

**CHARACTER GENERATORS AND GRAPHS FOR SIMPLE LIE
ALGEBRAS**

NNAMDI OKEKE

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Department of Physics
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NNAMDI OKEKE

Approved:

Mark Walton, Professor, Department of Physics, University of Lethbridge, Lethbridge, Alberta Date

Saurya Das, Assistant Professor, Department of Physics, University of Lethbridge, Lethbridge, Alberta Date

Dave Morris, Professor, Department of Mathematics and Computer Science, University of Lethbridge, Lethbridge, Alberta Date

External Examiner: Terry Gannon, Professor, Department of Mathematics, University of Alberta, Edmonton, Alberta Date

Chair of Thesis Defense Committee: Jonathan Seldin, Associate Professor, Department of Mathematic and computer Science, University of Lethbridge, Lethbridge, Alberta Date

Dedication

To my Parents

Abstract

We study character generating functions (character generators) of simple Lie algebras. The expression due to Patera and Sharp, derived from the Weyl character formula, is first reviewed. A new general formula is then found. It makes clear the distinct roles of “outside” and “inside” elements of the integrity basis, and helps determine their quadratic incompatibilities. We review, analyze and extend the results obtained by Gaskell using the Demazure character formulas. We find that the fundamental generalized-poset graphs underlying the character generators can be deduced from such calculations. These graphs, introduced by Baclawski and Towber, can be simplified for the purposes of constructing the character generator. The generating functions can be written easily using the simplified versions, and associated Demazure expressions. The rank-two algebras are treated in detail, but we believe our results are indicative of those for general simple Lie algebras.

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Chapter 1

Introduction

This thesis is a study of character generating functions (character generators) for simple Lie algebras. Our introduction will attempt to set the stage, by discussing the background and motivation in a general and, hopefully, comprehensible way.

We begin by introducing one of the most important concepts in physics, symmetry. Something is considered to be symmetric if there are transformations or changes that leave it invariant (unchanged). Different types of transformations exist: translation, rotation, permutations, reflections, and others. A symmetry operation takes a symmetrical object into itself. Symmetry is very important in physics because it can reduce complicated problems to much simpler ones.

Suppose we consider transformations which do change a certain class of objects. If for special cases, an object remains unchanged after undergoing a transformation, we say that the object is symmetric. As an example, an infinite straight line is symmetric under translation along its length because such a transformation does not change the straight line. Another example of a physical system possessing a symmetry is the water molecule. There is

symmetry between the two hydrogen ions: under their exchange the system is invariant, i.e. it does not change. There is also a translation symmetry because the interaction between any two ions depends only on their relative separation and not on their absolute position.

Symmetry can be discrete or continuous. For example, rotating an equilateral triangle through an angle $2\pi l/3$ ($l = 0, 1, 2$) about an axis normal to it and through its center does not change it. In this case, the angle of rotation takes values in a discrete set. Hence we have a discrete symmetry. In the case of a circle, however, the rotation angles that leave the system invariant (unchanged) take values in a continuous range. Therefore this symmetry is a continuous symmetry.

Symmetry transformations can be expressed precisely with group theory, the mathematics of transformation. For a fixed object, the set of symmetry transformations forms a group. A group is a set of elements with a multiplication or product. For transformations, the product is simply composition, i.e., doing one transformation after another. Groups must obey the following properties:

- the set is closed under multiplication: the product of any two members is again a member of the group. Since doing two successive transformations can just as well be achieved by performing one product transformation, transformations can fill out a group.
- the group product is associative. Composition of transformations has this property.
- there is the unit element or identity element *id*. For a group of transformations, this is the identity transformation. It simply implies doing nothing, and so clearly leaves the object invariant.

- for every element, there also exists an inverse element. An inverse transformation returns the object or physical system to its original state before the transformation. We can say that it undoes the transformation.

The number of elements of the group is the order of the group. Groups may be of finite or infinite order.

It is important to note here that in general $ab \neq ba$ for any two group elements a and b , which means that the operation of multiplication, in this case, is not commutative. So the order of the elements of the pair a, b is important here. But if $ab = ba$ for all $a, b \in G$, then the group is called Abelian. The set of integer numbers (... - 3, -2, -1, 0, 1, 2, 3, ...) under “addition” is a simple example of an Abelian group, where 0 is the unit element and the inverse is the same number with opposite sign.

Groups describing discrete symmetries are called discrete groups and the groups describing continuous symmetries are called continuous groups. We are interested in certain continuous, non-Abelian groups describing continuous symmetries called Lie groups.

A Lie group is a group with elements that are labelled by a set of continuous parameters with a multiplication law that depends smoothly on the parameters. For example, the rotation group is a Lie group. It is the set of all rotations about a fixed point, the origin, say, of 3-dimensional Euclidean space, \mathbb{R}^3 . A rotation about the origin is a linear transformation that preserves the length of vectors, and also preserves the orientation of space. The combination of two rotations gives us another rotation, and for every rotation there is a unique inverse which is again a rotation. These properties give the set of all rotations the mathematical structure of a group. Another important property of the rotation group is that it is non-Abelian. This means that the order in which rotations are composed

makes a difference. For example, a 90° turn around the positive x -axis followed by another 90° turn around the positive y -axis is a different rotation than the one obtained by first rotating around the y -axis and then around x -axis.

The three-dimensional rotation group is isomorphic to $SO(3)$, the group of special orthogonal 3×3 matrices. In three-dimensional space, a geometrical vector \mathbf{A} , can be represented by a column vector whose entries are the x, y , and z components of the vector:

$$\mathbf{A} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \quad (1.1)$$

The result of a rotation of this vector by angles ϕ, θ, ψ about the x, y , and z -axis, respectively, can be found by multiplication by 3×3 matrices:

$$\mathbf{R}_x(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi & \cos \phi \end{pmatrix} \quad (1.2)$$

$$\mathbf{R}_y(\theta) = \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix} \quad (1.3)$$

and

$$\mathbf{R}_z(\psi) = \begin{pmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (1.4)$$

One way the effect of a general rotation can be written is $\mathbf{R}(\phi, \theta, \psi) = \mathbf{R}_x(\phi)\mathbf{R}_y(\theta)\mathbf{R}_z(\psi)$.

These 3×3 rotation matrices are also special and orthogonal. Special implies that the matrices have determinants equal to one, orthogonal that their transposes equal their inverses, and 3 indicates they are 3×3 matrices. A rotation is continuously connected to the identity and so its determinant must be equal to 1, the determinant of the identity. A rotation also preserves the scalar product of two vectors. This implies that the product of a rotation matrix times its transpose gives the identity matrix. Therefore, a rotation matrix is orthogonal. The elements of the group must have a multiplication with all the properties required for a group. For rotation matrices, the product is just matrix multiplication; this means that by simply doing matrix multiplications we get the results of multiple transformations.

The $SO(3)$ group can be generalized to the group of rotations in N -dimensional space, $SO(N)$. $SO(3)$ therefore belongs to an infinite series of simple Lie groups. Another example of such a series results from the symplectic group, $Sp(2N)$, the symmetry group of the phase space of a particle in \mathbb{R}^N . There are four infinite series [27] of simple Lie groups: the A, B, C , and D series:

- A series (A_n): unitary transformations in n -dimensional complex space. They are therefore also denoted $SU(n + 1)$, and called the special unitary groups. They are important in physics; $SU(3)$, for example, is the colour group of the strong interaction.
- B series (B_n): rotations in odd-dimensional real space, corresponding to the special orthogonal group, $SO(2n + 1)$.
- C series (C_n): are called the symplectic groups, $Sp(2N)$.
- D series (D_n): rotations in even-dimensional real space corresponding to the spe-

cial orthogonal group, $SO(2N)$. The structure differs from $B_n = SO(2N + 1)$. For example, D_n has 2 fundamental spinor representations while B_n has one.

Apart from these four series, there are also five exceptional simple Lie groups: G_2, F_4, E_6, E_7 , and E_8 . In this thesis, G_2 is the sole exceptional Lie group that will be studied.

