A SURVEY OF BÜTHE'S METHOD FOR ESTIMATING PRIME COUNTING FUNCTIONS

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Abstract

This thesis provides explicit bounds for the Chebyshev prime counting function $\psi(x)$. This thesis aims to produce a detailed survey of the first part (from page 2483 to page 2494) of the paper, 'Estimating $\pi(x)$ and Related Functions Under Partial RH Assumptions' by Jan Büthe published in 2016. His article provides the best-known bounds for $\psi(x)$ for $x \leq e^{3000}$ using the Fourier Transform of the Logan Function, assuming the Riemann Hypothesis to be valid for all zeroes of the zeta function with $\Im(\rho) \in (0, T]$ for a specific T. The main theorem in Büthe's paper gives a bound for $|\psi(x) - x|$ using an equation with three major terms $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$ and provides bounds for each of these terms individually. The necessary lemmas, propositions and their proofs required to prove the main theorem are scattered throughout various papers such as [7], [8], [10], [18], and [6]. In this thesis, we have accumulated all these results, verified their proofs, and included various missing details. Several of the arguments in the original paper have been reworked, and necessary corrections, such as rectifying the error terms $\mathcal{E}_2$ and $\mathcal{E}_3$ in the main theorem, and other minor amendments have been made with the goal of turning this thesis into a self-contained research exposition of Büthe's work in [7].

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Contents

List of Tables

Chapter 1

Notation and Definitions

We will introduce some preliminary notations and definitions for reference:

- Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ be the set of natural numbers, integers, real numbers and complex numbers respectively.
- In this thesis p is used to denote a prime, n, q, a are used to denote an integer, x is a real number and $s = \sigma + it$ is a complex number.
- The notation $n \equiv a \pmod q$ indicates that there exists a $k \in \mathbb{Z}$ such that $n = kq + a$.
- $(a,q) = \gcd(a,q)$ is the greatest common divisor of a and q.
- A function $f(x)$ is said to be asymptotic to another function $g(x)$, denoted by $f(x) \sim g(x)$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$, where $g(x) \neq 0$ for sufficiently large x.
- For two function f and g, we write $f = O^*(g)$ if $|f| \leq g$.
- The convolution of two functions f and g is defined by $(f * g)(t) = \int_R f(y)g(t-y)dy$.
- The Fourier transform of a function f is given by:

$$\hat{f}(\omega) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\omega x} f(x) dx.$$

- For $\varepsilon > 0$ and a point y on the real line, $B_\varepsilon(y)$ denotes a ball of radius ε around y i.e. $B_\varepsilon(y) = \{x \in \mathbb{R} \mid |x-y| < \varepsilon\}$.

- p_n is the nth prime number.
- $\gamma_0 = 0.5772156\ldots$is the Euler-Mascheroni constant.
- We define the Logarithmic Integral as

$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t}.$$

- For a set $A \subset X$,
  - the *characteristic function* is defined as

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

  - the *normalized characteristic function* is defined as ;

$$\chi_A^*(x) = \begin{cases} 1 & x \in A \setminus \delta A, \\ \frac{1}{2} & x \in \delta A, \\ 0 & x \in X \setminus \overline{A}. \end{cases}$$

  where $\overline{A}$ denotes the closure of A and $\delta(A)$ denotes the boundary of the set A where $\delta(A) = \overline{A} \backslash A^{o}$.

- An *arithmetic function* is a function $f : \mathbb{N} \to \mathbb{C}$.
- The *von Mangoldt* function is an arithmetic function defined as :

$$\Lambda(n) = \begin{cases} \log p & \text{for } n = p^k \text{ for some prime } p \text{ and integer } k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

- $\pi(x)$ is the *prime counting function* defined as:

$$\pi(x) = \sum_{p \leq x} 1.$$

- We define $\theta(x)$ as follows :

$$\theta(x) = \sum_{p \leq x} \log p.$$

- The *Chebyshev function* is defined as:

$$\psi(x) = \sum_{n \leq x} \Lambda(n) = \sum_{p^m \leq x} \log p.$$

- The Riemann zeta function is defined on the complex variable $s = \sigma + it$, as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

  for $\sigma > 1$ and its analytic continuation elsewhere. The roots of the zeta function are denoted by $\rho = \beta + i\gamma$.

- $N(T)$ is defined to be the number of zeros $\rho = \beta + i\gamma$ (counted with multiplicity) of the Riemann zeta function $\zeta(s)$ for which $0 < \gamma < T$.

- $N(\sigma, T)$ denotes the number of nontrivial zeros $\rho = \beta + i\gamma$ of the Riemann zeta function with $\beta > \sigma$ and $0 < \gamma < T$. Explicit upper bounds for $N(\sigma, T)$ are commonly referred to as zero density results.

- The normalized Chebyshev function is defined as :

$$\psi_0(x) = \sum_{n \leq x} \Lambda(n) \chi^*_{[0,x]}(n) = \sum_{p^m \in [0,x]} \log p \cdot \chi^*_{[0,x]}(p^m)$$

- The von Mangoldt explicit formula for $\psi_0(x)$ is given by

$$\psi_0(x) = x - \sum_{\rho}^{*} \frac{x^\rho}{\rho} - \log(2\pi) - \frac{1}{2}\log(1 - x^{-2}),$$

  where the sum is taken over all non-trivial zeros (according to their multiplicities) of the Riemann zeta function and the $*$ indicates that the sum is computed as:

$$\lim_{T\to\infty} \sum_{|\Im(\rho)|<T} \frac{x^\rho}{\rho}.$$

Chapter 2

Introduction

The main focus of this thesis is to provide a survey of the first part of Jan Büthe's research paper, 'Estimating $\pi(x)$ and Related Functions under Partial RH Assumptions' and it deals with the bounds of Chebyshev's Prime Counting Function $\psi(x)$. In order for a complete discussion on estimates for prime counting functions it is essential to start with the history of the Prime Number Theorem.

2.1 The Prime Number Theorem

The Prime Number Theorem and its analogous results have a long and complex history. One of the most consequential theorems in the history of Number Theory, the Prime Number Theorem, deals with the asymptotic distribution of prime numbers among positive integers. We can state this theorem as follows :

Theorem 1 (Prime Number Theorem)**.** *Let $\pi(x)$ be the prime counting function defined as the number of primes less than or equal to x. The Prime Number Theorem states that*

$$\pi(x) \sim \frac{x}{\log x}$$

as x tends to infinity.

Legendre in 1798 conjectured that for large x, $\pi(x)$ is approximately of the form

$$\frac{x}{\log x - 1.08\ldots}.$$

The above theorem was historically the first published version of The Prime Number Theorem. The statement of the theorem can be more precisely written as $\pi(x) \sim \frac{x}{\log x - A(x)}$ where $A(x) \to 1.08\ldots$ as $x \to \infty$. This version has since been shown to be erroneous as it was later proved that $A(x) \to 1$ as $x \to \infty$.

There is evidence to suggest that Gauss had done extensive research on the density of primes as early as 1792-93, much before the publication of Legendre's conjecture. Although his results were never published, his notes contain compilations of extensive tables on the distribution of primes in various intervals of length 1000. From his observations, Gauss concluded that the density with which primes occur in the neighbourhood of an integer n is $\frac{1}{\log n}$. In his 1849 letter to the astronomer Encke, Gauss came up with an equivalent version of PNT where he claimed that a good approximation to $\pi(x)$ is given by the function $\mathrm{Li}(x)$ where $\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t}$, is the Logarithmic Integral.

In 1851 and 1852, Chebyshev, in his two memoirs introduced the functions: $\theta(x) = \sum_{p \le x} \log p$ and $\psi(x) = \sum_{p^m \le x} \log p$ on $\mathbb{R}$ where p denotes a prime and $m \in \mathbb{Z}^+$. He also proved that the Prime Number Theorem holds if each of the following statements :

$$\lim_{x\to\infty} \frac{\theta(x)}{x} = 1 \qquad \text{and} \qquad \lim_{x\to\infty} \frac{\psi(x)}{x} = 1$$

are true. In his first paper, Chebyshev provided a justification for Gauss's conjecture relating $\pi(x)$ and $\mathrm{Li}(x)$ by proving that

$$\liminf_{x\to\infty} \frac{\pi(x)}{\mathrm{Li}(x)} \le 1 \le \limsup_{x\to\infty} \frac{\pi(x)}{\mathrm{Li}(x)}$$

which implied that if $\lim_{x\to\infty} \frac{\pi(x)}{\mathrm{Li}(x)}$ exists, its limit must be 1. In his second paper, he provided definite inequalities for $\pi(x)$

$$(0.92\ldots)\frac{x}{\log x} \le \pi(x) \le (1.105\ldots)\frac{x}{\log x}$$

for all sufficiently large x.

In 1860, in his only paper on the theory of numbers, Riemann introduced the following function on the complex variable $s = \sigma + it$,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

now known as the Riemann zeta function. The series $\sum_{n=1}^{\infty} \frac{1}{n^s}$ converges absolutely and uniformly for s in a compact subset of the half-plane where $\sigma > 1$. Riemann proved that the function $\zeta(s)$ can be continued analytically to a function which is meromorphic in the whole complex plane, with only a simple pole at $s = 1$ with residue 1. The only zeros of $\zeta(s)$ for $\sigma < 0$ are at the points $s = -2, -4, -6 \cdots$, and are called the trivial zeros. The non-trivial zeroes of $\zeta(s)$ lie in the strip $0 \leq \sigma \leq 1$, also known as the critical strip. Riemann proved that $\zeta(s)$ has infinitely many zeroes in the critical strip which are placed symmetrically with respect to the real axis and also with respect to the central line $\sigma = 1/2$. Moreover, he conjectured that the real part of the non-trivial zeroes is $\frac{1}{2}$. This conjecture, known as the Riemann Hypothesis, has remained unproven to date.

Riemann set the background for the method used to ultimately prove the Prime Number Theorem. He showed that the prime number theorem would logically follow if one could prove that there were no zeros of the zeta function on the line where $\sigma = 1$; however, he was unable to successfully solve the Prime Number theorem.

By 1896, the analytical tools necessary for the proof of PNT had been developed. Working independently and almost simultaneously, Hadamard and de la Vallée Poussin provided two different methods to prove that there were no zeroes on the line $\sigma = 1$, which confirmed the validity of the Prime Number Theorem.

2.2 Bounds for certain arithmetic functions

The statement and subsequent proof of the Prime Number Theorem paved the way for extensive research on explicit bounds of certain arithmetic functions such as $\pi(x)$, $\theta(x)$, $\psi(x)$.

One of the first major works dealing with the bounds of these arithmetic functions was by Rosser in 1941, in the paper 'Explicit Bounds for Some Functions of Prime Numbers' [31], where he demonstrated the existence of the following inequalities:

- For each $A > 0$, there exists N, such that for $n \geq N$

$$\frac{x}{\log x - 1 + A} < \pi(x) < \frac{x}{\log x - 1 - A}.$$

- For each $B > 0$, there exists N, such that for $n \geq N$

$$n\log n + n\log\log n - n - Bn < p_n < n\log n + n\log\log n - n + Bn.$$

  where p_n is the nth prime.

- For each $C > 0$, there exits N, such that for $x \geq N$

$$\left(1 - \frac{C}{\log x}\right)x < \theta(x) < \left(1 + \frac{C}{\log x}\right),$$

thus providing explicit bounds for $\pi(x)$, $\theta(x)$ and p_n. Rosser's work contains numerous other explicit bounds for arithmetic functions, and made use of several key ideas, namely, averaging methods involving certain smoothing functions, an explicit zero free region for the Riemann zeta function, partial verification of the Riemann Hypothesis, and bounds for the zero-counting function, $N(t)$. The major result in this paper, regarding bounds for $\psi(x)$ can be stated as follows :

Theorem 2. *[31, Theorem 22, page 227] For $b > 0$, there exists an $\varepsilon = \varepsilon(b) > 0$ such that* *for all* $x \geq e^b$

$$(1-\varepsilon)x < \psi(x) < (1+\varepsilon)x. \tag{2.1}$$

Moreover, for $x \geq e^{4000}$,

$$\left|\frac{\psi(x)-x}{x}\right| \leq (\log x)^{\frac{1}{2}} e^{-\sqrt{(\log x/19)}}.$$

A table of values of b and corresponding upper bounds of ε as presented in [31], is given below:

Table 2.1: Table of values of b and corresponding ε as given in Theorem 2

b	ε	b	ε
13.8	0.0381	500	0.00511
15	0.0321	550	0.00467
20	0.0199	600	0.00427
30	0.0179	650	0.00392
40	0.0119	700	0.00356
60	0.0101	800	0.00294
80	0.00983	900	0.00235
90	0.00938	1000	0.00194
100	0.0932	1300	0.00104
300	0.00710	2000	0.000383
400	0.00609	2300	0.000255
450	0.00567	3000	0.000158

In their extensively cited 1962 paper ‘Approximate Formulas for Some Functions of Prime Numbers’, [32], Rosser and Schoenfeld provided a large number of results, including the following bounds on $\pi(x)$:

$$\frac{x}{\log x}\left(1+\frac{1}{2\log x}\right) < \pi(x) \qquad \text{for } x \geq 59,$$

$$\pi(x) < \frac{x}{\log x}\left(1+\frac{3}{2\log x}\right) \qquad \text{for } x > 1,$$

$$x/(\log x - 1/2) < \pi(x) \qquad \text{for } x \geq 67,$$

$$\pi(x) < x/(\log x - 3/2) \qquad \text{for } x > e^{\frac{3}{2}}.$$

Their paper also provided improvements to the bounds for $\psi(x)$ based on equation (2.1), given in Rosser's earlier paper [31], along with estimates for Merten's sums and products. [32] used similar arguments as those presented in Rosser's earlier paper, along with a better zero-free region, and an improved verification of the Riemann Hypothesis due to Lehmer. A table of values of b and corresponding values of ε as presented in [32] is given below : Later research works, for instance, 'Sharper Bounds for the Chebyshev Functions $\theta(x)$ and

Table 2.2: Table of values of b and corresponding ε in [32]

b	ε
20	$1.2880 \cdot 10^{-2}$
100	$9.99653 \cdot 10^{-4}$
500	$7.2240 \cdot 10^{-4}$
1000	$4.4744 \cdot 10^{-4}$
2000	$1.0274 \cdot 10^{-4}$

$\pi(x)$' [33], by Rosser and schoenfeld in 1975, and the 1998 thesis of Dusart [12] obtain successively smaller bounds for $\left|\frac{\psi(x)-x}{x}\right|$ using improved values for the verification of the Riemann Hypothesis up to a fixed height H, and an explicit zero free region of the form $\Re(s) \geq 1 - \frac{1}{R\log|\Im(s)|}$ for $|\Im(s)| \geq 2$ where R is a computable constant.

All of the preceding papers used the same smoothing function, however in 2014, Faber and Kadiri, in 'New bounds for $\psi(x)$'[16] provided explicit Chebyshev bounds for $\psi(x)$ by applying a general smoothing with Mellin Transform of which Rosser and Schoenfeld's work is a special case. Their discourse also utilised zero-density estimates for the Riemann zeta function, and bounds for the zero-density function, $N(\sigma, t)$, defined as

$$N(\sigma, T) = \#\{\rho = \beta + i\gamma \mid \zeta(\rho) = 0, \beta > \sigma, \text{ and } 0 < \gamma < T\}.$$

The results in [16] gives better bounds for $|\psi(x) - x|$ in the following theorem:

Theorem 3. *[16, Theorem 1.1, page 1339] Let* $b_0 \leq 9963$ *be a fixed positive constant. Then* *there exists* $\varepsilon_0 > 0$ *such that for all* $x \geq e^{b_0}$, $\left|\frac{\psi(x)-x}{x}\right| \leq \varepsilon_0$,

where the exact definition of ε_0 is given in equation 3.9 in [16]. This paper contained the best explicit bounds for $|\psi(x) - x|$, for small values of x prior to Büthe's results. A table of values of b_0 and the corresponding values of ε_0 are given in the following table:

Table 2.3: Table of values of b_0 and ε_0 as given in Theorem 3

b_0	ε_0
20	$5.3688 \cdot 10^{-4}$
25	$4.8208 \cdot 10^{-5}$
30	$5.6679 \cdot 10^{-6}$
35	$7.4457 \cdot 10^{-7}$
40	$8.6347 \cdot 10^{-8}$

Jan Büthe's research article 'Estimating $\pi(x)$ and Related Functions under Partial RH Assumptions'[7], in 2016, forgoes established methods of using zero density results and zero-free regions, in order to focus solely on the smoothing method, using a new function known as the Logan function, and provides the best known bounds for $\psi(x)$ for $x \leq e^{3000}$.

We present a table below, which is a modified version of the table found in [16] and shows the numerical improvements of ε over the years for $\left|\frac{\psi(x)-x}{x}\right| \leq \varepsilon x$ when $b = 50$, with improvements in the height of partial verification of RH and zero free region.

Table 2.4: Numerical Improvements of ε

Articles	H	R	ε
Rosser [31]	1467	17.72	1.1900×10^{-2}
Rosser and Schoefeld [32]	21943	17.5163 ...	1.7202×10^{-3}
Rosser and Schoenfeld[33]	1894438	9.645908801	1.7583×10^{-5}
Dusart[12]	545439823	9.645908801	9.0500×10^{-8}
Dusart[13]	2445999556030	5.69693	1.3010×10^{-9}
Faber and Kadiri [16]	2445999556030	5.69693	2.3643×10^{-9}
Büthe [7]	2.445×10^{12}		1.3765×10^{-9}

'The Error Term in the Prime Number Theorem' [29] by Platt and Trudgian in 2021, uses the explicit version of the argument by Pintz [27], forgoing smoothing, in favour of using Perron's formula, and a bound on $N(\sigma, t)$ due to [22], to get bounds for $\psi(x)$. Their method leads to a version of the prime number theorem where the new error term is roughly the square-root of what was previously known. The main theorem in this paper states that:

Theorem 4. *[29, Theorem 1, page 872] For* $R = 5.573412$*,*

$$|\psi(x) - x| \leq \left(A \left(\frac{\log x}{R} \right)^B \exp\left(-C\sqrt{\frac{\log x}{R}} \right) \right) x$$

and

$$|\psi(x) - x| \leq \varepsilon_0 x$$

for all $\log x \geq X$*, where A,B, X, C, and* ε_0 *are specific values which can be found in Table 2.5.*

The work in [29] improves on bounds of ε_0 given in [16], for larger values of x, in this instance, for $x \geq e^{3000}$.

A table of some of the values of $X, A, B, C, \varepsilon_0$ used in the above theorem is given in the table below :

Table 2.5: Table of values of A, B, C, and ε_0 used in Theorem 4

X	σ	A	B	C	ε_0
1000	0.98	461.9	1.52	1.89	$1.20 \cdot 10^{-5}$
3000	0.98	379.6	1.52	1.89	$4.51 \cdot 10^{-13}$
5000	0.99	713.0	1.51	1.94	$9.77 \cdot 10^{-19}$
7000	0.99	590.1	1.51	1.94	$3.09 \cdot 10^{-23}$
9000	0.99	552.3	1.51	1.94	$4.11 \cdot 10^{-27}$
10000	0.99	535.4	1.51	1.94	$6.78 \cdot 10^{-29}$

'Sharper bounds for the Chebyshev function $\psi(x)$' [17], by Fiori, Kadiri, and Swidinsky gives superior bounds using a refinement of Pintz's argument used in Platt and Trudgian's work in [29]. The significant enhancement in [17] stems from splitting the zeros of the zeta function into additional regions which leads to more elaborate estimates of sums over zeroes.

The article [17] improves the unconditional explicit bounds for the error term in the prime counting function $\psi(x)$ based on the theorem stated below.

Theorem 5. *The following bounds hold for all $x > 2$,*

$$|\psi(x) - x| < 9.22022x(\log x)^{3/2}\exp(-0.8476836\sqrt{\log x}). \tag{2.2}$$

For all $x \geq e^{3000}$

$$|\psi(x) - x| < 4.9678 \cdot 10^{-15}x \tag{2.3}$$

holds.

2.3 Survey on Büthe's results

This thesis deals with the survey of the paper 'Estimating $\pi(x)$ and Related Functions based on Partial RH Assumptions' by Jan Büthe. The main theorem of the paper deals with improving existing bounds of the Chebyshev type for the function $\psi(x)$. Büthe's work provides the current best-known bounds for $\psi(x)$ for x in the middle range $[e^{50}, e^{3000}]$, which provides the primary rationale for studying his research. [7] was published in 2016; however, the paper has not been improved since its publication. A significant reason for this appears to be the fact [7] is not a self-contained paper. This makes it a difficult paper to follow. Many of the results and their proofs required for the proof of the main theorem are scattered throughout other research papers, like [8], [18], [6] and [10]. This thesis aims to provide a report containing all the necessary lemmas, propositions and theorems found in [7]. The proof of all the results in [7] have been verified and expanded upon, which includes filling in missing details and correcting several results. These corrections have been indicated as required throughout the thesis. We have also improved specific results, such as the estimates in Lemma 14 and Lemma 15. Büthe's main theorem bounds $\psi(x)$ in terms of three error terms: $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$, and since his methods work best for the range $[e^{50}, e^{3000}]$, we have computed values for each of $\mathcal{E}_1$,$\mathcal{E}_2$, and $\mathcal{E}_3$ in order to understand the relative sizes of the error in various ranges of x in this interval.

Countless research has been done on bounds for $\psi(x)$, however the unique feature in Büthe's work is the application of a new smoothing function, the Logan function, which gives the current best bounds for $\psi(x)$ for smaller value of x in the range $[e^{50}, e^{3000}]$ even without the additional use of traditional methods such as zero density estimates and bounds for zero-free regions. Therefore, a survey of Büthe's work must begin with a motivation for his choice of smoothing function.

2.3.1 Motivation for choice of the Logan function

In 'Bounds for the Tails of Sharp-Cutoff Filter Kernel' [24] the author attempted to band limit time signals by convolving an arbitrary bounded function with a filter kernel $k(t : \alpha, \beta) \in L_1$, whose Fourier Transform is 1 over the interval $(-\alpha, \alpha)$, and vanishes outside the interval $(-\beta, \beta)$ for $0 < \alpha < \beta < \infty$. Rescaling the kernels to be defined using one variable λ, the collection of kernels $k(t)$ is denoted by $K_\lambda = K(\lambda - 1, \lambda + 1)$ where the Fourier transforms of the kernels satisfy

$$\hat{k}(\omega) = \int_{-\infty}^{\infty} k(t) e^{-i\omega t} dt = \begin{cases} 1 & \text{for } -(\lambda - 1) \leq \omega \leq \lambda - 1 \\ 0 & \text{for } \omega \geq \lambda + 1 \end{cases}, \tag{2.4}$$

for $\lambda > 1$.The objective in [24] was to obtain inequalities for the quantity defined by

$$\mu(\lambda, c) := \inf_{k \in K_\lambda} \int_{|t| > c} k(t) dt. \tag{2.5}$$

We note that for a function g whose Fourier transform vanishes outside $[-1,1]$ and $g(0)=1$, any kernel in L_1 of the form

$$k(t) = g(t)\frac{\sin\lambda t}{\pi t}, \tag{2.6}$$

where $\lambda > 1$, belongs to K_λ. If $B(1)$ denotes the collection of functions g which are restrictions of functions whose generalized Fourier transforms vanish outside $[-1,1]$, to the real line, from equations (2.5) and (2.6) the upper bound shown in the paper can be stated as

$$\mu(\lambda,c) \leq \inf_{\substack{g\in B(1)\\ g(0)=1}} \frac{1}{\pi}\int_{|t|>c} \frac{|g(t)|}{|t|}|\sin(\lambda t)|dt. \tag{2.7}$$

In another of his papers,'Optimal Trunction of the Hilbert Transform Kernel for Bounded High-pass functions' [25], the author B.F. Logan, has shown that for the extremal function

$$g(t) = g(t;c) = \frac{c}{\sinh c}\frac{\sin\sqrt{t^2-c^2}}{\sqrt{t^2-c^2}} \tag{2.8}$$

the identity

$$\inf_{\substack{g\in B(1)\\ g(0)=1}} \frac{1}{\pi}\int_{|t|>c} \frac{|g(t)|}{|t|}dt = \frac{2}{\pi}\log\frac{1+e^{-c}}{1-e^{-c}}, \tag{2.9}$$

holds. The absolutely convergent Fourier series expansion of $|\sin\lambda t|$ is

$$|\sin(\lambda t)| = \frac{2}{\pi} - a_1\cos(2\lambda t) - a_2\cos(4\lambda t) - \dots.$$

Since $\lambda \to \infty$, only the constant term $\frac{2}{\pi}$ will contribute to the integral in the inequality (2.7) which combined with the equality in equation (2.8) provides us with the inequality

$$\mu(\lambda,c) \leq \inf_{\substack{g\in B(1)\\ g(0)=1}} \frac{1}{\pi}\int_{|t|>c} \frac{|g(t)|}{|t|}|\sin(\lambda t)|dt \sim \frac{4}{\pi^2}\log\frac{1-e^{-c}}{1+e^{-c}},$$

for $c > 0$.

In 'Estimating $\pi(x)$ and Related Functions under Partial RH Assumptions', Büthe defines the Logan function as

$$\ell_{c,\varepsilon}(\xi) = \frac{c}{\sinh c} \frac{\sin(\sqrt{(\xi\varepsilon)^2 - c^2})}{\sqrt{(\xi\varepsilon)^2 - c^2}}, \tag{2.10}$$

which is a modified version of the function $g(t)$ defined in equation (2.8). This choice of Logan function allows the author to utilize its Fourier transform to flexibly control the truncation point and the size of the remainder term of the sum over zeros.

2.3.2 Survey Methods and Statement of Results

The major results in this survey are presented in Chapter 3 and Chapter 4. Chapter 3 contains definitions of all functions used in the thesis, along with preparatory lemmas that will be used throughout the rest of the survey. Chapter 4 comprises the major results in 'Estimating $\pi(x)$ and Related Functions for Partial RH Assumptions' and begins with a table which tabulates all the lemmas and propositions in [7], along with relevant details regarding their proofs.

Before starting the discussion on Büthe's work and the scope of this thesis, it is necessary to introduce certain functions:

The Logan function, $\ell_{c,\varepsilon}$ has already been defined in equation (2.10). It is worth noting that the Logan function is actually a real valued function even if intermediate steps are complex. The Fourier transform of the Logan function is denoted by $\eta_{c,\varepsilon}(t)$, where

$$\eta_{c,\varepsilon}(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-it\xi} \ell_{c,\varepsilon}(\xi)\, d\xi.$$

The auxilliary functions are denoted by $\mu_{c,\varepsilon}$ and $\nu_{c,\varepsilon}$ where

$$\mu_{c,\varepsilon}(t) = \begin{cases} -\int_{-\infty}^{t} \eta_{c,\varepsilon}(\tau)d\tau & \text{for } t < 0, \\ \int_{-\infty}^{-t} \eta_{c,\varepsilon}(\tau)d\tau = \int_{t}^{\infty} \eta_{c,\varepsilon}(\tau)d\tau & \text{for } t > 0, \\ 0 & \text{for } t = 0 \end{cases}$$

and

$$\nu_{c,\varepsilon}(t) = \int_{-\infty}^{t} \mu_{c,\varepsilon}(\tau)d\tau.$$

Note that μ_c and ν_c are the abbreviations of $\mu_{c,1}$ and $\nu_{c,1}$ respectively. Büthe constructs a continuous approximation to $\psi(x)$ using the Fourier Transform of the Logan function. For $\lambda_{c,\varepsilon} = \ell_{c,\varepsilon}(i/2)$, the author defines

$$\phi_{x,c,\varepsilon}(t) = \frac{1}{\lambda_{c,\varepsilon}} \left(\chi_{[0,\log x]} e^{(\bullet/2)} \right) * \eta_{c,\varepsilon}(t)$$

and the modified Chebyshev function as

$$\psi_{c,\varepsilon}(x) = \sum_{p^m} \frac{\log p}{p^{m/2}} \phi_{x,c,\varepsilon}(\log p^m).$$

To state the major theorem in Büthe's work, we start by defining $\mu_c^+(\alpha)$, which is any upper bound of $\mu_c(\alpha)$ i.e. $\mu_c(\alpha) < \mu_c^+(\alpha)$.

Theorem 6. *[7, Theorem 1] Let* $0 < \varepsilon < 10^{-3}$*,* $c \geq 3$*,* $x_0 \geq 100$ *and* $\alpha \in [0,1)$*. We define*

$$B_0(x_0) = \frac{\varepsilon e^{-\varepsilon} x_0 |\nu_c(\alpha)|}{2\mu_c^+(\alpha)}.$$

Assume that the inequality $B_0(x_0) > 1$ *holds. The zeroes of the Riemann zeta function are* *denoted by* $s = \beta + i\gamma$ *with* $\beta, \gamma \in \mathbb{R}$. *Then, if* $\beta = \frac{1}{2}$ *holds for* $0 < \gamma \leq c/\varepsilon$, *the inequality*

$$|\psi(x) - x| \leq x \cdot e^{\alpha\varepsilon}(\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3)$$

holds for all $x \geq e^{\alpha\varepsilon}x_0$, *where*

$$\mathcal{E}_1 = e^{2\varepsilon}\log(e^{\varepsilon}x_0)\left[\frac{2\varepsilon|\nu_c(\alpha)|}{\log B_0} + \frac{2.01\varepsilon}{\sqrt{x_0}} + \frac{\log\log(2x_0^2)}{2x_0}\right], \tag{2.11}$$

$$\mathcal{E}_2 = 0.16\frac{1+x_0^{-1}}{\sinh(c)}e^{0.71\sqrt{c\varepsilon}}\log\left(\frac{c}{\varepsilon}\right), \tag{2.12}$$

$$\mathcal{E}_3 = \frac{2}{\sqrt{x_0}}\sum_{0<\gamma\leq c/\varepsilon}\frac{\ell_{c,\varepsilon}(\gamma)}{\gamma} + \frac{2}{x_0}. \tag{2.13}$$

The following table has been reproduced from [7], Table 2 and provides the upper bound for $e^{\alpha\varepsilon}(\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3)$, as defined in Theorem 6. The value δ_0 denotes an upper bound for the term $e^{\alpha\varepsilon}(\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3)$, for $T \leq 2.445 \times 10^{12}$ where $\varepsilon = c/T$.

Table 2.6: Table of computations for δ_0 according to Theorem 6 as presented in [7, Table 2]

$e^{\alpha\varepsilon}x_0$	c	T	α	δ_0
e^{55}	39	8.5×10^{11}	0.1	1.12494×10^{-10}
e^{60}	33	2.445×10^{12}	0.11	1.22147×10^{-11}
e^{65}	33	2.445×10^{12}	0.1	3.57125×10^{-12}
e^{70}	33	2.445×10^{12}	0.09	2.79233×10^{-12}
e^{75}	32	2.445×10^{12}	0.08	2.70358×10^{-12}
e^{80}	33	2.445×10^{12}	0.08	2.61079×10^{-12}
e^{90}	33	2.445×10^{12}	0.07	2.52129×10^{-12}
e^{100}	33	2.445×10^{12}	0.06	2.45229×10^{-12}
e^{500}	33	2.445×10^{12}	0.012	1.99986×10^{-12}
e^{1000}	33	2.445×10^{12}	0.005	1.94751×10^{-12}
e^{2000}	33	2.445×10^{12}	0.003	1.92155×10^{-12}
e^{3000}	33	2.445×10^{12}	0.001	1.91298×10^{-12}
e^{4000}	33	2.445×10^{12}	0.001	1.90866×10^{-12}

In this thesis we have not been able to recover the bounds that are given in the above table. Theorem 6 contains certain errors which we have corrected. The corrected results are presented in the theorem below :

Theorem 7. *Let* $0 < \varepsilon < 10^{-3}$*,* $c \geq 3$*,* $x_0 \geq 100$ *and* $\alpha \in [0,1)$*. We define*

$$B_0(x_0) = \frac{\varepsilon e^{-\varepsilon} x_0 |\nu_c(\alpha)|}{2(\mu_c^+(\alpha))}.$$

Assume that the inequalities

$$B_0(x_0) > 1 \quad \text{and} \quad \frac{\varepsilon e^{-2\varepsilon}|\nu_c(\alpha)|}{2\mu_c(\alpha)} < 1$$

hold. The zeroes of the Riemann zeta function are denoted by $s = \beta + i\gamma$ *with* $\beta, \gamma \in \mathbb{R}$*. Then, if* $\beta = \frac{1}{2}$ *holds for* $0 < \gamma \leq c/\varepsilon$*, the inequality*

$$|\psi(x) - x| \leq x \cdot e^{\alpha\varepsilon}(\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3)$$

holds for all $x \geq e^{\alpha\varepsilon} x_0$*, where*

$$\mathcal{E}_1 = e^{2\varepsilon} \log(e^{\varepsilon} x_0) \left[\frac{2\varepsilon |\nu_c(\alpha)|}{\log B_0} + \frac{2.01\varepsilon}{\sqrt{x_0}} + \frac{\log\log(2x_0^2)}{2x_0} \right] + e^{\alpha\varepsilon} - 1 \tag{2.14}$$

$$\mathcal{E}_2 = 0.16 \frac{1 + 3x_0^{-1}}{\sinh(c)} e^{0.71\sqrt{c\varepsilon}} \left[\log\left(\frac{c}{\pi\varepsilon}\right) \log c + 2\log\left(\frac{c}{\pi\varepsilon}\right) + 18.75\varepsilon \log\left(\frac{c}{\varepsilon}\right) + 1 \right] \tag{2.15}$$

$$\mathcal{E}_3 = 2\left(\frac{1}{\sqrt{x_0}} + \frac{1}{x_0}\right) \sum_{0 < \gamma \leq c/\varepsilon} \frac{\ell_{c,\varepsilon}(\gamma)}{\gamma} + \frac{1.92}{x_0}. \tag{2.16}$$

To compute the numerical values in this thesis we used the upper bound $\mu_c(\alpha) < \mu_c^+(K,\alpha)$. For $\alpha \in (0,1), K \in \mathbb{N}$ and $h = \frac{1-\alpha}{K}$, $\mu_c^+(K,\alpha)$ is defined as

$$\mu_c^+(K,\alpha) = hc \sum_{k=1}^{K} \frac{I_0(c\sqrt{2kh - k^2h^2})}{2\sinh c}.$$

where I_0 is the modified Bessel function of the first kind. Below we present two tables which demonstrate the computations for values of error terms derived from the results in Theorem 7. Table 2.7 presents the values of B_0 and δ_0, the upper bound of $e^{\alpha\varepsilon}x_0(\mathcal{E}_1+\mathcal{E}_2+\mathcal{E}_3)$ for each $e^{\alpha\varepsilon}x_0$ used in Table 2.6.

For each $e^{\alpha\varepsilon}x_0$ given in Table 2.7, and corresponding values of the other parameters, T, α, c, Table 2.8 presents the individual values of $\mathcal{E}_1$,$\mathcal{E}_2$, and $\mathcal{E}_3$, computed using the corrected results in Theorem 7. We note here that the values of δ_0 given in Table 2.6, that

Table 2.7: Table of computations for upper bounds of δ_0 according to Theorem 7

$e^{\alpha\varepsilon}x_0$	c	T	α	δ_0	$B_0(x_0)$
e^{55}	39	8.5×10^{11}	0.1	1.12498×10^{-10}	1.68823×10^{11}
e^{60}	33	2.445×10^{12}	0.11	1.23991×10^{-11}	7.95903×10^{13}
e^{65}	33	2.445×10^{12}	0.1	3.75568×10^{-12}	1.21109×10^{16}
e^{70}	33	2.445×10^{12}	0.09	2.97677×10^{-12}	1.84320×10^{18}
e^{75}	33	2.445×10^{12}	0.08	2.85514×10^{-12}	2.76986×10^{20}
e^{80}	33	2.445×10^{12}	0.08	2.79522×10^{-12}	4.16409×10^{22}
e^{90}	33	2.445×10^{12}	0.07	2.70574×10^{-12}	9.40905×10^{26}
e^{100}	33	2.445×10^{12}	0.06	2.63677×10^{-12}	2.12642×10^{31}
e^{500}	33	2.445×10^{12}	0.012	2.18439×10^{-12}	1.25917×10^{205}
e^{1000}	33	2.445×10^{12}	0.005	2.13205×10^{-12}	1.80069×10^{422}
e^{2000}	33	2.445×10^{12}	0.003	2.10609×10^{-12}	3.56654×10^{856}
e^{3000}	33	2.445×10^{12}	0.001	2.09752×10^{-12}	7.06410×10^{1290}
e^{4000}	33	2.445×10^{12}	0.001	2.09320×10^{-12}	1.39167×10^{1725}

we obtain from [7, Table 2] are incorrect. The correct values for δ_0 have been computed in this thesis and presented in Table 2.7. The values of δ_0 at e^{55} and e^{60} are 0.003% and 1.4% worse respectively . All other values for δ_0 given in Table 2.7 are between 4.9% to 8.9% worse.

We now present an itemized list of all the errors that have been found in [7]:

- Limits of integration in $M_{x,c,\varepsilon}(t)$ in Proposition 1 (page 2484) are incorrect.
- The first line of page 2486 should be $\log x > \frac{2}{|\log\varepsilon|}$ instead of $x > 2/|\log(\varepsilon)|$.
- The first term of the penultimate equation in page 2487 should be $2|\Delta(0)|\int_{\varepsilon}^{\infty}\frac{e^{-\frac{t}{2}}}{1-e^{-2t}}dt$ in place of $2|\Delta(0)|\int_{\varepsilon}^{\infty}\frac{e^{-\frac{t}{2}}}{1-e^{-t}}dt$.

Table 2.8: Table of computations for $\mathcal{E}_1, \mathcal{E}_2$, and $\mathcal{E}_3$ according to Theorem 7

$e^{\alpha\varepsilon}x_0$	$\mathcal{E}_1$	$\mathcal{E}_2$	$\mathcal{E}_3$
e^{55}	9.14105×10^{-12}	5.54650×10^{-16}	1.19151×10^{-10} *
e^{60}	2.85861×10^{-12}	2.25860×10^{-13}	1.06037×10^{-11}*
e^{65}	2.76522×10^{-12} *	2.25860×10^{-13}	8.70409×10^{-13}
e^{70}	2.68815×10^{-12} *	2.25860×10^{-13}	7.14475×10^{-14}
e^{75}	2.58394×10^{-12} *	2.25860×10^{-13}	5.86477×10^{-15}
e^{80}	2.56894×10^{-12} *	2.25860×10^{-13}	4.81409×10^{-16}
e^{90}	2.47988×10^{-12} *	2.25860×10^{-13}	3.24371×10^{-18}
e^{100}	2.41091×20^{12} *	2.25860×10^{-13}	2.18559×10^{-20}
e^{500}	1.95853×10^{-12} *	2.25860×10^{-13}	3.02464×10^{-107}
e^{1000}	1.90619×10^{-12} *	2.25860×10^{-13}	8.07334×10^{-216}
e^{2000}	1.88023×10^{-12} *	2.25860×10^{-13}	5.75191×10^{-433}
e^{3000}	1.87166×10^{-12} *	2.25860×10^{-13}	4.09799×10^{-650}
e^{4000}	1.86734×10^{-12} *	2.25860×10^{-13}	2.91964×10^{-867}

where * indicates the dominant term for each $e^{\alpha\varepsilon}x_0$.

