log(1\,114\,759\, d_L^{\frac{44}{5}}))^2\right)$, then

$$E_\psi(x) \leq \frac{x^{\beta_0 - 1}}{\beta_0} + c_7' \exp\left(-\frac{7-4\sqrt{3}}{5}\sqrt{\frac{\log x}{n_L}}\right),$$

where $c_7' = 5\,805.17$ and the β_0 term is present only if the Dedekind ζ-function, ζ_L has an exceptional zero β_0.

In Theorem 1.10 and Theorem 1.11, we replace c_6 by $c_6' = 0.4958$ and $c_6'' = 0.4$ which is an improvement by a factor of 10^{15}. The larger x is, the smaller c_6, c_6', c_6'', c_7 and c_7' will become. Moreover, we correct errors in Winckler's work and as a consequence the value 150 867 in Theorem 1.8 should be corrected to 1 114 759 (the same constant that appears in Theorem 1.10). Note that Theorem 1.10 and Theorem 1.11 differ by the choice for one of the key parameters (labelled T_0) which occurs in the proofs. More precisely, in Theorem 1.10, we take $T_0 = 2$ and in Theorem 1.11, we take $T_0 = 44$.

The reasons behind our improvements to Winckler's result, although we followed the same techniques he employed, are explained in Section 2.10. We summarize corrections to Winckler's [30] article in Table 2.1, and improvements in Table 2.2.

In Chapter 3, we prove a new explicit version of Lagarias and Odlyzko's result on Chebotarev's density theorem. Instead of establishing an explicit formula relating $\psi_C(x)$ to the zeros of Dedekind ζ-function, ζ_L, we introduce a smooth weight in the definition of $\psi_C(x)$ and study the approximation $\tilde{\psi}_C(x)$. We appeal to the theory of inverse Mellin transform instead of using a Perron's formula (as used in Lagarias and Odlyzko [15] and Winckler [30]) and prove an original explicit formula for $\tilde{\psi}_C(x)$. We then proceed to study new weighted sums over the zeros of the Dedekind ζ-function. The sums over the zeros can be estimated by using bounds for the number of zeros in a box. We modify the original approach for the sums over the zeros closer to the real line and for the sums over the zeros of the large imaginary part. This leads to a better optimization of one of the key parameters and allows us to establish a result valid for more values of x than in Lagarias and Odlyzko [15] (and consequently of Winckler). For instance, we prove a new result for the number of non-trivial zeros of the Artin L-function, $L(s, \chi, L/E)$, where L/E is an abelian extension and χ is a character of $\mathrm{Gal}(L/E)$, and thus improve [30, Lemma 5.4]. Note that by class field theory, $L(s, \chi, L/E)$ is a Hecke L-function and therefore is holomorphic in the case $\chi \neq 1$.

Proposition 1.13. *Let $0 < \epsilon \leq 1$ and $a > 0$. Let $A(\chi)$ and $\delta(\chi)$ be the non-zero numbers defined in* (2.43) *and* (2.44) *respectively. Let T be a real number and let $n_{\chi,a,\epsilon}(T)$ denote the number of non-trivial zeros $\rho = \beta + i\gamma$ of $L(s,\chi,L/E)$ with $|T-\gamma| \leq a$. Then*

$$n_{\chi,a,\epsilon}(T) \leq n_E(c_1(a,\epsilon)\log(2+\epsilon+|T|) + c_2(a,\epsilon)) + c_1(a,\epsilon)\log A(\chi) + 4c_1(a,\epsilon)\delta(\chi),$$

where

$$c_1(a,\epsilon) = \frac{(1+\epsilon)^2 + a^2}{2\epsilon} \tag{1.5}$$

and

$$c_2(a,\epsilon) = \left(\frac{(1+\epsilon)^2 + a^2}{\epsilon}\right)\left(\frac{1}{\epsilon} + \frac{164}{14}\right). \tag{1.6}$$

As a consequence, we obtain two theorems. The first one has $E_\psi(x)$ dependent on both n_L and d_L:

Theorem 1.14. *Let C be a fixed conjugacy class of the Galois group,* $\mathrm{Gal}(L/K) = G$*. Let $R = 29.57$. Let $m \geq 2$ be an integer. If $\log x \geq 4mRn_L(\log 88 d_L^{1/n_L})^2$, then*

$$E_\psi(x) \leq \frac{x^{\beta_0 - 1}}{\beta_0} + \epsilon_1(m,x,n_L,d_L),$$

with

$$\epsilon_1(m,x,n_L,d_L) = \lambda(m)\max\left\{(\log d_L)n_L^{-\frac{1}{m+1}}, D^{\frac{m}{m+1}}\right\}(\log x)^{\frac{1}{m+1}}\exp\left(-\frac{2m^{\frac{1}{2}}}{m+1}\sqrt{\frac{\log x}{Rn_L}}\right) \tag{1.7}$$

where λ is defined in (3.163) *and the β_0 term is present only if the Dedekind ζ-function, ζ_L has an exceptional zero β_0.*

For the second one, we remove the dependence of d_L in $E_\psi(x)$:

Theorem 1.15. *Let C be a fixed conjugacy class of the Galois group,* $\mathrm{Gal}(L/K) = G$*. Let $R = 29.57$. Let $m \geq 2$ be an integer. If $\log x \geq 4mRn_L(\log 88 d_L^{1/n_L})^2$, then*

$$E_\psi(x) \leq \frac{x^{\beta_0 - 1}}{\beta_0} + \epsilon_2(m,x,n_L),$$

with

$$\epsilon_2(m,x,n_L) = \nu(m)\; n_L^{1-\frac{1}{m+1}} (\log x)^{\frac{1}{m+1}} \exp\left(-\frac{1.5 m^{\frac{1}{2}}}{m+1}\sqrt{\frac{\log x}{R n_L}}\right) \tag{1.8}$$

where ν is defined in (3.167) *and the β_0 term is present only if the Dedekind ζ-function, ζ_L has an exceptional zero β_0.*

As a corollary to Theorem 1.15 (taking $m = 2$), we have

Corollary 1.16. *Under the assumptions in Theorem 1.15, we have*

$$E_\psi(x) \le \frac{x^{\beta_0 - 1}}{\beta_0} + A_1 n_L^{\frac{2}{3}} (\log x)^{\frac{1}{3}} \exp\left(-0.13\sqrt{\frac{\log x}{n_L}}\right) \text{ for all } \log x \ge 19\,810\,\frac{(\log d_L)^2}{n_L},$$

where $A_1 = 0.0396$ if β_0 exists and 0.0249 otherwise.

We then deduce an explicit bound for $\pi_C(x)$ from $\psi_C(x)$:

Theorem 1.17. *Let C be a fixed conjugacy class of the Galois group, $\mathrm{Gal}(L/K) = G$. Let β_0 be the possible exceptional real zero of $\zeta_L(s)$. Then*

$$\left|\pi_C(x) - \frac{|C|}{|G|}\,\mathrm{Li}(x)\right| \le \frac{|C|}{|G|}\,\mathrm{Li}(x^{\beta_0}) + E_0 \frac{|C|}{|G|} n_L x \exp\left(-0.0919\sqrt{\frac{\log x}{n_L}}\right),$$

for all

$$\log x \ge 39\,620\frac{(\log d_L)^2}{n_L}$$

where $E_0 = 4.714 \times 10^{-6}$ if β_0 exists and 2.97×10^{-6} otherwise.

Corollary 1.18. *Under the assumptions in Theorem 1.17, we have*

$$\left|\pi_C(x) - \frac{|C|}{|G|}\,\mathrm{Li}(x)\right| \le \frac{|C|}{|G|}\,\mathrm{Li}(x^{\beta_0}) + E_1 \frac{|C|}{|G|} x \exp\left(-F_1\sqrt{\frac{\log x}{n_L}}\right)$$

for all

$$\log x \ge G_1 n_L (\log d_L)^2$$

where $E_1 = 1.65 \times 10^{-5}$, $F_1 = 0.09$ and $G_1 = 9\,906$. Another admissible value for (E_1, F_1, G_1) is $(1.23 \times 10^{-9}, 1/99, 9\,906)$.

Comparison: Notice that Winckler's result as shown in Theorem 1.8 gives $(E_1, F_1, G_1) = (7.84 \times 10^{14}, 1/99, 3\,090\,)$. Our corrections leads to $(E_1, F_1, G_1) = (0.4, 1/99, 16\,006)$ as shown in Theorem 1.11 and a new result for Corollary 1.18 gives $(1.23 \times 10^{-9}, 1/99, 9\,906)$. Therefore, we improve both the error term (factor of 10^{24}) and the range (factor of 1.6). Notice that the formula for range of $\log x$ given in Theorem 1.17 has n_L in the denominator whereas both Theorem 1.11 and Theorem 1.8 have n_L in the numerator. Therefore, as the degree of the field increases, our Theorem 1.17 improves the range of $\log x$ by a factor of n_L^2. A more detailed comparison with Winckler's result is given in Section 3.10.

Finally Table 1.1 lists best known explicit versions to Lagarias and Odlyzko's [15] lemmas used to establish their main theorem.

Table 1.1: Making Lagarias and Odlyzko explicit

Results from Lagarias and Odlyzko [15]	Best explicit version of their results
Lemma 5.2	[30, Lemma 4.3]
Lemma 5.3	[30, Lemma 4.5]
Lemma 5.4	Proposition 1.13
Lemma 5.5	Lemma 3.21
Lemma 5.6	Lemma 3.20
Lemma 6.1	[30, Lemma 4.4]
Lemma 6.2	Lemma 2.19
Lemma 6.3	Lemma 2.23
Lemma 8.1	Lemma 2.32
Lemma 8.2	[2, Theorem 1]
Theorem 9.2	Theorem 1.15

1.3 Compilation of ideas

Next, we mention the tools, ideas and motivations which form the basis of chapter 3.

- First of all, the idea for using smooth weight has been derived from the works of Faber and Kadiri in [8] who used smooth approximation of the prime counting functions in their work. In chapter 3, we introduce a smooth weight h as in (3.10) approximating the identity on $[0, x]$ and obtain a smooth version of $\psi_C(x)$, denoted as $\tilde{\psi}_C(x)$ as in (3.21). We introduce the smooth sum over all primes ideal, $I_{L/K}(x)$ as in (3.26) and relate it to $\tilde{\psi}_C(x)$ as in (3.25).

- Then using the inverse Mellin transform of h, we express $I_{L/K}(x)$ as a sum over Artin

L-functions as in (3.5.1). Since the holomorphicity of the Artin L-functions is yet to be proved, we use Deuring's reduction to express $I_{L/K}(X)$ as a sum over Hecke L-functions over an intermediate field (which are holomorphic) as in (3.5.1). This allows us to use Cauchy's residue theorem to obtain an explicit formula for the sum $I_{L/K}(x)$ which involves sums over the zeros of the Dedekind ζ-function, $\zeta_L(s)$ as shown in Proposition 3.16.

- Next we split this sum over zeros into four different regions as in (3.63), (3.64), (3.65), (3.66) and bound each one separately. This approach is different from Lagarias and Odlyzko's approach in [15] where they divided the sum over the zeros into three different sums. In particular, we add an extra region to better estimate the sum over the low-lying zeros (zeros close to real line).

- When bounding the sum over the zeros of $\zeta_L(s)$, two of the most important tools we require are :

  1. Zero-free regions (regions which can not contain a single zero of $\zeta_L(s)$). Inspiration for this comes from the works of Kadiri [13] on explicit zero-free regions for Riemann ζ-function generalized by Ahn and Kwon [2][3] to Dedekind ζ-function.
  2. Zero-density formulas (estimates for the number of zeros in a particular region). Recently, Hasanalizade et al. in [11] improved the bound given by Trudgian [28]. We use their result in this thesis as Theorem 3.10.

- Moreover, since there is no verification of GRH for number fields, a more cautious approach is taken to bound the sums over the low lying zeros of the Hecke L-functions. We prove a new result for the number of non-trivial zeros $\rho = \beta + i\gamma$ of the Hecke L-function, $L(s, \chi)$ with $|T - \gamma| \leq a$ denoted as $n_{\chi,a,\epsilon}(T)$ where T is any real number and ϵ is a constant taken close to 0. This is shown in Proposition 1.13 in this thesis. Note that $n_{\chi,a,\epsilon}(T)$ generalizes $n_\chi(T)$ as in [15, Lemma 5.4] which counts the number of non-trivial zeros $\rho = \beta + i\gamma$ of $L(s, \chi)$ with $|T - \gamma| \leq 1$. Following this, we also prove a new bound for the logarithmic derivative of $L(s, \chi)$ in a restricted range as given in Lemma 3.20.

- As mentioned earlier, to bound the sum over the non-trivial zeros above an arbitrary height $T \geq 44$, we use the zero density formula as proved by Hasanalizade et al. in [11] and additionally generalize the techniques used by Fiorelli and Martin in [10] for Dirichlet L-functions. This results in obtaining integrals which are called Bessel integrals and are given in (3.129) and (3.130).

- To compute the required bounds for the remaining sum over the non-trivial zeros of the Dedekind ζ-functions, we use types of incomplete Bessel functions as already introduced by Rosser and Schoenfeld [24], and recently studied in more detail by Kadiri and Lumley in [14] and by Bennett et al. in [4].

- Finally, we choose an appropriate T as in (3.148) to obtain the error term E_ψ as shown in Theorem 1.14 and Theorem 1.15. Our choice is different from Lagarias and Odlyzko [15] and from Winckler [30]. This allows an improvement on the range for $\log x$ for both papers. Then, using partial summation and integration by parts, we obtain Theorem 1.17 for $\pi_C(x)$.

1.4 Notation

This section introduces notation and definitions which are used in this thesis.

- Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ be the set of natural numbers, integers, real numbers and complex numbers respectively.

- Let d_F denote the absolute value of the discriminant of the number field F.

- Let n_F denote the degree of extension of F over $\mathbb{Q}$, $[F : \mathbb{Q}]$.

- Let $\mathcal{O}_F$ be the ring of integers of the number field F.

- Let N denote the absolute norm of an ideal I in $\mathcal{O}_F$ (i.e., $\mathrm{N}(I) = [\mathcal{O}_F : I]$).

- Let $\zeta_F(s)$ denote the Dedekind ζ-function corresponding to the field F and is defined for complex numbers s with real part $\Re(s) > 1$ by the Dirichlet series

$$\zeta_F(s) = \sum_{I \subseteq \mathcal{O}_F} \frac{1}{(\mathrm{N}(I))^s},$$

  where I denotes the non-zero ideals of the ring of integers, $\mathcal{O}_F$.

- Let L/K be a normal extension of number fields.
- Let G denote the Galois group, $\mathrm{Gal}(L/K)$ and let C denote a conjugacy class of G.
- Let $\rho : G \to \mathbb{GL}_n(\mathbb{C})$ be a group representation.
- Let $\mathfrak{p}$ denote a prime ideal in $\mathcal{O}_K$.
- Let $\mathfrak{q}$ be a prime ideal in $\mathcal{O}_L$ such that $\mathfrak{q}$ lies over $\mathfrak{p}$.
- Let $\sigma_{\mathfrak{p}}$ denote the Artin symbol at $\mathfrak{p}$.
- Let $L(s, \rho, L/K)$ denote the Artin L-function attached to ρ.
- Let ϕ denote the character of ρ and given by $\phi = \mathrm{tr}\,\rho$. We also write $L(s, \phi, L/K)$ to denote $L(s, \rho, L/K)$. Fixing L and K, we denote $L(s, \phi)$ for $L(s, \phi, L/K)$.
- Let C be a conjugacy class of $G = \mathrm{Gal}(L/K)$, $g \in C$, $G_0 =< g >$ be the cyclic group generated by g, E be the fixed field of G_0, and χ denote any irreducible character of G_0. Let $L(s, \chi, L/E)$ or $L(s, \chi)$ denote the Hecke L-function related to the character χ.
- We say f is big oh of g and denote it as $f(x) = \mathcal{O}(g(x))$ or $f(x) \ll g(x)$ if there is a positive real number c such that for all sufficiently large x, $|f(x)| \leq cg(x)$.
- The prime ideal counting functions studied in this thesis are:

$$\pi_C(x) = \sum_{\substack{\mathfrak{p} \text{ unramified} \\ \mathrm{N}\,\mathfrak{p} \leq x,\ \sigma_{\mathfrak{p}} = C}} 1, \quad \psi_C(x) = \sum_{\substack{\mathfrak{p} \text{ unramified} \\ \mathrm{N}\,\mathfrak{p}^m \leq x,\ \sigma_{\mathfrak{p}}^m = C}} \log(\mathrm{N}\,\mathfrak{p})$$

$$\theta_C(x) = \sum_{\substack{\mathfrak{p} \text{ unramified} \\ \mathrm{N}\,\mathfrak{p} \leq x,\ \sigma_{\mathfrak{p}} = C}} \log(\mathrm{N}\,\mathfrak{p}) \text{ and } \theta_0(x) = \sum_{\substack{\mathfrak{p} \text{ unramified} \\ \mathrm{N}\,\mathfrak{p} \leq x}} \log(\mathrm{N}\,\mathfrak{p}).$$

- Let h be a chosen smooth weight with its corresponding Mellin transform H defined as

$$H(s) = \int_0^\infty h(t) t^{s-1}\, dt.$$

- Let θ characterize the Artin symbol at $\mathfrak{p}$ coinciding with the conjugacy class C. More specifically, for $\mathfrak{p}$ unramified in L, we have

$$\theta(\mathfrak{p}^m) = \begin{cases} 1 & \text{if } \sigma_{\mathfrak{p}}^m = C, \\ 0 & \text{otherwise,} \end{cases}$$

and $|\theta(\mathfrak{p}^m)| \leq 1$ if $\mathfrak{p}$ ramifies in L.

- The smooth version of the prime ideal counting functions studied in the thesis are:

$$\begin{aligned}
\tilde{\psi}_C(x) &= \sum_{\substack{\mathfrak{p} \text{ unramified} \\ \sigma_{\mathfrak{p}}^m = C}} \sum_{m \geq 1} (\log(\mathrm{N}\,\mathfrak{p})) h\left(\frac{\mathrm{N}\,\mathfrak{p}^m}{x}\right) \\
I_{L/K}(x) &= \sum_{\mathfrak{p}} \sum_{m \geq 1} \theta(\mathfrak{p}^m) (\log(\mathrm{N}\,\mathfrak{p})) h\left(\frac{\mathrm{N}\,\mathfrak{p}^m}{x}\right) \\
\tilde{I}_{L/K}(x) &= \sum_{\mathfrak{p} \text{ ramified}} \sum_{m \geq 1} \theta(\mathfrak{p}^m) (\log(\mathrm{N}\,\mathfrak{p})) h\left(\frac{\mathrm{N}\,\mathfrak{p}^m}{x}\right).
\end{aligned}$$

- Let E_ψ denote the error term defined as

$$E_\psi(x) = \left| \frac{\psi_C(x) - \frac{|C|}{|G|}x}{\frac{|C|}{|G|}x} \right|,$$

and its smooth version, $E_{\tilde{\psi}}$ is defined as

$$E_{\tilde{\psi}}(x) = \left| \frac{\tilde{\psi}_C(x) - \frac{|C|}{|G|}x}{\frac{|C|}{|G|}x} \right|.$$

The following are related to the Dedekind ζ-function, $\zeta_L(s)$ and the Hecke L-function, $L(s, \chi)$:

- For $\beta, \gamma \in \mathbb{R}$, $\rho = \beta + i\gamma$ denote the zeros of $\zeta_L(s)$ or $L(s, \chi)$.
- Let β_0 denote the possible real exceptional zero of $\zeta_L(s)$.
- $Z(\zeta)$ denotes the set of non-trivial zeros of $\zeta_L(s)$, i.e.,

$$Z(\zeta) = \{\rho = \beta + i\gamma \mid \zeta_L(\rho) = 0,\ 0 < \beta < 1\}.$$

- $Z(\chi)$ denotes the set of non-trivial zeros of $L(s,\chi)$, i.e.,

$$Z(\chi) = \{\rho = \beta + i\gamma \mid L(\rho,\chi) = 0,\ 0 < \beta < 1\}.$$

- The zero counting functions used in the thesis are : For $t \in \mathbb{R}$ and $a > 0$,

$$n_\chi(t) = \#\{\rho = \beta + i\gamma \mid L(\rho,\chi) = 0,\ 0 < \beta < 1,\ |\gamma - t| \le 1\} \text{ and}$$
$$n_{\chi,a,\epsilon}(t) = \#\{\rho = \beta + i\gamma \mid L(\rho,\chi) = 0,\ 0 < \beta < 1,\ |\gamma - t| \le a\},$$

where ϵ is some real number taken close to 0 and for $T \ge 1$,

$$N_L(T) = \#\{\rho = \beta + i\gamma \mid \zeta_L(\rho) = 0,\ 0 < \beta < 1,\ |\gamma| \le T\}.$$

- We define the sums over the zeros, $\rho = \beta + i\gamma$ of the Dedekind ζ-function, $\zeta_L(s)$ as: Let $T \ge 1$ and $\alpha_4, R_L, D > 0$, then

$$\begin{aligned}
J^{(3)}(x) &= \sum_{\rho \ne 1-\beta_0, |\rho| < \frac{1}{2}} \left| x^\rho H(\rho) - \frac{1}{\rho} \right|, \\
J^{(4)}(x) &= \sum_{\rho \ne \beta_0, |\rho| \ge \frac{1}{2}, |\gamma| \le \frac{1}{\alpha_4 \log d_L}} x^{\beta-1} |H(\rho)|, \\
J^{(5)}(x,T) &= \sum_{|\rho| \ge \frac{1}{2}, \frac{1}{\alpha_4 \log d_L} < |\gamma| < T} x^{\beta-1} |H(\rho)|, \\
J^{(6)}(x,T) &= \sum_{|\gamma| \ge T} x^{\beta-1} |H(\rho)|, \\
S^{(1)}(m,T) &= \sum_{|\gamma| \ge T} \frac{1}{|\gamma|^{m+1}} \\
S^{(2)}(m,T,x) &= \sum_{|\gamma| \ge T} \frac{x^{-\frac{1}{R_L \log(D|\gamma|)}}}{|\gamma|^{m+1}}
\end{aligned}$$

Other special functions and notation used in the thesis:

- Let $M(\delta, m)$ be the function defined by:

$$M(\delta, m) = \max_{\alpha = 1-\delta, \delta} M(\alpha, \alpha + \delta, m)$$

where

$$M(\alpha, \alpha+\delta, m) = \int_0^1 |g^{(m+1)}(u)|(\delta u+\alpha)^{m+1} du,$$

with $0 < \delta < 0.01$, $\alpha = 1-\delta$, 1 and g being the smooth function h compressed to $[0,1]$. M_R and M_{FK} denote the function M corresponding to the smooth functions g_R and g_{FK}.

- For $m \in \mathbb{N}$, $m \geq 2$, $R_L > 0$ and $T \geq 1$, we denote

$$X_{m,T} = (m+1)R_L \log^2(DT), \text{ and } T_1 = \begin{cases} T & \text{if } \log x \leq X_{m,T}, \\ W = \frac{1}{D}\exp\left(\sqrt{\frac{\log x}{R_L(m+1)}}\right) & \text{if } \log x > X_{m,T}. \end{cases}$$

- For $\alpha_1, \alpha_2, \alpha_3, D > 0$, we define the function Q as

$$Q(t,u) = \frac{n_L u}{\pi}\log\left(\frac{Du}{4\pi e}\right) - \frac{n_L t}{\pi}\log\left(\frac{Dt}{4\pi e}\right) + 2\alpha_1 n_L\left(\log\frac{D\sqrt{ut}}{2}\right) + 2\alpha_2 n_L + 2\alpha_3.$$

- Given positive real numbers n, m, α, β and l, we define an incomplete modified Bessel function of the first kind as

$$I_{n,m}(\alpha,\beta;l) = \int_l^\infty \frac{(\log \beta u)^{n-1}}{u^{m+1}}\exp\left(-\frac{\alpha}{\log \beta u}\right)du.$$

- Given positive constants n, z, and y, we consider a variant of incomplete Bessel function of the second kind as

$$K_n(z;y) = \frac{1}{2}\int_y^\infty v^{n-1}\exp\left(-\frac{z}{2}\left(v+\frac{1}{v}\right)\right)dv.$$

- For $x > 0$, the complimentary error function, $erfc$ is given by

$$erfc(u) = \frac{2}{\sqrt{\pi}}\int_u^\infty e^{-t^2}dt.$$

Chapter 2

Studying different versions of Chebotarev's density theorem [15] and [30]

In this chapter, we give a survey of articles [15] and [30] and improve the results in [30]. As a consequence, we derive an asymptotic formula with an explicit error term for a weighted prime power counting function

$$\psi_C(x) = \psi_C(x, L/K) = \sum_{\substack{\mathrm{N}\,\mathfrak{p}^m \leq x \\ \mathfrak{p} \text{ unramified} \\ \sigma_{\mathfrak{p}}^m = C}} \log(\mathrm{N}\,\mathfrak{p}),$$

and prove the following theorem :

Theorem 1.12. *Let β_0 be the possible exceptional real zero of $\zeta_L(s)$, and χ_0 be the character (real) such that the L-function $L(\beta_0, \chi_0) = 0$. If $n_L \geq 2$ and $x \geq \exp\left(4n_L(\log(1\,114\,759\, d_L^{\frac{44}{5}}))^2\right)$, then*

$$\left|\psi_C(x) - \frac{|C|}{|G|}x + \frac{|C|}{|G|}\chi_0(g)\frac{x^{\beta_0}}{\beta_0}\right| \leq \frac{|C|}{|G|}\epsilon_\psi(x), \tag{2.1}$$

where $\epsilon_\psi(x) = 5805.17x \exp\left(-\frac{7-4\sqrt{3}}{5}\sqrt{\frac{\log x}{n_L}}\right)$ and the third term in the left hand side of (2.1) can be suppressed in the absence of the exceptional zero β_0.

The L-function mentioned in this theorem are introduced and studied in Section 2.1.

In general, this chapter can be summarized as the study of the five following steps which will be explained in detail:

1. $\psi_C(x)$ differs from a truncated inverse Mellin transform $I_C(x, \sigma_0, T)$ defined in (2.8) by a remainder term $R_1(x, \sigma_0, T)$ defined in 2.11 which is shown in Lemma 2.3. Further $R_1(x, \sigma_0, T)$ is bounded in Section 2.3.

2. $I_C(x, \sigma_0, T)$ can in fact be reduced to a linear combination of logarithmic derivatives of Hecke (abelian) L-functions using During reduction as shown in Section 2.4 and (2.42).

3. $I_C(x, \sigma_0, T)$ is expressed as a sum of $I_\chi(x, T)$'s defined in (2.60) which differs from a certain contour integral $I_\chi(x, T, U)$ defined in (2.61) by a remainder term $R_\chi(x, T, U)$ which is shown in Section 2.6. This step is traditionally labelled " shifting the line of integration to the left ". Certain results on the density of zeros of $\zeta_L(s)$ in the critical strip $0 < \mathrm{Re}(s) < 1$ are required to estimate $R_\chi(x, T, U)$.

4. The contour integral $I_\chi(x, T, U)$ is evaluated by Cauchy's residue theorem. The integrand has poles at the zeros and the poles of $\zeta_L(s)$, and the result is a main term $\frac{|C|}{|G|}x$ coming from the pole of $\zeta_L(s)$ at $s = 1$, together with a certain sum $S(x, T)$ over the zeros of $\zeta_L(s)$ within the contour $B_{T,U}$. The end result of these steps is a truncated explicit formula for $\psi_C(x)$ with an unconditional error term, which is stated as Theorem 2.29 and Theorem 2.30 depending on the position of T. This step is explained in detail in Section 2.7.

5. Finally Section 2.9 gives the required explicit estimates. The asymptotic formula $\psi_C(x) \sim \frac{|C|}{|G|}x$ with an explicit remainder term is derived by making an appropriate choice of T as a function of x, to minimize the accumulated error terms as shown in Theorem 1.12. The asymptotic formula $\pi_C(x) \sim \frac{|C|}{|G|}\,\mathrm{Li}(x)$ with an explicit remainder term is derived by partial summation from that for $\psi_C(x)$ as shown in Theorem 1.10.

Before commencing with the proof, we introduce Artin L-functions.

2.1 Artin L-functions

In this section, we give a definition for Artin L-function. This requires a lot of notation. Let L/K be a normal extension of number fields with the Galois group, $\mathrm{Gal}(L/K) = G$. Let $\rho : G \to \mathbb{GL}_n(\mathbb{C})$ be a group representation. Attached to this representation is a meromorphic L-function originally defined by Artin. We shall define this function, giving all required details.

Let d_L and d_K denote the absolute values of the discriminants of L and K, respectively, and let $n_L = [L : \mathbb{Q}]$ and $n_K = [K : \mathbb{Q}]$. Let $\mathcal{O}_L$ and $\mathcal{O}_K$ be the ring of integers of the number fields L and K respectively. Let $\mathfrak{p}$ be a prime ideal in $\mathcal{O}_K$. Let $\mathfrak{q}$ be a prime ideal in $\mathcal{O}_L$ such that $\mathfrak{q}$

lies over $\mathfrak{p}$ (denoted as $\mathfrak{q}|\mathfrak{p}$). Define the decomposition group as

$$D_{\mathfrak{q}} = \{\sigma \in G \mid \sigma(\mathfrak{q}) = \mathfrak{q}\}.$$

We know that $\mathcal{O}_K/\mathfrak{p}$ can be identified with a finite field $\mathbb{F}_q$ for some $q = p^m$ where p is prime and m is a positive integer. Furthermore, if $[\mathcal{O}_L/\mathfrak{q} : \mathcal{O}_K/\mathfrak{p}] = f$, then we can think of $\mathcal{O}_K/\mathfrak{q}$ as $\mathbb{F}_{q^f}$. Therefore we see that $\mathrm{Gal}(\mathcal{O}_L/\mathfrak{q}/\mathcal{O}_K/\mathfrak{p})$ is isomorphic to $\mathrm{Gal}(\mathbb{F}_{q^f}/\mathbb{F}_q)$. By Galois theory, we get that the group $\mathrm{Gal}(\mathbb{F}_{q^f}/\mathbb{F}_q)$ is cyclic of order f and is generated by the element $\tau_q : x \to x^q$ for $x \in \mathbb{F}_{q^f}$. Using the isomorphism of groups, we may assume τ_q to be an element of the group $\mathrm{Gal}(\mathcal{O}_L/\mathfrak{q}/\mathcal{O}_K/\mathfrak{p})$.

We now check that there exists a canonical map from $D_{\mathfrak{q}}$ to $\mathrm{Gal}(\mathcal{O}_L/\mathfrak{q}/\mathcal{O}_K/\mathfrak{p})$ which is defined by sending $\sigma \to \overline{\sigma}$ where $\overline{\sigma}(x + \mathfrak{q}) = \sigma(x) + \mathfrak{q}$. We define the inertia group to be $I_{\mathfrak{q}} = \ker(D_{\mathfrak{q}} \to \mathrm{Gal}(\mathcal{O}_L/\mathfrak{q}/\mathcal{O}_K/\mathfrak{p}))$. $I_{\mathfrak{q}}$ can also be described as

$$I_{\mathfrak{q}} = \{\sigma \in G \mid \sigma(x) \equiv x \pmod{\mathfrak{q}},\ \forall x \in \mathcal{O}_L\}.$$

Therefore, we obtain a canonical isomorphism

$$D_{\mathfrak{q}}/I_{\mathfrak{q}} \cong \mathrm{Gal}(\mathcal{O}_L/\mathfrak{q}/\mathcal{O}_K/\mathfrak{p}).$$

Therefore, we can now choose an element $\sigma_{\mathfrak{q}} \in D_{\mathfrak{q}}/I_{\mathfrak{q}}$ whose image in $\mathrm{Gal}(\mathcal{O}_L/\mathfrak{q}/\mathcal{O}_K/\mathfrak{p})$ is the generator τ_q as described above. Such an element $\sigma_{\mathfrak{q}}$ is called a Frobenius automorphism at $\mathfrak{q}$ and it is only well-defined modulo $I_{\mathfrak{q}}$. Notice that $\sigma_{\mathfrak{q}}$ is no longer an element of the Galois group G but a coset of $I_{\mathfrak{q}}$.

Now if $\mathfrak{p}$ is unramified, then one can show that $I_{\mathfrak{q}}$ is a trivial group for every $\mathfrak{q}|\mathfrak{p}$. Besides, since there are finitely many ramified prime ideals in $\mathcal{O}_K$, one can deduce that all but finitely many $I_{\mathfrak{q}}$ are trivial for $\mathfrak{q}|\mathfrak{p}$. Also, for unramified primes $\mathfrak{p}$, we can show that $\mathfrak{q}$ ranges over the prime ideals above $\mathfrak{p}$ and the $\sigma_{\mathfrak{q}}$'s form a conjugacy class. This class is called the Artin symbol at $\mathfrak{p}$, denoted as $\sigma_{\mathfrak{p}}$ and explicitly defined as

$$\sigma_{\mathfrak{p}} = \{\sigma_{\mathfrak{q}} \mid \mathfrak{q} \text{ divides } \mathfrak{p}\}.$$

Using the above theory and representations of finite groups, Artin introduced his L-functions that generalize Dirichlet L-function as follows. The Artin L-function attached to ρ is defined by

$$L(s, \rho, L/K) \;=\; \prod_{\mathfrak{p}} L_{\mathfrak{p}}(s, \rho, L/K),$$

where the local L-function at $\mathfrak{p}$ is defined as

$$L_{\mathfrak{p}}(s, \rho, L/K) = \det\left(I - \rho|^{V^{I_{\mathfrak{q}}}}(\sigma_{\mathfrak{q}})\,\mathrm{N}\,\mathfrak{p}^{-s}\right)^{-1}$$

for $\mathrm{Re}(s) > 1$. Here, the product runs over prime ideals in $\mathcal{O}_K$, N denotes the norm of a non-zero ideal, $\mathfrak{q}$ denotes a prime ideal above $\mathfrak{p}$, $V^{I_{\mathfrak{q}}} \;=\; \{v \in V \mid \rho(g)v = v \text{ for all } g \in I_{\mathfrak{q}}\}$, and for our case $V = \mathbb{C}$. Sometimes, we also write $L(s, \phi, L/K)$ for $L(s, \rho, L/K)$, where $\phi = \mathrm{tr}\,\rho$ denotes the character of ρ. Also, once we fix the fields K and L, we abbreviate $L(s, \rho, L/K)$ to $L(s, \rho)$. One can easily show that

$$L(s, \phi_1 + \phi_2) \;=\; L(s, \phi_1)\; L(s, \phi_2),$$

for any characters ϕ_1 and ϕ_2 of G.

2.2 Formula for $\psi_C(x)$

In this section, we will derive an explicit formula for $\psi_C(x)$. The proof follows the classical arguments for $\psi(x)$ as outlined in Davenport's book [6, Chapter 17].

Let ϕ be a irreducible character of $G = Gal(L/K)$. Let us define

$$\phi_K(\mathfrak{p}^m) = \frac{1}{|I|}\sum_{\alpha \in I} \phi(\tau^m \alpha), \tag{2.2}$$

where I is the inertia group of $\mathfrak{q}$, one of the prime ideal factors of $\mathfrak{p}$, and τ is one of the Frobenius automorphism corresponding to $\mathfrak{p}$.

If $L(s, \phi, L/K)$ is the Artin L-series associated to ϕ, then from [18, Proposition 2.3.1], we get that for $\mathrm{Re}(s) > 1$,

$$\log L(s, \phi, L/K) = \sum_{\mathfrak{p}, m} \frac{\phi_K(\mathfrak{p}^m)}{m(\mathrm{N}\,\mathfrak{p})^{ms}}. \tag{2.3}$$

Hence taking the derivatives on both side with respect to s, we get

$$-\frac{L'}{L}(s,\phi,L/K) = \sum_{\mathfrak{p},m} \phi_K(\mathfrak{p}^m)\frac{\log(\mathrm{N}\,\mathfrak{p})}{(\mathrm{N}\,\mathfrak{p})^{ms}},$$

where the outer sum is over all the prime ideals of K. To single out those $\mathfrak{p}^m$ with $\sigma_{\mathfrak{p}}^m = C$, we will use the characters ϕ. This is done as the following:-

Suppose that $g \in C$. We define a function $f_C : G \to \mathbb{C}$ by

$$f_C(h) = \sum_{\phi} \overline{\phi}(g)\phi(h), \tag{2.4}$$

where $\overline{\phi}$ is the complex conjugate of ϕ. By the orthogonality relation of characters, we get

$$f_C(h) = \begin{cases} |C_G(h)| & \text{if } h \in C, \\ 0 & \text{otherwise,} \end{cases} \tag{2.5}$$

where $|C_G(h)|$ denotes the centralizer of h in G. Hence if

$$F_C(s) = -\frac{|C|}{|G|}\sum_{\phi} \overline{\phi}(g)\frac{L'}{L}(s,\phi,L/K), \tag{2.6}$$

then for $\mathrm{Re}(s) > 1$, we have the Dirichlet series expansion

$$\begin{aligned} F_C(s) = -\frac{|C|}{|G|}\sum_{\phi}\overline{\phi}(g)\frac{L'}{L}(s,\phi,L/K) &= \frac{|C|}{|G|}\sum_{\phi}\overline{\phi}(g)\sum_{\mathfrak{p},m}\phi_K(\mathfrak{p}^m)\log(\mathrm{N}\,\mathfrak{p})(\mathrm{N}\,\mathfrak{p})^{-ms} \\ &= \frac{|C|}{|G|}\sum_{\phi}\overline{\phi}(g)\sum_{\mathfrak{p},m}\frac{1}{|I|}\sum_{\alpha\in I}\phi(\tau^m\alpha)\log(\mathrm{N}\,\mathfrak{p})(\mathrm{N}\,\mathfrak{p})^{-ms} \\ &= \sum_{\mathfrak{p},m}\left(\frac{|C|}{|I||G|}\sum_{\substack{\phi \\ \alpha\in I}}\overline{\phi}(g)\phi(\tau^m\alpha)\right)\log(\mathrm{N}\,\mathfrak{p})(\mathrm{N}\,\mathfrak{p})^{-ms}. \\ &= \sum_{\mathfrak{p},m}\theta(\mathfrak{p}^m)\log(\mathrm{N}\,\mathfrak{p})(\mathrm{N}\,\mathfrak{p})^{-ms}, \end{aligned} \tag{2.7}$$

where for $\mathfrak{p}$ unramified in L, we have

$$\theta(\mathfrak{p}^m) = \frac{|C|}{|I||G|} \sum_{\substack{\phi \\ \alpha \in I}} \overline{\phi}(g)\phi(\tau^m \alpha) = \begin{cases} 1 & \text{if } \sigma_{\mathfrak{p}}^m = C, \\ 0 & \text{otherwise,} \end{cases}$$

and $|\theta(\mathfrak{p}^m)| \leq 1$ if $\mathfrak{p}$ ramifies in L. Notice that if $\mathfrak{p}$ unramified in L and $\sigma_{\mathfrak{p}}^m = C$, then $I = \{1\}$ and $\sum_\phi \overline{\phi}(g)\phi(\tau^m) = |C_G(\tau^m)| = \frac{|G|}{|C|}$ which yields $\theta(\mathfrak{p}^m) = 1$. Now (2.7) shows us that except for the ramified prime factors, $\psi_C(x)$ is a partial sum of the coefficients of $F_C(s)$. Now let $\sigma_0 > 1, x \geq 2$ and define

$$I_C(x, \sigma_0, T) = \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} F_C(s) \frac{x^s}{s} ds. \tag{2.8}$$

2.2.1 Difference between $\psi_C(x)$ and $I_C(x, \sigma_0, T)$

Lagarias and Odlyzko in [15, (3.8)-(3.11)] proved that for $\text{Re}(s) > 1$ and $T > 0$,

$$\left| I_C(x, \sigma_0, T) - \sum_{\substack{\mathfrak{p}, m \\ \mathrm{N}(\mathfrak{p}^m) \leq x}} \theta(\mathfrak{p}^m) \log(\mathrm{N}\,\mathfrak{p}) \right| \leq n_K \sigma_0 T^{-1} + n_K(\log x) + R_0(x, \sigma_0, T),$$

and

$$\left| \sum_{\substack{\mathfrak{p}, m \\ \mathrm{N}(\mathfrak{p}^m) \leq x}} \theta(\mathfrak{p}^m) \log(\mathrm{N}\,\mathfrak{p}) - \psi_C(x) \right| \leq 2(\log x)(\log d_L),$$

where

$$R_0(x, \sigma_0, T) = \sum_{\substack{\mathfrak{p}, m \\ \mathrm{N}(\mathfrak{p}^m) \neq x}} \left(\frac{x}{\mathrm{N}\,\mathfrak{p}^m} \right)^{\sigma_0} \min\left(1, T^{-1} \left| \log \frac{x}{\mathrm{N}\,\mathfrak{p}^m} \right|^{-1} \right) \log(\mathrm{N}\,\mathfrak{p}). \tag{2.9}$$

Using this they obtained that

$$\psi_C(x) = I_C(x, \sigma_0, T) + R_1(x, \sigma_0, T),$$

where

$$R_1(x, \sigma_0, T) \leq 2(\log x)(\log d_L) + n_K \sigma_0 T^{-1} + n_K(\log x) + R_0(x, \sigma_0, T).$$

We prove an explicit version of their result. The key is to use Perron's formula [6, Page 109-110]:

Lemma 2.2. *If $y > 0, \sigma > 0$ and $T > 0$, then*

$$\begin{aligned}
\left|\frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT}\frac{y^s}{s}ds - 1\right| &\le y^\sigma \min(1, T^{-1}|\log y|^{-1}) \quad \text{if } y > 1,\\
\left|\frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT}\frac{y^s}{s}ds - \frac{1}{2}\right| &\le \sigma T^{-1} \quad \text{if } y = 1,\\
\left|\frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT}\frac{y^s}{s}ds\right| &\le y^\sigma \min(1, T^{-1}|\log y|^{-1}) \quad \text{if } 0 < y < 1.
\end{aligned}$$

Lemma 2.3. *Let $x \ge 2$ and $T > 0$. With the above notations, we have*

$$|\psi_C(x) - I_C(x, \sigma_0, T)| \le R_1(x, \sigma_0, T), \tag{2.10}$$

where

$$R_1(x, \sigma_0, T) \le \frac{2}{\log 2}\frac{(\log x)(\log d_L)}{|G|} + n_K(\log x)\left(\frac{1}{2} + \sigma_0 T^{-1}\right) + R_0(x, \sigma_0, T). \tag{2.11}$$

Proof. We notice that

$$|\psi_C(x) - I_C(x, \sigma_0, T)| \le \left|\sum_{\substack{\mathfrak{p}, m\\ \mathrm{N}(\mathfrak{p}^m)\le x}} \theta(\mathfrak{p}^m)\log(\mathrm{N}\,\mathfrak{p}) - \psi_C(x)\right| + \left|I_C(x, \sigma_0, T) - \sum_{\substack{\mathfrak{p}, m\\ \mathrm{N}(\mathfrak{p}^m)\le x}} \theta(\mathfrak{p}^m)\log(\mathrm{N}\,\mathfrak{p})\right|. \tag{2.12}$$

Since the Dirichlet Series $F_C(s)$ in (2.7) is absolutely convergent for $\mathrm{Re}(s) > 1$, we can integrate $I_C(x, \sigma_0, T)$ term by term to obtain

$$\begin{aligned}
I_C(x, \sigma_0, T) &= \frac{1}{2\pi i}\int_{\sigma_0 - iT}^{\sigma_0 + i + T} F_C(s)\frac{x^s}{s}ds\\
&= \frac{1}{2\pi i}\int_{\sigma_0 - iT}^{\sigma_0 + iT}\sum_{\mathfrak{p}, m}\theta(\mathfrak{p}^m)(\log(\mathrm{N}\,\mathfrak{p}))(\mathrm{N}\,\mathfrak{p})^{-ms}\frac{x^s}{s}ds\\
&= \sum_{\mathfrak{p}, m}\theta(\mathfrak{p}^m)(\log(\mathrm{N}\,\mathfrak{p}))\frac{1}{2\pi i}\int_{\sigma_0 - iT}^{\sigma_0 + iT}\left(\frac{x}{(\mathrm{N}\,\mathfrak{p})^m}\right)^s\frac{ds}{s}.
\end{aligned}$$

Now we split at $\mathrm{N}(\mathfrak{p}^m) = x$ and define:

$$\sum\nolimits_1 = \sum_{\substack{\mathfrak{p}, m\\ \mathrm{N}(\mathfrak{p}^m)=x}} \theta(\mathfrak{p}^m)(\log(\mathrm{N}\,\mathfrak{p}))\frac{1}{2\pi i}\int_{\sigma_0 - iT}^{\sigma_0 + iT}\left(\frac{x}{(\mathrm{N}\,\mathfrak{p})^m}\right)^s\frac{ds}{s} - \sum_{\substack{\mathfrak{p}, m\\ \mathrm{N}(\mathfrak{p}^m)=x}} \theta(\mathfrak{p}^m)\log(\mathrm{N}\,\mathfrak{p}),$$

and

$$\sum\nolimits_2 = \sum_{\substack{\mathfrak{p},m \\ \mathrm{N}(\mathfrak{p}^m)\neq x}} \theta(\mathfrak{p}^m)(\log(\mathrm{N}\,\mathfrak{p}))\frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT}\left(\frac{x}{(\mathrm{N}\,\mathfrak{p})^m}\right)^s\frac{ds}{s} - \sum_{\substack{\mathfrak{p},m \\ \mathrm{N}(\mathfrak{p}^m)<x}} \theta(\mathfrak{p}^m)\log(\mathrm{N}\,\mathfrak{p}).$$

Thus we obtain,

$$I_C(x,\sigma_0,T) - \sum_{\substack{\mathfrak{p},m \\ \mathrm{N}(\mathfrak{p}^m)\leq x}} \theta(\mathfrak{p}^m)\log(\mathrm{N}\,\mathfrak{p}) = \sum\nolimits_1 + \sum\nolimits_2 \tag{2.13}$$

and we study each sum separately. For $\sum_1$, we get

$$\begin{aligned}\left|\sum\nolimits_1\right| &\leq \left|\sum_{\substack{\mathfrak{p},m \\ \mathrm{N}(\mathfrak{p}^m)=x}} \theta(\mathfrak{p}^m)(\log(\mathrm{N}\,\mathfrak{p}))\frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT}\left(\frac{x}{(\mathrm{N}\,\mathfrak{p})^m}\right)^s\frac{ds}{s} - \sum_{\substack{\mathfrak{p},m \\ \mathrm{N}(\mathfrak{p}^m)=x}} \theta(\mathfrak{p}^m)\log(\mathrm{N}\,\mathfrak{p})\right| \\ &= \left|\sum_{\substack{\mathfrak{p},m \\ \mathrm{N}(\mathfrak{p}^m)=x}} \theta(\mathfrak{p}^m)(\log(\mathrm{N}\,\mathfrak{p}))\left(\frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT}\left(\frac{x}{(\mathrm{N}\,\mathfrak{p})^m}\right)^s\frac{ds}{s} - 1\right)\right| \\ &\leq \sum_{\substack{\mathfrak{p},m \\ \mathrm{N}(\mathfrak{p}^m)=x}} (\log\mathrm{N}\,\mathfrak{p})(\frac{1}{2} + \sigma_0 T^{-1}),\end{aligned} \tag{2.14}$$

by using Lemma 2.2 for $y = 1$ and $|\theta(\mathfrak{p}^m)| \leq 1$.