These simple Lie groups are important in physics. For example, in quantum mechanics, the propagator is special and unitary. Also, an $SU(N)$ matrix describes a transformation between different bases of an N -dimensional Hilbert space of states. Even the exceptional Lie groups occur in physics. They have been proposed as gauge groups in many theories of high-energy particle physics [2].

Now, let us go back to $SO(3)$ and look at infinitesimal rotations. Remarkably, they can tell us much of what is important about the whole group. For small angles ϕ , $\cos \phi \approx 1$ and $\sin \phi \approx \phi$. Therefore we can write

$$\mathbf{R}_{\mathbf{x}}(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi & \cos \phi \end{pmatrix} \approx I - i\phi \mathbf{T}_{\mathbf{x}} \quad (1.5)$$

where $\mathbf{T}_{\mathbf{x}}$ is the matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad (1.6)$$

Similarly, we can also find $\mathbf{T}_{\mathbf{y}}$ and $\mathbf{T}_{\mathbf{z}}$:

$$\mathbf{T}_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad \mathbf{T}_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (1.7)$$

$\mathbf{T}_x, \mathbf{T}_y,$ and \mathbf{T}_z are Hermitian (self-adjoint) matrices, determined by infinitesimal $SO(3)$ rotation matrices. We say they are the infinitesimal generators of the rotation matrices.

Now,

$$e^{i\phi\mathbf{T}} = 1 + i\phi\mathbf{T} - \frac{\phi^2\mathbf{T}^2}{2!} - i\frac{\phi^3\mathbf{T}^3}{3!} + \dots = \mathbf{I} \cos \phi + i\mathbf{T} \sin \phi \quad (1.8)$$

where $\mathbf{T}^2 = \mathbf{I}$ and \mathbf{I} is the identity matrix. Therefore we can write a rotation matrix as an exponential of another matrix:

$$\mathbf{R}_x(\phi) = e^{-i\phi\mathbf{T}_x} \quad (1.9)$$

This is a relation between the matrix for an element of the rotation group and the infinitesimal generators.

Likewise, the abstract Lie group element $g \in G$ can be expressed in terms of abstract generators, T_a . For any two such generators T_a, T_b , the commutation relation is defined as $[T_a, T_b] = T_a T_b - T_b T_a$. The product of two rotations, like $\exp(-i\theta T_y) \exp(-i\phi T_z)$, can always be written as a single exponential, $\exp(-i\psi \cdot T)$, say, where $\psi \cdot T = \psi_x T_x + \psi_y T_y + \psi_z T_z$.

Suppose we set $\exp(-i\theta \cdot T) \exp(-i\phi \cdot T) = \exp(-i\psi \cdot T)$, and try to calculate ψ in terms of θ and ϕ . We find

$$\begin{aligned}
& [1 - i\theta \cdot T - \frac{1}{2}(\theta \cdot T)^2 + \dots][1 - i\phi \cdot T - \frac{1}{2}(\phi \cdot T)^2 + \dots] \\
&= [1 - i(\theta + \phi) \cdot T - \frac{1}{2}((\theta + \phi) \cdot T)^2 - \frac{1}{2}[\theta \cdot T, \phi \cdot T] + \dots] \\
&= \exp\{-i(\theta + \phi) \cdot T - \frac{1}{2}[\theta \cdot T, \phi \cdot T] + \dots\}. \tag{1.10}
\end{aligned}$$

To this order in the expansion, to calculate ψ we need to know the value of the commutators like $[T_x, T_y]$. In fact, this is true for all orders and is known as the Campbell-Baker-Hausdorff theorem. It is for this reason that we can learn most of what we need to know about Lie groups by studying the commutation relations of the generators.

These commutation relations, obtained by considering geometrical rotations near the identity, form what is called an abstract Lie algebra. The Lie algebra defines the local geometric structure of the underlying Lie group. Formally [1], a Lie algebra, say \mathfrak{g} , can be defined as a vector space with an antisymmetric binary operation, $[\cdot, \cdot]$, known as a commutator, which maps $\mathfrak{g} \times \mathfrak{g}$ into \mathfrak{g} and which also satisfies the Jacobi identity

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0, \tag{1.11}$$

where A, B , and C belong to \mathfrak{g} . For the Lie algebra, the binary operation is given by commutation and for any two generators A, B the commutator $[A, B]$ is another generator. The commutator is non-associative, however. By “non-associative” we mean that in general $[[A, B], C]$ is not equal to $[A, [B, C]]$: the commutators satisfy the Jacobi identity instead.

The Jacobi identity is a consequence of the Lie group composition law: how two transformations combine to make a third. The exponential of \mathfrak{g} is the Lie group, G , say. More precisely, its connected component containing the unit element forms a Lie group: to any $A \in \mathfrak{g}$ there corresponds a group element $e^{i\theta A}$, where θ is some parameter, and

the exponential is defined by its series expansion. So if $A = \sum_a \alpha_a T_a$ belongs to \mathfrak{g} , there corresponds the group element $\exp i(\sum_a \alpha_a T_a)$ where the exponential is defined by the power expansion in the parameters α_a . This makes it clear that the algebra describes the group in the vicinity of the identity. Any group element, as an example, (1.9), which can be obtained from the identity by continuously changing the parameters can be written as $\exp(i \sum_a \alpha_a T_a)$. Here α_a are real parameters with $a = 1, \dots, \dim \mathfrak{g}$ and T_a are linearly independent Hermitian operators. The set of all linear combinations of $\sum \alpha_a T_a$ forms a vector space, with $\{T_a\}$ as a basis in the space. These basis vectors are called the group generators or the generators of the Lie algebra \mathfrak{g} . These generators and the commutation relations define the Lie algebra associated with the Lie group.

Let us return to the $so(3)$ Lie algebra. By direct computation, we can find the commutation relations of the matrices of (1.6) and (1.7) obtained above. They determine the Lie algebra as follows:

$$[T_x, T_y] = iT_z, \quad [T_y, T_z] = iT_x, \quad [T_z, T_x] = iT_y. \quad (1.12)$$

Since $[T, T] = 0$, for any T , and $[T, T'] = -[T', T]$, for any two T and T' , the last results determine all the $so(3)$ commutation relations. More concisely, for this Lie algebra, $so(3)$,

$$[T_a, T_b] = i \sum_c \varepsilon_{abc} T_c, \quad (1.13)$$

where $a, b, c, \dots \in \{1, 2, 3, \dots\}$ and ε_{abc} is the Levi-Civita tensor, with

$$\varepsilon_{abc} = \begin{cases} 1 & \text{if } abc = 123, 312, 231 \\ -1 & \text{if } abc = 213, 321, 132 \\ 0 & \text{if } abc = 112, 223, 331 \text{ or any 2 equal} \end{cases} \quad (1.14)$$

For example if $a = 1, b = 2$, and $c = 3$ then

$$[T_1, T_2] = i\varepsilon_{123}T_3 = iT_3 \quad (1.15)$$

More generally, for Lie algebras,

$$[T_a, T_b] = i \sum_c f_{abc} T_c, \quad (1.16)$$

where f_{abc} are known as the structure constants. The structure constant is determined by the group multiplication law: the product of group elements is another group element.

To introduce the concept of representation, let us re-consider the rotation group. Rotations and angular momentum are intimately related. For example, by Noether's theorem, rotational invariance of 3-dimensional space gives rise to the conservation of orbital angular momentum. The connection between rotation and $so(3)$ is most easily seen in quantum mechanics, however. The commutation relations between the x, y , and z components of the quantum angular momentum are

$$[L_x, L_y] = i\hbar L_z, \quad [L_y, L_z] = i\hbar L_x, \quad [L_z, L_x] = i\hbar L_y. \quad (1.17)$$

But putting $L_a = \hbar T_a$ gives us the commutation relations (1.12) of $so(3)$.