- The statement of Lemma 2 is incorrect. We prove that

$$\sum_{|\Im(\rho)|>\frac{c}{\varepsilon}} \frac{\left|\ell_{c,\varepsilon}\left(\frac{\rho}{i}-\frac{1}{2i}\right)\right|}{|\Im(\rho)|} \leq 0.32\frac{e^{0.71\sqrt{ce}}}{\sinh c}\left[\log\left(\frac{c}{\pi\varepsilon}\right)\log c + 2\log\left(\frac{c}{\pi\varepsilon}\right) + 18.75\varepsilon\log\left(\frac{c}{\varepsilon}\right)+1\right],$$

instead of

$$\sum_{|\Im(\rho)|>\frac{c}{\varepsilon}} \frac{\left|\ell_{c,\varepsilon}\left(\frac{\rho}{i}-\frac{1}{2i}\right)\right|}{|\Im(\rho)|} \leq 0.32\frac{e^{0.71\sqrt{ce}}}{\sinh c}\log(3c)\log\left(\frac{c}{\varepsilon}\right).$$

The proof of Lemma 2 given in [10, Lemma 1.2, page 27], which is a more flexible version of the proof in [18, Lemma 2.4] has an error in the calculation of $\int_c^\infty \frac{\log\left(\frac{t+c}{2\pi\varepsilon}\right)}{(t+c)t}dt$ and contains the term $\frac{\log\left(\frac{2c}{2\pi}\right)}{2\pi c}$ instead of $\frac{\log\left(\frac{2c}{2\pi\varepsilon}\right)}{2\pi c}$.

- The term $\log(3c)\log\left(\frac{c}{e}\right)$ in equation 4.1 of Proposition 3, has been replaced by $\log\left(\frac{c}{\pi\varepsilon}\right)\log c + 2\log\left(\frac{c}{\pi\varepsilon}\right) + \log\left(\frac{c}{\varepsilon}\right) + 1$.

- Lemma 1, equation 3.7 in page 2484 should be $|\Delta(t)| \leq \frac{1}{2}e^{\frac{\varepsilon}{2}}\sqrt{x}$ instead of $|\Delta(t)| \leq$

$\frac{1}{2}e^{\varepsilon}\sqrt{x}$.

- The expression for $\mathcal{E}_1$ given in equation (2.11) is incorrect as it is missing the term $e^{\alpha\varepsilon}-1$. the correct expression for $\mathcal{E}_1$ is given in (2.14).

- The expression for $\mathcal{E}_2$ given in equation (2.12) is incorrect. The term $\left[\log\left(\frac{c}{\pi\varepsilon}\right)\log c+2\log\left(\frac{c}{\pi\varepsilon}\right)+18.75\varepsilon\log\left(\frac{c}{\varepsilon}\right)+1\right]$ in the correct version given in equation (2.15) comes from a correction in Proposition 4. Note that we have the term $1+3x_0^{-1}$ instead of $1+x_0^{-1}$. The extra $2x_0^{-1}$ arises from bounding the constant term $\sum_{\rho}\left|a_{c,\varepsilon}(\rho)\frac{1}{\rho}\right|$ which appears not to be accounted for in the proof of Theorem 7 given in [7, Theorem 1]. In addition the version of $\mathcal{E}_2$ given in [7] appears to be missing $\log(3c)$ which comes from equation 4.1 of Proposition 3 given in [7].

- The expression for $\mathcal{E}_3$ given in equation (2.16) is slightly different from (2.13). Note that we obtain $2\left(\frac{1}{\sqrt{x_0}}+\frac{1}{x}\right)$ instead of $\frac{2}{\sqrt{x_0}}$. The extra $\frac{2}{x_0}$ arises from the constant term $\sum_{\rho}\left|a_{c,\varepsilon}(\rho)\frac{1}{\rho}\right|$ which appears not to be accounted for in the proof of Theorem 7 given in [7, Theorem 1].

Chapter 3

Definition and Properties of the Logan Function, its Fourier Transform and Other Related Functions

This Chapter contains all introductory results regarding the Logan Function, its Fourier Transformation and other preparatory lemmas. We begin by listing the required definitions:

3.1 Definitions

In this section, we define all of the functions that occur in Büthe's article.

Definition 1. (Logan Function)

For $c, \varepsilon > 0$, the Logan function is defined as follows:

$$\ell_{c,\varepsilon}(\xi) = \begin{cases} \frac{c}{\sinh c} \frac{\sin(\sqrt{(\xi\varepsilon)^2 - c^2})}{\sqrt{(\xi\varepsilon)^2 - c^2}} & \text{for } |\xi| \geq \frac{c}{\varepsilon} \\ \frac{c}{\sinh c} \frac{\sinh(\sqrt{(c^2 - \xi\varepsilon)^2})}{\sqrt{(c^2 - \xi\varepsilon)^2}} & \text{for } 0 \leq |\xi| < \frac{c}{\varepsilon}. \end{cases} \tag{3.1}$$

In [7], Büthe only defines $\ell_{c,\varepsilon}(z) = \frac{c}{\sinh c} \frac{\sin(\sqrt{(\xi\varepsilon)^2 - c^2})}{\sqrt{(\xi\varepsilon)^2 - c^2}}$ where $\sqrt{z}$ is defined by the principal branch of the logarithm. However for $0 \leq \xi < \frac{c}{\varepsilon}$, we notice that $((\xi\varepsilon)^2 - c^2) < 0$, and we show in Lemma 1(ii) that $\ell_{c,\varepsilon}(\xi)$ equals the expression in the second line of equation (3.1). The following two figures show the graphs for $\ell_{c,\varepsilon}$ for the ranges $0 \leq \xi < \frac{c}{\varepsilon}$ and $\xi \geq \frac{c}{\varepsilon}$ respectively.

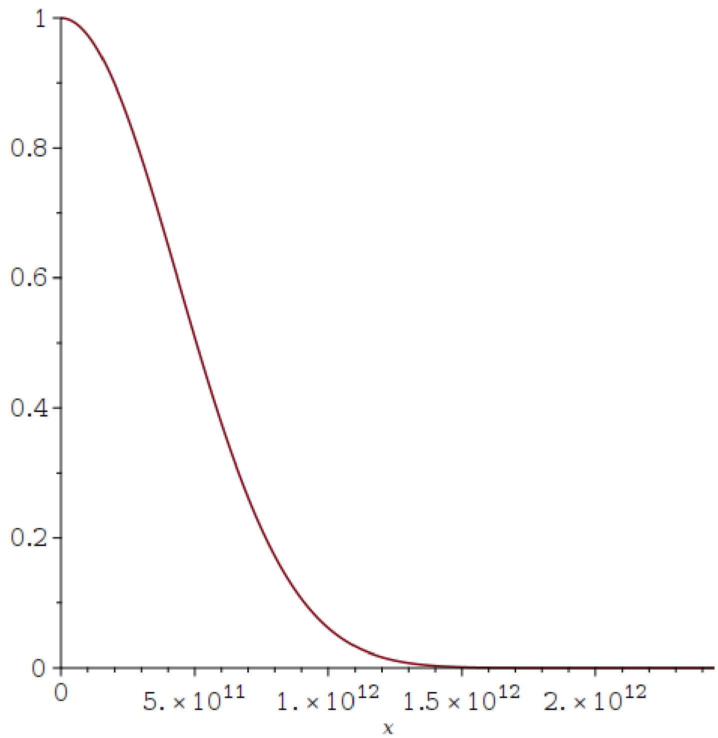


Figure 3.1: Graph of $\ell_{c,\varepsilon}(\xi)$ for $0 \leq \xi < \frac{c}{\varepsilon}$, where $c = 33$, $\varepsilon = \frac{c}{T}$ and $T = 2.445 \times 10^{12}$

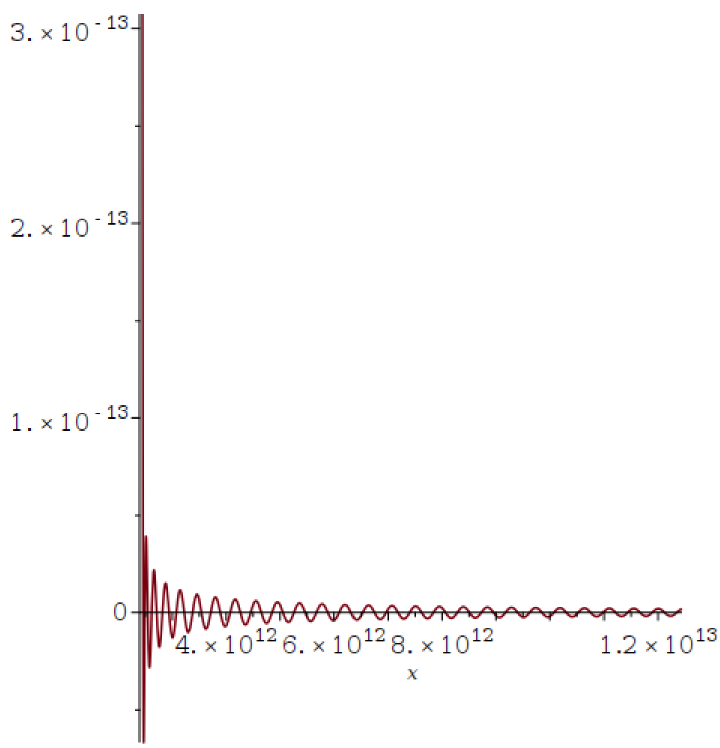


Figure 3.2: Graph of $\ell_{c,\varepsilon}(\xi)$ for $\frac{c}{\varepsilon} \leq \xi < \frac{c}{\varepsilon} + 10^{13}$, where $c = 33$, $\varepsilon = \frac{c}{T}$ and $T = 2.445 \times 10^{12}$

Definition 2. (Fourier transform of the Logan Function)

The Fourier transform of the Logan transform is given by :

$$\eta_{c,\varepsilon}(t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-it\xi}\ell_{c,\varepsilon}(\xi)\,d\xi. \tag{3.2}$$

An alternative definition for $\eta_{c,\varepsilon}(t)$ is be given by

$$\eta_{c,\varepsilon}(t) = \chi^*_{[-\varepsilon,\varepsilon]}(t)\frac{c}{2\varepsilon\sinh c}I_0\left(c\sqrt{1-(t/\varepsilon)^2}\right) \tag{3.3}$$

where I_0 is the modified Bessel function of the first kind defined by

$$I_0(z) = \sum_{n=0}^{\infty}\frac{\left(\frac{z}{2}\right)^{2n}}{(n!)^2}, \tag{3.4}$$

and

$$\chi^*_{[-\varepsilon,\varepsilon]}(t) = \begin{cases} 1 & \text{if } |t| < \varepsilon, \\ 0 & \text{if } |t| > \varepsilon, \\ \frac{1}{2} & \text{if } t = \pm\varepsilon. \end{cases}$$

Observe that $\eta_{c,\varepsilon}(t)$ is discontinuous at $t = \pm\varepsilon$, since $\lim_{t\to\varepsilon^+}\eta_{c,\varepsilon}(t) = 0$ and $\lim_{t\to\varepsilon^-}\eta_{c,\varepsilon}(t) = \frac{c}{2\varepsilon\sinh c}$, as $I_0(0) = 1$.

We prove later in Lemma 2(ii) that the definitions provided in equation (3.2) and equation (3.3) are equivalent. We also prove in Lemma 2 that $\eta_{c,\varepsilon}(t) = \eta_{c,\varepsilon}(-t)$, and thus it is an even function.

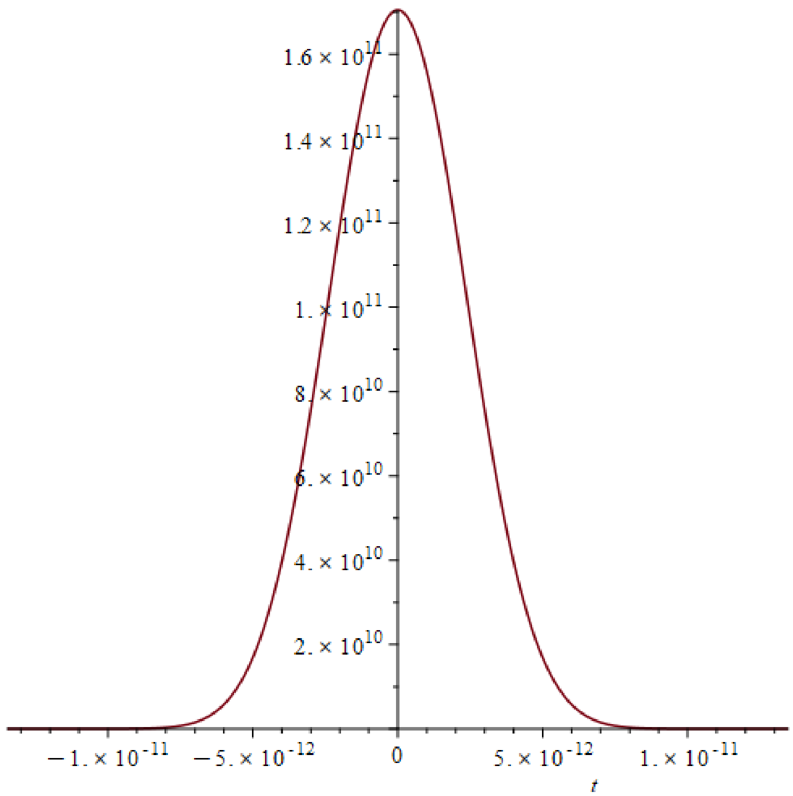


Figure 3.3: Graph of $\eta_{c,\varepsilon}(t)$ for $t \in (-\varepsilon, \varepsilon)$, where $c = 33$, $\varepsilon = \frac{c}{T}$ and $T = 2.445 \times 10^{12}$

Definition 3. We define $\lambda_{c,\varepsilon}$ as follows :

$$\lambda_{c,\varepsilon} = \ell_{c,\varepsilon}(i/2). \tag{3.5}$$

Therefore, expanding out the definition of $\ell_{c,\varepsilon}$ in the above equation gives us

$$\lambda_{c,\varepsilon} = \frac{c}{\sinh c} \frac{\sinh \sqrt{\frac{\varepsilon^2}{4} + c^2}}{\sqrt{\frac{\varepsilon^2}{4} + c^2}}$$

which indicates that $\lambda_{c,\varepsilon}$ is real, since $c, \varepsilon > 0$.

$\lambda_{c,\varepsilon}$ is used in the definition of the following function.

Definition 4. The function $\phi_{x,c,\varepsilon}(t)$ is defined as

$$\phi_{x,c,\varepsilon}(t) = \frac{1}{\lambda_{c,\varepsilon}}((\chi^{*}_{[0,\log x]}(\exp(./2))) * \eta_{c,\varepsilon})(t) \tag{3.6}$$

where $*$ indicates the convolution function.

Therefore, expanding the convolution, equation (3.6) can be re-written as:

$$\phi_{x,c,\varepsilon}(t) = \frac{1}{\lambda_{c,\varepsilon}} \int_{0}^{\log x} e^{\frac{\tau}{2}} \eta_{c,\varepsilon}(t-\tau)d\tau. \tag{3.7}$$

Definition 5. The following is the definition of the modified Chebyshev function:

$$\psi_{c,\varepsilon}(x) = \sum_{p^m} \frac{\log p}{p^{\frac{m}{2}}} \phi_{x,c,\varepsilon}(m \log p) = \sum_{p^m} \frac{\log p}{p^{\frac{m}{2}}} \phi_{x,c,\varepsilon}(\log p^m). \tag{3.8}$$

Note that $\psi_{c,\varepsilon}(x)$ is a smoothed version of $\psi(x)$. The following functions occur frequently in this thesis :

Definition 6. Let $c, \varepsilon > 0$, $x \geq 2$, $t \in \mathbb{R}$, and $s \in \mathbb{C}$, we can define the following functions:

$$f_x(t) = \chi^{*}_{[0,\log x]}(t) \exp(t/2). \tag{3.9}$$

$$a_{c,\varepsilon}(s) = \frac{1}{\lambda_{c,\varepsilon}} \ell_{c,\varepsilon}\left(\frac{s}{i} - \frac{1}{2i}\right) \tag{3.10}$$

$$\Delta(t) = \phi_{x,c,\varepsilon}(t) - f_x(t). \tag{3.11}$$

Related to the function $\eta_{c,\varepsilon}(\tau)$ are the functions $\mu_{c,\varepsilon}(t)$ and $\nu_{c,\varepsilon}(t)$. These two functions appear in the statement of the main theorem, Theorem 6 and are defined as follows :

Definition 7.

$$\mu_{c,\varepsilon}(t) = \begin{cases} -\int_{-\infty}^{t} \eta_{c,\varepsilon}(\tau)d\tau & \text{for } t < 0, \\ \int_{-\infty}^{-t} \eta_{c,\varepsilon}(\tau)d\tau = \int_{t}^{\infty} \eta_{c,\varepsilon}(\tau)d\tau & \text{for } t > 0, \\ 0 & \text{for } t = 0. \end{cases} \tag{3.12}$$

From the definition we can note that $\mu_{c,\varepsilon}(t) = -\mu_{c,\varepsilon}(-t)$, which implies that $\mu_{c,\varepsilon}(t)$ is an odd function. We note here that $\mu_{c,\varepsilon}(t) = \mu_{c,\varepsilon}(-t)$, and thus $\mu_{c,\varepsilon}$ is an odd function.

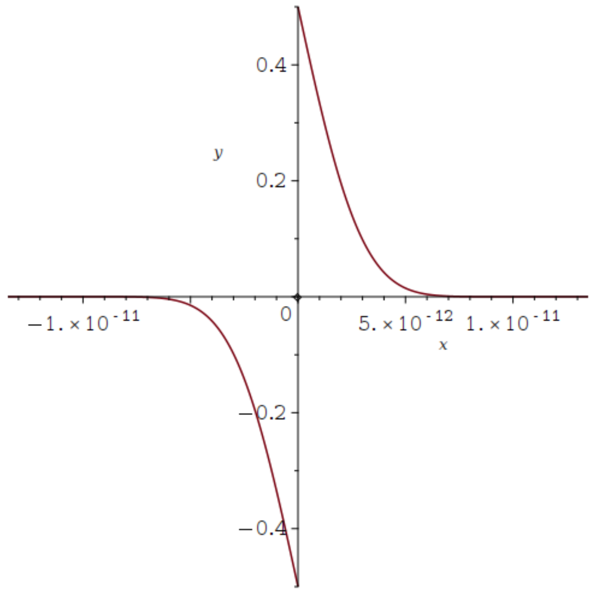


Figure 3.4: Graph of $\mu_{c,\varepsilon}(t)$, with $c = 33$, $\varepsilon = \frac{c}{T}$ and $T = 2.445 \times 10^{12}$

Definition 8.

$$\nu_{c,\varepsilon}(t) = \int_{-\infty}^{t} \mu_{c,\varepsilon}(\tau)d\tau. = \begin{cases} -\int_{-\varepsilon}^{t}(t-\tau)\eta_{c,\varepsilon}(\tau)d\tau & \text{for } t < 0, \\ \int_{t}^{\varepsilon}(t-\tau)\eta_{c,\varepsilon}(\tau)d\tau & \text{for } t > 0, \\ -\int_{0}^{\varepsilon}\tau\eta_{c,\varepsilon}(\tau)d\tau & \text{for } t = 0. \end{cases} \tag{3.13}$$

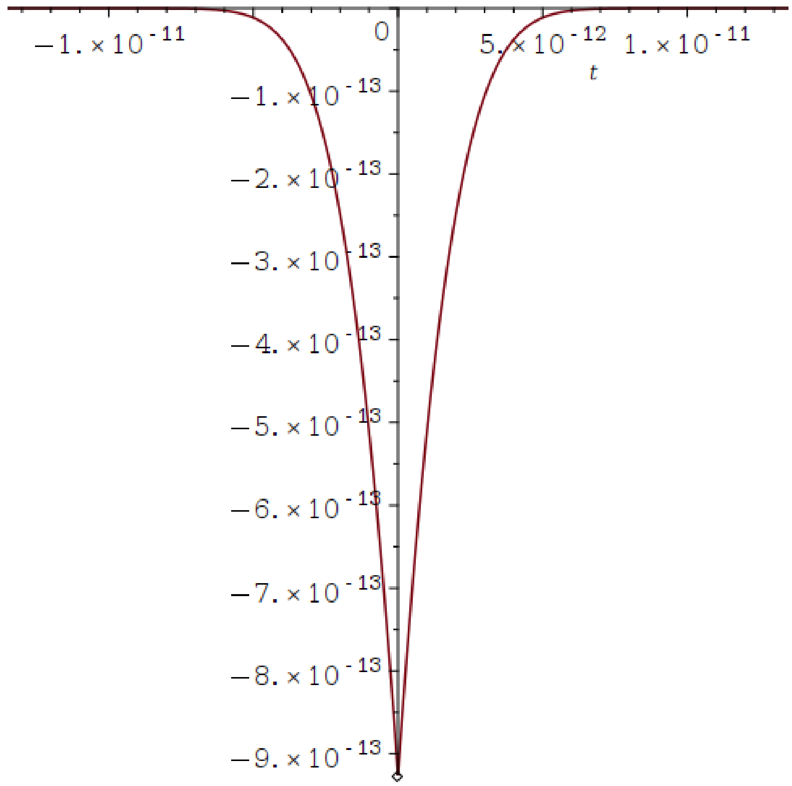


Figure 3.5: Graph of $\nu_{c,\varepsilon}(t)$ for $t \in (-\varepsilon, \varepsilon)$, with $c = 33$, $\varepsilon = \frac{c}{T}$ and $T = 2.445 \times 10^{12}$

3.2 Properties of the functions from Büthe's article

In the following lemma we record the properties of the Logan function that are used throughout this thesis.

Lemma 1. *(i) Let $c, \varepsilon > 0$, then*

$$\ell_{c,\varepsilon}(\xi) = \ell_{c,1}(\varepsilon\xi). \tag{3.14}$$

(ii) For $0 \leq \xi < \frac{c}{\varepsilon}$,

$$\frac{\sinh\left(\sqrt{c^2 - (\xi\varepsilon)^2}\right)}{\left(\sqrt{c^2 - (\xi\varepsilon)^2}\right)} = \frac{\sin\left(\sqrt{(\xi\varepsilon)^2 - c^2}\right)}{\left(\sqrt{(\xi\varepsilon)^2 - c^2}\right)},$$

where $\sqrt{z}$ is defined by the principal branch of the logarithm. This shows that case 2 *in equation* (3.1) *agrees with Büthe's definition of* $\ell_{c,\varepsilon}$ *given in [7].*

(iii) *We have*

$$\ell_{c,\varepsilon}(x) = \int_{\mathbb{R}} \eta_{c,\varepsilon}(\omega) e^{i\omega x} d\omega, \tag{3.15}$$

and

$$\ell_{c,\varepsilon}(0) = \int_{\mathbb{R}} \eta_{c,\varepsilon}(\omega) d\omega = 1. \tag{3.16}$$

(iv) *For* $c > 0$*, and* $z \in \mathbb{C}$*, with* $z = x + iy$ *such that* $|x| \geq c$ *and* $|y| < \frac{\varepsilon}{2}$*, we can claim*

$$|\ell_{c,1}(z)| \leq \frac{c}{\sinh(c)} e^{0.71\sqrt{c\varepsilon}} \min\left\{1, \frac{1}{|x| - c}\right\}. \tag{3.17}$$

(v) *The function* $\ell_{c,\varepsilon}$ *decreases monotonically in* $[0, \frac{c}{\varepsilon}]$.

Proof. (i) By equation (3.1), it follows that

$$\ell_{c,1}(\xi\varepsilon) = \frac{c}{\sinh c} \frac{\sin\left(\sqrt{(\xi\varepsilon \cdot 1)^2 - c^2}\right)}{\sqrt{(\xi\varepsilon \cdot 1)^2 - c^2}} = \ell_{c,\varepsilon}(\xi).$$

(ii) For $0 \leq \xi < \frac{c}{\varepsilon}$, we consider $\sqrt{z} := \exp(\frac{1}{2}\log(z))$ where $\log(z)$ is the principal branch of logarithm i.e. $\log(z) = \ln|z| + i\arg(z)$ with $\arg(z) \in (-\pi, \pi]$. Thus, we can write the term $\sqrt{c^2 - (\xi\varepsilon)^2}$ as

$$\begin{aligned}
\sqrt{c^2 - (\xi\varepsilon)^2} &= \exp\left(\frac{1}{2}\left(\ln|(\xi\varepsilon)^2 - c^2| + i\arg\left((\xi\varepsilon)^2 - c^2\right)\right)\right) \\
&= \exp\left(\frac{1}{2}\left(\ln|(\xi\varepsilon)^2 - c^2| + i\pi\right)\right) \\
&= |(\xi\varepsilon)^2 - c^2|^{\frac{1}{2}} e^{i\frac{\pi}{2}} = i|(\xi\varepsilon)^2 - c^2|^{\frac{1}{2}}.
\end{aligned}$$

Therefore the definition of $\ell_{c,\varepsilon}$ turns out to be

$$\ell_{c,\varepsilon}(\xi) = \frac{c}{\sinh c} \frac{\sin\left(i|(\xi\varepsilon)^2 - c^2|^{\frac{1}{2}}\right)}{i|(\xi\varepsilon)^2 - c^2|^{\frac{1}{2}}} = \frac{c}{\sinh c} \frac{\sinh\left(\sqrt{|(\xi\varepsilon)^2 - c^2|}\right)}{|\sqrt{(\xi\varepsilon)^2 - c^2}|}$$

which proves the identity in definition 1, equation (3.1).

(iii) From the definition of $\eta_{c,\varepsilon}$ as the Fourier Transform of $\ell_{c,\varepsilon}(\xi)$, we can write

$$\eta_{c,\varepsilon}(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-it\xi} \ell_{c,\varepsilon}(\xi) d\xi = \frac{1}{\sqrt{2\pi}} \hat{\ell_{c,\varepsilon}}(t),$$

using the Fourier transform of the unitary form in angular frequency. and thus

$$\hat{\ell_{c,\varepsilon}}(t) = \sqrt{2\pi}\eta_{c,\varepsilon}(t).$$

By Fourier inversion of the unitary form in angular frequency, we get

$$\ell_{c,\varepsilon}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{\ell_{c,\varepsilon}}(\omega) e^{i\omega x} d\omega = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \sqrt{2\pi}\eta_{c,\varepsilon}(\omega) e^{i\omega x} d\omega = \int_{\mathbb{R}} \eta_{c,\varepsilon}(\omega) e^{i\omega x} d\omega.$$

We have the following identity:

$$\ell_{c,\varepsilon}(x) = \int_{\mathbb{R}} \eta_{c,\varepsilon}(\omega) e^{i\omega x} d\omega.$$

Letting $x = 0$ in the above equation we obtain the following identity

$$\ell_{c,\varepsilon}(0) = \int_{\mathbb{R}} \eta_{c,\varepsilon}(\omega) d\omega.$$

From definition 1, equation 3.1, $\ell_{c,\varepsilon}$ is defined as

$$\ell_{c,\varepsilon}(\xi) = \frac{c}{\sinh c} \frac{\sin\left(\sqrt{(\xi\varepsilon)^2 - c^2}\right)}{\sqrt{(\xi\varepsilon)^2 - c^2}}.$$

We note that in the above definition $\sqrt{z} := \exp(\frac{1}{2}\mathrm{Log}(z))$ where $\mathrm{Log}(z)$ is the principal branch of logarithm. Using this definition,we get,

$$\begin{aligned}\ell_{c,\varepsilon}(0) &= \frac{c}{\sinh c}\cdot\frac{\sin\sqrt{-c^2}}{\sqrt{-c^2}} = \frac{c}{\sinh c}\cdot\frac{\sin(ic)}{ic} = \frac{c}{\sinh c}\cdot\frac{e^{-c}-e^{c}}{2i^2c}\\ &= \frac{c}{\sinh c}\cdot\frac{\sinh(c)}{c} = 1.\end{aligned}$$

(iv) This bound was first established in [10]. We reproduce his proof here.

Since $\ell_{c,1}$ is even and interchanges with complex conjugation, i.e. $\overline{\ell_{c,1}(z)} = \ell_{c,\varepsilon}(\bar{z})$, we can restrict ourselves to the case where $x, y \geq 0$. We begin by investigating the quotient $\frac{\sin(\sqrt{z^2-c^2})}{\sqrt{z^2-c^2}}$.

Firstly, it is true that

$$\left|\frac{\sin\omega}{\omega}\right| = \frac{1}{2}\left|\int_{-1}^{1} e^{i\omega t}dt\right| \leq e^{|\Im(\omega)|}. \tag{3.18}$$

On the other hand, it is also true that,

$$\left|\frac{\sin\omega}{\omega}\right| \leq \frac{e^{|\Im(\omega)|}}{|\omega|}. \tag{3.19}$$

Therefore, we can now estimate $|\Im(\sqrt{z^2-c^2})|$. To do this, we first show that for any $y > 0$ the mapping $x \mapsto \Im(\sqrt{z^2-c^2})$ is monotonically decreasing in the region $(0,\infty)$ if we choose the holomorphic root in $(0,\infty)\times i(0,\infty)$ with a positive imaginary part. Here we take the principal branch of logarithm to define the square root. This is true for $y = 0$, thus, we assume $y > 0$. Now, let $(a+ib)^2 = z^2 - c^2$, and we get that $b > 0$ since $y > 0$. Equating the real and imaginary parts on the left and right hand side of $(a+ib)^2 = z^2 - c^2$ we obtain the equation

$$\frac{x^2y^2}{b^2} - b^2 = x^2 - y^2 - c^2. \tag{3.20}$$

So b tends to $\sqrt{y^2+c^2} > y$ if $x \to 0$ and tends to y if $x \to \infty$. Furthermore, b is real analytic as a function of x and has no local maximum in $(0,\infty)$, since this would imply $b^2 = y^2$ which contradicts equation (3.20). Therefore, for $x \geq c$, and $0 \leq y \leq \frac{\varepsilon}{2}$, and using the inequality that $\arctan(t) \leq t$ for $t \geq 0$ and the monotonicity of sin in the interval $\left[0, \frac{\pi}{2}\right]$ we get the estimate,

$$0 \leq \Im(\sqrt{z^2 - c^2}) \leq \Im(\sqrt{(c+iy)^2 - c^2}) = \sqrt{|\, 2ciy - y^2 \,|} \sin\left(\frac{\pi}{4} + \frac{1}{2}\arctan\left(\frac{y}{2c}\right)\right)$$
$$\leq \sqrt{(2c+\varepsilon)y} \sin\left(\frac{\pi}{4} + \frac{1}{2000}\right) \leq 0.71\sqrt{2cy}. \tag{3.21}$$

Using this along with equation (3.18) , we get the inequality,

$$|\, \ell_{c,1}(z) \,| \leq \frac{e^{0.71\sqrt{c\varepsilon}}}{\sinh c}$$

for $x \geq c$. Letting $x \geq c+1$, for $y^2 \leq 2c$, it is obvious that

$$|\sqrt{z^2 - c^2}|^2 \geq |\Re(z^2 - c^2)| = (x-c)^2 + 2c(x-c) - y^2 \geq (x-c)^2. \tag{3.22}$$

Thus using equation (3.19), we get the second part,

$$|\ell_{c,1}(z)| \leq \frac{c}{\sinh c} e^{0.71\sqrt{c\varepsilon}} \frac{1}{|x| - c}$$

of the inequality (3.17).

(v) To prove that $\ell_{c,\varepsilon}$ is decreasing, it suffices to show that the derivative of $\ell_{c,\varepsilon}$ in the range $[0, \frac{c}{e}]$ is always negative. Since from definition 1, equation (3.1), we know that

$$\ell_{c,\varepsilon}(\xi) = \frac{c}{\sinh c} \frac{\sinh\left(\sqrt{c^2 - (\xi\varepsilon)^2}\right)}{\sqrt{c^2 - (\xi\varepsilon)^2}},$$

therefore,

$$\ell'_{c,\varepsilon}(\xi) = \frac{c}{\sinh(c)} \frac{-\cosh(\sqrt{c^2-(\xi\varepsilon)^2})(\varepsilon^2\xi) + \frac{\sinh(\sqrt{c^2-(\xi\varepsilon)^2})}{(\sqrt{c^2-(\xi\varepsilon)^2})}\varepsilon^2\xi}{c^2-(\xi\varepsilon)^2}.$$

We now want to show that $\ell'_{c,\varepsilon}(\xi) < 0$ for all $\xi \in [0, \frac{c}{\varepsilon}]$ i.e. we want to show

$$\frac{-\cosh(\sqrt{c^2-(\xi\varepsilon)^2-c^2})\varepsilon^2\xi + \frac{\sinh(\sqrt{c^2-(\xi\varepsilon)^2})}{(\sqrt{c^2-(\xi\varepsilon)^2})}\varepsilon^2\xi}{c^2-(\xi\varepsilon)^2} < 0,$$

which is true if and only if

$$\cosh(\sqrt{c^2-(\xi\varepsilon)^2}) > \frac{\sinh(\sqrt{c^2-(\xi\varepsilon)^2})}{(\sqrt{c^2-(\xi\varepsilon)^2})}.$$

Now, considering the Taylor series for $\sinh t$ and $\cosh t$ we can write

$$\frac{\sinh(t)}{t} = \sum_{n=0}^{\infty} \frac{\frac{t^{2n+1}}{(2n+1)!}}{t} = \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n+1)!} < \sum_{n=0}^{\infty} \frac{t^{2n}}{2n!} = \cosh(t).$$

Therefore, we prove the inequality

$$\frac{\sinh(\sqrt{c^2-(\xi\varepsilon)^2})}{(\sqrt{c^2-(\xi\varepsilon)^2})} < \cosh(\sqrt{c^2-(\xi\varepsilon)^2}),$$

which finishes the proof of Lemma 8(v).

□

In the next lemma we record the properties of the Fourier Transform of the Logan function , $\eta_{c,\varepsilon}(t)$.

Lemma 2. *(i) Since $\eta_{c,\varepsilon}$ is compactly supported and $*$ is convolution, then for $t \in \mathbb{R}$*

$$(e^{(\cdot/2)} * \eta_{c,\varepsilon})(t) = \lambda_{c,\varepsilon} e^{\frac{t}{2}}. \tag{3.23}$$

(ii) $\eta_{c,\varepsilon}$ *can be rewritten as*

$$\eta_{c,\varepsilon}(t) = \chi^*_{[-\varepsilon,\varepsilon]}(t)\frac{c}{2\varepsilon\sinh(c)}I_0(c\sqrt{1-(t/\varepsilon)^2}),$$

where $I_0(t) = \sum_{n=0}^{\infty}(t/2)^{2n}/(n!)^2$ *denotes the* $0-th$ *Bessel function of the first kind*

(iii) For $t \in \mathbb{R}$, $\eta_{c,\varepsilon}$ *is an even function.*

Proof. (i) Expanding the convolution in $(e^{(\cdot/2)} * \eta_{c,\varepsilon})(t)$, we can write

$$\begin{aligned}(e^{(\cdot/2)} * \eta_{c,\varepsilon})(t) &= \int_{-\infty}^{\infty}\eta_{c,\varepsilon}(\tau)e^{\frac{t-\tau}{2}}d\tau = e^{t/2}\int_{-\infty}^{\infty}\eta_{c,\varepsilon}(\tau)e^{\frac{-\tau}{2}}d\tau \\ &= e^{t/2}\int_{-\infty}^{\infty}\eta_{c,\varepsilon}(\tau)e^{i\frac{i}{2}\tau}d\tau = e^{t/2}\ell_{c,\varepsilon}(i/2)\end{aligned}$$

by Lemma 1(i). Since $\lambda_{c,\varepsilon} = \ell_{c,\varepsilon}(i/2)$, we have established that

$$(e^{(\cdot/2)} * \eta_{c,\varepsilon})(t) = e^{\frac{t}{2}}\lambda_{c,\varepsilon}. \tag{3.24}$$

(ii) The proof of this lemma can be found in [18, Proposition 4.1]. We recreate the proof here, adding relevant details as needed. We begin the proof by recalling the definition of $\eta_{c,\varepsilon}(t)$ given by

$$\eta_{c,\varepsilon}(t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-it\xi}\ell_{c,\varepsilon}(\xi)d\xi,$$

and equation (3.14) of Lemma 1(i) which states that $\ell_{c,\varepsilon}(\xi) = \ell_{c,1}(\varepsilon\xi)$. Thus, we can now write

$$\eta_{c,\varepsilon}(t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-it(\xi\varepsilon)\frac{1}{\varepsilon}}\ell_{c,1}(\xi\varepsilon)d(\xi).$$

Substituting $\xi\varepsilon = y$, we get $\frac{dy}{\varepsilon} = d\xi$ and

$$\eta_{c,\varepsilon}(t) = \frac{1}{2\pi}\frac{1}{\varepsilon}\int_{\mathbb{R}} e^{-i\frac{t}{\varepsilon}y}\ell_{c,1}(y)dy = \frac{1}{\varepsilon}\eta_{c,1}\left(\frac{t}{\varepsilon}\right). \tag{3.25}$$

To continue the proof, we assume that

$$i_c(t) = \begin{cases} \frac{c}{2\sinh c} I_0(c\sqrt{1-(t)^2}) & \text{for } |t| < 1 \\ 0 & \text{for } |t| > 1 \end{cases} \tag{3.26}$$

We now want to show that $i_c(t) = \eta_{c,1}(t)$ which implies $\hat{i}_c = \ell_{c,1}$.