Also, since the Galois group of $K/\mathbb{Q}$ has n_K elements, therefore any $\mathfrak{p}$ with $\mathrm{N}\,\mathfrak{p}^m = x$ is mapped to at most n_K distinct primes under action of the elements of the Galois group. Therefore we have at most n_K distinct pairs of $\mathfrak{p}$ and m such that $\mathrm{N}\,\mathfrak{p}^m = x$. Hence

$$\sum_{\substack{\mathfrak{p},m \\ \mathrm{N}\,\mathfrak{p}^m=x}} \log(\mathrm{N}\,\mathfrak{p}) \leq n_K \log x. \tag{2.15}$$

Using (2.15), we obtain

$$\left|\sum\nolimits_1\right| \leq n_K(\log x)\left(\frac{1}{2} + \sigma_0 T^{-1}\right). \tag{2.16}$$

For $\sum_2$, we split between $\mathrm{N}(\mathfrak{p}^m) > x$ and $\mathrm{N}(\mathfrak{p}^m) < x$ to obtain:

$$\begin{aligned}
\left|\sum\nolimits_2\right| &= \Bigg| \sum_{\substack{\mathfrak{p},m \\ \mathrm{N}(\mathfrak{p}^m)\neq x}} \theta(\mathfrak{p}^m)(\log(\mathrm{N}\,\mathfrak{p}))\frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT}\left(\frac{x}{(\mathrm{N}\,\mathfrak{p})^m}\right)^s\frac{ds}{s} - \sum_{\substack{\mathfrak{p},m \\ \mathrm{N}(\mathfrak{p}^m)<x}} \theta(\mathfrak{p}^m)\log(\mathrm{N}\,\mathfrak{p})\Bigg| \\
&\leq \Bigg| \sum_{\substack{\mathfrak{p},m \\ \mathrm{N}(\mathfrak{p}^m)<x}} \theta(\mathfrak{p}^m)(\log(\mathrm{N}\,\mathfrak{p}))\frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT}\left(\frac{x}{(\mathrm{N}\,\mathfrak{p})^m}\right)^s\frac{ds}{s} - \sum_{\substack{\mathfrak{p},m \\ \mathrm{N}(\mathfrak{p}^m)<x}} \theta(\mathfrak{p}^m)\log(\mathrm{N}\,\mathfrak{p})\Bigg| \\
&\quad + \Bigg| \sum_{\substack{\mathfrak{p},m \\ \mathrm{N}(\mathfrak{p}^m)>x}} \theta(\mathfrak{p}^m)(\log(\mathrm{N}\,\mathfrak{p}))\frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT}\left(\frac{x}{(\mathrm{N}\,\mathfrak{p})^m}\right)^s\frac{ds}{s}\Bigg| \\
&= \Bigg| \sum_{\substack{\mathfrak{p},m \\ \mathrm{N}(\mathfrak{p}^m)<x}} \theta(\mathfrak{p}^m)(\log(\mathrm{N}\,\mathfrak{p}))\frac{1}{2\pi i}\left(\int_{\sigma_0-iT}^{\sigma_0+iT}\left(\frac{x}{(\mathrm{N}\,\mathfrak{p})^m}\right)^s\frac{ds}{s} - 1\right)\Bigg| \\
&\quad + \Bigg| \sum_{\substack{\mathfrak{p},m \\ \mathrm{N}(\mathfrak{p}^m)>x}} \theta(\mathfrak{p}^m)(\log(\mathrm{N}\,\mathfrak{p}))\frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT}\left(\frac{x}{(\mathrm{N}\,\mathfrak{p})^m}\right)^s\frac{ds}{s}\Bigg|.
\end{aligned}$$

Now we use Lemma 2.2 for both $y > 1$ and $y < 1$ with $y = \frac{x}{(\mathrm{N}\,\mathfrak{p})^m}$ and the fact that $|\theta(\mathfrak{p}^m)| \leq 1$ to obtain

$$\left|\sum\nolimits_2\right| \leq \sum_{\substack{\mathfrak{p},m \\ \mathrm{N}(\mathfrak{p}^m)\neq x}} \left(\frac{x}{\mathrm{N}\,\mathfrak{p}^m}\right)^{\sigma_0} \min\left(1, T^{-1}\left|\log\frac{x}{\mathrm{N}\,\mathfrak{p}^m}\right|^{-1}\right)\log(\mathrm{N}\,\mathfrak{p}) = R_0(x,\sigma_0,T). \tag{2.17}$$

Therefore, using (2.13), (2.16) and (2.17), we get that for $\mathrm{Re}(s) > 1$,

$$\Bigg|I_C(x,\sigma_0,T) - \sum_{\substack{\mathfrak{p},m \\ \mathrm{N}(\mathfrak{p}^m)\leq x}} \theta(\mathfrak{p}^m)\log(\mathrm{N}\,\mathfrak{p})\Bigg| \leq n_K(\log x)\left(\frac{1}{2} + \sigma_0 T^{-1}\right) + R_0(x,\sigma_0,T). \tag{2.18}$$

Now we focus on the first difference in Lemma 2.12. We notice that

$$\Bigg|\sum_{\substack{\mathfrak{p},m \\ \mathrm{N}(\mathfrak{p}^m)\leq x}} \theta(\mathfrak{p}^m)\log(\mathrm{N}\,\mathfrak{p}) - \psi_C(x)\Bigg| \leq \sum_{\substack{\mathfrak{p},m \\ \mathfrak{p}\text{ ramified} \\ \mathrm{N}\,\mathfrak{p}^m\leq x}} \log(\mathrm{N}\,\mathfrak{p}) \leq \sum_{\mathfrak{p}\text{ ramified}} \log(\mathrm{N}\,\mathfrak{p}) \sum_{\substack{m \\ \mathrm{N}(\mathfrak{p}^m)\leq x}} 1.$$

Serre [26, Proposition 5] proved

$$\sum_{\mathfrak{p}\text{ ramified}} \log(\mathrm{N}\,\mathfrak{p}) \leq \frac{2}{|G|}\log d_L. \tag{2.19}$$

Also we know that for each prime ideal $\mathfrak{p}$, $\mathrm{N}\,\mathfrak{p} \geq 2$. Hence,

$$\sum_{\substack{m \\ \mathrm{N}(\mathfrak{p}^m)\leq x}} 1 \leq \frac{\log x}{\log 2}.$$

Therefore combining this with (2.19), we get

$$\left| \sum_{\substack{\mathfrak{p},m \\ \mathrm{N}(\mathfrak{p}^m)\leq x}} \theta(\mathfrak{p}^m)\log(\mathrm{N}\,\mathfrak{p}) - \psi_C(x) \right| \leq \frac{\log x}{\log 2} \sum_{\mathfrak{p} \text{ ramified}} \log(\mathrm{N}\,\mathfrak{p}) \leq \frac{2}{\log 2} \frac{(\log x)(\log d_L)}{|G|}. \tag{2.20}$$

Now combining (2.18) and (2.20), we obtain the required result. □

Revision 1. *Lagarias and Odlyzko in [15, (3.8)] and Winckler in [30, (1)] proved that for $x \geq 2$,*

$$\left| I_C(x,\sigma_0,T) - \sum_{\substack{\mathfrak{p},m \\ \mathrm{N}(\mathfrak{p}^m)\leq x}} \theta(\mathfrak{p}^m)\log(\mathrm{N}\,\mathfrak{p}) \right| \leq \sum_{\substack{\mathfrak{p},m \\ \mathrm{N}(\mathfrak{p}^m)=x}} (\log(\mathrm{N}(\mathfrak{p})) + \sigma_0 T^{-1}) + R_0(x,\sigma_0,T).$$

We corrected the sum on the right hand side of the above equation with

$$\sum_{\substack{\mathfrak{p},m \\ \mathrm{N}(\mathfrak{p}^m)=x}} (\log(\mathrm{N}\,\mathfrak{p}))\left(\frac{1}{2} + \sigma_0 T^{-1}\right)$$

as shown in (2.14).

2.3 Estimating remainder terms $R_0(x,T)$ and $R_1(x,T)$

In this section, we will establish an estimate for $R_0(x,\sigma_0,T)$ as defined in (2.9). From here onwards we will fix $\sigma_0 = \sigma_0(x) = 1 + (\log x)^{-1}$. We thus get $x^{\sigma_0} = ex$. Since this fixed σ_0 depends on x, we write $R_0(x,\sigma_0,T)$ as $R_0(x,T)$, $R_1(x,\sigma_0,T)$ as $R_1(x,T)$ and $I_C(x,\sigma_0,T)$ as $I_C(x,T)$ from now on.

We will write $R_0(x,T) = S_1 + S_2 + S_3$, where S_1 consists of those terms of (2.9) for which $|\,\mathrm{N}(\mathfrak{p}^m) - x| \geq \frac{1}{4}x$, S_2 consists of those terms of (2.9) for which $|\,\mathrm{N}(\mathfrak{p}^m) - x| \leq 1$, and S_3 consists of the remaining terms of (2.9).

Lagarias and Odlyzko in [15] proved bounds for S_1, S_2, S_3 and therefore proved a non-explicit bound for $R_0(x,T)$. They proved that

$$S_1 \ll n_K x T^{-1} \log x, \quad S_2 \ll n_K \log x \quad \text{and} \quad S_3 \ll n_K x T^{-1} (\log x)^2.$$

Putting all this together, they got, for $x \geq 2$ and $T \geq 1$, (see [15, (3.17)])

$$R_0(x,T) \ll n_K (\log x) + n_K x T^{-1} (\log x)^2.$$

Using this they also obtained a bound for $R_1(x, \sigma_0, T)$ as defined in (2.11). They proved (see [15, (3.18)])

$$R_1(x,T) \ll (\log x)(\log d_L) + n_K \log x + n_K x T^{-1} (\log x)^2.$$

We try to produce explicit bounds for the terms S_1, S_2, S_3, $R_0(x,T)$ and $R_1(x,T)$.

2.3.1 Bounding S_1

We study

$$S_1 = \sum_{\substack{\mathfrak{p},m \\ |\,\mathrm{N}(\mathfrak{p}^m) - x| \geq \frac{1}{4}x}} \left(\frac{x}{\mathrm{N}\,\mathfrak{p}^m}\right)^{\sigma_0} \min\left(1, T^{-1}\left|\log \frac{x}{\mathrm{N}\,\mathfrak{p}^m}\right|^{-1}\right) \log(\mathrm{N}\,\mathfrak{p}).$$

To bound S_1, we first prove a bound for the logarithmic derivative of the Dedekind ζ-function, $\zeta_K(s)$:

Lemma 2.4. *For $\sigma > 1$,*

$$-\frac{\zeta_K'}{\zeta_K}(\sigma) \leq -n_K \frac{\zeta'}{\zeta}(\sigma) \leq n_K (\sigma - 1)^{-1}.$$

Proof. By Euler product formula for the Dedekind ζ-function, $\zeta_K(s)$ and the Riemann ζ-function, $\zeta(s)$, we obtain their logarithmic derivatives respectively as

$$-\frac{\zeta_K'}{\zeta_K}(\sigma) = \sum_{\mathfrak{p}} \frac{\log(\mathrm{N}\,\mathfrak{p})}{(\mathrm{N}\,\mathfrak{p})^\sigma - 1} \quad \text{and} \quad -\frac{\zeta'}{\zeta}(\sigma) = \sum_{p} \frac{\log p}{p^\sigma - 1},$$

where in the second sum p runs through the rational primes. Also, for each prime ideal $\mathfrak{p}$,

$\mathrm{N}\,\mathfrak{p} = p^k$ for some rational prime p and some positive integer k. Therefore

$$\frac{\log(\mathrm{N}\,\mathfrak{p})}{(\mathrm{N}\,\mathfrak{p})^\sigma - 1} = \frac{k\log p}{p^{k\sigma} - 1} = \frac{k}{p^{(k-1)\sigma} + ... + 1} \times \frac{\log p}{p^\sigma - 1} \leq \frac{\log p}{p^\sigma - 1}.$$

Note that there are at most n_K distinct $\mathfrak{p}$ lying over a given rational prime p. Hence we get,

$$-\frac{\zeta_K'}{\zeta_K}(\sigma) = \sum_{\mathfrak{p}} \frac{\log(\mathrm{N}\,\mathfrak{p})}{(\mathrm{N}\,\mathfrak{p})^\sigma - 1} \leq n_K \sum_p \frac{\log p}{p^\sigma - 1} = -n_K \frac{\zeta'}{\zeta}(\sigma).$$

Using Abel's summation formula for $\zeta(\sigma)$, we obtain

$$\zeta(\sigma) = \frac{\sigma}{\sigma - 1} - \sigma I(\sigma), \tag{2.21}$$

where $I(\sigma) = \int_1^\infty (t - [t])t^{-\sigma-1}dt$. Taking the logarithmic derivatives on both sides, we get

$$\frac{\zeta'}{\zeta}(\sigma) = \frac{1}{\zeta(\sigma)}\bigg(\frac{-1}{(\sigma-1)^2} - I(\sigma) - \sigma I'(\sigma)\bigg). \tag{2.22}$$

Now using the (2.21), (2.22) and the fact that $I'(\sigma) \leq 0$ for $\sigma > 1$, we get

$$\begin{aligned} \frac{\zeta'}{\zeta}(\sigma) + \frac{1}{\sigma-1} &\geq \frac{1}{\zeta(\sigma)}\bigg(\frac{-1}{(\sigma-1)^2} - I(\sigma)\bigg) + \frac{1}{\sigma-1} \\ &= \frac{1}{\zeta(\sigma)}\bigg(\frac{-1}{(\sigma-1)^2} + \frac{\zeta(\sigma)}{\sigma} - \frac{1}{\sigma-1} + \frac{\zeta(\sigma)}{\sigma-1}\bigg) \\ &= \frac{1}{\zeta(\sigma)(\sigma-1)}\bigg(-\frac{\sigma}{\sigma-1} + \frac{2\sigma-1}{\sigma}\zeta(\sigma)\bigg). \end{aligned} \tag{2.23}$$

The Laurent series expansion of the Riemann zeta function about $s = 1$ is given by

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^\infty \frac{(-1)^n \gamma_n}{n!}(s-1)^n,$$

where the constants γ_n are called the Stieltjes constants and can be defined by the limit

$$\gamma_n = \lim_{m\to\infty}\bigg(\bigg(\sum_{k=1}^m \frac{(\log k)^n}{k}\bigg) - \frac{(\log m)^{n+1}}{n+1}\bigg).$$

Notice that for all $\sigma > 0$, $\gamma_0 \geq \frac{\sigma-1}{2\sigma-1}$. Now using this with the Laurent series expansion of ζ and writing $\sigma^2 = (\sigma-1)^2 + (2\sigma-1)$, we obtain that for $\sigma > 1$,

$$\begin{aligned}(\sigma-1)\zeta(\sigma) - \frac{\sigma^2}{2\sigma-1} &= 1 + \gamma_0(\sigma-1) + \sum_{n=1}^{\infty} \frac{(-1)^n\gamma_n}{n!}(\sigma-1)^{n+1} - \left(\frac{(\sigma-1)^2}{2\sigma-1} + 1\right) \\ &= \gamma_0(\sigma-1) + \sum_{n=1}^{\infty} \frac{(-1)^n\gamma_n}{n!}(\sigma-1)^{n+1} - \frac{(\sigma-1)^2}{2\sigma-1} \\ &\geq 0. \end{aligned} \tag{2.24}$$

Therefore we get,

$$\zeta(\sigma) \geq \frac{\sigma^2}{(\sigma-1)(2\sigma-1)},$$

and thus

$$\frac{2\sigma-1}{\sigma}\zeta(\sigma) \geq \frac{\sigma}{\sigma-1}. \tag{2.25}$$

Now using (2.25) in (2.23), we get that

$$\frac{\zeta'}{\zeta}(\sigma) + \frac{1}{\sigma-1} \geq 0.$$

Therefore,

$$-n_K\frac{\zeta'}{\zeta}(\sigma) \leq n_K(\sigma-1)^{-1}.$$

□

Lemma 2.5. *Let $x \geq x \geq 2$, $T \geq 1$ and $\sigma_0 = 1 + (\log x)^{-1}$. Then*

$$S_1 \leq \frac{e}{\log\left(\frac{5}{4}\right)} xT^{-1} n_K(\log x).$$

Proof. Since $|\,\mathrm{N}(\mathfrak{p}^m) - x| \geq \frac{1}{4}x$, thus

$$\left|\log\frac{x}{\mathrm{N}(\mathfrak{p}^m)}\right| \geq \log\frac{5}{4},$$

and therefore

$$\min\left(1, T^{-1}\left|\log\frac{x}{\mathrm{N}(\mathfrak{p}^m)}\right|^{-1}\right) \leq T^{-1}\left(\log\left(\frac{5}{4}\right)\right)^{-1}.$$

Now using this above result, $x^{\sigma_0} = ex$ and Lemma 2.4, we obtain that

$$S_1 \le \frac{xT^{-1}e}{\log\left(\frac{5}{4}\right)} \sum_{\mathfrak{p},m} \mathrm{N}(\mathfrak{p})^{-m\sigma_0} \log(\mathrm{N}\,\mathfrak{p}) = \frac{xT^{-1}e}{\log\left(\frac{5}{4}\right)} \left(-\frac{\zeta_K'}{\zeta_K}(\sigma_0) \right)$$
$$\le \frac{exT^{-1}n_K}{\log\left(\frac{5}{4}\right)} \left(-\frac{\zeta'}{\zeta}(\sigma_0) \right) \le \frac{exT^{-1}n_K}{\log\left(\frac{5}{4}\right)} (\sigma_0 - 1)^{-1} = \frac{e}{\log\left(\frac{5}{4}\right)} xT^{-1} n_K (\log x). \tag{2.26}$$

□

2.3.2 Bounding S_2

Now we will bound

$$S_2 = \sum_{\substack{\mathfrak{p},m \\ \mathrm{N}(\mathfrak{p}^m)\neq x \\ |\mathrm{N}(\mathfrak{p}^m)-x|\le 1}} \left(\frac{x}{\mathrm{N}(\mathfrak{p}^m)}\right)^{\sigma_0} \min\left(1, T^{-1}\left|\log\frac{x}{\mathrm{N}(\mathfrak{p}^m)}\right|^{-1}\right) \log(\mathrm{N}\,\mathfrak{p}).$$

Lemma 2.6. *Let $x \ge x_0 \ge 2$. Then*

$$S_2 \le 2a_1 n_K (\log x + a_2), \tag{2.27}$$

where $a_1 = \left(\frac{x_0}{x_0-1}\right)^{1+\frac{1}{\log(x_0)}}$ *and* $a_2 = \log\left(1 + \frac{1}{x_0}\right)$.

Proof. Notice that S_2 consists of those terms $\mathfrak{p}^m$ for which $0 < |\mathrm{N}(\mathfrak{p}^m) - x| \le 1$. Also, there are at most $2n_K$ of such $\mathfrak{p}^m$. Together with

$$\min\left(1, T^{-1}\left|\log\frac{x}{\mathrm{N}(\mathfrak{p}^m)}\right|^{-1}\right) \le 1 \quad \text{and} \quad x - 1 \le \mathrm{N}(\mathfrak{p}^m) \le x + 1,$$

we obtain,

$$S_2 \quad \le \quad 2n_K \log(x+1) \left(\frac{x}{x-1}\right)^{\sigma_0}. \tag{2.28}$$

Now we study the function $\log(x+1)\left(\frac{x}{x-1}\right)^{\sigma_0}$. Notice that

$$\begin{aligned}\log(x+1)\left(\frac{x}{x-1}\right)^{\sigma_0} &= \left(\log x + \log\left(1+\frac{1}{x}\right)\right)\left(\frac{x}{x-1}\right)^{\sigma_0} \\ &= (\log x)\left(\frac{x}{x-1}\right)^{\sigma_0} + \left(\log\left(1+\frac{1}{x}\right)\right)\left(\frac{x}{x-1}\right)^{\sigma_0}. \qquad (2.29)\end{aligned}$$

Now the second term

$$\left(\log\left(1+\frac{1}{x}\right)\right)\left(\frac{x}{x-1}\right)^{\sigma_0}$$

is a decreasing quantity for $x \geq x_0$ and achieves its maximum at $x = x_0$ which is $a_1 a_2$. Also, since $\left(\frac{x}{x-1}\right)^{\sigma_0}$ is decreasing for $x \geq x_0$ and its maximum value is at $x = x_0$ which is a_1, we get

$$\log(x+1)\left(\frac{x}{x-1}\right)^{\sigma_0} \leq a_1(\log x + a_2).$$

Now using this inequality in (2.28), we get the required result. □

2.3.3 Bounding S_3

Now we will bound

$$S_3 = \sum_{\substack{\mathfrak{p},m \\ \mathrm{N}(\mathfrak{p}^m)\neq x \\ 1<|\mathrm{N}\,\mathfrak{p}^m - x|<\frac{1}{4}x}} \left(\frac{x}{\mathrm{N}\,\mathfrak{p}^m}\right)^{\sigma_0} \min\left(1, T^{-1}\left|\log\frac{x}{\mathrm{N}\,\mathfrak{p}^m}\right|^{-1}\right)\log(\mathrm{N}\,\mathfrak{p}).$$

Lemma 2.7. *Let $x \geq x_0 \geq 2$ and $T \geq 1$. Then*

$$S_3 \leq \frac{5}{4} a_3 n_K T^{-1} x (\log x)^2, \qquad (2.30)$$

where $a_3 = \left(\frac{4}{3}\right)^{2+\frac{1}{\log(x_0)}}$.

Proof. We rewrite

$$S_3 = \sum_{\substack{\mathfrak{p},m \\ n=\mathrm{N}(\mathfrak{p}^m)\neq x \\ 1<|n-x|<\frac{1}{4}x}} \left(\frac{x}{n}\right)^{\sigma_0} \min\left(1, T^{-1}\left|\log\frac{x}{n}\right|^{-1}\right)\log(\mathrm{N}\,\mathfrak{p}).$$

Notice that in the region where $1 < |n-x| < \frac{1}{4}x$, we get $\frac{x}{n} < \frac{4}{3}$, $n < \frac{5x}{4}$ and $\max\{x,n\} < \frac{4n}{3}$. Using this and the mean value theorem, we get that for $1 < |n-x| < \frac{1}{4}x$,

$$\left|\log\frac{x}{n}\right|^{-1} \leq \frac{4}{3}\frac{n}{|x-n|}. \tag{2.31}$$

Now using $\left(\frac{x}{n}\right)^{\sigma_0} \leq \left(\frac{4}{3}\right)^{\sigma_0}$, $\min\left(1, T^{-1}\left|\log\frac{x}{n}\right|^{-1}\right) \leq T^{-1}\left|\log\frac{x}{n}\right|^{-1}$, $(\log \mathrm{N}\,\mathfrak{p}) \leq \log n \leq \log\left(\frac{5x}{4}\right)$, $x \geq x_0$ and (2.31), we get $\sigma_0 + 1 \leq 2 + \frac{1}{\log x_0}$ and

$$S_3 \leq T^{-1}\left(\frac{4}{3}\right)^{\sigma_0+1}\left(\log\frac{5x}{4}\right)\sum_{\substack{\mathfrak{p},m\\ n=\mathrm{N}(\mathfrak{p}^m)\neq x\\ 1<|n-x|<\frac{1}{4}x}}\frac{n}{|x-n|} \leq a_3T^{-1}\left(\log\frac{5x}{4}\right)\sum_{\substack{\mathfrak{p},m\\ n=\mathrm{N}(\mathfrak{p}^m)\neq x\\ 1<|n-x|<\frac{1}{4}x}}\frac{n}{|x-n|}. \tag{2.32}$$

Now we can bound $\frac{1}{|x-n|}$ by $\frac{1}{\lfloor |x-n|\rfloor}$ and make the variable change $\lfloor |x-n|\rfloor = k$. Considering $|n-x| \geq k$ where $k \in \mathbb{Z}$ and using $n < \frac{5x}{4}$ with the identity $\sum_{k=1}^{x}\frac{1}{k} \leq \log(x) + 1$ for $x \geq 1$, we get

$$\sum_{\substack{n\\ 1<|n-x|<\frac{1}{4}x}}\frac{n}{|x-n|} \leq 2\sum_{\substack{k\in\mathbb{Z}\\ 1\leq k<\frac{1}{4}x}}\frac{5}{4}\frac{x}{k} = \frac{5}{2}x\sum_{\substack{k\in\mathbb{Z}\\ 1\leq k<\frac{1}{4}x}}\frac{1}{k} \leq \frac{5}{2}x\left(\log\left(\frac{x}{4}\right)+1\right). \tag{2.33}$$

Recall that there are at most n_K prime ideals with $\mathrm{N}\,\mathfrak{p}^m = n$. Also it can be easily verified that $\left(\log\left(\frac{5x}{4}\right)\right)\left(\log\left(\frac{x}{4}\right)+1\right) \leq (\log x)^2$ for $x \geq 2$. Thus, using (2.33) we obtain

$$\begin{aligned} S_3 &\leq a_3T^{-1}\left(\log\left(\frac{5x}{4}\right)\right)\sum_{\substack{n\\ 1<|n-x|<\frac{1}{4}x}}\frac{n}{|x-n|}\sum_{\substack{\mathfrak{p},m\\ \mathrm{N}(\mathfrak{p}^m)=n\neq x}}1 \\ &\leq a_3n_KT^{-1}\left(\log\left(\frac{5x}{4}\right)\right)\left(\frac{5}{2}x\left(\log\left(\frac{x}{4}\right)+1\right)\right) \\ &\leq \frac{5}{2}a_3n_KT^{-1}x(\log x)^2. \end{aligned} \tag{2.34}$$

□

Revision 2. *Winckler in [30, (9)] showed that*

$$\sum_{\substack{n\\ 1<|n-x|<\frac{1}{4}x}}\frac{n}{|x-n|} \leq \sum_{\substack{k\\ 1<k<\frac{1}{4}x}}\left(1+\frac{x}{k}\right),$$

which is not correct (for $x = 12$, the inequality is false). We corrected this mistake by proving

$$\sum_{\substack{n \\ 1<|n-x|<\frac{1}{4}x}} \frac{n}{|x-n|} \leq \frac{5}{2} \sum_{\substack{k\in\mathbb{Z} \\ 1\leq k<\frac{1}{4}x}} \frac{x}{k},$$

as shown in (2.33).

Lemma 2.8. *If $x \geq x_0 \geq 2$ and $T \geq 1$, then*

$$R_0(x,T) \leq n_K T^{-1} x (\log x)^2 \left(\frac{e}{(\log \frac{5}{4})(\log 2)} + \frac{5}{2} a_3 \right) + 2a_1 n_K (\log x + a_2), \tag{2.35}$$

and

$$\begin{aligned} R_1(x,T) &\leq 2.89 \frac{(\log x)(\log d_L)}{|G|} + n_K (\log x) \left(\frac{1}{2} + \sigma_0 T^{-1} + 2a_1 \right) \\ &+ \left(\frac{e}{(\log \frac{5}{4})(\log 2)} + \frac{5}{2} a_3 \right) n_K T^{-1} x (\log x)^2 + 2a_1 a_2 n_K. \end{aligned} \tag{2.36}$$

Proof. Putting (2.26), (2.27) and (2.34) in (2.9), we get that,

$$\begin{aligned} R_0(x,T) = S_1 + S_2 + S_3 &\leq \frac{exT^{-1}n_K}{\log\left(\frac{5}{4}\right)} \log x + 2a_1 n_K (\log x + a_2) + \frac{5}{2} a_3 n_K T^{-1} x (\log x)^2 \\ &\leq n_K T^{-1} x (\log x)^2 \left(\frac{e}{(\log \frac{5}{4})(\log 2)} + \frac{5}{2} a_3 \right) + 2a_1 n_K (\log x + a_2). \end{aligned}$$

Now using (2.35) in (2.11), we get that,

$$\begin{aligned} R_1(x,T) &\leq \frac{2}{\log 2} \frac{(\log x)(\log d_L)}{|G|} + n_K (\log x) \left(\frac{1}{2} + \sigma_0 T^{-1} \right) + R_0(x,T) \\ &\leq 2.89 \frac{(\log x)(\log d_L)}{|G|} + n_K (\log x) \left(\frac{1}{2} + \sigma_0 T^{-1} + 2a_1 \right) \\ &+ \left(\frac{e}{(\log \frac{5}{4})(\log 2)} + \frac{5}{2} a_3 \right) n_K T^{-1} x (\log x)^2 + 2a_1 a_2 n_K. \end{aligned}$$

□

Remark 2.9. Bruno Winckler in [30] proved similar results for S_1, S_2, S_3 and $R_0(x,T)$. He proved that for $x \geq 2$ and $T \geq 1$, (see [30, (7),(8),(9)]

$$\begin{aligned} S_1 &\leq \frac{e}{\log\left(\frac{5}{4}\right)} n_K x T^{-1} (\log x) \\ S_2 &\leq \frac{4e\log(3)}{\log(2)} n_K (\log x), \\ S_3 &\leq \frac{8e^2}{3^{1+1/\log 2}} n_K T^{-1} x (\log x)^2. \end{aligned}$$

and therefore for $x \geq 2$ and $T \geq 1$, (see [30, (10)])

$$R_0(x, \sigma_0, T) \leq 17.25 n_K (\log x) + 21.67 n_K T^{-1} x (\log x)^2.$$

Using this he also proved that for $x \geq 2$ and $T \geq 1$, (see [30, (11)])

$$R_1(x, \sigma_0, T) \leq 2.89 \frac{(\log x)(\log d_L)}{|G|} + 18.25 n_K (\log x) + 24.17 n_K T^{-1} x (\log x)^2.$$

Remark 2.10. Now if we consider $x_0 = 2$, then we compute that $a_1 = 2e$, $a_2 = \log(3/2)$ and $a_3 = \frac{16e^2}{3^{2+(1/\log 2)}}$. Therefore, we compute further to get

$$\begin{aligned} S_2 &\leq 4e n_K (\log x + \log(3/2)), \\ S_3 &\leq \frac{32e^2}{3^{2+1/\log 2}} n_K T^{-1} x (\log x)^2, \\ R_0(x, T) &\leq 28.7 n_K T^{-1} x (\log x)^2 + 10.88 n_K \log x + 4.41 n_K, \\ R_1(x, T) &\leq 2.89 \frac{(\log x)(\log d_L)}{|G|} + 13.83 n_K (\log x) + 28.7 n_K T^{-1} x (\log x)^2 + 4.41 n_K, \end{aligned} \tag{2.37}$$

and observing these new constants, we see that these constants are better than the ones in Remark 2.9.

2.4 Reduction to the case of Hecke L-functions

In defining $F_C(s)$ in (2.7), we have already selected an element $g \in C$. Let $G_0 =< g >$ be the cyclic group generated by g, E be the fixed field of G_0, and let χ denote the irreducible characters of G_0. Since G_0 is cyclic and the irreducible representations of a cyclic group are one-dimensional (i.e., the corresponding vector space is $\mathbb{C}$), therefore the characters χ are one-dimensional (linear).

Lemma 2.11. *We have*

$$F_C(s) = -\frac{|C|}{|G|}\sum_{\chi}\overline{\chi}(g)\frac{L'}{L}(s,\chi,L/E). \tag{2.38}$$

Proof. Let $\tau : G_0 \to \mathbb{C}$ be the class function defined by

$$\tau(h) = \begin{cases} |G_0| & \text{if } \; h = g, \\ 0 & \text{if } \; h \neq g. \end{cases} \tag{2.39}$$

Now the orthogonality relations for characters of H imply that (since $\{g\}$ is a conjugacy class of the cyclic group G_0)

$$\tau = \sum_{\chi}\overline{\chi}(g)\chi.$$

Recall that a left transversal of G_0 is a subset of elements of G which contains exactly one element of each left coset of G_0. Let τ^* denote the class function on G induced by τ. Recall that τ^* is defined as

$$\tau^*(y) = \sum_{s\in S}\tau_0(s^{-1}ys)$$

where S is a left transversal of G_0 and τ_0 is defined as $\tau_0(h) = \tau(h)$ if $h \in G_0$ and 0 otherwise. Notice that with the above definition, if $y \notin C$, then $\tau^*(y) = 0$. Now if $y \in C$, then $y = a^{-1}ga$ for some $a \in G$. Thus $s^{-1}ys = (as)^{-1}g(as)$. Also if S is a transversal of G_0 in G, then aS is a transversal of G_0 in G. Moreover the class function τ^* is invariant under the transversals of G_0 in G. Using this we see that for $y \in C$,

$$\tau^*(y) = \sum_{s\in S}\tau_0(s^{-1}ys) = \sum_{s\in S}\tau_0((as)^{-1}g(as)) = \sum_{as\in aS}\tau_0((as)^{-1}g(as)) = \tau^*(g).$$

We know that τ_0 is only non-zero at g (with value $|G_0|$ at g) and $s^{-1}gs = g$ if and only if $s \in C_G(g)$ where $C_G(g)$ is the centralizer of g in G. Therefore

$$\tau^*(g) = \sum_{s\in S}\tau_0(s^{-1}gs) = \frac{|C_G(g)|}{|G_0|} \times |G_0| = |C_G(g)|.$$

Therefore we obtain

$$\tau^*(y) = \begin{cases} |C_G(g)| & \text{if } \ y \in C, \\ 0 & \text{if } \ y \notin C. \end{cases} \tag{2.40}$$

Now since G acts on itself by conjugation, therefore by orbit-stabilizer theorem, we get $|C_G(g)||C| = |G|$. This gives us that $\tau^* = f_C$ (see (2.4)). This implies that

$$\sum_{\chi} \overline{\chi}(g)\chi^* = \sum_{\phi} \overline{\phi}(g)\phi,$$

where χ^* is the character of G induced by G_0. Therefore for $\mathrm{Re}(s) > 1$, (2.6) modifies into

$$F_C(s) = -\frac{|C|}{|G|}\sum_{\chi} \overline{\chi}(g)\frac{L'}{L}(s, \chi^*, L/K). \tag{2.41}$$

From the properties of L-functions as described in [18, Theorem 2.3.2 (d)], we get that

$$L(s, \chi^*, L/K) = L(s, \chi, L/E).$$

Therefore, for $\mathrm{Re}(s) > 1$, (2.41) becomes

$$F_C(s) = -\frac{|C|}{|G|}\sum_{\chi} \overline{\chi}(g)\frac{L'}{L}(s, \chi, L/E),$$

and by analytic continuation of L-function, this holds for all s. □

2.5 Density of zeros of Hecke L-functions

Now combining (2.8) and (2.38), we obtain

$$I_C(x, \sigma_0, T) = -\frac{|C|}{|G|}\sum_{\chi} \overline{\chi}(g)\frac{1}{2\pi i}\int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{x^s}{s}\frac{L'}{L}(s, \chi, L/E)ds, \tag{2.42}$$

where $\sigma_0 = 1 + (\log x)^{-1}$ and χ runs through the (one-dimensional) irreducible characters of $H =< g >$. Our next goal is to evaluate each integrals in (2.42). To achieve this, we first prove some results relating to L'/L.

Since L and E are going to be fixed from now on, we will use $L(s, \chi)$ to denote $L(s, \chi, L/E)$.

We let $F(\chi)$ denote the conductor of χ and set

$$A(\chi) = d_E \,\mathrm{N}_{E/\mathbb{Q}}(F(\chi)) \tag{2.43}$$

and

$$\delta(\chi) = \begin{cases} 1 & \text{if } \chi = \chi_1, \text{ the principal character,} \\ 0 & \text{otherwise.} \end{cases} \tag{2.44}$$

For each χ there exists non-negative integers $a = a(\chi)$ and $b = b(\chi)$ such that

$$a(\chi) + b(\chi) = n_E, \tag{2.45}$$

such that if we define

$$\gamma_\chi(s) = \left(\pi^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right)\right)^{b(\chi)}\left(\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\right)^{a(\chi)} \tag{2.46}$$

and

$$\xi(s,\chi) = (s(s-1))^{\delta(\chi)}A(\chi)^{\frac{s}{2}}\gamma_\chi(s)L(s,\chi), \tag{2.47}$$

then $\xi(s,\chi)$ satisfies the functional equation (see [18, Theorem 2.2.1])

$$\xi(1-s,\overline{\chi}) = W(\chi)\xi(s,\chi), \tag{2.48}$$

where $W(\chi)$ is a certain constant of absolute value one. This $W(\chi)$ is known as the root number. Furthermore, ξ is an entire function of order 1 and does not vanish at $s = 0$, and hence by the Hadamard product theorem we have (see [18, Theorem 2.4.1.1])

$$\xi(s,\chi) = e^{(B_1(\chi)+B(\chi)s)}\prod_\rho\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}} \tag{2.49}$$

for some constants $B_1(\chi)$ and $B(\chi)$, where ρ runs through all the zeros of $\xi(s,\chi)$, which are precisely the non-trivial zeros of $L(s,\chi)$ (i.e., zeros of $L(s,\chi)$ with $0 < \Re(s) < 1$). Recall that $L(s,\chi)$ and hence $\xi(s,\chi)$ have no zeros ρ with $\mathrm{Re}(\rho) \geq 1$.

We are interested in the integral (2.42). Therefore we need to find identities involving L'/L.

We differentiate (2.47) logarithmically to obtain

$$\frac{\xi'}{\xi}(s,\chi) = \delta(\chi)\left(\frac{1}{s} + \frac{1}{s-1}\right) + \frac{1}{2}\log(A(\chi)) + \frac{\gamma'_\chi}{\gamma_\chi}(s) + \frac{L'}{L}(s,\chi). \tag{2.50}$$

Similarly we differentiate (2.49) logarithmically to obtain

$$\frac{\xi'}{\xi}(s,\chi) = B(\chi) + \sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right). \tag{2.51}$$

Now combining (2.50) and (2.51), we obtain

$$\frac{L'}{L}(s,\chi) = B(\chi) + \sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) - \delta(\chi)\left(\frac{1}{s} + \frac{1}{s-1}\right) - \frac{1}{2}\log(A(\chi)) - \frac{\gamma'_\chi}{\gamma_\chi}(s). \tag{2.52}$$

Theorem 2.12. *With the notation as above,*

$$\mathrm{Re}(B(\chi)) = -\sum_\rho \mathrm{Re}\,\frac{1}{\rho}, \tag{2.53}$$

and

$$\frac{L'}{L}(s,\chi) + \frac{L'}{L}(s,\overline{\chi}) = \sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{s-\overline{\rho}}\right) - \log(A(\chi)) - 2\delta(\chi)\left(\frac{1}{s} + \frac{1}{s-1}\right) - 2\frac{\gamma'_\chi}{\gamma_\chi}(s) \tag{2.54}$$

holds identically in the complex variable s, where ρ runs through the non-trivial zeros of $L(s,\chi)$.

Proof. Follows from [18, Corollary 2.4.1.2] and (2.54). □

Lagarias and Odlyzko in [15, Lemma 5.2] proved that if $\mathrm{Re}(s) > 1$, then $\left|\frac{L'}{L}(s,\chi)\right| \ll \frac{n_E}{\mathrm{Re}(s)-1}$. We give an explicit version to this as

Lemma 2.13. *If* $\mathrm{Re}(s) > 1$, *then* $\left|\frac{L'}{L}(s,\chi)\right| \leq \frac{n_E}{\mathrm{Re}(s)-1}$.

Proof. The comparison of the corresponding Dirichlet series gives us

$$\left|\frac{L'}{L}(s,\chi)\right| \leq -\frac{\zeta'_E}{\zeta_E}(\mathrm{Re}(s)).$$

Applying Lemma 2.4 to this, we obtain the required result. □

Lagarias and Odlyzko in [15, Lemma 6.1] proved that if $|z+k| \geq \frac{1}{8}$ for any non-negative integers k, then

$$\frac{\Gamma'}{\Gamma}(z) \ll \log(|z|) + 2.$$

As an explicit version to this, Winckler in [30, Lemma 4.4] proved that :

Lemma 2.14. *1. If* $\mathrm{Re}(z) \geq a$*, with* $a \geq 1$*, then*

$$\left|\frac{\Gamma'}{\Gamma}(z)\right| \leq \log(|z|) + \frac{\pi}{2} + \frac{1}{a}.$$

2. If $\mathrm{Im}(z) \geq b \geq 1$*, then*

$$\left|\frac{\Gamma'}{\Gamma}(z)\right| \leq \log(|z|) + \pi\left(1 + \frac{1}{2b}\right) + \frac{1}{2b}.$$

3. If $|z+k| \geq \frac{1}{8}$ *for every non-negative integers* k*, then*

$$\left|\frac{\Gamma'}{\Gamma}(z)\right| \leq \log(|z|) + c_1$$

where $c_1 = \frac{83}{5}$*.*

Lagarias and Odlyzko in [15, Lemma 5.3] proved that if $\mathrm{Re}(s) > -\frac{1}{2}$ and $|s| \geq \frac{1}{8}$, then

$$\left|\frac{\gamma_\chi'}{\gamma_\chi}(s)\right| \ll n_E \log(|s| + 2). \tag{2.55}$$

Again as a explicit version to this, Winckler in [30, Lemma 4.5] proved:

Lemma 2.15. *If* $\mathrm{Re}(s) > -\frac{1}{2}$ *and* $|s| \geq \frac{1}{8}$*, then*

$$\left|\frac{\gamma_\chi'}{\gamma_\chi}(s)\right| \leq \frac{n_E}{2}\left(\log(1+|s|) + c_2\right) \tag{2.56}$$

where $c_2 = \frac{164}{7}$*.*

Lagarias and Odlyzko in [15, Lemma 5.4] proved that if we let $n_\chi(t)$ be the number of zeros $\rho = \beta + i\gamma$ of $L(s,\chi)$ with $0 < \beta < 1$ and $|\gamma - t| \leq 1$, then for all t, we have

$$n_\chi(t) \ll \log A(\chi) + n_E \log(|t| + 2). \tag{2.57}$$

As an explicit version to this, Winckler in [30, Lemma 4.6] proved:

Lemma 2.16. *Let $n_\chi(t)$ be the number of zeros $\rho = \beta + i\gamma$ of $L(s,\chi)$ with $0 < \beta < 1$ and $|\gamma - t| \leq 1$. For all t, we have*

$$n_\chi(t) + n_\chi(-t) \leq c_3\bigg(\log(A(\chi)) + n_E\bigg(\log(|t|+3) + c_4\bigg)\bigg) \tag{2.58}$$

where $c_3 = \frac{5}{2}$ and $c_4 = \frac{1075}{134}$.

Revision 3. *While going through the proof demonstrated by Winckler for [30, Lemma 4.6], we notice that the proof gives $c_3 = 5$ instead of $\frac{5}{2}$.*

Lagarias and Odlyzko in [15, Lemma 5.5] proved that for all real ϵ such that $0 < \epsilon \leq 1$, we have

$$B(\chi) + \sum_{|\rho|<\epsilon} \frac{1}{\rho} \ll \epsilon^{-1}(\log A(\chi) + n_E).$$

As an explicit version to this, Winckler in [30, Lemma 4.7] proved:

Lemma 2.17. *For all real ϵ such that $0 < \epsilon \leq 1$, we have*

$$\left|B(\chi) + \sum_{|\rho|<\epsilon} \frac{1}{\rho}\right| \leq c_5 \log(A(\chi)) + c_6 n_E$$

where $c_5 = \frac{1}{8}\bigg(5\pi^2 + 34 + \frac{10}{\epsilon}\bigg)$ and $c_6 = \bigg(\frac{10842}{107} + \frac{1790}{157\epsilon}\bigg)$.