A representation is obtained when for each element of the algebra we have a linear transformation (a matrix) acting on a vector space (column vectors) in a way which preserves the commutation relations of the algebra.

If we define a set of matrices \mathbf{T}_a

$$(\mathbf{T}_a)_{bc} = -if_{abc}, \quad (1.18)$$

then (1.11) can be rewritten as

$$[\mathbf{T}_a, \mathbf{T}_b] = i \sum_c f_{abc} \mathbf{T}_c. \quad (1.19)$$

This implies that the structure constants themselves generate matrices that satisfy the commutation relations of the algebra. These matrices form a representation of the algebra, and the representation generated by the structure constants is known as the adjoint representation. For example, the $so(3)$ matrices, (1.6, 1.7).

Physical systems are described by fixed representations. Consequently, to describe a real physical system whose symmetry transformations form a group, representation theory must be used. It gives us exact mathematical expressions (matrices) describing given transformations in a chosen space. For instance, the matrix representing g in the representation R is $R(g)$, for any element, g , of a Lie group G . As an example, one $SO(3)$ representation is given by $R(T_a) = \mathbf{T}_a$, where the matrices \mathbf{T}_a are shown in (1.6, 1.7)

A group or algebra describes an abstract physical symmetry, while its representations describe how a particular system realizes the symmetry. What allows for diverse systems to realize a given symmetry group is the existence of different representations of the group. For example, to describe different size orbital angular momenta, different representations are needed; the larger the matrices, the larger is the angular momentum.

The importance and usefulness of group theory cannot be overemphasized. Group theory simplifies the physical analysis of systems possessing some degree of symmetry.

We can get some useful information about representations of Lie algebras by studying their characters. A character is a useful functional way of encoding the content of a representation. The character of a representation R of a group G is the trace of the relevant matrices. That is, the character equals $tr R(g)$, for $g \in G$. A character is basis independent

since $tr(MR(g)M^{-1}) = trR(g)$, so the character gives us the important information we need. Since the trace of a tensor product is a product of traces, we can learn about the tensor products of representations by multiplying their characters. Also, the relation of characters of a group (algebra) to those of a subgroup (subalgebra) can tell us much about group-subgroup (algebra-subalgebra) pairs.

For any element g of a Lie group G , we can write

$$g = \exp\left(i \sum_a \alpha_a T^a\right), \quad (1.20)$$

$a = 1, \dots, \dim G$. Remarkably, we can always relate g to a simpler group element by the transformation

$$g = f(e^{i\theta \cdot H})f^{-1}, \quad (1.21)$$

for some $f \in G$. Here $e^{i\theta \cdot H} = \exp\left(i \sum_{j=1}^r \theta_j H^j\right)$, and $j = 1, \dots, r$. H^j are the generators of the Cartan subalgebra of the Lie algebra \mathfrak{g} . The Cartan subalgebra is the largest subalgebra of the Lie algebra \mathfrak{g} with all elements commuting with each other, i.e., $[H^i, H^j] = 0$. its elements can be diagonalized simultaneously and so their eigenvalues can be used to label the states of representations. θ is known as the vector of Cartan angles.

Therefore, the character of a representation can be written as

$$trR(g) = trR(fgf^{-1}) = tr[R(f)R(e^{i\theta \cdot H})R(f^{-1})] = trR(e^{i\theta \cdot H}). \quad (1.22)$$

We therefore write the character as

$$ch_R(\theta) = trR(e^{i\theta \cdot H}). \quad (1.23)$$

If $|\mu\rangle$ denotes a state of the representation, of weight μ , then H^j acts as follows

$$H^j |\mu\rangle = \mu^j |\mu\rangle \quad (1.24)$$

$$e^{i\theta \cdot H} |\mu \rangle = e^{i\theta \cdot \mu} |\mu \rangle . \quad (1.25)$$

Representations can be labeled by their highest weight λ ; $R = R(\lambda)$, for example. Formally, the character of the representation of the highest weight λ is defined as [27]

$$\begin{aligned} \text{ch}_\lambda &= \sum_{|\mu \rangle \in R(\lambda)} \langle \mu | e^{i\theta \cdot H} | \mu \rangle = \sum_{\mu} e^{i\theta \cdot \mu} \text{mult}_\lambda(\mu) \\ &= \sum_{\mu} a^\mu \text{mult}_\lambda(\mu), \end{aligned} \quad (1.26)$$

where $\text{mult}_\lambda(\mu)$ is the multiplicity of the weight μ in the representation $R(\lambda)$ of highest weight λ . As an example, let us consider the character of the $SU(2)$ representation of highest weight $\lambda = \lambda_1 \Lambda^1$. Here, Λ^1 is the fundamental weight.

The algebra $so(3)$ and $su(2)$ are isomorphic (have the same form), i.e. $so(3) \cong su(2)$. The groups are distinct, however: $SO(3) = SU(2)/\mathbb{Z}_2$. $so(3)$ is the algebra of orbital angular momentum while $su(2)$ also describes spin. So for $so(3)$, we have $j = 0, 1, 2, \dots$; while $j = 0, \frac{1}{2}, 1, \dots$ for $su(2)$. Each value of j determines a representation, with a related highest weight. The highest weights are usually normalized to involve integers: $\lambda_1 = 2j$ for $su(2)$.

Consider some simple examples of $su(2)$ representations. For $\lambda = 0$, $2j = 0$ and we have

$$\text{ch}_\lambda(\theta) = \text{ch}_0(\theta) = \text{tr} e^{i\theta(0)} = \text{tr}(1) = 1 \quad (1.27)$$

If $\lambda_1 = 1$ i.e. $2j = 1$ we have

$$\begin{aligned} \text{ch}_{\Lambda^1}(\theta) &= \text{tr} \exp \left(i\theta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) = \text{tr} \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \\ &= e^{i\theta} + e^{-i\theta} = 2 \cos \theta = a + a^{-1} \end{aligned} \quad (1.28)$$

Here we have replaced $\theta_1 \rightarrow \theta$ because there is only one Cartan angle for $su(2)$. The a variable has been introduced because we will be working with such variables in this thesis. For $su(2)$, with highest weight $\lambda = \lambda_1 \Lambda^1$, μ_1 ranges from λ_1 to $-\lambda_1$. So for $\mu_1 = 2$ we have

$$\text{ch}_{2\Lambda^1}(\theta) = e^{2i\theta} + 1 + e^{-2i\theta} = 2 \cos 2\theta + 1 \quad (1.29)$$

In similar fashion, we can write the character of a general highest-weight representation as

$$\begin{aligned} \text{ch}_{\lambda_1 \Lambda^1}(\theta) &= e^{i\lambda_1 \theta} + e^{i(\lambda_1-2)\theta} + e^{i(\lambda_1-4)\theta} + \dots + e^{-i\lambda_1 \theta} \\ &= a^{\lambda_1} + a^{\lambda_1-2} + a^{\lambda_1-4} + \dots + a^{-\lambda_1} = \frac{a^{\lambda_1} - a^{-\lambda_1-2}}{1 - a^{-2}} \end{aligned} \quad (1.30)$$

As an example of their use, consider products of characters. They can encode the angular momentum addition rule thus:

$$\text{ch}_{2j_1}(\theta)\text{ch}_{2j_2}(\theta) = \text{ch}_{2(j_1+j_2)}(\theta) + \text{ch}_{2(j_1+j_2-1)}(\theta) + \dots + \text{ch}_{2|j_1-j_2|}(\theta) \quad (1.31)$$

For instance,

$$\{\text{ch}_1(\theta)\}^2 = 4 \cos^2 \theta = 2(1 + \cos 2\theta) = (2 \cos \theta + 1) + 1 = \text{ch}_1(\theta) + \text{ch}_0(\theta) \quad (1.32)$$

We will use two important general character formulas for the study of characters of representations, the Weyl character formula and the Demazure character formula. Both use the symmetry group of the weights of representations of the Lie algebra, known as the Weyl group. The Demazure character formula is a non-negative character formula that makes use of Demazure operators to compute the characters.