Since $\ell_{c,1} \in L^2(\mathbb{R})$, the Fourier Inversion formula is applicable. Let $\mathcal{F}^{-1}(\ell_{c,1})$ be the inverse Fourier transform of $\ell_{c,1}$. It is sufficient to prove that i_c is the inverse Fourier transform of $\ell_{c,1}$ i.e. $i_c(t) = \left(\mathcal{F}^{-1}(\ell_{c,1})\right)(t)$. Since $\ell_{c,1}$ is an even function its inverse Fourier Transform is also even and thus it is sufficient to prove the identity only for negative values of t. Then

$$\begin{aligned} \left(\mathcal{F}^{-1}(\ell_{c,1})\right)(t) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-it\xi}\ell_c(\xi)d\xi \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-it\xi}\frac{c}{\sinh c}\frac{\sin(\sqrt{(\xi)^2-c^2})}{\sqrt{(\xi)^2-c^2}}d\xi \\ &= \frac{c}{2\pi\sinh c}\int_{\mathbb{R}} e^{-it\xi}\frac{\sin(\sqrt{(\xi)^2-c^2})}{\sqrt{(\xi)^2-c^2}}d\xi. \end{aligned}$$

Let $\xi = z + \frac{c^2}{4z}$, then $d\xi = \left(1 - \frac{c^2}{4z^2}\right)dz$ and $\xi^2 - c^2 = \left(z - \frac{c^2}{4z}\right)^2$. Consider $\mathcal{C}$ to be the path which follows the real line from $-\infty$ to $\frac{c}{2}$ then follows the circumference of the half circle with centre at 0 and radius $\frac{c}{2}$ in the upper half plane and then follows the real line again from $\frac{c}{2}$ to ∞, then we can write

$$\int_{\mathbb{R}} e^{-it\xi}\frac{\sin(\sqrt{(\xi)^2-c^2})}{\sqrt{(\xi)^2-c^2}}d\xi = \int_{\mathcal{C}} e^{-it\left(z+\frac{c^2}{4z}\right)}\frac{\sin\left(z-\frac{c^2}{4z}\right)}{z}dz.$$

Expanding out the formula for sin gives us

$$\int_{\mathbb{R}} e^{-it\xi} \frac{\sin(\sqrt{(\xi)^2 - c^2})}{\sqrt{(\xi)^2 - c^2}} d\xi = \frac{1}{2i} \int_C e^{iz(1-t) - i\frac{c^2}{4z}(1+t)} \frac{dz}{z} - \frac{1}{2i} \int_C e^{-iz(1+t) + i\frac{c^2}{4z}(1-t)} \frac{dz}{z}.$$

Let

$$f_1(z) = e^{iz(1-t) - i\frac{c^2}{4z}(1+t)}$$

and

$$f_2(z) = e^{-iz(1+t) + i\frac{c^2}{4z}(1-t)}. \tag{3.27}$$

Now

$$i_c(t) = \frac{c}{2\sinh c} \left(\frac{1}{2\pi i} \int_C f_1(z) dz - \frac{1}{2\pi i} \int_C f_2(z) dz \right), \tag{3.28}$$

where

$$|f_1(z)| = \left| e^{iz(1-t) - i\frac{c^2}{4z}(1+t)} \right| = e^{\Re\left(iz(1-t) - i\frac{c^2}{4z}(1+t)\right)},$$

and

$$\begin{aligned} \Re\left(iz(1-t) - i\frac{c^2}{4z}(1+t)\right) &= (1-t)\Re(iz) - c^2(1+t)\Re\left(\frac{i}{4z}\right) \\ &= (t-1)\Im(z) + c^2(1+t)\frac{\Im(z)}{4z^2} \\ &= (t-1)\Im(z)\left[1 + c^2 \frac{1+t}{1-t} \frac{1}{4z^2}\right]. \end{aligned}$$

Similarly

$$|f_2(z)| = \left| e^{-iz(1+t)+i\frac{c^2}{4z}(1-t)} \right| = e^{\Re\left(-iz(1+t)+i\frac{c^2}{4z}(1-t)\right)},$$

and

$$\begin{aligned}\Re\left(-iz(1+t)+i\frac{c^2}{4z}(1-t)\right) &= -(1+t)\Re(iz)+c^2(1-t)\Re\left(\frac{i}{4z}\right) \\ &= (1+t)\Im(z)-c^2(1-t)\frac{\Im(z)}{4z^2} \\ &= (1+t)\Im(z)\left[1-c^2\frac{1-t}{1+t}\frac{1}{4z^2}\right].\end{aligned}$$

Therefore, for $z \to \infty$,

$$\left|\frac{f_1(z)}{z}\right| = \left|\frac{e^{iz(1-t)-i\frac{c^2}{4z}(1+t)}}{z}\right| \ll \frac{e^{\Im(z)(t-1)}}{|z|} \tag{3.29}$$

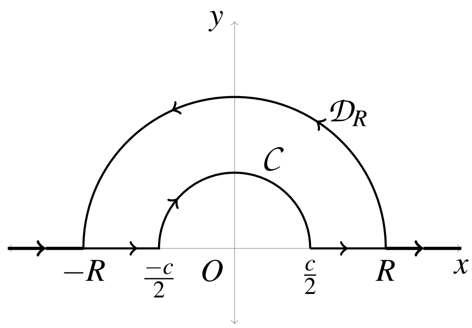
and

$$\left|\frac{f_2(z)}{z}\right| = \left|\frac{e^{-iz(1+t)+i\frac{c^2}{4z}(t-1)}}{z}\right| \ll \frac{e^{\Im(z)(1+t)}}{|z|}. \tag{3.30}$$

We now want to show that for $t < 0$,

$$\lim_{R\to\infty}\frac{1}{2\pi i}\int_{\mathcal{D}_R}\frac{f_1(z)}{z}dz = 0. \tag{3.31}$$

The following diagram shows the contour integral



From equation (3.29) we can write

$$\int_{\mathcal{D}_R} \frac{f_1(z)}{z} dz \ll \int_0^{\pi} \frac{e^{\Im(z)(t-1)}}{|z|} |dz|.$$

Let $z = Re^{i\theta}$, therefore $dz = iRe^{i\theta} d\theta$, $\Im(z) = R\sin\theta$, and $|z| = R$. Now we can write

$$\int_0^{\pi} \frac{e^{\Im(z)(t-1)}}{|z|} |dz| = \int_0^{\pi} \frac{e^{R\sin(\theta)(t-1)}}{R} R d\theta.$$

We need to show that

$$\int_0^{\pi} e^{R\sin(\theta)(t-1)} d\theta \to 0 \text{ as } R \to \infty. \tag{3.32}$$

Let $0 < \varepsilon < \frac{\pi}{2}$. We now divide the integral into three parts

$$\int_0^{\pi} e^{R\sin(\theta)(t-1)} d\theta = \int_0^{\varepsilon} e^{R\sin(\theta)(t-1)} d\theta + \int_{\varepsilon}^{\pi-\varepsilon} e^{R\sin(\theta)(t-1)} d\theta + \int_{\pi-\varepsilon}^{\pi} e^{R\sin(\theta)(t-1)} d\theta.$$

We see that

$$\int_0^{\varepsilon} e^{R\sin(\theta)(t-1)} d\theta \le \varepsilon(\max\{f(\theta) | \theta \in [0,\varepsilon]\}) \le c_1\varepsilon,$$

where c_1 is a constant and $f(\theta) = e^{R\sin(\theta)(t-1)}$. Similarly

$$\int_{\pi-\varepsilon}^{\pi} e^{R\sin(\theta)(t-1)} d\theta \le \varepsilon(\max\{f(\theta) | \theta \in [\pi-\varepsilon,\pi]\}) \le c_2\varepsilon,$$

where c_2 is a constant. For the second integral, since $\sin(\theta) \geq \sin(\varepsilon) \geq \frac{2}{\pi}\varepsilon$ for $\theta \in [\varepsilon, \pi-\varepsilon]$ and $t < 0$ which implies $t-1 < 0$, we can write

$$\int_{\varepsilon}^{\pi-\varepsilon} e^{R\sin(\theta)(t-1)} d\theta \leq \int_{\varepsilon}^{\pi-\varepsilon} e^{R\frac{2}{\pi}\varepsilon(t-1)} d\theta < \pi e^{R\varepsilon\frac{2}{\pi}(t-1)}.$$

Let $R > \left(\frac{1}{\varepsilon}\right)^2$. Therefore $R\varepsilon > \frac{1}{\varepsilon}$ which implies $(t-1)R\varepsilon < (t-1)\frac{1}{\varepsilon}$ and

$$\int_{\varepsilon}^{\pi-\varepsilon} e^{R\sin(\theta)(t-1)} d\theta < \pi e^{R\varepsilon\frac{2}{\pi}(t-1)} \ll e^{(t-1)\frac{2}{\pi}\frac{1}{\varepsilon}} \leq c_3\varepsilon$$

for some constant c_3 and sufficiently small ε. Thus, there exists $R_0(\varepsilon)$ such that

$$\int_0^{\pi} e^{R\sin(\theta)(t-1)} d\theta \leq (c_1 + c_2 + c_3)\varepsilon$$

for $R \geq R_0(\varepsilon)$, which establishes (3.32). Therefore we have proved that

$$\lim_{R\to\infty} \frac{1}{2\pi i} \int_{\mathcal{D}_R} \frac{f_1(z)}{z} dz = 0.$$

Similar arguments prove that for $|t| > 1$,

$$\lim_{R\to\infty} \frac{1}{2\pi i} \int_{\mathcal{D}_R} \frac{f_2(z)}{z} dz = 0,$$

since

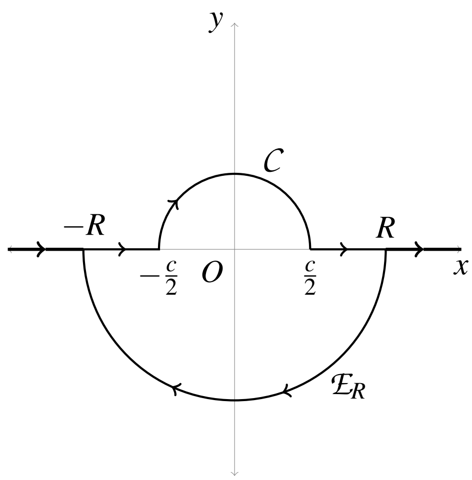
$$\int_{\mathcal{D}_R} \frac{f_2(z)}{z} dz \leq \int_0^{\pi} \frac{e^{\Im(z)(t+1)}}{z} dz,$$

and $t+1 < 0$ for $t < -1$. Now $\int_{C\cup\mathcal{D}_R} f_j(z)\frac{dz}{z} = 0$ by Cauchy's Integral Theorem and thus $\int_C f_j(z)\frac{dz}{z} = \lim_{R\to\infty} - \int_{\mathcal{D}_R} f_j(z)\frac{dz}{z}$. Hence we have proved that

$$\int_C f_j(z)\frac{dz}{z} = 0$$

for $|t| > 1$. Therefore from equation (3.28), and the fact that the Fourier transform is even, it follows that $i_c(t) = 0$ if $|t| > 1$.

For $-1 < t \leq 0$, to evaluate $\int_C f_2(z)\frac{dz}{z}$, we use the contour integral $\mathcal{E}_R$ given in the following diagram



where

$$\int_{\mathcal{E}_R} \frac{f_2(z)}{z} dz \ll \int_{\pi}^{2\pi} \frac{e^{\Im(z)(t+1)}}{z} dz.$$

We let $z = Re^{i\theta}$, which implies $dz = iRe^{i\theta}d\theta$, $\Im(z) = R\sin\theta$, and $|z| = R$. We can now write

$$\int_{\pi}^{2\pi} \frac{e^{\Im(z)(t+1)}}{|z|} |dz| = \int_{\pi}^{2\pi} \frac{e^{R\sin(\theta)(t+1)}}{R} R d\theta.$$

Our aim is to show that

$$\int_{\pi}^{2\pi} \frac{e^{R\sin(\theta)(t+1)}}{R} R d\theta \to 0 \text{ as } R \to \infty, \tag{3.33}$$

and thus we divide the integral into three parts as

$$\int_{\pi}^{2\pi} e^{R\sin(\theta)(t+1)} d\theta = \int_{\pi}^{\pi+\varepsilon} e^{R\sin(\theta)(t+1)} d\theta + \int_{\pi+\varepsilon}^{2\pi-\varepsilon} e^{R\sin(\theta)(t+1)} d\theta + \int_{2\pi-\varepsilon}^{2\pi} e^{R\sin(\theta)(t+1)} d\theta.$$

Note that

$$\int_{\pi}^{\pi+\varepsilon} e^{R\sin(\theta)(t+1)} d\theta \leq c_4\varepsilon,$$

and

$$\int_{2\pi-\varepsilon}^{2\pi} e^{R\sin(\theta)(t+1)} d\theta \leq c_5\varepsilon,$$

where c_4 and c_5 are constants. Since $\sin(\theta) \in [\pi+\varepsilon, 2\pi]$ and $\sin(\varepsilon) \geq \frac{2}{\pi}\varepsilon$, and we can write the second integrand as

$$\int_{\pi+\varepsilon}^{2\pi-\varepsilon} e^{R\sin(\theta)(t+1)} d\theta \leq \int_{\pi+\varepsilon}^{2\pi-\varepsilon} e^{-R\sin(\varepsilon)(t+1)} d\theta \leq \pi e^{-R\varepsilon\frac{2}{\pi}(t+1)}.$$

For $R > \left(\frac{1}{\varepsilon}\right)^2$, we obtain

$$\int_{\pi+\varepsilon}^{2\pi-\varepsilon} e^{R\sin(\theta)(t+1)} d\theta \ll e^{(t+1)\frac{2}{\pi}\frac{1}{\varepsilon}} \leq c_6\varepsilon,$$

where c_6 is a constant and ε is sufficiently small. Thus, there exists $R_1(\varepsilon)$ such that

$$\int_{\pi}^{2\pi} \frac{e^{R\sin(\theta)(t+1)}}{R} R d\theta \leq (c_4 + c_5 + c_6)\varepsilon$$

for $R \geq R_1(\varepsilon)$, which establishes (3.33). Thus, we have now proved that for $-1 < t \leq 0$

$$\lim_{R\to\infty} \frac{1}{2\pi i} \int_{\mathcal{E}_R} \frac{f_2(z)}{z} dz = 0$$

Therefore, we know that

$$\frac{1}{2\pi i} \int_{C\cup\mathcal{E}_R} f_2(z) \frac{dz}{z} = \text{Residue}\left(\frac{f_2(z)}{z}, z = 0\right)$$

and thus

$$\frac{1}{2\pi i}\int_C f_2(z)\frac{dz}{z} + \frac{1}{2\pi i}\int_{\mathcal{E}_{\mathcal{R}}} f_2(z)\frac{dz}{z} = \text{Residue}\left(\frac{f_2(z)}{z}, z=0\right).$$

It follows that

$$\frac{1}{2i}\int_C f_2(z)\frac{dz}{z} = \pi\text{Residue}\left(\frac{f_2(z)}{z}, z=0\right),$$

since $\lim_{R\to\infty}\frac{1}{2\pi i}\int_{\mathcal{E}_R} f_2(z)\frac{dz}{z} = 0$. We find that for each t

$$\mathcal{F}^{-1}(l_c)(t) = \frac{c}{2\sinh c}\text{Residue}\left(\frac{e^{-iz(1+t)+i\frac{c^2}{4z}(1-t)}}{z}, z=0\right).$$

The residue equals the constant term in the Laurent series expansion for $e^{-iz(1+t)+i\frac{c^2}{4z}(1-t)}$.

We can write

$$\begin{aligned} e^{-iz((1+t)+i\frac{c^2}{4z}(1-t)} &= \sum_{n=0}^{\infty}\frac{i^n}{n!}\left(-z(1+t)+\frac{c^2}{4z}(1-t)\right)^n \\ &= \sum_{n=0}^{\infty}\frac{i^n}{n!}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\left(\frac{c^2}{4z}(1-t)\right)^{n-k}(z(1+t))^k. \end{aligned}$$

Thus

$$\text{Residue}\left(e^{-iz(1+t)+i\frac{c^2}{4z}(1-t)}, z=0\right) = \sum_{n=0}^{\infty}\frac{1}{(n!)^2}\left(\frac{c\sqrt{1-t^2}}{2}\right)^{2n} = I_0(c\sqrt{1-t^2}),$$

which proves that

$$\eta_{c,1}(t) = \chi^*_{[-1,1]}\frac{c}{2\sinh c}I_0(c\sqrt{1-t^2}).$$

Using equation (3.25), which states that $\eta_{c,\varepsilon}(t) = \frac{1}{\varepsilon}\eta_{c,1}\left(\frac{t}{\varepsilon}\right)$ we now obtain

$$\eta_{c,\varepsilon}(t) = \chi^*_{[-1,1]}\frac{c}{2\varepsilon\sinh c}I_0(c\sqrt{1-(t/\varepsilon)^2}),$$

which concludes the proof.

For $t = -1$, one can show that $i_c(t) = \frac{c}{4\sinh c}$ using a well-known fact from Fourier analysis.

(iii) For $t \in [-\varepsilon,\varepsilon]$, we see that $-t \in [-\varepsilon,\varepsilon]$. So for $-t$ we can write

$$\begin{aligned}\eta_{c,\varepsilon}(-t) &= \chi^*_{[-\varepsilon,\varepsilon]}(-t)\frac{c}{2\varepsilon\sinh c}I_0(c\sqrt{1-(-t/\varepsilon)^2})\\ &= \chi^*_{[-\varepsilon,\varepsilon]}(-t)\frac{c}{2\varepsilon\sinh c}I_0(c\sqrt{1-(t/\varepsilon)^2}) = \eta_{c,\varepsilon}(t).\end{aligned}$$

Since $\eta_{c,\varepsilon}(t) = \eta_{c,\varepsilon}(-t)$, we can now claim that $\eta_{c,\varepsilon}(t)$ is an even function.

□

The following lemma proves the properties of $\lambda_{c,\varepsilon}$ introduced in 3, equation (3.5):

Lemma 3. *For* $\lambda_{c,\varepsilon} = \ell_{c,\varepsilon}(i/2)$

$$\lambda_{c,\varepsilon} = \int_{\mathbb{R}} \eta_{c,\varepsilon}(\omega)e^{-\frac{\omega}{2}}d\omega, \tag{3.34}$$

and

$$\lambda_{c,\varepsilon} \geq 1. \tag{3.35}$$

Proof. Substituting x by $i/2$ in equation (3.15), we get

$$\lambda_{c,\varepsilon} = \ell_{c,\varepsilon}(i/2) = \int_{\mathbb{R}} \eta_{c,\varepsilon}(\omega)e^{i\omega\frac{i}{2}}d\omega = \int_{\mathbb{R}} \eta_{c,\varepsilon}(\omega)e^{\frac{-\omega}{2}}d\omega.$$

We use the identity for $\lambda_{c,\varepsilon}$ given in the previous equation to prove the next part of the lemma:

$$\begin{aligned}
\lambda_{c,\varepsilon}e^{t/2} &= \int_{\mathbb{R}} \eta_{c,e}(\omega)e^{\frac{t-\omega}{2}}d\omega \\
&= \int_{-\varepsilon}^{\varepsilon} \eta_{c,e}(\omega)e^{\frac{t-\omega}{2}}d\omega \\
&= \int_{0}^{\varepsilon} \eta_{c,e}(\omega)e^{\frac{t-\omega}{2}}d\omega + \int_{0}^{\varepsilon} \eta_{c,e}(-\omega)e^{\frac{t+\omega}{2}}d\omega \\
&= e^{t/2}\int_{0}^{e} \eta_{c,\varepsilon}(\omega)(e^{\omega/2}+e^{-\omega/2})d\omega.
\end{aligned}$$

Since $e^{\omega/2}+e^{-\omega/2} \geq 2$, we can write the above equation as

$$e^{t/2}\int_{0}^{e} \eta_{c,\varepsilon}(\omega)(e^{\omega/2}+e^{-\omega/2})d\omega \geq e^{t/2}\int_{0}^{\varepsilon} 2\eta_{c,\varepsilon}(\omega)d\omega = e^{t/2}\int_{\mathbb{R}} \eta_{c,\varepsilon}(\omega)d\omega,$$

since by Lemma 2(iii), $\eta_{c,\varepsilon}$ is even. Thus we can write,

$$e^{t/2}\int_{0}^{e} \eta_{c,\varepsilon}(\omega)(e^{\omega/2}+e^{-\omega/2})d\omega \geq e^{t/2}\ell_{c,\varepsilon}(0) = 1e^{t/2}.$$

Therefore, we can claim that $\lambda_{c,\varepsilon} \geq 1$. □

The following lemma provides the properties of $\phi_{x,c,\varepsilon}(t)$, defined in definition 4, equation (3.6):

Lemma 4. *(i) For $c,\varepsilon > 0$ and $x > 1$, the support of $\phi_{x,c,\varepsilon}$ lies in $[-\varepsilon, \log x + \varepsilon]$.*

(ii) For all $t \in [-\varepsilon, \log x + \varepsilon]$

$$\phi_{x,c,\varepsilon}(t) = \lambda_{c,\varepsilon}^{-1} f_x * \eta_{c,\varepsilon}(t) \tag{3.36}$$

Proof. (i) For an arbitrary variable τ, $\chi^*_{[0,\log x]}(\tau) = 0$ for $\log x < \tau$ or $\tau < 0$, therefore for $\chi^*_{[0,\log x]}(\tau)$ to be non-zero, $0 \leq \tau \leq \log x$. Now, from the definition of $\eta_{c,\varepsilon}$ given by (3.3), we see that for $\eta_{c,\varepsilon}(t-\tau)$ to be non-zero, $-\varepsilon \leq t-\tau \leq \varepsilon$ which implies $t-\varepsilon \leq \tau \leq t+\varepsilon$. Therefore for $\phi_{x,c,\varepsilon}(t)$ to be non-zero, it is necessary for τ to be contained in $[0,\log x]\cap[t-\varepsilon,t+\varepsilon]$. To prove that the support of $\phi_{x,c,\varepsilon}(t)$ lies in the interval $[-\varepsilon,\log x+\varepsilon]$ it is sufficient to consider the following cases:

Case 1 : If $t+\varepsilon<0$, then $t<-\varepsilon$ which implies $\phi_{x,c,\varepsilon}(t)=0$.

Case 2: If $\log x<t-\varepsilon$ then $\log x+\varepsilon<t$,which implies $\phi_{x,c,\varepsilon}=0$.

Therefore $\phi_{x,c,\varepsilon}(t)$ is nonzero for $-\varepsilon\leq t\leq\log x+\varepsilon$ i.e. the support of $\phi_{x,c,\varepsilon}(t)$ is $[-\varepsilon,\log x+\varepsilon]$.

(ii) The proof of equation (3.36) follows directly from the definition of $\phi_{x,c,\varepsilon}(t)$ and $f_x(t)$.

□

The lemma given below has been used to simplify the formula for $\nu_{c,\varepsilon}(t)$ given by equation (3.13) in Definition 8.

Lemma 5. *(i)*

$$\nu_{c,\varepsilon}(t)=\begin{cases}-\int_{-\varepsilon}^{t}(t-\tau)\eta_{c,\varepsilon}(\tau)d\tau & for\ t<0,\\ \int_{t}^{\varepsilon}(t-\tau)\eta_{c,\varepsilon}(\tau)d\tau & for\ t>0,\\ -\int_{0}^{\varepsilon}\tau\eta_{c,\varepsilon}(\tau)d\tau & for\ t=0.\end{cases}$$

(ii)

$$\nu_c(t)=-\int_t^1\mu_c(\alpha)d\alpha.$$

Proof. (i) We recall that in definition 8, equation (3.13), $\nu_{c,\varepsilon}(t)$ is defined as :

$$\nu_{c,\varepsilon}(t)=\int_{-\infty}^{t}\mu_{c,\varepsilon}(\alpha)d\alpha.$$

Therefore, for $t\leq 0$, we get that α is also less than or equal to 0, and we can write,

$$\nu_{c,\varepsilon}(t)=\int_{-\infty}^{t}\left(-\int_{-\infty}^{\alpha}\eta_{c,\varepsilon}(\tau)d\tau\right)d\alpha.$$

We know that $-\infty \leq \alpha \leq t$ and $-\infty \leq \tau \leq \alpha$ which implies $\tau \leq \alpha \leq t$. Therefore,

$$\begin{aligned}\nu_{c,\varepsilon}(t) &= -\int_{-\infty}^{t} \eta_{c,\varepsilon}(\tau) \int_{\tau}^{t} 1 d\alpha d\tau \\ &= \int_{-\varepsilon}^{t} \eta_{c,\varepsilon}(\tau)(\tau - t) d\tau,\end{aligned}$$

since the support of $\eta_{c,\varepsilon}$ is $[-\varepsilon, \varepsilon]$. For $t = 0$, we see,

$$\nu_{c,\varepsilon}(0) = \int_{-\infty}^{0} \mu_{c,\varepsilon}(\alpha) d\alpha = \int_{-\varepsilon}^{0} \tau \eta_{c,\varepsilon}(\tau) d\tau.$$

Now, for $t > 0$, we write

$$\begin{aligned}\nu_{c,\varepsilon}(t) &= \int_{-\infty}^{0} \mu_{c,\varepsilon}(\alpha) d\alpha + \int_{0}^{t} \mu_{c,\varepsilon}(\alpha) d\alpha \\ &= \nu_{c,\varepsilon}(0) + \int_{0}^{t} \mu_{c,\varepsilon}(\alpha) d\alpha \\ &= \nu_{c,\varepsilon}(0) + \int_{0}^{t} \left(\int_{\alpha}^{\varepsilon} \eta_{c,\varepsilon}(\tau) d\tau \right) d\alpha.\end{aligned}$$

Now $0 \leq \alpha \leq t$ and $\alpha \leq \tau \leq \varepsilon$ which implies

$$\begin{aligned}\nu_{c,\varepsilon}(t) &= \nu_{c,\varepsilon}(0) + \int_{0}^{\varepsilon} \left(\int_{0}^{\min\{t,\tau\}} 1 d\alpha \right) \eta_{c,\varepsilon}(\tau) d\tau \\ &= \nu_{c,\varepsilon}(0) + \int_{0}^{t} \eta_{c,\varepsilon}(\tau) \left(\int_{0}^{\tau} 1 d\alpha \right) d\tau + \int_{t}^{\varepsilon} \eta_{c,\varepsilon}(\tau) \left(\int_{0}^{t} 1 d\alpha \right) d\tau \\ &= \int_{-\varepsilon}^{0} \tau \eta_{c,\varepsilon}(\tau) d\tau + \int_{0}^{t} \tau \eta_{c,\varepsilon}(\tau) d\tau + t \int_{t}^{\varepsilon} \eta_{c,\varepsilon}(\tau) d\tau.\end{aligned}$$

Since $\tau\eta_{c,\varepsilon}(\tau)$ is an odd function, $\int_{-\varepsilon}^{0} \tau\eta_{c,\varepsilon}(\tau) d\tau = -\int_{0}^{\varepsilon} \tau\eta_{c,\varepsilon}(\tau) d\tau$ and thus, we deduce

$$\begin{aligned}\nu_{c,\varepsilon}(t) &= -\int_0^{\varepsilon} \tau\eta_{c,\varepsilon}(\tau)d\tau + \int_0^{t} \tau\eta_{c,\varepsilon}(\tau)d\tau + t\int_t^{\varepsilon} \eta_{c,\varepsilon}(\tau)d\tau \\ &= -\int_t^{\varepsilon} \tau\eta_{c,\varepsilon}(\tau)d\tau + t\int_t^{\varepsilon} \eta_{c,\varepsilon}(\tau)d\tau \\ &= \int_t^{\varepsilon} (t-\tau)\eta_{c,\varepsilon}(\tau)d\tau.\end{aligned}$$

(ii) For $t \in (0,1)$, by the definition $\nu_c(t) = \int_{-\infty}^{t} \mu_c(\tau)d\tau$, we can write

$$\nu_c(t) + \int_t^1 \mu_c(\alpha)d\alpha = \nu_c(1) = \int_{-1}^{1} \mu_c(\alpha)d\alpha = 0$$

since $\mu_c(\alpha)$ is an odd function. Therefore,

$$\nu_c(t) = -\int_t^1 \mu_c(\alpha)d\alpha.$$

□

The next lemma has been used multiple times in this thesis to provide bounds for various functions.

Lemma 6. *For* $0 < \varepsilon < \frac{1}{10}$*, we have*

$$e^{t+\tau} = e^t + O^*(2|\tau|)$$

for $\max\{|t|, |\tau|\} \leq \varepsilon$.

Proof. We begin by writing $e^{t+\tau} = e^t \cdot e^{\tau}$. Now using the Taylor series expansion of e^{τ}, we get

$$e^{t+\tau} = e^t\left(1 + \tau + \frac{\tau^2}{2!} + \dots\right) = e^t + \tau e^t \sum_{j=1}^{\infty} \frac{\tau^{j-1}}{j!}.$$

Since e^x is monotonically increasing function, and $\max\{|t|,|\tau|\} \leq \varepsilon$ we can write that,

$$e^t \sum_{j=1}^{\infty} \frac{\tau^{j-1}}{j!} \leq e^{\varepsilon} \sum_{j=1}^{\infty} \frac{\varepsilon^{j-1}}{j!}.$$

Now, since $0 < \varepsilon < \frac{1}{10}$

$$\begin{aligned} e^t \sum_{j=1}^{\infty} \frac{\tau^{j-1}}{j!} \leq & e^{1/10} \sum_{j=1}^{\infty} \frac{1/10^{j-1}}{j!} \\ \leq & 10 e^{1/10} \sum_{j=1}^{\infty} \frac{1/10^{j}}{j!} \\ \leq & 10 e^{1/10}(e^{1/10}-1) \\ \leq & 2. \end{aligned}$$

This implies that $\left|\tau e^t \sum_{j=1}^{\infty} \frac{\tau^{j-1}}{j!}\right| \leq |2\tau|$, and therefore $e^{t+\tau} = e^t + O^*(2|\tau|)$. □

The following partial summation lemma which can be found in [20, Theorem A, page 18] has been used multiple times in this thesis, and thus we mention it at this juncture.

Lemma 7 (Ingham's Lemma)**.** *Let $\lambda_1, \lambda_2, \ldots$ be a non-decreasing real sequence which has the limit infinity and let*

$$C(x) = \sum_{\lambda_n \leq x} c_n,$$

where the c_n may be real or complex, and the notation indicates a summation over the (finite) set of positive integers n for which $\lambda_n \leq x$. Then, if $X \geq \lambda_1$ and $\phi(x)$ has a continuous derivative, we have

$$\sum_{\lambda_n \leq X} c_n \phi(\lambda_n) = -\int_{\lambda_1}^{X} C(x)\phi'(x)dx + C(X)\phi(X). \tag{3.37}$$

If, further, $C(X)\phi(X) \to 0$ as $X \to \infty$, then

$$\sum_{1}^{\infty} c_n \phi(\lambda_n) = -\int_{\lambda_1}^{\infty} C(x)\phi'(x)dx,$$

provided that either side is convergent.

As a consequence of Ingham's Lemma for $X < Y$ we can write

$$\sum_{X<\lambda_n\leq Y} c_n\phi(\lambda_n) = -\int_X^Y C(x)\phi'(x)dx + C(Y)\phi(Y) - C(X)\phi(X). \tag{3.38}$$

Chapter 4

Main Results in Büthe's Work

4.1 A Modified Chebyshev Function

This Chapter consists of all the main lemmas and propositions that are present in Büthe's paper with necessary modifications and corrections where needed. We tabulate a list of these propositions and lemmas in the chronological order that they are presented in the original paper, along with a brief description and references for their proofs:

Table 4.1: List of Results

Proposition 1 (Prop 1)	Shows the $\psi_{x,c,\varepsilon}(x)$ can be written as a sum of $\psi(x)$ and a small error term. Also gives inequalities for the error term .	Proof can be found in [10] and [7]
Proposition 2 (Prop 3)	Provides explicit formula for $\psi_{x,c,\varepsilon}(x)$	Proof can be found in [3] and [7]
Lemma 1 (Lemma 8)	Shows that $\Delta(t) = \phi_{x,c,\varepsilon}(t) - \chi^*_{0,\log x}(t)e^{t/2}$ vanishes for $t \in B_\varepsilon(0) \cup B_\varepsilon(\log x)$. Also proves some inequalities involving $\Delta(t)$	Proof can be found in [10, Lemma 1.1] and [7]
Proposition 3 (Prop 4) (Lemma 13)	Provides bound for sum over zeros with weight $a_{c,\varepsilon}(\rho) = \frac{1}{\lambda_{c,\varepsilon}}\ell_{c,\varepsilon}\left(\frac{\rho}{i} - \frac{1}{2i}\right)$ for $\gamma > H$	Proof can be found in [8, Lemma 5.5] and [7]

Lemma 2 (Lemma 12)	Gives bound for sum over zeros $\ell_{c,\varepsilon}$ weight with $\gamma > H$	Proof can be found in [18, Lemma 2.4], [10, Lemma 1.2]() and [7]
Lemma 3 (Lemma 14)	Gives bound for $\frac{1}{\gamma}$ in finite range	Proof can be found in [7]
Proposition 4 (Prop 5)	Relates $\psi(x)$ to smoothed version $\psi_{c,\varepsilon}(x)$	Proof can be found in [8] and [7]
Lemma 4 (Lemma 15)	Provides bound for a weighted sum over prime powers	Proof can be found in [8, Lemma 3.5]
Lemma 5 (Lemma 16)	Bounds for weighted prime sums of $\mu_{c,\varepsilon}$	Proof can be found in [8, Lemma 5.8], [6, Theorem 4.3]
Proposition 5 (Prop 6)	Gives upper and lower bounds for $\|\nu_c(0)\|$	Proof can be found in [18, page 15]
Lemma 6 (Lemma 17)	Proves properties of ratios of Modified Bessel functions	Proof can be found in [7]
Theorem 1 (Theorem 6)	Main theorem	Proof can be found in [7]
Lemma 7 (Lemma 20)	Provides bounds for $\frac{\ell_{c,\varepsilon}(\gamma)}{\gamma}$ in finite ranges	Proof can be found in [18]
Lemma 8 (Lemma 18)	Provides bounds for $\mu_c(\alpha)$	Proof can be found in [7]

Proposition 1 demonstrates how to approximate $\psi(x)$ by a smooth variant $\psi_{c,\varepsilon}(x)$.

Proposition 1. *[7, Proposition 1]*

Let $0 < \varepsilon < \frac{1}{10}$ *and let*

$$M_{x,c,\varepsilon}(t) = \frac{\log t}{\lambda_{c,\varepsilon}} \Big[\chi^*_{[x,e^{\varepsilon}x]}(t) \int_{\log t/x}^{\varepsilon} \eta_{c,\varepsilon}(\tau) e^{-\tau/2} d\tau - \chi^*_{[e^{-\varepsilon}x,x]}(t) \int_{-\varepsilon}^{\log t/x} \eta_{c,\varepsilon}(\tau) e^{-\tau/2} d\tau \Big]. \tag{4.1}$$

Then we have

$$\psi(x) = \psi_{c,\varepsilon}(x) - \sum_{e^{-\varepsilon}x < p^m < e^{\varepsilon}x} \frac{1}{m} M_{x,c,\varepsilon}(p^m). \tag{4.2}$$

Moreover we have,

$$\psi(e^{-\alpha\varepsilon}x) \leq \psi_{c,\varepsilon}(x) - \sum_{e^{-\varepsilon}x < p^m < e^{-\alpha\varepsilon}x} \frac{1}{m} M_{x,c,\varepsilon}(p^m), \tag{4.3}$$

$$\psi(e^{\alpha\varepsilon}x) \geq \psi_{c,\varepsilon}(x) - \sum_{e^{\alpha\varepsilon}x < p^m < e^{\varepsilon}x} \frac{1}{m} M_{x,c,\varepsilon}(p^m) \tag{4.4}$$

for every $0 < \alpha < 1$.

Proof. We know that the normalized Chebyshev function is of the form

$$\psi(x) = \sum_{p^m} \chi^*_{[0,x]}(p^m) \log p,$$

and the modified Chebyshev function is defined in equation (3.8) as

$$\psi_{c,\varepsilon}(x) = \sum_{p^m} \frac{\log p}{p^{m/2}} \phi_{x,c,\varepsilon}(m \log p).$$

We begin the proof by considering the difference $(\psi(x) - \psi_{c,\varepsilon}(x))$ and see that

$$\begin{aligned}\psi(x) - \psi_{c,\varepsilon}(x) &= \sum_{p^m} \chi^*_{[0,x]}(p^m)\log p - \sum_{p^m} \frac{\log p}{p^{m/2}} \phi_{x,c,\varepsilon}(m\log p) \\ &= \sum_{p^m} \left[\chi^*_{[0,x]}(p^m)\log p - \frac{\log p}{p^{m/2}} \phi_{x,c,\varepsilon}(\log p^m) \right].\end{aligned}$$

Lemma 4(i) tells us that the support of $\phi_{x,c,\varepsilon}(t)$ is $[-\varepsilon, \log x + \varepsilon]$. Therefore, $\phi_{x,c,\varepsilon}(m\log p)$ vanishes for $\log p^m > \log x + \varepsilon$ or $\log p^m < -\varepsilon$. We note that $\log p^m < -\varepsilon$ which implies that $p^m < e^{-\varepsilon}$ and $\log p^m > \log x + \varepsilon$ implies $p^m > e^{\varepsilon}x$. Therefore,

$$\psi_{c,\varepsilon}(x) = \sum_{e^{-\varepsilon} \le p^m \le e^{\varepsilon}x} \frac{\log p}{p^{m/2}} \phi_{x,c,\varepsilon}(\log p^m).$$

Now, by definition, $\chi^*_{[0,x]}(p^m)\log p = 0$ for $p^m > x$ or $p^m < 0$. So $\psi(x) = \sum_{0 \le p^m \le x} \log p$. However, for $0 < p^m < e^{-\varepsilon}$, we see that $\log p = 0$ since e^{-x} is a decreasing function with $e^0 = 1$ and $e^{-\varepsilon} < 1$ for all given ε. Thus $\sum\limits_{0 < p^m < e^{-\varepsilon}} \log p = 0$ which implies

$$\psi(x) = \sum_{e^{-\varepsilon} \le p^m \le x} \log p$$

We can now write the difference as

$$\psi(x) - \psi_{c,\varepsilon}(x) = \sum_{e^{-\varepsilon} \le p^m \le x} \log p - \sum_{e^{-\varepsilon} \le p^m \le e^{\varepsilon}x} \frac{\log p}{p^{m/2}} \phi_{x,c,\varepsilon}(\log p^m)$$

To finish the proof, we divide the remaining interval, $[e^{-\varepsilon}, e^{\varepsilon}x]$ into sub-intervals in order to evaluate $\psi(x)$ and $\psi_{c,\varepsilon}(x)$. We also note that $0 < e^{-\varepsilon} < e^{-\varepsilon}x < x < e^{\varepsilon}x$.