Lagarias and Odlyzko in [15, Lemma 5.6] proved that if $s = \sigma + it$ with $-\frac{1}{2} \leq \sigma \leq 3$ and $|s| \geq \frac{1}{8}$, then

$$\left|\frac{L'}{L}(s,\chi) + \frac{\delta(\chi)}{s-1} - \sum_{\substack{\rho \\ |\gamma - t| \leq 1}} \frac{1}{s-\rho}\right| \ll \log A(\chi) + n_E \log(|t|+2).$$

Winckler in [30, Lemma 4.8] proved an explicit version to this as :

Lemma 2.18. *If $s = \sigma + it$ with $-\frac{1}{2} \leq \sigma \leq 3$ and $|s| \geq \frac{1}{8}$, then*

$$\begin{aligned}\left|\frac{L'}{L}(s,\chi) + \frac{\delta(\chi)}{s-1} - \sum_{\substack{\rho \\ |\gamma - t| \leq 1}} \frac{1}{s-\rho}\right| &\leq c_7 \log(A(\chi)) + \frac{n_E}{2}\log(|t|+5)\bigg(c_8 + \frac{c_9}{|t|+4}\bigg) \\ &+ c_{10} n_E + c_{11}, \end{aligned} \tag{2.59}$$

where $c_7 = \frac{5}{4}\left(1 + \frac{7}{4}\pi^2\right), c_8 = 57, c_9 = 35, c_{10} = \frac{50096}{255}$ *and* $c_{11} = \frac{53}{6}$.

2.6 The contour integral

In this section, we evaluate $I_C(x,T)$ by evaluating

$$I_\chi(x,T) = \frac{1}{2\pi i}\int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{x^s}{s}\left(\frac{L'}{L}(s,\chi)\right)ds, \tag{2.60}$$

for each character χ of $H =< g >$. From here onwards we impose an additional condition on $T \geq 1$. T should not coincide with the ordinate of a zero of any of the $L(s,\chi)$. We also introduce a new parameter, U, which satisfies $U = j + \frac{1}{2}$ for some non-negative integer j (with the aim of letting $U \to \infty$). We define

$$I_\chi(x,T,U) = \frac{1}{2\pi i}\int_{B_{T,U}} \frac{x^s}{s}\left(\frac{L'}{L}(s,\chi)\right)ds, \tag{2.61}$$

where $B_{T,U}$ is the positively oriented rectangle with vertices at $\sigma_0 - iT$, $\sigma_0 + iT$, $-U + iT$ and $-U - iT$. In this section we show that the difference

$$R_\chi(x,T,U) = I_\chi(x,T,U) - I_\chi(x,T) \tag{2.62}$$

is small. To do this, we first divide the remainder $R_\chi(x,T,U)$ into one vertical integral and two horizontal integral. The vertical integral is given by

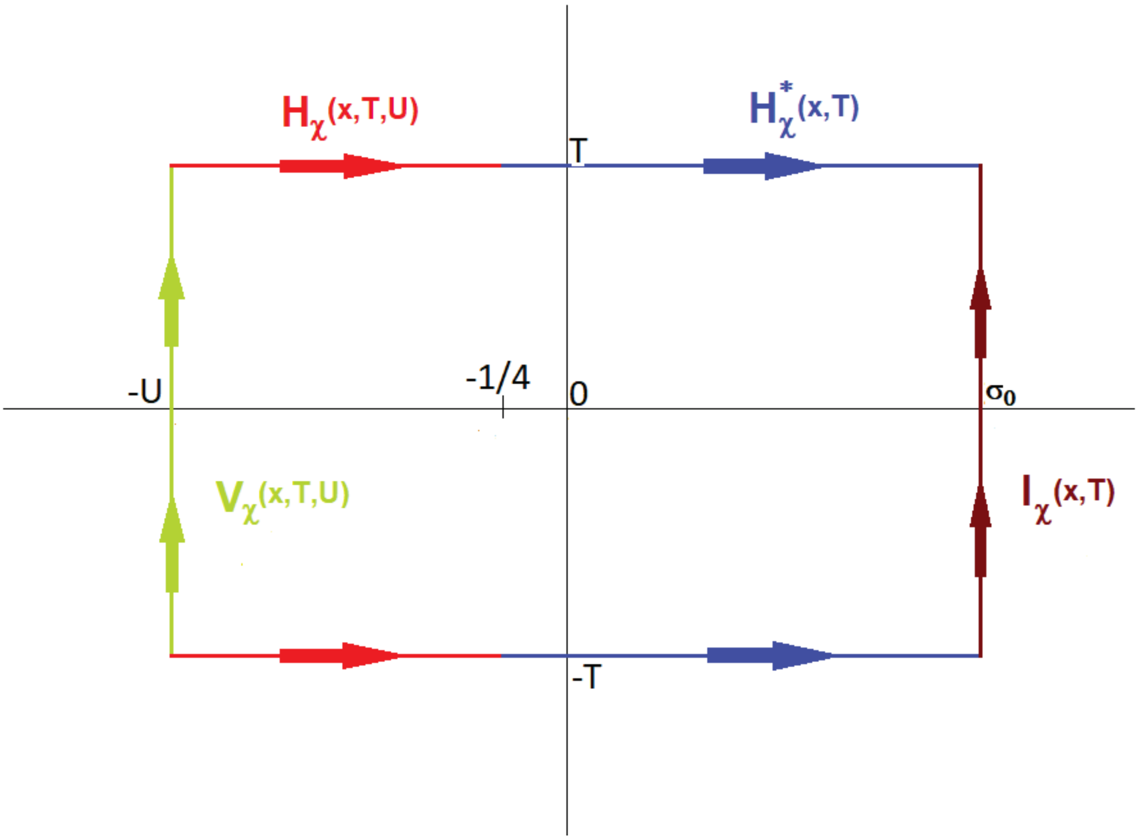
$$V_\chi(x,T,U) = \frac{1}{2\pi}\int_{T}^{-T} \frac{x^{-U+it}}{-U+it}\left(\frac{L'}{L}(-U+it,\chi)\right)dt \tag{2.63}$$

and the two horizontal integrals are given by

$$H_\chi(x,T,U) = \frac{1}{2\pi i}\int_{-U}^{-1/4} \left(\frac{x^{\sigma - iT}}{\sigma - iT}\left(\frac{L'}{L}(\sigma - iT,\chi)\right) - \frac{x^{\sigma + iT}}{\sigma + iT}\left(\frac{L'}{L}(\sigma + iT,\chi)\right)\right)d\sigma, \tag{2.64}$$

and

$$H^*_\chi(x,T) = \frac{1}{2\pi i}\int_{-1/4}^{\sigma_0} \left(\frac{x^{\sigma - iT}}{\sigma - iT}\left(\frac{L'}{L}(\sigma - iT,\chi)\right) - \frac{x^{\sigma + iT}}{\sigma + iT}\left(\frac{L'}{L}(\sigma + iT,\chi)\right)\right)d\sigma. \tag{2.65}$$



In order to bound these integrals, we need the result relating to L'/L. Lagarias and Odlyzko in [15, Lemma 6.2] proved that if $s = \sigma + it$ with $\sigma \leq -1/4$, and $|s+m| \geq 1/4$ for all non-negative integer m, then

$$\frac{L'}{L}(s,\chi) \ll \log(A(\chi)) + n_E \log(|s|+2). \tag{2.66}$$

As an explicit version to this, we prove

Lemma 2.19. *If $s = \sigma + it$ with $\sigma \leq -1/4$, and $|s+m| \geq 1/4$ for all non-negative integer m, then*

$$\left|\frac{L'}{L}(s,\chi)\right| \leq \log(A(\chi)) + n_E\Big(\log(1+|s|) + 4.452 + c_1\Big), \tag{2.67}$$

with $c_1 = \frac{83}{5}$.

Proof. Now combining the logarithmic derivative of (2.47) and (2.48), we get

$$\frac{L'}{L}(s,\chi) = -\frac{L'}{L}(1-s,\overline{\chi}) - \log(A(\chi)) - \frac{\gamma'_\chi}{\gamma_\chi}(1-s) - \frac{\gamma'_\chi}{\gamma_\chi}(s). \tag{2.68}$$

Since $\sigma \leq -1/4$, therefore $\mathrm{Re}(1-s) \geq 5/4$. Thus we use Lemma 2.13 to obtain

$$\left|\frac{L'}{L}(1-s,\overline{\chi})\right| \leq \frac{n_E}{(\mathrm{Re}(1-s))-1} \leq \frac{n_E}{\frac{5}{4}-1} = 4n_E. \tag{2.69}$$

Now since $|s+m| \geq 1/4$ for all non-negative integer m, therefore $\left|\frac{s}{2}+m\right| \geq \frac{1}{8}$ and $\left|\frac{s+1}{2}+m\right| \geq \frac{1}{8}$. Thus from Lemma 2.14, it follows that

$$\left|\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right)\right| \leq \log\left(\left|\frac{s}{2}\right|\right) + c_1, \quad \text{and} \quad \left|\frac{\Gamma'}{\Gamma}\left(\frac{s+1}{2}\right)\right| \leq \log\left(\left|\frac{s+1}{2}\right|\right) + c_1. \tag{2.70}$$

Also, the logarithmic derivative of (2.46) gives us

$$\left|\frac{\gamma_\chi'}{\gamma_\chi}(s)\right| = \frac{n_E}{2}\log(\pi) + \frac{b}{2}\left|\frac{\Gamma'}{\Gamma}\left(\frac{s+1}{2}\right)\right| + \frac{a}{2}\left|\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right)\right|. \tag{2.71}$$

Similarly, since $|1-s+m| \geq 1/4$ for all non-negative integer m, therefore $\left|\frac{1-s}{2}+m\right| \geq \frac{1}{8}$ and $\left|\frac{s}{2}+m\right| = \left|\frac{(1-s)+1}{2}+m\right| \geq \frac{1}{8}$. Thus from Lemma 2.14, it follows that

$$\left|\frac{\Gamma'}{\Gamma}\left(\frac{1-s}{2}\right)\right| \leq \log\left(\left|\frac{1-s}{2}\right|\right) + c_1. \tag{2.72}$$

Also, the logarithmic derivative of (2.46) gives us

$$\left|\frac{\gamma_\chi'}{\gamma_\chi}(1-s)\right| = \frac{n_E}{2}\log(\pi) + \frac{b}{2}\left|\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right)\right| + \frac{a}{2}\left|\frac{\Gamma'}{\Gamma}\left(\frac{1-s}{2}\right)\right|. \tag{2.73}$$

Now combining (2.68) with (2.69), (2.70), (2.71), (2.72) and (2.73), we get

$$\begin{aligned}
\left|\frac{L'}{L}(s,\chi)\right| &\le 4n_E + \log(A(\chi)) + \left|\frac{\gamma_\chi'}{\gamma_\chi}(1-s)\right| + \left|\frac{\gamma_\chi'}{\gamma_\chi}(s)\right| \\
&\le 4n_E + \log(A(\chi)) + \frac{n_E}{2}\log(\pi) + \frac{b}{2}\left|\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right)\right| + \frac{a}{2}\left|\frac{\Gamma'}{\Gamma}\left(\frac{1-s}{2}\right)\right| \\
&\quad + \frac{n_E}{2}\log(\pi) + \frac{b}{2}\left|\frac{\Gamma'}{\Gamma}\left(\frac{s+1}{2}\right)\right| + \frac{a}{2}\left|\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right)\right| \\
&\le n_E(4+\log(\pi)) + \log(A(\chi)) + \frac{n_E}{2}\left(\log\left(\left|\frac{s}{2}\right|\right) + c_1\right) \\
&\quad + \frac{a}{2}\left(\log\left(\left|\frac{1-s}{2}\right|\right) + c_1\right) + \frac{b}{2}\left(\log\left(\left|\frac{1+s}{2}\right|\right) + c_1\right) \\
&\le n_E(4+c_1+\log(\pi)) + \log(A(\chi)) + n_E\left(\log\left(\frac{1+|s|}{2}\right)\right) \\
&= n_E(4+c_1+\log(\pi)-\log 2) + \log(A(\chi)) + n_E\left(\log(1+|s|)\right) \\
&\le \log(A(\chi)) + n_E\left(\log(1+|s|) + 4.452 + c_1\right).
\end{aligned}$$

□

Remark 2.20. Winckler in [30, Lemma 5.1] proved that under the same condition as that of Lemma 2.19,

$$\left|\frac{L'}{L}(s,\chi)\right| \le \log(A(\chi)) + n_E\left(\log(2+|s|) + \frac{19683}{812}\right). \tag{2.74}$$

2.6.1 Bounding $V_\chi(x,T,U)$ and $H_\chi(x,T,U)$

Now we observe that since $U = \frac{1}{2}+j$ for some non-negative integer j, we get $|-U+it+m| \ge 1/4$. Thus we employ Lemma 2.19 to bound $V_\chi(x,T,U)$ and $H_\chi(x,T,U)$. Lagarias and Odlyzko in [15, (6.8),(6.9)] showed that

$$V_\chi(x,T,U) \ll \frac{x^{-U}}{U}T(\log(A(\chi)) + n_E\log(T+U)), \tag{2.75}$$

and

$$H_\chi(x,T,U) \ll \frac{x^{-1/4}}{T}(\log(A(\chi)) + n_E\log T). \tag{2.76}$$

We prove explicit versions of (2.75) and (2.76). We have:

Lemma 2.21. *If $x \ge 2$, $T \ge 1$ and T does not coincide with the ordinate of a zero of any of* *the $L(s,\chi)$, then*

$$|V_\chi(x,T,U)| \leq \frac{x^{-U}T}{\pi U}\Bigg(\log(A(\chi)) + n_E(\log(U+T+1) + 4.452 + c_1)\Bigg), \tag{2.77}$$

and

$$\begin{aligned} |H_\chi(x,T,U)| &\leq \frac{x^{-1/4}}{\pi T(\log x)}\Bigg[\log(A(\chi)) + n_E\Bigg(\log\left(\frac{5}{4}+T\right) + 4.452 + c_1\Bigg)\Bigg] \\ &\quad + \frac{n_E}{\pi T\left(\frac{5}{4}+T\right)}\frac{x^{-1/4}}{(\log x)^2}, \end{aligned} \tag{2.78}$$

where c_1 is defined in Lemma 2.19.

Proof. Using Lemma 2.19, we have

$$\begin{aligned} |V_\chi(x,T,U)| &= \left|\frac{1}{2\pi}\int_T^{-T}\frac{x^{-U+it}}{-U+it}\left(\frac{L'}{L}(-U+it,\chi)\right)dt\right| \\ &\leq \frac{x^{-U}}{2\pi U}\int_{-T}^{T}\left|\frac{L'}{L}(-U+it,\chi)\right|dt \\ &\leq \frac{x^{-U}}{2\pi U}\int_{-T}^{T}\Bigg(\log(A(\chi)) + n_E\Bigg(\log(1+|s|) + 4.452 + c_1\Bigg)\Bigg)dt \\ &\leq \frac{x^{-U}T}{\pi U}\Bigg(\log(A(\chi)) + n_E(\log(U+T+1) + 4.452 + c_1)\Bigg), \end{aligned}$$

and

$$\begin{aligned} |H_\chi(x,T,U)| &= \left|\frac{1}{2\pi i}\int_{-U}^{-1/4}\left(\frac{x^{\sigma-iT}}{\sigma-iT}\left(\frac{L'}{L}(\sigma-iT,\chi)\right) - \frac{x^{\sigma+iT}}{\sigma+iT}\left(\frac{L'}{L}(\sigma+iT,\chi)\right)\right)d\sigma\right| \\ &\leq \frac{1}{\pi T}\int_{-U}^{-1/4}x^\sigma\left|\frac{L'}{L}(\sigma+iT,\chi)\right|d\sigma \\ &\leq \frac{1}{\pi T}\int_{-U}^{-1/4}x^\sigma\Bigg(\log(A(\chi)) + n_E\Bigg(\log(1+|s|) + 4.452 + c_1\Bigg)\Bigg)d\sigma \\ &\leq \frac{1}{\pi T}\int_{-U}^{-1/4}x^\sigma\Bigg(\log(A(\chi)) + n_E\Bigg(\log(1+|\sigma|+T) + 4.452 + c_1\Bigg)\Bigg)d\sigma \\ &\leq \frac{1}{\pi T}\int_{-\infty}^{-1/4}x^\sigma\Bigg(\log(A(\chi)) + n_E\Bigg(\log(1+|\sigma|+T) + 4.452 + c_1\Bigg)\Bigg)d\sigma. \end{aligned}$$

Now

$$(\log(A(\chi)) + n_E(4.452 + c_1)) \int_{-\infty}^{-1/4} x^{\sigma} d\sigma = (\log(A(\chi)) + n_E(4.452 + c_1)) \left(\frac{x^{-1/4}}{\log x} \right), \tag{2.79}$$

and we employ integration by parts to obtain

$$\int_{-\infty}^{-1/4} x^{\sigma} (\log(1 + T + |\sigma|)) d\sigma \quad \leq \quad \left(\log\left(\frac{5}{4} + T\right)\right) \frac{x^{-1/4}}{\log x} + \frac{1}{\left(\frac{5}{4} + T\right)} \frac{x^{-1/4}}{(\log x)^2} \tag{2.80}$$

Now combining (2.79), (2.79) and (2.80), we obtain

$$|H_{\chi}(x, T, U)| \leq \frac{x^{-1/4}}{\pi T(\log x)} \left[\log(A(\chi)) + n_E \left(\log\left(\frac{5}{4} + T\right) + 4.452 + c_1 \right) \right] + \frac{n_E}{\pi T \left(\frac{5}{4} + T\right)} \frac{x^{-1/4}}{(\log x)^2}.$$

□

Remark 2.22. Winckler in [30, (30),(31)] proved that

$$|V_{\chi}(x, T, U)| \leq \frac{x^{-U} T}{\pi U} \left(\log(A(\chi)) + n_E \left(\log(U + T + 2) + \frac{19683}{812} \right) \right), \tag{2.81}$$

and

$$|H_{\chi}(x, T, U)| \leq \frac{x^{-1/4}}{\pi T(\log x)} \left[\log(A(\chi)) + n_E \left(\log\left(\frac{9}{4} + T\right) + \frac{19683}{812} \right) \right] + \frac{n_E}{\pi T \left(\frac{9}{4} + T\right)} \frac{x^{-1/4}}{(\log x)^2}. \tag{2.82}$$

2.6.2 Bounding $H^*_{\chi}(x, T)$

Now we give an estimate for $H^*_{\chi}(x, T)$. Lagarias and Odlyzko in [15, (6.12)] proved that

$$H^*_{\chi}(x, T) \ll \frac{x \log x}{T} (\log(A(\chi)) + n_E \log T), \tag{2.83}$$

and as a result, they showed in [15, (6.13)] that,

$$\begin{aligned} I_\chi(x,T) - I_\chi(x,T,U) &= -V_\chi(x,T,U) - H_\chi(x,T,U) - H^*_\chi(x,T) \\ &\ll \frac{x(\log x)}{T}(\log(A(\chi)) + n_E \log T) + \frac{Tx^{-U}}{U}(\log(A(\chi)) + n_E(\log(T+U)). \end{aligned} \tag{2.84}$$

We prove explicit versions of (2.83) and (2.84).

Lemma 2.23. *Suppose $\rho = \beta + i\gamma$, with $0 < \beta < 1$ and $\gamma \neq t$. If $|t| \geq 2$, $x \geq 2$ and $1 < \sigma_1 \leq 3$, then*

$$\left| \int_{-1/4}^{\sigma_1} \frac{x^{\sigma+it}}{(\sigma+it)(\sigma+it-\rho)} d\sigma \right| \leq \frac{(12\sigma_1 + 27)x^{\sigma_1}}{(|t|-1)(\sigma_1 - \beta)}. \tag{2.85}$$

Proof. First we suppose that $\gamma > t$. Let B be the rectangle with vertices at $\sigma_1 + i(t-1)$, $\sigma_1 + it$, $-\frac{1}{4} + it$ and $-\frac{1}{4} + i(t-1)$, oriented counterclockwise. By Cauchy's theorem,

$$\int_B \frac{x^s}{s(s-\rho)} ds = 0,$$

since the integrand has no singularities inside the contour. Also, on the three sides of the rectangle other than the segment from $-\frac{1}{4} + it$ to $\sigma_1 + it$, since $0 < \sigma_1 - \beta \leq 3$ and $\frac{1}{4} < \frac{1}{4} + \beta < \frac{5}{4}$, therefore the integrand is majorized by

$$\frac{12x^{\sigma_1}}{(|t|-1)(\sigma_1 - \beta)} \tag{2.86}$$

which proves the result for $\gamma > t$. A similar proof for $\gamma < t$ uses the rectangle with vertices at $\sigma_1 + i(t+1)$, $\sigma_1 + it$, $-\frac{1}{4} + it$ and $-\frac{1}{4} + i(t+1)$. □

Revision 4. *Winckler in [30, Lemma 5.2] stated that the $\frac{x^{\sigma_1}}{(|t|-1)(\sigma_1-\beta)}$ is enough to majorize the corresponding integrands majorized by $\frac{12x^{\sigma_1}}{(|t|-1)(\sigma_1-\beta)}$ as in* (2.86) *which is not correct and we corrected it by multiplying a factor of 12 to the numerator as shown in* (2.86)*. For example, taking $\sigma_1 = 3$, $\left|\frac{x^s}{s(s-\rho)}\right|$ in the segment $-\frac{1}{4} + i(t-1)$ to $-\frac{1}{4} + it$, is majorized by $\frac{x^3}{(|t|-1)(\frac{1}{4}+\beta)}$. Using Winckler's assumption, we have $(3-\beta) \leq \frac{1}{4} + \beta$ equivalent to $2\beta \geq \frac{11}{4}$ which is incorrect because $\beta < 1$.*

Lemma 2.24. *If $x \geq 2$, $T \geq 1$ and T does not coincide with the ordinate of a zero of any of the $L(s,\chi)$, then*

$$|H_\chi^*(x,T)| \leq \frac{ex - x^{-1/4}}{\pi T(\log x)}\left(c_7 \log(A(\chi)) + \frac{n_E}{2}\log(T+5)\left(c_8 + \frac{c_9}{T+4}\right) + \frac{\delta(\chi)}{T}\right) + \frac{e}{2\pi}\frac{(12\sigma_0+27)x}{(T-1)(\sigma_0-1)}\left(c_3\left(\log(A(\chi)) + n_E\left(\log(T+3)+c_4\right)\right)\right). \tag{2.87}$$

Proof. Notice that for $\sigma \in [-\frac{1}{4}, \sigma_0]$, $x \geq 2$ and $T \geq 2$, Lemma 2.18 gives us

$$\left|\frac{L'}{L}(\sigma+iT,\chi) - \sum_{\substack{\rho \\ |\gamma-T|\leq 1}} \frac{1}{\sigma+iT-\rho}\right| \leq c_7 \log(A(\chi)) + \frac{n_E}{2}\log(T+5)\left(c_8 + \frac{c_9}{T+4}\right) + \frac{\delta(\chi)}{T}, \tag{2.88}$$

and

$$\left|\frac{L'}{L}(\sigma-iT,\chi) - \sum_{\substack{\rho \\ |\gamma+T|\leq 1}} \frac{1}{\sigma-iT-\rho}\right| \leq c_7 \log(A(\chi)) + \frac{n_E}{2}\log(T+5)\left(c_8 + \frac{c_9}{T+4}\right) + \frac{\delta(\chi)}{T}. \tag{2.89}$$

Therefore using (2.88) and (2.89), we get

$$\begin{aligned}
&\left|H_\chi^*(x,T) - \frac{1}{2\pi i}\int_{-1/4}^{\sigma_0}\left(\frac{x^{\sigma-iT}}{\sigma-iT}\sum_{\substack{\rho\\|\gamma+T|\le 1}}\frac{1}{\sigma-iT-\rho} - \frac{x^{\sigma+iT}}{\sigma+iT}\sum_{\substack{\rho\\|\gamma-T|\le 1}}\frac{1}{\sigma+iT-\rho}\right)d\sigma\right| \\
&= \left|\frac{1}{2\pi i}\int_{-1/4}^{\sigma_0}\left(\frac{x^{\sigma-iT}}{\sigma-iT}\left(\frac{L'}{L}(\sigma-iT,\chi) - \sum_{\substack{\rho\\|\gamma+T|\le 1}}\frac{1}{\sigma-iT-\rho}\right)\right.\right. \\
&\qquad\left.\left. - \frac{x^{\sigma+iT}}{\sigma+iT}\left(\frac{L'}{L}(\sigma+iT,\chi) - \sum_{\substack{\rho\\|\gamma-T|\le 1}}\frac{1}{\sigma+iT-\rho}\right)\right)d\sigma\right| \\
&\le \frac{1}{2\pi}\int_{-1/4}^{\sigma_0}\frac{x^\sigma}{T}\left(\left|\frac{L'}{L}(\sigma-iT,\chi) - \sum_{\substack{\rho\\|\gamma+T|\le 1}}\frac{1}{\sigma-iT-\rho}\right|\right. \\
&\qquad\left. + \left|\frac{L'}{L}(\sigma+iT,\chi) - \sum_{\substack{\rho\\|\gamma-T|\le 1}}\frac{1}{\sigma+iT-\rho}\right|\right)d\sigma \\
&\le \frac{1}{\pi T}\left(c_7\log(A(\chi)) + \frac{n_E}{2}\log(T+5)\left(c_8+\frac{c_9}{T+4}\right) + \frac{\delta(\chi)}{T}\right)\int_{-1/4}^{\sigma_0}x^\sigma d\sigma \\
&= \frac{ex - x^{-1/4}}{\pi T(\log x)}\left(c_7\log(A(\chi)) + \frac{n_E}{2}\log(T+5)\left(c_8+\frac{c_9}{T+4}\right) + \frac{\delta(\chi)}{T}\right). \qquad (2.90)
\end{aligned}$$

Now using Lemma 2.23, we get

$$\begin{aligned}
\left|\frac{1}{2\pi i}\int_{-1/4}^{\sigma_0}\frac{x^{\sigma-iT}}{\sigma-iT}\left(\sum_{\substack{\rho\\|\gamma+T|\le 1}}\frac{1}{\sigma-iT-\rho}\right)d\sigma\right| &= \left|\frac{1}{2\pi i}\sum_{\substack{\rho\\|\gamma+T|\le 1}}\left(\int_{-1/4}^{\sigma_0}\frac{x^{\sigma-iT}}{(\sigma-iT)(\sigma-iT-\rho)}d\sigma\right)\right| \\
&\le \frac{1}{2\pi}\sum_{\substack{\rho\\|\gamma+T|\le 1}}\left|\int_{-1/4}^{\sigma_0}\frac{x^{\sigma-iT}}{(\sigma-iT)(\sigma-iT-\rho)}d\sigma\right| \\
&\le \frac{1}{2\pi}\sum_{\substack{\rho\\|\gamma+T|\le 1}}\frac{(12\sigma_0+27)x^{\sigma_0}}{(T-1)(\sigma_0-\beta)} \\
&\le \frac{1}{2\pi}\frac{(12\sigma_0+27)x^{\sigma_0}}{(T-1)(\sigma_0-1)}\sum_{\substack{\rho\\|\gamma+T|\le 1}}1 \\
&= \frac{1}{2\pi}\frac{(12\sigma_0+27)x^{\sigma_0}}{(T-1)(\sigma_0-1)}n_\chi(-T). \qquad (2.91)
\end{aligned}$$

Similarly, we get

$$\left|\frac{1}{2\pi i}\int_{-1/4}^{\sigma_0}\frac{x^{\sigma+iT}}{\sigma+iT}\left(\sum_{\substack{\rho\\|\gamma-T|\le 1}}\frac{1}{\sigma+iT-\rho}\right)d\sigma\right| \le \frac{1}{2\pi}\frac{(12\sigma_0+27)x^{\sigma_0}}{(T-1)(\sigma_0-1)}n_\chi(T). \qquad (2.92)$$

Now combining (2.90), (2.91), (2.92) and Lemma 2.16, we get

$$\begin{aligned}|H_\chi^*(x,T)| &\le \frac{ex-x^{-1/4}}{\pi T(\log x)}\left(c_7\log(A(\chi))+\frac{n_E}{2}\log(T+5)\left(c_8+\frac{c_9}{T+4}\right)+\frac{\delta(\chi)}{T}\right)\\ &+\frac{1}{2\pi}\frac{(12\sigma_0+27)x^{\sigma_0}}{(T-1)(\sigma_0-1)}(n_\chi(T)+n_\chi(-T))\\ &\le \frac{ex-x^{-1/4}}{\pi T(\log x)}\left(c_7\log(A(\chi))+\frac{n_E}{2}\log(T+5)\left(c_8+\frac{c_9}{T+4}\right)+\frac{\delta(\chi)}{T}\right)\\ &+\frac{e}{2\pi}\frac{(12\sigma_0+27)x}{(T-1)(\sigma_0-1)}\left(c_3\left(\log(A(\chi))+n_E\left(\log(T+3)+c_4\right)\right)\right).\end{aligned}$$

□

2.6.3 Bounding $R_\chi(x,T,U)$

Lemma 2.25. *If $x\ge 2$, $T\ge 1$ and T does not coincide with the ordinate of a zero of any of the $L(s,\chi)$, then*

$$\begin{aligned}|R_\chi(x,T,U)| &\le \frac{x^{-U}T}{\pi U}\left(\log(A(\chi))+n_E(\log(U+T+1)+4.452+c_1)\right)\\ &+\frac{x^{-1/4}}{\pi T(\log x)}\Bigg[(1-c_7)\log(A(\chi))+n_E\Bigg(\log\left(\frac{5}{4}+T\right)+4.452\\ &\qquad +c_1-\frac{1}{2}\log(T+5)\left(c_8+\frac{c_9}{T+4}\right)\Bigg)-\frac{\delta(\chi)}{T}\Bigg]\\ &+\frac{n_E}{\pi T\left(\frac{5}{4}+T\right)}\frac{x^{-1/4}}{(\log x)^2}\\ &+\frac{e}{\pi}\frac{x}{T(\log x)}\left(c_7\log(A(\chi))+\frac{n_E}{2}\log(T+5)\left(c_8+\frac{c_9}{T+4}\right)+\frac{\delta(\chi)}{T}\right)\\ &+\frac{39e}{2\pi}\frac{x(\log x)}{T-1}\left(c_3\left(\log(A(\chi))+n_E\left(\log(T+3)+c_4\right)\right)\right)\\ &+\frac{6e}{\pi}\frac{x}{T-1}\left(c_3\left(\log(A(\chi))+n_E\left(\log(T+3)+c_4\right)\right)\right).\end{aligned}\tag{2.93}$$

Proof. From (2.62), remember that

$$|R_\chi(x,T,U)|=|I_\chi(x,T,U)-I_\chi(x,T)|\le|V_\chi(x,T,U)|+|H_\chi(x,T,U)|+|H_\chi^*(x,T)|.$$

Now combining (2.77), (2.78) and (2.87) with the fact that $\sigma_0=1+(\log x)^{-1}$, we get the required result. □

Remark 2.26. Winckler in [30, (32)] showed that

$$|H_\chi^*(x,T)| \le \left(\frac{5(13(\log x)+4)ex}{8(T-1)} + \frac{571}{25}\frac{ex - x^{-1/4}}{T(\log x)}\right)\frac{\log(A(\chi))}{\pi} + \frac{n_E}{\pi}\left(\left(\frac{57}{2}\frac{ex - x^{-1/4}}{T(\log x)} + \frac{5(13(\log x)+4)ex}{8(T-1)}\right)\log(T+5) + \frac{5375(13(\log x)+4)ex}{2144(T-1)} + \frac{5921}{28}\frac{ex - x^{-1/4}}{T(\log x)}\right),$$

and thus in [30, (33)] showed that

$$\begin{aligned}|I_\chi(x,T) - I_\chi(x,T,U)| \le{}& \frac{65e}{8\pi}\frac{x(\log x)}{(T-1)}\left[\log(A(\chi)) + n_E\left(\log(T+5) + \frac{1075}{268}\right)\right]\\ &+ \frac{5e}{2\pi}\frac{x}{(T-1)}\left[\log(A(\chi)) + n_E\left(\log(T+5) + \frac{1075}{268}\right)\right]\\ &+ \frac{e}{\pi}\frac{x}{T(\log x)}\left[\frac{571}{25}\log(A(\chi)) + n_E\left(\frac{57}{2}\log(T+5) + \frac{5921}{28}\right)\right]\\ &+ \frac{x^{-U}T}{\pi U}\left[\log(A(\chi)) + n_E\left(\log(U+T+2) + \frac{19683}{812}\right)\right]\\ &+ \frac{4n_E x^{-1/4}}{17\pi T(\log x)^2}.\end{aligned}$$

2.7 The explicit formula

2.7.1 Estimating $I_\chi(x,T,U)$

Recall that the integral $I_\chi(x,T,U)$ is defined in (2.61). We first evaluate this integral. We recall that $x \ge 2$, $U = j + \frac{1}{2}$ for some non-negative integer j, and $T \ge 2$ does not equal the ordinate of any zero of any of the $L(s,\chi)$. By Cauchy's residue theorem, $I_\chi(x,T,U)$ equals the sum of the residues of the integrand at poles inside $B_{T,U}$. Now if $\chi = \chi_1$, the principal character, then L'/L has a first order pole of residue -1 at $s = 1$, and hence (this term being absent if $\chi \ne \chi_1$) we obtain a contribution of $-\delta(\chi)x$ from the possible pole at $s = 1$. Further, L'/L has a first order pole with residues $+1$ at each non-trivial zero ρ of $L(s,\chi)$ (the ρ's are counted according to their multiplicity), and so such ρ's contribute $\sum_\rho \frac{x^\rho}{\rho}$. In addition, L'/L has first order poles at the so-called trivial zeros, which are real and non-positive. Also, the functional equation of $L(s,\chi)$ in (2.47) and study of Γ-function shows that L'/L has first order poles at $s = -(2m-1)$, $m = 1, 2, ...$ where the residue is $b(\chi)$, and first order poles at $s = -2m$, $m = 0, 1, 2, ...$ where the residue is $a(\chi)$. Therefore, the residues at points s with $\mathrm{Re}(s) < 0$

contribute

$$-b(\chi)\sum_{m=1}^{\left[\frac{U+1}{2}\right]}\frac{x^{-(2m-1)}}{2m-1}-a(\chi)\sum_{m=1}^{\left[\frac{U}{2}\right]}\frac{x^{-2m}}{2m}.$$

Now the only remaining residue is at $s=0$, where we have the complication that both x^s/s and L'/L may have first order poles. The Laurent series expansions of $\frac{x^s}{s}$ about $s=0$ gives us

$$\frac{x^s}{s}=\frac{1}{s}+\log x+sh_1(s),$$

where $h_1(s)$ is a function that is analytic at $s=0$. Similarly, the Laurent series expansion of L'/L as defined in (2.52) about $s=0$ shows that

$$\frac{L'}{L}(s,\chi)=\frac{a(\chi)-\delta(\chi)}{s}+r(\chi)+sh_2(s),$$

where $h_2(s)$ is a function that is analytic at $s=0$ and

$$r(\chi)=B(\chi)-\frac{1}{2}(\log A(\chi))+\frac{n_E}{2}(\log\pi)+\delta(\chi)-\frac{b(\chi)}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}\right)-\frac{a(\chi)}{2}\frac{\Gamma'}{\Gamma}(1). \tag{2.94}$$

Therefore the residue of $\frac{x^s}{s}\frac{L'}{L}(s,\chi)$ at $s=0$ is $r(\chi)+(a(\chi)-\delta(\chi))\log(x)$. Now, combining all these residue terms, we get

$$I_\chi(x,T,U)=-\delta(\chi)x+\sum_{\substack{\rho\\|\gamma|<T}}\frac{x^\rho}{\rho}-b(\chi)\sum_{m=1}^{\left[\frac{U+1}{2}\right]}\frac{x^{1-2m}}{2m-1}-a(\chi)\sum_{m=1}^{\left[\frac{U}{2}\right]}\frac{x^{-2m}}{2m}+r(\chi)+(a(\chi)-\delta(\chi))\log x. \tag{2.95}$$

From now on going forward, we let $U\to\infty$.

Lemma 2.27. *If $x\geq 2$, $T\geq 2$ and T does not coincide with the ordinate of a zero of any of* *the $L(s,\chi)$, then*

$$\begin{aligned}
&\Bigg|I_\chi(x,T)+\delta(\chi)x-\sum_{\substack{\rho\\|\gamma|<T}}\frac{x^\rho}{\rho}-r(\chi)-(a(\chi)-\delta(\chi))\log x-\frac{n_E}{2}\log(1-x^{-1})\\
&\qquad+\left(\frac{b(\chi)-a(\chi)}{2}\right)\log(1+x^{-1})\Bigg|\\
\le&\frac{x^{-1/4}}{\pi T(\log x)}\Bigg[(1-c_7)\log(A(\chi))+n_E\Bigg(\log\left(\frac{5}{4}+T\right)+4.452\\
&\qquad+c_1-\frac{1}{2}\log(T+5)\Bigg(c_8+\frac{c_9}{T+4}\Bigg)\Bigg)-\frac{\delta(\chi)}{T}\Bigg]\\
&+\frac{n_E}{\pi T\left(\frac{5}{4}+T\right)}\frac{x^{-1/4}}{(\log x)^2}\\
&+\frac{e}{\pi}\frac{x}{T(\log x)}\left(c_7\log(A(\chi))+\frac{n_E}{2}\log(T+5)\left(c_8+\frac{c_9}{T+4}\right)+\frac{\delta(\chi)}{T}\right)\\
&+\frac{39e}{2\pi}\frac{x(\log x)}{T-1}\left(c_3\left(\log(A(\chi))+n_E\left(\log(T+3)+c_4\right)\right)\right)\\
&+\frac{6e}{\pi}\frac{x}{T-1}\left(c_3\left(\log(A(\chi))+n_E\left(\log(T+3)+c_4\right)\right)\right).
\end{aligned}\tag{2.96}$$

Proof. Notice that

$$\begin{aligned}
a(\chi)\sum_{m=1}^{\infty}\frac{x^{-2m}}{2m}+b(\chi)\sum_{m=1}^{\infty}\frac{x^{1-2m}}{2m-1}&=\left(\frac{a(\chi)+b(\chi)}{2}\right)\sum_{n=1}^{\infty}\frac{(x^{-1})^n}{n}+\left(\frac{a(\chi)-b(\chi)}{2}\right)\sum_{n=1}^{\infty}\frac{(-x^{-1})^n}{n}\\
&=-\frac{n_E}{2}\log(1-x^{-1})+\left(\frac{b(\chi)-a(\chi)}{2}\right)\log(1+x^{-1}).
\end{aligned}\tag{2.97}$$

Therefore combining this above equation with (2.95) and (2.93) and using $U\to\infty$, we get the required result. □

Lemma 2.28. *Using the notation as above,*

$$\Bigg|r(\chi)+\sum_{|\rho|<\frac{1}{2}}\frac{1}{\rho}\Bigg|\le\left(c_5+\frac{1}{2}\right)(\log A(\chi))+n_E\left(\frac{\log\pi}{2}+\frac{\gamma}{2}+\log 2+c_6\right)+\delta(\chi).\tag{2.98}$$

Proof. We remember that $b(\chi)\le n_E$, $\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}\right)=-2(\log 2)-\gamma$ and $\frac{\Gamma'}{\Gamma}(1)=-\gamma$. Now using

Lemma 2.17, we get

$$\begin{aligned}
&\left|r(\chi)+\sum_{|\rho|<\frac{1}{2}}\frac{1}{\rho}\right| \\
=&\left|B(\chi)-\frac{1}{2}(\log A(\chi))+\frac{n_E}{2}(\log\pi)+\delta(\chi)-\frac{b(\chi)}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}\right)-\frac{a(\chi)}{2}\frac{\Gamma'}{\Gamma}(1)+\sum_{|\rho|<\frac{1}{2}}\frac{1}{\rho}\right| \\
=&\left|B(\chi)-\frac{1}{2}(\log A(\chi))+\frac{n_E}{2}(\log\pi)+\delta(\chi)+\frac{b(\chi)}{2}(2(\log 2)+\gamma)+\frac{a(\chi)}{2}\gamma+\sum_{|\rho|<\frac{1}{2}}\frac{1}{\rho}\right| \\
\leq&\left|B(\chi)+\sum_{|\rho|<\frac{1}{2}}\frac{1}{\rho}\right|+\frac{\log A(\chi)}{2}+\delta(\chi)+n_E\left(\frac{\log\pi}{2}+\frac{\gamma}{2}+\log 2\right) \qquad (2.99)\\
\leq&c_5\log(A(\chi))+c_6n_E+\frac{\log A(\chi)}{2}+\delta(\chi)+n_E\left(\frac{\log\pi}{2}+\frac{\gamma}{2}+\log 2\right) \\
=&\left(c_5+\frac{1}{2}\right)\log(A(\chi))+n_E\left(\frac{\log\pi}{2}+\frac{\gamma}{2}+\log 2+c_6\right)+\delta(\chi).
\end{aligned}$$

□

Revision 5. *Lagarias and Odlyzko in the proof of [15, Theorem 7.1] state that Lemma 5.5 and (5.4) show*

$$r(\chi)-\sum_{\substack{\rho\\|\rho|<\frac{1}{2}}}\frac{1}{\rho}\ll\log A(\chi)+n_E,$$

which is not correct and we correct it by using the estimate for

$$r(\chi)+\sum_{\substack{\rho\\|\rho|<\frac{1}{2}}}\frac{1}{\rho}$$

as shown in Lemma 2.28.

2.7.2 Explicit bounds depending on T

Lagarias and Odlyzko in [15, Theorem 7.1] proved that for $x\geq 2$ and $T\geq 2$,

$$\psi_C(x)-\frac{|C|}{|G|}x+S(x,T)\ll\frac{|C|}{|G|}\left(\frac{x(\log x)+T}{T}(\log d_L)+n_L(\log x)+\frac{n_Lx(\log x)(\log T)}{T}\right)$$
$$+(\log x)(\log d_L)+n_KxT^{-1}(\log x)^2,$$

where

$$S(x,T) = \frac{|C|}{|G|}\sum_{\chi}\overline{\chi}(g)\Bigg(\sum_{\substack{\rho\\|\gamma|<T}}\frac{x^{\rho}}{\rho} - \sum_{\substack{\rho\\|\rho|<\frac{1}{2}}}\frac{1}{\rho}\Bigg), \tag{2.100}$$

and the inner sums in (2.100) are over the nontrivial zeros ρ of $L(s,\chi)$. We prove an explicit version of this theorem.

Theorem 2.29. *If $x \geq 2$, $T \geq 2$ and T does not coincide with the ordinate of a zero of any of the $L(s,\chi)$, then*

$$\begin{aligned}
&\left|\psi_C(x) - \frac{|C|}{|G|}x + S(x,T)\right| \\
\leq & 2.89\frac{(\log x)(\log d_L)}{|G|} + 13.83 n_K(\log x) + 28.7 n_K T^{-1} x(\log x)^2 + 4.41 n_K \\
&+ \frac{|C|}{|G|}\Bigg(\frac{x^{-1/4}}{\pi T(\log x)}\Bigg[(1-c_7)\log d_L + n_L\Bigg(\log\left(\frac{5}{4}+T\right) + 4.452 \\
&+ c_1 - \frac{1}{2}\log(T+5)\left(c_8 + \frac{c_9}{T+4}\right)\Bigg) - \frac{1}{T}\Bigg] + \frac{n_L}{\pi T\left(\frac{5}{4}+T\right)}\frac{x^{-1/4}}{(\log x)^2} \\
&+ \frac{e}{\pi}\frac{x}{T(\log x)}\left(c_7\log d_L + \frac{n_L}{2}\log(T+5)\left(c_8 + \frac{c_9}{T+4}\right) + \frac{1}{T}\right) \\
&+ \frac{39e}{2\pi}\frac{x(\log x)}{T-1}\Big(c_3\Big(\log d_L + n_L\Big(\log(T+3) + c_4\Big)\Big)\Big) \\
&+ \frac{6e}{\pi}\frac{x}{T-1}\Big(c_3\Big(\log d_L + n_L\Big(\log(T+3) + c_4\Big)\Big)\Big) \\
&+ n_L(\log x) + \left(c_5 + \frac{1}{2}\right)\log d_L + n_L\left(\frac{\log \pi}{2} + \frac{\gamma}{2} + \log 2 + \frac{\log 3}{2} + c_6\right) + 1\Bigg).
\end{aligned} \tag{2.101}$$

where $S(x,T)$ is defined as in (2.100).