The study of characters can be complicated because we need to keep track of the weights and the multiplicities of the representations. This problem can be aided by the use of the generating function technique.

This brings us to the subject of generating functions. Sometimes a series of seemingly complicated quantities is simplified by introducing a variable (or variables) and summing the series. This is the generating function technique. Let us consider a simple example; we will prove that

$$2^N = \sum_{m=0}^N \binom{N}{m}, \quad (1.33)$$

where the binomial coefficient $\binom{N}{m}$ is the number of ways of choosing m objects from a larger set of N objects. We construct a generating function for the binomial coefficients by introducing a variable, x , thus:

$$\begin{aligned} G_N(x) &= \sum_{m=0}^N \binom{N}{m} x^m \\ &= 1 + Nx + \frac{N(N-1)x^2}{2!} + \dots + x^N \\ &= (1+x)^N. \end{aligned} \quad (1.34)$$

It is clearly simpler to deal with the last form of the equation than with the binomial coefficient itself. Putting $x = 1$ in equation (1.34), we find (1.33).

The introduction of a continuous variable also allows us to take derivatives, which can be very important. As a simple example, take derivatives with respect to x in (1.34), to find

$$G'_N(x) = N(1+x)^{N-1} = N \sum_{l=0}^{N-1} \binom{N-1}{l} x^l \quad (1.35)$$

and,

$$G'_N(x) = \sum_{m=1}^N \binom{N}{m} m x^{m-1}. \quad (1.36)$$

Comparing these results, we can easily see that

$$m \binom{N}{m} = N \binom{N-1}{m-1}. \quad (1.37)$$

Thus, nontrivial identities can be obtained from generating functions by taking derivatives. In more complicated cases, by simply using the generating function technique, we prove identities which otherwise might be difficult to prove.

Generating functions occur in many branches of physics. Many special functions used in physics are most easily treated using their generating functions. The concept is used in many areas of theoretical physics. The partition function $\sum_r e^{-\beta E_r} = \sum_E g(E) e^{-\beta E}$ is essentially a generating function for the energies and degeneracies of a canonical ensemble [5], for example. In Hamiltonian mechanics, canonical transformations are often treated using generating functions [29].

The generating function technique is a very important mathematical tool for the study of Lie groups and Lie algebras. The character generator (generating function of characters) is a special kind of generating function.

Finding character generating functions is interesting but often involves lengthy calculations, especially for higher rank algebras. Much work has been done, but the motivation for this project is finding a general formula for the character generator that is as simple as possible and also applies generally, i.e, to any simple Lie algebra. Sometimes, the term universal is also used to mean properties common to any simple Lie algebra.

Patera and Sharp [23] derived an interesting formula, (2.10), for the character generating function from the well-known Weyl character formula, (2.8). But as one goes to higher rank algebras using this formula becomes more complicated, and the calculations

very lengthy. In the chapter that follows, we will review the Patera and Sharp formula, (2.10), and derive a new formula for the character generator, (2.27).

Stanley [24] and Baclawski [6] pioneered the use of posets underlying character generators. The graphs that are the Hasse diagrams of the posets are very helpful in understanding the expressions for the character generators. In their later work, Baclawski and Towber [7] generalized the notion of posets and derived the G_2 generalized poset graph, also using a non-universal method.

A more general approach which applies only to classical Lie algebras ($su(N)$, $so(N)$, $sp(2N)$) was developed by King and collaborators [19, 20]. Though an important step in the right direction, it does not apply to all simple Lie algebras.

Gaskell, in his classic work [14], applied a universal approach to the calculation of character generators for simple Lie algebras. His use of Demazure operators turns out to be an interesting and convenient way of doing the calculations. Though his paper did not address the issue of the underlying poset graphs, his methods can be used to find them.

Knowing about the generalized posets of Baclawski and Towber, we apply Gaskell's approach and arrive at a conjecture for a general method of constructing character generators for simple Lie algebras. We believe our approach indicates a general way of calculating character generators and the related graphs for higher rank simple Lie algebras.

The succeeding chapters are organized as follows. Chapter 2 is a review of the formula due to Patera and Sharp [23], derived from the well known Weyl character formula, (2.8), followed by the derivation of a new general formula, (2.27). Chapter 3 uses the new formula to look at the integrity basis and incompatibilities. In Chapter 4, we take a lead from the work of Gaskell [14] and apply the Demazure character formula, (4.8), to calculate

character generators. And Chapter 5 shows the underlying graphs and possible extension of our method to an arbitrary simple Lie algebra.

Finally, we hope the examples in this thesis help in the understanding of our work.

Chapter 2

Character generators from the Weyl character formula

2.1 Character generators from the Weyl character formula

Let $X(L, a)$ denote the generator (generating function) for the characters of a fixed complex, simple Lie algebra X_r , of rank r . It is defined [23]

$$X(L, a) := \sum_{\lambda \in P_{\geq}} L^{\lambda} \text{ch}_{\lambda}(a) \ , \quad (2.1)$$

where the character of the integrable, irreducible representation $R(\lambda)$ of highest weight λ is

$$\text{ch}_{\lambda}(a) = \sum_{\sigma \in P} \text{mult}_{\lambda}(\sigma) a^{\sigma} \ . \quad (2.2)$$

Two sets of indeterminate variables are used. We write

$$L^{\lambda} = L^{\sum_i \lambda_i \Lambda^i} := L_1^{\lambda_1} \dots L_r^{\lambda_r} \quad (2.3)$$

to keep track of the highest weights of representations, and $a^{\mu} := a_1^{\mu_1} \dots a_r^{\mu_r}$ to record the weights with nonvanishing multiplicities in those representations. In (2.1), $\text{mult}_{\lambda}(\sigma)$ is the

multiplicity of weight σ in $R(\lambda)$.

The fundamental weights are the Λ^j , and the set thereof will be denoted F . The set of integral weights of X_r is

$$P := \left\{ \sum_{i=1}^r \lambda_i \Lambda^i \mid \lambda_i \in \mathbb{Z} \right\}, \quad (2.4)$$

i.e., the set of weights with integer Dynkin labels λ_i . $P_{\geq} \subset P$ will be the set of dominant weights

$$P_{\geq} := \left\{ \sum_{i=1}^r \lambda_i \Lambda^i \mid \lambda_i \in \mathbb{Z}_{\geq} \right\}, \quad (2.5)$$

with non-negative integer (semi-natural) Dynkin labels. We will also sometimes use the notation

$$\lambda = \sum_{i=1}^r \lambda_i \Lambda^i =: (\lambda_1, \lambda_2, \dots, \lambda_r). \quad (2.6)$$

The set of weights of representation $R(\lambda)$ will be indicated by

$$P_{\lambda} := \{ \mu \in P \mid \text{mult}_{\lambda}(\mu) \geq 1 \}. \quad (2.7)$$

The Weyl formula for the character of $R(\lambda)$ is

$$\begin{aligned} \text{ch}_{\lambda}(a) &= \sum_{w \in W} a^{w\lambda} \prod_{\alpha \in \Delta_+} (1 - a^{-w\alpha})^{-1} \\ &= \prod_{\alpha \in \Delta_+} (1 - a^{-\alpha})^{-1} \sum_{w \in W} (\det w) a^{w\lambda}. \end{aligned} \quad (2.8)$$

W is the Weyl group of X_r , Δ_+ the set of its positive roots, and $w.\lambda = w(\lambda + \rho) - \rho$ is the shifted action of the Weyl group element $w \in W$. The Weyl vector is denoted

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha = \Lambda^1 + \Lambda^2 + \dots + \Lambda^r = \sum_{\Lambda \in F} \Lambda. \quad (2.9)$$

The key observation of [23] is that if the Weyl character formula (2.8) is used, the sum over P_{\geq} in (2.1) can be done, yielding the Patera-Sharp formula

$$X(L, a) = \prod_{\alpha \in \Delta_+} (1 - a^{-\alpha})^{-1} \sum_{w \in W} a^{w\rho - \rho} (\det w) \prod_{\Lambda \in F} (1 - L^{\Lambda} a^{w\Lambda})^{-1}. \quad (2.10)$$

This is already a nice, general result. However, the sum over the Weyl group is daunting for any but the smallest Lie algebras. Furthermore, division by the Weyl denominator $\prod_{\alpha \in \Delta_+} (1 - a^{-\alpha})^{-1}$ makes direct computation quite difficult.