First Interval: For $e^{-\varepsilon} < p^m < e^{-\varepsilon}x$,

$$\chi^*_{[0,x]}(p^m)\log p = \log p$$

Also, we know that,

$$\phi_{x,c,\varepsilon}(\log p^m) = \frac{1}{\lambda_{c,\varepsilon}}\left[\int_{\mathbb{R}}\left(\chi_{[0,\log x]}(\tau)e^{\tau/2}\right)\eta_{c,\varepsilon}(\log p^m-\tau)d\tau\right].$$

Since $p^m < e^{-\varepsilon}x$, this implies $\log p^m < \log x - \varepsilon$. We also see that $0 \leq \tau \leq \log x$ and $-\varepsilon \leq \log p^m - \tau \leq \varepsilon$. Therefore, $\log p^m - \varepsilon \leq \tau \leq \varepsilon + \log p^m < \varepsilon + (\log x - \varepsilon) = \log x$. Thus,

$$\phi_{x,c,\varepsilon}(\log p^m) = \frac{1}{\lambda_{c,\varepsilon}}\left[\int_{\mathbb{R}} e^{\tau/2}(\eta_{c,\varepsilon}(\log p^m-\tau))d\tau\right] = \frac{1}{\lambda_{c,\varepsilon}}\left(\lambda_{c,\varepsilon}e^{\log p^{m/2}}\right) = p^{m/2},$$

Second Interval: For $x \leq (p^m) \leq e^{\varepsilon}x$, we note that $\chi^*_{[0,x]}(p^m)\log p = 0$. Now, since $\eta_{c,\varepsilon}$ is compactly supported on $[-\varepsilon,\varepsilon]$, we can assert that $-\varepsilon \leq \log p^m - \tau \leq \varepsilon$, which implies that $-\varepsilon + \log p^m \leq \tau \leq \varepsilon + \log p^m$. Therefore, for $x \leq (p^m) \leq e^{\varepsilon}x$,

$$\begin{aligned}\phi_{x,c,\varepsilon}(t) &= \frac{1}{\lambda_{c,\varepsilon}}\left[\int_{[0,\log x]\cap[-\varepsilon+\log p^m,\varepsilon+\log p^m]} e^{\tau/2}(\eta_{c,\varepsilon}(\log p^m-\tau))d\tau\right]\\ &= \frac{1}{\lambda_{c,\varepsilon}}\left[\int_{-\varepsilon+\log p^m}^{\log x} e^{\tau/2}(\eta_{c,\varepsilon}(\log p^m-\tau))d\tau\right].\end{aligned}$$

Doing a variable change with $\log p^m - \tau = u$, we get $\tau = \log p^m - u$ and $-du = d\tau$, so

$$\begin{aligned}\phi_{x,c,\varepsilon}(t) &= \frac{1}{\lambda_{c,\varepsilon}}\left[\int_{\varepsilon}^{\log p^m-\log x} -e^{(\log p^m-u)/2}\eta_{c,\varepsilon}(u)du\right]\\ &= \frac{p^{\frac{m}{2}}}{\lambda_{c,\varepsilon}}\left[\int_{\log\left(\frac{p^m}{x}\right)}^{\varepsilon} e^{\frac{-u}{2}}\eta_{c,\varepsilon}(u)du\right]\end{aligned}$$

Third Interval: For $e^{-\varepsilon}x \leq p^m \leq x$,

$$\chi^*_{[0,x]}(p^m)\log p = \log p.$$

Since $\eta_{c,\varepsilon}(t)$ is compactly supported on $[-\varepsilon,\varepsilon]$, we deduce that $-\varepsilon \leq \log p^m - \tau \leq \varepsilon$

which implies $-\varepsilon+\log p^m \leq \tau \leq \varepsilon+\log p^m$. We can now compute $\phi_{x,c,\varepsilon}(t)$ in the interval $[e^{-\varepsilon}x, x]$ as

$$\phi_{x,c,\varepsilon}(t) = \frac{1}{\lambda_{c,\varepsilon}}\left[\int_{-\varepsilon+\log p^m}^{\log x} e^{\tau/2}(\eta_{c,\varepsilon}(\log p^m-\tau))d\tau\right]$$

Substituting $\log p^m - \tau = u$ in the above equation, we get

$$\begin{aligned}\phi_{x,c,\varepsilon}(t) &= \frac{1}{\lambda_{c,\varepsilon}}\left(\int_{\varepsilon}^{\log p^m-\log x} -e^{(\log p^m-u)/2}\eta_{c,\varepsilon}(u)du\right)\\ &= \frac{p^{\frac{m}{2}}}{\lambda_{c,\varepsilon}}\left(\int_{\log\left(\frac{p^m}{x}\right)}^{\varepsilon} e^{\frac{-u}{2}}\eta_{c,\varepsilon}(u)du\right).\end{aligned}$$

With $\tau=\omega$ in Lemma 1(iv), equation (3.34), we get $\lambda_{c,\varepsilon}=\int_{-\varepsilon}^{\varepsilon}\eta_{c,\varepsilon}(\tau)e^{\frac{-\tau}{2}}d\tau$, and so we can write $\frac{1}{\lambda_{c,\varepsilon}}\left(\int_{-\varepsilon}^{\varepsilon}\eta_{c,\varepsilon}(\tau)e^{\frac{-\tau}{2}}d\tau\right)=1$. Thus we obtain,

$$\begin{aligned}\phi_{x,c,\varepsilon}(t) =& p^{\frac{m}{2}}\left[\frac{1}{\lambda_{c,\varepsilon}}\left(\int_{-\varepsilon}^{\varepsilon} e^{\frac{-u}{2}}\eta_{c,\varepsilon}(u)du\right)+\frac{1}{\lambda_{c,\varepsilon}}\left(\int_{-\varepsilon}^{\log\left(\frac{p^m}{x}\right)}\eta_{c,\varepsilon}(u)e^{\frac{-u}{2}}du\right)\right]\\ &= p^{\frac{m}{2}}\left[\frac{1}{\lambda_{c,\varepsilon}}\left(\int_{-\varepsilon}^{\log\left(\frac{p^m}{x}\right)} e^{\frac{-u}{2}}\eta_{c,\varepsilon}(u)du\right)+1\right]\end{aligned}$$

Considering all three of the above intervals, we can write that $\psi_{c,\varepsilon}(x)-\psi(x)$ equals

$$\begin{aligned}\psi_{c,\varepsilon}(x)-\psi(x) =& \sum_{e^{-\varepsilon}<p^m<e^{-\varepsilon}x}[\log p-\log p] + \sum_{e^{-\varepsilon}x<p^m<x}\left[0-\frac{\log p}{\lambda_{c,\varepsilon}}\left[\int_{\log\left(\frac{p^m}{x}\right)}^{\varepsilon} e^{\frac{-u}{2}}\eta_{c,\varepsilon}(u)du\right]\right]\\ &+\sum_{x<p^m<e^{\varepsilon}x}\left[\log p\left[1-\frac{1}{\lambda_{c,\varepsilon}}\left(\int_{-\varepsilon}^{\log\left(\frac{p^m}{x}\right)} e^{\frac{-u}{2}}\eta_{c,\varepsilon}(u)du\right)-1\right]\right]\\ &= \sum_{e^{-\varepsilon}x<p^m<e^{\varepsilon}x}\frac{1}{m}\frac{\log p^m}{\lambda_{c,\varepsilon}}\left[\chi^*_{[x,e^{\varepsilon}x]}(p^m)\int_{\log\left(\frac{p^m}{x}\right)}^{\varepsilon} e^{\frac{-u}{2}}\eta_{c,\varepsilon}(u)du\right.\\ &\left.-\chi^*_{[e^{-\varepsilon}x,x]}(p^m)\int_{-\varepsilon}^{\log\left(\frac{p^m}{x}\right)} e^{\frac{-u}{2}}\eta_{c,\varepsilon}(u)du\right]\\ &= \sum_{e^{-\varepsilon}x<p^m<e^{\varepsilon}x}\frac{1}{m}M_{x,c,\varepsilon}(p^m),\end{aligned}$$

and we can now write

$$\psi(x) = \psi_{c,\varepsilon}(x) - \sum_{e^{-\varepsilon}x < p^m < e^{\varepsilon}x} \frac{1}{m} M_{x,c,\varepsilon}(p^m),$$

which finishes the proof.

To prove the inequality given in (4.3), we note that

$$\psi(x) = \psi(e^{-\alpha\varepsilon}x) + \sum_{e^{-\alpha\varepsilon}x \le p^m \le x} \log p.$$

Using the above identity in equation (4.2), we obtain,

$$\psi(e^{-\alpha\varepsilon}x) + \sum_{e^{-\alpha\varepsilon}x \le p^m \le x} \log p = \psi_{c,\varepsilon}(x) - \sum_{e^{-\varepsilon}x \le p^m \le e^{\varepsilon}x} \frac{1}{m} M_{x,c,\varepsilon}(p^m)$$

which implies

$$\psi(e^{-\alpha\varepsilon}x) = \psi_{c,\varepsilon}(x) - \sum_{e^{-\varepsilon}x \le p^m \le e^{\varepsilon}x} \frac{1}{m} M_{x,c,\varepsilon}(p^m) - \sum_{e^{-\alpha\varepsilon}x \le p^m \le x} \log p.$$

In order to prove that $\psi(e^{-\alpha\varepsilon}x) \le \psi_{c,\varepsilon}(x) - \sum_{e^{-\varepsilon}x < p^m < e^{-\alpha\varepsilon}x} \frac{1}{m} M_{x,c,\varepsilon}(p^m)$, we need to show

$$\psi_{c,\varepsilon}(x) - \sum_{e^{-\varepsilon}x \le p^m \le e^{\varepsilon}x} \frac{1}{m} M_{x,c,\varepsilon}(p^m) - \sum_{e^{-\alpha\varepsilon}x \le p^m \le x} \log p \le \psi_{c,\varepsilon}(x) - \sum_{e^{-\varepsilon}x \le p^m \le e^{-\alpha\varepsilon}x} \frac{1}{m} M_{x,c,\varepsilon}(p^m)$$

which is true if and only if

$$- \sum_{e^{-\alpha\varepsilon}x \le p^m \le e^{\varepsilon}x} \frac{1}{m} M_{x,c,\varepsilon}(p^m) \le \sum_{e^{-\alpha\varepsilon}x \le p^m \le x} \log p.$$

The sum $\sum_{e^{-\alpha\varepsilon}x \le p^m \le e^{\varepsilon}x} \frac{1}{m} M_{x,c,\varepsilon}(p^m)$ can be separated into two intervals to get the inequality,

$$-\sum_{e^{-\alpha\varepsilon}x \le p^m \le x} \frac{1}{m} M_{x,c,\varepsilon}(p^m) \le \sum_{e^{-\alpha\varepsilon}x \le p^m \le x} \log p + \sum_{x \le p^m \le e^{\varepsilon}x} \frac{1}{m} M_{x,c,\varepsilon}(p^m).$$

To complete the proof, it is sufficient to show that

$$-\sum_{e^{-\alpha\varepsilon}x \le p^m \le x} \frac{1}{m} M_{x,c,\varepsilon}(p^m) \le \sum_{e^{-\alpha\varepsilon}x \le p^m \le x} \log p$$

In other words it is sufficient to show that for $e^{-\alpha\varepsilon}x \le p^m \le x$,

$$-\frac{1}{m} M_{x,c,\varepsilon}(p^m) \le \log p.$$

For $e^{-\alpha\varepsilon}x \le p^m \le x$, we know,

$$\begin{aligned}
-\frac{1}{m} M_{x,c,\varepsilon}(p^m) &= \frac{1}{m} \frac{\log p^m}{\lambda_{c,\varepsilon}} \int_{-\varepsilon}^{\log(p^m/x)} \eta_{c,\varepsilon}(\tau) e^{-\tau/2} d\tau \\
&= \frac{\log p}{\lambda_{c,\varepsilon}} \int_{-\varepsilon}^{\log(p^m/x)} \eta_{c,\varepsilon}(\tau) e^{-\tau/2} d\tau \\
&\le \frac{\log p}{\lambda_{c,\varepsilon}} \int_{-\varepsilon}^{\varepsilon} \eta_{c,\varepsilon}(\tau) e^{-\tau/2} d\tau.
\end{aligned}$$

Since equation (3.34) states that $\lambda_{c,\varepsilon} = \int_{-\varepsilon}^{\varepsilon} \eta_{c,\varepsilon}(\tau) e^{-\tau/2} d\tau$, we can now claim that

$$-\frac{1}{m} M_{x,c,\varepsilon}(p^m) \le \frac{\log p}{\lambda_{c,\varepsilon}} \cdot \lambda_{c,\varepsilon} = \log p.$$

Therefore we have proved that

$$-\frac{1}{m} M_{x,c,\varepsilon}(p^m) \le \log p.$$

Similarly, to prove the inequality (4.4), we note that

$$\psi(x) = \psi(e^{\alpha\varepsilon}x) - \sum_{x \leq p^m \leq e^{\alpha\varepsilon}x} \log p.$$

Using the above identity in equation (4.2), we write $\psi(x)$ as :

$$\psi(e^{\alpha\varepsilon}x) - \sum_{x \leq p^m \leq e^{\alpha\varepsilon}x} \log p = \psi_{c,\varepsilon}(x) - \sum_{e^{-\varepsilon}x \leq p^m \leq e^{\varepsilon}x} \frac{1}{m} M_{x,c,\varepsilon}(p^m)$$

which implies

$$\psi(e^{\alpha\varepsilon}x) = \psi_{c,\varepsilon}(x) - \sum_{e^{-\varepsilon}x \leq p^m \leq e^{\varepsilon}x} \frac{1}{m} M_{x,c,\varepsilon}(p^m) + \sum_{x \leq p^m \leq e^{\alpha\varepsilon}x} \log p$$

Now, to prove that $\psi(e^{\alpha\varepsilon}x) \geq \psi_{c,\varepsilon}(x) - \sum_{e^{\alpha\varepsilon}x \leq p^m < e^{\varepsilon}x} \frac{1}{m} M_{x,c,\varepsilon}(p^m)$, we need to show

$$\psi_{c,\varepsilon}(x) - \sum_{e^{-\varepsilon}x \leq p^m \leq e^{\varepsilon}x} \frac{1}{m} M_{x,c,\varepsilon}(p^m) + \sum_{x \leq p^m \leq e^{\alpha\varepsilon}x} \log p \geq \psi_{c,\varepsilon}(x) - \sum_{e^{\alpha\varepsilon}x \leq p^m < e^{\varepsilon}x} \frac{1}{m} M_{x,c,\varepsilon}(p^m),$$

which is true if and only if

$$- \sum_{e^{-\varepsilon}x \leq p^m \leq e^{\varepsilon}x} \frac{1}{m} M_{x,c,\varepsilon}(p^m) + \sum_{e^{\alpha\varepsilon}x \leq p^m < e^{\varepsilon}x} \frac{1}{m} M_{x,c,\varepsilon}(p^m) + \sum_{x \leq p^m \leq e^{\alpha\varepsilon}x} \log p \geq 0.$$

The left hand side of the above inequality can be modified to obtain

$$- \sum_{e^{-\varepsilon}x \leq p^m \leq x} \frac{1}{m} M_{x,c,\varepsilon}(p^m) + \sum_{x \leq p^m \leq e^{\alpha\varepsilon}x} \left(\log p - \frac{1}{m} M_{x,c,\varepsilon}(p^m) \right) \geq 0.$$

For $e^{-\varepsilon}x < p^m < x$, we know $M_{x,c,\varepsilon}(p^m) \leq 0$ which implies

$$- \sum_{e^{-\varepsilon}x \leq p^m \leq x} \frac{1}{m} M_{x,c,\varepsilon}(p^m) \geq 0.$$

Therefore to finish the proof, it is sufficient to show that

$$\sum_{x \le p^m \le e^{\alpha\varepsilon}x} \left(\log p - \frac{1}{m} M_{x,c,\varepsilon}(p^m) \right) \ge 0.$$

For $x < p^m < e^{\alpha\varepsilon}x$, we see

$$\begin{aligned} \frac{1}{m} M_{x,c,\varepsilon}(p^m) &= \frac{\log p}{\lambda_{c,\varepsilon}} \int_{-\varepsilon}^{\log\left(\frac{p^m}{x}\right)} \eta_{c,\varepsilon}(\tau) e^{-\frac{\tau}{2}} d\tau \\ &\le \frac{\log p}{\lambda_{c,\varepsilon}} \int_{-\varepsilon}^{\varepsilon} \eta_{c,\varepsilon}(\tau) e^{-\frac{\tau}{2}} d\tau. \end{aligned}$$

Since from equation (3.34) we know that $\lambda_{c,\varepsilon} = \int_{-\varepsilon}^{\varepsilon} \eta_{c,\varepsilon}(\tau) e^{-\tau/2} d\tau$, we can now obtain

$$\frac{1}{m} M_{x,c,\varepsilon}(p^m) \le \frac{\log p}{\lambda_{c,\varepsilon}} \cdot \lambda_{c,\varepsilon} = \log p.$$

Therefore,

$$-\frac{1}{m} M_{x,c,\varepsilon}(p^m) \ge -\log p,$$

and adding $\log p$ to both sides of the above inequality gives us

$$\log p - \frac{1}{m} M_{x,c,\varepsilon}(p^m) \ge \log p - \log p \ge 0.$$

So we can now claim that

$$\sum_{x \le p^m \le e^{\alpha\varepsilon}x} \left(\log p - \frac{1}{m} M_{x,c,\varepsilon}(p^m) \right) \ge 0,$$

and thus we have concluded the proof. □

4.2 The Explicit Formula

We begin this section by providing the following lemma which proves results about $\Delta(t)$ where $\Delta(t) = \phi_{x,c,\varepsilon}(t) - f_x(t)$ as given in definition 6, equation (3.11).

Lemma 8. *[7, Lemma 1] Let* $0 < \varepsilon < \frac{1}{10}$ *and* $\log x > \frac{2}{|\log \varepsilon|}$.

(i) $\Delta(t)$ *vanishes for* $t \notin B_\varepsilon(0) \cup B_\varepsilon(\log x)$.

(ii) For $0 \leq t \leq \varepsilon$,

$$\Delta(t) + \Delta(-t) = 2\Delta(0) + O^*(2t). \tag{4.5}$$

(iii) For $t \in B_\varepsilon(\log x)$,

$$|\Delta(t)| \leq \frac{1}{2} e^{\frac{\varepsilon}{2}} \sqrt{x}. \tag{4.6}$$

(iv)

$$|\Delta(0)| \leq \varepsilon \tag{4.7}$$

Proof. (i) From Lemma 4(i) we know that $\phi_{x,c,\varepsilon}(t) = 0$, for $t < -\varepsilon$ and $t > \log x + \varepsilon$. Definition 6, equation (3.9), tells us that $f_x(t) = 0$ for $t < 0$ and $t > \log x$. Also, from definition 6, equation (3.11), we know $\Delta(t) = \phi_{x,c,\varepsilon}(t) - f_x(t)$. Therefore, for $t < -\varepsilon$ and $t > \log x + \varepsilon$, we can deduce that $\Delta(t) = 0$.

To finish the proof it is necessary to show that $\Delta(t) = 0$ for $\varepsilon < t < \log x - \varepsilon$, and to do so it is sufficient to show that $\phi_{x,c,\varepsilon}(t) = f_x(t)$ for $\varepsilon < t < \log x - \varepsilon$.

Using equation (3.6) in definition 4, we can write

$$\phi_{x,c,\varepsilon}(t) = \frac{1}{\lambda_{c,\varepsilon}} (f_x * \eta_{c,\varepsilon})(t).$$

Now, expanding the convolution gives us

$$\begin{aligned}\phi_{x,c,\varepsilon}(t) &= \frac{1}{\lambda_{c,\varepsilon}} \int_{\mathbb{R}} \eta_{c,\varepsilon}(\tau) f_x(t-\tau) d\tau \\ &= \frac{1}{\lambda_{c,\varepsilon}} \int_{-\varepsilon}^{\varepsilon} \eta_{c,\varepsilon}(\tau) f_x(t-\tau) d\tau\end{aligned}$$

since $\eta_{c,\varepsilon}(\tau)$ vanishes for $\tau \notin (-\varepsilon, \varepsilon)$. For $\varepsilon < \tau < -\varepsilon$, when $\varepsilon < t < \log x - \varepsilon$ we note that $0 < t - \tau < \log x$. From definition 6, equation (3.9) , we know $f_x(t-\tau) = e^{\frac{t-\tau}{2}}$

for $0 < t - \tau < \log x$. Thus we can write

$$\begin{aligned}\phi_{x,c,\varepsilon}(t) &= \frac{1}{\lambda_{c,\varepsilon}} \int_{-\varepsilon}^{\varepsilon} \eta_{c,\varepsilon}(\tau) e^{\frac{t-\tau}{2}} d\tau \\ &= \frac{1}{\lambda_{c,\varepsilon}} \left(e^{(\cdot/2)} * \eta_{c,\varepsilon} \right)(t).\end{aligned}$$

However, we have already proved in Lemma 2(i) that $e^{(\bullet/2)} * \eta_{c,\varepsilon} = \lambda_{c,\varepsilon} e^{t/2}$. Therefore, for $\varepsilon < t < \log x - \varepsilon$

$$\phi_{x,c,\varepsilon}(t) = e^{t/2}.$$

Now, since $f_x(t) = \chi^*_{[0,\log(x)]}(t) e^{t/2} = e^{t/2}$ for $\varepsilon < t < \log x - \varepsilon$, we have proved that $\phi_{x,c,\varepsilon}(t) = f_x(t)$ and therefore $\Delta(t)$ vanishes for $t \notin B_\varepsilon(0) \cup B_\varepsilon(\log x)$.

(ii) From definition 4, equation (3.6) we can write,

$$\phi_{x,c,\varepsilon}(t) = \frac{1}{\lambda_{c,\varepsilon}} ((\chi^*_{[0,\log x]}(\exp(\cdot/2))) * \eta_{c,\varepsilon})(t).$$

However, since $f * g = g * f$, where $*$ denotes the convolution of two functions, we can deduce that

$$\begin{aligned}\phi_{x,c,\varepsilon}(t) &= \frac{1}{\lambda_{c,\varepsilon}} \int_{\mathbb{R}} \left(\chi_{[0,\log x]}(t-\tau) e^{\frac{t-\tau}{2}} \right) \cdot \eta_{c,\varepsilon}(\tau) d\tau \\ &= \frac{1}{\lambda_{c,\varepsilon}} \int_{-\varepsilon}^{\varepsilon} \left(\chi_{[0,\log x]}(t-\tau) e^{\frac{t-\tau}{2}} \right) \cdot \eta_{c,\varepsilon}(\tau) d\tau,\end{aligned}$$

since the support of $\eta_{c,\varepsilon}(t)$ lies in $[-\varepsilon, \varepsilon]$. Substituting $t = 0$ in the definition of $\phi_{x,c,\varepsilon}(t)$, we get

$$\begin{aligned}\phi_{x,c,\varepsilon}(0) &= \frac{1}{\lambda_{c,\varepsilon}} \int_{-\varepsilon}^{0} \left(\chi_{[0,\log x]}(-\tau) e^{\frac{-\tau}{2}} \right) \cdot \eta_{c,\varepsilon}(\tau) d\tau \\ &= \frac{1}{\lambda_{c,\varepsilon}} \int_{-\varepsilon}^{0} e^{\frac{-\tau}{2}} \eta_{c,\varepsilon}(\tau) d\tau,\end{aligned}$$

as $\chi_{[0,\log x]}(-\tau) = 0$ for $\tau \in [0,\varepsilon]$.

Now, since $e^{(\bullet/2)} * \eta_{c,\varepsilon}(t) = \lambda_{c,\varepsilon} e^{t/2}$, from Lemma 2(i), we can write

$$e^{t/2} = \frac{1}{\lambda_{c,\varepsilon}} \int_{-\varepsilon}^{\varepsilon} \eta_{c,\varepsilon}(\tau) e^{\frac{t-\tau}{2}} d\tau.$$

From definition 6, equation (3.9) $f_x(t) = \chi^*[0,\log x](t)e^{t/2}$. So, using the above identity, we write

$$f_x(t) = \chi^*[0,\log x](t) \frac{1}{\lambda_{c,\varepsilon}} \left[\int_{-\varepsilon}^{\varepsilon} \eta_{c,\varepsilon}(\tau) e^{\frac{t-\tau}{2}} d\tau \right],$$

and thus

$$f_x(0) = \frac{1}{2\lambda_{c,\varepsilon}} \left[\int_{-\varepsilon}^{\varepsilon} \eta_{c,\varepsilon}(\tau) e^{\frac{-\tau}{2}} d\tau \right].$$

Recalling that $\Delta(t) = \phi_{x,c,\varepsilon}(t) - f_x(t)$, it follows that

$$\begin{aligned} \Delta(t) =& \frac{1}{\lambda_{c,\varepsilon}} \int_{-\varepsilon}^{\varepsilon} \left(\chi_{[0,\log x]}(t-\tau) e^{\frac{t-\tau}{2}} \right) \cdot \eta_{c,\varepsilon}(\tau) d\tau \\ & - \chi^*[0,\log x](t) \frac{1}{\lambda_{c,\varepsilon}} \left[\int_{-\varepsilon}^{\varepsilon} \eta_{c,\varepsilon}(\tau) e^{\frac{t-\tau}{2}} d\tau \right]. \end{aligned} \tag{4.8}$$

The first integrand in equation (4.8) is nonzero for $0 \le t-\tau \le \log x$ which implies $t - \log x \le \tau \le t$. Therefore $\Delta(t)$ can now be written as

$$\Delta(t) = \frac{1}{\lambda_{c,\varepsilon}} \int_{[-\varepsilon.\varepsilon]\cap[t-\log x,t]} e^{\frac{t-\tau}{2}} \cdot \eta_{c,\varepsilon}(\tau) d\tau - \frac{1}{\lambda_{c,\varepsilon}} \left[\int_{-\varepsilon}^{\varepsilon} \eta_{c,\varepsilon}(\tau) e^{\frac{t-\tau}{2}} d\tau \right],$$

which implies

$$\Delta(t) = \frac{1}{\lambda_{c,\varepsilon}} \int_{-\varepsilon}^{t} e^{\frac{t-\tau}{2}} \cdot \eta_{c,\varepsilon}(\tau) d\tau - \frac{1}{\lambda_{c,\varepsilon}} \left[\int_{-\varepsilon}^{\varepsilon} \eta_{c,\varepsilon}(\tau) e^{\frac{t-\tau}{2}} d\tau \right] = -\frac{1}{\lambda_{c,\varepsilon}} \int_{t}^{\varepsilon} \eta_{c,\varepsilon}(\tau) e^{\frac{t-\tau}{2}} d\tau. \tag{4.9}$$

Similarly $\Delta(-t)$ can be written as

$$\Delta(-t) = \frac{1}{\lambda_{c,\varepsilon}} \int_{-\varepsilon}^{\varepsilon} \left(\chi_{[0,\log x]}(-t-\tau) e^{\frac{-t-\tau}{2}} \right) \cdot \eta_{c,\varepsilon}(\tau) d\tau \\ - \chi^*[0, \log x](-t) \frac{1}{\lambda_{c,\varepsilon}} \left[\int_{-\varepsilon}^{\varepsilon} \eta_{c,\varepsilon}(\tau) e^{\frac{-t-\tau}{2}} d\tau \right]. \tag{4.10}$$

We note that the first integrand in equation (4.10) is nonzero for $0 \le -t-\tau \le \log x$ which implies $-t-\log x \le \tau \le -t$. Also, since $t \ge 0$, $-t < 0$, and the second integrand in $\Delta(-t)$ is always 0, and we can write $\Delta(-t)$ as

$$\Delta(-t) = \frac{1}{\lambda_{c,\varepsilon}} \int_{[-\varepsilon,\varepsilon]\cap[-t-\log x,-t]} \eta_{c,\varepsilon}(\tau) e^{\frac{-t-\tau}{2}} d\tau = \frac{1}{\lambda_{c,\varepsilon}} \int_{-\varepsilon}^{-t} \eta_{c,\varepsilon}(\tau) e^{\frac{-t-\tau}{2}} d\tau.$$

Substituting $-\tau = u$, we get

$$\Delta(-t) = \frac{1}{\lambda_{c,\varepsilon}} \int_{t}^{\varepsilon} \eta_{c,\varepsilon}(u) e^{\frac{-t+u}{2}} du$$

since $\eta_{c,\varepsilon}(t)$ is an even function. Now, replacing u by τ in the above equation, we can write

$$\Delta(-t) = \frac{1}{\lambda_{c,\varepsilon}} \int_{t}^{\varepsilon} \eta_{c,\varepsilon}(\tau) e^{\frac{\tau-t}{2}} d\tau. \tag{4.11}$$

Furthermore, for $t = 0$ we can compute $\Delta(0)$ as

$$\begin{aligned} \Delta(0) &= \frac{1}{\lambda_{c,\varepsilon}} \int_{-\varepsilon}^{0} e^{\frac{-\tau}{2}} \eta_{c,\varepsilon}(\tau) d\tau - \frac{1}{2\lambda_{c,\varepsilon}} \left[\int_{-\varepsilon}^{0} \eta_{c,\varepsilon}(\tau) e^{\frac{-\tau}{2}} d\tau + \int_{0}^{\varepsilon} \eta_{c,\varepsilon}(\tau) e^{\frac{-\tau}{2}} d\tau \right] \\ &= \frac{1}{2\lambda_{c,\varepsilon}} \int_{-\varepsilon}^{0} e^{\frac{-\tau}{2}} \eta_{c,\varepsilon}(\tau) d\tau - \frac{1}{2\lambda_{c,\varepsilon}} \int_{0}^{\varepsilon} \eta_{c,\varepsilon}(\tau) e^{\frac{-\tau}{2}} d\tau \\ &= \frac{1}{2\lambda_{c,\varepsilon}} \int_{0}^{\varepsilon} \eta_{c,\varepsilon}(\tau)(e^{\tau/2} - e^{-\tau/2}) d\tau. \end{aligned}$$

Adding the identities for $\Delta(\tau)$ and $\Delta(-\tau)$ in equations (4.9) and (4.11) respectively, we obtain

$$\Delta(t)+\Delta(-t)=\frac{1}{\lambda_{c,\varepsilon}}\int_{t}^{\varepsilon}\eta_{c,\varepsilon}(\tau)(e^{\frac{\tau-t}{2}})d\tau-\frac{1}{\lambda_{c,\varepsilon}}\int_{t}^{\varepsilon}\eta_{c,\varepsilon}(\tau)(e^{\frac{t-\tau}{2}})d\tau. \tag{4.12}$$

Lemma 6 states that $e^{t+\tau}=e^{t}+O^{*}(2|\tau|)$. We substitute $t=\frac{\tau}{2}$ and $\tau=\frac{-t}{2}$ in Lemma 6 to get $e^{\frac{\tau-t}{2}}=e^{\frac{\tau}{2}}+O^{*}(|t|)$ and apply the result to the first term of equation (4.12). Similarly, we replace $t=\frac{-\tau}{2}$ and $\tau=\frac{t}{2}$ in Lemma 6 and apply it to the second term of equation (4.12) to obtain,

$$\Delta(t)+\Delta(-t)=\frac{1}{\lambda_{c,\varepsilon}}\int_{t}^{\varepsilon}\eta_{c,\varepsilon}(\tau)(e^{\frac{\tau}{2}}-e^{\frac{-\tau}{2}})d\tau+O^{*}(t).$$

Now, since putting $t=\frac{-\tau}{2}$ in Lemma 6 gives us $e^{\frac{\tau}{2}}=e^{-\frac{\tau}{2}}+O^{*}(2|\tau|)$, we can write

$$\left|\int_{0}^{t}\eta_{c,\varepsilon}(\tau)(e^{\frac{\tau}{2}}-e^{\frac{-\tau}{2}})d\tau\right|\leq 2\int_{0}^{t}\eta_{c,\varepsilon}(\tau)\tau d\tau\leq 2t\int_{0}^{t}\eta_{c,\varepsilon}(\tau)d\tau\leq 2t\int_{0}^{\varepsilon}\eta_{c,\varepsilon}(\tau)d\tau.$$

We note that $\eta_{c,\varepsilon](t)}$ is even, $\int_{-\varepsilon}^{\varepsilon}\eta_{c,\varepsilon}(\tau)d\tau=1$ from equation (3.16) of Lemma 1 , and $\frac{1}{\lambda_{c,\varepsilon}}\leq 1$ by Lemma 3,equation (3.35), which implies

$$\left|\int_{0}^{t}\eta_{c,\varepsilon}(\tau)(e^{\frac{\tau}{2}}-e^{\frac{-\tau}{2}})d\tau\right|\leq 2\frac{1}{2}t\int_{-\varepsilon}^{\varepsilon}\eta_{c,\varepsilon}(\tau)d\tau\leq t.$$

Therefore, we now deduce that

$$\begin{aligned}\Delta(t)+\Delta(-t)&=\frac{1}{\lambda_{c,\varepsilon}}\int_{0}^{\varepsilon}\eta_{c,\varepsilon}(\tau)(e^{\frac{\tau}{2}}-e^{\frac{-\tau}{2}})d\tau+O^{*}(2t)\\&=2\Delta(0)+O^{*}(2t),\end{aligned}$$

which proves Lemma 8(ii).

(iii) We recall from equation (4.8) that

$$\Delta(t) = \frac{1}{\lambda_{c,\varepsilon}} \int_{-\varepsilon}^{\varepsilon} \left(\chi_{[0,\log x]}(t-\tau)e^{\frac{t-\tau}{2}}\right) \cdot \eta_{c,\varepsilon}(\tau)d\tau \\ - \chi^*[0,\log x](t)\frac{1}{\lambda_{c,\varepsilon}}\left[\int_{-\varepsilon}^{\varepsilon} \eta_{c,\varepsilon}(\tau)e^{\frac{t-\tau}{2}}d\tau\right].$$

Therefore for $t > \log x$ we can write $\Delta(t)$ as

$$\Delta(t) = \frac{1}{\lambda_{c,\varepsilon}} \int_{-\varepsilon}^{\varepsilon} \left(\chi_{[0,\log x]}(t-\tau)e^{\frac{t-\tau}{2}}\right) \cdot \eta_{c,\varepsilon}(\tau)d\tau.$$

Now, for $\Delta(t)$ to be non-zero, we need $0 \leq t-\tau \leq \log x$. Note that this implies $t - \log x \leq \tau \leq t$. So for $t \in B_\varepsilon \log x$ such that $t > \log x$, we have

$$\Delta(t) = \frac{1}{\lambda_{c,\varepsilon}} \int_{\tau\in[-\varepsilon,\varepsilon]\cap[t-\log x,t]} e^{\frac{t-\tau}{2}} \cdot \eta_{c,\varepsilon}(\tau)d\tau = \frac{1}{\lambda_{c,\varepsilon}} \int_{t-\log x}^{\varepsilon} \eta_{c,\varepsilon}(\tau)e^{\frac{t-\tau}{2}}d\tau, \quad (4.13)$$

since $t - \log x > -\varepsilon$ and $t > \varepsilon$ because $t > \log x$.

For $t \in B_\varepsilon(\log x)$ such that $t \leq \log x$, we know $\log x - \varepsilon \leq t \leq \log x$, and we can write ,

$$\begin{aligned}\Delta(t) &= \frac{1}{\lambda_{c,\varepsilon}}\left(\int_{\tau\in[-\varepsilon,\varepsilon]\cap[t-\log x,t]} e^{\frac{t-\tau}{2}} \cdot \eta_{c,\varepsilon}(\tau)d\tau\right) - \frac{1}{\lambda_{c,\varepsilon}}\left(\int_{-\varepsilon}^{\varepsilon} \eta_{c,\varepsilon}(\tau)e^{\frac{t-\tau}{2}}d\tau\right) \\ &= \frac{1}{\lambda_{c,\varepsilon}}\left(\int_{t-\log x}^{\varepsilon} e^{\frac{t-\tau}{2}} \cdot \eta_{c,\varepsilon}(\tau)d\tau\right) - \frac{1}{\lambda_{c,\varepsilon}}\left(\int_{-\varepsilon}^{\varepsilon} \eta_{c,\varepsilon}(\tau)e^{\frac{t-\tau}{2}}d\tau\right) \\ &= -\frac{1}{\lambda_{c,\varepsilon}}\left(\int_{-\varepsilon}^{t-\log x} \eta_{c,\varepsilon}(\tau)e^{\frac{t-\tau}{2}}d\tau\right) \qquad (4.14)\end{aligned}$$

Using equations (4.13) and (4.14), we see, that for $t \in B_\varepsilon(\log x)$, we have

$$\Delta(t) = \frac{\chi_{(\log x,\infty)}(t)}{\lambda_{c,\varepsilon}} \int_{t-\log x}^{\varepsilon} \eta_{c,\varepsilon}(\tau)e^{\frac{t-\tau}{2}}d\tau - \frac{\chi_{(0,\log x)}(t)}{\lambda_{c,\varepsilon}} \int_{-\varepsilon}^{t-\log x} \eta_{c,\varepsilon}(\tau)e^{\frac{t-\tau}{2}}d\tau. \quad (4.15)$$

For $t \in (\log x, \infty)$ the second term in equation (4.15) is 0. To compute the first term we note that $t - \log x \leq \tau$, which implies $\frac{t-\tau}{2} \leq \frac{\log x}{2}$. Therefore, for $\log x < t < \infty$, we can write

$$\frac{\chi_{(\log x,\infty)}(t)}{\lambda_{c,\varepsilon}}\int_{t-\log x}^{\varepsilon}\eta_{c,\varepsilon}(\tau)e^{\frac{t-\tau}{2}}d\tau \leq \int_{t-\log x}^{\varepsilon}\eta_{c,\varepsilon}(\tau)e^{\frac{\log x}{2}}d\tau \leq \sqrt{x}\int_{0}^{\varepsilon}\eta_{c,\varepsilon}(\tau)d\tau,$$

since we know, $\lambda_{c,\varepsilon} \geq 1$ from Lemma 3, equation (3.35). From Lemma 1, equation (3.16), we know $\int_{-\varepsilon}^{\varepsilon}\eta_{c,\varepsilon}(\tau)d\tau = 1$ and since $\eta_{c,\varepsilon}$ is an even function, we can write that $\int_{0}^{\varepsilon}\eta_{c,\varepsilon}(\tau)d\tau = \frac{1}{2}$. Thus, we can deduce that

$$\frac{\chi_{(\log x,\infty)}(t)}{\lambda_{c,\varepsilon}}\int_{t-\log x}^{\varepsilon}\eta_{c,\varepsilon}(\tau)e^{\frac{t-\tau}{2}}d\tau \leq \frac{1}{2}\sqrt{x}.$$

which implies that for $t \in (\log x, \infty)$, $|\Delta(t)| \leq \frac{1}{2}\sqrt{x}$.

For $t \in (0, \log x)$, the first term in equation (4.15) is 0. To compute the second term, we note that $\tau \geq -\varepsilon$ which implies $\frac{t-\tau}{2} \leq \frac{t+\varepsilon}{2}$. Moreover, for t in the given interval, $t - \log x > 0$ and thus $\frac{t+\varepsilon}{2} > \frac{\log x+\varepsilon}{2}$. Therefore we obtain the inequality

$$\frac{\chi_{(0,\log x)}(t)}{\lambda_{c,\varepsilon}}\int_{-\varepsilon}^{t-\log x}\eta_{c,\varepsilon}(\tau)e^{\frac{t-\tau}{2}}d\tau \leq \int_{-\varepsilon}^{0}\eta_{c,\varepsilon}(\tau)e^{\frac{\log x+\varepsilon}{2}}d\tau \leq \frac{1}{2}\sqrt{x}e^{\frac{\varepsilon}{2}},$$

which proves that for $t \in (0, \log x)$, $|\Delta(t)| \leq \frac{1}{2}\sqrt{x}e^{\frac{\varepsilon}{2}}$. Thus we have proved that for all t, $|\Delta(t)| \leq \frac{1}{2}e^{\frac{\varepsilon}{2}}\sqrt{x}$ which improves the inequality in Lemma 8(iii), equation (4.6).

(iv) We write,

$$\Delta(0) = \frac{1}{2\lambda_{c,\varepsilon}}\int_{0}^{\varepsilon}\eta_{c,\varepsilon}(\tau)(e^{\tau/2} - e^{-\tau/2})d\tau.$$

Using $t = -\frac{\tau}{2}$ in Lemma 6, we obtain $e^{\frac{\tau}{2}} = e^{-\frac{\tau}{2}} + O^{*}(2|\tau|)$.

Using this result in the equation for $\Delta(0)$ we are able to write,

$$\Delta(0) \leq \frac{1}{2\lambda_{c,\varepsilon}}\int_{0}^{\varepsilon}\eta_{c,\varepsilon}(\tau)2\tau d\tau \leq \varepsilon\cdot\frac{1}{\lambda_{c,\varepsilon}}\int_{0}^{\varepsilon}\eta_{c,\varepsilon}(\tau)d\tau.$$

Since $\int_{-\varepsilon}^{\varepsilon}\eta_{c,\varepsilon}(\tau)d(\tau) = 1$ and η is an even function, we know $\int_{0}^{\varepsilon}\eta_{c,\varepsilon}(\tau)d(\tau) = \frac{1}{2}$.

Now since $\lambda_{c,\varepsilon} \geq 1$ i.e. $\frac{1}{\lambda_{c,\varepsilon}} \leq 1$, we have $\Delta(0) \leq \frac{\varepsilon}{2} \leq \varepsilon$, as desired.

□

The following result is a variant of the Weil's explicit formula.

Proposition 2. *[Weil-Barner] [3] Let $g : \mathbb{R} \to \mathbb{C}$ satisfy the following conditions:*

- *There is an $a > \frac{1}{2}$ such that the function $e^{a|t|}g(t)$ is Lebesgue Integrable and is of limited variation.*
- *The function g is normalized, i.e. $\forall t \in \mathbb{R}$*

$$g(t) = \frac{1}{2}\lim_{h\to 0}(g(t+h)+g(t-h)).$$

- *There exists an $\varepsilon > 0$, such that,*

$$2g(0) = g(t)+g(-t)+O(|t|^{\varepsilon}).$$

Define

$$w_s(\hat{g}) = \sum_{\rho}^{*} \hat{g}(i/2-is) - \hat{g}(i/2) - \hat{g}(-i/2), \tag{4.16}$$

$$w_f(g) = -\sum_{p}\sum_{m=0}^{\infty}\frac{\log p}{p^{m/2}}\left(g(m\log p)+g(-m\log p)\right), \tag{4.17}$$

$$w_\infty(g) = \left(\frac{\Gamma'}{\Gamma}(1/4)-\log\pi\right)g(0) - \int_0^\infty \frac{g(t)+g(-t)-2g(0)}{1-e^{-2t}}e^{-t/2}dt. \tag{4.18}$$

Then we have

$$w_s(\hat{g}) = w_f(g)+w_\infty(g). \tag{4.19}$$

Before we begin the proof of Proposition 2, we need to prove some lemmas. The next lemma evaluates a certain integral that appears in the proof of Lemma 10 .