Proof. Using (2.10) and triangle inequality, we get

$$\left|\psi_C(x) - \frac{|C|}{|G|}x + S(x,T)\right| \leq R_1(x,T) + \left|I_C(x,T) - \frac{|C|}{|G|}x + S(x,T)\right|. \tag{2.102}$$

Recall that

$$I_C(x,T) = -\frac{|C|}{|G|}\sum_{\chi}\overline{\chi}(g)I_{\chi}(x,T).$$

Therefore using (2.100),

$$I_C(x,T) - \frac{|C|}{|G|}x + S(x,T) = -\frac{|C|}{|G|}\Bigg(\sum_{\chi}\overline{\chi}(g)\Big(I_\chi(x,T) + \delta(\chi)x - \sum_{\substack{\rho\\|\gamma|<T}}\frac{x^\rho}{\rho} + \sum_{\substack{\rho\\|\rho|<\frac{1}{2}}}\frac{1}{\rho}\Big)\Bigg).$$

Also, remember that $|\overline{\chi}(g)| = 1$ for all χ. Thus (2.102) turns out to be

$$\left|\psi_C(x) - \frac{|C|}{|G|}x + S(x,T)\right| \le R_1(x,T) + \frac{|C|}{|G|}\sum_{\chi}\left|I_\chi(x,T) + \delta(\chi)x - \sum_{\substack{\rho\\|\gamma|<T}}\frac{x^\rho}{\rho} + \sum_{\substack{\rho\\|\rho|<\frac{1}{2}}}\frac{1}{\rho}\right|. \tag{2.103}$$

Using triangle inequality, we obtain

$$\begin{aligned}
&\left|I_\chi(x,T) + \delta(\chi)x - \sum_{\substack{\rho\\|\gamma|<T}}\frac{x^\rho}{\rho} + \sum_{\substack{\rho\\|\rho|<\frac{1}{2}}}\frac{1}{\rho}\right| \\
\le &\Bigg|I_\chi(x,T) + \delta(\chi)x - \sum_{\substack{\rho\\|\gamma|<T}}\frac{x^\rho}{\rho} - r(\chi) - (a(\chi)-\delta(\chi))\log x - \frac{n_E}{2}\log(1-x^{-1}) \\
&\qquad + \left(\frac{b(\chi)-a(\chi)}{2}\right)\log(1+x^{-1})\Bigg| \\
+ &\left|r(\chi) + \sum_{\substack{\rho\\|\rho|<\frac{1}{2}}}\frac{1}{\rho}\right| + \Big|(a(\chi)-\delta(\chi))\log x\Big| + \left|\frac{n_E}{2}\log(1-x^{-1})\right| + \left|\left(\frac{b(\chi)-a(\chi)}{2}\right)\log(1+x^{-1})\right|.
\end{aligned} \tag{2.104}$$

Now we recall that $|b(\chi) - a(\chi)| \le n_E$ and $|\log(1-x^{-1})| = \log\Big((1-x^{-1})^{-1}\Big)$. Hence

$$\left|\frac{n_E}{2}\log(1-x^{-1})\right| + \left|\left(\frac{b(\chi)-a(\chi)}{2}\right)\log(1+x^{-1})\right| \le \frac{n_E}{2}\left(\log\left(\frac{x+1}{x-1}\right)\right),$$

which has a maximum value at $x = 2$ for $x \ge 2$ and the value is $\frac{n_E}{2}(\log 3)$. Now combining this result with (2.104), Lemma 2.27, Lemma 2.28 and the fact that $|a(\chi) - \delta(\chi)| \le n_E$, we get

$$\left| I_\chi(x,T) + \delta(\chi)x - \sum_{\substack{\rho \\ |\gamma|<T}} \frac{x^\rho}{\rho} + \sum_{\substack{\rho \\ |\rho|<\frac{1}{2}}} \frac{1}{\rho} \right|$$

$$\begin{aligned}
&\le \frac{x^{-1/4}}{\pi T(\log x)} \Bigg[(1-c_7)\log(A(\chi)) + n_E\Bigg(\log\left(\frac{5}{4}+T\right) + 4.452 \\
&+ c_1 - \frac{1}{2}\log(T+5)\left(c_8 + \frac{c_9}{T+4}\right)\Bigg) - \frac{\delta(\chi)}{T}\Bigg] + \frac{n_E}{\pi T\left(\frac{5}{4}+T\right)} \frac{x^{-1/4}}{(\log x)^2} \\
&+ \frac{e}{\pi}\frac{x}{T(\log x)}\left(c_7\log(A(\chi)) + \frac{n_E}{2}\log(T+5)\left(c_8+\frac{c_9}{T+4}\right) + \frac{\delta(\chi)}{T}\right) \\
&+ \frac{39e}{2\pi}\frac{x(\log x)}{T-1}\left(c_3\left(\log(A(\chi)) + n_E\left(\log(T+3)+c_4\right)\right)\right) \\
&+ \frac{6e}{\pi}\frac{x}{T-1}\left(c_3\left(\log(A(\chi)) + n_E\left(\log(T+3)+c_4\right)\right)\right) \\
&+ n_E(\log x) + \left(c_5+\frac{1}{2}\right)\log(A(\chi)) + n_E\left(\frac{\log\pi}{2} + \frac{\gamma}{2} + \log 2 + \frac{\log 3}{2} + c_6\right) + \delta(\chi). \qquad (2.105)
\end{aligned}$$

Now the conductor-discriminant formula [15, Page 451] gives us

$$\sum_\chi \log A(\chi) = \log d_L. \qquad (2.106)$$

We know that the number of irreducible representations of the cyclic group G_0 is $|G_0|$. Also, by the Fundamental theorem of Galois, $|G_0| = [L:E]$ where E is the fixed field of G_0. Hence, $\sum_\chi n_E = n_E \times [L:E] = n_L$. Also, $\sum_\chi \delta(\chi) = 1$. Now combining this with (2.103), (2.105) and

(2.37), we get

$$\begin{aligned}
&\left|\psi_C(x) - \frac{|C|}{|G|}x + S(x,T)\right| \\
\leq& 2.89\frac{(\log x)(\log d_L)}{|G|} + 13.83n_K(\log x) + 28.7n_K T^{-1}x(\log x)^2 + 4.41n_K \\
&+ \frac{|C|}{|G|}\left(\frac{x^{-1/4}}{\pi T(\log x)}\left[(1-c_7)\log d_L + n_L\left(\log\left(\frac{5}{4}+T\right) + 4.452\right.\right.\right. \\
&\left.\left.\left. + c_1 - \frac{1}{2}\log(T+5)\left(c_8 + \frac{c_9}{T+4}\right)\right) - \frac{1}{T}\right] + \frac{n_L}{\pi T\left(\frac{5}{4}+T\right)}\frac{x^{-1/4}}{(\log x)^2} \\
&+ \frac{e}{\pi}\frac{x}{T(\log x)}\left(c_7\log d_L + \frac{n_L}{2}\log(T+5)\left(c_8 + \frac{c_9}{T+4}\right) + \frac{1}{T}\right) \\
&+ \frac{39e}{2\pi}\frac{x(\log x)}{T-1}(c_3\left(\log d_L + n_L\left(\log(T+3) + c_4\right)\right)) \\
&+ \frac{6e}{\pi}\frac{x}{T-1}(c_3\left(\log d_L + n_L\left(\log(T+3) + c_4\right)\right)) \\
&+ n_L(\log x) + \left(c_5 + \frac{1}{2}\right)\log d_L + n_L\left(\frac{\log\pi}{2} + \frac{\gamma}{2} + \log 2 + \frac{\log 3}{2} + c_6\right) + 1\Bigg).
\end{aligned}$$

□

Theorem 2.30. *If $x \geq 2$, $T \geq 2$, T coincide with the ordinate of a zero of any of the $L(s,\chi)$ and $\epsilon > 0$ such that $T+\epsilon$ does not coincide with the ordinate of a zero of any of the $L(s,\chi)$,* *then*

$$\begin{aligned}
&\left|\psi_C(x) - \frac{|C|}{|G|}x + S(x,T)\right| \\
&\leq 2.89\frac{(\log x)(\log d_L)}{|G|} + 13.83n_K(\log x) + 28.7n_K(T+\epsilon)^{-1}x(\log x)^2 + 4.41n_K \\
&+ \frac{|C|}{|G|}\Bigg(\frac{x^{-1/4}}{\pi(T+\epsilon)(\log x)}\Bigg[(1-c_7)\log d_L + n_L\Bigg(\log\left(\frac{5}{4}+(T+\epsilon)\right) + 4.452 \\
&+ c_1 - \frac{1}{2}\log((T+\epsilon)+5)\left(c_8 + \frac{c_9}{(T+\epsilon)+4}\right)\Bigg) - \frac{1}{T+\epsilon}\Bigg] + \frac{n_L}{\pi(T+\epsilon)\left(\frac{5}{4}+(T+\epsilon)\right)}\frac{x^{-1/4}}{(\log x)^2} \\
&+ \frac{e}{\pi}\frac{x}{(T+\epsilon)(\log x)}\left(c_7\log d_L + \frac{n_L}{2}\log((T+\epsilon)+5)\left(c_8+\frac{c_9}{(T+\epsilon)+4}\right) + \frac{1}{T+\epsilon}\right) \\
&+ \frac{39e}{2\pi}\frac{x(\log x)}{(T+\epsilon)-1}(c_3\Big(\log d_L + n_L\Big(\log((T+\epsilon)+3)+c_4\Big)\Big)) \\
&+ \frac{6e}{\pi}\frac{x}{(T+\epsilon)-1}(c_3\Big(\log d_L + n_L\Big(\log((T+\epsilon)+3)+c_4\Big)\Big)) \\
&+ \frac{x}{T}\Big(c_3\Big(\log d_L + n_L\Big(\log(T+3)+c_4\Big)\Big)\Big) \\
&+ n_L(\log x) + \left(c_5+\frac{1}{2}\right)\log d_L + n_L\left(\frac{\log\pi}{2}+\frac{\gamma}{2}+\log 2+\frac{\log 3}{2}+c_6\right)+1\Bigg).
\end{aligned} \tag{2.107}$$

where $S(x,T)$ *is defined as in* (2.100).

Proof. Let $T = |\gamma|$ for some $\rho = \beta + i\gamma$ where $0 < \beta < 1$. Now we evaluate (2.101) with T replaced by $T+\epsilon$ for the ϵ very close to zero and $T+\epsilon$ not coinciding with the ordinate of a zero of any of the $L(s,\chi)$ to obtain $\left|\psi_C(x) - \frac{|C|}{|G|}x + S(x,T+\epsilon)\right|$. We also notice that

$$S(x,T+\epsilon) = \frac{|C|}{|G|}\sum_{\chi}\overline{\chi}(g)\Bigg(\sum_{\substack{\rho \\ |\gamma|<T+\epsilon}}\frac{x^\rho}{\rho} - \sum_{\substack{\rho \\ |\rho|<\frac{1}{2}}}\frac{1}{\rho}\Bigg) = S(x,T) + \frac{|C|}{|G|}\sum_{\chi}\overline{\chi}(g)\sum_{\substack{\rho \\ |\gamma|=T}}\frac{x^\rho}{\rho}. \tag{2.108}$$

Recall that the number of zeros $\rho = \beta + i\gamma$ with $|\gamma| = T$ is less than or equal to $\sum_\chi(n_\chi(T) + n_\chi(-T))$. Combining this with $|\overline{\chi}(g)| = 1$, (2.58), (2.106), $\sum_\chi n_E = n_L$ and the triangle inequality we get

$$\begin{aligned}
&\left|\psi_C(x) - \frac{|C|}{|G|}x + S(x,T)\right| \\
&\leq \left|\psi_C(x) - \frac{|C|}{|G|}x + S(x,T+\epsilon)\right| + \frac{|C|}{|G|}\left|\sum_{\chi}\overline{\chi}(g)\sum_{\substack{\rho \\ |\gamma|=T}}\frac{x^\rho}{\rho}\right| \\
&= \left|\psi_C(x) - \frac{|C|}{|G|}x + S(x,T+\epsilon)\right| + \frac{|C|}{|G|}\frac{x}{T}\sum_{\chi}(n_\chi(T) + n_\chi(-T)) \\
&\leq \left|\psi_C(x) - \frac{|C|}{|G|}x + S(x,T+\epsilon)\right| + \frac{|C|}{|G|}\frac{x}{T}\sum_{\chi}\left(c_3\left(\log(A(\chi)) + n_E\left(\log(|t|+3) + c_4\right)\right)\right) \\
&\leq \left|\psi_C(x) - \frac{|C|}{|G|}x + S(x,T+\epsilon)\right| + \frac{|C|}{|G|}\frac{x}{T}\left(c_3\left(\log d_L + n_L\left(\log(T+3) + c_4\right)\right)\right). \qquad (2.109)
\end{aligned}$$

Now combining (2.101) with T replaced by $T+\epsilon$ and (2.109), we get the required result. □

Remark 2.31. We can combine Theorem 2.29 and Theorem 2.30 to provide a bound which works independent of the position of T. This is done by taking $\epsilon \to 0$ in (2.107). Thus

$$\begin{aligned}
&\left|\psi_C(x) - \frac{|C|}{|G|}x + S(x,T)\right| \\
&\leq 2.89\frac{(\log x)(\log d_L)}{|G|} + 13.83 n_K(\log x) + 28.7 n_K T^{-1} x(\log x)^2 + 4.41 n_K \\
&+ \frac{|C|}{|G|}\left(\frac{x^{-1/4}}{\pi T(\log x)}\left[(1-c_7)\log d_L + n_L\left(\log\left(\frac{5}{4}+T\right) + 4.452\right.\right.\right. \\
&\left.\left.\left. + c_1 - \frac{1}{2}\log(T+5)\left(c_8 + \frac{c_9}{T+4}\right)\right) - \frac{1}{T}\right] + \frac{n_L}{\pi T\left(\frac{5}{4}+T\right)}\frac{x^{-1/4}}{(\log x)^2} \\
&+ \frac{e}{\pi}\frac{x}{T(\log x)}\left(c_7\log d_L + \frac{n_L}{2}\log(T+5)\left(c_8 + \frac{c_9}{T+4}\right) + \frac{1}{T}\right) \\
&+ \frac{39e}{2\pi}\frac{x(\log x)}{T-1}(c_3\left(\log d_L + n_L\left(\log(T+3) + c_4\right)\right)) \\
&+ \left(\frac{6e}{\pi} + 1\right)\frac{x}{T-1}(c_3\left(\log d_L + n_L\left(\log(T+3) + c_4\right)\right)) \\
&+ n_L(\log x) + \left(c_5 + \frac{1}{2}\right)\log d_L + n_L\left(\frac{\log\pi}{2} + \frac{\gamma}{2} + \log 2 + \frac{\log 3}{2} + c_6\right) + 1\Bigg).
\end{aligned}$$

Revision 6. *Lagarias and Odlyzko in the proof of [15, Theorem 7.1] and Winckler in the proof of [30, Theorem 6.1] used* $-\sum_{|\rho|<\frac{1}{2}}$ *throughout the entire proofs which is not correct since they used the results about* $\sum_{|\rho|<\frac{1}{2}}$ *(check [30, Page 18, last equation]). We corrected this mistake as* *depicted in the proofs of Theorem 2.29 and Theorem 2.30.*

2.8 Zero-free regions

In this section, we prove a zero-free region for $\zeta_L(s)$. A well-known result as stated in [15, (8.1)] shows

$$\zeta_L(s) = \prod_{\chi} L(s, \chi). \tag{2.110}$$

Since L is the fixed field of the abelian group G_0, it has already been proved that each $L(s, \chi)$ is analytic for $s \neq 1$. Also (2.110) implies that any zero-free region for $\zeta_L(s)$ immediately gives a zero-free region for each of the $L(s, \chi)$.

2.8.1 Zero-free region for $\zeta_L(s)$

Lagarias and Odlyzko in [15, Lemma 8.1] proved that there is an absolute, effectively computable positive constant c such that $\zeta_L(s)$ has no zeros $\rho = \beta + i\gamma$ in the region

$$|\gamma| \geq (1 + 4\log d_L)^{-1}$$
$$\beta \geq 1 - c(\log d_L + n_L \log(|\gamma| + 2))^{-1}.$$

We prove an explicit version to this as:

Lemma 2.32. *The ζ_L function has no zeros $\rho = \beta + i\gamma$ in the region*

$$|\gamma| \geq (1 + 4\log d_L)^{-1}$$
$$\beta \geq 1 - (7 - 4\sqrt{3})\left(22(\log d_L) + n_L\left(\frac{5}{2}(\log(|\gamma| + 3)) + 4c_2 + \frac{\log 54}{2}\right) + \frac{25}{2}\right)^{-1}.$$

Proof. We know that

$$-\frac{\zeta_L'}{\zeta_L}(s) = \sum_{m=1}^{\infty} \alpha(m) m^{-s} \tag{2.111}$$

for $\sigma = \mathrm{Re}(s) > 1$, where $\alpha(m) \geq 0$ for all m. Hence

$$\mathrm{Re}\left(-3\frac{\zeta_L'}{\zeta_L}(\sigma) - 4\frac{\zeta_L'}{\zeta_L}(\sigma + it) - \frac{\zeta_L'}{\zeta_L}(\sigma + 2it)\right) = \sum_{m=1}^{\infty} \frac{\alpha(m)}{m^{\sigma}}(3 + 4\cos(t\log m) + \cos(2t\log m)).$$

We also know that

$$3 + 4\cos\theta + \cos 2\theta = 2(1 + \cos\theta)^2 \geq 0.$$

Therefore combining this with the fact that $\alpha(m) \geq 0$ and $m^\sigma > 0$, we obtain that

$$\operatorname{Re}\left(-3\frac{\zeta_L'}{\zeta_L}(\sigma) - 4\frac{\zeta_L'}{\zeta_L}(\sigma + it) - \frac{\zeta_L'}{\zeta_L}(\sigma + 2it)\right) \geq 0. \tag{2.112}$$

Now if we consider the trivial normal extension L of L, then $\zeta_L(s)$ is the Artin L-function associated to the principal character. Let $\gamma_L(s)$ denote the gamma factor associated to $\zeta_L(s)$. Then, (2.54) gives us

$$2\frac{\zeta_L'}{\zeta_L}(s) = \sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{s-\overline{\rho}} \right) - \log d_L - \frac{2}{s} - \frac{2}{s-1} - 2\frac{\gamma_L'}{\gamma_L}(s), \tag{2.113}$$

where the summation is over the non-trivial zeros $\rho = \beta + i\gamma$ of $\zeta_L(S)$. Since $0 < \beta < 1$, therefore if $\operatorname{Re}(s) > 1$, then $\operatorname{Re}(s-\rho)^{-1} > 0$ for each zero ρ. Therefore, for $2 \geq \sigma > 1$, using Lemma 2.15 and (2.113), we obtain

$$\begin{aligned} -\operatorname{Re}\left(\frac{\zeta_L'}{\zeta_L}(\sigma) \right) &= -\frac{1}{2}\operatorname{Re}\left(\sum_\rho \left(\frac{1}{\sigma-\rho} + \frac{1}{\sigma-\overline{\rho}} \right)\right) + \frac{1}{2}\log d_L + \frac{1}{\sigma} + \frac{1}{\sigma-1} + \frac{\gamma_L'}{\gamma_L}(\sigma) \\ &\leq \frac{1}{2}\log d_L + \frac{1}{\sigma} + \frac{1}{\sigma-1} + \frac{\gamma_L'}{\gamma_L}(\sigma) \\ &\leq \frac{1}{2}\log d_L + 1 + \frac{1}{\sigma-1} + \frac{n_L}{2}\Big((\log 3) + c_2 \Big). \end{aligned} \tag{2.114}$$

Let $\rho' = \beta + i\gamma$ be some particular zero with $|\gamma| \geq (1 + 4\log d_L)^{-1}$. Now combining (2.15) and

(2.113), we get that for $2 \geq \sigma > 1$,

$$\begin{aligned}
-\operatorname{Re}\left(\frac{\zeta_L'}{\zeta_L}(\sigma+2i\gamma)\right) =& -\frac{1}{2}\operatorname{Re}\left(\sum_\rho\left(\frac{1}{\sigma+2i\gamma-\rho}+\frac{1}{\sigma+2i\gamma-\overline{\rho}}\right)\right)+\frac{1}{2}\log d_L \\
&+\operatorname{Re}\left(\frac{1}{\sigma+2i\gamma}+\frac{1}{\sigma+2i\gamma-1}\right)+\operatorname{Re}\left(\frac{\gamma_L'}{\gamma_L}(\sigma)\right) \\
\leq& \frac{1}{2}\log d_L+\operatorname{Re}\left(\frac{1}{\sigma+2i\gamma}+\frac{1}{\sigma+2i\gamma-1}\right)+\operatorname{Re}\left(\frac{\gamma_L'}{\gamma_L}(\sigma)\right) \\
\leq& \frac{1}{2}\log d_L+\operatorname{Re}\left(\frac{1}{\sigma+2i\gamma}+\frac{1}{\sigma+2i\gamma-1}\right)+\frac{n_L}{2}\Big((\log(2|\gamma|+\sigma+1))+c_2\Big) \\
\leq& \frac{1}{2}\log d_L+\operatorname{Re}\left(\frac{1}{\sigma+2i\gamma}+\frac{1}{\sigma+2i\gamma-1}\right)+\frac{n_L}{2}\Big((\log(2|\gamma|+3))+c_2\Big).
\end{aligned} \tag{2.115}$$

Since $|\gamma| \geq (1+4\log d_L)^{-1}$, hence for $2 \geq \sigma > 1$,

$$\operatorname{Re}\left(\frac{1}{\sigma+2i\gamma}+\frac{1}{\sigma+2i\gamma-1}\right) \leq \frac{1}{\sigma}+\frac{1}{2|\gamma|} \leq 1+\frac{1}{2}+2\log d_L = \frac{3}{2}+2\log d_L.$$

Now combining this with (2.115), we obtain

$$-\operatorname{Re}\left(\frac{\zeta_L'}{\zeta_L}(\sigma+2i\gamma)\right) = \frac{5}{2}\log d_L+\frac{n_L}{2}\Big((\log(2|\gamma|+3))+c_2\Big)+\frac{3}{2}. \tag{2.116}$$

Now, again combining (2.15) and (2.113), we also get that for $2 \geq \sigma > 1$,

$$\begin{aligned}
&-\operatorname{Re}\left(\frac{\zeta_L'}{\zeta_L}(\sigma+i\gamma)\right) \\
&= -\frac{1}{2}\operatorname{Re}\left(\sum_\rho\left(\frac{1}{\sigma+i\gamma-\rho}+\frac{1}{\sigma+i\gamma-\overline{\rho}}\right)\right)+\frac{1}{2}\log d_L+\operatorname{Re}\left(\frac{1}{\sigma+i\gamma}+\frac{1}{\sigma+i\gamma-1}\right)+\operatorname{Re}\left(\frac{\gamma_L'}{\gamma_L}(\sigma)\right) \\
&\leq \frac{1}{2}\log d_L+\operatorname{Re}\left(\frac{1}{\sigma+i\gamma}+\frac{1}{\sigma+i\gamma-1}\right)+\operatorname{Re}\left(\frac{\gamma_L'}{\gamma_L}(\sigma)\right)-\operatorname{Re}\left(\frac{1}{\sigma+i\gamma-\rho'}\right) \\
&= \frac{1}{2}\log d_L+\operatorname{Re}\left(\frac{1}{\sigma+i\gamma}+\frac{1}{\sigma+i\gamma-1}\right)+\operatorname{Re}\left(\frac{\gamma_L'}{\gamma_L}(\sigma)\right)-\frac{1}{\sigma-\beta}.
\end{aligned} \tag{2.117}$$

Again since $|\gamma| \geq (1+4\log d_L)^{-1}$, hence for $2 \geq \sigma > 1$,

$$\operatorname{Re}\left(\frac{1}{\sigma+i\gamma}+\frac{1}{\sigma+i\gamma-1}\right) \leq \frac{1}{\sigma}+\frac{1}{|\gamma|} \leq 1+1+4\log d_L = 2+4\log d_L.$$

Now combining this with (2.117) and (2.15), we obtain

$$-\operatorname{Re}\left(\frac{\zeta_L'}{\zeta_L}(\sigma+i\gamma)\right)=\frac{9}{2}\log d_L+\frac{n_L}{2}\Big((\log(|\gamma|+3))+c_2\Big)+2-\frac{1}{\sigma-\beta}. \tag{2.118}$$

Now combining (2.112), (2.114), (2.116) and (2.118), we get

$$\frac{4}{\sigma-\beta}<\frac{3}{\sigma-1}+22(\log d_L)+n_L\left(\frac{5}{2}(\log(|\gamma|+3))+4c_2+\frac{\log 54}{2}\right)+\frac{25}{2}. \tag{2.119}$$

Now suppose we take $l=22(\log d_L)+n_L\left(\frac{5}{2}(\log(|\gamma|+3))+4c_2+\frac{\log 54}{2}\right)+\frac{25}{2}$ and let $\sigma=1+\frac{a}{l}$. Thus (2.119) turns out to be

$$\frac{4l}{(1-\beta)l+a}<\frac{3l}{a}+l,$$

which on further computing turns out to be equivalent to

$$\beta<1-\frac{a-a^2}{(3+a)l}.$$

Further, the evaluation of the function $f(a)=\frac{a-a^2}{(3+a)l}$ shows that f is maximum at $2\sqrt{3}-3$ and $f(2\sqrt{3}-3)=\frac{7-4\sqrt{3}}{l}$. Now using $a=2\sqrt{3}-3$, we obtain

$$\sigma=1+(2\sqrt{3}-3)\left(22(\log d_L)+n_L\left(\frac{5}{2}(\log(|\gamma|+3))+4c_2+\frac{\log 54}{2}\right)+\frac{25}{2}\right)^{-1},$$

and using this σ in (2.119), we obtain the required result. □

Remark 2.33. Winckler in [30, Lemma 7.1] proved that the ζ_L function has no zeros $\rho=\beta+i\gamma$ in the region

$$|\gamma|\geq(1+4\log d_L)^{-1}$$
$$\beta\geq 1-(7-4\sqrt{3})\left(22(\log d_L)+n_L\left(\frac{5}{2}(\log(|\gamma|+3))+\frac{1078}{67}+2(\log 3)\right)+\frac{15}{2}\right)^{-1}.$$

Revision 7. *The constant $\frac{15}{2}$ as given in [30, Lemma 7.1] by Winckler is not correct and we corrected this mistake by replacing that value with $\frac{25}{2}$ as shown in Lemma 2.32.*

Lemma 2.34. *[15, Lemma 8.2]. If $n_L>1$, then $\zeta_L(s)$ has at most one zero $\rho=\beta+i\gamma$ in the* *region*

$$|\gamma| \leq (4\log d_L)^{-1} \text{ and } \beta \geq 1-(4\log d_L)^{-1}. \tag{2.120}$$

Proof. (2.113) shows that $1<\sigma\leq 2$,

$$\sum_{\rho} \frac{\sigma-\beta}{(\sigma-\beta)^2+\gamma^2} = \frac{1}{\sigma-1} + \frac{1}{2}\log d_L + \frac{\zeta_L'}{\zeta_L}(\sigma) + \frac{1}{\sigma} + \frac{\gamma_L'}{\gamma_L}(\sigma). \tag{2.121}$$

Notice that $\frac{\zeta'}{\zeta}(\sigma) \leq 0$. Also, for $1<\sigma\leq 2$, $\frac{\Gamma'}{\Gamma}\left(\frac{\sigma}{2}\right) < 0$, $\frac{\Gamma'}{\Gamma}\left(\frac{\sigma+1}{2}\right) < 0$ and since $n_L>1$, therefore $\frac{1}{\sigma} - \frac{n_L}{2}\log\pi < 0$. Hence using the logarithmic derivative of (2.46), we get

$$\frac{1}{\sigma} + \frac{\gamma_L'}{\gamma_L}(\sigma) = \left(\frac{1}{\sigma} - \frac{n_L}{2}\log\pi\right) + \frac{a(L)}{2}\frac{\Gamma'}{\Gamma}\left(\frac{\sigma}{2}\right) + \frac{b(L)}{2}\frac{\Gamma'}{\Gamma}\left(\frac{\sigma+1}{2}\right) < 0. \tag{2.122}$$

Now using this, (2.121) gives us

$$\sum_{\rho} \frac{\sigma-\beta}{(\sigma-\beta)^2+\gamma^2} < \frac{1}{\sigma-1} + \frac{1}{2}\log d_L. \tag{2.123}$$

Due to symmetry of zeros about the real axis for $\zeta_L(s)$, presence of one complex zero of $\zeta_L(s)$, $\rho=\beta+i\gamma$ with $\gamma\neq 0$ guarantees the presence of another zero. If $\rho=\beta+i\gamma$ is a complex zero (i.e. $|\gamma|\neq 0$) in the region described by (2.120), then there are at least two zeros in that region and (2.121) gives us

$$2\frac{\sigma-\beta}{(\sigma-\beta)^2+\gamma^2} \leq \frac{1}{\sigma-1} + \frac{1}{2}\log d_L. \tag{2.124}$$

Now let us choose $\sigma = 1+(\log d_L)^{-1} \leq 2$. Then $\frac{1}{\sigma-1} + \frac{1}{2}\log d_L = \frac{3}{2}\log d_L$. Let $\rho=\beta+i\gamma$ be in the region described by (2.120) and $\gamma\neq 0$. We thus set $\beta = 1-\frac{a}{\log d_L}$ with $0<a\leq 1/4$ and $|\gamma| = \frac{b}{\log d_L}$ with $0<b\leq 1/4$. We now consider the function f given by $f(a,b) = \frac{2(a+1)}{(a+1)^2+b^2}$ and use multi-variable calculus to find out that f attains the minimum value in the region $0<a\leq 1/4$ and $0<b\leq 1/4$ at $a=1/4$ and $b=1/4$ and the minimum value is approximately

1.53846. Therefore,

$$2\frac{\sigma-\beta}{(\sigma-\beta)^2+\gamma^2} = 2\times\frac{\frac{a+1}{\log d_L}}{\left(\frac{a+1}{\log d_L}\right)^2+\left(\frac{b}{\log d_L}\right)^2} = \frac{2(a+1)}{(a+1)^2+b^2}\log d_L$$
$$\geq 1.53846\log d_L > 1.5\log d_L = \frac{1}{\sigma-1}+\frac{1}{2}\log d_L,$$

and thus we get a contradiction to (2.123) in the case where $\gamma \neq 0$. Hence $\zeta_L(s)$ cannot have a complex zero in the region described by (2.120).

Now suppose $\zeta_L(s)$ has more than one real zero (i.e. $\gamma = 0$) or a double real zero in the region defined by (2.120), say $\rho_1 = \beta_1$ and $\rho_2 = \beta_2$. Let $\beta = \min\{\beta_1, \beta_2\}$. Then (2.123) together with $\frac{5}{4\log d_L} \geq \sigma - \beta > \frac{1}{\log d_L}$ gives

$$\sum_\rho \frac{\sigma-\beta}{(\sigma-\beta)^2+\gamma^2} > \frac{\sigma-\beta_1}{(\sigma-\beta_1)^2}+\frac{\sigma-\beta_2}{(\sigma-\beta_2)^2} = \frac{1}{\sigma-\beta_1}+\frac{1}{\sigma-\beta_2}$$
$$\geq \frac{2}{\sigma-\beta} \geq \frac{2\times 4}{5}\log d_L = 1.6\log d_L > 1.5\log d_L,$$

which is again a contradiction to (2.123). Thus there cannot be more than one zero in the region described in (2.120) and if such a zero exists, then it must be real and simple. □

Remark 2.35. If the possible zero described by the above lemma exists, we denote it by β_0 and call it the exceptional (Siegel) zero. We also note that if $n_L = 1$ (so that $L = \mathbb{Q}$, $\log d_L = 0$), then ζ_L has no non-trivial zeros $\rho = \beta + i\gamma$ with $|\gamma| < 14$. If β_0 exists, then (2.110) shows that there exists a unique χ_0 such that $L(\beta_0, \chi_0) = 0$. This χ_0 must then be a real character, since $L(\beta_0, \overline{\chi}_0) = \overline{L(\beta_0, \chi_0)} = 0$.

Remark 2.36. Many number theorists have proved several results related to the zeros of Dedekind ζ-function. For example, Stark [27] in 1974 also proved that $\zeta_L(s)$ has at most one zero $\rho = \beta + i\gamma$ with $\beta > 1 - (4\log d_L)^{-1}$ and $|\gamma| < (4\log d_L)^{-1}$. Habiba Kadiri [13, Theorem 1.1] in 2012 proved that for sufficiently large d_L, $\zeta_L(s)$ has at most one zero $\rho = \beta + i\gamma$ with $\beta \geq 1 - (12.74\log d_L)^{-1}$ and $|\gamma| < 1$. But the result that we will use is the one proved by Ahn and Kwon in [2, Theorem 1]. They proved:

Theorem 2.37. *[2, Theorem 1] If $n_L \geq 2$, then $\zeta_L(s)$ has at most one zero $\rho = \beta + i\gamma$ with*

$$\beta > 1 - (2\log d_L)^{-1} \text{ and } |\gamma| < (2\log d_L)^{-1}$$

and if this zero exists, then it has to be real and simple.

2.9 Final Estimates

We finally conclude by applying Remark 2.31 to estimate $\psi_C(x)$ and $\pi_C(x)$.

2.9.1 Estimating $\psi_C(x)$

Lagarias and Odlyzko in [15, Theorem 9.2] proved that there is an effectively computable positive absolute constant c_{13} such that if $x \geq \exp(4n_L(\log d_L)^2)$, then

$$\psi_C(x) = \frac{|C|}{|G|}x - \frac{|C|}{|G|}\chi_0(g)\frac{x^{\beta_0}}{\beta_0} + \epsilon_\psi(x), \tag{2.125}$$

where

$$|\epsilon_\psi(x)| \leq x\exp(-c_{13}n_L^{-\frac{1}{2}}(\log x)^{\frac{1}{2}}),$$

and where the second term on the right side of (2.125) occurs only if $\zeta_L(s)$ has an exceptional zero β_0, and χ_0 is the (real) character of $G_0 = \mathrm{Gal}(L/E) =< g >$ for which $L(s, \chi_0)$ has β_0 as a zero. We prove an explicit result to this as Theorem 1.12.

Proof of Theorem 1.12. Let $a = e^{\frac{25}{44}}$, $b = 3^{\frac{4}{5}}e^{\frac{2156}{335}}$ and $c = 2 \times \frac{7-4\sqrt{3}}{5}$. If $\rho = \beta + i\gamma$ with $\rho \neq \beta_0$ be the non-trivial zeros of some $L(s, \chi)$ with $|\gamma| < T$, then by Lemma 2.32 and checking

$$(7-4\sqrt{3})\left(22(\log d_L)+n_L\left(\frac{5}{2}(\log(|\gamma|+3))+4c_2+\frac{\log 54}{2}\right)+\frac{25}{2}\right)^{-1} \geq \frac{c}{\log((ad_L)^{44/5}(b(T+3))^{n_L})},$$

we get

$$
\begin{aligned}
|x^{\rho}| = x^{\beta} &\le x^{1-(7-4\sqrt{3})\left(22(\log d_L)+n_L\left(\frac{5}{2}(\log(|\gamma|+3))+4c_2+\frac{\log 54}{2}\right)+\frac{25}{2}\right)^{-1}} \\
&= x x^{-(7-4\sqrt{3})\left(22(\log d_L)+n_L\left(\frac{5}{2}(\log(|\gamma|+3))+4c_2+\frac{\log 54}{2}\right)+\frac{25}{2}\right)^{-1}} \\
&= x e^{-(\log x)(7-4\sqrt{3})\left(22(\log d_L)+n_L\left(\frac{5}{2}(\log(|\gamma|+3))+4c_2+\frac{\log 54}{2}\right)+\frac{25}{2}\right)^{-1}} \\
&\le x e^{-\frac{c(\log x)}{\log((ad_L)^{44/5}(b(T+3))^{n_L})}}, \qquad (2.126)
\end{aligned}
$$

for $x \ge 2$ and $T \ge T_0 \ge 2$. Also, by Lemma 2.16, $\sum_{\chi} \log A(\chi) = \log d_L$ and $\sum_{\chi} n_E = n_L$, we get,

$$
\begin{aligned}
&\sum_{\chi} \sum_{\substack{|\rho|\ge\frac{1}{2} \\ |\gamma|\le T}} \left|\frac{1}{\rho}\right| \le \sum_{\chi}\Bigg(\sum_{0\le j\le \frac{T-1}{2}} \frac{n_{\chi}(2j+2)+n_{\chi}(-(2j+2))}{2j+1} + 2n_{\chi}(0)\Bigg) \\
&\le \sum_{\chi}\Bigg(\sum_{0\le j\le \frac{T-1}{2}} \frac{c_3\Big(\log(A(\chi))+n_E\Big(\log(2j+5)+c_4\Big)\Big)}{2j+1} + c_3\Big(\log(A(\chi))+n_E\Big((\log 3)+c_4\Big)\Big)\Bigg) \\
&\le c_3\Big(\log d_L + n_L\Big((\log(T+4))+c_4\Big)\Big)\Bigg(\sum_{0\le j\le \frac{T-1}{2}} \frac{1}{2j+1}+1\Bigg). \qquad (2.127)
\end{aligned}
$$

Recall that $\log(K+1) < \sum_{1\le j\le K} \frac{1}{j} \le (\log K)+1$. Therefore

$$
\begin{aligned}
\sum_{0}^{\frac{T-1}{2}} \frac{1}{2j+1} &= \sum_{1}^{T}\frac{1}{j} - \frac{1}{2}\sum_{1}^{\left\lfloor\frac{T}{2}\right\rfloor}\frac{1}{j} \le (\log T)+1-\frac{1}{2}\Big(\log\Big(\Big\lfloor\frac{T}{2}\Big\rfloor+1\Big)\Big) \\
&\le (\log T)+1-\frac{1}{2}\Big(\log\Big(\frac{T}{2}\Big)\Big) \le \frac{\log T}{2}+1.35. \qquad (2.128)
\end{aligned}
$$

Therefore using (2.128) in (2.127), we get

$$
\sum_{\chi} \sum_{\substack{|\rho|\ge\frac{1}{2} \\ |\gamma|\le T}} \left|\frac{1}{\rho}\right| \le c_3\Big(\frac{\log T}{2}+2.35\Big)\Big(\log d_L + n_L\Big((\log(T+4))+c_4\Big)\Big). \qquad (2.129)
$$

Now for $\rho \ne 1-\beta_0$, using $\zeta_L(s) = \prod_{\chi} L(s,\chi)$ and the result of Ahn and Kwon as given in

Theorem 2.37, we get $|\gamma| \geq \frac{1}{2(\log d_L)}$ and hence $|\rho| \geq \frac{1}{2(\log d_L)}$. We use this result, Lemma 2.16, $\sum_\chi \log A(\chi) = \log d_L$ and $\sum_\chi n_E = n_L$ to obtain:

$$\begin{aligned}
\sum_{\chi} \sum_{\substack{\rho \neq 1-\beta_0 \\ |\rho| < \frac{1}{2}}} \left(\left| \frac{x^\rho}{\rho} \right| + \left| \frac{1}{\rho} \right| \right) &\leq (\sqrt{x}+1) \sum_{\chi} \sum_{\substack{\rho \neq 1-\beta_0 \\ |\rho| < \frac{1}{2}}} \left| \frac{1}{\rho} \right| \\
&\leq (\sqrt{x}+1)(2\log d_L) \sum_{\chi} \sum_{\substack{\rho \neq 1-\beta_0 \\ |\rho| < \frac{1}{2}}} 1 \\
&\leq (\sqrt{x}+1)(2\log d_L) \sum_{\chi} n_\chi(0) \\
&\leq (\sqrt{x}+1)(2\log d_L) \sum_{\chi} \left(\frac{c_3}{2} \left(\log(A(\chi)) + n_E \left((\log 3) + c_4 \right) \right) \right) \\
&\leq (\sqrt{x}+1)(\log d_L) \left(c_3 \left((\log d_L) + n_L \left((\log 3) + c_4 \right) \right) \right) \\
&= c_3 (\log d_L)(\sqrt{x}+1) \left((\log d_L) + n_L \left((\log 3) + c_4 \right) \right).
\end{aligned} \tag{2.130}$$

Now by Hermite-Minkowski identity, we have $n_L \leq \frac{\log d_L}{\log \sqrt{3}} = \frac{2\log d_L}{\log 3}$ for $n_L > 1$. Using this in (2.130), we get

$$\begin{aligned}
\sum_{\chi} \sum_{\substack{\rho \neq 1-\beta_0 \\ |\rho| < \frac{1}{2}}} \left(\left| \frac{x^\rho}{\rho} \right| + \left| \frac{1}{\rho} \right| \right) &\leq c_3 (\log d_L)(\sqrt{x}+1) \left((\log d_L) + \frac{\log d_L}{\log \sqrt{3}} \left((\log 3) + c_4 \right) \right) \\
&= c_3 (\log d_L)^2 (\sqrt{x}+1) \left(1 + \frac{(\log 3) + c_4}{\log \sqrt{3}} \right).
\end{aligned} \tag{2.131}$$

Note that if $n_L = 1$ (and $\log d_L = 0$), then (2.131) is trivially true since ζ_L has no non-trivial zeros $\rho = \beta + i\gamma$ with $|\gamma| < 14$ as mentioned in Remark 2.35. We remember that $0 < 1 - \beta_0 \leq \frac{1}{2}$. Therefore by Mean Value Theorem,

$$\frac{x^{1-\beta_0}}{1-\beta_0} - \frac{1}{1-\beta_0} = \frac{x^{1-\beta_0} - 1}{1-\beta_0} \leq \max_{0 < \Omega \leq 1-\beta_0} x^\Omega (\log x) \leq x^{1-\beta_0} (\log x) \leq \sqrt{x} (\log x). \tag{2.132}$$

Now using (2.100), we have

$$\left|S(x,T)-\frac{|C|}{|G|}\chi_0(g)\frac{x^{\beta_0}}{\beta_0}\right| = \frac{|C|}{|G|}\left|\sum_{\chi}\overline{\chi}(g)\Bigg(\sum_{\substack{|\rho|\geq\frac{1}{2}\\|\gamma|<T}}\frac{x^\rho}{\rho}+\sum_{\substack{|\rho|<\frac{1}{2}\\|\gamma|<T}}\frac{x^\rho}{\rho}-\sum_{|\rho|<\frac{1}{2}}\frac{1}{\rho}\Bigg)-\chi_0(g)\frac{x^{\beta_0}}{\beta_0}\right|. \tag{2.133}$$

Now using the fact that

$$\left|\sum_{\chi}\overline{\chi}(g)\sum_{\substack{|\rho|\geq\frac{1}{2}\\|\gamma|<T}}\frac{x^\rho}{\rho}-\chi_0(g)\frac{x^{\beta_0}}{\beta_0}\right|\leq\sum_{\chi}\sum_{\substack{|\rho|\geq\frac{1}{2},\rho\neq\beta_0\\|\gamma|<T}}\left|\frac{x^\rho}{\rho}\right|,$$

along with $n_L\leq\frac{\log d_L}{\log\sqrt{3}}$, (2.126), (2.129), (2.131), (2.132) and triangle inequality, (2.133) becomes

$$\begin{aligned}
&\left|S(x,T)-\frac{|C|}{|G|}\chi_0(g)\frac{x^{\beta_0}}{\beta_0}\right|\\
&\leq\frac{|C|}{|G|}\Bigg[\sum_{\chi}\sum_{\substack{|\rho|\geq\frac{1}{2},\rho\neq\beta_0\\|\gamma|<T}}\left|\frac{x^\rho}{\rho}\right|+\sum_{\chi}\sum_{\substack{\rho\neq1-\beta_0\\|\rho|<\frac{1}{2}}}\left(\left|\frac{x^\rho}{\rho}\right|+\left|\frac{1}{\rho}\right|\right)+\frac{x^{1-\beta_0}}{1-\beta_0}-\frac{1}{1-\beta_0}\Bigg]\\
&\leq\frac{|C|}{|G|}\Bigg[xe^{-\frac{c(\log x)}{\log((ad_L)^{44/5}(b(T+3))^{n_L})}}\Bigg(c_3\Bigg(\frac{\log T}{2}+2.35\Bigg)\Bigg(\log d_L+n_L\Big((\log(T+4))+c_4\Big)\Bigg)\Bigg)\\
&\quad+c_3(\log d_L)^2(\sqrt{x}+1)\Bigg(1+\frac{(\log 3)+c_4}{\log\sqrt{3}}\Bigg)+\sqrt{x}(\log x)\Bigg]\\
&\leq\frac{|C|}{|G|}\Bigg[xe^{-\frac{c(\log x)}{\log((ad_L)^{44/5}(b(T+3))^{n_L})}}\Bigg(c_3(\log d_L)\Bigg(\frac{\log T}{2}+2.35\Bigg)\Bigg(1+\Bigg(\frac{(\log(T+4))+c_4}{\log\sqrt{3}}\Bigg)\Bigg)\Bigg)\\
&\quad+\sqrt{x}\Bigg(c_3(\log d_L)^2\Bigg(1+\frac{1}{\sqrt{x}}\Bigg)\Bigg(\frac{\frac{3}{2}(\log 3)+c_4}{\log\sqrt{3}}\Bigg)+\log x\Bigg)\Bigg].
\end{aligned}\tag{2.134}$$

Let $T=\frac{1}{b(ad_L)^{\frac{44}{5}}}e^{\sqrt{\frac{\log x}{n_L}}}-3\geq T_0$ and consider $x\geq 2$ such that

$$\log x\geq 4n_L(\log((T_0+3)b(ad_L)^{\frac{44}{5}}))^2.$$

Set $T_0=2$. Thus we have $T\geq 2$, $\frac{\log x}{\log((ad_L)^{44/5}(b(T+3))^{n_L})}\geq\sqrt{\frac{\log x}{n_L}}$ for $n_L\geq 2$, $\log d_L\leq\frac{5}{88}\sqrt{\frac{\log x}{n_L}}$ and therefore using $T+4\leq 4T$ (2.134) turns out to be

$$\left|S(x,T)-\frac{|C|}{|G|}\chi_0(g)\frac{x^{\beta_0}}{\beta_0}\right|\leq\frac{|C|}{|G|}(E_1+E_2), \tag{2.135}$$

where

$$E_1 = xe^{-c\sqrt{\frac{\log x}{n_L}}}\left(c_3(\log d_L)\left(\frac{1}{2}\sqrt{\frac{\log x}{n_L}} + 2.35\right)\left(1 + \left(\frac{\sqrt{\frac{\log x}{n_L}} + (\log 4) + c_4}{\log\sqrt{3}}\right)\right)\right) \tag{2.136}$$

and

$$E_2 = \sqrt{x}\left(c_3\left(\log d_L\right)^2\left(1 + \frac{1}{\sqrt{x}}\right)\left(\frac{\frac{3}{2}(\log 3) + c_4}{\log\sqrt{3}}\right) + \log x\right). \tag{2.137}$$

Now we use Remark 2.31 and (2.135) to obtain

$$\begin{aligned}
&\left|\psi_C(x) - \frac{|C|}{|G|}x + \frac{|C|}{|G|}\chi_0(g)\frac{x^{\beta_0}}{\beta_0}\right| \\
&= \left|\psi_C(x) - \frac{|C|}{|G|}x + S(x,T) - S(x,T) + \frac{|C|}{|G|}\chi_0(g)\frac{x^{\beta_0}}{\beta_0}\right| \\
&\leq \left|\psi_C(x) - \frac{|C|}{|G|}x + S(x,T)\right| + \left|S(x,T) - \frac{|C|}{|G|}\chi_0(g)\frac{x^{\beta_0}}{\beta_0}\right| \\
&\leq \frac{|C|}{|G|}\left(E_1 + E_2 + D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7\right),
\end{aligned} \tag{2.138}$$

where

$$D_1 = 28.7n_L\frac{x(\log x)^2}{T}, \tag{2.139}$$

$$\begin{aligned}
D_2 = \frac{x^{-1/4}}{\pi T(\log x)}\Bigg[&(1-c_7)\log d_L + n_L\left(\log\left(\frac{5}{4} + T\right) + 4.452 + c_1\right. \\
&\left.- \frac{1}{2}\log(T+5)\left(c_8 + \frac{c_9}{T+4}\right)\right) - \frac{1}{T}\Bigg],
\end{aligned} \tag{2.140}$$

$$D_3 = \frac{n_L}{\pi T\left(\frac{5}{4} + T\right)}\frac{x^{-1/4}}{(\log x)^2}, \tag{2.141}$$

$$D_4 = \frac{e}{\pi}\frac{x}{T(\log x)}\left(5\log d_L + \frac{n_L}{2}\log(T+5)\left(c_8 + \frac{c_9}{T+4}\right) + \frac{1}{T}\right), \tag{2.142}$$

$$D_5 = \frac{39e}{2\pi}\frac{x(\log x)}{T-1}\left(c_3\left(\log d_L + n_L\left(\log(T+3) + c_4\right)\right)\right), \tag{2.143}$$

$$D_6 = \left(\frac{6e}{\pi} + 1\right)\frac{x}{T-1}\left(c_3\left(\log d_L + n_L\left(\log(T+3) + c_4\right)\right)\right), \tag{2.144}$$

$$D_7 = 0.165\frac{(\log x)^{\frac{3}{2}}}{\sqrt{n_L}} + 14.83 n_L(\log x) + \left(c_5 + \frac{1}{2}\right)\log d_L + n_L\left(\frac{\log \pi}{2} + \frac{\gamma}{2} + \log 2 + \frac{\log 3}{2} + c_6 + 4.41\right) + 1. \tag{2.145}$$

Notice that by using $n_L \geq 2$ and $\log d_L \geq \log 3$, the condition $x \geq \exp\left(4n_L(\log(1\,114\,759\, d_L^{\frac{44}{5}}))^2\right)$ gives $\frac{\log x}{n_L} \geq 2226$ and $\log x \geq 4452$. Moreover using $T = \frac{1}{b(ad_L)^{\frac{44}{5}}} e^{\sqrt{\frac{\log x}{n_L}}} - 3 \geq 2$ with the identities

$$\frac{44}{5}\log d_L \leq \log(b(ad_L)^{\frac{44}{5}}) \leq \frac{1}{2}\sqrt{\frac{\log x}{n_L}}, \quad T - 1 \geq \frac{e^{\sqrt{\frac{\log x}{n_L}}}}{2b(ad_L)^{\frac{44}{5}}} \geq \frac{1}{2}e^{\frac{1}{2}\sqrt{\frac{\log x}{n_L}}}, \tag{2.146}$$

and the Hermite- Minkowski's inequality which gives

$$2 \leq n_L \leq \frac{5}{88(\log\sqrt{3})}\sqrt{\frac{\log x}{n_L}}, \tag{2.147}$$

we compute each term in (2.138) using MAPLE to obtain

$$E_2 + D_2 + D_3 + D_4 + D_6 + D_7 \leq 10^{-7} x e^{-\frac{c}{2}\sqrt{\frac{\log x}{n_L}}},$$

and

$$E_1 \leq 5801.4 x e^{-\frac{c}{2}\sqrt{\frac{\log x}{n_L}}}, \ D_1 \leq 3.71 x e^{-\frac{c}{2}\sqrt{\frac{\log x}{n_L}}}, D_5 \leq 0.06 x e^{-\frac{c}{2}\sqrt{\frac{\log x}{n_L}}}.$$

Inserting these values and $c = 2 \times \frac{7-4\sqrt{3}}{5}$ into (2.138), we complete the proof. □

Remark 2.38. Winckler in [30, Theorem 8.2] proved that if β_0 is the possible exceptional real zero of $\zeta_L(s)$, and χ_0 is the character (real) such that $L(\beta_0, \chi_0) = 0$, then for $x \geq \exp\left(4n_L(\log(150867 d_L^{\frac{44}{5}}))^2\right)$,

$$\psi_C(x) = \frac{|C|}{|G|}x - \frac{|C|}{|G|}\chi_0(g)\frac{x^{\beta_0}}{\beta_0} + \frac{|C|}{|G|}\epsilon_\psi(x), \tag{2.148}$$

where $|\epsilon_\psi(x)| \leq 1505234280710 x \exp\left(-\frac{7-4\sqrt{3}}{5}\sqrt{\frac{\log x}{n_L}}\right)$ and the second term in (2.148) can be suppressed in the absence of the exceptional zero β_0.