However, if we factor out a common denominator, call it Z , things improve somewhat. As is usual, let $W\lambda$ indicate the set of weights in the Weyl orbit of λ . Then we can write

$$X(L, a) = \prod_{\Lambda \in F} \prod_{\varphi \in W\Lambda} (1 - L^\Lambda a^\varphi)^{-1} Y =: Z^{-1} Y, \quad (2.11)$$

with

$$Y = \prod_{\alpha \in \Delta_+} (1 - a^{-\alpha})^{-1} \times \sum_{w \in W} a^{w\rho - \rho} (\det w) \prod_{\Lambda \in F} \prod_{\sigma \in W\Lambda \setminus \{w\Lambda\}} (1 - L^\Lambda a^\sigma). \quad (2.12)$$

It is well known that the characters may be written as integer polynomials of the fundamental characters. We therefore expect that the integrity basis I_X will be

$$I_X = \{ L^\Lambda a^\varphi \mid \Lambda \in F, \varphi \in P_\Lambda \}. \quad (2.13)$$

That is, we expect that the character generator X can be written as a rational function of the elements of I_X . For the integrity-basis element $L^\Lambda a^\varphi$, Λ and φ will be known as its shape (or its highest weight) and its weight, respectively.

Clearly, it is the numerator Y that encodes the truly nontrivial information carried by a character generator X . The denominator Z tells us only that the “outside weights” of the fundamental representations determine a subset

$$I_{\text{out}} = \{ L^\Lambda a^\varphi \mid \Lambda \in F, \varphi \in W\Lambda \} \quad (2.14)$$

of the integrity basis I_X for the terms of X . The elements of I_{out} and $I_{\text{in}} := I_X \setminus I_{\text{out}}$ will be called outside and inside generators, respectively.

Another helpful observation is that the terms in the sum of (2.12) are simply related to each other. Let \hat{w} denote an “operator” with action

$$\hat{w} (L^\nu a^\mu) := L^\nu a^{w\mu}, \quad (2.15)$$

for any weights ν, μ . Then we can write

$$Y = \prod_{\alpha \in \Delta_+} (1 - a^{-\alpha})^{-1} \sum_{w \in W} a^{w\rho - \rho} (\det w) \hat{w} (\mathcal{Y}) \quad (2.16)$$

where we have defined

$$\mathcal{Y} := \prod_{\Lambda \in F} \prod_{\sigma \in \overline{W\Lambda}} (1 - L^\Lambda a^\sigma) \quad (2.17)$$

and the shorthand

$$\overline{W\Lambda} := W\Lambda \setminus \{\Lambda\}. \quad (2.18)$$

Now, comparing with the Weyl formula (2.8), we see that

$$\prod_{\alpha \in \Delta_+} (1 - a^{-\alpha})^{-1} \sum_{w \in W} a^{w\rho - \rho} (\det w) \hat{w} =: \hat{\text{ch}} \quad (2.19)$$

acts as follows:

$$\hat{\text{ch}} (a^\lambda) = \prod_{\alpha \in \Delta_+} (1 - a^{-\alpha})^{-1} \sum_{w \in W} a^{w\rho - \rho} (\det w) \hat{w} (a^\lambda) = \text{ch}_\lambda(a). \quad (2.20)$$

Therefore, we get

$$Y = \hat{\text{ch}} (\mathcal{Y}) = \hat{\text{ch}} \left(\prod_{\Lambda \in F} \prod_{\sigma \in \overline{W\Lambda}} (1 - L^\Lambda a^\sigma) \right). \quad (2.21)$$

This formula shows that we can decompose Y into characters,

$$Y = \sum_{\mu \in P_\geq} y_\mu(L) \text{ch}_\mu. \quad (2.22)$$

The coefficients $y_\mu(L)$ will be polynomials in the $L_j = L^{\Lambda_j}$, with integer coefficients. To evaluate Y in this form, we use the shifted-Weyl (anti-)symmetry of the characters

$$\text{ch}_\lambda = (\det w) \text{ch}_{w.\lambda}. \quad (2.23)$$

If we define a partition function $K_\mu(L)$ as follows:*

$$\mathcal{Y}(L, a) = \prod_{\Lambda \in F} \prod_{\sigma \in \overline{W\Lambda}} (1 - L^\Lambda a^\sigma) =: \sum_{\tau \in P} K_\tau(L) a^\tau , \quad (2.24)$$

then the desired coefficients can be computed using

$$y_\mu(L) = \sum_{w \in W} (\det w) K_{w.\mu}(L) . \quad (2.25)$$

This equation says that the y_μ can be calculated by first expanding \mathcal{Y} , using its definition (see eqn. (2.21)). Each term obtained with a -dependence $a^{\varphi+\rho}$ can be Weyl-transformed using the shifted action, either into the dominant sector, or onto its boundary. In the latter case, the term should be dropped. Otherwise, it contributes with an extra factor of $\det w$, where w is the Weyl group element used. All terms $a^{\nu+\rho}$ so collected, with $\nu \in P_\geq$, signal a contribution of ch_ν to Y . We hope that the examples worked through in the following section will make the procedure clear.

Formally, then, the answer is

$$Y(L, a) = \sum_{\nu \in P_\geq} \text{ch}_\nu(a) \sum_{w \in W} (\det w) K_{w.\nu}(L) , \quad (2.26)$$

so that the character generator is

$$X(L, a) = \left\{ \prod_{\substack{\Lambda \in F \\ \varphi \in W\Lambda}} (1 - L^\Lambda a^\varphi) \right\}^{-1} \sum_{\nu \in P_\geq} \text{ch}_\nu(a) \sum_{w \in W} (\det w) K_{w.\nu}(L) . \quad (2.27)$$

2.2 Examples

2.2.1 A_1

For $X_r = A_1$, there is only one fundamental weight, Λ^1 , and we have

$$Z = (1 - L^{\Lambda^1} a^{\Lambda^1})(1 - L^{\Lambda^1} a^{-\Lambda^1}) . \quad (2.28)$$

*Here we imitate the definition of the Kostant partition function. See Sect. 25.2 of [11], for example.

Since $\mathcal{Y} = (1 - L^{\Lambda^1} a^{-\Lambda^1})$, we have

$$Y = 1 - L^{\Lambda^1} \text{ch}_{-\Lambda^1} , \quad (2.29)$$

by (2.21). But $\text{ch}_{-\Lambda^1} = -\text{ch}_{\Lambda^1} = 0$, by (2.23), so that $Y = 1$. Finally, we have the well-known result

$$X(L, a) = \left[(1 - L^{\Lambda^1} a^{\Lambda^1})(1 - L^{\Lambda^1} a^{-\Lambda^1}) \right]^{-1} . \quad (2.30)$$

2.2.2 A_2

$$\begin{aligned} Z = & (1 - L^{\Lambda^1} a^{\Lambda^1})(1 - L^{\Lambda^1} a^{-\Lambda^1 + \Lambda^2})(1 - L^{\Lambda^1} a^{-\Lambda^2}) \\ & \times (1 - L^{\Lambda^2} a^{\Lambda^2})(1 - L^{\Lambda^2} a^{\Lambda^1 - \Lambda^2})(1 - L^{\Lambda^2} a^{-\Lambda^1}) , \end{aligned} \quad (2.31)$$

and

$$\mathcal{Y} = (1 - L^{\Lambda^1} a^{-\Lambda^1 + \Lambda^2})(1 - L^{\Lambda^1} a^{-\Lambda^2})(1 - L^{\Lambda^2} a^{\Lambda^1 - \Lambda^2})(1 - L^{\Lambda^2} a^{-\Lambda^1}) . \quad (2.32)$$