Lemma 9. *For a real number t, the following identity holds:*

$$\int_0^{\log x} \left(\frac{e^{-2t}}{2t} - \frac{e^{-2t}}{1-e^{-2t}} \right) dt = -\frac{\gamma_0}{2} - \frac{E_1(2\log x)}{2} - \frac{\log(1-e^{-2\log x})}{2} \tag{4.20}$$

where γ_0 is the Euler-Mascheroni constant and E_1 denotes the first exponential integral and is defined as

$$E_1(y) = \int_y^{\infty} \frac{e^{-t}}{t} dt.$$

Proof. We start our proof by noting that

$$\int_0^{\log x} \left(\frac{e^{-2t}}{2t} - \frac{e^{-2t}}{1-e^{-2t}} \right) dt = -\int_0^{\log x} \left(\frac{1}{e^{2t}-1} - \frac{1}{e^{2t}2t} \right) dt.$$

From [34, 12.3, page 255], we know the Gauss-Digamma function satisfies

$$\frac{\Gamma'}{\Gamma}(z) = \int_0^{\infty} \left(\frac{1}{e^t t} - \frac{e^{-zt}}{1-e^{-t}} \right) dt$$

and since $\frac{\Gamma'}{\Gamma}(1) = -\gamma_0$, we get the identity

$$\gamma_0 = \int_0^{\infty} \left(\frac{1}{e^t-1} - \frac{1}{e^t t} \right) dt$$

holds, we make a variable change by replacing t with $2t$ to get

$$\begin{aligned} -\frac{\gamma_0}{2} &= -\int_0^{\infty} \left(\frac{1}{e^{2t}-1} - \frac{1}{e^{2t}2t} \right) dt \\ &= -\left[\int_0^{\log x} \left(\frac{1}{e^{2t}-1} - \frac{1}{e^{2t}2t} \right) dt + \int_{\log x}^{\infty} \left(\frac{1}{e^{2t}-1} - \frac{1}{e^{2t}2t} \right) dt \right]. \end{aligned}$$

This gives us the following equation for $-\int_0^{\log x}\left(\frac{1}{e^{2t}-1}-\frac{1}{e^{2t}2t}\right)dt$:

$$\begin{aligned}\int_0^{\log x}\left(\frac{1}{e^{2t}-1}-\frac{1}{e^{2t}2t}\right)dt &= -\frac{\gamma_0}{2}+\int_{\log x}^{\infty}\left(\frac{1}{e^{2t}-1}-\frac{1}{e^{2t}2t}\right)dt\\ &= -\frac{\gamma_0}{2}-\frac{E_1(2\log x)}{2}+\int_{\log x}^{\infty}\frac{1}{e^{2t}-1}dt\\ &= -\frac{\gamma_0}{2}-\frac{E_1(2\log x)}{2}-\frac{\log(1-e^{-2\log x})}{2}\end{aligned}$$

and thus we have proved equation (4.20) of Lemma 9. □

Using the formulas for $w_s(\hat{g})$, $w_f(g)$ and $w_\infty(g)$ given in Proposition 2, we prove the subsequent lemma:

Lemma 10. *For* $\log x > \frac{2}{|\log(\varepsilon)|}$, *and* $0<\varepsilon<\frac{1}{10^2}$ *we have the following identities:*

(i)

$$w_f(\phi_{x,c,\varepsilon}) = -\psi_{c,\varepsilon}(x) \tag{4.21}$$

(ii)

$$w_s(\hat{\phi}_{x,c,\varepsilon}) = \sum_{\rho}^{*} a_{c,\varepsilon}(\rho)\frac{x^{\rho}-1}{\rho} - x - \log x + 1 \tag{4.22}$$

(iii)

$$w_\infty(f_x) = -\log x\frac{\gamma}{2}-\frac{1}{2}\log\pi-\frac{1}{2}\log(1-x^2) \tag{4.23}$$

(iv)

$$w_\infty(\Delta) = O^*(8\varepsilon|\log\varepsilon|) \tag{4.24}$$

Proof. (i) From the definition of $w_f(g)$ given in equation (4.17), we know

$$w_f(g) = -\sum_p\sum_{m=0}^{\infty}\frac{\log p^m}{p^{m/2}}(g(m\log p)+g(-m\log p)).$$

Putting $g = \phi_{x,c,\varepsilon}$ in the above definition of $w_f(g)$, we get

$$w_f(\phi_{x,c,\varepsilon}) = -\sum_p \sum_{m=0}^{\infty} \frac{\log p^m}{p^{m/2}} \left(\phi_{x,c,\varepsilon}(m\log p) + \phi_{x,c,\varepsilon}(-m\log p)\right).$$

Since $\chi_{[0,\log x]}(-m\log p) = 0$, the above equation now reduces to

$$w_f(\phi_{x,c,\varepsilon}) = -\sum_p \sum_{m=0}^{\infty} \frac{\log p^m}{p^{m/2}} (\phi_{x,c,\varepsilon}(m\log p)) = -\psi_{c,\varepsilon}(x),$$

which proves equation (4.21).

(ii) From the definition of $w_s(\hat{g})$ given in equation (4.16), we know

$$w_s(\hat{g}) = \sum_{\rho}^{*} \hat{g}(i/2 - i\rho) - \hat{g}(i/2) - \hat{g}(-i/2).$$

Using $g = \phi_{x,c,\varepsilon}$ in the above definition, we obtain,

$$w_s(\hat{\phi}_{x,c,\varepsilon}) = \sum_{\rho}^{*} \hat{\phi}_{x,c,\varepsilon}(i/2 - i\rho) - \hat{\phi}_{x,c,\varepsilon}(i/2) - \hat{\phi}_{x,c,\varepsilon}(-i/2). \tag{4.25}$$

Since $\phi_{x,c,\varepsilon} = \lambda_{c,\varepsilon}^{-1} f_x * \eta_{c,\varepsilon}$,from Lemma 4(ii), equation (3.36), by the Convolution Theorem [4, page 117], we can state:

$$\hat{\phi}_{x,c,\varepsilon} = \lambda_{c,\varepsilon}^{-1}(\hat{f}_x)(\hat{\eta}_{c,\varepsilon}) = \lambda_{c,\varepsilon}^{-1}(\hat{f}_x)\ell_{c,\varepsilon}.$$

Now, using Fourier Transform on the function $f_x(t)$, we write

$$\begin{aligned}\hat{f}_x(\xi) &= \int_{-\infty}^{\infty} e^{i\xi t} f_x(t)dt = \int_{-\infty}^{\infty} e^{i\xi t} \chi_{[0,\log x]}^{*}(t) e^{t/2} dt = \int_0^{\log x} e^{i\xi t} e^{t/2} dt \\ &= \int_0^{\log x} e^{t(i\xi + \frac{1}{2})} dt. \end{aligned} \tag{4.26}$$

If $\xi = (i/2 - i\rho)$, we get $i\xi + \frac{1}{2} = \rho$. Therefore, putting these values in the equation

(4.26), we get

$$\hat{f}_x(\xi) = \int_0^{\log x} e^{\rho t} dt = \left. \frac{e^{\rho t}}{\rho} \right|_0^{\log x} = \frac{x^\rho - 1}{\rho},$$

and since $\left(\frac{i}{2} - i\rho\right) = \left(\frac{-1}{2i} + \frac{\rho}{i}\right)$, we have

$$\begin{aligned} \hat{\phi}_{x,c,\varepsilon}(i/2 - i\rho) &= \lambda_{c,\varepsilon}^{-1} \ell_{c,\varepsilon} \left(\frac{i}{2} - i\rho\right) \frac{x^\rho - 1}{\rho} \\ &= \frac{1}{\lambda_{c,\varepsilon}} \ell_{c,\varepsilon} \left(\frac{\rho}{i} - \frac{1}{2i}\right) \frac{x^\rho - 1}{\rho} = a_{c,\varepsilon}(\rho) \frac{x^\rho - 1}{\rho}, \end{aligned}$$

as $a_{c,\varepsilon}(\rho) = \ell_{c,\varepsilon}\left(\frac{\rho}{i} - \frac{1}{2i}\right)$ by definition 6, equation (3.10).

Again, if $\xi = i/2$, we get $i\xi + \frac{1}{2} = 0$, and using these values in equation (4.26), we see that

$$\hat{f}_x(\xi) = \int_0^{\log x} 1 dt = \log x.$$

Therefore, we can write

$$\hat{\phi}_{x,c,\varepsilon}(i/2) = \lambda_{c,\varepsilon}^{-1} \ell_{c,\varepsilon}(i/2)(\log x) = \log x,$$

since by definition 3, equation (3.5), $\ell_{c,\varepsilon}(i/2) = \lambda_{c,\varepsilon}$.

For $\xi = -i/2$, we get $i\xi + \frac{1}{2} = 1$, and putting these values in equation (4.26) gives us

$$\hat{f}_x(\xi) = \int_0^{\log x} e^t dt = x - 1.$$

Since $\ell_{c,\varepsilon}$ is an even function, we can write,

$$\hat{\phi}_{x,c,\varepsilon}(-i/2) = \lambda_{c,\varepsilon}^{-1} \ell_{c,\varepsilon}(-i/2)(x-1) = \lambda_{c,\varepsilon}^{-1} \ell_{c,\varepsilon}(i/2)(x-1) = x - 1.$$

Putting the necessary values of $\phi_{x,c,\varepsilon}$ in equation (4.25) we get

$$\begin{aligned} w_s(\hat{\phi}_{x,c,\varepsilon}) &= \sum_{\rho}^{*} \hat{\phi}_{x,c,\varepsilon}(i/2 - i\rho) - \hat{\phi}_{x,c,\varepsilon}(i/2) - \hat{\phi}_{x,c,\varepsilon}(-i/2) \\ &= \sum_{\rho}^{*} a_{c,\varepsilon}(\rho)\frac{x^{\rho}-1}{\rho} - \log x - x + 1 \end{aligned}$$

which proves the identity (4.22).

(iii) Since we know from equation (4.18) that

$$w_{\infty}(g) = \left(\frac{\Gamma'}{\Gamma}(1/4) - \log\pi\right) g(0) - \int_0^{\infty} \frac{g(t)+g(-t)-2g(0)}{1-e^{-2t}} e^{-t/2} dt,$$

and from definition 6, equation (3.9) that $f_x(t) = \chi^*_{[0,\log x]}(t)\exp(t/2)$; we can write that

$$\begin{aligned} w_{\infty}(f_x) &= \left(\frac{\Gamma'}{\Gamma}(1/4) - \log\pi\right)\frac{1}{2} - \int_0^{\infty} \frac{\chi^*_{[0,\log x]}(t)e^{t/2} + \chi^*_{[0,\log x]}(-t)e^{-t/2} - 1}{1-e^{-2t}} e^{-t/2} dt \\ &= \frac{1}{2}\frac{\Gamma'}{\Gamma}(1/4) - \frac{1}{2}\log\pi - \int_0^{\log x} \frac{1}{1-e^{-2t}} dt - \int_0^{\infty} \frac{e^{-t/2}}{1-e^{-2t}} dt \\ &= \frac{1}{2}\frac{\Gamma'}{\Gamma}(1/4) - \frac{1}{2}\log\pi - \int_0^{\log x} \frac{1-e^{-t/2}}{1-e^{-2t}} dt + \int_{\log x}^{\infty} \frac{e^{-t/2}}{1-e^{-2t}} dt. \end{aligned} \tag{4.27}$$

We see that the term $-\int_0^{\log x} \frac{1-e^{-t/2}}{1-e^{-2t}}$ on the right hand side of the above equation can be simplified as :

$$\begin{aligned} -\int_0^{\log x} \frac{1-e^{-t/2}}{1-e^{-2t}} &= \int_0^{\log x} \left(\frac{e^{-t/2} - 1 + 1 - e^{-2t}}{1-e^{-2t}} - 1\right) dt \\ &= \int_0^{\log x} \frac{e^{-t/2} - e^{-2t}}{1-e^{-2t}} dt - \log x. \end{aligned}$$

Now, we note from [34, 12.3, page 255], that the Gauss-Digamma function satisfies the identity

$$\frac{\Gamma'}{\Gamma}(z) = \int_0^\infty \left(\frac{e^{-u}}{u} - \frac{e^{-zu}}{1-e^{-u}} \right) du,$$

and so it follows that

$$\frac{\Gamma'}{\Gamma}(1/4) = \int_0^\infty \left(\frac{e^{-u}}{u} - \frac{e^{-u/4}}{1-e^{-u}} \right) du = \int_0^\infty \left(\frac{e^{-2t}}{2t} - \frac{e^{-t/2}}{1-e^{-2t}} \right) 2dt,$$

by the substitution $u/2 = t$. Therefore,

$$\frac{1}{2}\frac{\Gamma'}{\Gamma}(1/4) = \int_0^\infty \frac{e^{-2t}}{2t} - \frac{e^{-t/2}}{1-e^{-2t}} dt.$$

Substituting the value of $\frac{1}{2}\frac{\Gamma'}{\Gamma}(1/4)$ in equation (4.27), we obtain

$$\begin{aligned}
w_\infty(f_x) = &-\frac{1}{2}\log\pi - \log x + \int_0^{\log x} \frac{e^{-t/2}-e^{-2t}}{1-e^{-2t}} dt + \int_0^\infty \frac{e^{-2t}}{2t} - \frac{e^{-t/2}}{1-e^{-2t}} dt \\
&+ \int_{\log x}^\infty \frac{e^{-t/2}}{1-e^{-2t}} dt \\
= &-\log x - -\frac{1}{2}\log\pi + \int_0^{\log x} \frac{e^{-t/2}}{1-e^{-2t}} dt - \int_0^{\log x} \frac{e^{-2t}}{1-e^{-2t}} dt + \int_0^\infty \frac{e^{-2t}}{2t} dt \\
&- \int_0^\infty \frac{e^{-t/2}}{1-e^{-2t}} dt + \int_{\log x}^\infty \frac{e^{-t/2}}{1-e^{-2t}} dt \\
= &-\log x - -\frac{1}{2}\log\pi + \int_0^{\log x} \left(\frac{e^{-2t}}{2t} - \frac{e^{-2t}}{1-e^{-2t}} \right) dt + \int_{\log x}^\infty \frac{e^{-2t}}{2t} dt.
\end{aligned}$$

Equation (4.20) in Lemma 9 proves that

$$\int_0^{\log x} \left(\frac{e^{-2t}}{2t} - \frac{e^{-2t}}{1-e^{-2t}} \right) dt = -\frac{\gamma}{2} - \frac{E_1(2\log x)}{2} - \frac{\log(1-e^{-2\log x})}{2}.$$

Since $\int_{\log x}^{\infty} \frac{e^{-2t}}{2t} dt = \frac{E_1(2\log x)}{2}$, we can claim that

$$\begin{aligned} w_\infty(f_x) &= -\log x - \frac{1}{2}\log \pi - \frac{\gamma}{2} - \frac{E_1(2\log x)}{2} - \frac{\log(1-x^{-2})}{2} + \frac{E_1(2\log x)}{2} \\ &= -\log x - \frac{\gamma}{2} - \frac{1}{2}\log \pi - \frac{1}{2}\log(1-x^{-2}), \end{aligned}$$

thus completing the proof of equation (4.23).

(iv) Substituting $g = \Delta$ in Proposition 2, equation (4.18), we can write that

$$w_\infty(\Delta) = \left(\frac{\Gamma'}{\Gamma}(1/4) - \log \pi\right)\Delta(0) - \int_0^\infty \frac{\Delta(t)+\Delta(-t)-2\Delta(0)}{1-e^{-2t}} e^{-t/2} dt. \qquad (4.28)$$

The integral in $w_\infty(\Delta)$ can be divided as follows:

$$\begin{aligned} \int_0^\infty \frac{\Delta(t)+\Delta(-t)-2\Delta(0)}{1-e^{-2t}} e^{-t/2} dt = &\int_0^\varepsilon \frac{\Delta(t)+\Delta(-t)-2\Delta(0)}{1-e^{-2t}} e^{-t/2} dt \\ &+ \int_\varepsilon^\infty \frac{\Delta(t)+\Delta(-t)}{1-e^{-2t}} e^{-t/2} dt \\ &- 2\int_\varepsilon^\infty \frac{\Delta(0)}{1-e^{-2t}} e^{-t/2} dt. \end{aligned}$$

Since $\Delta(t)$ vanishes for $t \notin (B_\varepsilon(0) \cup B_\varepsilon(\log x))$, the integral can be rewritten as

$$\begin{aligned} \int_0^\infty \frac{\Delta(t)+\Delta(-t)-2\Delta(0)}{1-e^{-2t}} e^{-t/2} dt = &\int_0^\varepsilon \frac{\Delta(t)+\Delta(-t)-2\Delta(0)}{1-e^{-2t}} e^{-t/2} dt \\ &+ \int_{B_\varepsilon(\log x)} \frac{\Delta(t)}{1-e^{-2t}} e^{-t/2} dt \qquad (4.29) \\ &- 2\int_\varepsilon^\infty \frac{\Delta(0)}{1-e^{-2t}} e^{-t/2} dt. \qquad (4.30) \end{aligned}$$

Noting that $t \to \frac{1-e^{-2t}}{t}$ is monotonically decreasing in $(0,\infty)$, for $0 \le t \le \varepsilon \le 1/10$, we can see that

$$1 - e^{-2t} \ge \left(1 - e^{(-\frac{2}{10})}\right) t \ge 1.8t. \qquad (4.31)$$

Lemma 8(ii), equation (4.5) implies that $\Delta(t)+\Delta(-t)-2\Delta(0) \le 2t$. Using this inequality along with the inequality (4.31) in the first integral of equation (4.29), we obtain,

$$\int_0^{\varepsilon} \frac{\Delta(t)+\Delta(-t)-2\Delta(0)}{1-e^{-2t}} e^{-t/2} dt \le \int_0^{\varepsilon} \frac{2t}{1.8t} dt \le 1.2\varepsilon, \tag{4.32}$$

since $e^{-t/2} < 1$ for $0 \le t \le \varepsilon \le 1/10$.

Next we try to bound the second integral in equation (4.29), and see that

$$1-e^{-2t} \ge 1-e^{2\varepsilon-2\log x} \ge 1-e^{-\frac{4}{|\log\varepsilon|}} \ge \frac{2}{|\log\varepsilon|} \tag{4.33}$$

for $t \in B_{\varepsilon}(\log x)$. Using the inequality (4.33) and the inequality $|\Delta(t)| \le \frac{1}{2} e^{\frac{\varepsilon}{2}} \sqrt{x}$ given in Lemma 8, equation (4.6), we obtain,

$$\begin{aligned}\int_{B_{\varepsilon}(\log x)} \frac{|\Delta(t)|}{1-e^{-2t}} e^{-t/2} dt \le & \frac{1}{2} e^{\varepsilon/2} \sqrt{x} \cdot e^{\frac{-(\log x-\varepsilon)}{2}} \int_{B_{\varepsilon}(\log x)} \frac{dt}{1-e^{-2t}} \\ \le & \frac{1}{2} e^{\varepsilon/2} \sqrt{x} \frac{e^{\varepsilon/2}}{\sqrt{x}} \int_{B_{\varepsilon}(\log x)} \frac{dt}{1-e^{-2t}}.\end{aligned}$$

From equation (4.33), since $1-e^{-2t} \ge \frac{2}{|\log\varepsilon|}$, we can now write

$$\frac{1}{2} e^{\varepsilon/2} \sqrt{x} \frac{e^{\varepsilon/2}}{\sqrt{x}} \int_{B_{\varepsilon}(\log x)} \frac{dt}{1-e^{-2t}} \le \frac{1}{2} e^{\varepsilon} \frac{|\log\varepsilon|}{2} \int_{B_{\varepsilon}(\log x)} dt \le \frac{e^{\varepsilon}}{2} \varepsilon |\log\varepsilon| \le \varepsilon |\log\varepsilon|. \tag{4.34}$$

To estimate the third integral we begin by evaluating $\int_{\varepsilon}^{\infty} \frac{e^{-t/2}}{1-e^{-2t}} dt$ to get

$$\begin{aligned}\int_{\varepsilon}^{\infty} \frac{e^{-t/2}}{1-e^{-2t}} dt = & \left[-\arctan(e^{-\frac{t}{2}}) + \frac{1}{2} \log \left| \frac{e^{-\frac{t}{2}}-1}{e^{-\frac{t}{2}}+1} \right| \right]_{\varepsilon}^{\infty} \\ = & \arctan(e^{-\frac{\varepsilon}{2}}) - \frac{1}{2} \log \left| \frac{e^{-\frac{\varepsilon}{2}}-1}{e^{-\frac{\varepsilon}{2}}+1} \right|,\end{aligned}$$

and thus we can deduce

$$\left|\int_{\varepsilon}^{\infty}\frac{e^{-t/2}}{1-e^{-2t}}dt\right| = \left|\frac{1}{2}\log\left|\frac{e^{-\frac{\varepsilon}{2}}-1}{e^{-\frac{\varepsilon}{2}}+1}\right| - \arctan(e^{-\frac{\varepsilon}{2}})\right| = \left|\frac{1}{2}\log\left|\frac{1-e^{\frac{\varepsilon}{2}}}{1+e^{\frac{\varepsilon}{2}}}\right| - \arctan(e^{-\frac{\varepsilon}{2}})\right|$$
$$= \left|\frac{1}{2}\log\left(\frac{e^{\frac{\varepsilon}{2}}-1}{e^{\frac{\varepsilon}{2}}+1}\right) - \arctan(e^{-\frac{\varepsilon}{2}})\right|. \tag{4.35}$$

From the equation (4.35), we get the inequality

$$\left|\int_{\varepsilon}^{\infty}\frac{e^{-t/2}}{1-e^{-2t}}dt\right| \leq \frac{1}{2}\left|\log\left(\frac{e^{\frac{\varepsilon}{2}}-1}{e^{\frac{\varepsilon}{2}}+1}\right)\right| + \left|\arctan\left(e^{\frac{-\varepsilon}{2}}\right)\right|$$
$$\leq \frac{1}{2}\left|\log\left(\frac{e^{\frac{\varepsilon}{2}}-1}{e^{\frac{\varepsilon}{2}}+1}\right)\right| + \frac{\pi}{2},$$

since $\arctan(x)$ is bounded above by $\frac{\pi}{2}$. For $0<\varepsilon<\frac{1}{100}$, $e^{\frac{\varepsilon}{2}}-1 = \frac{\varepsilon}{2}+\frac{1}{2}\left(\frac{\varepsilon}{2}\right)^2+\ldots$. Therefore, $e^{\frac{\varepsilon}{2}}-1<1$ which implies $\log\left(\frac{e^{\frac{\varepsilon}{2}}-1}{e^{\frac{\varepsilon}{2}}+1}\right)<0$. Moreover, $e^{\frac{\varepsilon}{2}}-1>\frac{\varepsilon}{2}$ and so we obtain $\frac{\frac{\varepsilon}{2}}{e^{\frac{\varepsilon}{2}}+1}<\frac{e^{\frac{\varepsilon}{2}}-1}{e^{\frac{\varepsilon}{2}}+1}$, which implies

$$\log\left(\frac{\frac{\varepsilon}{2}}{e^{\frac{\varepsilon}{2}}-1}\right) \leq \log\left(\frac{e^{\frac{\varepsilon}{2}}-1}{e^{\frac{\varepsilon}{2}}-1}\right).$$

Thus we get the inequality

$$-\log\left(\frac{e^{\frac{\varepsilon}{2}}-1}{e^{\frac{\varepsilon}{2}}+1}\right) \leq -\log\left(\frac{\frac{\varepsilon}{2}}{e^{\frac{\varepsilon}{2}}+1}\right) = -\log(\varepsilon)+\log(2(1+e^{\frac{\varepsilon}{2}})).$$

Since $\left|\log\left(\frac{e^{\frac{\varepsilon}{2}}-1}{e^{\frac{\varepsilon}{2}}+1}\right)\right| = -\log\left(\frac{e^{\frac{\varepsilon}{2}}-1}{e^{\frac{\varepsilon}{2}}+1}\right)$, we obtain

$$\left|\int_{\varepsilon}^{\infty}\frac{e^{-t/2}}{1-e^{-2t}}dt\right| \leq -\frac{1}{2}\log\left(\frac{e^{\frac{\varepsilon}{2}}-1}{e^{\frac{\varepsilon}{2}}+1}\right)+\frac{\pi}{2} \leq \frac{1}{2}\log\left(\frac{1}{\varepsilon}\right)+\frac{1}{2}\log(2(1+e^{\frac{\varepsilon}{2}}))+\frac{\pi}{2}$$
$$\leq 0.5|\log\varepsilon|+\frac{1}{2}\log(2(1+e^{\frac{\varepsilon}{2}}))+\frac{\pi}{2} \leq 0.5|\log\varepsilon|+2.27$$
$$\leq 1.7|\log\varepsilon|,$$

using the bound $|\log\varepsilon| \geq 2.3$, which is true since $\varepsilon < \frac{1}{10^2}$. Hence we now deduce

$$\left|2\Delta(0)\int_{\varepsilon}^{\infty}\frac{e^{-t/2}}{1-e^{-2t}}dt\right| = 2|\Delta(0)|\left|\int_{\varepsilon}^{\infty}\frac{e^{-t/2}}{1-e^{-2t}}dt\right| \leq 2\varepsilon(1.7|\log\varepsilon|) = 3.4\varepsilon|\log\varepsilon|.$$

since $|\Delta(0)| \leq \varepsilon$ by Lemma 8, equation (4.7). Now, we know from [2, Theorem 1.2.7], that

$$\frac{\Gamma'}{\Gamma}\left(\frac{p}{q}\right) = -\gamma_0 - \frac{\pi}{2}\cot\frac{\pi p}{q} - \log q + 2\sum_{n=1}^{\lfloor \frac{q}{2} \rfloor}{}' \cos\frac{2\pi np}{q}\log\left(2\sin\frac{\pi n}{q}\right),$$

where $\sum'$ implies that when q is even, the term with index $n = q/2$ is divided by 2. Therefore,

$$\begin{aligned}\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}\right) &= -\gamma_0 - \frac{\pi}{2}\cot\frac{\pi}{4} - \log 4 + 2\sum_{n=1}^{2}{}' \cos\frac{2\pi n}{4}\log\left(2\sin\frac{\pi n}{4}\right) \\ &= -\gamma_0 - \frac{\pi}{2} - 3\log 2. \qquad (4.36)\end{aligned}$$

Using the values of the Euler-Mascheroni constant and π, we can compute an upper bound for $\left|\left(\frac{\Gamma'}{\Gamma}(1/4) - \log\pi\right)\right|$ as

$$\left|\left(\frac{\Gamma'}{\Gamma}(1/4) - \log\pi\right)\right| \leq 5.38.$$

Since Lemma 8(iv), equation (4.7) proves that $|\Delta(0)| \leq \varepsilon$, we can sate that

$$\left|\left(\frac{\Gamma'}{\Gamma}(1/4) - \log\pi\right)\Delta(0)\right| \leq 5.4\varepsilon. \qquad (4.37)$$

Now adding all the bounds together, we get

$$|w_{\infty}(\Delta)| \leq \varepsilon(5.4 + 1.2 + (3.4 + 1)|\log\varepsilon|) \leq 8\varepsilon|\log\varepsilon|, \qquad (4.38)$$

which proves equation (4.24) of Lemma 10(iv).

□

Lemma 11. *For two function f and g*

$$w_\infty(f+g) = w_\infty(f) + w_\infty(g) \tag{4.39}$$

Proof. Using the definition of w_∞ in Proposition 2, equation (4.18), we can write

$$\begin{aligned}
w_\infty(f+g) =& \left(\frac{\Gamma'}{\Gamma}(1/4) - \log\pi\right)(f+g)(0) \\
&- \int_0^\infty \frac{(f+g)(t) + (f+g)(-t) - 2(f+g)(0)}{1-e^{-2t}} e^{-t/2} dt. \\
=& \left(\frac{\Gamma'}{\Gamma}(1/4) - \log\pi\right)(f(0)+g(0)) - \\
& \int_0^\infty \frac{(f(t)+g(t)) + (f(-t)+g(-t)) - 2(f(0)+g(0))}{1-e^{-2t}} e^{-t/2} dt \\
=& \left[\left(\frac{\Gamma'}{\Gamma}(1/4) - \log\pi\right) f(0) - \int_0^\infty \frac{(f(t)+f(-t)-2f(0)}{1-e^{-2t}} e^{-t/2} dt\right] + \\
& \left[\left(\frac{\Gamma'}{\Gamma}(1/4) - \log\pi\right) g(0) - \int_0^\infty \frac{(g(t)+g(-t)-2g(0)}{1-e^{-2t}} e^{-t/2} dt\right] \\
=& w_\infty(f) + w_\infty(g),
\end{aligned}$$

which proves equation (4.39). □

The following Proposition provides a smoothed version of the explicit formula for $\psi(x)$.

Proposition 3. *([7, Proposition 2]) Let* $0 < \varepsilon < 1/10$ *and let* $\log(x) > 2/|\log\varepsilon|$. *We define*

$$C_1 = -\gamma_0/2 - 1 - \log(\pi)/2 \tag{4.40}$$

where $\gamma_0 = 0.577\ldots$ *is the Euler-Mascheroni constant. Then we have*

$$\psi_{c,\varepsilon}(x) = x - \sum_\rho^* a_{c,\varepsilon}(\rho)\frac{x^\rho - 1}{\rho} + C_1 - \frac{1}{2}\log(1-x^{-2}) + O^*(8\varepsilon|\log\varepsilon|) \tag{4.41}$$

Proof. Since from definition 6, equation (3.11), we know $\Delta = \phi_{x,c,\varepsilon} - f_x$, we can write that

$$w_\infty(\phi_{x,c,\varepsilon}) = w_\infty(f_x) + w_\infty(\Delta).$$

From Proposition 2, equation (4.19), we know that

$$w_s(\hat{g}) = w_f(g) + w_\infty(g),$$

so substituting the values of $w_s(\hat{g}), w_f(g), w_\infty(g)$ that we derive from Lemma 10 in the above equation, we obtain,

$$\sum_\rho^* a_{c,\varepsilon}(\rho)\frac{x^\rho - 1}{\rho} - x - \log x + 1 = -\psi_{c,\varepsilon}(x) - \log x - \frac{\gamma_0}{2} - \frac{1}{2}\log\pi - \frac{1}{2}\log(1-x^2) + O^*(8\varepsilon|\log\varepsilon|).$$

Therefore we can deduce that

$$\psi_{c,\varepsilon}(x) = x - \sum_\rho^* a_{c,\varepsilon}(\rho)\frac{x^\rho - 1}{\rho} + \log x - 1 - \log x - \frac{\gamma_0}{2} - \frac{1}{2}\log\pi - \frac{1}{2}\log(1-x^2) + O^*(8\varepsilon|\log\varepsilon|),$$

which implies

$$\psi_{c,\varepsilon}(x) = x - \sum_\rho^* a_{c,\varepsilon}(\rho)\frac{x^\rho - 1}{\rho} + C_1 - \frac{1}{2}\log(1-x^2) + O^*(8\varepsilon|\log\varepsilon|),$$

and proves Proposition 3. □

4.3 Bounding the Sum over zeroes

In this section we bound a sum over the zeroes of the zeta function weighted by the Logan function.

Lemma 12. *(Lemma 2, [7]) Let* $0 < \varepsilon < 10^{-3}$ *and let* $c \geq 3$*. Then we have,*

$$\sum_{|\Im(\rho)| > \frac{c}{\varepsilon}} \frac{\left|\ell_{c,\varepsilon}\left(\frac{\rho}{i} - \frac{1}{2i}\right)\right|}{|\Im(\rho)|} \leq 0.32 \frac{e^{0.71\sqrt{c\varepsilon}}}{\sinh c} \left[\log\left(\frac{c}{\pi\varepsilon}\right)\log c + 2\log\left(\frac{c}{\pi\varepsilon}\right) + 18.75\varepsilon \log\left(\frac{c}{\varepsilon}\right) + 1\right]. \tag{4.42}$$

Proof. We consider z to be a complex number of the form $z = x + iy$ with $|y| \leq \frac{\varepsilon}{2}$. Equation (3.17) of Lemma 1 gives us the following bounds for the Logan Function:

$$|\ell_{c,1}(z)| \leq \frac{c}{\sinh(c)} e^{0.71\sqrt{c\varepsilon}} \min\left\{1, \frac{1}{|x| - c}\right\}. \tag{4.43}$$

To continue the proof, we need to find an estimate for the zero counting function $N(t)$. For this estimate we use Rosser's bounds given in [31] according to which, for $T \geq 2$

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + O^*(0.137\log T + 0.443\log\log T + 1.588). \tag{4.44}$$

Currently, better estimates for the remainder are known, however the effect on the estimate of the considered sum over zeroes is extremely small, so we utilize a more conservative choice. In some parts of this proof, we will use an even weaker result which can be obtained from equation (4.44) for $T \geq 100$, given by

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi e} + O^*(0.82\log T). \tag{4.45}$$

We first want to estimate the contribution of the zeroes with imaginary parts in $\left(\frac{c}{\varepsilon}, \frac{c+1}{\varepsilon}\right]$. We let $T \geq 100$ and $M \geq 0$. Then, since $0.137\log T + 0.443\log\log T + 1.588 = O^*(0.82\log T)$ for $T \geq 100$, from the equation (4.44) we obtain

$$\begin{aligned} N(T+M) - N(T) &= \frac{T+M}{2\pi}\log\frac{T+M}{2\pi} - \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{M}{2\pi} + O^*(1.64\log(T+M)) \\ &= \frac{M}{2\pi}\log\frac{T+M}{2\pi} + \frac{T}{2\pi}\log\frac{T+M}{T} - \frac{M}{2\pi} + O^*(1.64\log(T+M)), \end{aligned}$$

where we can use the inequality $\log\frac{T+M}{T} \leq \frac{M}{T}$, to state that

$$N(T+M)-N(T) < \frac{M}{2\pi}\log\frac{T+M}{2\pi} + 1.64\log(T+M).$$

Now, since $T+M \geq 100$, we can compute the value of $\log 2\pi$ and $\frac{18}{2\pi}$ to claim that

$$\frac{18}{2\pi}\log(2\pi) \leq \left(\frac{18}{2\pi} - 1.64\right)\log 100 \leq \left(\frac{18}{2\pi} - 1.64\right)\log(T+M),$$

which implies

$$1.64\log(T+M) \leq \frac{18}{2\pi}\log\left(\frac{T+M}{2\pi}\right).$$

Therefore we now get the inequality ,

$$N(T+M)-N(T) < \frac{M+18}{2\pi}\log\frac{T+M}{2\pi}. \tag{4.46}$$

Furthermore, we define

$$f(z) = \frac{\sinh c}{c}e^{-0.71\sqrt{c\varepsilon}}\ell_{c,\varepsilon}(z).$$

Using equation (3.17) and the inequality in equation (4.46) applied with $T = \frac{c}{\varepsilon}$ and $M = \frac{1}{\varepsilon}$ we get

$$\sum_{\substack{\rho \\ \frac{c}{\varepsilon} < \Im(\rho) \leq \frac{c+1}{\varepsilon}}} \frac{\left|f\left(\frac{\rho}{i} - \frac{1}{2i}\right)\right|}{|\Im(\rho)|} \leq \frac{\varepsilon}{c}\frac{\varepsilon^{-1}+18}{2\pi}\log\frac{c+1}{2\pi\varepsilon} \leq \frac{0.16}{c}\log\frac{c}{\pi\varepsilon} \tag{4.47}$$

We now try to estimate the contribution of the zeroes with imaginary part greater than $\frac{c+1}{\varepsilon}$

by noting that

$$\begin{aligned}\sum_{\substack{\rho\\ \Im(\rho)>\frac{c+1}{e}}} \frac{\left|f\left(\frac{\rho}{i}-\frac{1}{2i}\right)\right|}{|\Im(\rho)|} &= \sum_{\substack{\rho\\ \gamma>\frac{c+1}{e}}} \frac{\sinh c}{c} e^{-0.71\sqrt{c\varepsilon}} \frac{\left|\ell_{c,\varepsilon}\left(\frac{\rho}{i}-\frac{1}{2i}\right)\right|}{|\gamma|}\\ &= \sum_{\substack{\rho\\ \gamma>\frac{c+1}{e}}} \frac{\sinh c}{c} e^{-0.71\sqrt{c\varepsilon}} \frac{\left|\ell_{c,\varepsilon}\left(\gamma-i\left(\beta-\frac{1}{2}\right)\right)\right|}{|\gamma|}\\ &= \sum_{\substack{\rho\\ \gamma>\frac{c+1}{e}}} \frac{\sinh c}{c} e^{-0.71\sqrt{c\varepsilon}} \frac{\left|\ell_{c,1}\left(\varepsilon\left(\gamma-i\left(\beta-\frac{1}{2}\right)\right)\right)\right|}{|\gamma|}.\end{aligned}$$

Applying the inequality (4.42) to the RHS of the above equation, we get

$$\begin{aligned}\sum_{\substack{\rho\\ \Im(\rho)>\frac{c+1}{e}}} \frac{\left|f\left(\frac{\rho}{i}-\frac{1}{2i}\right)\right|}{|\Im(\rho)|} &\leq \sum_{\substack{\rho\\ \gamma>\frac{c+1}{e}}} \frac{1}{\gamma}\min\left(1, \frac{1}{|\varepsilon\gamma|-c}\right)\\ &= \sum_{\substack{\rho\\ \gamma>\frac{c+1}{e}}} \frac{1}{\gamma}\left(\frac{1}{|\varepsilon\gamma|-c}\right),\end{aligned}$$

since for $\gamma > \frac{c+1}{\varepsilon}$, $|\varepsilon\gamma| - c > 1$. Using the partial summation formula in equation (3.38) as a consequence of Lemma 7, we can now write that

$$\begin{aligned}\sum_{\substack{\rho\\ \Im(\rho)>\frac{c+1}{e}}} \frac{\left|f\left(\frac{\rho}{i}-\frac{1}{2i}\right)\right|}{|\Im(\rho)|} \leq& \frac{N(t)}{t(\varepsilon t-c)}\Bigg|_{\frac{c+1}{\varepsilon}}^{\infty} - \int_{\frac{c+1}{\varepsilon}}^{\infty} N(t)\frac{d}{dt}\left(\frac{1}{t(\varepsilon t-c)}\right)dt\\ =& -\int_{\frac{c+1}{\varepsilon}}^{\infty} N(t)\frac{d}{dt}\left(\frac{1}{t(\varepsilon t-c)}\right)dt - \frac{\varepsilon}{c+1}N\left(\frac{c+1}{\varepsilon}\right)\\ \leq& -\int_{\frac{c+1}{\varepsilon}}^{\infty} N(t)\frac{d}{dt}\left(\frac{1}{t(\varepsilon t-c)}\right)dt - \frac{1}{2\pi}\log\frac{c+1}{2\pi\varepsilon e}\\ &+ O^*\left(0.82\frac{\varepsilon}{c}\log\frac{c+1}{\varepsilon}\right). \qquad (4.48)\end{aligned}$$

We use equation (4.45) to further estimate the integral on the first term in equation (4.48).