2.9.2 Estimating $\pi_C(x)$

Lagarias and Odlyzko in [15] showed that there exists absolute effectively computable constants c_3 and c_4 such that if $x \geq \exp(10 n_L (\log d_L)^2)$, then

$$\left| \pi_C(x) - \frac{|C|}{|G|}\,\mathrm{Li}(x) \right| \leq \frac{|C|}{|G|}\,\mathrm{Li}(x^{\beta_0}) + c_3 x \exp(-c_4 n_L^{-\frac{1}{2}} (\log x)^{\frac{1}{2}}), \tag{2.149}$$

where the β_0 term is present only when β_0 exists. Now we prove an explicit version to this. To go from $\psi_C(x)$ to $\pi_C(x)$, we introduce new quantities as

$$\theta_C(x) = \sum_{\substack{\mathfrak{p} \text{ unramified} \\ \mathrm{N}\,\mathfrak{p} \leq x \\ \sigma_{\mathfrak{p}} = C}} \log(\mathrm{N}\,\mathfrak{p}) \text{ and } \theta_0(x) = \sum_{\substack{\mathfrak{p} \text{ unramified} \\ \mathrm{N}\,\mathfrak{p} \leq x}} \log(\mathrm{N}\,\mathfrak{p}). \tag{2.150}$$

Theorem 2.39. *Let β_0 be the possible exceptional real zero of $\zeta_L(s)$, and χ_0 be the character (real) such that $L(\beta_0, \chi_0) = 0$. If $n_L \geq 2$ and $x \geq \exp\left(4 n_L (\log(1\,114\,759\, d_L^{\frac{44}{5}}))^2\right)$, then*

$$\left| \theta_C(x) - \frac{|C|}{|G|} x + \frac{|C|}{|G|} \chi_0(g) \frac{x^{\beta_0}}{\beta_0} \right| \leq \frac{|C|}{|G|} \epsilon_\theta(x), \tag{2.151}$$

$\epsilon_\theta(x) = 5805.17 x \exp\left(-\frac{7-4\sqrt{3}}{5} \sqrt{\frac{\log x}{n_L}} \right)$. *The third term in the left side of (2.151) can be suppressed in the absence of the exceptional zero β_0.*

Proof. Notice that there are at most n_K ideals $\mathfrak{p}^m$ with $\mathfrak{p}$ prime whose norm has a given value. Also $\mathrm{N}(\mathfrak{p}) \geq 2$ and for $\mathrm{N}(\mathfrak{p}^m) \leq x$, $m \leq \lfloor \frac{\log x}{\log(\mathrm{N}\,\mathfrak{p})} \rfloor \leq \lfloor \frac{\log x}{\log 2} \rfloor$. Using this, we obtain

$$\sum_{\substack{\mathfrak{p}, m \geq 2 \\ \mathfrak{p} \text{ unramified} \\ \mathrm{N}(\mathfrak{p}^m) \leq x}} \log(\mathrm{N}\,\mathfrak{p}) = \theta_0(x^{\frac{1}{2}}) + \theta_0(x^{\frac{1}{3}}) + \cdots + \theta_0\left(x^{\frac{1}{\lfloor \frac{\log x}{\log 2} \rfloor}} \right). \tag{2.152}$$

Clearly, $\theta_C(x) \leq \theta_0(x)$. Also, it can be easily verified that $\theta_0(x) \leq n_K \theta(x)$ where $\theta(x) = \sum_{p \leq x} \log p$. Rosser and Schoenfeld in [24, Theorem 9] showed that for $x \geq 2$, $\theta(x) < 1.01624x$.

Combining all this we obtain, $\theta_0(x) < 1.01624 n_K x$ and

$$|\psi_C(x) - \theta_C(x)| \leq \sum_{\substack{\mathfrak{p}, m\geq 2\\ \mathfrak{p}\text{ unramified}\\ \mathrm{N}(\mathfrak{p}^m)\leq x}} \log(\mathrm{N}\,\mathfrak{p}) = \theta_0(x^{\frac{1}{2}}) + \theta_0(x^{\frac{1}{3}}) + \cdots + \theta_0\left(x^{\frac{1}{\lfloor \frac{\log x}{\log 2}\rfloor}}\right)$$
$$\leq \frac{\log x}{\log 2}\theta_0(x^{\frac{1}{2}}) < \frac{\log x}{\log 2} \times (1.01624 n_K x^{\frac{1}{2}}) < \frac{22}{15} n_K \sqrt{x}(\log x). \tag{2.153}$$

Therefore, using Theorem 1.12, (2.147), $|G| \times n_K = n_L$ for $x \geq \exp\left(4n_L(\log(1\,114\,759\, d_L^{\frac{44}{5}}))^2\right)$, we obtain

$$\left|\theta_C(x) - \frac{|C|}{|G|}x + \frac{|C|}{|G|}\chi_0(g)\frac{x^{\beta_0}}{\beta_0}\right| \leq |\theta_C(x) - \psi_C(x)| + \left|\psi_C(x) - \frac{|C|}{|G|}x + \frac{|C|}{|G|}\chi_0(g)\frac{x^{\beta_0}}{\beta_0}\right|$$
$$< \frac{22}{15} n_K \sqrt{x}(\log x) + \frac{|C|}{|G|}\left(5805.17 x e^{-\frac{c}{2}\sqrt{\frac{\log x}{n_L}}}\right)$$
$$< \frac{|C|}{|G|}\frac{22}{15} n_L \sqrt{x}(\log x) + \frac{|C|}{|G|}\left(5805.17 x e^{-\frac{c}{2}\sqrt{\frac{\log x}{n_L}}}\right)$$
$$\leq \frac{|C|}{|G|}\left(5805.17 x e^{-\frac{c}{2}\sqrt{\frac{\log x}{n_L}}}\right), \tag{2.154}$$

since we compute using MAPLE that $\frac{22}{15} n_L \sqrt{x}(\log x) \leq 10^{-961} x \exp\left(-\frac{c}{2}\sqrt{\frac{\log x}{n_L}}\right)$ for $\log x \geq 4452$ with $c = 2 \times \frac{7-4\sqrt{3}}{5}$. □

Remark 2.40. Winckler in [30] proved that if β_0 is the possible exceptional real zero of $\zeta_L(s)$, and χ_0 is the character (real) such that $L(\beta_0, \chi_0) = 0$, then for $x \geq \exp\left(4n_L(\log(150867 d_L^{\frac{44}{5}}))^2\right)$,

$$\theta_C(x) = \frac{|C|}{|G|}x - \frac{|C|}{|G|}\chi_0(g)\frac{x^{\beta_0}}{\beta_0} + \frac{|C|}{|G|}\epsilon_\theta(x), \tag{2.155}$$

where $|\epsilon_\theta(x)| \leq 1505234280719 x \exp\left(-\frac{7-4\sqrt{3}}{5}\sqrt{\frac{\log x}{n_L}}\right)$ and the second term in (2.155) can be suppressed in the absence of the exceptional zero β_0.

Proof of Theorem 1.10. Now using partial summation and integration by parts, we obtain

$$\begin{aligned}\pi_C(x) - \frac{|C|}{|G|}\operatorname{Li}(x) &= \frac{\theta_C(x)}{\log x} + \int_2^x \frac{\theta_C(t)}{t(\log t)^2}dt - \frac{|C|}{|G|}\left(\frac{x}{\log x} + \int_2^x \frac{dt}{(\log t)^2}\right)\\ &= \frac{\theta_C(x) - \frac{|C|}{|G|}x}{\log x} + \int_2^x \frac{\theta_C(t) - \frac{|C|}{|G|}t}{t(\log t)^2}dt. \end{aligned} \tag{2.156}$$

(Note that the equality defined in (2.156) is true up to a constant $\leq n_K$ which depends on $\#\{\mathfrak{p}|\mathfrak{p}$ unramified, $\mathrm{N}\,\mathfrak{p} = 2$ and $\sigma_{\mathfrak{p}} = C\}$. This quantity is very small and so we neglect this in our computations.)

Now let $a = \exp\left(4n_L(\log(1\,114\,759\, d_L^{\frac{44}{5}}))^2\right)$. Therefore for $x \geq a^2$, using triangle inequality, $|\chi_0(g)| = 1$ and Theorem 2.39, we obtain:

$$\begin{aligned}\left|\pi_C(x) - \frac{|C|}{|G|}\operatorname{Li}(x)\right| &\leq \left|\frac{\theta_C(x) - \frac{|C|}{|G|}x}{\log x}\right| + \left|\int_2^x \frac{\theta_C(t) - \frac{|C|}{|G|}t}{t(\log t)^2}dt\right|\\ &< \frac{|C|}{|G|}\frac{x^{\beta_0}}{\log(x^{\beta_0})} + \frac{|C|}{|G|}\frac{\epsilon_\theta(x)}{\log x} + \int_2^{\sqrt{x}} \frac{\left|\theta_C(t) - \frac{|C|}{|G|}t\right|}{t(\log t)^2}dt + \int_{\sqrt{x}}^x \frac{\frac{|C|}{|G|}\frac{t^{\beta_0}}{\beta_0} + \frac{|C|}{|G|}R'(t)}{t(\log t)^2}dt. \end{aligned} \tag{2.157}$$

Now since $\theta_C(x) < 1.01624 n_K x$, thus $\left|\theta_C(t) - \frac{|C|}{|G|}t\right| < 2.01624 n_K t$. Using this and $|G| \times n_K = n_L$ we obtain

$$\begin{aligned}\left|\pi_C(x) - \frac{|C|}{|G|}\operatorname{Li}(x)\right| &< \frac{|C|}{|G|}\frac{x^{\beta_0}}{\log(x^{\beta_0})} + \frac{|C|}{|G|}\frac{\epsilon_\theta(x)}{\log x} + n_K\int_2^{\sqrt{x}} \frac{2.01624}{(\log t)^2}dt + \int_{\sqrt{x}}^x \frac{\frac{|C|}{|G|}\frac{t^{\beta_0}}{\beta_0} + \frac{|C|}{|G|}R'(t)}{t(\log t)^2}dt\\ &< \frac{|C|}{|G|}\left(\frac{x^{\beta_0}}{\log(x^{\beta_0})} + \frac{1}{\beta_0}\int_{\sqrt{x}}^x \frac{t^{\beta_0-1}}{(\log t)^2}dt\right) + \frac{|C|}{|G|}\frac{\epsilon_\theta(x)}{\log x} + 4.2\frac{|C|}{|G|}n_L\sqrt{x} + \frac{|C|}{|G|}\int_{\sqrt{x}}^x \frac{R'(t)}{t(\log t)^2}dt. \end{aligned} \tag{2.158}$$

Now it can be easily checked that

$$\frac{x^{\beta_0}}{\log(x^{\beta_0})} + \frac{1}{\beta_0}\int_{\sqrt{x}}^x \frac{t^{\beta_0-1}}{(\log t)^2}dt \leq \operatorname{Li}(x^{\beta_0}). \tag{2.159}$$

We know from Theorem 2.39 that $\epsilon_\theta(x) = d_0 x e^{-\frac{c}{2}\sqrt{\frac{\log x}{n_L}}}$ with $d_0 = 5805.17$. Using MAPLE, we compute that $e^{-\frac{c}{2}\sqrt{\frac{\log x}{n_L}}} < 0.76 e^{-\frac{c}{2\sqrt{2}}\sqrt{\frac{\log x}{n_L}}}$ for the given range

$$\frac{\log x}{n_L} \geq 8(\log(1\,114\,759\, d_L^{\frac{44}{5}}))^2 \geq 4452.$$

Therefore using $\log x \geq n_L \times 4452 \geq 2 \times 4452$, we compute that

$$\frac{\epsilon_\theta(x)}{\log x} < (8.54 \times 10^{-5}) d_0 x e^{-\frac{c}{2\sqrt{2}}\sqrt{\frac{\log x}{n_L}}}.$$

Using (2.147) and computing with MAPLE for $\log x \geq 2 \times 4452$, we also obtain that $(4.2) n_L \sqrt{x} \leq (1.9 \times 10^{-1932}) x e^{-\frac{c}{2\sqrt{2}}\sqrt{\frac{\log x}{n_L}}}$. Moreover in the same range,

$$\int_{\sqrt{x}}^{x} \frac{R'(t)}{t(\log t)^2} dt \leq \int_{\sqrt{x}}^{x} \frac{d_0 e^{-\frac{c}{2}\sqrt{\frac{\log t}{n_L}}}}{(\log t)^2} dt \leq d_0 \left(4x \frac{e^{-\frac{c}{2\sqrt{2}}\sqrt{\frac{\log x}{n_L}}}}{(\log x)^2} \right) \leq (5.05 \times 10^{-8}) d_0 x e^{-\frac{c}{2\sqrt{2}}\sqrt{\frac{\log x}{n_L}}}.$$

Now combining all this, we get

$$\left| \pi_C(x) - \frac{|C|}{|G|} \operatorname{Li}(x) \right| \leq \frac{|C|}{|G|} \operatorname{Li}(x^{\beta_0}) + \frac{|C|}{|G|} (8.54 \times 10^{-5}) d_0 x e^{-\frac{c}{2\sqrt{2}}\sqrt{\frac{\log x}{n_L}}}. \tag{2.160}$$

Now using $d_0 = 5805.17$ and $\frac{c}{2\sqrt{2}} \geq \frac{1}{99}$, we complete the proof. □

Remark 2.41. Winckler in [30, Theorem 1.1] proved that $\epsilon_\pi(x) < 7.84 \times 10^{14} x e^{-\frac{1}{99}\sqrt{\frac{\log x}{n_L}}}$ for $x \geq \exp\left(8 n_L (\log(150867 d_L^{\frac{44}{5}}))^2 \right)$.

Proof of Theorem 1.11. In the proof of Theorem 1.12, we set $T_0 = 44$ instead of $T_0 = 2$. As a result, we obtain the corresponding regions :

$$x \geq \exp\left(4 n_L (\log(10478733 d_L^{\frac{44}{5}}))^2 \right) \geq \exp\left(4 n_L (\log((T_0 + 3) b (a d_L)^{\frac{44}{5}}))^2 \right),$$

which gives $\frac{\log x}{n_L} \geq 2670$ and $\log x \geq 5340$. We use this in MAPLE to compute

$$E_2 + D_2 + D_3 + D_4 + D_6 + D_7 \leq 10^{-7} x e^{-\frac{c}{2}\sqrt{\frac{\log x}{n_L}}},$$

and

$$E_1 \leq 5801.38 x e^{-\frac{c}{2}\sqrt{\frac{\log x}{n_L}}}, \ D_1 \leq 0.793 x e^{-\frac{c}{2}\sqrt{\frac{\log x}{n_L}}}, D_5 \leq 0.011 x e^{-\frac{c}{2}\sqrt{\frac{\log x}{n_L}}},$$

where $c = 2 \times \frac{7-4\sqrt{3}}{5}$. As a result we obtain, $\epsilon_\psi(x) = d_0 x e^{-\frac{c}{2}\sqrt{\frac{\log x}{n_L}}}$ with $d_0 = 5802.19$ which also implies $\epsilon_\theta(x) = d_0 x e^{-\frac{c}{2}\sqrt{\frac{\log x}{n_L}}}$ for $\log x \geq 5340$. We also check that the condition $\log x \geq 16\,006 n_L (\log d_L)^2$ ensures that $\log x \geq 8 n_L (\log(10478733 d_L^{\frac{44}{5}}))^2$, since $\log d_L \geq \log 3$. Using

MAPLE for $\log x \geq 2 \times 5340$, we compute (other terms are smaller in size and can be neglected in computation)

$$\frac{\epsilon_\theta(x)}{\log x} \leq (6.89 \times 10^{-5}) d_0 x e^{-\frac{c}{2\sqrt{2}}\sqrt{\frac{\log x}{n_L}}},$$

and putting everything together as in the proof of Theorem 1.10, we obtain,

$$\left|\pi_C(x) - \frac{|C|}{|G|}\operatorname{Li}(x)\right| \leq \frac{|C|}{|G|}\operatorname{Li}(x^{\beta_0}) + \frac{|C|}{|G|} 0.4 x e^{-\frac{1}{99}\sqrt{\frac{\log x}{n_L}}}.$$

for $\log x \geq 16\,006 n_L (\log d_L)^2$. □

Remark 2.42. The reason for using $T_0 = 44$ as shown in the proof of Theorem 1.11 is to compare the values of the error term as well as the corresponding range for x (the larger x is, the better the bound for the error term will become). The next chapter uses the assumption $T \geq T_0 \geq 44$ to estimate the error terms. Hence, we compare the values by fixing $T_0 = 44$.

2.10 Reasons for improvements to Winckler's results and their impact

This section provides a list of the changes made in comparison to the work in [30].

1. We recall the list of revisions.

Table 2.1: Revising Lagarias and Odlyzko's and Winckler's results

Results from Winckler [30] and Lagarias and Odlyzko [15]	Corrections/Revisions
[15, (3.8)] and [30, (1)]	Revision 1 (Page 28)
[30, (9)]	Revision 2 (Page 34)
[30, Lemma 4.6]	Revision 3 (Page 42)
[30, Lemma 5.2]	Revision 4 (Page 49)
Proof of [15, Theorem 7.1]	Revision 5 (Page 56)
Proof of [15, Theorem 7.1] and [30, Theorem 6.1]	Revision 6 (Page 62)
[30, Lemma 7.1]	Revision 7 (Page 66)

2. In this chapter, certain new and improved estimates for the zeros of Hecke L-functions and Dedekind ζ-function have been used rather than the ones used by Winckler. For instance, the coefficient of n_L in [30, Lemma 5.1] has been improved from $\log(2+|s|) + \frac{19683}{812}$ to $\log(1+|s|) + 4.452 + \frac{83}{5}$ in Lemma 2.19. Moreover, Winckler uses the zero-free region for Dedekind ζ-function given in Lemma 2.34 which states that if $n_L > 1$, then $\zeta_L(s)$ has at

most one zero $\rho = \beta + i\gamma$ in the region

$$|\gamma| \leq (4 \log d_L)^{-1} \text{ and } \beta \geq 1 - (4 \log d_L)^{-1}.$$

Instead, we use the result proved by Ahn and Kwon and shown in Theorem 2.37 which gives us the zero-free region

$$|\gamma| \leq (2 \log d_L)^{-1} \text{ and } \beta \geq 1 - (2 \log d_L)^{-1},$$

which is clearly an improvement to the one Winckler used. Additionally, Winckler uses the Hermite-Minkowski inequality $n_L \leq \frac{\log d_L}{\log \frac{\pi}{3}}$ whereas we use $n_L \leq \frac{2 \log d_L}{\log 3}$ instead. All these changes contribute towards the improvements we obtain.

3. For computing the bound for the error term $E_\psi(x)$, Winckler used $x \geq 2$, which is a trivial lower bound on x. But we notice that the condition $x \geq \exp\left(4n_L(\log(1\,114\,759\, d_L^{\frac{44}{5}}))^2\right)$ given in Theorem 1.12 for $E_\psi(x)$ ensures that $\frac{\log x}{n_L} \geq 2226$ and thus $\log x \geq 4452$. Similarly, the condition $x \geq \exp\left(8n_L(\log(1\,114\,759\, d_L^{\frac{44}{5}}))^2\right)$ given in Theorem 1.10 ensures that $\frac{\log x}{n_L} \geq 4452$ and thus $\log x \geq 2 \times 4452$. Since the bounds for the error terms are decreasing in x, large values of x will give improved results. This is the main factor which makes such a significant change in our values than that of Winckler's.

Table 2.2: Improving Winckler's results

Results from Winckler [30]	Results in this thesis
Theorem 1.1	Theorem 1.10 Theorem 1.11
Lemma 5.1	Lemma 2.19
Theorem 8.2	Theorem 1.12

Chapter 3

A new explicit version of Chebotarev's density theorem

3.1 Introduction

The objective of this chapter is to provide a new explicit version of Lagarias and Odlyzko's result on Chebotarev's density theorem. We derive an asymptotic formula with an explicit error term for a weighted prime power counting function

$$\psi_C(x) = \sum_{\substack{\mathrm{N}\,\mathfrak{p}^m \le x \\ \mathfrak{p} \text{ unramified} \\ \sigma_{\mathfrak{p}}^m = C}} \log(\mathrm{N}\,\mathfrak{p}).$$

Our main objective going ahead will be to study the error term E_ψ defined by

$$E_\psi(x) = \left| \frac{\psi_C(x) - \frac{|C|}{|G|}x}{\frac{|C|}{|G|}x} \right|. \tag{3.1}$$

Recall that d_L denotes the absolute discriminant of the number field L and $n_L = [L : \mathbb{Q}]$. In this chapter, we prove two main theorems based on the dependency of $E_\psi(x)$ on d_L. The first one has $E_\psi(x)$ dependent on d_L :

Theorem 1.14. *Let C be a fixed conjugacy class of the Galois group,* $\mathrm{Gal}(L/K) = G$. *Let β_0 be the possible exceptional real zero of $\zeta_L(s)$. Let $m \ge 2$ be an integer. Let $R = 29.57$. If $\log x \ge 4mRn_L(\log 88d_L^{1/n_L})^2$, then*

$$E_\psi(x) \le \frac{x^{\beta_0 - 1}}{\beta_0} + \epsilon_1(m, x, n_L, d_L),$$

with

$$\epsilon_1(m,x,n_L,d_L)=\lambda(m)\max\left\{(\log d_L)n_L^{-\frac{1}{m+1}},D^{\frac{m}{m+1}}\right\}(\log x)^{\frac{1}{m+1}}\exp\left(-\frac{2m^{\frac{1}{2}}}{m+1}\sqrt{\frac{\log x}{Rn_L}}\right) \quad (3.2)$$

where λ *is defined in* (3.163).

Corollary 3.2. *Under the assumptions in Theorem 1.14, we have*

$$E_\psi(x)\leq\frac{x^{\beta_0-1}}{\beta_0}+A_0\max\{(\log d_L)n_L^{-\frac{1}{3}},2^{\frac{2}{3}}d_L^{\frac{2}{3n_L}}\}(\log x)^{\frac{1}{3}}\exp\left(-B_0\sqrt{\frac{\log x}{n_L}}\right)$$

for all

$$\log x\geq C_0\frac{(\log d_L)^2}{n_L}$$

where $A_0=0.782$ *if* β_0 *exists and* 0.493 *otherwise,* $B_0=0.173$ *and* $C_0=19\,810$.

The second theorem has $E_\psi(x)$ independent of d_L:

Theorem 1.15. *Let* C *be a fixed conjugacy class of the Galois group,* $\mathrm{Gal}(L/K)=G$. *Let* β_0 *be the possible exceptional real zero of* $\zeta_L(s)$. *Let* $m\geq 2$ *be an integer. Let* $R=29.57$. *If* $\log x\geq 4mRn_L(\log 88d_L^{1/n_L})^2$, *then*

$$E_\psi(x)\leq\frac{x^{\beta_0-1}}{\beta_0}+\epsilon_2(m,x,n_L),$$

with

$$\epsilon_2(m,x,n_L)=\nu(m)\ n_L^{1-\frac{1}{m+1}}(\log x)^{\frac{1}{m+1}}\exp\left(-\frac{1.5m^{\frac{1}{2}}}{m+1}\sqrt{\frac{\log x}{Rn_L}}\right) \quad (3.3)$$

where ν *is defined in* (3.167).

Corollary 1.16. *Under the assumptions in Theorem 1.15, we have*

$$E_\psi(x)\leq\frac{x^{\beta_0-1}}{\beta_0}+A_1n_L^{\frac{2}{3}}(\log x)^{\frac{1}{3}}\exp\left(-B_1\sqrt{\frac{\log x}{n_L}}\right)\ \text{for all}\ \log x\geq C_1\frac{(\log d_L)^2}{n_L}$$

where $A_1=0.0396$ *if* β_0 *exists and* 0.0249 *otherwise,* $B_1=0.13$ *and* $C_1=19\,810$.

Corollary 3.5. *Under the assumptions in Theorem 1.15, we have*

$$E_\psi(x) \le \frac{x^{\beta_0-1}}{\beta_0} + A_2 \exp\left(- B_2\sqrt{\frac{\log x}{n_L}}\right) \text{ for all } \log x \ge C_2 n_L (\log d_L)^2$$

where $A_2 = 3.11$*,* $B_2 = 0.125$ *and* $C_2 = 4953$*.*

Remark 3.6. Another admissible value for (A_2, B_2, C_2) is $(1.84 \times 10^{-4}, 0.014, 4953)$. Winckler in [30, Theorem 8.2] proved $(A_2, B_2, C_2) = (1.51 \times 10^{12}, 0.014, 1545)$.

Theorem 1.17. *Let C be a fixed conjugacy class of the Galois group,* $\mathrm{Gal}(L/K) = G$*. Let β_0 be the possible exceptional real zero of $\zeta_L(s)$. Then*

$$\left|\pi_C(x) - \frac{|C|}{|G|}\,\mathrm{Li}(x)\right| \le \frac{|C|}{|G|}\,\mathrm{Li}(x^{\beta_0}) + E_0 \frac{|C|}{|G|} n_L x \exp\left(- F_0\sqrt{\frac{\log x}{n_L}}\right)$$

for all

$$\log x \ge G_0 \frac{(\log d_L)^2}{n_L}$$

where $E_0 = 4.714 \times 10^{-6}$ *if β_0 exists and* 2.97×10^{-6} *otherwise,* $F_0 = 0.0919$ *and* $G_0 = 39\,620$*.*

Corollary 1.18. *Under the assumptions in Theorem 1.17, we have*

$$\left|\pi_C(x) - \frac{|C|}{|G|}\,\mathrm{Li}(x)\right| \le \frac{|C|}{|G|}\,\mathrm{Li}(x^{\beta_0}) + E_1 \frac{|C|}{|G|} x \exp\left(- F_1\sqrt{\frac{\log x}{n_L}}\right)$$

for all

$$\log x \ge G_1 n_L (\log d_L)^2$$

where $E_1 = 1.65 \times 10^{-5}$*,* $F_1 = 0.09$ *and* $G_1 = 9\,906$*.*

Remark 3.9. Another admissible value for (E_1, F_1, G_1) is $(1.23 \times 10^{-9}, 1/99, 9\,906)$. Winckler in [30, Theorem 1.1] proved $(E_1, F_1, G_1) = (7.84 \times 10^{14}, 1/99, 3\,090\,)$.

Next, we give some pre-requisites concerning the zeros of the Dedekind ζ-function, $\zeta_L(s)$.

Theorem 3.10. *([11, Corollary 1.2]). Let $T \ge 1$ and $N_L(T)$ be the number of zeros $\rho = \beta + i\gamma$ of $\zeta_L(s)$ in the region $0 < \beta < 1$ and $|\gamma| \le T$. Then*

$$\left|N_L(T) - P(T)\right| \le E(T), \tag{3.4}$$

where

$$P(T) = \frac{T}{\pi}\log\left(d_L\left(\frac{T}{2\pi e}\right)^{n_L}\right) \quad \text{and} \quad E(T) = \alpha_1(\log d_L + n_L \log T) + \alpha_2 n_L + \alpha_3,$$

with

$$\alpha_1 = 0.228, \ \alpha_2 = 23.108 \ \text{ and } \ \alpha_3 = 4.520. \tag{3.5}$$

Theorem 3.11. *Let L be a number field with $n_L \geq 2$. Then there exists $\alpha_4 > 0$ such that the Dedekind zeta function, $\zeta_L(s)$ has at most one zero $\rho = \beta + i\gamma$ with*

$$\beta > 1 - \frac{1}{\alpha_4 \log d_L} \quad \text{and} \quad |\gamma| < \frac{1}{\alpha_4 \log d_L}. \tag{3.6}$$

This zero, if it exists, has to be real and simple, and denoted as β_0.

Table 3.1: Improvements to α_4

Author	Region	α_4
Stark [27, Lemma 3]	All d_L	4
Kadiri [12, Corollary 1.2]	d_L sufficiently large	2
Ahn and Kwon [2, Theorem 1]	All d_L	2

In this thesis, we will use

$$\alpha_4 = 2. \tag{3.7}$$

Theorem 3.12. *Let L be a number field with $n_L \geq 2$. Let $\rho = \beta + i\gamma$ be non-trivial zero of $\zeta_L(s)$ with $\rho \neq \beta_0$ and $\tau = |\gamma| + 2$. Then there exists $R > 0$ such that*

$$1 - \beta > (R_L \log(D\tau/2))^{-1}, \tag{3.8}$$

where $R_L = Rn_L$, $D = 2d_L^{\frac{1}{n_L}}$.

Ahn and Kwon in [3, Proposition 6.1] proved that

$$R = 29.57. \tag{3.9}$$

3.2 Introducing a smooth weight

Let $0 < \delta \leq 1, \alpha = 1 - \delta$ or 1, $m \in \mathbb{N}$ and $m \geq 2$. We define a function h on $[0, \infty)$ by

$$h(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq \alpha, \\ g(\frac{x-\alpha}{\delta}) & \text{if } \alpha \leq x \leq \alpha + \delta, \\ 0 & \text{if } x \geq \alpha + \delta, \end{cases} \tag{3.10}$$

where g is a function defined on $[0, 1]$ satisfying

1. (Condition 1) $0 \leq g(x) \leq 1$ for $0 \leq x \leq 1$,

2. (Condition 2) g is an m-times differentiable function on $(0, 1)$ such that for all $k = 1, ..., m$,

$$g^{(k)}(0) = g^{(k)}(1) = 0,$$

and there exist positive constants a_k such that

$$|g^{(k)}(x)| \leq a_k \text{ for all } 0 < x < 1.$$

The Mellin transform of h is given by

$$H(s) = \int_0^\infty h(t)t^{s-1}dt. \tag{3.11}$$

Also if H is analytic in $\Re(s) > 0$, the inverse Mellin transform formula is given by

$$h(t) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} H(s)t^{-s}ds.$$

By the definition of h and g as above, we can check that

$$\int_\alpha^{\alpha+\delta} |h^{(m+1)}(t)|t^{m+1}dt = \frac{1}{\delta^m} M(\delta, m),$$

where for any non-negative integer m, we define

$$M(\delta, m) = \max_{\alpha=1-\delta,\ 1} M(\alpha, \alpha+\delta, m)$$

with

$$M(\alpha, \alpha+\delta, m) = \int_0^1 |g^{(m+1)}(u)|(\delta u + \alpha)^{m+1} du. \tag{3.12}$$

Let $0 < \delta \leq 0.01$, $\alpha = 1-\delta$ or 1, $m \in \mathbb{N}$ and $m \geq 2$. [8, Lemma 2.2] gives us

1. The Mellin transform H of h has a single pole at $s = 0$ with residue 1 and is analytic everywhere else.

2. Let $s \in \mathbb{C}$ such that $\Re(s) \leq 1$. Then H satisfies

$$H(1) = \alpha + \delta \int_0^1 g(u)du, \tag{3.13}$$

$$|H(s)| \leq \frac{M(\alpha, \alpha+\delta, k)}{\delta^k |s|^{k+1}}, \text{ for all } k = 0, 1, ..., m. \tag{3.14}$$

3.2.1 Choice of weight $g(x)$

In this subsection, we define two choices of smooth weights, i.e, $g(x) = g_R(x)$ or $g_{FK}(x)$.

1. The first choice of smooth weight, g_R is such that it is equivalent to the approach of Rosser in [23] and is demonstrated in [8, Section 3.4] giving the smooth function

$$h_R(x) = \frac{1}{m!}\sum_{j=0}^{m}(-1)^{j+m}\binom{m}{j}\left(\frac{(1+(\delta+2(\alpha-1))j/m)-x}{\delta/m}\right)^m \mathbb{1}\left(\frac{x}{1+(\delta+2(\alpha-1))j/m}\right), \tag{3.15}$$

where $\mathbb{1}$ is the indicator function on $(0,1)$, with its Mellin transform

$$H_R(s) = \frac{\sum_{j=0}^{m}(-1)^{j+m+1}\binom{m}{j}(1+(\delta+2(\alpha-1))j/m)^{m+s}}{(\delta/m)^m s(s+1)\cdots(s+m)}. \tag{3.16}$$

From (3.15), we deduce that

$$g_R(x) = \frac{1}{m!}\sum_{j=0}^{m}(-1)^{j+m}\binom{m}{j}(j-x)^m \mathbb{1}\left(\frac{x}{j}\right) \text{ with } \int_\alpha^{\alpha+\delta} h_R(x)(x)\ dt = \frac{\delta}{2}, \tag{3.17}$$

and using [23, Theorem 15], we obtain

$$M_R(\delta, m) = \begin{cases} (m((1+\delta/m)^{m+1}+1))^m & \text{for } m \geq 1, \text{ and} \\ 1+\frac{\delta}{2} & \text{for } m = 0. \end{cases} \tag{3.18}$$

2. The second choice of smooth weight is as chosen by Faber and Kadiri [8], that is

$$g_{FK}(x) = 1 - \frac{(2m+1)!}{(m!)^2} \int_0^x t^m (1-t)^m dt, \tag{3.19}$$

and some associated properties are

- $\int_0^1 g_{FK}(u)du = \frac{1}{2}$,
- $M_{FK}(\alpha, \alpha+\delta, 0) = \frac{2\alpha+\delta}{2}$, $M_{FK}(\delta, 0) = 1 + \frac{\delta}{2} \leq \frac{2.01}{2}$ since $\delta \leq 0.01$, and
- $M_{FK}(\alpha, \alpha+\delta, m)$ for this choice of smooth weight is bounded as shown in [8, Equation 3.5] with

$$M_{FK}(\alpha, \alpha+\delta, m) \leq \sqrt{\frac{(\alpha+\delta)^{2m+3} - \alpha^{2m+3}}{\delta(2m+3)}} \frac{\sqrt{(2m)!(2m+1)!}}{m!}.$$

For $1-\delta \leq \alpha \leq 1$, we obtain

$$M_{FK}(\delta, m) = \sqrt{\frac{(1+\delta)^{2m+3} - 1}{\delta(2m+3)}} \frac{\sqrt{(2m)!(2m+1)!}}{m!}. \tag{3.20}$$

We check that $M_R(\delta, m) \approx (2m)^m$ and $M_{FK}(\delta, m) \approx \frac{\sqrt{(2m)!(2m+1)!}}{m!}$ as δ is close to 0. Also $(2m)^m < \frac{\sqrt{(2m)!(2m+1)!}}{m!}$ as soon as $m \leq 5$. We also check that for $0 < \delta \leq 10^{-100}$, $M_R(\delta, m) \leq M_{FK}(\delta, m)$ for $m \leq 5$ and $M_R(\delta, m) > M_{FK}(\delta, m)$ otherwise. In this chapter, we use only the first choice of smooth weight, since it produces better bounds for smaller m values, in particular, $m = 2$.

3.3 Introducing smoothed version of $\psi_C(x)$

We introduce the smooth version of $\psi_C(x)$ as

$$\tilde{\psi}_C(x) = \sum_{\substack{\mathfrak{p} \text{ unramified} \\ \sigma_{\mathfrak{p}}^m = C}} \sum_{m \geq 1} (\log(\mathrm{N}\,\mathfrak{p})) h\left(\frac{\mathrm{N}\,\mathfrak{p}^m}{x}\right), \tag{3.21}$$

where $\mathfrak{p}$ runs over all the prime ideals of K and h is defined in (3.10). Recall our definition of E_ψ in (3.1). Similarly $E_{\tilde{\psi}}$ denote the error term corresponding to $\tilde{\psi}_C(x)$ and defined as

$$E_{\tilde{\psi}}(x) = \left| \frac{\tilde{\psi}_C(x) - \frac{|C|}{|G|}x}{\frac{|C|}{|G|}x} \right|. \tag{3.22}$$

We define ψ_C^- and ψ_C^+ as the sums $\tilde{\psi}_C$ associated to the weights h defined by $\alpha = 1 - \delta$ and $\alpha = 1$ respectively. We also denote E_ψ^- and E_ψ^+ the respective error terms. Observe that

$$\psi_C^-(x) \leq \psi_C(x) \leq \psi_C^+(x), \tag{3.23}$$

and

$$E_\psi(x) \leq \max(E_\psi^-(x), E_\psi^+(x)). \tag{3.24}$$

We write

$$\tilde{\psi}_C(x) = I_{L/K}(x) - \tilde{I}_{L/K}(x), \tag{3.25}$$

with

$$I_{L/K}(x) = \sum_{\mathfrak{p}} \sum_{m \geq 1} \theta(\mathfrak{p}^m)(\log(\mathrm{N}\,\mathfrak{p})) h\left(\frac{\mathrm{N}\,\mathfrak{p}^m}{x}\right), \tag{3.26}$$

and

$$\tilde{I}_{L/K}(x) = \sum_{\mathfrak{p} \text{ ramified}} \sum_{m \geq 1} \theta(\mathfrak{p}^m)(\log(\mathrm{N}\,\mathfrak{p})) h\left(\frac{\mathrm{N}\,\mathfrak{p}^m}{x}\right), \tag{3.27}$$

where θ is the indicator function characterizing the Artin symbol at $\mathfrak{p}$ coinciding with the conjugacy class C. More specifically, for $\mathfrak{p}$ unramified in L, we have

$$\theta(\mathfrak{p}^m) = \begin{cases} 1 & \text{if } \sigma_{\mathfrak{p}}^m = C, \\ 0 & \text{otherwise,} \end{cases} \tag{3.28}$$

and $|\theta(\mathfrak{p}^m)| \leq 1$ if $\mathfrak{p}$ ramifies in L.