From (2.21), expanding \mathcal{Y} and applying $\widehat{\text{ch}}$ gives

$$\begin{aligned} Y = & 1 - L^{\Lambda^1} (\text{ch}_{-\Lambda^1 + \Lambda^2} + \text{ch}_{-\Lambda^2}) - L^{\Lambda^2} (\text{ch}_{\Lambda^1 - \Lambda^2} + \text{ch}_{-\Lambda^1}) \\ & + L^{2\Lambda^1} \text{ch}_{-\Lambda^2} + L^{2\Lambda^2} \text{ch}_{-\Lambda^2} \\ & + L^{\Lambda^1 + \Lambda^2} (1 + \text{ch}_{-2\Lambda^1 + \Lambda^2} + \text{ch}_{\Lambda^1 - 2\Lambda^2} + \text{ch}_{-\Lambda^1 - \Lambda^2}) \\ & - L^{2\Lambda^1 + \Lambda^2} (\text{ch}_{-\Lambda^2} + \text{ch}_{-2\Lambda^1}) - L^{\Lambda^1 + 2\Lambda^2} (\text{ch}_{-\Lambda^1} + \text{ch}_{-2\Lambda^2}) \\ & + L^{2\Lambda^1 + 2\Lambda^2} \text{ch}_{-\Lambda^1 - \Lambda^2} . \end{aligned} \quad (2.33)$$

Any term ch_μ with a Dynkin label $\mu_i = -1$ vanishes, since if r_i denotes the primitive reflection related to the simple root α_i , then $r_i \cdot \mu = \mu$. Eqn. (2.23) then tells us that

$\text{ch}_\mu = -\text{ch}_\mu = 0$. The expression immediately simplifies to

$$\begin{aligned} Y = & 1 + L^{\Lambda^1 + \Lambda^2} (1 + \text{ch}_{-2\Lambda^1 + \Lambda^2} + \text{ch}_{\Lambda^1 - 2\Lambda^2}) \\ & - L^{2\Lambda^1 + \Lambda^2} \text{ch}_{-2\Lambda^1} - L^{\Lambda^1 + 2\Lambda^2} \text{ch}_{-2\Lambda^2} . \end{aligned} \quad (2.34)$$

But $r_1.(-2\Lambda^1) = -\Lambda^2$ and $r_2.(-2\Lambda^2) = -\Lambda^1$, so the last 2 terms vanish. Also, $r_1.(-2\Lambda^1 + \Lambda^2) = r_2.(\Lambda^1 - 2\Lambda^2) = 0$, so that we obtain

$$Y = 1 - L^{\Lambda^1 + \Lambda^2} . \quad (2.35)$$

Finally, we can write

$$X = Z^{-1} \left[1 - L^{\Lambda^1 + \Lambda^2} \right] , \quad (2.36)$$

with Z given by (2.31), in agreement with the known result [23].

2.2.3 B_2

The simple roots are $\alpha_1 = 2\Lambda^1 - 2\Lambda^2$ and $\alpha_2 = -\Lambda^1 + 2\Lambda^2$. The Weyl orbits of the fundamental weights,

$$\begin{aligned} W\Lambda^1 &= \{ \pm\Lambda^1, \pm(-\Lambda^1 + 2\Lambda^2) \} \\ W\Lambda^2 &= \{ \pm\Lambda^2, \pm(\Lambda^1 - \Lambda^2) \} , \end{aligned} \quad (2.37)$$

determine both Z , by (2.11), and Y , by (2.21).

For convenience, we write explicitly

$$\begin{aligned} Z = & (1 - L^{\Lambda^1} a^{\Lambda^1})(1 - L^{\Lambda^1} a^{-\Lambda^1 + 2\Lambda^2})(1 - L^{\Lambda^1} a^{\Lambda^1 - 2\Lambda^2}) \\ & \times (1 - L^{\Lambda^1} a^{-\Lambda^1})(1 - L^{\Lambda^2} a^{\Lambda^2})(1 - L^{\Lambda^2} a^{\Lambda^1 - \Lambda^2}) \\ & \times (1 - L^{\Lambda^2} a^{-\Lambda^1 + \Lambda^2})(1 - L^{\Lambda^2} a^{-\Lambda^2}) , \end{aligned} \quad (2.38)$$

and

$$\begin{aligned}
Y &= \widehat{\text{ch}} (1 - L^{\Lambda^1} a^{-\Lambda^1+2\Lambda^2})(1 - L^{\Lambda^1} a^{\Lambda^1-2\Lambda^2})(1 - L^{\Lambda^1} a^{-\Lambda^1}) \\
&\times (1 - L^{\Lambda^2} a^{\Lambda^1-\Lambda^2})(1 - L^{\Lambda^2} a^{-\Lambda^1+\Lambda^2})(1 - L^{\Lambda^2} a^{-\Lambda^2}) .
\end{aligned} \tag{2.39}$$

As a first step in evaluating Y , we expand the last expression, dropping any characters with Dynkin labels equalling -1 . The result is

$$\begin{aligned}
Y &= \widehat{\text{ch}} \{ 1 - L^{\Lambda^1} a^{\Lambda^1-2\Lambda^2} + L^{2\Lambda^1} (1 + a^{-2\Lambda^2} + a^{-2\Lambda^1+2\Lambda^2}) \\
&\quad + L^{\Lambda^1+\Lambda^2} (a^{-2\Lambda^1+\Lambda^2} + a^{-2\Lambda^1+3\Lambda^2} + a^{2\Lambda^1-3\Lambda^2} + a^{\Lambda^2} + a^{\Lambda^1-3\Lambda^2}) \\
&\quad + L^{2\Lambda^2} (1 + a^{\Lambda^1-2\Lambda^2}) - L^{2\Lambda^1+\Lambda^2} (a^{-3\Lambda^2} + a^{-3\Lambda^1+3\Lambda^2} + a^{\Lambda^1-3\Lambda^2} \\
&\quad\quad + a^{-2\Lambda^1+\Lambda^2}) - L^{\Lambda^1+2\Lambda^2} (1 + a^{\Lambda^1-2\Lambda^2} \\
&\quad\quad + a^{-2\Lambda^1} + a^{-2\Lambda^1+2\Lambda^2} + a^{2\Lambda^1-4\Lambda^2} + 2a^{-2\Lambda^2}) + L^{3\Lambda^1+\Lambda^2} a^{-2\Lambda^1+\Lambda^2} \\
&\quad + L^{2\Lambda^1+2\Lambda^2} (a^{-2\Lambda^2} + a^{-2\Lambda^1+2\Lambda^2} + 1 + a^{\Lambda^1-4\Lambda^2} + a^{\Lambda^1-2\Lambda^2} + a^{-3\Lambda^1+2\Lambda^2}) \\
&\quad\quad + L^{\Lambda^1+3\Lambda^2} a^{\Lambda^1-3\Lambda^2} - L^{3\Lambda^1+2\Lambda^2} (a^{-2\Lambda^1} + a^{-2\Lambda^2}) \\
&\quad\quad - L^{2\Lambda^1+3\Lambda^2} (a^{-3\Lambda^2} + a^{-2\Lambda^1+\Lambda^2}) \} .
\end{aligned} \tag{2.40}$$

Not only will the contributions of weights $(-1, \mu_2)$ and $(\mu_1, -1)$ vanish, but so will those of any in their shifted Weyl orbits, $W.(-1, \mu_2)$:

$$\{(-1, \mu_2), (\mu_2, -\mu_2 - 2), (-\mu_2 - 2, \mu_2), (-1, -\mu_2 - 2)\} , \tag{2.41}$$

and $W.(\mu_1, -1)$:

$$\{(\mu_1, -1), (-\mu_1 - 2, 2\mu_1 + 1), (\mu_1, -2\mu_1 - 3), (-\mu_1 - 1, -1)\} . \tag{2.42}$$

We can therefore also eliminate any terms a^μ with μ of forms $(a, -a - 2)$ and $(a, -2a - 3)$.