The main term $\frac{t}{2\pi}\log\frac{t}{2\pi e}$ on the RHS of (4.45) provides the contribution

$$\frac{-1}{2\pi}\int_{\frac{c+1}{\varepsilon}}^{\infty} t\log\left(\frac{t}{2\pi e}\right)\frac{d}{dt}\left(\frac{1}{t(\varepsilon t-c)}\right)dt=\frac{1}{2\pi}\log\frac{c+1}{2\pi\varepsilon e}+\frac{1}{2\pi}\int_{\frac{c+1}{\varepsilon}}^{\infty}\frac{\log\left(\frac{t}{2\pi}\right)}{t(\varepsilon t-c)}dt \qquad (4.49)$$

to the integral involving $N(t)$ in equation (4.48). We see that the first term on the right hand side of equation (4.49) cancels the second term on the right hand side of equation (4.48) and thus, to bound the sum in (4.48), evaluating the integral in (4.49) is sufficient. We can now estimate the integral in equation (4.49) to get,

$$\frac{1}{2\pi}\int_{\frac{c+1}{\varepsilon}}^{\infty}\frac{\log\left(\frac{t}{2\pi}\right)}{t(\varepsilon t-c)}dt=\frac{1}{2\pi}\int_{1}^{\infty}\frac{\log\left(\frac{t+c}{2\pi\varepsilon}\right)}{(t+c)t}dt.$$

Splitting the integral into two intervals, and noting that $\displaystyle\int_1^c\frac{\log\left(\frac{(t+c)}{2\pi\varepsilon}\right)}{(t+c)t}dt\leq\frac{\log\left(\frac{2c}{2\pi\varepsilon}\right)}{c+1}\int_1^c\frac{dt}{t}$, we can write

$$\begin{aligned}\frac{1}{2\pi}\int_{\frac{c+1}{\varepsilon}}^{\infty}\frac{\log\left(\frac{t}{2\pi}\right)}{t(\varepsilon t-c)}dt\leq&\frac{1}{2\pi}\left(\frac{\log\left(\frac{c}{\pi\varepsilon}\right)}{c}\int_1^c\frac{dt}{t}+\int_c^{\infty}\frac{\log\left(\frac{(t+c)}{2\pi\varepsilon}\right)}{t^2}dt\right)\\=&\frac{1}{2\pi c}\log\left(\frac{c}{\pi\varepsilon}\right)\log c+\frac{\log\left(\frac{2c}{2\pi\varepsilon}\right)}{2\pi c}+\frac{1}{2\pi}\int_c^{\infty}\frac{dt}{t(t+c)}\\\leq&\frac{0.16}{c}\left(\log\left(\frac{c}{\pi\varepsilon}\right)\log c+\log\left(\frac{c}{\pi\varepsilon}\right)+1\right),\end{aligned}$$

since $t\leq c$ and $\frac{1}{t+c}\geq\frac{1}{c+1}$. To estimate the contribution of the remainder O^* term in the estimate of $N(t)$ in equation (4.45) to the integral in equation (4.48), we use the inequality,

$$\begin{aligned}-\int_{\frac{c+1}{\varepsilon}}^{\infty}\log t\frac{d}{dt}\left(\frac{1}{t(\varepsilon t-c)}\right)&=\varepsilon\int_1^{\infty}\frac{\log\frac{t+c}{\varepsilon}}{(t+c)^2t}dt+\varepsilon\int_1^{\infty}\frac{\log\frac{t+c}{\varepsilon}}{(t+c)t^2}dt\\&\leq 2\varepsilon\int_1^{\infty}\frac{\log\frac{t+c}{\varepsilon}}{(t+c)t^2}dt\\&=2\varepsilon\int_1^{\infty}\frac{\log(t+c)-\log\varepsilon}{(t+c)t^2}dt\\&\leq 2\varepsilon\int_1^{\infty}\frac{dt}{t^2}-2\varepsilon\log\varepsilon\int_1^{\infty}\frac{dt}{t^3}=2\varepsilon-\varepsilon\log\varepsilon\leq\varepsilon\log\frac{c}{\varepsilon}.\end{aligned}$$

The above contribution is thus bounded by $\varepsilon\log\frac{c}{\varepsilon}$, and we can generously bound the O^* term on the right side of equation (4.48) by $2\varepsilon\log\frac{c}{\varepsilon}$. Since the zeroes with a positive imaginary part are in bijection with the zeroes with a negative imaginary part via the mapping $\rho\mapsto 1-\rho$, using (4.47), and equation (4.48) along with the above estimates,we get

$$\begin{aligned}\sum_{\substack{\rho\\ |\Im(\rho)|>\frac{c}{\varepsilon}}}\frac{\left|f\left(\frac{\rho}{i}-\frac{1}{2i}\right)\right|}{|\Im(\rho)|} &\leq\frac{2}{c}\left[0.16\log\left(\frac{c}{\pi\varepsilon}\right)+0.16\left(\log\left(\frac{c}{\pi\varepsilon}\right)\log c+\log\left(\frac{c}{\pi\varepsilon}\right)+1\right)\right.\\ &\quad\left.+3\varepsilon\log\left(\frac{c}{\varepsilon}\right)\right]\\ &=\frac{0.32}{c}\left[\log\left(\frac{c}{\pi\varepsilon}\right)\log c+2\log\left(\frac{c}{\pi\varepsilon}\right)+18.75\varepsilon\log\left(\frac{c}{\varepsilon}\right)+1\right]\end{aligned}$$

which estimates equation (4.42). □

Lemma 13. *Let $c>0$, $0<\varepsilon<10^{-3}$and $a\in(0,1)$ satisfy $\frac{ac}{\varepsilon}\geq 10^3$. Then we have*

$$\sum_{\frac{ac}{\varepsilon}<|\Im(\rho)|\leq\frac{c}{\varepsilon}}\left|\frac{\ell_{c,\varepsilon}(\Im(\rho))}{\Im(\rho)}\right|\leq\frac{1+11c\varepsilon}{\pi c a^2}\log\left(\frac{c}{\varepsilon}\right)\frac{\cosh(c\sqrt{1-a^2})}{\sinh(c)}. \tag{4.50}$$

Proof. We start by denoting the zeroes of the Riemann zeta function by $\rho=\beta+i\gamma$ with $\beta,\gamma\in\mathbb{R}$. Recall that $N(t)$ is the zero counting function which counts the number of zeroes of the Riemann zeta function (counted according to their multiplicities) with imaginary part in $(0,t]$. We assume that

$$\tilde{N}(t)=\frac{t}{2\pi}\log\frac{t}{2\pi e}+\frac{7}{8}\quad\text{and}\quad R(t)=N(t)-\tilde{N}(t), \tag{4.51}$$

and note that Rosser's estimate given in [31] implies

$$R(t)=O^*(0.5\log t), \tag{4.52}$$

for $t\geq 10^3$.

Since the zeroes of the Riemann zeta function are symmetric with respect to the real axis, it is sufficient to treat the sum on the left hand side of equation (4.50) over $\gamma>0$, which gives us

$$\begin{aligned}\sum_{\frac{ac}{\varepsilon}<\gamma\leq\frac{c}{\varepsilon}}\frac{\ell_{c,\varepsilon}(\gamma)}{\gamma} &= \int_{\frac{ac}{\varepsilon}}^{\frac{c}{\varepsilon}}\frac{\ell_{c,\varepsilon}(t)}{t}d(\tilde{N}(t)+R(t)) \\ &= \frac{1}{2\pi}\int_{\frac{ac}{\varepsilon}}^{\frac{c}{\varepsilon}}\ell_{c,\varepsilon}(t)\log\frac{t}{2\pi}\frac{dt}{t}+\int_{\frac{ac}{\varepsilon}}^{\frac{c}{\varepsilon}}\frac{\ell_{c,\varepsilon}(t)}{t}dR(t).\end{aligned} \tag{4.53}$$

We continue the proof by estimating the first integral on the right hand side of (4.53). We note that for $t\in(\frac{ac}{\varepsilon},\frac{c}{\varepsilon}]$, we can claim the inequality

$$0<\frac{1}{t}\log\frac{t}{2\pi}\leq\frac{\varepsilon^2}{(ac)^2}\log\left(\frac{c}{2\pi\varepsilon}\right). \tag{4.54}$$

Definition 1, equation (3.1) and the above inequality implies

$$\begin{aligned}0<\frac{1}{2\pi}\int_{\frac{ac}{\varepsilon}}^{\frac{c}{\varepsilon}}\ell_{c,\varepsilon}(t)\log\left(\frac{t}{2\pi}\right)\frac{dt}{t} &\leq \frac{1}{2\pi(ac)^2}\log\left(\frac{c}{2\pi\varepsilon}\right)\int_{\frac{ac}{\varepsilon}}^{\frac{c}{\varepsilon}}\ell_{c,\varepsilon}(t)dt \\ &= \frac{1}{2\pi(ac)^2}\log\left(\frac{c}{2\pi\varepsilon}\right)\frac{c}{\sinh c}\int_{\frac{ac}{\varepsilon}}^{\frac{c}{\varepsilon}}\frac{\sin\left(\sqrt{(t\varepsilon)^2-c^2}\right)}{\sqrt{(t\varepsilon)^2-c^2}}dt.\end{aligned}$$

Applying the substitution $u=\sqrt{c^2-(\varepsilon t)^2}$ we get the bound,

$$\begin{aligned}0<\frac{1}{2\pi}\int_{\frac{ac}{\varepsilon}}^{\frac{c}{\varepsilon}}\ell_{c,\varepsilon}(t)\log\left(\frac{t}{2\pi}\right)\frac{dt}{t} &\leq \frac{1}{2\pi(ac)^2}\log\left(\frac{c}{2\pi\varepsilon}\right)\frac{c}{\sinh(c)}\int_0^{c\sqrt{1-a^2}}\sinh(u)du \\ &\leq \frac{1}{2\pi ca^2}\log\left(\frac{c}{2\pi\varepsilon}\right)\frac{\cosh(c\sqrt{1-a^2})}{\sinh(c)}.\end{aligned} \tag{4.55}$$

To evaluate the second integral in equation (4.53) we use partial integration and write,

$$\begin{aligned}\int_{\frac{ac}{\varepsilon}}^{\frac{c}{\varepsilon}}\frac{\ell_{c,\varepsilon}(t)}{t}dR(t) &= \left[\frac{\ell_{c,\varepsilon}(t)}{t}R(t)\right]_{\frac{ac}{\varepsilon}}^{\frac{c}{\varepsilon}}-\int_{\frac{ac}{\varepsilon}}^{\frac{c}{\varepsilon}}\frac{d}{dt}\left(\frac{\ell_{c,\varepsilon}(t)}{t}\right)R(t)dt \\ &\leq \left|\frac{\ell_{c,\varepsilon}\left(\frac{c}{\varepsilon}\right)}{\frac{c}{\varepsilon}}R\left(\frac{c}{\varepsilon}\right)-\frac{\ell_{c,\varepsilon}\left(\frac{ac}{\varepsilon}\right)}{\frac{ac}{\varepsilon}}R\left(\frac{ac}{\varepsilon}\right)\right|-\int_{\frac{ac}{\varepsilon}}^{\frac{c}{\varepsilon}}\frac{d}{dt}\left(\frac{\ell_{c,\varepsilon}(t)}{t}\right)R(t)dt.\end{aligned}$$

Now, applying the estimate of $R(t)$ from equation (4.52), and using that $\ell_{c,\varepsilon}$ is decreasing in the range $\left[0, \frac{c}{\varepsilon}\right]$ from Lemma 1(v), we obtain,

$$\begin{aligned}\int_{\frac{ac}{\varepsilon}}^{\frac{c}{\varepsilon}} \frac{\ell_{c,\varepsilon}(t)}{t} dR(t) \leq & \frac{1}{\frac{ac}{\varepsilon}} \max\left\{\ell_{c,\varepsilon}\left(\frac{c}{\varepsilon}\right), \ell_{c,\varepsilon}\left(\frac{ac}{\varepsilon}\right)\right\}\left(\left|0.5\log\left(\frac{c}{\varepsilon}\right)\right| + \left|0.5\log\left(\frac{ac}{\varepsilon}\right)\right|\right) \\ & - 0.5\int_{\frac{ac}{\varepsilon}}^{\frac{c}{\varepsilon}} \frac{d}{dt}\left(\frac{\ell_{c,\varepsilon}(t)}{t}\right)\log(t)dt \\ \leq & \frac{\varepsilon}{ac}\ell_{c,\varepsilon}\left(\frac{ac}{\varepsilon}\right)\left[0.5\log\left(\frac{c}{\varepsilon}\right) + 0.5\log\left(\frac{c}{\varepsilon}\right)\right] \\ & - 0.5\int_{\frac{ac}{\varepsilon}}^{\frac{c}{\varepsilon}} \frac{d}{dt}\left(\frac{\ell_{c,\varepsilon}(t)}{t}\right)\log(t)dt,\end{aligned}$$

since $a \in (0,1)$. Noting that $\ell_{c,\varepsilon}(\xi) = \ell_{c,1}(\varepsilon\xi)$ from Lemma 1(i), we obtain the inequality

$$\begin{aligned}\int_{\frac{ac}{\varepsilon}}^{\frac{c}{\varepsilon}} \frac{\ell_{c,\varepsilon}(t)}{t} dR(t) &\leq \frac{\varepsilon}{ac}\ell_{c,1}(ac)\log\left(\frac{c}{\varepsilon}\right) - 0.5\int_{\frac{ac}{\varepsilon}}^{\frac{c}{\varepsilon}} \frac{d}{dt}\left(\frac{\ell_{c,\varepsilon}(t)}{t}\right)\log(t)dt \\ &\leq 1.5\frac{\varepsilon}{ac}\ell_{c,1}(ac)\log\left(\frac{c}{\varepsilon}\right) + 0.5\int_{\frac{ac}{\varepsilon}}^{\frac{c}{\varepsilon}} \frac{\ell_{c,\varepsilon}(t)}{t^2}dt.\end{aligned}$$

We can now apply that $\ell_{c,\varepsilon}(\xi)$ is monotonically decreasing for $\xi \in [\frac{ac}{\varepsilon} \frac{c}{\varepsilon}]$ to the term $0.5\int_{\frac{ac}{\varepsilon}}^{\frac{c}{\varepsilon}} \frac{\ell_{c,\varepsilon}(t)}{t^2}dt$ to get the following inequality:

$$0.5\int_{\frac{ac}{\varepsilon}}^{\frac{c}{\varepsilon}} \frac{\ell_{c,\varepsilon}(t)}{t^2}dt \leq 0.5\ell_{c,1}(ac)\int_{\frac{ac}{\varepsilon}}^{\frac{c}{\varepsilon}} \frac{dt}{t^2} \leq 0.5\ell_{c,1}(ac)\left[\frac{-\varepsilon}{c} + \frac{\varepsilon}{ac}\right] \leq 0.1\ell_{c,1}(ac)\frac{\varepsilon}{ac}(5(1-a)).$$

We now note that $\frac{c}{\varepsilon} \geq \frac{ac}{\varepsilon} \geq 1000$, which implies that $\log\left(\frac{c}{\varepsilon}\right) \geq \log(1000) > 6$. Hence, we can now write

$$0.5\int_{\frac{ac}{\varepsilon}}^{\frac{c}{\varepsilon}} \frac{\ell_{c,\varepsilon}(t)}{t^2}dt \leq 0.1\ell_{c,1}(ac)\frac{\varepsilon}{ac}\log\left(\frac{c}{\varepsilon}\right).$$

Therefore we can now deduce that

$$\int_{\frac{ac}{\varepsilon}}^{\frac{c}{\varepsilon}} \frac{\ell_{c,\varepsilon}(t)}{t} dR(t) \leq 1.6\frac{\varepsilon}{ac}\ell_{c,1}(ac)\log\left(\frac{c}{\varepsilon}\right). \tag{4.56}$$

Since the function $t \mapsto \frac{t\cosh t}{\sinh t}$ is monotonically increasing in $[0,\infty)$, we know that

$$c\sqrt{1-a^2}\frac{\cosh(c\sqrt{1-a^2})}{\sinh(c\sqrt{1-a^2})} \geq 1.$$

Combining the above inequality with the inequality (4.56), we get

$$1.6\frac{\varepsilon}{ac}\log\left(\frac{c}{\varepsilon}\right)\ell_{c,1}(ac) \leq 1.6\frac{\varepsilon c}{ac}\log\left(\frac{c}{\varepsilon}\right)\frac{\cosh(c\sqrt{1-a^2})}{\sinh(c)}. \tag{4.57}$$

Furthermore, we note that,

$$\begin{aligned}\frac{1}{2\pi ca^2}\log\left(\frac{c}{2\pi\varepsilon}\right)+1.6\frac{\varepsilon c}{ac}\log\left(\frac{c}{\varepsilon}\right) =&\frac{1}{2\pi ca^2}\left[\log\left(\frac{c}{\varepsilon}\right)-\log(2\pi)+1.6(2\pi ac\varepsilon)\log\left(\frac{c}{\varepsilon}\right)\right]\\ \leq&\frac{1}{2\pi ca^2}\log\left(\frac{c}{\varepsilon}\right)(1+11c\varepsilon). \end{aligned}\tag{4.58}$$

Now, using equations (4.58), (4.57) and equation (4.55) we get

$$\begin{aligned}\sum_{\frac{ac}{\varepsilon}<\gamma\leq\frac{c}{\varepsilon}}\frac{\ell_{c,\varepsilon}(\gamma)}{\gamma} =&\frac{1}{2\pi}\int_{\frac{ac}{\varepsilon}}^{\frac{c}{\varepsilon}}\ell_{c,\varepsilon}(t)\log\frac{t}{2\pi}\frac{dt}{t}+\int_{\frac{ac}{\varepsilon}}^{\frac{c}{\varepsilon}}\frac{\ell_{c,\varepsilon}(t)}{t}dR(t)\\ \leq&\frac{1}{2\pi ca^2}\log\left(\frac{c}{2\pi\varepsilon}\right)\frac{\cosh(c\sqrt{1-a^2})}{\sinh(c)}+1.6\frac{\varepsilon c}{ac}\log\left(\frac{c}{\varepsilon}\right)\frac{\cosh(c\sqrt{1-a^2})}{\sinh(c)}\\ \leq&\frac{1}{2}\frac{1+11c\varepsilon}{\pi ca^2}\log\left(\frac{c}{\varepsilon}\right)\frac{\cosh(c\sqrt{1-a^2})}{\sinh(c)},\end{aligned}$$

which proves Lemma 13. □

The Proposition given below provides bounds for the truncated sum of zeroes in the explicit formula for $\psi_{c,\varepsilon}(x)$.

Proposition 4. *(Proposition 3, [7]) Let $x \geq 1, \varepsilon \leq 10^{-3}$ and $c \geq 3$. Then we have*

$$\sum_{|\Im(\rho)|>\frac{c}{\varepsilon}}\left|a_{c,\varepsilon}(\rho)\frac{x^\rho}{\rho}\right| \leq 0.16\frac{x+1}{\sinh(c)}e^{0.71\sqrt{c\varepsilon}}\left[\log\left(\frac{c}{\pi\varepsilon}\right)\log c+2\log\left(\frac{c}{\pi\varepsilon}\right)+\log\left(\frac{c}{\varepsilon}\right)+1\right] \tag{4.59}$$

Furthermore, if $a \in (0,1)$, such that $\frac{ac}{\varepsilon} \geq 10^3$ holds, and if the Riemann Hypothesis holds *for all zeros with imaginary part in* $(0, \frac{c}{\varepsilon}]$, *then we have*

$$\sum_{\frac{ac}{\varepsilon} < |\Im(\rho)| \leq \frac{c}{\varepsilon}} \left| a_{c,\varepsilon}(\rho) \frac{x^\rho}{\rho} \right| \leq \frac{1 + 11c\varepsilon}{\pi c a^2} \log\left(\frac{c}{\varepsilon}\right) \frac{\cosh(c\sqrt{1-a^2})}{\sinh(c)} \sqrt{x}. \tag{4.60}$$

Proof. From Definition 6, equation ((3.10)), we write

$$\left| a_{c,\varepsilon}(\rho) \frac{x^\rho}{\rho} \right| = \left| \frac{1}{\lambda_{c,\varepsilon}} \ell_{c,\varepsilon}\left(\frac{\rho}{i} - \frac{1}{2i}\right) \frac{x^\rho}{\rho} \right|.$$

It has been previously proved in Lemma 3, equation (3.35), that $\lambda_{c,\varepsilon} \geq 1$. From this, we deduce,

$$\left| a_{c,\varepsilon}(\rho) \frac{x^\rho}{\rho} \right| \leq \left| \ell_{c,\varepsilon}\left(\frac{\rho}{i} - \frac{1}{2i}\right) \frac{x^\rho}{\rho} \right| \leq \left| \ell_{c,\varepsilon}\left(\frac{\rho}{i} - \frac{1}{2i}\right) \right| \frac{|x^\rho|}{|\rho|} \leq \left| \ell_{c,\varepsilon}\left(\frac{\rho}{i} - \frac{1}{2i}\right) \right| \frac{x^{\Re(\rho)}}{|\Im(\rho)|}. \tag{4.61}$$

Note that this implies

$$\sum_{|\Im(\rho)| > \frac{c}{\varepsilon}} \left| a_{c,\varepsilon}(\rho) \frac{x^\rho}{\rho} \right| \leq \sum_{|\Im(\rho)| > \frac{c}{\varepsilon}} x^{\Re(\rho)} \frac{\left|\ell_{c,\varepsilon}\left(\frac{\rho}{i} - \frac{1}{2i}\right)\right|}{|\Im(\rho)|}. \tag{4.62}$$

Now using Lemma 12 and equation (4.62) we get

$$\begin{aligned} \sum_{|\Im(\rho)| > \frac{c}{\varepsilon}} \left| a_{c,\varepsilon}(\rho) \frac{x^\rho}{\rho} \right| &\leq \sum_{|\Im(\rho)| > \frac{c}{\varepsilon}} x^{\Re(\rho)} \frac{\left|\ell_{c,\varepsilon}\left(\frac{\rho}{i} - \frac{1}{2i}\right)\right|}{|\Im(\rho)|} \\ &\leq \sum_{\substack{|\Im(\rho)| > \frac{c}{\varepsilon} \\ \Re(\rho) = \frac{1}{2}}} x^{\Re(\rho)} \frac{\left|\ell_{c,\varepsilon}\left(\frac{\rho}{i} - \frac{1}{2i}\right)\right|}{|\Im(\rho)|} + \sum_{\substack{|\Im(\rho)| > \frac{c}{\varepsilon} \\ \Re(\rho) > \frac{1}{2}}} x^{\Re(\rho)} \frac{\left|\ell_{c,\varepsilon}\left(\frac{\rho}{i} - \frac{1}{2i}\right)\right|}{|\Im(\rho)|} \\ &\quad + \sum_{\substack{|\Im(\rho)| > \frac{c}{\varepsilon} \\ \Re(\rho) < \frac{1}{2}}} x^{\Re(\rho)} \frac{\left|\ell_{c,\varepsilon}\left(\frac{\rho}{i} - \frac{1}{2i}\right)\right|}{|\Im(\rho)|}. \end{aligned} \tag{4.63}$$

We know, that for each ρ such that $\Re(\rho) < \frac{1}{2}$, there exists a zero of the zeta function, $\rho' = 1 - \rho$ which implies $\Re(\rho') = 1 - \Re(\rho)$ and $\Re(\rho') > \frac{1}{2}$. Now we can rewrite the right hand side of the inequality in (4.63) as

$$\sum_{\substack{|\Im(\rho)|>\frac{c}{\varepsilon}\\ \beta=\frac{1}{2}}} x^{1/2}\frac{\left|\ell_{c,\varepsilon}\left(\frac{\rho}{i}-\frac{1}{2i}\right)\right|}{|\Im(\rho)|}+\sum_{\substack{|\Im(\rho)|>\frac{c}{\varepsilon}\\ \beta>\frac{1}{2}}}\frac{x^{\beta}\left|\ell_{c,\varepsilon}\left(\frac{\rho-1/2}{i}\right)\right|+x^{1-\beta}\left|\ell_{c,\varepsilon}\left(\frac{1-\rho-1/2}{i}\right)\right|}{|\Im(\rho)|}$$
$$=\sum_{\substack{|\Im(\rho)|>\frac{c}{\varepsilon}\\ \beta=\frac{1}{2}}} x^{1/2}\frac{\left|\ell_{c,\varepsilon}\left(\frac{\rho}{i}-\frac{1}{2i}\right)\right|}{|\Im(\rho)|}+\sum_{\substack{|\Im(\rho)|>\frac{c}{\varepsilon}\\ \beta>\frac{1}{2}}}\frac{x^{\beta}\left|\ell_{c,\varepsilon}\left(\frac{\rho-1/2}{i}\right)\right|+x^{1-\beta}\left|\ell_{c,\varepsilon}\left(\frac{-(\rho-1/2)}{i}\right)\right|}{|\Im(\rho)|}.$$

Since $\ell_{c,\varepsilon}$ is an even function, therefore, $\ell_{c,\varepsilon}\left(\frac{\rho-1/2}{i}\right)=\ell_{c,\varepsilon}\left(\frac{-(\rho-1/2)}{i}\right)$ and the above equation can be written as

$$\sum_{\substack{|\Im(\rho)|>\frac{c}{\varepsilon}\\ \beta=\frac{1}{2}}} x^{1/2}\frac{\left|\ell_{c,\varepsilon}\left(\frac{\rho}{i}-\frac{1}{2i}\right)\right|}{|\Im(\rho)|}+\sum_{\substack{|\Im(\rho)|>\frac{c}{\varepsilon}\\ \beta>\frac{1}{2}}}(x^{\beta}+x^{1-\beta})\frac{\left|\ell_{c,\varepsilon}\left(\frac{\rho-1/2}{i}\right)\right|}{|\Im(\rho)|}$$
$$\leq\sum_{\substack{|\Im(\rho)|>\frac{c}{\varepsilon}\\ \beta=\frac{1}{2}}} x^{1/2}\frac{\left|\ell_{c,\varepsilon}\left(\frac{\rho}{i}-\frac{1}{2i}\right)\right|}{|\Im(\rho)|}+\sum_{\substack{|\Im(\rho)|>\frac{c}{\varepsilon}\\ \beta>\frac{1}{2}}}(x+1)\frac{\left|\ell_{c,\varepsilon}\left(\frac{\rho-1/2}{i}\right)\right|}{|\Im(\rho)|},$$

since $x^{\beta}+x^{1-\beta}$ attains its maximum value for $\beta=0,1$. We note that the zeroes of the Riemann zeta function are symmetrical with respect to the half-line, and the above equation is thus equal to

$$=\frac{1}{2}\left[\sum_{\substack{|\Im(\rho)|>\frac{c}{\varepsilon}\\ \beta=\frac{1}{2}}} 2x^{1/2}\frac{\left|\ell_{c,\varepsilon}\left(\frac{\rho}{i}-\frac{1}{2i}\right)\right|}{|\Im(\rho)|}+\sum_{\substack{|\Im(\rho)|>\frac{c}{\varepsilon}\\ \beta\neq\frac{1}{2}}}(x+1)\frac{\left|\ell_{c,\varepsilon}\left(\frac{\rho-1/2}{i}\right)\right|}{|\Im(\rho)|}\right]$$
$$\leq\frac{1}{2}\left[\sum_{\substack{|\Im(\rho)|>\frac{c}{\varepsilon}\\ \beta=\frac{1}{2}}}(x+1)\frac{\left|\ell_{c,\varepsilon}\left(\frac{\rho}{i}-\frac{1}{2i}\right)\right|}{|\Im(\rho)|}+\sum_{\substack{|\Im(\rho)|>\frac{c}{\varepsilon}\\ \beta\neq\frac{1}{2}}}(x+1)\frac{\left|\ell_{c,\varepsilon}\left(\frac{\rho-1/2}{i}\right)\right|}{|\Im(\rho)|}\right]$$
$$=\frac{1}{2}(x+1)\sum_{|\Im(\rho)|>\frac{c}{\varepsilon}}\frac{\left|\ell_{c,\varepsilon}\left(\frac{\rho}{i}-\frac{1}{2i}\right)\right|}{|\Im(\rho)|}.$$

From Lemma 12, equation (4.42), we know that

$$\sum_{|\Im(\rho)|>\frac{c}{\varepsilon}} \frac{\left|\ell_{c,\varepsilon}\left(\frac{\rho}{i}-\frac{1}{2i}\right)\right|}{|\Im(\rho)|} \leq 0.32\frac{e^{0.71\sqrt{c\varepsilon}}}{\sinh(c)}\left[\log\left(\frac{c}{\pi\varepsilon}\right)\log c + 2\log\left(\frac{c}{\pi\varepsilon}\right)+\log\left(\frac{c}{\varepsilon}\right)+1\right]$$

. Therefore we can deduce that

$$\begin{aligned}\sum_{|\Im(\rho)|>\frac{c}{\varepsilon}} \left|a_{c,\varepsilon}(\rho)\frac{x^{\rho}}{\rho}\right| \leq & \frac{1}{2}(x+1)0.32\frac{e^{0.71\sqrt{c\varepsilon}}}{\sinh(c)}\left[\log\left(\frac{c}{\pi\varepsilon}\right)\log c + 2\log\left(\frac{c}{\pi\varepsilon}\right)+\log\left(\frac{c}{\varepsilon}\right)+1\right] \\ \leq & 0.16\frac{x+1}{\sinh(c)}e^{0.71\sqrt{c\varepsilon}}\left[\log\left(\frac{c}{\pi\varepsilon}\right)\log c + 2\log\left(\frac{c}{\pi\varepsilon}\right)+\log\left(\frac{c}{\varepsilon}\right)+1\right]\end{aligned}$$

which proves (4.59).

For the zeroes of the Riemann zeta function with $\Re(\rho)=\frac{1}{2}$, we see that $\Im(\rho)=\frac{\rho}{i}-\frac{1}{2i}$. Therefore, equation (4.61) can be written as

$$\left|a_{c,\varepsilon}(\rho)\frac{x^{\rho}}{\rho}\right| \leq \left|\frac{\ell_{c,\varepsilon}(\Im(\rho))}{\Im(\rho)}\right|\sqrt{x}. \tag{4.64}$$

Equation (4.64) now implies that

$$\sum_{\frac{ac}{\varepsilon}<|\Im(\rho)|\leq\frac{c}{\varepsilon}} \left|a_{c,\varepsilon}(\rho)\frac{x^{\rho}}{\rho}\right| \leq \sqrt{x}\sum_{\frac{ac}{\varepsilon}<|\Im(\rho)|\leq\frac{c}{\varepsilon}} \left|\frac{\ell_{c,\varepsilon}(\Im(\rho))}{\Im(\rho)}\right|. \tag{4.65}$$

Since Lemma 13 states that

$$\sum_{\frac{ac}{\varepsilon}<\gamma\leq\frac{c}{\varepsilon}} \frac{\ell_{c,\varepsilon}(\gamma)}{\gamma} \leq \frac{1}{2}\frac{1+11c\varepsilon}{\pi c a^2}\log\left(\frac{c}{\varepsilon}\right)\frac{\cosh(c\sqrt{1-a^2})}{\sinh(c)}.$$

pairing ρ and $1-\rho$ for every zero off the critical line, and putting $\gamma=\Im(\rho)$,we get,

$$\sum_{\frac{ac}{\varepsilon}<|\Im(\rho)|\leq\frac{c}{\varepsilon}} \left|\frac{\ell_{c,\varepsilon}(\Im(\rho))}{\Im(\rho)}\right| \leq \frac{1+11c\varepsilon}{\pi c a^2}\log\left(\frac{c}{\varepsilon}\right)\frac{\cosh(c\sqrt{1-a^2})}{\sinh(c)}. \tag{4.66}$$

Now, combining equation (4.66) and equation (4.65), we obtain,

$$\sum_{\frac{ac}{\varepsilon}<|\Im(\rho)|\leq\frac{c}{\varepsilon}} \left| a_{c,\varepsilon}(\rho)\frac{x^{\rho}}{\rho} \right| \leq \sqrt{x} \sum_{\frac{ac}{\varepsilon}<|\Im(\rho)|\leq\frac{c}{\varepsilon}} \left| \frac{\ell_{c,\varepsilon}(\Im(\rho))}{\Im(\rho)} \right|$$
$$\leq \frac{1+11c\varepsilon}{\pi c a^2} \log\left(\frac{c}{\varepsilon}\right) \frac{\cosh(c\sqrt{1-a^2})}{\sinh(c)} \sqrt{x},$$

proving Proposition 4. □

We will need the following lemma to compute the remainder of the sum over zeroes in the explicit formula for $\psi_{c,\varepsilon}(x)$.

Lemma 14. *(Lemma 3, [7])*

Let $t_2 > t_1 \geq 14$. *Then we have*

$$\sum_{t_1\leq\Im(\rho)\leq t_2} \frac{1}{\Im(\rho)} \leq \frac{1}{4\pi}\left[\log\left(\frac{t_2}{2\pi}\right)^2 - \log\left(\frac{t_1}{2\pi}\right)^2\right] + O^*\left(5\frac{\log t_1}{t_1}\right), \tag{4.67}$$

and for $t_2 \geq 5000$, *we have*

$$\sum_{0<\Im(\rho)<t_2} \frac{1}{\Im(\rho)} \leq \frac{1}{4\pi}\log\left(\frac{t_2}{2\pi}\right)^2. \tag{4.68}$$

Proof. We recall that $N(t)$, is defined in Chapter 1 as

$$N(t) = \#\{\rho = \beta + i\gamma \mid 0 < \gamma < t\}.$$

Ingham's Lemma [20, page 18], states the following Riemann- Stieltjes integral

$$\sum_{t_1\leq\gamma<t_2} \frac{1}{\gamma} = \int_{t_1}^{t_2} \frac{dN(t)}{dt}. \tag{4.69}$$

where we assume that t_1 and t_2 are not ordinates of ρ. Denoting $N(t) = \tilde{N}(t) + R(t)$, where

$\tilde{N}(t) = \frac{t}{2\pi}\log\left(\frac{t}{2\pi e}\right) + \frac{7}{8}$ we can rewrite (4.69) as

$$\sum_{t_1 \leq \gamma < t_2} \frac{1}{\Im(\rho)} = \int_{t_1}^{t_2} \frac{d\tilde{N}(t)}{t} + \int_{t_1}^{t_2} \frac{dR(t)}{t}$$
$$= \int_{t_1}^{t_2} \frac{\tilde{N}'(t)}{t} + \int_{t_1}^{t_2} \frac{dR(t)}{t},$$

since $\tilde{N}(t)$ is differentiable.

Evaluating the first integral, we get $\frac{\tilde{N}'(t)}{t} = \frac{1}{2\pi t}\left(\log\left(\frac{t}{2\pi e}\right) + 1\right)$, which when simplified gives us

$$\frac{\tilde{N}'(t)}{t} = \frac{1}{2\pi t}\log\left(\frac{t}{2\pi}\right).$$

Therefore, we can now state that

$$\int_{t_1}^{t_2} \frac{\tilde{N}'(t)}{t} = \frac{1}{2\pi}\int_{t_1}^{t_2} \frac{1}{t}\log\left(\frac{t}{2\pi}\right) dt$$
$$= \frac{1}{4\pi}\left[\log\left(\frac{t_2}{2\pi}\right)^2 - \log\left(\frac{t_1}{2\pi}\right)^2\right], \tag{4.70}$$

which gives us the first part of equation (4.67).

Using Integration by parts on the second integral, we can write $\int_{t_1}^{t_2} \frac{dR(t)}{t}$ as

$$\int_{t_1}^{t_2} \frac{dR(t)}{t} = \left.\frac{R(t)}{t}\right|_{t_1}^{t_2} + \int_{t_1}^{t_2} \frac{R(t)}{t^2} dt = \frac{R(t_2)}{t_2} - \frac{R(t_1)}{t_1} + \int_{t_1}^{t_2} \frac{R(t)}{t^2} dt.$$

Now, using the Triangle inequality, we can claim,

$$\int_{t_1}^{t_2} \frac{dR(t)}{t} \leq \frac{|R(t_2)}{t_2} + \frac{|R(t_1)|}{t_1} + \int_{t_1}^{t_2} \frac{R(t)}{t^2} dt.$$

Rosser's estimate in (4.69) implies $|R(t)| \leq \log t$ for $t \geq 14$ and thus, for $t_2 \geq t_1 \geq 14$

$$\int_{t_1}^{t_2} \frac{dR(t)}{t} \leq \frac{\log(t_2)}{t_2} + \frac{\log t_1}{t_1} + \int_{t_1}^{t_2} \frac{\log t}{t^2} dt \leq 2\frac{\log t_1}{t_1} + \int_{t_1}^{t_2} \frac{\log t}{t^2} dt,$$

since $\frac{\log x}{x}$ is a decreasing function for $x > e$. Using integration by parts on the second integral, we obtain

$$\int_{t_1}^{t_2} \frac{dR(t)}{t} \leq 2\frac{\log t_1}{t_1} + \left(-\frac{\log t_2}{t_2} - \frac{1}{t_2}\right) - \left(-\frac{\log t_1}{t_1} - \frac{1}{t_1}\right) \leq 3\frac{\log t_1}{t_1} + \frac{1}{t_1} \leq 4\frac{\log t_1}{t_1}. \tag{4.71}$$

The equation (4.70) and the inequality (4.71) together gives us the following inequality :

$$\int_{t_1}^{t_2} \frac{\tilde{N}'(t)}{t} + \int_{t_1}^{t_2} \frac{dR(t)}{t} \leq \frac{1}{4\pi}\left[\log\left(\frac{t_2}{2\pi}\right)^2 - \log\left(\frac{t_1}{2\pi}\right)^2\right] + O^*\left(4\frac{\log t_1}{t_1}\right),$$

which implies

$$\sum_{t_1 \leq \gamma < t_2} \frac{1}{\Im(\rho)} \leq \frac{1}{4\pi}\left[\log\left(\frac{t_2}{2\pi}\right)^2 - \log\left(\frac{t_1}{2\pi}\right)^2\right] + O^*\left(5\frac{\log t_1}{t_1}\right),$$

and gives us a proof of (4.67).

To prove the inequality (4.68), we begin by writing

$$\sum_{0<\Im(\rho)<t_2} \frac{1}{\Im(\rho)} = \sum_{0<\Im(\rho)<5000} \frac{1}{\Im(\rho)} + \sum_{5000<\Im(\rho)<t_2} \frac{1}{\Im(\rho)}.$$

By equation (4.67) we know

$$\sum_{5000<\Im(\rho)<t_2} \frac{1}{\Im(\rho)} \leq \frac{1}{4\pi}\log\left(\frac{t_2}{2\pi}\right)^2 - \frac{1}{4\pi}\log\left(\frac{5000}{2\pi}\right)^2 + 5\frac{\log 5000}{5000}.$$

Evaluating $\sum_{0<\Im(\rho)<5000} \frac{1}{\Im(\rho)}$, $\frac{1}{4\pi}\log\left(\frac{5000}{2\pi}\right)^2$, and $5\frac{\log 5000}{5000}$, we can now claim,

$$\sum_{0<\Im(\rho)<t_2} \frac{1}{\Im(\rho)} \leq \frac{1}{4\pi}\log\left(\frac{t_2}{2\pi}\right)^2 - 3.55 + 0.0086 + 3.54 \leq \frac{1}{4\pi}\log\left(\frac{t_2}{2\pi}\right)^2$$

for $t_2 \geq 5000$, which proves the lemma. □

4.4 Bounding the Sum over Prime Powers

The modified Chebyshev function $\psi_{c,\varepsilon}(x)$ can be used to trivially bound $\psi(x)$, using $\alpha = 1$ in Proposition 1. However, choosing α close to 0 and bounding the sum over prime powers yield significantly better answers.