3.4 Controlling the smoothed sum over ramified prime ideals

Lemma 3.13. *Let C be a fixed conjugacy class of $G = \mathrm{Gal}(L/K)$. For $x \geq x_0 \geq \exp(4484)$, $0 \leq \delta \leq \delta_0 \leq 10^{-10}$ and $\alpha = 1 - \delta$ or 1, we have*

$$\left| \frac{\tilde{I}_{L/K}(x)}{\frac{|C|}{|G|} x} \right| \leq \ell_0 (\log d_L) \frac{\log x}{x}$$

where ℓ_0 depends on x_0 and δ_0, and is given by

$$\ell_0 = \frac{2}{\log 2} \left(1 + \frac{\log(1 + \delta_0)}{\log x_0} \right) \leq 2.89. \tag{3.29}$$

Proof. By definition of θ in (3.28) and h in (3.10), we get

$$|\tilde{I}_{L/K}(x)| \leq \sum_{\mathfrak{p} \text{ ramified}} \sum_{\substack{m \geq 1 \\ \mathrm{N}\,\mathfrak{p}^m < x(\alpha+\delta)}} (\log(\mathrm{N}\,\mathfrak{p})) \leq \sum_{\mathfrak{p} \text{ ramified}} \log(\mathrm{N}\,\mathfrak{p}) \sum_{\substack{m \geq 1 \\ \mathrm{N}(\mathfrak{p}^m) < x(\alpha+\delta)}} 1. \tag{3.30}$$

Serre [26, Proposition 5] proved

$$\sum_{\mathfrak{p} \text{ ramified}} \log(\mathrm{N}\,\mathfrak{p}) \leq \frac{2}{|G|} \log d_L. \tag{3.31}$$

Also we know that for each prime ideal $\mathfrak{p}$, $\mathrm{N}\,\mathfrak{p} \geq 2$. Hence,

$$\sum_{\substack{m \geq 1 \\ \mathrm{N}(\mathfrak{p}^m) < x(\alpha+\delta)}} 1 \leq \frac{\log(x(\alpha + \delta))}{\log 2}. \tag{3.32}$$

Putting together (3.30), (3.31) and (3.32), we get

$$|\tilde{I}_{L/K}(x)| \leq \frac{2}{\log 2} \frac{(\log d_L)(\log(x(\alpha + \delta)))}{|G|}. \tag{3.33}$$

Using $\delta \leq \delta_0$, $\alpha \leq 1$ and $|C| \geq 1$ in (3.33), we complete the proof. □

3.5 Explicit formula for smoothed sum over all prime ideals ($I_{L/K}$)

To obtain explicit formula for the prime counting function, $\psi_C(x)$, Lagarias and Odlyzko in [15] and Winckler in [30] use the classical method of Perron's formula. We instead use inverse Mellin transform by generalizing the approach for Riemann ζ-function taken by Faber and Kadiri in [8], and for Dirichlet L-functions taken by Kadiri and Lumley in [14].

3.5.1 Expressing $I_{L/K}$ in terms of Hecke L-functions

Lemma 3.14. *Let $g \in C$, $G_0 =< g >$ be the cyclic group generated by g, E be the fixed field of G_0, and χ denotes the irreducible characters of G_0. Let H be defined in* (3.11)*. Then*

$$I_{L/K}(x) = \frac{|C|}{|G|}\sum_{\chi}\overline{\chi}(g)\bigg(\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} H(s)x^s\Big(-\frac{L'}{L}(s,\chi,L/E)\Big)ds\bigg).$$

Proof. Let ϕ be a irreducible characters of $G = Gal(L/K)$. Recall from (2.2) that

$$\phi_K(\mathfrak{p}^m) = \frac{1}{|I_0|}\sum_{\alpha\in I_0}\phi(\tau^m\alpha),$$

where I_0 is the inertia group of $\mathfrak{q}$, one of the prime ideal factors of $\mathfrak{p}$, and τ is one of the Frobenius automorphism corresponding to $\mathfrak{p}$. If $L(s,\phi,L/K)$ is the Artin L-series associated to ϕ, then from [15, (3.2)], we get that for $\Re(s) > 1$,

$$-\frac{L'}{L}(s,\phi,L/K) = \sum_{\mathfrak{p},m}\phi_K(\mathfrak{p}^m)\log(\mathrm{N}\,\mathfrak{p})(\mathrm{N}\,\mathfrak{p})^{-ms},$$

where the outer sum is over all the prime ideals of K. Using [15, (3.1),(3.2),(3.5),(3.6)], θ defined in (3.28) can be redefined in a general way as

$$\theta(\mathfrak{p}^m) = \frac{|C|}{|I_0||G|}\sum_{\substack{\phi\\ \alpha\in I_0}}\overline{\phi}(g)\phi(\tau^m\alpha). \tag{3.34}$$

Using (3.34) and the inverse Mellin transform of h in (3.26), we obtain

$$I_{L/K}(x) = \frac{|C|}{|G|}\sum_{\phi\in G}\overline{\phi}(g)\bigg(\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} H(s)x^s\Big(-\frac{L'}{L}(s,\phi,L/K)\Big)ds\bigg).$$

Deuring reduction as shown in [15, Lemma 4.1] is the process of reduction of Artin L-functions, $L(s,\phi,L/K)$ to the case of Hecke L-functions, $L(s,\chi,L/E)$ where intermediate field extension L/E has a cyclic Galois group. Hecke L-functions have been proven to be holomorphic on the entire complex plane whereas the same has not been proven for the Artin L-functions. Following Deuring reduction, we obtain, $\sum_\chi \overline{\chi}(g)\chi^* = \sum_\phi \overline{\phi}(g)\phi$ where χ^* is the character of G induced by G_0. Also [18, Theorem 2.3.2(d)] gives us $L(s,\chi^*,L/K) = L(s,\chi,L/E)$. Therefore,

$$I_{L/K}(x) = \frac{|C|}{|G|}\sum_\chi \overline{\chi}(g)\bigg(\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} H(s)x^s\Big(-\frac{L'}{L}(s,\chi,L/E)\Big)ds\bigg),$$

which holds for $\Re(s) > 1$, and hence by analytic continuation, for all s. □

3.5.2 Obtaining formula for $I_{L/K}$

From now on, we will denote $L(s,\chi)$ for $L(s,\chi,L/E)$. We let $F(\chi)$ denote the conductor of χ and set $A(\chi) = d_E\mathbb{N}_{E/\mathbb{Q}}(F(\chi))$. Let $\delta(\chi)$ be defined as

$$\delta(\chi) = \begin{cases} 1 & \text{if } \chi = \chi_1, \text{ the principal character,} \\ 0 & \text{otherwise.} \end{cases} \tag{3.35}$$

From (2.44) - (2.48), we recall that, for each χ, there exist non-negative integers $a(\chi)$ and $b(\chi)$ such that $a(\chi)+b(\chi) = n_E$, and such that if we define $\gamma_\chi(s)$ as in (2.46) and $\xi(s,\chi)$ as in (2.47), then $\xi(s,\chi)$ satisfies the functional equation (2.48). From (2.52), we recall that, for all complex number s,

$$\frac{L'}{L}(s,\chi) = B(\chi) + \sum_\rho\Big(\frac{1}{s-\rho}+\frac{1}{\rho}\Big) - \delta(\chi)\Big(\frac{1}{s}+\frac{1}{s-1}\Big) - \frac{1}{2}\log(A(\chi)) - \frac{\gamma'_\chi}{\gamma_\chi}(s), \tag{3.36}$$

where $B(\chi)$ is a constant which depends on χ but not defined explicitly and ρ denotes the non-trivial zeros of $L(s,\chi)$, i.e., $\rho = \beta + i\gamma$ with $0 < \beta < 1$. Theorem 2.12 showed that

$$\Re B(\chi) = -\sum_\rho \Re\Big(\frac{1}{\rho}\Big). \tag{3.37}$$

Also, from (2.94), recall that, $r(\chi)$ is a constant defined as

$$r(\chi) = B(\chi) - \frac{1}{2}(\log A(\chi)) + \frac{n_E}{2}(\log \pi) + \delta(\chi) - \frac{b(\chi)}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}\right) - \frac{a(\chi)}{2}\frac{\Gamma'}{\Gamma}(1). \tag{3.38}$$

Lemma 3.15. *Let $s = \sigma + it$ and $\rho = \beta + i\gamma$ denotes a zero of $L(s, \chi)$ with $0 < \beta < 1$.*

1. *For $\sigma \leq -1/4$ and $|s + m| \geq 1/4$ for all non-negative integer m,*

$$\frac{L'}{L}(s, \chi) \ll \log A(\chi) + n_E \log(|s| + 2). \tag{3.39}$$

2. *For $-1/2 \leq \sigma \leq 3$ and $|s| \geq 1/8$,*

$$\left|\frac{L'}{L}(s, \chi) + \frac{\delta(\chi)}{s-1} - \sum_{\substack{\rho \\ |\gamma - t| \leq 1}} \frac{1}{s - \rho}\right| \ll \log A(\chi) + n_E \log(|t| + 2). \tag{3.40}$$

3. *Suppose $n_\chi(t) = \#\{\rho \mid L(\rho, \chi) = 0,\ |\gamma - t| \leq 1\}$. Then for all t,*

$$n_\chi(t) \ll \log A(\chi) + n_E \log(|t| + 2). \tag{3.41}$$

4. *Suppose $\gamma \neq t$, $|t| \geq 2$, $x \geq 2$ and $1 < \sigma_1 \leq 3$, then*

$$\int_{-1/4}^{\sigma_1} \frac{x^{\sigma + it}}{(\sigma + it)(\sigma + it - \rho)} d\sigma \ll |t|^{-1} x^{\sigma_1} (\sigma_1 - \beta)^{-1}. \tag{3.42}$$

Proof. See [15, Lemma 6.2, Lemma 5.6, Lemma 5.4, Lemma 6.3]. □

Proposition 3.16. *Let C be a conjugacy class of $G = \mathrm{Gal}(L/K)$, $g \in C$, $G_0 =< g >$ be the cyclic group generated by g, E be the fixed field of G_0, and χ runs through the irreducible characters of G_0. Let $0 < \delta \leq 0.01$, $\alpha = 1 - \delta$ or 1, h be defined in* (3.10) *and H be defined in* (3.11)*. Then for $x \geq 2$,*

$$I_{L/K} = xH(1) + \frac{|C|}{|G|}\sum_{\chi} \overline{\chi}(g)\Bigg(- r(\chi) - (a(\chi) - \delta(\chi))\left((\log \alpha x) + \int_{\alpha}^{\alpha+\delta} \frac{h(t)}{t} dt\right) - \sum_{\rho \in Z(\chi)} x^{\rho} H(\rho) - b(\chi) \sum_{m \geq 1} x^{-(2m-1)} H(1 - 2m) - a(\chi) \sum_{m \geq 1} x^{-2m} H(-2m)\Bigg), \tag{3.43}$$

where $Z(\chi)$ denote the set of non-trivial zeros of $L(s,\chi)$, which are precisely those zeros $\rho = \beta + i\gamma$ of $L(s,\chi)$ for which $0 < \beta < 1$.

Proof. Let $x \geq 2$ and $T \geq 2$ be such that it does not equal the ordinate of any zero of any of the $L(s,\chi)$. We rewrite (3.5.1) as

$$I_{L/K}(x) = \frac{|C|}{|G|} \sum_{\chi} \overline{\chi}(g) \bigg(\lim_{T\to\infty} J_\chi(x,T) \bigg), \tag{3.44}$$

where

$$J_\chi(x,T) = \frac{1}{2\pi i} \int_{2-iT}^{2+iT} Y_\chi(s) ds \quad \text{with} \quad Y_\chi(s) = H(s) x^s \bigg(\frac{L'}{L}(s,\chi) \bigg). \tag{3.45}$$

Let $U = j + \frac{1}{2}$ for some non-negative integer j and $B_{T,U}$ be the positively oriented rectangle with vertices at $2 - iT$, $2 + iT$, $-U + iT$ and $-U - iT$. We define

$$J_\chi(x,T,U) = \frac{1}{2\pi i} \int_{B_{T,U}} Y_\chi(s) ds. \tag{3.46}$$

The next step is to study the poles for $Y_\chi(s)$, followed by using Cauchy's theorem and then take the limit as $T, U \to \infty$. The poles of $Y_\chi(s)$ inside $B_{T,U}$ and their contributions are:

1. **At $s = 0$.** The Laurent series expansion of $\frac{L'}{L}(s,\chi)$ about $s = 0$ as defined in [15, Page 448, (7.1)] shows that

$$\frac{L'}{L}(s,\chi) = \frac{a(\chi) - \delta(\chi)}{s} + r(\chi) + s f(s,\chi),$$

where $f(s,\chi)$ is a function that is analytic at $s = 0$. Also, Kadiri and Lumley in [14, (3.3),(3.4)] showed that $H(s)x^s$ has a simple pole at $s = 0$ and its Laurent series expansion at this point is

$$H(s)x^s = \frac{1}{s}(1 + (\log x + G'(0))s + \mathcal{O}(s^2)) \text{ with } G'(0) = \log\alpha + \int_\alpha^{\alpha+\delta} \frac{h(t)}{t} dt. \tag{3.47}$$

Therefore the residue of $Y_\chi(s)$ at $s = 0$ is

$$-r(\chi) - (a(\chi) - \delta(\chi)) \bigg((\log \alpha x) + \int_\alpha^{\alpha+\delta} \frac{h(t)}{t} dt \bigg).$$

2. **At $s = 1$.** We know that $H(s)x^s$ is analytic at $s = 1$, as well as $L(s, \chi)$ unless $\chi = \chi_1$, the principal character, in which case $\frac{L'}{L}(s, \chi)$ has a first order pole of residue -1 at $s = 1$. Hence, the residue of $Y_\chi(s)$ at $s = 1$ is

$$\delta(\chi)xH(1).$$

3. **At non-trivial zeros of $L(s, \chi)$.** $\frac{L'}{L}(s, \chi)$ has a first order pole with residues $+1$ at each non-trivial zero ρ of $L(s, \chi)$ (counted with multiplicity). Also $F(s)x^s$ is analytic at such points. Hence, the residue of $Y_\chi(s)$ at $Z(\chi)$ is

$$-\sum_{\substack{\rho\in Z(\chi)\\ |\gamma|<T}} x^\rho H(\rho).$$

4. **At trivial zeros of $L(s, \chi)$.** $\frac{L'}{L}(s, \chi)$ has first order poles at the so-called trivial zeros, i.e., at $s = -(2m-1)$, $m = 1, 2, ...$ with residue $b(\chi)$, and at $s = -2m$, $m = 0, 1, 2, ...$ with residue $a(\chi)$. Therefore, the residue of $Y_\chi(s)$ at trivial zeros with $\Re(s) < 0$ is:

$$-b(\chi) \sum_{1\le m\le \frac{U+1}{2}} x^{-(2m-1)}H(-(2m-1)) - a(\chi) \sum_{1\le m\le \frac{U}{2}} x^{-2m}H(-2m).$$

Now using Cauchy's theorem on $J_\chi(x, T, U)$, we obtain

$$\begin{aligned}
&J_\chi(x, T)\\
&= V_{1,\chi}(x, T, U) + V_{2,\chi}(x, T, U) + V_{3,\chi}(x, T) - r(\chi) - (a(\chi) - \delta(\chi))\left((\log \alpha x) + \int_\alpha^{\alpha+\delta} \frac{h(t)}{t} dt\right)\\
&+ \delta(\chi)xH(1) - \sum_{\substack{\rho\in Z(\chi)\\ |\gamma|<T}} x^\rho H(\rho) - b(\chi) \sum_{1\le m\le \frac{U+1}{2}} x^{-(2m-1)}H(1-2m) - a(\chi) \sum_{1\le m\le \frac{U}{2}} x^{-2m}H(-2m),
\end{aligned} \tag{3.48}$$

where $V_{1,\chi}(x, T, U)$ is the vertical integral defined as

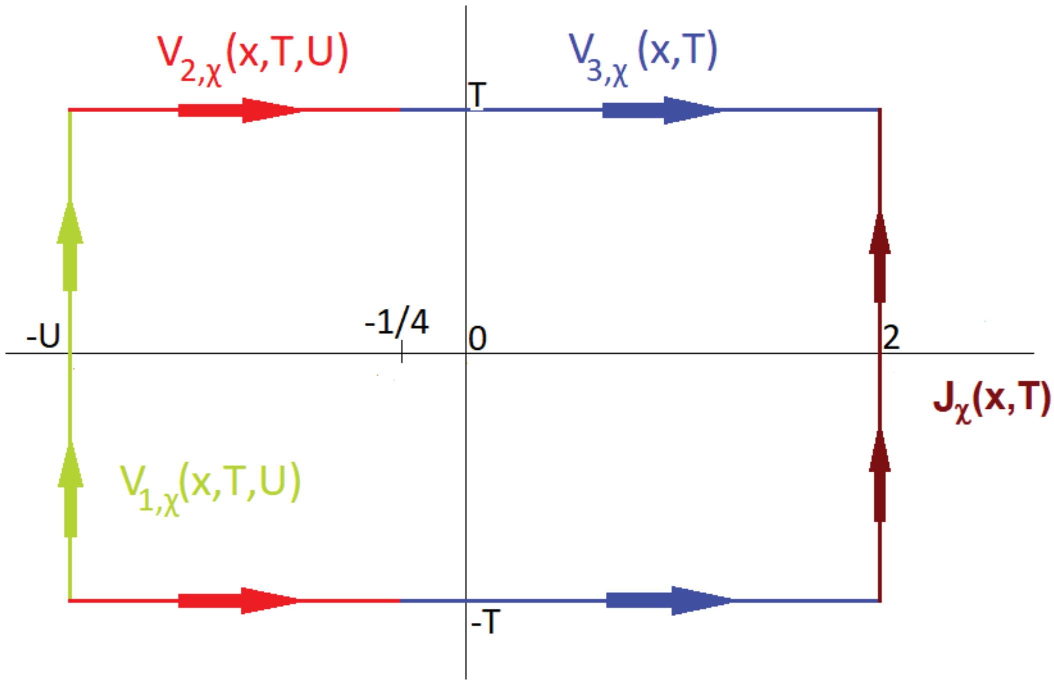
$$V_{1,\chi}(x, T, U) = -\frac{1}{2\pi} \int_{-T}^{T} Y_\chi(-U + it)dt, \tag{3.49}$$

and $V_{2,\chi}(x,T,U)$ and $V_{3,\chi}(x,T)$ are two horizontal integrals defined respectively as

$$V_{2,\chi}(x,T,U) = \frac{1}{2\pi i}\int_{-U}^{-1/4}(Y_\chi(\sigma - iT) - Y_\chi(\sigma + iT))d\sigma, \tag{3.50}$$

and

$$V_{3,\chi}(x,T) = \frac{1}{2\pi i}\int_{-1/4}^{2}(Y_\chi(\sigma - iT) - Y_\chi(\sigma + iT))d\sigma. \tag{3.51}$$



We use the bounds (2.67) for the L-function and (3.14) for the Mellin transform H. For $V_{1,\chi}$, we use

$$Y_\chi(-U \pm it) \ll \begin{cases} \frac{\log U}{U^{m+1}} x^{-U} & \text{when } 0 < |t| < \min(U,T), \\ \frac{\log |t|}{|t|^{m+1}} x^{-U} & \text{when } U < \sigma < T, \end{cases}$$

giving

$$V_{1,\chi} \ll \frac{\log U}{U^m} x^{-U} + \frac{\log T}{T^m} x^{-U}. \tag{3.52}$$

For $V_{2,\chi}$, we use

$$Y_\chi(-\sigma \pm iT) \ll \begin{cases} \frac{\log T}{T^{m+1}} x^{-\sigma} & \text{when } 1/4 < \sigma < \min(U,T), \\ \frac{\log \sigma}{\sigma^{m+1}} x^{-\sigma} & \text{when } T < \sigma < U, \end{cases}$$

giving

$$V_{2,\chi} \ll \frac{\log T}{T^{m+1}} \frac{x^{-1/4}}{\log x} + \frac{x^{-T}}{T^{m-1}}. \tag{3.53}$$

For $V_{3,\chi}$, we use (3.40) to control the size of the L-function inside the critical strip: since $\frac{L'}{L}(s,\chi) - \sum_{\substack{\rho \in Z(\chi) \\ |\gamma \mp T| \le 1}} \frac{1}{s-\rho} \ll \log T$, then

$$\int_{-1/4}^{2} x^{\sigma \pm iT} H(\sigma \pm iT) \left(\frac{L'}{L}(\sigma \pm iT, \chi) - \sum_{\substack{\rho \in Z(\chi) \\ |\gamma \mp T| \le 1}} \frac{1}{\sigma \pm iT - \rho} \right) d\sigma \ll \frac{\log T}{T^{m+1}} \frac{x^2}{\log x}. \tag{3.54}$$

On the other hand, it follows from (3.42) that

$$\int_{-1/4}^{2} \frac{x^{\sigma \pm iT}}{(\sigma \pm iT)(\sigma \pm it - \rho)} d\sigma \ll \frac{x^2}{T}, \tag{3.55}$$

and together with (3.41) , we obtain

$$\int_{-1/4}^{2} x^{\sigma \pm iT} H(\sigma \pm iT) \left(\sum_{\substack{\rho \in Z(\chi) \\ |\gamma \mp T| \le 1}} \frac{1}{\sigma \pm iT - \rho} \right) d\sigma \ll \frac{x^2}{T^{m+1}} n_\chi(\pm T) \ll x^2 \frac{\log T}{T^{m+1}}. \tag{3.56}$$

Adding (3.54)and (3.56) gives

$$V_{3,\chi} \ll x^2 \frac{\log T}{T^{m+1}}. \tag{3.57}$$

Adding (3.52), (3.53), and (3.57) and taking the limit as $T, U \to \infty$, we obtain, that for any fixed $x > 0$,

$$\lim_{T,U \to \infty} V_{1,\chi}(x,T,U) + V_{2,\chi}(x,T,U) + V_{3,\chi}(x,T) = 0. \tag{3.58}$$

Combining (3.44), (3.48), (3.58) and $\sum_\chi \delta(\chi) = 1$, we obtain the announced identity (3.43). □

Now since $\prod_\chi L(s,\chi) = \zeta_L(s)$, therefore any sum over the non-trivial zeros of all the $L(s,\chi)$'s can be considered as a sum over the non-trivial zeros of $\zeta_L(s)$. Thus rewriting

$$r(\chi)+\sum_{\rho\in Z(\chi)}x^\rho H(\rho)=r(\chi)+\sum_{\substack{\rho\in Z(\chi)\\|\rho|<\frac12}}\frac{1}{\rho}-\sum_{\substack{\rho\in Z(\chi)\\|\rho|<\frac12}}\frac{1}{\rho}+\sum_{\rho\in Z(\chi)}x^\rho H(\rho)$$

and using $\sum_\chi |a(\chi)-\delta(\chi)|\le\sum_\chi n_E=n_L$ in (3.43), we obtain :

Corollary 3.17. *Let $0<\delta\le 0.01$, $\alpha=1-\delta$ or 1, h be defined in* (3.10) *and H be defined in* (3.11)*. Then for $x\ge 2$ and $T\ge 2$,*

$$\left|I_{L/K}-\frac{|C|}{|G|}xH(1)\right|\le\frac{|C|}{|G|}\Bigg(x^{\beta_0}H(\beta_0)+n_L\left|(\log\alpha x)+\int_\alpha^{\alpha+\delta}\frac{h(t)}{t}dt\right|+J^{(0)}(x)+J^{(1)}(x)+J^{(2)}(x)$$
$$+J^{(3)}(x)+x\Big(J^{(4)}(x)+J^{(5)}(x,T)+J^{(6)}(x,T)\Big)\Bigg),\qquad(3.59)$$

where

$$J^{(0)}(x)=\sum_\chi\Big(b(\chi)\sum_{m\ge1}x^{-(2m-1)}|H(1-2m)|+a(\chi)\sum_{m\ge1}x^{-2m}|H(-2m)|\Big),\qquad(3.60)$$
$$J^{(1)}=\sum_\chi\left|r(\chi)+\sum_{|\rho|<\frac12}\frac{1}{\rho}\right|,\qquad(3.61)$$
$$J^{(2)}(x)=x^{1-\beta_0}H(1-\beta_0)-\frac{1}{1-\beta_0},\qquad(3.62)$$
$$J^{(3)}(x)=\sum_{\rho\ne1-\beta_0,|\rho|<\frac12}\left|x^\rho H(\rho)-\frac{1}{\rho}\right|,\qquad(3.63)$$
$$J^{(4)}(x)=\sum_{\rho\ne\beta_0,|\rho|\ge\frac12,|\gamma|\le\frac{1}{\alpha_4\log d_L}}x^{\beta-1}|H(\rho)|,\qquad(3.64)$$
$$J^{(5)}(x,T)=\sum_{|\rho|\ge\frac12,\frac{1}{\alpha_4\log d_L}<|\gamma|<T}x^{\beta-1}|H(\rho)|,\qquad(3.65)$$
$$J^{(6)}(x,T)=\sum_{|\gamma|\ge T}x^{\beta-1}|H(\rho)|,\qquad(3.66)$$

with α_4 defined in (3.6)*, β_0 being the possible real exceptional real zero of $\zeta_L(s)$, $J^{(1)}$ summing over the non-trivial zeros, ρ of $L(s,\chi)$ for all χ and $J^{(3)},J^{(4)},J^{(5)},J^{(6)}$ summing over the non-trivial zeros, ρ of the Dedekind ζ-function, $\zeta_L(s)$.*

Assume for the rest of the article,

$$
\begin{aligned}
&\bullet\ m \in \mathbb{N},\ m \geq 2,\\
&\bullet\ T \geq T_0 \geq 44,\\
&\bullet\ x \geq x_0 \geq \exp(11954),\\
&\bullet\ \alpha = 1-\delta \text{ or } 1 \text{ with } 0 < \delta \leq \delta_0 \leq 10^{-10},\\
&\bullet\ M(\delta, 0) \leq \frac{2+\delta_0}{2},\\
&\bullet\ \int_0^1 g(u)\ du = \frac{1}{2},\\
&\bullet\ R_L = Rn_L \text{ with } R = 29.57,\\
&\bullet\ D = 2d_L^{\frac{1}{n_L}},\\
&\bullet\ 2 \leq n_L \leq \frac{\log d_L}{\log\sqrt{3}} \text{ and } \log d_L \geq \log 3.
\end{aligned}
\tag{3.67}
$$

Note that the above assumptions on M and h are satisfied for the weights we consider in this article and that the inequality on n_L is the classical Hermite-Minkowski's inequality. Moreover, following three facts are going to be used repeatedly in this article :

$$\sum_{\chi} \log(A(\chi)) = \log d_L,\ \sum_{\chi} n_E = n_L \text{ and } \sum_{\chi} \delta(\chi) = 1. \tag{3.68}$$

3.6 Expressing $\left|I_{L/K} - \frac{|C|}{|G|}xH(1)\right|$ in terms of sum over zeros

3.6.1 Preliminary results about sums over zeros of $L(s,\chi)$

Lagarias and Odlyzko in [15, Lemma 5.4] proved a non-explicit result concerning the zero counting function, $n_\chi(T)$ which counts the number of non-trivial zeros $\rho = \beta + i\gamma$ of $L(s,\chi)$ with $|T-\gamma| \leq a$ where $a = 1$ and Winckler in [30, Lemma 4.6] made their result explicit. We prove a more general result for $n_\chi(T)$ with any $a > 0$ in this section. This is achieved by generalizing a new technique employed for Dirichlet L-functions by Fiorilli and Martin in [10, Section 5]. In [10, Lemma 5.3], they used $2+iT$ in their sum to obtain their result. We, on the other hand, use $1+\epsilon+iT$ instead to get closer to the $\Re(s) = 1$ line and also provide a general result depended on both a and ϵ. First, from [30, Lemma 4.3, Lemma 4.5], we recall that

Lemma 3.18. *If $\Re(s) > 1$, then*

$$\left|\frac{L'}{L}(s,\chi)\right| \leq \frac{n_E}{\Re(s)-1}. \tag{3.69}$$

and if $\Re(s) > -\frac{1}{2}$ and $|s| \geq \frac{1}{8}$, then

$$\left|\frac{\gamma'_\chi}{\gamma_\chi}(s)\right| \leq \frac{n_E}{2}\left(\log(1+|s|) + \frac{164}{7}\right). \tag{3.70}$$

Lemma 3.19. *Let $0 < \epsilon \leq 1$ and let $\rho = \beta + i\gamma$ be the non-trivial zero of $L(s,\chi)$. For any real number T,*

$$\sum_{\rho \in Z(\chi)} \frac{1}{|1+\epsilon+iT-\rho|^2} < n_E\left(\frac{1}{2\epsilon}\log(2+\epsilon+|T|) + \left(\frac{1}{\epsilon^2} + \frac{164}{14\epsilon}\right)\right) + \frac{\log A(\chi)}{2\epsilon} + \frac{2\delta(\chi)}{\epsilon}.$$

Proof. By the classical explicit formula for $\frac{L'}{L}(s,\chi)$ given in (2.52) and (3.37), we notice that

$$\Re\left(\sum_{\rho \in Z(\chi)} \frac{1}{1+\epsilon+iT-\rho}\right) = \Re\left(\frac{L'}{L}(1+\epsilon+iT,\chi)\right) + \frac{\log A(\chi)}{2}$$
$$+ \delta(\chi)\left(\Re\left(\frac{1}{1+\epsilon+iT} + \frac{1}{\epsilon+iT}\right)\right) + \Re\left(\frac{\gamma'_\chi}{\gamma_\chi}(1+\epsilon+iT)\right).$$

Using Lemma 3.18 in the above equation, we obtain

$$\Re\left(\sum_{\rho \in Z(\chi)} \frac{1}{1+\epsilon+iT-\rho}\right) < n_E\left(\frac{1}{2}\log(2+\epsilon+|T|) + \left(\frac{1}{\epsilon} + \frac{164}{14}\right)\right) + \frac{\log A(\chi)}{2} + 2\delta(\chi). \tag{3.71}$$

Using this with the fact that $\sum_{\rho \in Z(\chi)} \frac{1}{|1+\epsilon+iT-\rho|^2} < \frac{1}{\epsilon}\Re\left(\sum_{\rho \in Z(\chi)} \frac{1}{1+\epsilon+iT-\rho}\right)$, we obtain the required result. $\square$

Proof of Proposition 1.13. Since $0 < \beta < 1$, therefore $\epsilon < 1+\epsilon-\beta < 1+\epsilon$. Using this we get

$$\sum_{\substack{\rho\in Z(\chi)\\ |T-\gamma|\le a}} 1 \le \frac{(1+\epsilon)^2+a^2}{\epsilon}\sum_{\rho\in Z(\chi)}\frac{1+\epsilon-\beta}{(1+\epsilon-\beta)^2+(T-\gamma)^2}$$
$$= \frac{(1+\epsilon)^2+a^2}{\epsilon}\sum_{\rho\in Z(\chi)}\Re\left(\frac{1}{1+\epsilon+iT-\rho}\right).$$

Thus using (3.71), we obtain the required result. □

Lemma 3.20. *Let $0<\epsilon\le 1$. If $s=\sigma+iT$ with $-\frac{1}{2}<\sigma\le 2$ and $|s|\ge\frac{1}{8}$, then*

$$\left|\frac{L'}{L}(s,\chi)+\frac{\delta(\chi)}{s-1}-\sum_{\substack{\rho\in Z(\chi)\\ |\gamma-T|\le a}}\frac{1}{s-\rho}\right| \le n_E(c_5(a,\epsilon)\log(3+|T|)+c_6(a,\epsilon))+(c_5(a,\epsilon)-1)\log A(\chi)$$
$$+c_7(a,\epsilon)\delta(\chi),$$

where

$$c_3(a,\epsilon)=\frac{1}{2\epsilon}\left(\frac{3}{2}+\epsilon\right)\sqrt{1+\left(\frac{1+\epsilon}{a}\right)^2}, \tag{3.72}$$
$$c_4(a,\epsilon)=\left(\frac{3}{2}+\epsilon\right)\sqrt{1+\left(\frac{1+\epsilon}{a}\right)^2}\left(\frac{1}{\epsilon^2}+\frac{164}{14\epsilon}\right), \tag{3.73}$$
$$c_5(a,\epsilon)=\frac{c_1(a,\epsilon)}{\epsilon}+c_3(a,\epsilon)+1, \tag{3.74}$$
$$c_6(a,\epsilon)=\frac{1}{\epsilon}+\frac{c_2(a,\epsilon)}{\epsilon}+\frac{164}{7}+c_4(a,\epsilon), \tag{3.75}$$
$$c_7(a,\epsilon)=\frac{4c_1(a,\epsilon)}{\epsilon}+4c_3(a,\epsilon)+10. \tag{3.76}$$

Proof. Using the formula for $\frac{L'}{L}(s,\chi)$ as in (2.52) and studying the difference $\frac{L'}{L}(s,\chi)-\frac{L'}{L}(1+$

$\epsilon + iT, \chi)$, we obtain

$$\begin{aligned}
&\left|\frac{L'}{L}(s,\chi) + \frac{\delta(\chi)}{s-1} - \sum_{\substack{\rho\in Z(\chi)\\|\gamma-T|\le a}} \frac{1}{s-\rho}\right| \\
&\le \left|\frac{L'}{L}(1+\epsilon+iT,\chi)\right| + \left|\frac{\gamma'_\chi}{\gamma_\chi}(1+\epsilon+iT) - \frac{\gamma'_\chi}{\gamma_\chi}(s)\right| + \sum_{\substack{\rho\in Z(\chi)\\|\gamma-T|>a}} \left|\frac{1}{s-\rho} - \frac{1}{1+\epsilon+iT-\rho}\right| \\
&+ \sum_{\substack{\rho\in Z(\chi)\\|\gamma-T|\le a}} \frac{1}{|1+\epsilon+iT-\rho|} + \delta(\chi)\left|\frac{1}{1+\epsilon+iT} + \frac{1}{\epsilon+iT} - \frac{1}{s}\right|. \qquad (3.77)
\end{aligned}$$

We bound the terms on the right side individually. By Lemma 2.13, we obtain

$$\left|\frac{L'}{L}(1+\epsilon+iT,\chi)\right| \le \frac{n_E}{\epsilon}.$$

By Lemma 2.15 and using $\epsilon \le 1$, we obtain

$$\left|\frac{\gamma'_\chi}{\gamma_\chi}(1+\epsilon+iT) - \frac{\gamma'_\chi}{\gamma_\chi}(s)\right| \le n_E\left(\log(3+|T|) + \frac{164}{7}\right).$$

Remember that $-\frac{1}{2} < \sigma < 2$, thus $1+\epsilon-\sigma \le \epsilon + \frac{3}{2}$. Using this, we get

$$\begin{aligned}
\sum_{\substack{\rho\in Z(\chi)\\|\gamma-T|>a}} \left|\frac{1}{s-\rho} - \frac{1}{1+\epsilon+iT-\rho}\right| &= \sum_{\substack{\rho\in Z(\chi)\\|\gamma-T|>a}} \frac{1+\epsilon-\sigma}{|s-\rho||1+\epsilon+iT-\rho|} \\
&< \left(\epsilon + \frac{3}{2}\right) \sum_{\substack{\rho\in Z(\chi)\\|\gamma-T|>a}} \frac{|1+\epsilon+iT-\rho|}{|s-\rho|}\frac{1}{|1+\epsilon+iT-\rho|^2}.
\end{aligned}$$

Using Lemma 3.19 and the fact that $\frac{|1+\epsilon+iT-\rho|}{|s-\rho|} \le \sqrt{1+\left(\frac{1+\epsilon}{a}\right)^2}$, we get

$$\begin{aligned}
&\sum_{\substack{\rho\in Z(\chi)\\|\gamma-T|>a}} \left|\frac{1}{s-\rho} - \frac{1}{1+\epsilon+iT-\rho}\right| \\
&< n_E(c_3(a,\epsilon)\log(2+\epsilon+|T|) + c_4(a,\epsilon)) + c_3(a,\epsilon)\log A(\chi) + 4c_3(a,\epsilon)\delta(\chi).
\end{aligned}$$

Since $0 < \beta < 1$, therefore, $|1+\epsilon+iT-\beta-i\gamma| > \epsilon$. Using this, we get

$$\sum_{\substack{\rho\in Z(\chi)\\ |\gamma-T|\le a}} \frac{1}{|1+\epsilon+iT-\rho|} \le \frac{1}{\epsilon} \sum_{\substack{\rho\in Z(\chi)\\ |\gamma-T|\le a}} 1 = \frac{n_{\chi,a,\epsilon}(T)}{\epsilon}.$$

Now using Proposition 1.13 in the above equation, we get

$$\sum_{\substack{\rho\in Z(\chi)\\ |\gamma-T|\le a}} \frac{1}{|1+\epsilon+iT-\rho|} \le n_E\left(\frac{c_1(a,\epsilon)}{\epsilon}\log(2+\epsilon+|T|)+\frac{c_2(a,\epsilon)}{\epsilon}\right)+\frac{c_1(a,\epsilon)}{\epsilon}\log A(\chi)+\frac{4c_1(a,\epsilon)}{\epsilon}\delta(\chi).$$

Finally, since $|s| \ge \frac{1}{8}$, thus

$$\delta(\chi)\left|\frac{1}{1+\epsilon+iT}+\frac{1}{\epsilon+iT}-\frac{1}{s}\right| \le 10\delta(\chi).$$

Thus using $\epsilon \le 1$ in (3.77), we obtain the required result. □

Recall that $B(\chi)$ is the undefined constant in the expression for $\frac{L'}{L}(s,\chi)$ given in (2.52). Lagarias and Odlyzko in [15, Lemma 5.5] proved that, for any ϵ with $0 < \epsilon \le 1$, we have

$$B(\chi) + \sum_{\substack{\rho\in Z(\chi)\\ |\rho|<\epsilon}} \frac{1}{\rho} \ll \frac{\log A(\chi) + n_E}{\epsilon}, \tag{3.78}$$

and Winckler made their result explicit which is shown in Lemma 2.17. We prove :

Lemma 3.21. *For any $\epsilon \in (0,1]$, we have*

$$\sum_{\chi}\left|B(\chi)+\sum_{\substack{\rho\in Z(\chi)\\ |\rho|<\epsilon}}\frac{1}{\rho}\right| \le \left(3.19+\frac{0.547}{\epsilon}\right)\log d_L + \left(79.251+\frac{22.205}{\epsilon}\right)n_L + \left(15.06+\frac{4.520}{\epsilon}\right).$$

Proof. Let r be a positive real number. We split the sum over the non-trivial zeros of $L(s,\chi)$ as the following

$$\sum_{\substack{\rho\in Z(\chi)\\ |\rho|<\epsilon}} \frac{1}{\rho} = \sum_{\rho\in Z(\chi)}\left(\frac{1}{\rho}+\frac{1}{1+r-\rho}\right) - \sum_{\substack{\rho\in Z(\chi)\\ |\rho|\ge 1}}\left(\frac{1}{\rho}+\frac{1}{1+r-\rho}\right) - \sum_{\substack{\rho\in Z(\chi)\\ |\rho|<1}}\frac{1}{1+r-\rho} - \sum_{\substack{\rho\in Z(\chi)\\ \epsilon\le|\rho|<1}}\frac{1}{\rho}. \tag{3.79}$$

Note that the second sum can be compared to the convergent sum $\sum_{|\rho|\geq 1} \frac{1}{|\rho|^2}$. In addition, in the last two finite sums, we have, $\frac{1}{|1+r-\rho|} < \frac{1}{r}$ and $\frac{1}{|\rho|} \leq \frac{1}{\epsilon}$. Using (3.79), we obtain

$$\sum_{\chi}\left|B(\chi) + \sum_{\substack{\rho\in Z(\chi)\\|\rho|<\epsilon}} \frac{1}{\rho}\right| \leq \sum_{\chi}\left|B(\chi) + \sum_{\rho\in Z(\chi)}\left(\frac{1}{1+r-\rho} + \frac{1}{\rho}\right)\right| + \sum_{\chi}\left|\sum_{\substack{\rho\in Z(\chi)\\|\rho|\geq 1}}\left(\frac{1}{1+r-\rho} + \frac{1}{\rho}\right)\right|$$
$$+ \sum_{\chi}\left|\sum_{\substack{\rho\in Z(\chi)\\|\rho|<1}} \frac{1}{1+r-\rho}\right| + \sum_{\chi}\left|\sum_{\substack{\rho\in Z(\chi)\\ \epsilon\leq|\rho|<1}} \frac{1}{\rho}\right|. \tag{3.80}$$

Let $Z(\zeta)$ be the set of zeros of $\zeta_L(s)$, $\rho = \beta + i\gamma$ with $0 < \beta < 1$. Since $\prod_\chi L(s,\chi) = \zeta_L(s)$, thus $Z(\zeta) = \cup_\chi Z(\chi)$. For $|\rho| \geq 1/2$, we use $\left|\frac{1}{1+r-\rho} + \frac{1}{\rho}\right| = \frac{1+r}{|(1+r-\rho)\rho|} \leq \frac{1+r}{|\gamma|^2}$ for $|\gamma| > 1$, and $\left|\frac{1}{1+r-\rho} + \frac{1}{\rho}\right| \leq \frac{1+r}{r}$ for $|\gamma| \leq 1$ to obtain

$$\begin{aligned}
\sum_{\chi}\left|\sum_{\substack{\rho\in Z(\chi)\\|\rho|\geq 1}}\left(\frac{1}{1+r-\rho} + \frac{1}{\rho}\right)\right| &\leq \frac{1+r}{r}N_L(1) + \sum_{\substack{\rho\in Z(\zeta)\\|\gamma|>1}} \frac{1+r}{|\gamma|^2}\\
&\leq \frac{1+r}{r}N_L(1) + (1+r)\sum_{k=1}^{\infty}\frac{N_L(k+1) - N_L(k)}{k^2}\\
&= \left(\frac{1}{r} - r\right)N_L(1) + (1+r)\sum_{k=2}^{\infty} N_L(k)\frac{2k-1}{(k(k-1))^2}.
\end{aligned} \tag{3.81}$$

Using definition of $N_L(k)$ as in (3.4), we obtain

$$N_L(k) \leq \left(\frac{1}{\pi}\log\left(\frac{d_L}{(2\pi e)^{n_L}}\right)\right)k + \frac{n_L}{\pi}(k\log k) + \alpha_1 n_L \log k + \alpha_1 \log d_L + \alpha_2 n_L + \alpha_3. \tag{3.82}$$

We compute that

$$2.645 \leq \sum_{k=2}^{\infty}\frac{k(2k-1)}{(k(k-1))^2} \leq 2.65,\ \sum_{k=2}^{\infty}\frac{k(\log k)(2k-1)}{(k(k-1))^2} \leq 3.06,\ \sum_{k=2}^{\infty}\frac{2k-1}{(k(k-1))^2} = 1 \text{ and}$$
$$\sum_{k=2}^{\infty}\frac{(\log k)(2k-1)}{(k(k-1))^2} \leq 0.87. \tag{3.83}$$

Note that $\frac{1}{r} - r$ changes sign at $r = 1$. Thus using (3.83) and (3.82) in (3.81), we obtain

$$\begin{aligned}&\sum_{\chi}\Bigg|\sum_{\substack{\rho\in Z(\chi)\\|\rho|\geq 1}}\left(\frac{1}{1+r-\rho}+\frac{1}{\rho}\right)\Bigg|\\&\leq \max\left\{0,\frac{1}{r}-r\right\}\left(\frac{1}{\pi}\log\left(\frac{d_L}{(2\pi e)^{n_L}}\right)+\alpha_1\log d_L+\alpha_2 n_L+\alpha_3\right)\\&+(1+r)\left(\frac{2.65}{\pi}\log d_L-\frac{2.645\log(2\pi e)}{\pi}n_L+\frac{3.06}{\pi}n_L+0.87\alpha_1 n_L+\alpha_1\log d_L+\alpha_2 n_L+\alpha_3\right).\end{aligned} \tag{3.84}$$

Note that $|1 + r - \rho| > 1 + r - \beta > r$. Again using (3.4), we get

$$\begin{aligned}\sum_{\chi}\Bigg|\sum_{\substack{\rho\in Z(\chi)\\|\rho|<1}}\frac{1}{1+r-\rho}\Bigg|+\sum_{\chi}\Bigg|\sum_{\substack{\rho\in Z(\chi)\\ \epsilon\leq|\rho|<1}}\frac{1}{\rho}\Bigg| &\leq \left(\frac{1}{r}+\frac{1}{\epsilon}\right)N_L(1)\\&\leq\left(\frac{1}{r}+\frac{1}{\epsilon}\right)\left(\frac{1}{\pi}\log\left(\frac{d_L}{(2\pi e)^{n_L}}\right)+\alpha_1\log d_L+\alpha_2 n_L+\alpha_3\right).\end{aligned} \tag{3.85}$$

For the last part, we use (2.52) with $s = 1 + r$. Notice that

$$\begin{aligned}&\sum_{\chi}\Bigg|B(\chi)+\sum_{\rho\in Z(\chi)}\left(\frac{1}{1+r-\rho}+\frac{1}{\rho}\right)\Bigg|\\&=\sum_{\chi}\left|\frac{L'}{L}(1+r,\chi)+\frac{1}{2}\log(A(\chi))+\delta(\chi)\left(\frac{1}{1+r}+\frac{1}{1+r-1}\right)+\frac{\gamma'_\chi}{\gamma_\chi}(1+r)\right|.\end{aligned}$$

Now using Lemma 2.13 and Lemma 2.15, we obtain $\left|\frac{L'}{L}(1+r,\chi)\right| \leq \frac{n_E}{r}$ and $\left|\frac{\gamma'_\chi}{\gamma_\chi}(1+r)\right| \leq \frac{n_E}{2}\left(\log(2+r)+\frac{164}{7}\right)$. Thus using (3.68), we have

$$\sum_{\chi}\Bigg|B(\chi)+\sum_{\rho\in Z(\chi)}\left(\frac{1}{1+r-\rho}+\frac{1}{\rho}\right)\Bigg|\leq\frac{1}{2}\log d_L+\left(\frac{1}{r}+\frac{\log(2+r)}{2}+\frac{164}{14}\right)n_L+\frac{2r+1}{r(r+1)}. \tag{3.86}$$

Finally, we insert (3.84), (3.85) and (3.86) into (3.80) to obtain

$$\sum_{\chi}\left|B(\chi)+\sum_{\substack{\rho\in Z(\chi)\\|\rho|<\epsilon}}\frac{1}{\rho}\right| \le \left(m_1+\frac{m_2}{\epsilon}\right)\log d_L+\left(m_3+\frac{m_4}{\epsilon}\right)n_L+\left(m_5+\frac{m_6}{\epsilon}\right), \tag{3.87}$$

where

$$\begin{aligned}
m_1 &= \max\left\{0,\frac{1}{r}-r\right\}\left(\frac{1}{\pi}+\alpha_1\right)+\frac{2.65(1+r)}{\pi}+\frac{1}{r\pi}+\left(1+r+\frac{1}{r}\right)\alpha_1+0.5,\\
m_2 &= \frac{1}{\pi}+\alpha_1,\\
m_3 &= \max\left\{\frac{1}{r},\frac{2}{r}-r\right\}\left(-\frac{\log(2\pi e)}{\pi}+\alpha_2\right)+(1+r)\left(-\frac{2.645\log(2\pi e)}{\pi}+\frac{3.06}{\pi}+0.87\alpha_1+\alpha_2\right)\\
&\quad+\frac{1}{r}+\frac{\log(2+r)}{2}+\frac{164}{14},\\
m_4 &= -\frac{\log(2\pi e)}{\pi}+\alpha_2,\\
m_5 &= \max\left\{1+r+\frac{1}{r},1+\frac{2}{r}\right\}\alpha_3+\frac{2r+1}{r(r+1)},\\
m_6 &= \alpha_3.
\end{aligned} \tag{3.88}$$

Finally we insert $r = 1$ and $(\alpha_1,\alpha_2,\alpha_3) = (0.228, 23.108, 4.520)$ as given in Theorem 3.10 into (3.87) to obtain the required result. □

Remark 3.22. We use the Hermite-Minkowski's bound, $n_L \le \frac{\log d_L}{\log\sqrt{3}}$ and check that $m_1 + \frac{m_3}{\log\sqrt{3}}$ is minimized at $r = 1$. Hence we choose $r = 1$ in the proof of Lemma 3.21.