Doing this, we find

$$\begin{aligned}
Y = \widehat{\text{ch}} \{ & 1 - L^{\Lambda^1} a^{\Lambda^1 - 2\Lambda^2} + L^{2\Lambda^1} (1 + a^{-2\Lambda^1 + 2\Lambda^2}) \\
& + L^{\Lambda^1 + \Lambda^2} (a^{-2\Lambda^1 + 3\Lambda^2} + a^{2\Lambda^1 - 3\Lambda^2} + a^{\Lambda^2}) \\
& + L^{2\Lambda^2} (1 + a^{\Lambda^1 - 2\Lambda^2}) - L^{2\Lambda^1 + \Lambda^2} a^{-3\Lambda^1 + 3\Lambda^2} \\
& - L^{\Lambda^1 + 2\Lambda^2} (1 + a^{\Lambda^1 - 2\Lambda^2} + a^{-2\Lambda^1 + 2\Lambda^2}) \\
& + L^{2\Lambda^1 + 2\Lambda^2} (a^{-2\Lambda^1 + 2\Lambda^2} + 1 + a^{\Lambda^1 - 4\Lambda^2} + a^{\Lambda^1 - 2\Lambda^2} + a^{-3\Lambda^1 + 2\Lambda^2}) \} . \tag{2.43}
\end{aligned}$$

Calculating

$$\begin{aligned}
r_1 \cdot (-2, 3) &= r_2 \cdot (2, -3) = (0, 1) , \\
r_2 \cdot (1, -2) &= r_1 \cdot (-2, 2) = (r_1 r_2) \cdot (1, -4) = (r_2 r_1) \cdot (-3, 2) = 0 , \tag{2.44}
\end{aligned}$$

we find

$$Y = 1 + L^{\Lambda^1} - L^{\Lambda^1 + \Lambda^2} \text{ch}_{\Lambda^2} + L^{\Lambda^1 + 2\Lambda^2} + L^{2\Lambda^1 + 2\Lambda^2} . \tag{2.45}$$

Using

$$\text{ch}_{\Lambda^2} = a^{\Lambda^2} + a^{-\Lambda^1 + \Lambda^2} + a^{-\Lambda^2} + a^{\Lambda^1 - \Lambda^2} , \tag{2.46}$$

we have checked that this answer agrees with the known result [23].

2.2.4 G_2

The simple roots are $\alpha_1 = 2\Lambda^1 - 3\Lambda^2$ and $\alpha_2 = -\Lambda^1 + 2\Lambda^2$. The Weyl orbits of the fundamental weights are:

$$\begin{aligned}
W\Lambda^1 &= \{\pm\Lambda^1, \pm(\Lambda^1 - 3\Lambda^2), \pm(2\Lambda^1 - 3\Lambda^2)\} , \\
W\Lambda^2 &= \{\pm\Lambda^2, \pm(\Lambda^1 - \Lambda^2), \pm(\Lambda^1 - 2\Lambda^2)\} . \tag{2.47}
\end{aligned}$$

Therefore,

$$\begin{aligned}
Z &= (1 - L^{\Lambda^1} a^{\Lambda^1})(1 - L^{\Lambda^1} a^{\Lambda^1 - 3\Lambda^2})(1 - L^{\Lambda^1} a^{-\Lambda^1 + 3\Lambda^2}) \\
&\times (1 - L^{\Lambda^1} a^{2\Lambda^1 - 3\Lambda^2})(1 - L^{\Lambda^1} a^{-2\Lambda^1 + 3\Lambda^2})(1 - L^{\Lambda^1} a^{-\Lambda^1}) \\
&\quad \times (1 - L^{\Lambda^2} a^{\Lambda^2})(1 - L^{\Lambda^2} a^{\Lambda^1 - \Lambda^2})(1 - L^{\Lambda^2} a^{\Lambda^1 - 2\Lambda^2}) \\
&\times (1 - L^{\Lambda^2} a^{-\Lambda^1 + \Lambda^2})(1 - L^{\Lambda^2} a^{-\Lambda^1 + 2\Lambda^2})(1 - L^{\Lambda^2} a^{-\Lambda^2})
\end{aligned} \tag{2.48}$$

and

$$\begin{aligned}
\mathcal{Y} &= (1 - L^{\Lambda^1} a^{\Lambda^1 - 3\Lambda^2})(1 - L^{\Lambda^1} a^{-\Lambda^1 + 3\Lambda^2})(1 - L^{\Lambda^1} a^{2\Lambda^1 - 3\Lambda^2}) \\
&\quad \times (1 - L^{\Lambda^1} a^{-2\Lambda^1 + 3\Lambda^2})(1 - L^{\Lambda^1} a^{-\Lambda^1})(1 - L^{\Lambda^2} a^{\Lambda^1 - \Lambda^2}) \\
&\times (1 - L^{\Lambda^2} a^{\Lambda^1 - 2\Lambda^2})(1 - L^{\Lambda^2} a^{-\Lambda^1 + \Lambda^2})(1 - L^{\Lambda^2} a^{-\Lambda^1 + 2\Lambda^2})(1 - L^{\Lambda^2} a^{-\Lambda^2}).
\end{aligned} \tag{2.49}$$

The Maple program in Appendix *B* can be used to generate \mathcal{Y} , for any rank two simple Lie algebra.

By (2.21), expanding \mathcal{Y} and applying $\widehat{\text{ch}}$ gives

$$\begin{aligned}
Y = & 1 + L^{\Lambda^1} + L^{\Lambda^2} + L^{3\Lambda^1+3\Lambda^2} + L^{\Lambda^1+4\Lambda^2} \\
& + L^{3\Lambda^1} + L^{4\Lambda^1+3\Lambda^2} + L^{\Lambda^1+\Lambda^2} + L^{\Lambda^1+3\Lambda^2} \\
& + L^{\Lambda^1+2\Lambda^2} + L^{3\Lambda^1+2\Lambda^2} + L^{4\Lambda^1+4\Lambda^2} + L^{2\Lambda^1+4\Lambda^2} \\
& + L^{3\Lambda^1+4\Lambda^2} + L^{3\Lambda^1+\Lambda^2} + L^{2\Lambda^1} + (L^{3\Lambda^1+4\Lambda^2} + L^{2\Lambda^1+4\Lambda^2} \\
& + L^{\Lambda^1} + L^{2\Lambda^1} + 2L^{2\Lambda^1+2\Lambda^2} - L^{3\Lambda^1+\Lambda^2} - L^{\Lambda^1+3\Lambda^2}) \text{ch}_{\Lambda^2} \\
& + (L^{3\Lambda^1+2\Lambda^2} + L^{\Lambda^1+2\Lambda^2} - L^{2\Lambda^1+\Lambda^2} - L^{2\Lambda^1+3\Lambda^2}) \text{ch}_{\Lambda^1} \\
& + L^{2\Lambda^1+2\Lambda^2} \text{ch}_{\Lambda^1+\Lambda^2} \\
& - (L^{\Lambda^1+\Lambda^2} + L^{2\Lambda^1+\Lambda^2} + L^{3\Lambda^1+3\Lambda^2} + L^{2\Lambda^1+3\Lambda^2}) \text{ch}_{2\Lambda^2} \tag{2.50}
\end{aligned}$$

after applying (2.23). To help save time, a Maple program, listed in Appendix C, was used to work out the Weyl reflections for G_2 .