Lemma 15. *Let $x \geq 100$, $\varepsilon \leq \frac{1}{100}$ and $I = [e^{-\varepsilon}x, e^{\varepsilon}x]$. Then we have,*

$$\sum_{\substack{p^m \in I \\ m \geq 2}} \frac{1}{m} \leq 1.29\varepsilon\sqrt{x} + \log\left(\frac{\log(e^{\varepsilon}x)}{\log 2}\right). \tag{4.72}$$

Equation (4.72) of Lemma 15 is an improvement on Lemma 4 of [7], which states for the same given conditions that

$$\sum_{\substack{p^m \in I \\ m \geq 2}} \frac{1}{m} \leq 4.01\varepsilon\sqrt{x} + \log\log(2x^2). \tag{4.73}$$

Proof. We begin by considering the sum

$$\mathcal{S} = \sum_{\substack{p;m \geq 2 \\ p^m \in I}} \frac{1}{m}.$$

If we let M_0 be the maximum integer such that $p^{M_0} \in I$, then $2^{M_0} \leq p^{M_0} \leq e^{\varepsilon}x \implies 2^{M_0} \leq e^{\varepsilon}x$. Now, taking the logarithm on both sides of the previous inequality we obtain,

$$M_0 \leq \frac{\log(e^{\varepsilon}x)}{\log 2}.$$

Also, we observe that

$$\mathcal{S} \leq \sum_{m=2}^{M_0} \frac{1}{m} \#\{p \mid p^m \in I\}.$$

To continue the proof we start by denoting $X-Y=e^{-\varepsilon}x$ and $X=e^{\varepsilon}x$, and see that

$$\begin{aligned}
\#\{p \mid p^m \in I\} &= \#\{p \mid p^m \in [X-Y,X]\} \\
&\leq \#\{k\in\mathbb{N} \mid k^m \in [X-Y,X]\} = \#\{k\in\mathbb{N} \mid (X-Y)^{\frac{1}{m}} \leq k \leq X^{\frac{1}{m}}\} \\
&= X^{\frac{1}{m}} - (X-Y)^{\frac{1}{m}} + 1.
\end{aligned}$$

Therefore, by the Mean Value Theorem,

$$\#\{p \mid p^m \in I\} \leq \frac{Y}{m}X^{\frac{1}{m}-1} + 1. \tag{4.74}$$

Now, solving for Y using the equations $X=e^{\varepsilon}x$ and $X-Y=e^{-\varepsilon}x$ gives us $e^{\varepsilon}x-Y=e^{-\varepsilon}x$ which implies that $Y=2\frac{x(e^{\varepsilon}-e^{-\varepsilon})}{2}$. Therefore $Y=2x\sinh\varepsilon$. Substituting the values for X and Y in the equation (4.74), we see that

$$\begin{aligned}
\#\{p \mid p^m \in I\} &\leq \frac{Y}{m}X^{\frac{1}{m}-1}+1 = \frac{2x\sinh\varepsilon}{m}(e^{\varepsilon}x)^{\frac{1}{m}-1}+1 = \frac{2\sinh\varepsilon}{m}e^{\varepsilon(\frac{1}{m}-1)}x^{\frac{1}{m}}+1 \\
&\leq \frac{2\sinh\varepsilon}{m}e^{-\frac{\varepsilon}{2}}\sqrt{x}+1 \leq \frac{2\varepsilon\sqrt{x}}{m}\left(\frac{\sinh\varepsilon}{\varepsilon}e^{\frac{-\varepsilon}{2}}\right)+1 \leq \frac{2\varepsilon\sqrt{x}}{m}\left(\max_{0\leq\varepsilon\leq\frac{1}{100}}\left(\frac{\sinh\varepsilon}{\varepsilon}e^{\frac{-\varepsilon}{2}}\right)\right)+1 \\
&\leq \frac{2\varepsilon\sqrt{x}}{m}+1,
\end{aligned}$$

and thus we have shown that $\#\{p \mid p^m \in I\}$ is bounded by $\frac{2\varepsilon}{m}\sqrt{x}+1$. We now have the following inequality for $\mathcal{S}$:

$$\begin{aligned}
\mathcal{S} &\leq \sum_{m=2}^{M_0}\frac{1}{m}\left(\frac{2\varepsilon}{m}\sqrt{x}+1\right) \\
&= 2\varepsilon\sqrt{x}\sum_{m=2}^{M_0}\frac{1}{m^2}+\sum_{m=2}^{M_0}\frac{1}{m} \\
&\leq 2\varepsilon\sqrt{x}\left(\sum_{m=1}^{\infty}\frac{1}{m^2}-1\right)+\left(\sum_{m=1}^{M_0}\frac{1}{m}-1\right).
\end{aligned}$$

Lemma 2.8 of [11] and Lemma 2.1 of [30] provides us with the bound $\sum_{n\leq X}\frac{1}{n}$, which when applied to the rightmost term of the previous inequality gives us,

$$\mathcal{S} \leq 2\varepsilon\sqrt{x}(\zeta(2)-1) + \left(\log x + \gamma_0 + \frac{7}{12\sqrt{x}} - 1\right).$$

Since $x \geq 100$ we can write

$$\begin{aligned}
\mathcal{S} &\leq 2\varepsilon\sqrt{x}(0.644\ldots) + (\log x - 0.32) \\
&\leq 2\varepsilon\sqrt{x}(0.644\ldots) + (\log x) \\
&\leq 2\varepsilon\sqrt{x}(0.644\ldots) + \log\left(\frac{\log(e^{\varepsilon}x)}{\log 2}\right) \\
&\leq 1.29\varepsilon\sqrt{x} + \log\left(\frac{\log(e^{\varepsilon}x)}{\log 2}\right).
\end{aligned}$$

We have thus proved equation (4.72) and improved Büthe's result for Lemma 4 in [7]. □

Lemma 16. *([7, Lemma 5]) Let* $x > 1$, $\varepsilon < 1$ *and* $\alpha \in (0,1)$ *such that*

$$B_0(x) = \frac{\varepsilon x e^{-\varepsilon}|\nu_c(\alpha)|}{2\mu_c(\alpha)} > 1$$

holds. Furthermore, let $I_\alpha^+ = [e^{\alpha\varepsilon}x, e^{\varepsilon}x]$ *and* $I_\alpha^- = [e^{-\varepsilon}x, e^{-\varepsilon\alpha}x]$*. Then we have*

$$\sum_{p^m \in I_\alpha^\pm} \left|\mu_{c,\varepsilon}\left(\log\frac{p}{x}\right)\right| \leq 2\frac{\varepsilon x e^{\varepsilon}|\nu_c(\alpha)|}{\log(B_0(x))}. \tag{4.75}$$

Proof. We give proof for $I = I_\alpha^+$. For $t \in I$, let

$$f(t) = \mu_c\left(\frac{1}{\varepsilon}\log\frac{t}{x}\right),$$

and note that

$$f(t) \geq 0 \text{ for } t \in I_\alpha^+. \tag{4.76}$$

We shall establish below, that f satisfies the conditions of [6, Theorem 4.3], and thus

$$\sum_{p\in I} f(p) \leq 2\|f\|_{1,I}\left(\log \frac{\|f\|_{1,I}}{\|f\|_{\infty,I}+\|f'\|_{1,I}}\right)^{-1}. \tag{4.77}$$

We know from the definition that

$$\|f\|_{\infty,I} = \sup_{x\in I}|f(x)|,$$

and since $f(t)$ decreases on $I = I_\alpha^+$ and thus is maximized at the left end point, we can write

$$\|f\|_{\infty,I} = |f(e^{\alpha\varepsilon}x)| = \mu_c\left(\frac{1}{\varepsilon}\log\left(\frac{e^{\alpha\varepsilon}x}{x}\right)\right) = \mu_c(\alpha). \tag{4.78}$$

Next, considering the definition of $\|f'\|_{1,I}$, we get the following identity:

$$\begin{aligned}
\|f'\|_{1,I} &= \int_I \frac{d}{dt}|f(t)|dt = \int_I \frac{d}{dt}\left|\mu_c\left(\frac{1}{\varepsilon}\log\frac{t}{x}\right)\right|dt \\
&= \int_I \left(\mu_c'\left(\frac{1}{\varepsilon}\log\frac{t}{x}\right)\right)\frac{1}{\varepsilon}\frac{1}{t}dt \\
&= \int_{e^{\alpha\varepsilon}x}^{e^{\varepsilon}x} \left(\eta_c\left(\frac{1}{\varepsilon}\log\frac{t}{x}\right)\right)\frac{1}{\varepsilon}\frac{1}{t}dt,
\end{aligned}$$

where, substituting $\left(\frac{1}{\varepsilon}\log\frac{t}{x}\right) = \tau$, we obtain

$$\|f'\|_{1,I} = \int_\alpha^1 \eta_c(\tau)d\tau = -\int_1^\alpha \eta_c(\tau)d\tau.$$

Now letting $u = -\tau$, the above identity can be written as,

$$\|f'\|_{1,I} = \int_{-1}^{-\alpha} \eta_c(u)du = \mu_c(\alpha), \tag{4.79}$$

since η_c is an even function.

Similarly, doing the same substitution $\left(\frac{1}{\varepsilon}\log\frac{t}{x}\right) = \tau$, we can write $\varepsilon t d\tau = dt$ and $t = e^{\varepsilon\tau}x$

which gives us

$$\begin{aligned}\|f\|_{1,I} = \left(\int_I |f(t)|dt\right) &= \int_{e^{\alpha\varepsilon}x}^{e^{\varepsilon}x} f(t)dt \\ &= \int_{e^{\alpha\varepsilon}x}^{e^{\varepsilon}x} \mu_c\left(\frac{1}{\varepsilon}\log\frac{t}{x}\right)dt \\ &= \varepsilon x\int_{\alpha}^{1} \mu_c(\tau)e^{\varepsilon\tau}d\tau \\ &\le \varepsilon x e^{\varepsilon}\int_{\alpha}^{1} \mu_c(\tau)d\tau,\end{aligned}$$

since $e^{\varepsilon\tau}$ is maximized for $\tau = 1$. From Lemma 5(ii), we know $\nu_c(t) = -\int_t^1 \mu_c(\alpha)d\alpha$, and we can now claim the following inequality:

$$\|f\|_{1,I} \le \varepsilon x e^{\varepsilon}|\nu_c(\alpha)|. \tag{4.80}$$

Similarly, since $e^{\varepsilon\tau}$ is minimized for $\tau = \alpha$, we can also claim that

$$\|f\|_{1,I} \ge \varepsilon x e^{\varepsilon\alpha}|\nu_c(\alpha)| \ge \varepsilon x e^{-\varepsilon}|\nu_c(\alpha)|. \tag{4.81}$$

Putting in the bounds from equations (4.79), (4.80), (4.81) and (4.78) in the inequality (4.77), we get

$$\begin{aligned}\sum_{p\in I} f(p) = \sum_{p\in I}\mu_c\left(\frac{1}{\varepsilon}\log\frac{p}{x}\right) &\le 2\varepsilon x e^{\varepsilon}|\nu_c(\alpha)|\frac{1}{\left(\log\frac{\varepsilon x e^{-\varepsilon}|\nu_c(\alpha)|}{2\mu_c(\alpha)}\right)} \\ &\le 2\frac{\varepsilon x e^{\varepsilon}|\nu_c(\alpha)|}{\log(B_0(x))}.\end{aligned}$$

Since $\mu_c\left(\frac{1}{\varepsilon}\log\frac{t}{x}\right)$ is positive in I, it is true that

$$\sum_{p\in I}\left|\mu_c\left(\frac{1}{\varepsilon}\log\frac{p}{x}\right)\right| = \sum_{p\in I}\mu_c\left(\frac{1}{\varepsilon}\log\frac{p}{x}\right) \le 2\frac{\varepsilon x e^{\varepsilon}|\nu_c(\alpha)|}{\log(B_0(x))}.$$

Now, considering the subset I_{α}^{-}, we see that for f restricted to I_{α}^{-}, f is monotonically increasing, and thus attains maximum at $e^{-\varepsilon\alpha}$. Therefore, we can state that

$$\|f\|_{\infty,I_\alpha^-} = |f(e^{-\alpha\varepsilon}x)| = \left|\mu_c\left(\frac{1}{\varepsilon}\log\left(\frac{e^{-\alpha\varepsilon}x}{x}\right)\right)\right| = |\mu_c(-\alpha)| = |-\mu_c(\alpha)| = \mu_c(\alpha).$$

Using calculations similar to the ones for $I = I_\alpha^+$ also gives the inequality

$$\sum_{p\in I_\alpha^-}\left|\mu_c\left(\frac{1}{\varepsilon}\log\frac{p}{x}\right)\right| \leq 2\frac{\varepsilon x e^{\varepsilon}|\nu_c(\alpha)|}{\log(B_0(x))},$$

from which the inequality (4.75) follows. □

The two previous lemmas give rise to Proposition 5.

Proposition 5. *[7, Proposition 4] Let* $0 \leq \alpha < 1$, $x > 100$*, and* $\varepsilon < 10^{-2}$*, such that*

$$B_0(x) = \frac{\varepsilon x e^{-\varepsilon}|\nu_c(\alpha)|}{2\mu_c(\alpha)} > 1$$

holds. We define

$$A(x,c,\varepsilon,\alpha) = e^{2\varepsilon}\log(e^{\varepsilon}x)\left[\frac{2\varepsilon x|\mu_c(\alpha)|}{\log B_0(x)} + 2.01\varepsilon\sqrt{x} + \frac{1}{2}\log\log(2x^2)\right]. \tag{4.82}$$

Then we have

$$\psi(e^{-\alpha\varepsilon}x) \leq \psi_{c,\varepsilon}(x) + A(x,c,\varepsilon,\alpha) \tag{4.83}$$

and

$$\psi(e^{\alpha\varepsilon}x) \geq \psi_{c,\varepsilon}(x) - A(x,c,\varepsilon,\alpha) \tag{4.84}$$

Proof. From Proposition 1, equation (4.3) we know

$$\psi(e^{-\alpha\varepsilon}x) \leq \psi_{c,\varepsilon}(x) - \sum_{e^{-\varepsilon}x<p^m<e^{\alpha\varepsilon}x}\frac{1}{m}M_{x,c,\varepsilon}(p^m).$$

Therefore, if

$$A(x,c,\varepsilon,\alpha) \geq -\sum_{e^{-\varepsilon}x<p^m<e^{\alpha\varepsilon}x} \frac{1}{m} M_{x,c,\varepsilon}(p^m),$$

equation (4.83) holds.

We also know from Proposition 1, equation (4.4), that

$$\psi(e^{\alpha\varepsilon}x) \geq \psi_{c,\varepsilon}(x) - \sum_{e^{\alpha\varepsilon}x<p^m<e^{\varepsilon}x} \frac{1}{m} M_{x,c,\varepsilon}(p^m).$$

Hence, if

$$\sum_{e^{\alpha\varepsilon}x<p^m<e^{\varepsilon}x} \frac{1}{m} M_{x,c,\varepsilon}(p^m) \leq A(x,c,\varepsilon,\alpha),$$

equation (4.84) holds. So to prove this proposition it suffices to show that

$$\left| \sum_{p^m \in I_\alpha^\pm} \frac{1}{m} M_{x,c,\varepsilon}(p^m) \right| \leq A(x,c,\varepsilon,\alpha).$$

Now from Proposition 1 we know that

$$M_{x,c,\varepsilon}(t) = \frac{\log t}{\lambda_{c,\varepsilon}} \Big[\chi^*_{[x,e^{\varepsilon}x]}(t) \int_{-\varepsilon}^{\log t/x} \eta_{c,\varepsilon}(\tau) e^{-\tau/2} d\tau \\ - \chi^*_{[e^{-\varepsilon}x,x]}(t) \int_{\log t/x}^{\varepsilon} \eta_{c,\varepsilon}(\tau) e^{-\tau/2} d\tau \Big].$$

Applying Lemma 6, with $t = 0$ and $\tau = \frac{-\tau}{2}$ to the above identity gives us

$$\begin{aligned} M_{x,c,\varepsilon}(t) &= \frac{\log t}{\lambda_{c,\varepsilon}} \Big[\chi^*_{[x,e^{\varepsilon}x]}(t) \int_{-\varepsilon}^{\log t/x} \eta_{c,\varepsilon}(\tau)(1+O^*(|\tau|)) d\tau \\ &\qquad - \chi^*_{[e^{-\varepsilon}x,x]}(t) \int_{\log t/x}^{\varepsilon} \eta_{c,\varepsilon}(\tau)(1+O^*(|\tau|)) d\tau \Big] \\ &= \frac{\log t}{\lambda_{c,\varepsilon}} \Big[\chi^*_{[x,e^{\varepsilon}x]}(t) \int_{-\varepsilon}^{\log t/x} \eta_{c,\varepsilon}(\tau)(1+O^*(|\varepsilon|)) d\tau \\ &\qquad - \chi^*_{[e^{-\varepsilon}x,x]}(t) \int_{\log t/x}^{\varepsilon} \eta_{c,\varepsilon}(\tau)(1+O^*(|\varepsilon|)) d\tau \Big]. \end{aligned}$$

Therefore we can state that

$$M_{x,c,\varepsilon}(t) = \frac{\log t}{\lambda_{c,\varepsilon}}(1+O^*(|\varepsilon|))\Big[\chi^*_{[x,e^{\varepsilon}x]}(t)\int_{-\varepsilon}^{\log t/x}\eta_{c,\varepsilon}(\tau)d\tau - \chi^*_{[e^{-\varepsilon}x,x]}(t)\int_{\log t/x}^{\varepsilon}\eta_{c,\varepsilon}(\tau)d\tau\Big]. \tag{4.85}$$

Now, since $[x,e^{\varepsilon}x]$ and $[e^{-\varepsilon}x,x]$ are two disjoint sets, we can consider two distinct cases for t in equation (4.85) :

- If $t\in[x,e^{\varepsilon}x]$, then $\chi^*_{[e^{-\varepsilon}x,x]}(t)=0$, and we can write

$$\begin{aligned}
M_{x,c,\varepsilon}(t) &= \frac{\log t}{\lambda_{c,\varepsilon}}(1+O^*(|\varepsilon|))\Big[\chi^*_{[x,e^{\varepsilon}x]}(t)\int_{-\varepsilon}^{\log t/x}\eta_{c,\varepsilon}(\tau)d\tau\Big]\\
&= \frac{\log t}{\lambda_{c,\varepsilon}}(1+O^*(|\varepsilon|))\Big[\int_{-\varepsilon}^{\log t/x}\eta_{c,\varepsilon}(\tau)d\tau\Big]\\
&= \frac{\log t}{\lambda_{c,\varepsilon}}(1+O^*(|\varepsilon|))\mu_{c,\varepsilon}\left(\log\frac{t}{x}\right)
\end{aligned}$$

- If $t\in[e^{-\varepsilon}x,x]$, then $\chi^*_{[x,e^{\varepsilon}x]}(t)=0$, and we can write

$$\begin{aligned}
M_{x,c,\varepsilon}(t) &= \frac{\log t}{\lambda_{c,\varepsilon}}(1+O^*(|\varepsilon|))\Big[-\chi^*_{[e^{-\varepsilon}x,x]}(t)\int_{\log t/x}^{\varepsilon}\eta_{c,\varepsilon}(\tau)d\tau\Big]\\
&= \frac{\log t}{\lambda_{c,\varepsilon}}(1+O^*(|\varepsilon|))\Big[-\int_{\log t/x}^{\varepsilon}\eta_{c,\varepsilon}(\tau)d\tau\Big]
\end{aligned}$$

  Substituting $\tau=-\tau_0$, we get

$$M_{x,c,\varepsilon}(t) = \frac{\log t}{\lambda_{c,\varepsilon}}(1+O^*(|\varepsilon|))\mu_{c,\varepsilon}\left(\log\frac{t}{x}\right).$$

Therefore, $M_{x,c,\varepsilon}(t)=\frac{\log t}{\lambda_{c,\varepsilon}}\mu_{c,\varepsilon}\Big(\log\frac{t}{x}\Big)(1+O^*(\varepsilon))$, $\forall t\in[e^{-\varepsilon}x,e^{\varepsilon}x]$. We can now write,

$$\begin{aligned}
|M_{x,c,\varepsilon}(t)| &\le \frac{\log t}{\lambda_{c,\varepsilon}}\left|\mu_{c,\varepsilon}\Big(\log\Big(\frac{t}{x}\Big)\Big)\right|(1+\varepsilon)\\
&\le \log(e^{\varepsilon}x)\frac{\left|\mu_{c,\varepsilon}\left(\log\left(\frac{t}{x}\right)\right)\right|}{\lambda_{c,\varepsilon}}(1+\varepsilon).
\end{aligned}$$

We consider the case where $x \le t \le e^{\varepsilon}x$ which implies $0 \le \log\left(\frac{t}{x}\right) \le \varepsilon$. Hence, we can write

$$\frac{\mu_{c,\varepsilon}\left(\log\left(\frac{t}{x}\right)\right)}{\lambda_{c,\varepsilon}} = \frac{\int_{\log\left(\frac{t}{x}\right)}^{1} \eta_{c,\varepsilon}(\tau)d\tau}{\lambda_{c,\varepsilon}} \le \frac{\frac{1}{2}\int_{-\infty}^{\infty}\eta_{c,\varepsilon}(\tau)d\tau}{\ell_{c,\varepsilon}(i/2)} = \frac{\frac{1}{2}\ell_{c,\varepsilon}(0)}{\ell_{c,\varepsilon}(i/2)} = \frac{1}{2\ell_{c,\varepsilon}(i/2)}.$$

Therefore we now see

$$|M_{x,c,\varepsilon}(t)| \le \log(e^{\varepsilon}x)\frac{1}{2\ell_{c,\varepsilon}(i/2)}(1+\varepsilon) \le \log(e^{\varepsilon}x)\frac{e^{\varepsilon}}{2}, \tag{4.86}$$

since $e^{\varepsilon} \ge 1+\varepsilon$ and $\lambda_{c,\varepsilon} > 1$. The proof for the other case, for $e^{-\varepsilon}x \le t \le x$ follows similarly. Furthermore, considering $\sum_{p^m \in I_\alpha^{\pm}} \frac{1}{m} M_{x,c,\varepsilon}(p^m)$, for $I_\alpha^{\pm}$ defined in Lemma16, we see that

$$\sum_{p^m \in I_\alpha^{\pm}} \frac{1}{m} M_{x,c,\varepsilon}(p^m) = \sum_{p \in I_\alpha^{\pm}} M_{x,c,\varepsilon}(p) + \sum_{\substack{p^m \in I_\alpha^{\pm} \\ m \ge 2}} \frac{1}{m} M_{x,c,\varepsilon}(p^m) \tag{4.87}$$

Considering the first term in equation (4.87), and using the fact that $\log p \le \log(e^{\varepsilon}x)$, and the Lemma 3, equation (3.35), which states that $\lambda_{c,\varepsilon} \ge 1$, we deduce

$$\left|\sum_{p \in I_\alpha^{\pm}} M_{x,c,\varepsilon}(p)\right| \le \sum_{p \in I_\alpha^{\pm}} \frac{\log p}{\lambda_{c,\varepsilon}} \left|\mu_{c,\varepsilon}\left(\log\frac{p}{x}\right)\right| \le (\log(e^{\varepsilon}x)) \sum_{p \in I_\alpha^{\pm}} \left|\mu_{c,\varepsilon}\left(\log\frac{p}{x}\right)\right|.$$

Applying equation (4.75) of Lemma 16 leads to the inequality

$$\left|\sum_{p \in I_\alpha^{\pm}} M_{x,c,\varepsilon}(p)\right| \le (\log(e^{\varepsilon}x))\frac{2\varepsilon x e^{\varepsilon}|\nu_c(\alpha)|}{\log(B_0(x))} \le 2e^{\varepsilon}\log(e^{\varepsilon}x)\frac{\varepsilon x|\nu_c(\alpha)|}{\log(B_0(x))}.$$

Considering the second term in equation (4.87), we have

$$\left|\sum_{\substack{p^m \in I_\alpha^{\pm} \\ m \ge 2}} \frac{1}{m} M_{x,c,\varepsilon}(p^m)\right| \le \sum_{\substack{p^m \in I_\alpha^{\pm} \\ m \ge 2}} \frac{|M_{x,c,\varepsilon}(p^m)|}{m} \le \left(\sum_{\substack{p^m \in I_\alpha^{\pm} \\ m \ge 2}} \frac{1}{m}\right) \log(e^{\varepsilon}x)\frac{e^{\varepsilon}}{2},$$

by using the inequality in (4.86). Now using equation (4.73) of Lemma 15 we have

$$\left|\sum_{\substack{p^m\in I_\alpha^\pm\\ m\geq 2}} \frac{1}{m} M_{x,c,\varepsilon}(p^m)\right| \leq [4.01\varepsilon\sqrt{x}+\log\log 2x^2]\log(e^\varepsilon x)\frac{e^\varepsilon}{2}.$$

Finally, returning to the original sum, we get

$$\begin{aligned}\left|\sum_{p^m\in I_\alpha^\pm} \frac{1}{m} M_{x,c,\varepsilon}(p^m)\right| &\leq 2e^\varepsilon \log(e^\varepsilon x)\frac{\varepsilon x|\nu_c(\alpha)|}{\log(B_0(x))} + [4.01\varepsilon\sqrt{x}+\log\log 2x^2]\log(e^\varepsilon x)\frac{e^\varepsilon}{2}\\ &\leq e^\varepsilon\log(e^\varepsilon x)\left[2\frac{\varepsilon x|\nu_c(\alpha)|}{\log(B_0(x))}+2.01\varepsilon\sqrt{x}+\frac{1}{2}\log\log 2x^2\right] = A(x,c,\varepsilon,\alpha).\end{aligned}$$

□

4.5 Bounds for $\mu_c(\alpha)$ and $\nu_c(\alpha)$

Analysing the asymptotic behaviour of $\mu_c(\alpha)$ and $\nu_c(\alpha)$ are difficult for arbitrary α. However the analysis is plausible for $\alpha = 0$ which is close to the optimal choice. To accomplish this, it is prudent to introduce the modified Bessel function of the first kind for real parameters $\gamma \geq 0$.

Lemma 17. *[7, Lemma 6] The modified Bessel function of the first kind for real parameters $\gamma \geq 0$ is defined as*

$$I_\gamma(x) = \left(\frac{x}{2}\right)^\gamma \sum_{n=0}^{\infty} \frac{(x/2)^{2n}}{n!\Gamma(\gamma+n+1)}$$

Let $\alpha, \beta \in [0,\infty)$ be such that $\alpha < \beta$ holds. Then the function

$$\frac{I_\beta(x)}{I_\alpha(x)}$$

is positive and monotonically increasing in $(0,\infty)$ and converges to 1 for $x \to \infty$.

Proof. The proof is based on Sturm Monotony principle. We define the auxilliary function as $f_\gamma(x) = \sqrt{x} I_\gamma(x)$, and thus $\frac{f_\gamma(x)}{\sqrt{x}} = I_\gamma(x)$. The modified Bessel function of the first kind satisfies the following differential equation (see [1], page 358):

$$\frac{d^2}{dx^2}(I_\gamma) + \frac{1}{x}\frac{d}{dx}(I_\gamma) - \left(1 + \frac{\gamma^2}{x^2}\right) I_\gamma = 0. \tag{4.88}$$

Replacing I_γ by $\frac{f_\gamma(x)}{\sqrt{x}}$, in equation (4.88) we get

$$\frac{d^2}{dx^2}\left(\frac{f_\gamma(x)}{\sqrt{x}}\right) + \frac{1}{x}\frac{d}{dx}\left(\frac{f_\gamma(x)}{\sqrt{x}}\right) - \left(1 + \frac{\gamma^2}{x^2}\right)\left(\frac{f_\gamma(x)}{\sqrt{x}}\right) = 0.$$

Differentiating with respect to x gives us the equation

$$\frac{d}{dx}\frac{\left(\frac{d}{dx}(f_\gamma)\sqrt{x} - \frac{1}{2\sqrt{x}}f_\gamma\right)}{x} + \frac{1}{x}\left(\frac{\frac{d}{dx}(f_\gamma)\sqrt{x} - \frac{1}{2\sqrt{x}}f_\gamma}{x}\right) - \left(1 + \frac{\gamma^2}{x^2}\right)\frac{f_\gamma}{\sqrt{x}} = 0.$$

We now differentiate the above equation again to obtain

$$\frac{\frac{d}{dx}\left(\frac{d}{dx}(f_\gamma)\sqrt{x} - \frac{1}{2\sqrt{x}}f_\gamma\right)x - \left(\frac{d}{dx}(f_\gamma)\sqrt{x} - \frac{1}{2\sqrt{x}}f_\gamma\right)}{x^2} + \frac{\frac{d}{dx}f_\gamma}{x\sqrt{x}} - \frac{1}{2x^2\sqrt{x}}f_\gamma - \frac{f_\gamma}{\sqrt{x}} - \frac{\gamma^2 f_\gamma}{x^2\sqrt{x}} = 0,$$

which simplifies to

$$\frac{d^2}{dx^2}(f_\gamma) - \left(1 - \frac{1}{4x^2} + \frac{\gamma^2}{x^2}\right) f_\gamma = 0. \tag{4.89}$$

Substituting the value of γ by α and then by β respectively, we get

$$\frac{d^2}{dx^2}(f_\alpha) - \left(1 - \frac{1}{4x^2} + \frac{\alpha^2}{x^2}\right) f_\alpha = 0,$$

which implies

$$f_\beta f''_\alpha - \left(1 - \frac{1}{4x^2} + \frac{\alpha^2}{x^2}\right) f_\alpha f_\beta = 0, \tag{4.90}$$

and

$$\frac{d^2}{dx^2}(f_\beta) - \left(1 - \frac{1}{4x^2} + \frac{\beta^2}{x^2}\right) f_\beta = 0$$

which implies

$$f_\alpha f_\beta'' - \left(1 - \frac{1}{4x^2} + \frac{\beta^2}{x^2}\right) f_\alpha f_\beta = 0. \tag{4.91}$$

Subtracting equation (4.90) from (4.91), we get

$$f_\alpha f_\beta'' - f_\beta f_\alpha'' = \frac{\beta^2 - \alpha^2}{x^2} f_\alpha f_\beta > 0$$

in $(0, \infty)$. Thus we can write

$$\int_\varepsilon^x f_\alpha f_\beta'' - f_\beta f_\alpha'' > 0 \implies \Big[f_\beta' f_\alpha - f_\alpha' f_\beta \Big]_\varepsilon^x > 0$$

for $x > \varepsilon$ and every $\varepsilon > 0$. Since

$$f_\beta' f_\alpha - f_\alpha' f_\beta = x I_\beta' I_\alpha - x I_\alpha' I_\beta$$

vanishes for $x \to 0$, we get

$$f_\beta' f_\alpha - f_\alpha' f_\beta \geq 0 \implies \frac{d}{dx}\left(\frac{f_\beta}{f_\alpha}\right) \geq 0.$$

Consequently, the function $\frac{f_\beta}{f_\alpha}$ is increasing monotonously in $(0, \infty)$ and since

$$I_\gamma(x) \sim \frac{e^x}{\sqrt{2\pi x}}$$

holds for every $\gamma \geq 0$, it converges to 1 for $x \to \infty$. □

Proposition 6. *[7, Proposition 5] For $c_0 > 0$, let*

$$D(c_0) = \sqrt{\frac{\pi c_0}{2}} \frac{I_1(c_0)}{\sinh(c_0)}. \tag{4.92}$$

Then the inequalities,

$$\frac{D(c_0)}{\sqrt{2\pi c}} \leq |\nu_c(0)| \leq \frac{1}{2\pi c}$$

holds for all $c \geq c_0$. Furthermore, we have $D(c_0)$ approaches 1 i.e. $D(c_0) \nearrow 1$ for $c_0 \to \infty$.

Proof. $D(c_0)$ is defined as follows:

$$D(c_0) = \sqrt{\frac{\pi c_0}{2}} \frac{I_1(c_0)}{\sinh(c_0)} = \sqrt{2\pi c_0} \frac{I_1(c_0)}{2\sinh(c_0)}.$$

Since $|\nu_c(0)| = \frac{I_1(c)}{2\sinh c}$, we can write $D(c_0) = \sqrt{2\pi c_0}|\nu_{c_0}(0)|$ or $\frac{D(c_0)}{\sqrt{2\pi c_0}} = |\nu_{c_0}(0)|$.
As $c \geq c_0$ for all c , we need to show that $D(c)$ is an increasing function to prove that

$$\frac{D(c_0)}{\sqrt{2\pi c}} \leq |\nu_c(0)| = \frac{D(c)}{\sqrt{2\pi c}}.$$

Since $I_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sinh x$, we see that $D(c)$ can be written as

$$D(c) = \frac{I_1(c)}{\sqrt{\frac{2}{\pi x}} \sinh c} = \frac{I_1(c)}{I_{1/2}(c)}.$$

From Lemma 17, we know that when $\beta > \alpha$ for $\alpha, \beta \in [0, \infty)$, $\frac{I_\beta}{I_\alpha}$ is positive and monotonically increasing in $(0, \infty)$ and converges to 1 for $x \to \infty$. Therefore $D(c) = \frac{I_1(c)}{I_{1/2}(c)}$ is positive and monotonically increasing. We have therefore proved that $\frac{D(c_0)}{\sqrt{2\pi c}} \leq |\nu_c(0)|$.

Moreover, as $D(c) \to 1$ and $D(c)$ is monotonically increasing , it follows that $D(c) \leq 1$. Therefore $\frac{D(c)}{\sqrt{2\pi c}} \leq \frac{1}{\sqrt{2\pi c}}$, and thus $|\nu_c(0)| \leq \frac{1}{\sqrt{2\pi c}}$. This completes the proof of equation (4.92) of Proposition 6.

$D(c_0)$ approaches 1 for c_0 approaching ∞, follows from using $\alpha = \frac{1}{2}$, and $\beta = 1$ in

Lemma 17. □

Deriving the upper and lower bounds for the auxiliary functions are significant in finding upper and lower bounds for $|\psi(x)-x|$.

Lemma 18. *[7, Lemma 8]] Let $\alpha \in (0,1)$, $K \in \mathbb{N}$ and $h = \frac{1-\alpha}{K}$. Then we have*

$$hc\sum_{k=0}^{K-1}\frac{I_0(c\sqrt{2kh-k^2h^2})}{2\sinh c} \leq \mu_c(\alpha) \leq hc\sum_{k=1}^{K}\frac{I_0(c\sqrt{2kh-k^2h^2})}{2\sinh c}, \tag{4.93}$$

and

$$h^2c\sum_{k=0}^{K-1}\sum_{j=0}^{k}\frac{I_0(c\sqrt{2jh-j^2h^2})}{2\sinh(c)} \leq |\nu_c(\alpha)| \leq h^2c\sum_{k=1}^{K}\sum_{j=1}^{k}\frac{I_0(c\sqrt{2jh-j^2h^2})}{2\sinh(c)}. \tag{4.94}$$

Proof. We know $\mu_c(\alpha) = \mu_{c,1}(\alpha)$, and for $\alpha \in (0,1)$, we can write

$$\begin{aligned}\mu_c(\alpha) &= \int_{-1}^{-\alpha}\eta_{c,\varepsilon}(\tau)d\tau = \int_{\alpha}^{1}\eta_{c,1}(\tau)d\tau \\ &= \frac{c}{2\sinh c}\int_{\alpha}^{1}I_0(c\sqrt{1-\tau^2})d\tau.\end{aligned}$$

We divide the interval $[\alpha,1]$ into K sub-intervals of length h with $x_0 = 1$ and $x_K = \alpha$ such that for any k, $x_k = 1-kh$. Therefore, we can state,

$$1-x_k^2 = 1-(1-kh)^2 = 1-(1-2kh+k^2h^2) = 2kh-k^2h^2.$$

Since $\eta_{c,1}$ is a decreasing function, we can bound the integral $\int_\alpha^1 \eta_{c,1}(\tau)d\tau$, by upper and lower Riemann sums and get

$$\sum_{k=0}^{K-1}\eta_{c,1}(x_k)\cdot h \leq \int_{\alpha}^{1}\eta_{c,1}(\tau)d\tau \leq \sum_{k=1}^{K}\eta_{c,1}(x_k)\cdot h.$$

This implies that

$$h\sum_{k=0}^{K-1}\frac{c}{2\sinh c}I_0(c\sqrt{1-(x_k)^2}) \le \int_\alpha^1 \eta_{c,1}(\tau)d\tau \le h\sum_{k=1}^{K}\frac{c}{2\sinh c}I_0(c\sqrt{1-(x_k)^2}).$$

Using the definition of $\mu_c(\alpha) = \int_\alpha^1 \eta_{c,1}(\tau)d\tau$ we can claim the following inequality:

$$h\frac{c}{2\sinh c}\sum_{k=0}^{K-1}I_0(c\sqrt{2kh-k^2h^2}) \le \mu_c(\alpha) \le h\frac{c}{2\sinh c}\sum_{k=1}^{K}I_0(c\sqrt{2kh-k^2h^2}),$$

which proves Lemma 20, equation (4.93).

From Lemma 5(ii), we know that for $\alpha \in (0,1)$,

$$\nu_c(\alpha) = -\int_\alpha^1 \mu_c(\tau)d\tau.$$

We now divide the interval $[\alpha, 1]$ into K sub-intervals with $y_0 = \alpha$, $y_K = 1$ such that for any t, $y_t = \alpha + th$. Since $\mu_c(\alpha)$ is a decreasing function

$$\begin{aligned} |\nu_c(\alpha)| &\le \sum_{r=1}^{K}\mu_c(y_r)h = h\sum_{k=1}^{K}\left(\int_{y_r}^1 \eta_{c,1}(\tau)d\tau\right) \\ &\le h\sum_{r=1}^{K}\left(\sum_{t=r}^{K}h\eta_c(y_t)\right) = h^2\sum_{r=1}^{K}\left(\sum_{t=r}^{K}\eta_c(y_t)\right) \\ &\le h^2\sum_{t=1}^{K}\sum_{r=1}^{t}\frac{c}{\sinh c}I_0\left(c\sqrt{2th-(th)^2}\right). \end{aligned} \tag{4.95}$$

Similarly, we also get the inequality,

$$\begin{aligned} |\nu_c(\alpha)| &\ge \sum_{r=0}^{K-1}\mu_c(y_r)h \ge \sum_{r=0}^{K-1}\left(\sum_{t=r}^{K}\eta_{c,1}(y_t)h\right)h \\ &\ge h^2\sum_{t=0}^{K-1}\sum_{r=0}^{t}\frac{c}{\sinh c}I_0\left(c\sqrt{2th-(th)^2}\right). \end{aligned} \tag{4.96}$$

Combining the inequalities in equations (4.95) and (4.96), we can claim

$$h^2 \sum_{t=0}^{K-1} \sum_{r=0}^{t} \frac{c}{\sinh c} I_0\left(c\sqrt{2th-(th)^2}\right) \leq |\nu_c(\alpha)| \leq h^2 \sum_{t=1}^{K} \sum_{r=1}^{t} \frac{c}{\sinh c} I_0\left(c\sqrt{2th-(th)^2}\right),$$

which proves equation (4.94) of Lemma 18. □

4.6 Proof of the Main Theorem

The lemma given below is applied in the proof of the main Theorem.