Remark 3.23. Both [15, Lemma 5.5] and [30, Lemma 4.7] used the zero counting function, $n_\chi(t)$ which denotes the number of non-trivial zeros, $\rho = \beta + i\gamma$ of $L(s,\chi)$ in the region $|\gamma - t| \le 1$. We instead use $N_L(t)$ defined in (3.4), i.e., the zero-counting function which counts all the zeros, $\rho = \beta + i\gamma$ of all the characters χ in the region $0 < \beta < 1$ and $|\gamma| \le T$. We notice that for $T \ge 2$, $\sum_\chi(n_\chi(T)+n_\chi(-T))$ is equivalent to $N_L(T+1) - N_L(T-1)$. Also, Winckler's result regarding $n_\chi(T)$ in [30, Lemma 4.6] gives

$$\sum_{\chi}(n_\chi(T)+n_\chi(-T)) \le 2.5\log d_L + 2.5 n_L\log(T+3) + 20.06 n_L, \tag{3.89}$$

whereas using [11, Corollary 1.2] regarding $N_L(T)$, we obtain

$$N_L(T+1) - N_L(T-1) \leq 1.093 \log d_L + 1.093 n_L \log T + 45.11 n_L + 9.04. \tag{3.90}$$

We use $N_L(T)$ to prove Lemma 3.21 with coefficient of $\log d_L$ as $3.19 + \frac{0.547}{\epsilon}$ and coefficient of n_L as $79.242 + \frac{22.205}{\epsilon}$. A similar result can be obtained from Lemma 2.17 as proved by Winckler which uses $n_\chi(T)$ to give $10.42 + \frac{1.25}{\epsilon}$ and $101.33 + \frac{11.41}{\epsilon}$ as the coefficient of $\log d_L$ and n_L respectively.

3.6.2 Bounding the $J^{(i)}$'s

In this subsection, we bound all $J^{(i)}$'s defined in Corollary 3.17.

Lemma 3.24. *Under the assumptions in* (3.67),*we have*

$$\left|(\log \alpha x) + \int_\alpha^{\alpha+\delta} \frac{h(t)}{t} dt\right| n_L + J^{(0)}(x) + J^{(1)} \leq \ell_1 (\log d_L)(\log x), \tag{3.91}$$

where ℓ_1 depends on x_0 and δ_0, and is given by

$$\ell_1 = \frac{4.782}{\log x_0} + \frac{1}{\log \sqrt{3}} \left(1 + \frac{\delta_0}{2(1-\delta_0)\log x_0} + \frac{2+\delta_0}{2x_0^2 \log x_0} + \frac{125.214}{\log x_0}\right) + \frac{25.10}{(\log 3)(\log x_0)} \leq 1.842. \tag{3.92}$$

Proof. First of all, using $\delta \leq \delta_0$, $\alpha = 1-\delta$ or 1, $h = h_R$ and $\int_\alpha^{\alpha+\delta} h_R(t, \alpha, \delta)\, dt = \delta/2$, we have

$$\left|(\log \alpha x) + \int_\alpha^{\alpha+\delta} \frac{h(t)}{t} dt\right| \leq \log x + \frac{\delta_0}{2(1-\delta_0)}. \tag{3.93}$$

Now using (3.14) for $k = 0$, we get $|H(s)| \leq \frac{M(\delta,0)}{|s|}$ which we insert in (3.60) to obtain

$$J^{(0)}(x) \leq M(\delta, 0) \sum_\chi \left(b(\chi) \sum_{m \geq 1} \frac{x^{-(2m-1)}}{2m-1} + a(\chi) \sum_{m\geq 1} \frac{x^{-2m}}{2m}\right).$$

Using $\sum_{m\geq 1} \frac{x^{-(2m-1)}}{2m-1} = \sum_{m\geq 1} \frac{x^{-m}}{m} - \sum_{m \geq 1} \frac{x^{-2m}}{2m}$, $b(\chi) = n_E - a(\chi)$ and the Taylor series expansion of $\log x$, we obtain

$$\begin{aligned} b(\chi) \sum_{m \geq 1} \frac{x^{-(2m-1)}}{2m-1} + a(\chi) \sum_{m \geq 1} \frac{x^{-2m}}{2m} &= \frac{n_E}{2} \log\left(\frac{x+1}{x-1}\right) + a(\chi) \log\left(\frac{x}{x+1}\right) \\ &\leq \frac{n_E}{2} \log\left(\frac{x^2}{x^2-1}\right) \leq \frac{n_E}{x^2}. \end{aligned}$$

Thus

$$J^{(0)}(x) \leq M(\delta, 0) \sum_{\chi} \frac{n_E}{x^2} \leq \frac{2+\delta_0}{2} \frac{n_L}{x^2}. \tag{3.94}$$

Inserting $\frac{\Gamma'}{\Gamma}(\frac{1}{2}) = -2(\log 2) - \gamma$, $\frac{\Gamma'}{\Gamma}(1) = -\gamma$ and $a(\chi) + b(\chi) = n_E$ into (2.94), we obtain

$$\left| r(\chi) + \sum_{\substack{\rho \in Z(\chi) \\ |\rho| < \frac{1}{2}}} \frac{1}{\rho} \right| \leq \left| B(\chi) + \sum_{\substack{\rho \in Z(\chi) \\ |\rho| < \frac{1}{2}}} \frac{1}{\rho} \right| + \frac{\log A(\chi)}{2} + \delta(\chi) + n_E \left(\frac{\log \pi}{2} + \frac{\gamma}{2} + \log 2 \right).$$

We conclude with Lemma 3.21 applied to $\epsilon = \frac{1}{2}$, and with (3.68) :

$$J^{(1)} \leq 4.782 \log d_L + 125.214 n_L + 25.10. \tag{3.95}$$

The announced bound is obtained by putting together (3.93), (3.94), (3.95) and (3.67). □

Lemma 3.25. *Under the assumptions in* (3.67)*, we have*

$$J^{(2)}(x) \leq \ell_2 (\log x) x^{\frac{1}{2}}, \tag{3.96}$$

where ℓ_2 depends on x_0 and δ_0, and is given by

$$\ell_2 = 1 + \frac{\delta_0}{2\sqrt{1-\delta_0} \log x_0} \leq 1.001. \tag{3.97}$$

Proof. Using the equation for $H(s)$ given in (3.11) with $h(t) = 1$ for $0 \leq t \leq \alpha$, we have

$$x^s H(s) - \frac{1}{s} = \frac{1}{s}\Big((\alpha x)^s - 1 \Big) + x^s \int_{\alpha}^{\alpha+\delta} h(t) t^{s-1} dt. \tag{3.98}$$

For $s = 1-\beta_0 \le \frac{1}{2}$, the Mean Value Theorem with $\alpha \le 1$ gives

$$\frac{(\alpha x)^{1-\beta_0}-1}{1-\beta_0} \le x^{1-\beta_0}\log(\alpha x) \le \sqrt{x}\log x. \tag{3.99}$$

Since $\int_\alpha^{\alpha+\delta} h(t)dt = \frac{\delta}{2}$, thus we get

$$x^{1-\beta_0}\int_\alpha^{\alpha+\delta} h(t)t^{-\beta_0}dt \le \frac{\sqrt{x}}{\sqrt{\alpha}}\frac{\delta}{2}. \tag{3.100}$$

Now combining (3.98), (3.99), (3.100) with the assumptions in (3.67), we obtain the required result. □

To provide bounds for $J^{(3)}(x)$ defined in (3.63) and $J^{(4)}(x)$ defined in (3.64), we use the zero-free region of $\zeta_L(s)$ as described in Theorem 3.11 with α_4 defined in (3.7).

Lemma 3.26. *Under the assumptions in* (3.67)*, we have*

$$J^{(3)}(x) \le \ell_3(\log d_L)^2 x^{\frac{1}{2}}, \tag{3.101}$$

where ℓ_3 depends on x_0 and δ_0, and is given by

$$\ell_3 = \alpha_4\left(\frac{2+\delta_0}{2} + \frac{1}{\sqrt{x_0}}\right)\left(c_{11} + \frac{1}{\log\sqrt{3}}\left(c_{11}\log 3 + c_{21}\right) + \frac{4c_{11}}{\log 3}\right) \le 224.97, \tag{3.102}$$

with $c_{11} = \frac{17}{8}$ and $c_{21} = \frac{1513}{28}$ and α_4 defined in (3.6).

Proof. Notice that using (3.14), we have

$$J^{(3)}(x) \le \sum_{\substack{\rho\in\mathcal{Z}(\zeta),\rho\neq 1-\beta_0\\ |\rho|<\frac{1}{2}}}\left(M(\delta,0)\left|\frac{x^\rho}{\rho}\right| + \left|\frac{1}{\rho}\right|\right) \le (\sqrt{x}M(\delta,0)+1)\sum_{\substack{\rho\in\mathcal{Z}(\zeta),\rho\neq 1-\beta_0\\ |\rho|<\frac{1}{2}}}\left|\frac{1}{\rho}\right|. \tag{3.103}$$

In the region $|\rho| < \frac{1}{2}$ with $\rho \neq 1-\beta_0$, (3.6) tells us that either $|\gamma| > \frac{1}{\alpha_4\log d_L}$ or $\beta > \frac{1}{\alpha_4\log d_L}$, which implies that $|\rho| > \frac{1}{\alpha_4\log d_L}$. Thus,

$$\sum_{\substack{\rho\in\mathcal{Z}(\zeta),\rho\neq 1-\beta_0\\ |\rho|<\frac{1}{2}}}\left|\frac{1}{\rho}\right| \le \alpha_4(\log d_L)\sum_\chi\sum_{\substack{\rho\in Z(\chi),\rho\neq 1-\beta_0\\ |\rho|<\frac{1}{2}}} 1 \le \alpha_4(\log d_L)\sum_\chi n_{\chi,\frac{1}{2},1}(0). \tag{3.104}$$

We apply Proposition 1.13 to bound $n_{\chi,\frac{1}{2},1}$. Denoting c_{i1} instead of $c_i(1/2,1)$, we get

$$\sum_{\chi} n_{\chi,\frac{1}{2},1}(0) \leq (c_{11}\log(3) + c_{21})n_L + c_{11}\log d_L + 4c_{11}. \tag{3.105}$$

Combining (3.103), (3.104), (3.105) with (3.67) completes the proof. □

Lemma 3.27. *Under the assumptions in* (3.67)*, we have*

$$J^{(4)}(x) \leq \ell_4(\log d_L)x^{-\frac{1}{\alpha_4 \log d_L}}, \tag{3.106}$$

with ℓ_4 depends on δ_0 and is given by

$$\ell_4 = (2+\delta_0)\left(\left(3 + \frac{4}{\log 3}\right)c_{12} + \frac{1}{\log\sqrt{3}}c_{22}\right) \leq 222.70, \tag{3.107}$$

where $c_{i2} = c_i(1/(\alpha_4\log 3), 1)$ are defined in (1.5)(1.6) *and with α_4 defined in* (3.6).

Proof. Recall from (3.14) that $|H(\rho)| \leq \frac{M(\delta,0)}{|\rho|}$. As a result

$$J^{(4)}(x) \leq M(\delta,0) \sum_{\substack{\rho\in\mathcal{Z}(\zeta),\rho\neq\beta_0,|\rho|\geq\frac{1}{2}\\ |\gamma|\leq\frac{1}{\alpha_4\log d_L}}} \frac{x^{\beta-1}}{|\rho|}. \tag{3.108}$$

For $\rho = \beta + i\gamma$, (3.6) implies $\beta - 1 \leq -\frac{1}{\alpha_4 \log d_L}$ when $|\gamma| \leq \frac{1}{\alpha_4\log d_L}$ and $\rho \neq \beta_0$. Thus using $|\rho| \geq 1/2$, we have

$$\sum_{\substack{\rho\in\mathcal{Z}(\zeta),\rho\neq\beta_0,|\rho|\geq\frac{1}{2}\\ |\gamma|\leq\frac{1}{\alpha_4\log d_L}}} \frac{x^{\beta-1}}{|\rho|} \leq 2x^{-\frac{1}{\alpha_4\log d_L}} \sum_{\chi} \sum_{\substack{\rho\in\mathcal{Z}(\chi),\rho\neq\beta_0,|\rho|\geq\frac{1}{2}\\ |\gamma|\leq\frac{1}{\alpha_4\log d_L}}} 1 \leq 2x^{-\frac{1}{\alpha_4\log d_L}} \sum_{\chi} n_{\chi,\frac{1}{\alpha_4\log d_L},1}(0). \tag{3.109}$$

Similar to (3.105), using Proposition 1.13 with $a = \frac{1}{\alpha_4 \log d_L}$, $\epsilon = 1$, we get

$$\sum_{\chi} n_{\chi,\frac{1}{\alpha_4\log d_L},1}(0) \leq \left(c_{12}\log 3 + c_{22}\right)n_L + c_{11}\log d_L + 4c_{12}. \tag{3.110}$$

We conclude by combining (3.108), (3.109), (3.110) with (3.67). □

To provide bounds for $J^{(5)}(x,T)$ defined in (3.65) and $J^{(6)}(x,T)$ defined in (3.66), we use the explicit zero free region for $\zeta_L(s)$ as shown in (3.8) with R defined in (3.9).

Lemma 3.28. *Under the assumptions in* (3.67), *we have*

$$J^{(5)}(x,T) \leq \ell_5 (\log d_L)(\log T)^2 x^{-\frac{1}{R_L \log(D(T+2)/2)}}, \tag{3.111}$$

where ℓ_5 depends on T_0 and δ_0, and is given by

$$\begin{aligned}\ell_5 = \frac{2+\delta_0}{2}\Bigg[&\frac{\log(T_0-1)}{\pi(\log T_0)^2} + \frac{1}{(\log T_0)^2}\left(\frac{2}{\pi} + 2\alpha_1\right) + \frac{\alpha_1}{(\log T_0)^2(T_0-1)} + \frac{T_0+1}{\pi(T_0-1)(\log T_0)^2} \\ &+ \frac{1}{\log\sqrt{3}}\left(\frac{1}{\pi} + \frac{(T_0+1)\log(T_0+1)}{\pi(T_0-1)(\log T_0)^2} + \frac{\alpha_1 \log(T_0+1)}{(T_0-1)(\log T_0)^2} + \frac{1}{(\log T_0)^2}\left(2\alpha_2 + \frac{0.22}{\pi} + 1.3\alpha_1\right.\right. \\ &\left.\left.+ \frac{\alpha_2}{T_0-1} - \frac{2\log(2\pi e)}{\pi}\right)\right) + \frac{1}{(\log 3)(\log T_0)^2}\left(2\alpha_3 + \frac{\alpha_3}{T_0-1}\right)\Bigg] \leq 7.27,\end{aligned} \tag{3.112}$$

with $\alpha_1, \alpha_2, \alpha_3$ defined in (3.5).

Proof. For all $\rho = \beta + i\gamma$ with $|\gamma| \leq T$ and $\rho \neq \beta_0$, by (3.8), we have $\beta - 1 \leq -(R_L(\log(D(T+2)/2))^{-1}$. Also, (3.12) and (3.14) with $k=0$ gives $|H(\rho)| \leq \frac{M(\delta,0)}{|\rho|}$. Therefore,

$$J^{(5)}(x,T) \leq M(\delta,0) x^{-\frac{1}{R_L \log(D(T+2)/2)}} \sum_{\substack{\rho \in \mathcal{Z}(\zeta), |\rho| \geq \frac{1}{2} \\ \frac{1}{\alpha_4 \log d_L} < |\gamma| < T}} \frac{1}{|\rho|}. \tag{3.113}$$

We split the interval $[0,T)$ into blocks $[k,k+1)$ with k ranging from 0 to $\lceil T-1 \rceil$. For $k=0$, we use $|\rho| \geq 1/2$ and $|\rho| \geq k$ otherwise to obtain

$$\sum_{\substack{\rho \in \mathcal{Z}(\zeta), |\rho| \geq \frac{1}{2} \\ \frac{1}{\alpha_4 \log d_L} < |\gamma| \leq T}} \frac{1}{|\rho|} \leq 2N_L(1) + \sum_{k=1}^{\lceil T-1 \rceil} \frac{N_L(k+1) - N_L(k)}{k} = N_L(1) + \frac{N_L(T+1)}{T-1} + \sum_{k=2}^{\lceil T-1 \rceil} \frac{N_L(k)}{k(k-1)}. \tag{3.114}$$

Using the bound for N_L defined in (3.82), and the inequalities

$$\log(T-1) \leq \sum_{k=2}^{\lceil T-1 \rceil} \frac{1}{k-1} \leq \log(T-1) + 1, \quad \sum_{k=2}^{\lceil T-1 \rceil} \frac{\log k}{k-1} \leq (\log T)^2 + 0.22, \quad \sum_{k=2}^{\lceil T-1 \rceil} \frac{\log k}{k(k-1)} \leq 1.3$$
$$\text{and} \quad \sum_{k=2}^{\lceil T-1 \rceil} \frac{1}{k(k-1)} \leq 1, \tag{3.115}$$

we have

$$N_L(1) + \frac{N_L(T+1)}{T-1} + \sum_{k=2}^{\lceil T-1\rceil} \frac{N_L(k)}{k(k-1)} \leq \left(\frac{\log(T-1)}{\pi} + \frac{2}{\pi} + 2\alpha_1 + \frac{\alpha_1}{T-1} + \frac{T+1}{\pi(T-1)}\right)\log d_L$$
$$+ \left(\frac{1}{\pi}((\log T)^2 + 0.22) + \frac{(T+1)\log(T+1)}{\pi(T-1)} + \frac{\alpha_1 \log(T+1)}{T-1} + 1.3\alpha_1 - \frac{\log(2\pi e)}{\pi}\frac{T+1}{T-1}\right.$$
$$\left. - \frac{\log(2\pi e)}{\pi} + 2\alpha_2 - \frac{\log(2\pi e)}{\pi}\log(T-1) + \frac{\alpha_2}{T-1}\right)n_L + \left(2\alpha_3 + \frac{\alpha_3}{T-1}\right). \tag{3.116}$$

We complete the proof by using (3.67). □

To obtain a bound for $J^{(6)}(x,T)$ defined in (3.66), we generalize the techniques for Dirichlet L-functions as used by Bennett et al. in [4, Section 3.1].

Lemma 3.29. *Under the assumptions in* (3.67), *we have*

$$J^{(6)}(x,T) \leq \frac{M(\delta,m)}{2\delta^m}\left(x^{-1+\frac{1}{R_L \log(D(T+2)/2)}} S^{(1)}(m,T) + S^{(2)}(m,T,x)\right), \tag{3.117}$$

with M *defined in* (3.12) *and*

$$S^{(1)}(m,T) = \sum_{\substack{\rho\in Z(\zeta)\\ |\gamma|\geq T}} \frac{1}{|\gamma|^{m+1}}, \quad S^{(2)}(m,T,x) = \sum_{\substack{\rho\in Z(\zeta)\\ |\gamma|\geq T}} \frac{x^{-\frac{1}{R_L \log(D|\gamma|)}}}{|\gamma|^{m+1}}.$$

Proof. Since $|H(\rho)| \leq \frac{M(\delta,m)}{\delta^m |\rho|^{m+1}}$ and $|\rho| = |\beta + i\gamma| \geq |\gamma|$, we get

$$xJ^{(6)}(x,T) = \frac{M(\delta,m)}{\delta^m} \sum_{\substack{\rho\in Z(\zeta)\\ |\gamma|\geq T}} \frac{x^\beta}{|\gamma|^{m+1}}. \tag{3.118}$$

Using symmetry of zeros of $\zeta_L(s)$ across $\Re(s) = 1/2$ line with (3.8) and the fact that $x^\beta + x^{1-\beta}$ increases as β moves away from $\frac{1}{2}$ in either direction, we have

$$\sum_{\substack{\rho\in Z(\zeta)\\ |\gamma|\geq T}} \frac{x^\beta}{|\gamma|^{m+1}} = \frac{1}{2}\sum_{\substack{\rho\in Z(\zeta)\\ |\gamma|\geq T}} \frac{x^\beta + x^{1-\beta}}{|\gamma|^{m+1}} \leq \frac{1}{2}\sum_{\substack{\rho\in Z(\zeta)\\ |\gamma|\geq T}} \frac{x^{\frac{1}{R_L\log(D\tau/2)}}}{|\gamma|^{m+1}} + \frac{1}{2}\sum_{\substack{\rho\in Z(\zeta)\\ |\gamma|\geq T}} \frac{x^{1-\frac{1}{R_L\log(D\tau/2)}}}{|\gamma|^{m+1}}, \tag{3.119}$$

where $\tau = |\gamma| + 2$. Since $|\gamma| > T \geq 2$, $x^{\frac{1}{R_L \log(D\tau/2)}} \leq x^{\frac{1}{R_L \log(D(T+2)/2)}}$ and $x^{-\frac{1}{R_L\log(D\tau/2)}} \leq x^{-\frac{1}{R_L\log(D|\gamma|)}}$. Finally, we conclude by combining (3.118) and (3.119). □

Proposition 3.30. *Under the assumptions in* (3.67), *we have*

$$\left|I_{L/K} - \frac{|C|}{|G|}xH(1)\right|$$
$$\leq \frac{|C|}{|G|}\Bigg(x^{\beta_0}H(\beta_0) + \ell_1(\log d_L)(\log x) + \ell_2(\log x)x^{\frac{1}{2}} + \ell_3(\log d_L)^2 x^{\frac{1}{2}} + \ell_4(\log d_L)x^{1-\frac{1}{\alpha_4 \log d_L}}$$
$$+ \ell_5(\log d_L)(\log T)^2 x^{1-\frac{1}{R_L \log(D(T+2)/2)}} + \frac{M(\delta,m)}{2\delta^m}\left(x^{\frac{1}{R_L \log(D(T+2)/2)}} S^{(1)}(m,T) + xS^{(2)}(m,T,x)\right)\Bigg),$$

where $\ell_1, \ell_2, \ell_3, \ell_4$ *and* ℓ_5 *are defined in* (3.92), (3.97), (3.102), (3.107) *and* (3.112) *respectively.*

Proof. We combine Corollary 3.17 with Lemma 3.24, Lemma 3.25, Lemma 3.26, Lemma 3.27, Lemma 3.28 and Lemma 3.29. □

Therefore, it remains to estimate $S^{(1)}$ and $S^{(2)}$, which are the sums over the larger zeros.

3.7 Study of the sums over the larger zeros

We recall the zero-free region obtained for all Dedekind zeta functions as described in Theorem 3.12: Let $\rho = \beta + i\gamma$ be non-trivial zero of $\zeta_L(s)$ with $\rho \neq \beta_0$ and $\tau = |\gamma| + 2$. Then there exists $R > 0$ such that

$$\beta < 1 - \frac{1}{R_L(\log(D\tau/2))}, \tag{3.120}$$

where $R_L = Rn_L$ with R defined in (3.9) and $D = 2d_L^{\frac{1}{n_L}}$. Introducing the function

$$\phi_{m,x}(u) = \frac{x^{-\frac{1}{R_L(\log Du)}}}{u^{m+1}} = \frac{1}{u^{m+1}} \exp\left(\frac{-\log x}{R_L(\log Du)}\right), \tag{3.121}$$

we rewrite

$$S^{(1)}(m,T) = \sum_{\substack{\rho \in Z(\rho) \\ |\gamma| \geq T}} \phi_{m,1}(|\gamma|) \quad \text{and} \quad S^{(2)}(m,T,x) = \sum_{\substack{\rho \in Z(\rho) \\ |\gamma| \geq T}} \phi_{m,x}(|\gamma|). \tag{3.122}$$

For the rest of the article, we denote

$$X_{m,T} = (m+1)R_L \log^2(DT), \text{ and } T_1 = \begin{cases} T & \text{if } \log x \leq X_{m,T}, \\ W = \frac{1}{D}\exp\left(\sqrt{\frac{\log x}{R_L(m+1)}}\right) & \text{if } \log x > X_{m,T}. \end{cases} \tag{3.123}$$

Note that if $\log x > X_{m,T}$, $W > T$. We also recall the estimate for the number of zeros $N_L(T)$ given by Theorem 3.10 and define $Q(t,u) = P(u) - P(T) + E(u) + E(T)$ so that $Q(t,u)$ is an upper bound for $N_L(u) - N_L(t)$:

$$Q(t,u) = \frac{n_L u}{\pi} \log\left(\frac{Du}{4\pi e}\right) - \frac{n_L t}{\pi} \log\left(\frac{Dt}{4\pi e}\right) + 2\alpha_1 n_L \left(\log \frac{D\sqrt{ut}}{2}\right) + 2\alpha_2 n_L + 2\alpha_3, \quad (3.124)$$

with α_i's defined in (3.5). The next lemma provides a bound for $Q(t,u)$ similar to [4, Lemma 2.15] for Dirichlet L-functions.

Lemma 3.31. *Let $Q(t,u)$ be defined as in* (3.124)*. If $44 \le t \le u$, then*

$$Q(t,u) < \frac{u n_L}{\pi} \log(Du).$$

Proof. Let $\epsilon(t,u,D) = \frac{\pi}{n_L}\left(\frac{u n_L}{\pi}\log(Du) - Q(t,u)\right)$. We have $\frac{\partial \epsilon}{\partial u} = \log(4\pi e) - \frac{\alpha_1 \pi}{u}$ and $\frac{\partial \epsilon}{\partial D} = \frac{t - 2\alpha_1\pi}{D}$, which are both positive as long as $u > \alpha_1\pi/\log(2\pi e)$ and $t > 2\alpha_1\pi$. Thus, for $u \ge t \ge 44$, $\epsilon(t,u,D)$ increases with both u and D, and $\epsilon(t,u,D) \ge \epsilon(t,t,1)$ with $\epsilon(t,t,1) = (t - 2\alpha_1\pi)\log t + 2\alpha_1\pi\log 2 - 2\alpha_2\pi - \frac{2\alpha_3\pi}{n_L} > 0$. □

Remark 3.32. Here, the condition $t \ge 44$ is implied by $\epsilon(t,t,1) > 0$ and thus depends on the values of the α_i's.

3.7.1 Estimating the sum over inverse of zeros $S^{(1)}(m,T)$

Lemma 3.33. *Under the assumptions in* (3.67)*, we have,*

$$S^{(1)}(m,T) \le \ell_6(m)(\log d_L)\frac{\log T}{T^m},$$

where $\ell_6(m)$ depends on m and T_0, and is given by

$$\ell_6(m) = \frac{1}{m\pi}\left(\frac{2}{\log 3} + \frac{1}{\log T_0}\right) + \frac{4\alpha_1}{(\log 3)T_0} + \frac{2\alpha_1 + \frac{2}{\log 3}\left(\frac{\alpha_1}{m+1} + 2\alpha_2 + 2\alpha_3\right)}{(\log T_0)T_0} \le 1.38, \quad (3.125)$$

where α_1, α_2 and α_3 are as defined in (3.5).

Proof. By partial summation,

$$S^{(1)}(m,T) = \sum_{|\gamma|>T} \frac{1}{|\gamma|^{m+1}} = \int_T^\infty \frac{d(N_L(u)-N_L(T))}{u^{m+1}}du = (m+1)\int_T^\infty \frac{N_L(u)-N_L(T)}{u^{m+2}}du.$$

Together with Lemma 3.31, we get

$$S^{(1)}(m,T) \le (m+1)\Bigg(\frac{n_L}{\pi}\log\left(\frac{D}{4\pi e}\right)\int_T^\infty \frac{du}{u^{m+1}} + \frac{n_L}{\pi}\int_T^\infty \frac{\log u}{u^{m+1}}du + \alpha_1 n_L \int_T^\infty \frac{\log u}{u^{m+2}}du$$
$$+\left(2\alpha_1\log d_L + \alpha_1 n_L \log T + 2\alpha_2 n_L + 2\alpha_3 - \frac{T}{\pi}\log d_L - \frac{n_L T}{\pi}\log\left(\frac{T}{2\pi e}\right)\right)\int_T^\infty \frac{du}{u^{m+2}}\Bigg).$$

We calculate the integrals

$$\int_T^\infty \frac{1}{u^{k+1}}du = \frac{1}{kT^k}, \int_T^\infty \frac{\log u}{u^{k+1}}du = \frac{\log T}{kT^k} + \frac{1}{k^2T^k},$$

and finally obtain

$$S^{(1)}(m,T) \le \frac{n_L}{\pi}\frac{1}{m}\frac{\log T}{T^m} + \left(\frac{\log d_L}{m\pi} + \frac{n_L}{m\pi}\left(\frac{(m+1)}{m} - \log\left(2\pi e\right)\right)\right)\frac{1}{T^m}$$
$$+ 2\alpha_1 n_L \frac{\log T}{T^{m+1}} + \left(\frac{\alpha_1 n_L}{(m+1)} + 2\alpha_1 \log d_L + 2\alpha_2 n_L + 2\alpha_3\right)\frac{1}{T^{m+1}}.$$

We conclude with the bounds on n_L, d_L from (3.67) and the fact that $\frac{m+1}{m^2\pi} - \frac{\log(2\pi e)}{m\pi} \le 0$. □

3.7.2 Estimating $S_L^{(2)}(m,T,x)$

Recall that

$$S^{(2)}(m,T,x) = \sum_{|\gamma|>T} \frac{x^{-\frac{1}{R_L \log(D|\gamma|)}}}{|\gamma|^{m+1}}.$$

Lemma 3.34. *Under the assumptions* (3.67), *we have*

$$S^{(2)}(m,T,x) \le Q(T,T_1)\phi_{m,x}(T_1) + \int_T^\infty \left(\frac{\partial}{\partial u}Q(t,u)\right)\phi_{m,x}(u)du,$$

where $\phi_{m,x}, Q$, and T_1 are defined as in (3.121), (3.124), *and* (3.123) *respectively.*

Proof. Partial summation, $N_L(u) - N_L(T) \ll u \log u$ and $\phi_{m,x}(u) \ll u^{-2}$ as $u \to \infty$, give

$$S^{(2)}(m,T,x) = \int_T^\infty \phi_{m,x}(u)\, d(N_L(u) - N_L(T)) = \int_T^\infty (N_L(u) - N_L(T))(-\phi'_{m,x}(u))du. \quad (3.126)$$

Note that $-\phi'_{m,x}(u) \le 0$ for $u \le W$ and that $-\phi'_{m,x}(u) > 0$ for $u > W$. Using this and $N_L(u) - N_L(T) \le Q(t,u)$ in (3.126), we get

$$S^{(2)}(m,T,x) < \int_{T_1}^\infty Q(t,u)(-\phi'_{m,x}(u))du.$$

Again integrating by parts yields

$$S^{(2)}(m,T,x) \le Q(T,T_1)\phi_{m,x}(T_1) + \int_{T_1}^\infty \left(\frac{\partial}{\partial u} Q(t,u)\right)\phi_{m,x}(u)du.$$

Since the last integrand is positive, we can conclude by pushing T_1 to T. □

Lemma 3.35. *Under the assumptions* (3.67)*, we have*

$$Q(T,T_1)\phi_{m,x}(T_1) \le \begin{cases} 2E(T)\phi_{m,x}(T) = \frac{2E(T)}{T^{m+1}} \exp\left(\frac{-\log x}{R_L(\log DT)}\right) & \text{if } 0 < \log x \le X_{m,T}, \\ \frac{n_L D^m}{\pi}\sqrt{\frac{\log x}{R_L(m+1)}} \exp\left(-(2m+1)\sqrt{\frac{\log x}{R_L(m+1)}}\right) & \text{if } \log x > X_{m,T}, \end{cases}$$

where T_1 *and* $X_{m,T}$ *are defined in* (3.123).

Proof. In the case $0 < \log x \le X_{m,T}$, $T_1 = T$ and thus $Q(T,T) = 2E(T)$, giving the first inequality:

$$Q(T,T_1)\phi_{m,x}(T_1) = 2E(T)\phi_{m,x}(T).$$

In the other case, $\log x > X_{m,T}$, then $T_1 = W > T \ge 44$, and Lemma 3.31 gives

$$Q(T,T_1) = Q(T,W) < \frac{Wn_L}{\pi}\log(DW) = \frac{n_L}{\pi D}\sqrt{\frac{\log x}{R_L(m+1)}} \exp\left(\sqrt{\frac{\log x}{R_L(m+1)}}\right). \quad (3.127)$$

Similarly,

$$\phi_{m,x}(T_1) = \phi_{m,x}(W) = \frac{1}{W^{m+1}} \exp\left(-\frac{\log x}{R_L(\log DW)}\right) = D^{m+1}\exp\left(-2\sqrt{\frac{(m+1)\log x}{R_L}}\right). \quad (3.128)$$

We conclude by putting together (3.127), (3.128) and $2\sqrt{m+1} - \frac{1}{\sqrt{m+1}} = \frac{2m+1}{\sqrt{m+1}}$. □

To complete the bound for $S^{(2)}(m,T,x)$, it remains to estimate the integral

$$\int_T^\infty \left(\frac{\partial}{\partial u}Q(t,u)\right)\phi_{m,x}(u)du.$$

To do so, we introduce the following integral functions. Given positive real numbers n, m, α, β and l, we define an incomplete modified Bessel function of the first kind as

$$I_{n,m}(\alpha,\beta;l) = \int_l^\infty \frac{(\log\beta u)^{n-1}}{u^{m+1}} \exp\left(-\frac{\alpha}{\log\beta u}\right)du. \tag{3.129}$$

Moreover, given positive constants n, z, and y, we call the "imposter" Bessel function of the second kind the integral

$$K_n(z;y) = \frac{1}{2}\int_y^\infty v^{n-1}\exp\left(-\frac{z}{2}\left(v+\frac{1}{v}\right)\right)dv. \tag{3.130}$$

Both integrals are related through the change of variable $v = (\log(\beta u))\sqrt{\frac{m}{\alpha}}$

$$I_{n,m}(\alpha,\beta;l) = 2\beta^m\left(\frac{\alpha}{m}\right)^{n/2} K_n\left(2\sqrt{\alpha m}, \sqrt{\frac{m}{\alpha}}\log(\beta l)\right). \tag{3.131}$$

In particular, in our context,

$$I_{n,m}\left(\frac{\log x}{R_L}, D; T\right) = 2D^m\left(\frac{\log x}{mR_L}\right)^{n/2} K_n\left(z_m, w_m\right), \tag{3.132}$$

where

$$z_m = 2\sqrt{\frac{m\log x}{R_L}},\ w_m = \sqrt{\frac{mR_L}{\log x}}\log(DT). \tag{3.133}$$

Lemma 3.36. *Under the assumptions* (3.67), *we have*

$$\int_T^\infty \left(\frac{\partial}{\partial u}Q(t,u)\right)\phi_{m,x}(u)du \le \frac{2n_L}{\pi}D^m\frac{\log x}{mR_L}K_2(z_m,w_m),$$

where K_2 and (z_m, w_m) are defined in (3.130) *and* (3.133) *respectively.*

Proof. It follows from (3.124) that

$$\frac{\partial}{\partial u}Q(t,u) = \frac{n_L}{\pi}\log\left(\frac{Du}{4\pi}\right) + \frac{\alpha_1 n_L}{u} \le \frac{n_L}{\pi}\log(Du),$$

since $\frac{\log(4\pi)}{\pi} - \frac{\alpha_1}{T} \ge 0$ for $u \ge T \ge 44$. We recognize

$$\int_T^\infty \log(Du)\phi_{m,x}(u)du = I_{2,m}\left(\frac{\log x}{R_L}, D, T\right),$$

and conclude with (3.132). □

Combining Lemma 3.34, Lemma 3.35 and Lemma 3.36 leads to a bound for $S^{(2)}$ in terms of K_2:

Lemma 3.37. *Under the assumptions* (3.67)*, we have*

$$S^{(2)}(m,T,x) \le B^{(2)}(m,T,x),$$

where

$$B^{(2)}(m,T,x) = \begin{cases} \frac{2E(T)}{T^{m+1}}\exp\left(\frac{-\log x}{R_L(\log DT)}\right) + \frac{2n_L}{\pi}D^m\frac{\log x}{mR_L}K_2(z_m,w_m) & \text{if } 0 < \log x \le X_{m,T}, \\ \\ \frac{n_L D^m}{\pi}\sqrt{\frac{\log x}{R_L(m+1)}}\exp\left(-(2m+1)\sqrt{\frac{\log x}{R_L(m+1)}}\right) & \\ +\frac{2n_L}{\pi}D^m\frac{\log x}{mR_L}K_2(z_m,w_m) & \text{if } \log x > X_{m,T}, \end{cases} \tag{3.134}$$

where $X_{m,T}$, K_2 and (z_m,w_m) are defined in (3.123), (3.130) *and* (3.133) *respectively.*

The last section investigates those “Bessel” integrals.

3.7.3 Study of imposter Bessel function $K_2(z_m, w_m)$

Lemma 3.38. *If $w_m < \sqrt{\frac{m}{m+1}}$, then*

$$K_2(z_m,w_m) \le \frac{1}{2}\left(\frac{1}{w_m} - w_m\right)k^\flat\left(z_m,\frac{1}{w_m}\right) + J_{2a}\left(z_m,\frac{1}{w_m}\right) + J_{2b}\left(z_m,\frac{1}{w_m}\right), \tag{3.135}$$

where

$$k^\flat\left(z_m, \frac{1}{w_m}\right) = \begin{cases} \frac{1}{w_m}\exp\left(-\frac{z_m}{2}\left(\frac{1}{w_m}+w_m\right)\right) & \text{if } 1 \le \frac{1}{w_m} < \frac{1+\sqrt{z_m^2+1}}{z_m}, \\ \frac{1+\sqrt{z_m^2+1}}{z_m}\exp\left(-\sqrt{z_m^2+1}\right)\Big) & \text{if } \frac{1}{w_m} \ge \frac{1+\sqrt{z_m^2+1}}{z_m}, \end{cases} \tag{3.136}$$

$$J_{2a}(z;y) = \frac{(35y^{3/2}+128y+135y^{1/2}+128y^{-1})z+105y^{1/2}+256}{256z^2e^{z(y+1/y)/2}}, \tag{3.137}$$

$$J_{2b}(z;y) = \sqrt{\pi}erfc\left(\sqrt{\frac{z}{2}}\left(\sqrt{y}-\frac{1}{\sqrt{y}}\right)\right)\frac{128z^2+240z+105}{256\sqrt{2}z^{5/2}e^z}, \tag{3.138}$$

with

$$erfc(u) = \frac{2}{\sqrt{\pi}}\int_u^\infty e^{-t^2}dt. \tag{3.139}$$

Proof. We assume $w_m < \sqrt{\frac{m}{m+1}}$, i.e. $\log x > X_{m,T}$. Modifying (3.130) for $n = 2$, we have

$$K_2(z_m, w_m) = \frac{1}{2}\int_{w_m}^{1/w_m} u\exp\left(-\frac{z_m}{2}\left(u+\frac{1}{u}\right)\right)du + K_2\left(z_m, \frac{1}{w_m}\right). \tag{3.140}$$

Kadiri and Lumley in [14, Lemma A.4] proved that

$$\frac{1}{2}\int_{w_m}^{1/w_m} u\exp\left(-\frac{z_m}{2}\left(u+\frac{1}{u}\right)\right)du \le \frac{1}{2}\left(\frac{1}{w_m}-w_m\right)k^\flat\left(z_m, \frac{1}{w_m}\right), \tag{3.141}$$

where $k^\flat$ is defined in (3.142). Also, since $\frac{1}{w_m} > 1$, [4, Proposition 4.7] gives us

$$K_2\left(z_m, \frac{1}{w_m}\right) \le J_{2a}\left(z_m, \frac{1}{w_m}\right) + J_{2b}\left(z_m, \frac{1}{w_m}\right), \tag{3.142}$$

where J_{2a} and J_{2b} are defined in (3.137) and (3.138) respectively. We conclude by putting together (3.140), (3.141) and (3.142). □

Remark 3.39. The other case, $w_m \ge \sqrt{\frac{m}{m+1}}$, which is the same as $\log x \le X_{m,T}$, will be discussed in following articles.

Lemma 3.40. *If* $\log x > X_{m,T}$, *then*

$$B^{(2)}(m,T,x) \leq \frac{D^m}{\pi}\sqrt{\frac{n_L \log x}{R(m+1)}} \exp\left(-(2m+1)\sqrt{\frac{\log x}{R_L(m+1)}}\right)$$
$$+ \frac{2}{\pi} D^m \frac{\log x}{mR}\left(\frac{1}{2}\left(\frac{1}{w_m} - w_m\right) k^{\flat}\left(z_m, \frac{1}{w_m}\right) + J_{2a}\left(z_m, \frac{1}{w_m}\right) + J_{2b}\left(z_m, \frac{1}{w_m}\right)\right).$$

Proof. Assuming $\log x > X_{m,T}$ is the same as having $w_m < \sqrt{\frac{m}{m+1}}$. We use (3.134) and Lemma 3.38 to complete the proof. □

3.8 Explicit formula for the error term in the case $\log x > X_{m,T}$

Let us introduce a_{β_0} as the following:

$$a_{\beta_0} = \begin{cases} 1 & \text{if } \beta_0 \text{ exists,} \\ 2 & \text{otherwise.} \end{cases} \tag{3.143}$$

Theorem 3.41. *Let* C *be a conjugacy class of* $G = \mathrm{Gal}(L/K)$. *Let* β_0 *be the possible exceptional real zero of* $\zeta_L(s)$. *Under the assumptions in* (3.67), *we have*

$$E_\psi(x) = \left| \frac{\psi_C(x) - \frac{|C|}{|G|}x}{\frac{|C|}{|G|}x} \right| \leq \frac{x^{\beta_0 - 1}}{\beta_0} + \epsilon_\psi(\delta, m, T, x),$$

where

$$\epsilon_\psi(\delta,m,T,x) = \frac{\delta}{a_{\beta_0}} + \ell_7(\delta,m,T,x)(\log T)^2(\log d_L)x^{-\frac{1}{R_L \log(D(T+2)/2)}} + \frac{M(\delta,m)}{2\delta^m}B^{(2)}(m,T,x), \tag{3.144}$$

$$\ell_7(\delta,m,T,x) = \frac{\ell_0+\ell_1}{(\log T)^2}(\log x)x^{-1+\frac{1}{R_L\log(D(T+2)/2)}} + \frac{\ell_2}{(\log T)^2(\log d_L)}(\log x)x^{-\frac{1}{2}+\frac{1}{R_L\log(D(T+2)/2)}}$$
$$+ \frac{\ell_3}{(\log T)^2}(\log d_L)x^{-\frac{1}{2}+\frac{1}{R_L\log(D(T+2)/2)}} + \frac{\ell_4}{(\log T)^2}x^{-\frac{1}{\alpha_4\log d_L}+\frac{1}{R_L\log(D(T+2)/2)}}$$
$$+ \ell_5 + \frac{M(\delta,m)}{2\delta^m}\ell_6(m)\frac{1}{(\log T)T^m}x^{-1+\frac{2}{R_L\log(D(T+2)/2)}} \tag{3.145}$$

and where H, M *and* $B^{(2)}$ *are defined in* (3.11), (3.12) *and* (3.134) *respectively, and where the* l_i*'s are defined in* (3.29)(3.92)(3.97)(3.102)(3.107)(3.112)(3.125).