Lemma 19. *Let* $A(x,c,\varepsilon,\alpha)$ *be defined as in Proposition 5, equation* (4.82). *Assume the conditions*

$$\frac{\varepsilon e^{-2\varepsilon}|\nu_c(\alpha)|}{2\mu_c(\alpha)} < 1,$$

$0<\varepsilon<10^{-3}$, $\alpha\in[0,1)$ *and* $x_0 \geq 100$ *hold. Then for* $x \geq x_0$, $\frac{A(x,c,\varepsilon,\alpha)}{x}$ *is a decreasing function.*

Proof. From equation (4.82), in Proposition 5, we can write $A(x,c,\varepsilon,\alpha)$, as

$$\begin{aligned}\frac{A(x,c,\varepsilon,\alpha)}{x} &= \frac{e^{2\varepsilon}\log(e^{\varepsilon}x)}{x}\left[\frac{2\varepsilon x|\mu_c(\alpha)|}{\log(B_0(x))} + 2.01\varepsilon\sqrt{x} + \frac{1}{2}\log\log(2x^2)\right] \\ &= e^{2\varepsilon}\left[2\varepsilon|\mu_c(\alpha)|\frac{\log(e^{\varepsilon}x)}{\log(B_0(x))} + 2.01\varepsilon\frac{\log(e^{\varepsilon}x)}{\sqrt{x}} + \frac{1}{2}\frac{\log(e^{\varepsilon}x)\log\log(2x^2)}{x}\right].\end{aligned} \tag{4.97}$$

To show that $\frac{A(x,c,\varepsilon,\alpha)}{x}$ is decreasing under the conditions given in the Lemma 19 , it is sufficient to show that each of the fractions on the right hand side of equation (4.97) is decreasing. We begin by considering the first fraction $\frac{\log(e^{\varepsilon}x)}{\log(B_0(x))}$. Since $B_0(x) = \frac{\varepsilon x e^{-\varepsilon}|\nu_c(\alpha)|}{2\mu_c(\alpha)}$, we write $B_0(x) = Dx$ where $D = \frac{\varepsilon e^{-\varepsilon}|\nu_c(\alpha)|}{2\mu_c(\alpha)}$. Differentiating with respect to x, we get

$$\frac{d}{dx}\frac{\log(e^{\varepsilon}x)}{\log(B_0(x))} = \frac{\log(B_0(x))\frac{1}{x} - \log(e^{\varepsilon}x)\frac{1}{Dx}\frac{d}{dx}Dx}{\log(B_0(x))^2} = \frac{1}{x}\frac{(\log(B_0(x)) - \log(e^{\varepsilon}x))}{\log(B_0(x))^2}$$
$$= \frac{\log\left(\frac{D}{e^{\varepsilon}}\right)}{x(\log(B_0(x))^2} = \frac{\log\left(\frac{\varepsilon e^{-2\varepsilon}|\nu_c(\alpha)|}{2\mu_c(\alpha)}\right)}{x(\log(B_0(x))^2}.$$

So for $\frac{\log(e^{\varepsilon}x)}{\log(B_0(x))}$ to be decreasing, we need to show that $\frac{\varepsilon e^{-2\varepsilon}|\nu_c(\alpha)|}{2\mu_c(\alpha)} < 1$ which is a given condition.

Next, we consider the second fraction, $\frac{\log(e^{\varepsilon}x)}{\sqrt{x}} = \frac{\varepsilon+\log x}{\sqrt{x}}$. Differentiating with respect to x we get,

$$\frac{d}{dx}\frac{\log(e^{\varepsilon}x)}{\sqrt{x}} = \frac{\frac{1}{\sqrt{x}} - (\varepsilon + \log x)\frac{1}{2\sqrt{x}}}{x} = \frac{\left(1 - \frac{\varepsilon+\log x}{2}\right)}{x\sqrt{x}}.$$

Therefore, $\frac{\log(e^{\varepsilon}x)}{\sqrt{x}}$ is decreasing for $1 - \frac{\varepsilon+\log x}{2} < 0$ i.e. for $x > e^{2-\varepsilon}$ which holds since $x \geq x_0 \geq 100$. So we can conclude the $\frac{\log(e^{\varepsilon}x)}{\sqrt{x}}$ is decreasing in the specified region.

Finally, considering the third fraction $\frac{\log(e^{\varepsilon}x)\log\log(2x^2)}{x}$ and differentiating with respect to x, we get

$$\frac{d}{dx}\left(\frac{\log(e^{\varepsilon}x)\log\log(2x^2)}{x}\right) = \frac{x\left(\frac{\log\log 2x^2}{x} + \frac{4x\log(e^{\varepsilon}x)}{2x^2\log 2x^2}\right) - \log(e^{\varepsilon}x)\log\log(2x^2)}{x^2}$$
$$= \frac{\log\log(2x^2)}{x^2} + \frac{2\log(e^{\varepsilon}x)}{x^2\log(2x^2)} - \frac{\log(e^{\varepsilon}x)\log\log(2x^2)}{x^2}.$$

To show that $\frac{\log(e^{\varepsilon}x)\log\log(2x^2)}{x}$ is a decreasing function, we need to show that

$$\frac{\log\log(2x^2)}{x^2} + \frac{2\log(e^{\varepsilon}x)}{x^2\log(2x^2)} - \frac{\log(e^{\varepsilon}x)\log\log(2x^2)}{x^2} < 0,$$

which is true if and only if

$$\log\log(2x^2) + \frac{2\log(e^{\varepsilon}x)}{\log(2x^2)} < \log(e^{\varepsilon}x)\log\log(2x^2).$$

The above inequality can be simplified to obtain

$$\frac{1}{\log(e^{\varepsilon}x)} + \frac{2}{\log(2x^2)\log\log(2x^2)} < 1. \tag{4.98}$$

Since $x > 100$ and both the terms in the LHS of (4.98) are decreasing functions, we have proved the inequality in (4.98), and thus $\frac{A(x,c,\varepsilon,\alpha)}{x}$ is decreasing. □

The previous results contribute to a method for calculating sharp bounds of the form $|\psi(x) - x| \leq \delta_0 x$, for $x \geq x_0$ when x_0 lies in the range $[\exp(50), \exp(3000)]$. We now give the proof of Theorem 7, which is the main theorem of this thesis. This is the corrected version of [7, Theorem 1].

Theorem 7. *Let* $0 < \varepsilon < 10^{-3}$, $c \geq 3$, $x_0 \geq 100$, $\alpha \in [0,1)$. *We define* $B_0(x_0)$ *by*

$$B_0(x_0) = \frac{\varepsilon e^{-\varepsilon} x_0 |\nu_c(\alpha)|}{2(\mu_c^+(\alpha))}.$$

Assume that the inequalities

$$B_0(x_0) > 1 \quad \textit{and} \quad \frac{\varepsilon e^{-2\varepsilon}|\nu_c(\alpha)|}{2\mu_c(\alpha)} < 1$$

hold. The zeroes of the Riemann zeta function are denoted by $s = \beta + i\gamma$ *with* $\beta, \gamma \in \mathbb{R}$. *Then, if* $\beta = \frac{1}{2}$ *holds for* $0 < \gamma \leq c/\varepsilon$, *the inequality*

$$|\psi(x) - x| \leq x \cdot e^{\alpha\varepsilon}(\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3)$$

holds for all $x \geq e^{\alpha\varepsilon}x_0$*, where*

$$\mathcal{E}_1 = e^{2\varepsilon}\log(e^{\varepsilon}x_0)\left[\frac{2\varepsilon|\nu_c(\alpha)|}{\log B_0} + \frac{2.01\varepsilon}{\sqrt{x_0}} + \frac{\log\log(2x_0^2)}{2x_0}\right] + e^{\alpha\varepsilon} - 1, \tag{4.99}$$

$$\mathcal{E}_2 = 0.16\frac{1+3x_0^{-1}}{\sinh(c)}e^{0.71\sqrt{c\varepsilon}}\left[\log\left(\frac{c}{\pi\varepsilon}\right)\log c + 2\log\left(\frac{c}{\pi\varepsilon}\right) + 18.75\varepsilon\log\left(\frac{c}{\varepsilon}\right) + 1\right], \tag{4.100}$$

$$\mathcal{E}_3 = 2\left(\frac{1}{\sqrt{x_0}} + \frac{1}{x_0}\right)\sum_{0<\gamma\leq c/\varepsilon}\frac{\ell_{c,\varepsilon}(\gamma)}{\gamma} + \frac{2}{x_0}. \tag{4.101}$$

Proof. We begin the proof of the main theorem by recalling that equation (4.41) of Proposition 3 states that

$$\psi_{c,\varepsilon}(x) - x = -\sum_{\rho} a_{c,\varepsilon}(\rho)\frac{x^{\rho}-1}{\rho} + C_1 - \frac{1}{2}\log(1-x^{-2}) + O^{*}8\varepsilon|\log\varepsilon|,$$

as long as $x \geq e^{\frac{2}{|\log\varepsilon|}}$. Since $\varepsilon \leq 10^{-3}$ and $x \geq x_0 \geq 100 \geq e^{\frac{2}{|\log(10^{-3})|}} \geq e^{\frac{2}{|\log\varepsilon|}}$, the identity holds. Applying the triangle inequality to both sides of the above equation, we get

$$\begin{aligned}
|\psi_{c,\varepsilon}(x) - x| &\leq \sum_{\rho}\left|a_{c,\varepsilon}(\rho)\frac{x^{\rho}-1}{\rho}\right| + |C_1| + \left|\frac{1}{2}\log(1-x^{-2})\right| + 8\varepsilon|\log\varepsilon| \\
&\leq \sum_{\rho}\left|a_{c,\varepsilon}(\rho)\frac{x^{\rho}}{\rho}\right| + \sum_{\rho}\left|a_{c,\varepsilon}(\rho)\frac{1}{\rho}\right| + |C_1| + \left|\frac{1}{2}\log(1-x^{-2})\right| + 8\varepsilon|\log\varepsilon| \\
&\leq \sum_{|\Im(\rho)|>\frac{c}{\varepsilon}}\left|a_{c,\varepsilon}(\rho)\frac{x^{\rho}}{\rho}\right| + \sum_{|\Im(\rho)|\leq\frac{c}{\varepsilon}}\left|a_{c,\varepsilon}(\rho)\frac{x^{\rho}}{\rho}\right| + \sum_{|\Im(\rho)|>\frac{c}{\varepsilon}}\left|a_{c,\varepsilon}(\rho)\frac{1}{\rho}\right| \\
&\quad + \sum_{|\Im(\rho)|\leq\frac{c}{\varepsilon}}\left|a_{c,\varepsilon}(\rho)\frac{1}{\rho}\right| + |C_1| + \left|\frac{1}{2}\log(1-x^{-2})\right| + 8\varepsilon|\log\varepsilon|.
\end{aligned}$$

Now, Proposition 4, equation (4.59) provides us with an upper bound for $\sum_{|\Im(\rho)|>\frac{c}{\varepsilon}}\left|a_{c,\varepsilon}(\rho)\frac{x^{\rho}}{\rho}\right|$,

$$\begin{aligned}
\sum_{|\Im(\rho)|>\frac{c}{\varepsilon}}\left|a_{c,\varepsilon}(\rho)\frac{x^{\rho}}{\rho}\right| \leq 0.16\frac{x+1}{\sinh(c)}e^{0.71\sqrt{c\varepsilon}}&\left[\log\left(\frac{c}{\pi\varepsilon}\right)\log c + 2\log\left(\frac{c}{\pi\varepsilon}\right)\right. \\
&\left. + 18.75\varepsilon\log\left(\frac{c}{\varepsilon}\right) + 1\right]
\end{aligned}$$

for $x \geq 1$, $\varepsilon < 10^{-3}$ and $c \geq 3$. Putting $x = 1$ in the above equation yields

$$\sum_{\rho} \left| a_{c,\varepsilon}(\rho) \frac{1}{\rho} \right| \leq +0.32 \frac{e^{0.71\sqrt{c\varepsilon}}}{\sinh(c)} \left[\log\left(\frac{c}{\pi\varepsilon}\right) \log c + 2\log\left(\frac{c}{\pi\varepsilon}\right) + 18.75\varepsilon \log\left(\frac{c}{\varepsilon}\right) + 1 \right]$$

which gives us the inequality

$$\begin{aligned} |\ \psi_{c,\varepsilon}(x) - x\ | \leq & \left(0.16 \frac{e^{0.71\sqrt{c\varepsilon}}(x+3)}{\sinh(c)} \left[\log\left(\frac{c}{\pi\varepsilon}\right) \log c + 2\log\left(\frac{c}{\pi\varepsilon}\right) \right. \right. \\ & + \left. \left. 18.75\varepsilon \log\left(\frac{c}{\varepsilon}\right) + 1 \right] \right) + \sum_{|\Im(\rho)| \leq \frac{c}{\varepsilon}} \left| a_{c,\varepsilon}(\rho) \frac{1}{\rho} \right| + \sum_{|\Im(\rho)| \leq \frac{c}{\varepsilon}} \left| a_{c,\varepsilon}(\rho) \frac{x^{\rho}}{\rho} \right| \\ & + |C_1| + \left| \frac{1}{2} \log(1 - x^{-2}) \right| + 8\varepsilon |\log \varepsilon| \\ \leq & \left(0.16 \frac{1 + 3x_0^{-1}}{\sinh(c)} e^{0.71\sqrt{c\varepsilon}} \left[\log\left(\frac{c}{\pi\varepsilon}\right) \log c + 2\log\left(\frac{c}{\pi\varepsilon}\right) \right. \right. \\ & + \left. \left. 18.75\varepsilon \log\left(\frac{c}{\varepsilon}\right) + 1 \right] \right) x + \sum_{|\Im(\rho)| \leq \frac{c}{\varepsilon}} \left| a_{c,\varepsilon}(\rho) \frac{x^{\rho}}{\rho} \right| + \sum_{|\Im(\rho)| \leq \frac{c}{\varepsilon}} \left| a_{c,\varepsilon}(\rho) \frac{1}{\rho} \right| \\ & + |C_1| + \left| \frac{1}{2} \log(1 - x^{-2}) \right| + 8\varepsilon |\log \varepsilon|, \end{aligned}$$

valid for $x \geq x_0$. Using equation (4.40), we have

$$\begin{aligned} |C_1| + \left| \frac{1}{2} \log(1 - x^{-2}) \right| + 8\varepsilon |\log \varepsilon| = & \left| -\frac{\gamma_0}{2} - 1 - \frac{\log(\pi)}{2} \right| + \left| \frac{1}{2} \log(1 - x^{-2}) \right| \\ & + 8\varepsilon |\log \varepsilon| \\ \leq & \left| -\frac{\gamma_0}{2} - 1 - \frac{\log(\pi)}{2} \right| + \left| \frac{1}{2} \log(1 - 100^{-2}) \right| \\ & + 8(10^{-3}) \log(10^{-3}) \\ < & 1.9162 < 1.92, \end{aligned}$$

since $x \geq x_0 \geq 100$ and $\varepsilon \leq 10^{-3}$. Hence for $x \geq x_0$, we can write

$$|\ \psi_{c,\varepsilon}(x) - x\ | \leq \mathcal{E}_2 x + \sum_{|\Im(\rho)| \leq \frac{c}{\varepsilon}} \left(\left| a_{c,\varepsilon}(\rho) \frac{x^{\rho}}{\rho} \right| + \left| a_{c,\varepsilon}(\rho) \frac{1}{\rho} \right| \right) + 1.92 \qquad (4.102)$$

where

$$\mathcal{E}_2 = 0.16\frac{1+3x_0^{-1}}{\sinh(c)}e^{0.71\sqrt{c\varepsilon}}\left[\log\left(\frac{c}{\pi\varepsilon}\right)\log c + 2\log\left(\frac{c}{\pi\varepsilon}\right) + 18.75\varepsilon\log\left(\frac{c}{\varepsilon}\right) + 1\right].$$

Furthermore, equation (4.64), in the proof of Proposition 4 states that

$$\left|a_{c,\varepsilon}(\rho)\frac{x^\rho}{\rho}\right| \leq \frac{|\ell_{c,\varepsilon}(\Im(\rho))|}{|\Im(\rho)|}\sqrt{x},$$

where setting $x = 1$ gives us

$$\left|a_{c,\varepsilon}(\rho)\frac{1}{\rho}\right| \leq \frac{|\ell_{c,\varepsilon}(\Im(\rho))|}{|\Im(\rho)|}. \tag{4.103}$$

Thus adding both the inequalities in (4.64) and (4.103), we can write

$$\begin{aligned}\sum_{|\Im(\rho)|\leq\frac{c}{\varepsilon}}\left(\left|a_{c,\varepsilon}(\rho)\frac{x^\rho}{\rho}\right| + \left|a_{c,\varepsilon}(\rho)\frac{1}{\rho}\right|\right) + 1.92 &\leq \sum_{|\gamma|\leq\frac{c}{\varepsilon}}(\sqrt{x}+1)\frac{|\ell_{c,\varepsilon}(\gamma)|}{|\gamma|} + 1.92\\ &\leq x\left[\left(\frac{1}{\sqrt{x_0}} + \frac{1}{x_0}\right)\sum_{|\gamma|\leq\frac{c}{\varepsilon}}\frac{|\ell_{c,\varepsilon}(\gamma)|}{|\gamma|} + \frac{1.92}{x_0}\right].\end{aligned}$$

Since the zeros of the zeta function are symmetrical with respect to the real axis we get the inequality

$$\sum_{|\Im(\rho)|\leq\frac{c}{\varepsilon}}\left(\left|a_{c,\varepsilon}(\rho)\frac{x^\rho}{\rho}\right| + \left|a_{c,\varepsilon}(\rho)\frac{1}{\rho}\right|\right) + 1.92 \leq \mathcal{E}_3 x, \tag{4.104}$$

where $\mathcal{E}_3 = \left[2\left(\frac{1}{\sqrt{x_0}} + \frac{1}{x_0}\right)\sum_{0\leq\gamma\leq\frac{c}{\varepsilon}}\frac{\ell_{c,\varepsilon}(\gamma)}{\gamma} + \frac{1.92}{x_0}\right]$. Combining equations (4.102) and (4.104) gives us the inequality

$$|\ \psi_{c,\varepsilon}(x) - x\ | \leq \mathcal{E}_2 x + \mathcal{E}_3 x \tag{4.105}$$

for $x \geq x_0$. Now, we know from Proposition 5, equation (4.83), that

$$\psi(e^{-\alpha\varepsilon}x) \leq \psi_{c,\varepsilon}(x) + A(x,c,\varepsilon,\alpha).$$

Subtracting $e^{-\alpha\varepsilon}x$ from both sides of the above inequality gives us the following inequality

$$\psi(e^{-\alpha\varepsilon}x) - e^{-\alpha\varepsilon}x \leq \psi_{c,\varepsilon}(x) - e^{-\alpha\varepsilon}x + A(x,c,\varepsilon,\alpha).$$

Since $x \geq x_0$, Lemma 19 proves that $\frac{A(x,c,\varepsilon,\alpha)}{x}$ is a monotonically decreasing function, and thus we have

$$\begin{aligned}\psi(e^{-\alpha\varepsilon}x) - e^{-\alpha\varepsilon}x &\leq \psi_{c,\varepsilon}(x) - e^{-\alpha\varepsilon}x + \frac{A(x_0,c,\varepsilon,\alpha)}{x_0}x \\ &\leq \psi_{c,\varepsilon}(x) - x + \frac{A(x_0,c,\varepsilon,\alpha)}{x_0}x + (x - e^{-\alpha\varepsilon}x). \end{aligned} \tag{4.106}$$

Recalling the definition of $A(x_0,c,\varepsilon,\alpha)$ given in Proposition 5, equation (4.82), we have

$$\frac{A(x_0,c,\varepsilon,\alpha)}{x_0} = \frac{1}{x_0}\left[e^{2\varepsilon}\log(e^{\varepsilon}x_0)\left[\frac{2\varepsilon x_0|\mu_c(\alpha)|}{\log(B_0(x_0))} + 2.01\varepsilon\sqrt{x_0} + \frac{1}{2}\log\log(2x_0^2)\right]\right].$$

Using this in (4.106), we obtain

$$\begin{aligned}\psi(e^{-\alpha\varepsilon}x) - e^{-\alpha\varepsilon}x \leq &\psi_{c,\varepsilon}(x) - x + \frac{1}{x_0}\left[e^{2\varepsilon}\log(e^{\varepsilon}x_0)\left[\frac{2\varepsilon x_0|\mu_c(\alpha)|}{\log(B_0(x_0))} + 2.01\varepsilon\sqrt{x_0}\right.\right. \\ &\left.\left. + \ \frac{1}{2}\log\log(2x_0^2)\right]\right]x + (1 - e^{-\alpha\varepsilon})x \\ \leq &\psi_{c,\varepsilon}(x) - x + \mathcal{E}_1 x,\end{aligned}$$

for $x \geq x_0$, where $\mathcal{E}_1$ is defined in equation (4.99). Similarly from Proposition 5, equation (4.84), we know

$$\psi(e^{\alpha\varepsilon}x) \geq \psi_{c,\varepsilon}(x) - A(x,c,\varepsilon,\alpha),$$

and subtracting $e^{\alpha\varepsilon}x$ from both sides of the above inequality gives us

$$\psi(e^{\alpha\varepsilon}x) - e^{\alpha\varepsilon}x \geq \psi_{c,\varepsilon}(x) - e^{\alpha\varepsilon}x - A(x,c,\varepsilon,\alpha).$$

Since $x \geq e^{\alpha\varepsilon}x_0$ and $-\frac{A(x_0,c,\varepsilon,\alpha)}{x_0}x \leq -A(x,c,\varepsilon,\alpha)$, which follows from $\frac{A(x,c,\varepsilon,\alpha)}{x}$ being a monotonically decreasing function, we can now write

$$\begin{aligned}\psi(e^{\alpha\varepsilon}x) - e^{\alpha\varepsilon}x &\geq \psi_{c,\varepsilon}(x) - e^{\alpha\varepsilon}x - \frac{A(x_0,c,\varepsilon,\alpha)}{x_0}x \\ &\geq \psi_{c,\varepsilon}(x) - x - \frac{A(x_0,c,\varepsilon,\alpha)}{x_0}x + (x - e^{\alpha\varepsilon}x).\end{aligned}$$

Substituting the formula for $\frac{A(x_0,c,\varepsilon,\alpha)}{x_0}$, we obtain

$$\begin{aligned}\psi(e^{\alpha\varepsilon}x) - e^{\alpha\varepsilon}x \geq& \psi_{c,\varepsilon}(x) - x - \frac{1}{x_0}\left[e^{2\varepsilon}\log(e^{\varepsilon}x_0)\left[\frac{2\varepsilon x_0|\mu_c(\alpha)|}{\log B} + 2.01\varepsilon\sqrt{x_0}\right.\right. \\ &\left.\left. + \frac{1}{2}\log\log(2x_0^2)\right]\right]x + (1 - e^{\alpha\varepsilon})x, \\ \geq& \psi_{c,\varepsilon}(x) - x - \mathcal{E}_1 x\end{aligned}$$

for $x \geq x_0$. To finish the proof, we define $b(x) = \mathcal{E}_2 x + \mathcal{E}_3 x$, and $\tilde{b}(x) = \mathcal{E}_1 x + b(x) = (\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3)x$. Therefore by equation (4.105)

$$b(x) \geq (\psi_{c,\varepsilon}(x) - x).$$

Then,

$$\begin{aligned}\psi(e^{-\alpha\varepsilon}x) - e^{-\alpha\varepsilon}x &\leq (\psi_{c,\varepsilon}(x) - x) + \mathcal{E}_1 x \leq b(x) + \mathcal{E}_1 x, \\ \psi(e^{\alpha\varepsilon}x) - e^{\alpha\varepsilon}x &\geq (\psi_{c,\varepsilon}(x) - x) - \mathcal{E}_1 x \geq -b(x) - \mathcal{E}_1 x.\end{aligned}$$

Also, if we let $a(x) = \psi(x) - x$, then

$$a(e^{-\alpha\varepsilon}x) \leq \tilde{b}(x) \quad \text{and} \quad a(xe^{\alpha\varepsilon}) \geq -\tilde{b}(x) \tag{4.107}$$

for $x \geq x_0$. Now, if we denote $u = xe^{-\alpha\varepsilon}$ and $v = xe^{\alpha\varepsilon}$, then $x = ue^{\alpha\varepsilon}$ and $x = ve^{-\alpha\varepsilon}$, and we see

$$a(u) \leq \tilde{b}(ue^{\alpha\varepsilon}) \tag{4.108}$$

for $u \geq x_0 e^{-\alpha\varepsilon}$, and

$$a(v) \geq -\tilde{b}(ve^{-\alpha\varepsilon}) \tag{4.109}$$

for $v \geq x_0 e^{\alpha\varepsilon}$. Since $\tilde{b}$ is increasing, so $\tilde{b}(ue^{\alpha\varepsilon}) \geq \tilde{b}(ue^{-\alpha\varepsilon})$. Therefore, combining (4.108) and (4.109) and making a variable change, it follows that

$$-\tilde{b}(ue^{\alpha\varepsilon}) \leq -\tilde{b}(ue^{-\alpha\varepsilon}) \leq a(u) \leq \tilde{b}(ue^{\alpha\varepsilon})$$

for $u \geq x_0 e^{\alpha\varepsilon}$. Thus,

$$|a(u)| \leq \tilde{b}(ue^{\alpha\varepsilon})$$

for $u \geq x_0 e^{\alpha\varepsilon}$. In other words

$$|\psi(u) - u| \leq \tilde{b}(ue^{\alpha\varepsilon}) = e^{\alpha\varepsilon} u(\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3) \tag{4.110}$$

for $u \geq e^{\alpha\varepsilon} x_0$. □

The following table provides the upper bound for $e^{\alpha\varepsilon}(\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3)$ denoted by δ_0 such that $|\psi(x) - x| < e^{\alpha\varepsilon}(\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3)x \leq \delta_0(x)$, for $T \leq 2.445 \times 10^{12}$ which is used in [7,

Table 2]. Here $\varepsilon = c/T$.

Table 4.2: Table of computations for δ_0.

$e^{\alpha\varepsilon}x_0$	c	T	α	δ_0
e^{55}	39	8.5×10^{11}	0.1	1.12498×10^{-10}
e^{60}	33	2.445×10^{12}	0.11	1.23991×10^{-11}
e^{65}	33	2.445×10^{12}	0.1	3.75568×10^{-12}
e^{70}	33	2.445×10^{12}	0.09	2.97677×10^{-12}
e^{75}	33	2.445×10^{12}	0.08	2.85514×10^{-12}
e^{80}	33	2.445×10^{12}	0.08	2.79522×10^{-12}
e^{90}	33	2.445×10^{12}	0.07	2.70574×10^{-12}
e^{100}	33	2.445×10^{12}	0.06	2.63677×10^{-12}
e^{500}	33	2.445×10^{12}	0.012	2.18439×10^{-12}
e^{1000}	33	2.445×10^{12}	0.005	2.13205×10^{-12}
e^{2000}	33	2.445×10^{12}	0.003	2.10609×10^{-12}
e^{3000}	33	2.445×10^{12}	0.001	2.09752×10^{-12}
e^{4000}	33	2.445×10^{12}	0.001	2.09320×10^{-12}

4.6.1 Numerical estimates for $\mathcal{E}_3$

Lemma 20. *[7, Lemma 7] Let $c, \varepsilon > 0$ and let $14 \le T_0 < T_1 < c/\varepsilon$. Then we have*

$$\sum_{T_0\le\gamma<T_1} \frac{\ell_{c,\varepsilon}(\gamma)}{\gamma} \le \frac{\ell_{c,\varepsilon}(T_0)}{4\pi}\left[\log\left(\frac{T_1}{2\pi}\right)^2 - \log\left(\frac{T_0}{2\pi}\right)^2 + 20\pi\frac{\log(T_1)}{T_1}\right]. \tag{4.111}$$

Proof. Since $\ell_{c,\varepsilon}$ is monotonically decreasing in $[0, c/\varepsilon]$, we get the inequality :

$$\sum_{T_0\le\gamma\le T_1} \frac{\ell_{c,\varepsilon}(\gamma)}{\gamma} \le \ell_{c,\varepsilon}(T_0)\left(\sum_{T_0\le\gamma\le T_1}\frac{1}{\gamma}\right).$$

From Lemma 14, equation (4.67), we know that

$$\begin{aligned}\sum_{T_0\le\gamma\le T_1} \frac{\ell_{c,\varepsilon}(\gamma)}{\gamma} &\le \frac{\ell_{c,\varepsilon}(T_0)}{4\pi}\left(\left[\log\left(\frac{T_1}{2\pi}\right)^2 - \log\left(\frac{T_0}{2\pi}^2\right)\right] + O^*\left(5\frac{\log T_0}{T_0}\right)\right)\\ &= \frac{\ell_{c,\varepsilon}(T_0)}{4\pi}\left[\log\left(\frac{T_1}{2\pi}\right)^2 - \log\left(\frac{T_0}{2\pi}^2\right) + 20\pi\frac{\log T_0}{T_0}\right],\end{aligned}$$

which proves Lemma 20. □

The values for the error term $\mathcal{E}_3$ have been computed in Table 2.8 by using the exact values of γ in $\sum \frac{\ell_{c,\varepsilon}(\gamma)}{\gamma}$ for the first hundred thousand zeroes of the Riemann zeta function. For the remaining zeroes, the term $\sum \frac{\ell_{c,\varepsilon}(\gamma)}{\gamma}$ has been estimated using the bounds in Lemma 20, equation (4.111) piece-wise, where each piece is of length 10^6. The equation for computing $\sum \frac{\ell_{c,\varepsilon}(\gamma)}{\gamma}$ has been given below .

$$\begin{aligned}
\sum_{0<\gamma\leq\frac{c}{\varepsilon}} \frac{\ell_{c,\varepsilon}(\gamma)}{\gamma} &\leq \sum_{0<\gamma\leq t_0} \frac{\ell_{c,\varepsilon}(\gamma)}{\gamma} + \sum_{j=0}^{J} \left(\sum_{t_0+10^6 j<\gamma\leq t_0+10^6(j+1)} \frac{\ell_{c,\varepsilon}(\gamma)}{\gamma} \right) \\
&\leq \sum_{0<\gamma\leq t_0} \frac{\ell_{c,\varepsilon}(\gamma)}{\gamma} + \sum_{j=0}^{J} \left(\frac{\ell_{c,\varepsilon}(t_0+10^6 j)}{4\pi} \left[\log\left(\frac{t_0+10^6(j+1)}{2\pi} \right)^2 \right.\right. \\
&\quad \left.\left. - \log\left(\frac{t_0+10^6 j}{2\pi} \right)^2 + 20\pi \frac{\log(t_0+10^6(j+1))}{t_0+10^6(j+1)} \right] \right)
\end{aligned}$$

where $N(t_0) = 10^5$ and J is the greatest integer such that $t_0 + 10^6 J \leq \frac{c}{\varepsilon}$. For each j, Lemma 20 is applied to the term $\sum_{t_0+10^6 j<\gamma\leq t_0+10^6(j+1)} \frac{\ell_{c,\varepsilon}(\gamma)}{\gamma}$.

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Appendix A

PARI Code

This Appendix contains the PARI code we have used to get explicit bounds in this thesis. The code is divided into three parts, each part corresponding to an error term : $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$ obtained from Theorem 6. In the Table A.1 we provide the the names we have used in our code for each of the relevant parameters, variables and functions of this thesis.

Table A.1: Names used in PARI for functions, parameters or variables

Name of variable or function in code	Corresponding variable/function /parameter in thesis	Name of variable or function in code	Corresponding variable/function /parameter in thesis
I0(t)	$I_0(t)$: 0th modified Bessel function of the first kind	Ncub	Upper bound for $\nu_c(\alpha)$
B0	B_0	Mc	Upper bound for $\mu_c(\alpha)$
ep	ε	Eps1	$\mathcal{E}_1$
Nclb	Lower bound for $\nu_c(\alpha)$	Eps2	$\mathcal{E}_2$
Eps3	$\mathcal{E}_3$	alpha	$\alpha \in (0,1)$

A.1 Code for computing $\mathcal{E}_1$

```
I0(t)=suminf(n=0,(t/2)∧ (2*n)/(n!)∧2);
B0(ep,x,Nclb,Mc)=ep*exp(-ep)*x*Nclb/2/Mc;
Eps1(alpha,ep,x,Nclb,Ncub,Mc)=exp(2*ep)*log(exp(ep)*x)*
(2*ep*Ncub/log(B0(ep,x,Nclb,Mc))+2.01*ep/sqrt(x)+log(log(2*x∧
2))/2/x)+exp(ep*alpha)-1;
I0precomp=List();
K=10000;
h=(1-alpha)/K;
I0func(h,c,j)=I0(c*sqrt(2*j*h-j∧ 2*h∧2));
for(j=0,K,listput(I0precomp,I0func(h,c,j)));
computeEps1(b,c,alpha,T)={
```

```
ep=c/T;
I0precomp=List();
K=10000;
h=(1-alpha)/K;
for(j=0,K,listput(I0precomp,I0func(h,c,j)));
x=exp(b-alpha*ep);
Mc=sum(k=1,K,I0precomp[k+1])*h*c/sinh(c)/2;
Nclb=sum(k=0,K-1,sum(j=0,k,I0precomp[j+1]))*h^2*c/2/sinh(c);
Ncub=sum(k=1,K,sum(j=1,k,I0precomp[j+1]))*h^2*c/2/sinh(c);
listkill(I0precomp);
printf("[%d, %d, %.3f, %s]\n",b,c,ceil(alpha*1000)/1000.0,
Eps1(alpha,ep,x,Nclb,Ncub,Mc)); }
```

A.2 Code for computing $\mathcal{E}_2$

```
Eps2(ep,c,x)=0.16*(1+3*x^(-1))/sinh(c)*exp(0.71*sqrt(c*ep))
(log(c/(Pi*ep))* log (c) + 2*log (c/(Pi* ep))+ (18.75*ep)*log(c/ep)+1)
```

A.3 Code for computing $\mathcal{E}_3$

```
a=readvec("C:\\Users\\sbchr\\Dropbox\\Sreerupa\\Computerprograms
\\firsthundredthousandzeros.txt");
Logan(c,ep,xi)=c/sinh(c)*sin(sqrt((xi*ep)^2-c^2))/
sqrt((xi*ep)^2-c^2);
logansum(c,ep)= sum(i=1,length(a), Logan(c,ep,a[i])/a[i]);
zsum=0;
rvalsum(T,t0,t1,width,c,ep)={while(t1 < T,
zsum+=Logan(c,ep,t0)/4/Pi*((log(t1/2/Pi))^2-
(log(t0/2/Pi))^2+20*Pi*log(t1)/t1);
t0 = t1 ;
t1 = min(T,t1+width);
);
return(zsum);}
Eps3(T,ep,c,x)= 2*(1/sqrt(x)+1/x)*(logansum(c,ep))+ 2*(1/sqrt(x)+1/x)*
(rvalsum(T,74920.827498994, +10^6,10^6,c,ep))+1.92/x;
computeEps3(b,c,alpha,T)={
ep=c/T;
x=exp(b-alpha*ep);
printf("[%d,%d, %s, %.3f, %s]\n",b,c,'H0',ceil(alpha*1000)/1000.0,
Eps3(T,ep,c,x)); zsum=0; }
```

Appendix B

Table B.1: Table of computations for upper bounds of δ_0 according to Theorem 7

$e^{\alpha\varepsilon}x_0$	c	T	α	δ_0
e^{55}	39	8.5×10^{11}	0.1	1.12498×10^{-10}
e^{60}	33	2.445×10^{12}	0.11	1.23991×10^{-11}
e^{65}	33	2.445×10^{12}	0.1	3.75568×10^{-12}
e^{70}	33	2.445×10^{12}	0.09	2.97677×10^{-12}
e^{75}	33	2.445×10^{12}	0.08	2.85514×10^{-12}
e^{80}	33	2.445×10^{12}	0.08	2.79522×10^{-12}
e^{85}	33	2.445×10^{12}	0.07	2.74742×10^{-12}
e^{90}	33	2.445×10^{12}	0.07	2.70574×10^{-12}
e^{95}	33	2.445×10^{12}	0.06	2.67000×10^{-12}
e^{100}	33	2.445×10^{12}	0.06	2.63677×10^{-12}
e^{200}	33	2.445×10^{12}	0.04	2.34981×10^{-12}
e^{300}	33	2.445×10^{12}	0.02	2.25542×10^{-12}
e^{400}	33	2.445×10^{12}	0.02	2.21168×10^{-12}
e^{500}	33	2.445×10^{12}	0.012	2.18439×10^{-12}
e^{600}	33	2.445×10^{12}	0.01	2.16685×10^{-12}
e^{700}	33	2.445×10^{12}	0.009	2.15437×10^{-12}
e^{800}	33	2.445×10^{12}	0.008	2.14504×10^{-12}
e^{900}	33	2.445×10^{12}	0.007	2.13780×10^{-12}
e^{1000}	33	2.445×10^{12}	0.005	2.13205×10^{-12}
e^{1100}	33	2.445×10^{12}	0.005	2.12729×10^{-12}
e^{1200}	33	2.445×10^{12}	0.005	2.12335×10^{-12}
e^{1300}	33	2.445×10^{12}	0.005	2.12003×10^{-12}
e^{1400}	33	2.445×10^{12}	0.004	2.11718×10^{-12}
e^{1500}	33	2.445×10^{12}	0.004	2.11471×10^{-12}
e^{1600}	33	2.445×10^{12}	0.004	2.11255×10^{-12}
e^{1700}	33	2.445×10^{12}	0.004	2.11065×10^{-12}
e^{1800}	33	2.445×10^{12}	0.003	2.10897×10^{-12}
e^{1900}	33	2.445×10^{12}	0.003	2.10745×10^{-12}
e^{2000}	33	2.445×10^{12}	0.003	2.10609×10^{-12}

Table B.2: Table of computations for upper bounds of δ_0 according to Theorem 7

$e^{\alpha\varepsilon}x_0$	c	T	α	δ_0
e^{2100}	33	2.445×10^{12}	0.003	2.10486×10^{-12}
e^{2200}	33	2.445×10^{12}	0.003	2.10374×10^{-12}
e^{2300}	33	2.445×10^{12}	0.003	2.10272×10^{-12}
e^{2400}	33	2.445×10^{12}	0.003	2.10179×10^{-12}
e^{2500}	33	2.445×10^{12}	0.002	2.10093×10^{-12}
e^{2600}	33	2.445×10^{12}	0.002	2.10014×10^{-12}
e^{2700}	33	2.445×10^{12}	0.002	2.09940×10^{-12}
e^{2800}	33	2.445×10^{12}	0.002	2.09872×10^{-12}
e^{2900}	33	2.445×10^{12}	0.002	2.09808×10^{-12}
e^{3000}	33	2.445×10^{12}	0.001	2.09752×10^{-12}
e^{3100}	33	2.445×10^{12}	0.001	2.09696×10^{-12}
e^{3200}	33	2.445×10^{12}	0.001	2.09644×10^{-12}
e^{3300}	33	2.445×10^{12}	0.001	2.09595×10^{-12}
e^{3400}	33	2.445×10^{12}	0.001	2.09549×10^{-12}
e^{3500}	33	2.445×10^{12}	0.001	2.09505×10^{-12}
e^{3600}	33	2.445×10^{12}	0.001	2.09464×10^{-12}
e^{3700}	33	2.445×10^{12}	0.001	2.09425×10^{-12}
e^{3800}	33	2.445×10^{12}	0.001	2.09388×10^{-12}
e^{3900}	33	2.445×10^{12}	0.001	2.09353×10^{-12}
e^{4000}	33	2.445×10^{12}	0.001	2.09320×10^{-12}