Proof. We recall that $I_{L/K} = \tilde{I}_{L/K}(x) + \tilde{\psi}_C(x)$ as defined in (3.25). Therefore

$$E_{\tilde{\psi}}(x) = \left|\frac{\tilde{\psi}_C(x) - \frac{|C|}{|G|}x}{\frac{|C|}{|G|}x}\right| \le \left|\frac{I_{L/K} - \frac{|C|}{|G|}x}{\frac{|C|}{|G|}x}\right| + \left|\frac{\tilde{I}_{L/K}(x)}{\frac{|C|}{|G|}x}\right| \le \left|\frac{I_{L/K} - \frac{|C|}{|G|}xH(1)}{\frac{|C|}{|G|}x}\right| + |H(1)-1| + \left|\frac{\tilde{I}_{L/K}(x)}{\frac{|C|}{|G|}x}\right|. \tag{3.146}$$

Using (3.14) with $k = 0$ and $\frac{x^{\beta_0 - 1}}{\beta_0} \le 1$, we obtain

$$x^{\beta_0 - 1}H(\beta_0) \le \frac{x^{\beta_0-1}}{\beta_0} + \frac{\delta}{2}. \tag{3.147}$$

Combining (3.146), (3.147), $|H(1) - 1| = \frac{\delta}{2}$ and Lemma 3.13 with Proposition 3.30, Lemma 3.33 and Lemma 3.37, we obtain

$$E_{\tilde{\psi}}(x) \le \frac{x^{\beta_0-1}}{\beta_0} + \epsilon_{\tilde{\psi}}(\delta, m, T, x),$$

with

$$\begin{aligned}\epsilon_{\tilde{\psi}}(\delta, m, T, x) = {} & \frac{\delta}{a_{\beta_0}} + (\ell_0 + \ell_1)(\log d_L)(\log x)x^{-1} + \ell_2(\log x)x^{-\frac{1}{2}} + \ell_3(\log d_L)^2 x^{-\frac{1}{2}} \\ & + \ell_4(\log d_L)x^{-\frac{1}{\alpha_4 \log d_L}} + \ell_5(\log d_L)(\log T)^2 x^{-\frac{1}{R_L \log(D(T+2)/2)}} \\ & + \frac{M(\delta, m)}{2\delta^m}\left(\ell_6(m)\frac{(\log d_L)(\log T)}{T^m}x^{-1+\frac{1}{R_L \log(D(T+2)/2)}} + B^{(2)}(m, T, x)\right).\end{aligned}$$

Note that $\epsilon_{\tilde{\psi}}$ is independent of α. Hence $E_{\psi}(x) \le \max\{E^-(x), E^+(x)\} \le E_{\tilde{\psi}}(x)$ completes the proof. □

Remark 3.42. Assume (3.67) with $\log d_L \le \frac{\log x}{4mR(\log\sqrt{3})}$ which satisfies the condition for Theorem 1.15 given in (3.165). Then $\ell_7(\delta, m, T, x) \le \ell_7(\delta, m, T_0, x_0) \le 7.27$ for $\delta = 10^{-19}$, $m = 2$, $T_0 = 44$, $x_0 = \exp(4484)$, $\alpha_4 = 2$ and $M = M_R$ as defined in (3.18).

Going forward, we are going to use ℓ_6 and ℓ_7 to denote the functions $\ell_6(m)$ and $\ell_7(\delta, m, T, x)$ respectively.

3.8.1 Explicit bounds for $E_\psi(x)$ independent of T and δ but dependent on d_L

We choose T as the following:

$$T = \frac{1}{D}\exp\left(\frac{1}{2\sqrt{m}}\sqrt{\frac{\log x}{R_L}}\right). \tag{3.148}$$

Thus

$$w_m = \sqrt{\frac{mR_L}{\log x}}\log(DT) = \frac{1}{2} \quad \text{and} \quad \frac{1}{2}\left(\frac{1}{w_m} - w_m\right) = \frac{3}{4}. \tag{3.149}$$

We determine which case this corresponds to in order to define $k^\flat$. We have

$$\frac{1}{w_m} \geq \frac{1+\sqrt{z_m^2+1}}{z_m} \iff (2z_m-1)^2 \geq z_m^2+1 \iff 3z_m^2 - 4z_m \geq 0 \iff z_m \geq \frac{4}{3}.$$

Remember that we are in the region $\log x > X_{m,T}$, which by using $D \geq 2\sqrt{3}$, $m \geq 2$ and $T \geq T_0 \geq 44$ gives

$$\frac{\log x}{R_L} > (m+1)(\log(DT))^2 \geq (m+1)(\log(2\sqrt{3}T_0))^2 \geq 3(\log 88\sqrt{3})^2 \tag{3.150}$$

and ensures that $z_m > 2\sqrt{6}(\log 88\sqrt{3}) > \frac{4}{3}$. Hence we are in the case where by (3.142),

$$k^\flat\left(z_m, \frac{1}{w_m}\right) = \frac{1+\sqrt{z_m^2+1}}{z_m}\exp\left(-\sqrt{z_m^2+1}\right) \leq \frac{1+\sqrt{\frac{4m\log x}{R_L}+1}}{2\sqrt{\frac{m\log x}{R_L}}}\exp\left(-2\sqrt{\frac{m\log x}{R_L}}\right).$$

Using (3.150) and $\frac{1+\sqrt{\frac{4m\log x}{R_L}+1}}{2\sqrt{\frac{m\log x}{R_L}}} \leq \frac{1}{\sqrt{4m(m+1)(\log(2\sqrt{3}T_0))^2}} + \sqrt{1+\frac{1}{4m(m+1)(\log(2\sqrt{3}T_0))^2}} \leq 1.042$, it follows

$$\frac{2}{\pi}D^m\frac{\log x}{mR}\frac{1}{2}\left(\frac{1}{w_m} - w_m\right)k^\flat\left(z_m, \frac{1}{w_m}\right) \leq \eta_{m,1}D^m\frac{\log x}{R}\exp\left(-2\sqrt{\frac{m\log x}{R_L}}\right), \tag{3.151}$$

where for $m \geq 2$ and $T_0 \geq 44$,

$$\eta_{m,1} = \frac{3}{2\pi m}\left(\frac{1}{\sqrt{4m(m+1)(\log(2\sqrt{3}T_0))^2}} + \sqrt{1+\frac{1}{4m(m+1)(\log(2\sqrt{3}T_0))^2}}\right) \leq 0.249. \tag{3.152}$$

Now using the definition of J_{2a} as in (3.137), we find

$$J_{2a}\left(2\sqrt{\frac{m\log x}{R_L}},2\right) = \frac{(205\sqrt{2}+320)2\sqrt{\frac{m\log x}{R_L}}+105\sqrt{2}+256}{(256)(4)\frac{m\log x}{R_L}\exp\left(\frac{5}{2}\sqrt{\frac{m\log x}{R_L}}\right)}$$
$$= \left[\frac{205\sqrt{2}+320}{(256)(2)}+\frac{105\sqrt{2}+256}{(256)(4)}\sqrt{\frac{R_L}{m\log x}}\right]\sqrt{\frac{R_L}{m\log x}}\exp\left(-\frac{5}{2}\sqrt{\frac{m\log x}{R_L}}\right).$$

Using (3.150), we obtain

$$\frac{2}{\pi}D^m\frac{\log x}{mR}J_{2a}\left(2\sqrt{\frac{m\log x}{R_L}},2\right) \leq \eta_{m,2}D^m\sqrt{\frac{n_L\log x}{R}}\exp\left(-\frac{5}{2}\sqrt{\frac{m\log x}{R_L}}\right), \tag{3.153}$$

where

$$\eta_{m,2} = \frac{2}{\pi m^{\frac{3}{2}}}\left[\frac{205\sqrt{2}+320}{(256)(2)}+\frac{105\sqrt{2}+256}{(256)(4)}\frac{1}{\sqrt{m(m+1)}\log(2\sqrt{3}T_0)}\right] \leq 0.275\dots. \tag{3.154}$$

Using the definition (3.138) for J_{2b}, together with $\sqrt{\frac{z_m}{2}}\left(\sqrt{2}-\frac{1}{\sqrt{2}}\right) = \frac{\sqrt{z_m}}{2}$ and the bound $\sqrt{\pi}erfc(x) < \exp(-x^2)x^{-1}$(see [1, 7.1.13]), we get

$$J_{2b}(z_m,2) < \frac{2}{\sqrt{z_m}}\frac{128z_m^2+240z_m+105}{256\sqrt{2}z_m^{5/2}}\frac{2}{\sqrt{z_m}}e^{-\frac{5}{4}z_m} < \frac{128+240z_m^{-1}+105z_m^{-2}}{256\sqrt{2}}\frac{2}{z_m}e^{-\frac{5}{4}z_m}.$$

For $z_m = 2\sqrt{\frac{m\log x}{R_L}} > 2\sqrt{m(m+1)}\log(2\sqrt{3}T_0)$, we have

$$\frac{2}{\pi}D^m\frac{\log x}{mR}J_{2b}(z_m,2) \leq \eta_{m,3}D^m\sqrt{\frac{n_L\log x}{R}}\exp\left(-\frac{5}{2}\sqrt{\frac{m\log x}{R_L}}\right), \tag{3.155}$$

with

$$\eta_{m,3} = \frac{1}{128\sqrt{2}\pi m^{3/2}}\left(128+\frac{240}{(2\sqrt{m(m+1)}(\log(2\sqrt{3}T_0))}+\frac{105}{(2\sqrt{m(m+1)}(\log(2\sqrt{3}T_0)))^2}\right) \leq 0.086. \tag{3.156}$$

Inserting (3.151), (3.153) and (3.155) in Lemma 3.40, we obtain

$$B^{(2)}(m,T,x) \leq \frac{D^m}{\pi}\sqrt{\frac{n_L \log x}{R(m+1)}}\exp\left(-\frac{2m+1}{(m+1)^{\frac{1}{2}}}\sqrt{\frac{\log x}{R_L}}\right) + \eta_{m,1}D^m\frac{\log x}{R}\exp\left(-2\sqrt{\frac{m\log x}{R_L}}\right) + (\eta_{m,2}+\eta_{m,3})D^m\sqrt{\frac{n_L\log x}{R}}\exp\left(-\frac{5}{2}\sqrt{\frac{m\log x}{R_L}}\right).$$

As $\frac{5}{2}\sqrt{m} \geq \frac{2m+1}{\sqrt{m+1}} \geq 2\sqrt{m}$, and $\frac{\log x}{R_L} > (m+1)(\log DT)^2 \geq (m+1)(\log(2\sqrt{3}T_0))^2$, then

$$B^{(2)}(m,T,x) < \kappa_m D^m (\log x)\exp\left(-2\sqrt{m}\sqrt{\frac{\log x}{R_L}}\right), \tag{3.157}$$

$$\text{with } \kappa_m = \frac{1}{\pi R(m+1)(\log(2\sqrt{3}T_0))}\exp\left(-\left(2m+1-2\sqrt{m(m+1)}\right)(\log(2\sqrt{3}T_0))\right) + \frac{\eta_{m,1}}{R} + \frac{\eta_{m,2}+\eta_{m,3}}{R\sqrt{m+1}(\log(2\sqrt{3}T_0))}\exp\left(-\frac{1}{2}\sqrt{m(m+1)}(\log(2\sqrt{3}T_0))\right) \leq 0.009. \tag{3.158}$$

Proof of Theorem 1.14. We apply Theorem 3.41 with the choice (3.148) for $T \geq T_0 \geq 44$ and $D \geq 2\sqrt{3}$. More precisely we use

$$\log x \geq 4mR_L(\log DT_0)^2 \geq 4mR_L(\log 88\sqrt{3})^2 \geq 11954, \tag{3.159}$$

$$(\log T)^2 e^{-\frac{\log x}{R_L \log(D(T+2)/2)}} \leq (\log(DT))^2 e^{-\frac{\log x}{R_L \log(DT)}} \leq \frac{1}{4m}\frac{\log x}{R_L}e^{-2\sqrt{m}\frac{\sqrt{\log x}}{\sqrt{R_L}}},$$

so that we get for ϵ_ψ as defined in (3.144):

$$\epsilon_\psi(\delta,m,T,x) \leq \frac{\delta}{a_{\beta_0}} + \ell_7(\log d_L)\frac{1}{4m}\frac{\log x}{Rn_L}e^{-2\sqrt{m}\sqrt{\frac{\log x}{Rn_L}}} + \frac{M(\delta,m)}{2\delta^m}\kappa_m D^m(\log x)e^{-2\sqrt{m}\sqrt{\frac{\log x}{Rn_L}}}. \tag{3.160}$$

Denoting

$$A = \frac{M(\delta,m)}{2}\kappa_m D^m(\log x)e^{-2\sqrt{m}\sqrt{\frac{\log x}{Rn_L}}}, \tag{3.161}$$

we choose δ such that $\frac{\delta}{a_{\beta_0}} + A\delta^{-m}$ with is as small as possible. We check that $\frac{\delta}{a_{\beta_0}} + A\delta^{-m}$

minimizes at $\delta = (a_{\beta_0} m A)^{\frac{1}{m+1}}$. With this choice of δ, we obtain

$$\begin{aligned}\frac{\delta}{a_{\beta_0}} + A\delta^{-m} &= \left(m^{\frac{1}{m+1}} + m^{-\frac{m}{m+1}}\right) a_{\beta_0}^{-\frac{m}{m+1}} A^{\frac{1}{m+1}} \\ &= (1+m^{-1})\left(\frac{mM(\delta,m)\kappa_m D^m}{2a_{\beta_0}^m}\right)^{\frac{1}{m+1}} (\log x)^{\frac{1}{m+1}} e^{-2\frac{\sqrt{m}}{m+1}\sqrt{\frac{\log x}{Rn_L}}},\end{aligned}$$

and thus

$$\begin{aligned}\epsilon_\psi(\delta,m,T,x) \le (1+m^{-1})\Bigg(&\left(\frac{mM(\delta,m)\kappa_m D^m}{2a_{\beta_0}^m}\right)^{\frac{1}{m+1}} \\ &+ \frac{\ell_7}{4mR}\frac{\log d_L}{n_L}(\log x)^{1-\frac{1}{m+1}} e^{-2\frac{m\sqrt{m}}{m+1}\sqrt{\frac{\log x}{Rn_L}}}\Bigg)(\log x)^{\frac{1}{m+1}} e^{-2\frac{\sqrt{m}}{m+1}\sqrt{\frac{\log x}{Rn_L}}}. \quad (3.162)\end{aligned}$$

Since $\log x \ge 4mRn_L(\log DT_0)^2 \ge \frac{Rn_L}{m}$, then $(\log x)^{1-\frac{1}{m+1}} e^{-\frac{2m^{\frac{3}{2}}}{m+1}\sqrt{\frac{\log x}{Rn_L}}}$ decreases with x. Hence

$$(\log x)^{1-\frac{1}{m+1}} e^{-\frac{2m^{\frac{3}{2}}}{m+1}\sqrt{\frac{\log x}{Rn_L}}} \le \left(4mRn_L\big(\log(2\sqrt{3}T_0)\big)^2\right)^{1-\frac{1}{m+1}} (2\sqrt{3}T_0)^{-\frac{4m^2}{m+1}},$$

as $D = 2d_L^{\frac{1}{n_L}} \ge 2\sqrt{3}$. We deduce

$$\begin{aligned}\epsilon_\psi(\delta,m,T,x) &\le \Bigg((1+m^{-1})\left(\frac{mM(\delta,m)\kappa_m D^m}{2a_{\beta_0}^m}\right)^{\frac{1}{m+1}} \\ &\quad + \ell_7(\log d_L)\frac{\big(\log(2\sqrt{3}T_0)\big)^{2-\frac{2}{m+1}}(2\sqrt{3}T_0)^{-\frac{4m^2}{m+1}}}{(4mRn_L)^{\frac{1}{m+1}}}\Bigg)(\log x)^{\frac{1}{m+1}} e^{-2\frac{\sqrt{m}}{m+1}\sqrt{\frac{\log x}{Rn_L}}} \\ &\le \max\left(D^{\frac{m}{m+1}}, (\log d_L) n_L^{-\frac{1}{m+1}}\right)\lambda(m)(\log x)^{\frac{1}{m+1}} e^{-2\frac{\sqrt{m}}{m+1}\sqrt{\frac{\log x}{Rn_L}}}\end{aligned}$$

with

$$\lambda(m) = (1+m^{-1})\left(\frac{mM(\delta_0,m)\kappa_m}{2a_{\beta_0}^m}\right)^{\frac{1}{m+1}} + \ell_7\frac{\big(\log(2\sqrt{3}T_0)\big)^{2-\frac{2}{m+1}}(2\sqrt{3}T_0)^{-\frac{4m^2}{m+1}}}{(4mR)^{\frac{1}{m+1}}}, \quad (3.163)$$

where $\lambda(m) \le 0.787$ if β_0 exists and $\lambda(m) \le 0.496$ otherwise. □

Proof of Corollary 3.2. We set $R = 29.57$, $T_0 = 44$ and $M_R(\delta,m) = (m((1+\delta_0/m)^{m+1} + 1))^m$. Using GP/Pari, we optimize m and δ_0 to obtain $m = 2$, $\delta_0 = 10^{-23}$, $\frac{2\sqrt{m}}{m+1} \ge 0.942$ and the explicit constant $\lambda(2) = 0.782$ if β_0 exists and obtain $m = 2$, $\delta_0 = 10^{-19}$, $\frac{2\sqrt{m}}{m+1} \ge 0.942$ and the explicit constant $\lambda(2) = 0.493$ if β_0 does not exist. Using this, we obtain the required result. □

3.8.2 Explicit bounds for $E_\psi(x)$ independent of T, δ and d_L

Proof of Theorem 1.15. Using our choice of T as in (3.148) and $T \geq T_0$, we obtain:

$$D = 2d_L^{\frac{1}{n_L}} \leq \frac{1}{T_0}\exp\left(\frac{1}{2m^{\frac{1}{2}}}\sqrt{\frac{\log x}{Rn_L}}\right), \tag{3.164}$$

and

$$\log d_L \leq \frac{1}{2m^{\frac{1}{2}}}\sqrt{\frac{n_L\log x}{R}}. \tag{3.165}$$

Thus, (3.162) is modified into

$$\begin{aligned}
\epsilon_\psi(\delta,m,T,x) \leq &\Bigg((1+m^{-1})\left(\frac{mM(\delta,m)\kappa_m}{2a_{\beta_0}^m}\frac{1}{T_0^m}e^{\frac{\sqrt{m}}{2}\sqrt{\frac{\log x}{Rn_L}}}\right)^{\frac{1}{m+1}}\\
&+\frac{\ell_7}{4mRn_L}\frac{1}{2m^{\frac{1}{2}}}\sqrt{\frac{n_L\log x}{R}}(\log x)^{1-\frac{1}{m+1}}e^{-2\frac{m\sqrt{m}}{m+1}\sqrt{\frac{\log x}{Rn_L}}}\Bigg)(\log x)^{\frac{1}{m+1}}e^{-2\frac{\sqrt{m}}{m+1}\sqrt{\frac{\log x}{Rn_L}}}\\
\leq &\Bigg((1+m^{-1})\left(\frac{mM(\delta,m)\kappa_m}{2a_{\beta_0}^mT_0^m}\right)^{\frac{1}{m+1}}\\
&+\frac{\ell_7}{8(mR)^{\frac{3}{2}}n_L^{\frac{1}{2}}}(\log x)^{\frac{3}{2}-\frac{1}{m+1}}e^{-\frac{2m^{\frac{3}{2}}+0.5m^{\frac{1}{2}}}{m+1}\sqrt{\frac{\log x}{Rn_L}}}\Bigg)(\log x)^{\frac{1}{m+1}}e^{-1.5\frac{\sqrt{m}}{m+1}\sqrt{\frac{\log x}{Rn_L}}}.
\end{aligned} \tag{3.166}$$

Since $\log x \geq 4mR_L\big(\log(DT_0)\big)^2 \geq \frac{Rn_L}{m}$, then $(\log x)^{\frac{3}{2}-\frac{1}{m+1}}e^{-\frac{2m^{\frac{3}{2}}+0.5m^{\frac{1}{2}}}{m+1}\sqrt{\frac{\log x}{Rn_L}}}$ decreases with x. Hence

$$(\log x)^{\frac{3}{2}-\frac{1}{m+1}}e^{-\frac{2m^{\frac{3}{2}}+0.5m^{\frac{1}{2}}}{m+1}\sqrt{\frac{\log x}{Rn_L}}} \leq \left(4mR_L\big(\log(2\sqrt{3}T_0)\big)^2\right)^{\frac{3}{2}-\frac{1}{m+1}}(2\sqrt{3}T_0)^{-\frac{4m^2+m}{m+1}},$$

as $D = 2d_L^{\frac{1}{n_L}} \geq 2\sqrt{3}$. We deduce

$$\begin{aligned}
\epsilon_\psi(\delta,m,T,x) \leq &\Bigg((1+m^{-1})\left(\frac{mM(\delta,m)\kappa_m}{2a_{\beta_0}^mT_0^m}\right)^{\frac{1}{m+1}}n_L^{-1+\frac{1}{m+1}}\\
&+\frac{\ell_7}{(4mR)^{\frac{1}{m+1}}}\big(\log(2\sqrt{3}T_0)\big)^{3-\frac{2}{m+1}}(2\sqrt{3}T_0)^{-\frac{4m^2+m}{m+1}}\Bigg)n_L^{1-\frac{1}{m+1}}(\log x)^{\frac{1}{m+1}}e^{-1.5\frac{\sqrt{m}}{m+1}\sqrt{\frac{\log x}{Rn_L}}}\\
\leq &\,\nu(m)n_L^{1-\frac{1}{m+1}}(\log x)^{\frac{1}{m+1}}e^{-1.5\frac{\sqrt{m}}{m+1}\sqrt{\frac{\log x}{Rn_L}}}
\end{aligned}$$

with

$$\nu(m) = \frac{1+m^{-1}}{2}\left(\frac{mM(\delta_0,m)\kappa_m}{a_{\beta_0}^m T_0^m}\right)^{\frac{1}{m+1}} + \frac{\ell_7}{(4mR)^{\frac{1}{m+1}}}\left(\log(2\sqrt{3}T_0)\right)^{3-\frac{2}{m+1}}(2\sqrt{3}T_0)^{-\frac{4m^2+m}{m+1}}, \tag{3.167}$$

where $\nu(m) \le 0.0398$ if β_0 exists and $\nu(m) = 0.0251$ otherwise. □

Proof of Corollary 1.16. We set $R = 29.57$, $T_0 = 44$ and $M(\delta_0, m) = M_R(\delta_0, m) = (m((1 + \delta/m)^{m+1} + 1))^m$. Using GP/Pari, we optimize m and δ to obtain $m = 2$, $\delta = 10^{-19}$, $\frac{1.5m^{\frac{1}{2}}}{m+1} \ge 0.707$ and the explicit constant $\nu(m) = 0.0396$ if β_0 exists and obtain $m = 2$, $\delta = 10^{-23}$, $\frac{1.5m^{\frac{1}{2}}}{m+1} \ge 0.707$ and the explicit constant $\nu(m) = 0.0249$ if β_0 does not exist. Using this, we obtain the required result. □

3.9 Explicit bounds for prime ideal counting functions in Chebotarev's density theorem equivalent to θ and π

Remember that:

$$\pi_C(x) = \sum_{\substack{\mathfrak{p} \text{ unramified} \\ \sigma_{\mathfrak{p}}=C,\ \mathrm{N}\,\mathfrak{p}\le x}} 1. \tag{3.168}$$

To go from $\psi_C(x)$ to $\pi_C(x)$, we introduce new quantities as

$$\theta_C(x) = \sum_{\substack{\mathfrak{p} \text{ unramified} \\ \sigma_{\mathfrak{p}}=C,\ \mathrm{N}\,\mathfrak{p}\le x}} \log(\mathrm{N}\,\mathfrak{p}) \text{ and } \theta_0(x) = \sum_{\substack{\mathfrak{p} \text{ unramified} \\ \mathrm{N}\,\mathfrak{p}\le x}} \log(\mathrm{N}\,\mathfrak{p}). \tag{3.169}$$

Lemma 3.43. *For $x \ge x_0$,*

$$|\psi_C(x) - \theta_C(x)| < \frac{22}{15} n_K \sqrt{x}(\log x).$$

Proof. Since $\mathrm{N}\,\mathfrak{p} \ge 2$, therefore for $\mathrm{N}\,\mathfrak{p}^m \le x$, $m \le \lfloor\frac{\log x}{\log 2}\rfloor$. Using this, we notice that

$$\sum_{\substack{\mathfrak{p} \text{ unramified} \\ m\ge 2,\ \mathrm{N}\,\mathfrak{p}^m\le x}} \log(\mathrm{N}\,\mathfrak{p}) = \theta_0(x^{\frac{1}{2}}) + \theta_0(x^{\frac{1}{3}}) + \cdots + \theta_0\left(x^{\frac{1}{\lfloor\frac{\log x}{\log 2}\rfloor}}\right). \tag{3.170}$$

Clearly, $\theta_C(x) \le \theta_0(x)$. Also, it can be easily verified that $\theta_C(x) \le \theta_0(x) \le n_K\theta(x)$ where $\theta(x) =$

$\sum_{p\leq x}\log p$. Rosser and Schoenfeld in [24, Theorem 9] showed that for $x\geq 2$, $\theta(x)<1.01624x$. Combining this we obtain, $\theta_0(x)<1.01624n_Kx$ and

$$|\psi_C(x)-\theta_C(x)|\leq \sum_{\substack{\mathfrak{p}\text{ unramified}\\ m\geq 2,\ \mathrm{N}\,\mathfrak{p}^m\leq x}}\log(\mathrm{N}\,\mathfrak{p})=\theta_0(x^{\frac{1}{2}})+\theta_0(x^{\frac{1}{3}})+\cdots+\theta_0\Big(x^{\frac{1}{\lfloor\frac{\log x}{\log 2}\rfloor}}\Big)$$
$$\leq\frac{\log x}{\log 2}\theta_0(x^{\frac{1}{2}})<\frac{\log x}{\log 2}\times(1.01624n_Kx^{\frac{1}{2}})<\frac{22}{15}n_K\sqrt{x}(\log x). \tag{3.171}$$

□

Lemma 3.44. *Let C be a fixed conjugacy class of the Galois group,* $\mathrm{Gal}(L/K)=G$. *Let β_0 be the possible exceptional real zero of $\zeta_L(s)$. Let H be defined in* (3.11)*. For* $\log x\geq\frac{19\,810}{n_L}(\log d_L)^2$, *we have*

$$E_\theta(x)=\left|\frac{\theta_C(x)-\frac{|C|}{|G|}x}{\frac{|C|}{|G|}x}\right|<\frac{x^{\beta_0-1}}{\beta_0}+\epsilon_\theta(x,n_L), \tag{3.172}$$

where

$$\epsilon_\theta(x,n_L)=\frac{22}{15}n_L\frac{\log x}{\sqrt{x}}+A_1(\log x)^{\frac{1}{3}}n_L^{\frac{2}{3}}\exp\Big(-B_1\sqrt{\frac{\log x}{n_L}}\Big). \tag{3.173}$$

Proof. Using Lemma 3.43, we obtain

$$\left|\theta_C(x)-\frac{|C|}{|G|}x\right|\leq|\theta_C(x)-\psi_C(x)|+\left|\psi_C(x)-\frac{|C|}{|G|}x\right|<\frac{22}{15}n_K\sqrt{x}(\log x)+\left|\psi_C(x)-\frac{|C|}{|G|}x\right|. \tag{3.174}$$

We complete the proof using $|C|\geq 1$, $n_K\times|G|=n_L$ and Remark 1.16. □

Proof of Theorem 1.17. Using partial summation and integration by parts, we obtain

$$\pi_C(x)-\frac{|C|}{|G|}\,\mathrm{Li}(x)=\frac{\theta_C(x)}{\log x}+\int_2^x\frac{\theta_C(t)}{t(\log t)^2}dt-\frac{|C|}{|G|}\Big(\frac{x}{\log x}+\int_2^x\frac{dt}{(\log t)^2}\Big)$$
$$=\frac{\theta_C(x)-\frac{|C|}{|G|}x}{\log x}+\int_2^x\frac{\theta_C(t)-\frac{|C|}{|G|}t}{t(\log t)^2}dt. \tag{3.175}$$

Note that the first equality defined in (3.175) is true up to a constant $\leq n_K$ which depends on $\#\{\mathfrak{p}|\mathfrak{p}$ unramified, $\mathbb{N}\mathfrak{p}=2$ and $\sigma_{\mathfrak{p}}=C\}$. Since this quantity is very small, we neglect this in our computations. For $\log x\geq\frac{2\times 19\,810}{n_L}(\log d_L)^2$, using the triangle inequality and Lemma 3.44, we obtain:

$$\begin{aligned}
&\left|\pi_C(x) - \frac{|C|}{|G|}\operatorname{Li}(x)\right| \\
&\leq \left|\frac{\theta_C(x) - \frac{|C|}{|G|}x}{\log x}\right| + \int_2^x \left|\frac{\theta_C(t) - \frac{|C|}{|G|}t}{t(\log t)^2}\right| dt \leq \frac{|C|}{|G|}\frac{x}{\log x}\left(\frac{x^{\beta_0 - 1}}{\beta_0} + \epsilon_\theta(x, n_L)\right) + \int_2^x \left|\frac{\theta_C(t) - \frac{|C|}{|G|}t}{t(\log t)^2}\right| dt \\
&= \frac{|C|}{|G|}\frac{x}{\log x}\left(\frac{x^{\beta_0 - 1}}{\beta_0} + \frac{22}{15}n_L\frac{\log x}{\sqrt{x}} + A_1(\log x)^{\frac{1}{3}} n_L^{\frac{2}{3}}\exp\left(-B_1\sqrt{\frac{\log x}{n_L}}\right)\right) + \int_2^x \left|\frac{\theta_C(t) - \frac{|C|}{|G|}t}{t(\log t)^2}\right| dt
\end{aligned} \tag{3.176}$$

We know $\theta_C(t) \leq \theta_0(t) < 1.01624 n_K t$. Therefore $\left|\theta_C(t) - \frac{|C|}{|G|}t\right| < 2.01624 n_K t$. Using this and $n_K \times |G| = n_L$ after splitting the above integral at $\sqrt{x}$, we obtain:

$$\begin{aligned}
\int_2^x \left|\frac{\theta_C(t) - \frac{|C|}{|G|}t}{t(\log t)^2}\right| dt &< n_K \int_2^{\sqrt{x}} \frac{2.01624}{(\log t)^2} dt + \int_{\sqrt{x}}^x \left|\frac{\theta_C(t) - \frac{|C|}{|G|}t}{t(\log t)^2}\right| dt \\
&< 4.2\frac{|C|}{|G|} n_L \sqrt{x} + \int_{\sqrt{x}}^x \left|\frac{\theta_C(t) - \frac{|C|}{|G|}t}{t(\log t)^2}\right| dt.
\end{aligned} \tag{3.177}$$

For the integral on the right side, we can use Lemma 3.44 as $\log\sqrt{x} \geq \frac{19\,810}{n_L}(\log d_L)^2$. Thus, we obtain:

$$\int_{\sqrt{x}}^x \left|\frac{\theta_C(t) - \frac{|C|}{|G|}t}{t(\log t)^2}\right| dt < \frac{|C|}{|G|}\int_{\sqrt{x}}^x \frac{\frac{t^{\beta_0 - 1}}{\beta_0}}{(\log t)^2} dt + \frac{|C|}{|G|}\int_{\sqrt{x}}^x \frac{\epsilon_\theta(t, n_L)}{(\log t)^2} dt. \tag{3.178}$$

Now,

$$\begin{aligned}
\int_{\sqrt{x}}^x \frac{\epsilon_\theta(t, n_L)}{(\log t)^2} dt &= \int_{\sqrt{x}}^x \left(\frac{22}{15}n_L\frac{\log t}{\sqrt{t}} + A_1(\log t)^{\frac{1}{3}} n_L^{\frac{2}{3}}\exp\left(-B_1\sqrt{\frac{\log t}{n_L}}\right)\right)\frac{1}{(\log t)^2} dt \\
&= \frac{22 n_L}{15}\int_{\sqrt{x}}^x \frac{1}{\sqrt{t}(\log t)} dt + A_1 n_L^{\frac{2}{3}}\int_{\sqrt{x}}^x (\log t)^{-\frac{5}{3}}\exp\left(-B_1\sqrt{\frac{\log t}{n_L}}\right) dt \\
&\leq \frac{88 n_L}{15}\frac{\sqrt{x}}{\log x} + 2^{\frac{5}{3}} A_1 n_L^{\frac{2}{3}}(\log x)^{-\frac{5}{3}} x \exp\left(-B_1\sqrt{\frac{\log x}{2n_L}}\right).
\end{aligned} \tag{3.179}$$

Also we can easily check that

$$\frac{x^{\beta_0}}{\log x^{\beta_0}} + \frac{1}{\beta_0}\int_{\sqrt{x}}^x \frac{t^{\beta_0 - 1}}{(\log t)^2} dt \leq \operatorname{Li}(x^{\beta_0}). \tag{3.180}$$

Now combining (3.176), (3.177), (3.178), (3.179) and (3.180), we obtain

$$\begin{aligned}
&\left|\pi_C(x) - \frac{|C|}{|G|}\operatorname{Li}(x)\right| \\
&< \frac{|C|}{|G|}\operatorname{Li}(x^{\beta_0}) + \frac{|C|}{|G|} n_L \sqrt{x}\left(\frac{22}{15} + 4.2 + \frac{88}{15(\log x)}\right) \\
&+ \frac{|C|}{|G|} A_1 n_L^{\frac{2}{3}} x \exp\left(-\frac{B_1}{\sqrt{2}}\sqrt{\frac{\log x}{n_L}}\right)\left(2^{\frac{5}{3}}(\log x)^{-\frac{5}{3}} + (\log x)^{-\frac{2}{3}}\exp\left(-\left(B_1 - \frac{B_1}{\sqrt{2}}\right)\sqrt{\frac{\log x}{n_L}}\right)\right) \\
&= \frac{|C|}{|G|}\operatorname{Li}(x^{\beta_0}) + \frac{|C|}{|G|} n_L x \exp\left(-\frac{B_1}{\sqrt{2}}\sqrt{\frac{\log x}{n_L}}\right)\left(\frac{1}{\sqrt{x}}\left(\frac{22}{15} + 4.2 + \frac{88}{15(\log x)}\right)\times\right. \\
&\quad \left.\exp\left(\frac{B_1}{\sqrt{2}}\sqrt{\frac{\log x}{n_L}}\right) + A_1 n_L^{-\frac{1}{3}}\left(2^{\frac{5}{3}}(\log x)^{-\frac{5}{3}} + (\log x)^{-\frac{2}{3}}\exp\left(-\left(B_1 - \frac{B_1}{\sqrt{2}}\right)\sqrt{\frac{\log x}{n_L}}\right)\right)\right).
\end{aligned} \tag{3.181}$$

Using $\frac{\log x}{n_L} > 2 \times 8R(\log 88\sqrt{3})^2$, $\log x \geq 11954$, $n_L \geq 2$,

$$\frac{1}{\sqrt{x}}\exp\left(\frac{B_1}{\sqrt{2}}\sqrt{\frac{\log x}{n_L}}\right) \leq \frac{1}{\exp(16R(\log 88\sqrt{3})^2))}\exp\left(\frac{B_1}{\sqrt{2}}(\sqrt{16R}(\log 88\sqrt{3}))\right),$$

with $R = 29.57$, $A_1 = 0.0249$, $B_1 = 0.13$ (β_0 does not exist) we obtain

$$\left|\pi_C(x) - \frac{|C|}{|G|}\operatorname{Li}(x)\right| \leq 2.97 \times 10^{-6}\frac{|C|}{|G|} n_L x \exp\left(-\frac{0.13}{\sqrt{2}}\sqrt{\frac{\log x}{n_L}}\right), \tag{3.182}$$

and with $R = 29.57$, $A_1 = 0.0396$, $B_1 = 0.13$ (β_0 exists) we obtain

$$\left|\pi_C(x) - \frac{|C|}{|G|}\operatorname{Li}(x)\right| \leq \frac{|C|}{|G|}\operatorname{Li}(x^{\beta_0}) + 4.714 \times 10^{-6}\frac{|C|}{|G|} n_L x \exp\left(-\frac{0.13}{\sqrt{2}}\sqrt{\frac{\log x}{n_L}}\right). \tag{3.183}$$

□

3.10 Comparison to results from Winckler

We use $R = 29.57$, $\frac{\log d_L}{n_L} \geq \log\sqrt{3}$, $n_L \geq 2$ and $m = 2$ to obtain

$$\begin{aligned}
&4mRn_L(\log 88 d_L^{\frac{1}{n_L}})^2 = 8Rn_L\left(\log 88 + \frac{1}{n_L}\log d_L\right)^2 = n_L\left(\frac{\log d_L}{n_L}\right)^2\left(8(29.57)\left(\frac{\log 88 n_L}{\log d_L} + 1\right)^2\right) \\
&\leq \frac{(\log d_L)^2}{n_L}\left(8(29.57)\left(\frac{\log 88}{\log\sqrt{3}} + 1\right)^2\right) < \frac{19\,810}{n_L}(\log d_L)^2.
\end{aligned}$$

Thus we obtain the form $\log x \geq \frac{19\,810}{n_L}(\log d_L)^2 > 4mRn_L(\log 88 d_L^{\frac{1}{n_L}})^2$ as given in Corollary 3.2 and Corollary 1.16. Now to compare our results with Winckler's [30], we look at the case which takes into account existence of β_0 and work with another form for the above expression. We use $R = 29.57$, $\log d_L \geq \log 3$, $n_L \geq 2$ and $m = 2$ to obtain

$$4mRn_L(\log 88 d_L^{\frac{1}{n_L}})^2 = 8Rn_L\left(\log 88 + \frac{1}{n_L}\log d_L\right)^2 = n_L(\log d_L)^2\left(8(29.57)\left(\frac{\log 88}{\log d_L} + \frac{1}{n_L}\right)^2\right)$$
$$\leq n_L(\log d_L)^2\left(8(29.57)\left(\frac{\log 88}{\log 3} + \frac{1}{2}\right)^2\right) < 4953 n_L(\log d_L)^2. \tag{3.184}$$

Thus $\log x \geq 4953 n_L(\log d_L)^2$ implies $\log x > 4mRn_L(\log 88 d_L^{\frac{1}{n_L}})^2$. Thus using $\log d_L \geq n_L \log\sqrt{3}$, we obtain

$$\log x \geq 4953 n_L(n_L\log\sqrt{3})^2 = 4953(\log\sqrt{3})^2 n_L^3.$$

Using this and then computing in MAPLE, for $\log x \geq 11954$, we get,

$$0.0396(\log x)^{\frac{1}{3}} n_L^{\frac{2}{3}}\exp\left(-(0.130 - B_2)\sqrt{\frac{\log x}{n_L}}\right)$$
$$\leq 0.0396(\log x)^{\frac{1}{3}}\frac{(\log x)^{\frac{2}{9}}}{(4953(\log\sqrt{3})^2)^{\frac{2}{9}}}\exp\left(-(0.130 - B_2)\sqrt{\frac{\log x}{(\log x)^{\frac{1}{3}}}(4953(\log\sqrt{3})^2)^{\frac{1}{3}}}\right)$$
$$= \frac{0.0396}{(4953(\log\sqrt{3})^2)^{\frac{2}{9}}}(\log x)^{\frac{5}{9}}\exp(-(0.130 - B_2)(4953(\log\sqrt{3})^2)^{\frac{1}{6}}(\log x)^{\frac{1}{3}})$$
$$\leq A_2,$$

where (A_2, B_2) as given in Corollary 3.5 have the admissible values $(1.84 \times 10^{-4}, 0.014)$ and $(3.11, 0.125)$. Thus

$$0.0396(\log x)^{\frac{1}{3}} n_L^{\frac{2}{3}}\exp\left(-0.707\sqrt{\frac{\log x}{Rn_L}}\right) \leq 0.0396(\log x)^{\frac{1}{3}} n_L^{\frac{2}{3}}\exp\left(-0.13\sqrt{\frac{\log x}{n_L}}\right)$$
$$\leq A_2\exp\left(-B_2\sqrt{\frac{\log x}{n_L}}\right).$$

Winckler in [30] proved that for $\log x \geq 1545 n_L(\log d_L)^2$,

$$E_\psi(x) \leq \frac{x^{\beta_0 - 1}}{\beta_0} + 1.51 \times 10^{12}\exp\left(-0.014\sqrt{\frac{\log x}{n_L}}\right).$$

We instead prove that for $\log x \geq 4953 n_L (\log d_L)^2$,

$$E_\psi(x) \leq \frac{x^{\beta_0 - 1}}{\beta_0} + 1.84 \times 10^{-4} \exp\left(- 0.014 \sqrt{\frac{\log x}{n_L}} \right).$$

Using (3.184), we have

$$2 \times 4mRn_L(\log 88 d_L^{\frac{1}{n_L}})^2 < 9\,906 n_L (\log d_L)^2. \tag{3.185}$$

Thus $\log x \geq 9\,906 n_L (\log d_L)^2$ implies $\log x > 8mRn_L(\log 88 d_L^{\frac{1}{n_L}})^2$. Thus using $\log d_L \geq n_L \log \sqrt{3}$, we obtain

$$\log x \geq 9\,906 n_L (n_L \log \sqrt{3})^2 = 9\,906 (\log \sqrt{3})^2 n_L^3.$$

Using this and then computing in MAPLE, for $\log x \geq 2 \times 11954$, we get,

$$\begin{aligned}
&4.714 \times 10^{-6} n_L \exp\left(- \left(\frac{0.13}{\sqrt{2}} - F_1 \right) \sqrt{\frac{\log x}{n_L}} \right) \\
&\leq 4.714 \times 10^{-6} \frac{(\log x)^{\frac{1}{3}}}{(9\,906 (\log \sqrt{3})^2)^{\frac{1}{3}}} \exp\left(- \left(\frac{0.13}{\sqrt{2}} - F_1 \right) \sqrt{\frac{\log x}{(\log x)^{\frac{1}{3}}} (9\,906 (\log \sqrt{3})^2)^{\frac{1}{3}}} \right) \\
&\leq E_1,
\end{aligned}$$

where admissible values of (E_1, F_1) as given in Corollary 1.18 are $(1.23 \times 10^{-9}, 1/99)$ and $(1.65 \times 10^{-5}, 0.09)$. Thus

$$4.714 \times 10^{-6} \frac{|C|}{|G|} n_L x \exp\left(- \frac{0.13}{\sqrt{2}} \sqrt{\frac{\log x}{n_L}} \right) \leq E_1 \frac{|C|}{|G|} x \exp\left(- F_1 \sqrt{\frac{\log x}{n_L}} \right).$$

Winckler in [30] proved that for $\log x \geq 3\,090\, n_L (\log d_L)^2$,

$$\left| \pi_C(x) - \frac{|C|}{|G|} \operatorname{Li}(x) \right| \leq \frac{|C|}{|G|} \operatorname{Li}(x^{\beta_0}) + 7.84 \times 10^{14} x \exp\left(- \frac{1}{99} \sqrt{\frac{\log x}{n_L}} \right).$$

We instead prove that for $\log x \geq 9\,906 n_L (\log d_L)^2$,

$$\left| \pi_C(x) - \frac{|C|}{|G|} \operatorname{Li}(x) \right| \leq \frac{|C|}{|G|} \operatorname{Li}(x^{\beta_0}) + 1.23 \times 10^{-9} \frac{|C|}{|G|} x \exp\left(- \frac{1}{99} \sqrt{\frac{\log x}{n_L}} \right).$$

Chapter 4

Future work

This chapter provides a brief introduction into plans for future works based on this thesis.

1. Recall that in this thesis, I use the zero-free region for the Dedekind ζ-function as proved by Ahn and Kwon in [3, Proposition 6.1]:

   Theorem 3.12. *Let L be a number field with $n_L \geq 2$. Let $\rho = \beta + i\gamma$ be non-trivial zero of $\zeta_L(s)$ with $\rho \neq \beta_0$ and $\tau = |\gamma| + 2$. Then*

   $$\beta < 1 - (Rn_L \log(D\tau/2))^{-1}, \tag{4.1}$$

   where $R = 29.57$, $D = 2d_L^{\frac{1}{n_L}}$.

   Recently, Lee in [16, Theorem 1] has improved the zero-free region for the Dedekind ζ-function, $\zeta_L(s)$ by proving that Theorem 3.12 holds true for

   $$R = 12.2411.$$

   I plan to incorporate this new result into my current research to improve the bounds for the error term $E_\psi(x)$ as well as increasing the range of x for which the bounds are valid.

2. In this thesis, I have proved unconditional bounds for $E_\psi(x)$. However, stronger bounds are known under the assumption of Artin's Holomorphy Conjecture. This conjecture asserts that:

   Let L/K be a normal extension with Galois group G. If ρ is a non-trivial irreducible representation of G, then the Artin L-function $L(s, \rho, L/K)$ is a holomorphic function.

I plan to use ideas from Ng [18, Chapter 3] to provide conditional bounds for $E_\psi(x)$ assuming Artin's Holomorphy Conjecture.

3. The study of distribution of zeros of L-functions is critical to finding explicit estimates related to the distribution of prime ideals in number fields. The distribution of zeros is affected by several factors including the Deuring-Heilbronn phenomenon which states that a counterexample to the generalized Riemann hypothesis for one L-function affects the location of the zeros of other L-functions. In particular, the existence of an exceptional zero for the Dedekind ζ-function yields a larger zero-free region. I would like to explore whether the existence of an exceptional zero leads to an improved bound for $E_\psi(x)$.

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