Using the following characters

$$\begin{aligned}
\text{ch}_{\Lambda^1} = & 2 + a^{\Lambda^1} + a^{-\Lambda^1+3\Lambda^2} + a^{2\Lambda^1-3\Lambda^2} + a^{-2\Lambda^1+3\Lambda^2} + a^{\Lambda^1-3\Lambda^2} \\
& + a^{-\Lambda^1} + a^{\Lambda^2} + a^{-\Lambda^1+2\Lambda^2} + a^{-\Lambda^1+\Lambda^2} + a^{-\Lambda^2} \\
& + a^{\Lambda^1-2\Lambda^2} + a^{\Lambda^1-\Lambda^2}, \tag{2.51}
\end{aligned}$$

$$\begin{aligned}
\text{ch}_{\Lambda^2} = & 1 + a^{\Lambda^2} + a^{\Lambda^1-\Lambda^2} + a^{-\Lambda^1+2\Lambda^2} \\
& + a^{\Lambda^1-2\Lambda^2} + a^{-\Lambda^1+\Lambda^2} + a^{-\Lambda^2}, \tag{2.52}
\end{aligned}$$

$$\begin{aligned}
\text{ch}_{\Lambda^1+\Lambda^2} &= 4 + 2a^{2\Lambda^2} + 4a^{\Lambda^2} + 2a^{\Lambda^1} + 2a^{2\Lambda^1-3\Lambda^2} \\
&+ 2a^{-\Lambda^1+3\Lambda^2} + 2a^{-2\Lambda^1+3\Lambda^2} + 2a^{\Lambda^1-3\Lambda^2} + 2a^{-\Lambda^1} \\
&+ 4a^{-\Lambda^1+2\Lambda^2} + 4a^{-\Lambda^1+\Lambda^2} + 4a^{\Lambda^2} + 4a^{\Lambda^1-2\Lambda^2} \\
&+ 4a^{\Lambda^1-\Lambda^2} + 2a^{2\Lambda^1-2\Lambda^2} + 2a^{2\Lambda^1-4\Lambda^2} + 2a^{-2\Lambda^2} \\
&+ 2a^{-2\Lambda^1+2\Lambda^2} + 2a^{-2\Lambda^1+4\Lambda^2} + a^{\Lambda^1+\Lambda^2} \\
&+ a^{2\Lambda^1-\Lambda^2} + a^{3\Lambda^1-5\Lambda^2} + a^{2\Lambda^1-5\Lambda^2} \\
&+ a^{\Lambda^1-4\Lambda^2} + a^{-2\Lambda^1+\Lambda^2} + a^{-3\Lambda^1+4\Lambda^2} \\
&+ a^{-3\Lambda^1+5\Lambda^2} + a^{-2\Lambda^1+5\Lambda^2} \\
&+ a^{-\Lambda^1+4\Lambda^2} + a^{-\Lambda^1-\Lambda^2} + a^{3\Lambda^1-4\Lambda^2},
\end{aligned} \tag{2.53}$$

and

$$\begin{aligned}
\text{ch}_{2\Lambda^2} &= 3 + 2a^{\Lambda^2} + 2a^{\Lambda^1-\Lambda^2} + 2a^{-\Lambda^1+2\Lambda^2} \\
&+ 2a^{\Lambda^1-2\Lambda^2} + 2a^{-\Lambda^1+\Lambda^2} + 2a^{-\Lambda^2} + a^{\Lambda^1} \\
&+ a^{-\Lambda^1+3\Lambda^2} + a^{2\Lambda^1-3\Lambda^2} + a^{-2\Lambda^1+3\Lambda^2} \\
&+ a^{\Lambda^1-3\Lambda^2} + a^{-\Lambda^1} + a^{2\Lambda^1-2\Lambda^2} \\
&+ a^{2\Lambda^1-4\Lambda^2} + a^{-2\Lambda^2} + a^{-2\Lambda^1+2\Lambda^2} \\
&+ a^{-2\Lambda^1+4\Lambda^2} + a^{2\Lambda^2},
\end{aligned} \tag{2.54}$$

we have verified that this expression agrees with the known result [15].

Chapter 3

Integrity basis and incompatibilities

3.1 Integrity basis and incompatibilities

The characters can be generated by an integrity basis subject to certain relations. The basis is given in (2.13). For a fixed simple Lie algebra, the character of any irreducible representation can be written as a non-negative integer polynomial in these basis elements.

Important relations can be expressed as incompatibilities, quadratic products of basis elements that do not appear in any of the monomials just mentioned. Here we show how the new formula (2.27) can be used as a guide to the incompatibilities. A different method was described in [16].

Since the outside and inside generators ($I_{\text{in}} := I_X \setminus I_{\text{out}}$) play different roles in generating characters, it will be useful to split the fundamental characters into inside and

outside parts, by writing

$$\text{ch}_\Lambda(a) =: \mathcal{O}_\Lambda(a) + \mathcal{I}_\Lambda(a) . \quad (3.1)$$

Here we denote the orbit sum by

$$\mathcal{O}_\lambda(a) = \sum_{\sigma \in W_\lambda} a^\sigma = c_\lambda \sum_{v \in W} a^{v\lambda} . \quad (3.2)$$

The numerator Y contains the required information. Using (2.21), we will expand Y up to terms quadratic in the L_i :

$$Y = Y^{(0)} + Y^{(1)} + Y^{(2)} + \dots = \widehat{\text{ch}}(\mathcal{Y}^{(0)} + \mathcal{Y}^{(1)} + \mathcal{Y}^{(2)} + \dots) . \quad (3.3)$$

Clearly,

$$Y^{(0)} = \mathcal{Y}^{(0)} = 1 . \quad (3.4)$$

The linear term is

$$Y^{(1)} = -\widehat{\text{ch}}\left(\sum_{\Lambda \in F} \sum_{\phi \in \overline{W\Lambda}} L^\Lambda a^\phi\right) = \widehat{\text{ch}}\left(\sum_{\Lambda \in F} L^\Lambda (a^\Lambda - \mathcal{O}_\Lambda(a))\right) ; \quad (3.5)$$

see (3.2). We now use the identity

$$\widehat{\text{ch}}(a^\mu \mathcal{O}_\lambda(a)) = \text{ch}_\mu(a) \mathcal{O}_\lambda(a) , \quad (3.6)$$

proved in the Appendix, with $\mu = 0$. We find

$$\begin{aligned} Y^{(1)} &= \sum_{\Lambda \in F} L^\Lambda (\text{ch}_\Lambda(a) - \mathcal{O}_\Lambda(a)) \\ &=: \sum_{\Lambda \in F} L^\Lambda \mathcal{I}_\Lambda . \end{aligned} \quad (3.7)$$

The result shows explicitly that the inside generators all appear linearly in X :

$$Y^{(1)} = \sum_{\iota \in I_{\text{in}}} \iota , \quad (3.8)$$

for all simple Lie algebras.

The quadratic term can be expressed as

$$\mathcal{Y}^{(2)} = \sum_{\Lambda \in F} L^{2\Lambda} \sum_{\substack{\phi, \phi' \in \overline{W\Lambda} \\ \phi' \neq \phi}} a^{\phi+\phi'} + \sum_{\substack{\Lambda, \Lambda' \in F \\ \Lambda \neq \Lambda'}} L^{\Lambda+\Lambda'} \sum_{\substack{\phi \in \overline{W\Lambda} \\ \phi' \in \overline{W\Lambda'}}} a^{\phi+\phi'}. \quad (3.9)$$

This leads to the expression

$$\begin{aligned} Y^{(2)} &= \sum_{\substack{\Lambda, \Lambda' \in F \\ \Lambda' \neq \Lambda}} \left(\mathcal{I}_\Lambda \mathcal{I}_{\Lambda'} - S_{\Lambda, \Lambda'} \right) L^{\Lambda+\Lambda'} \\ &+ \sum_{\Lambda \in F} \left(\text{ch}_{2\Lambda} - S_{\Lambda, \Lambda} + \mathcal{I}_\Lambda^2 - \mathcal{O}_{2\Lambda} \right) L^{2\Lambda}. \end{aligned} \quad (3.10)$$

Here we have defined

$$S_{\Lambda, \Lambda'} := \text{ch}_\Lambda \text{ch}_{\Lambda'} - \text{ch}_{\Lambda+\Lambda'}. \quad (3.11)$$

Similar expressions for terms $Y^{(n)}$ with $n > 2$ are complicated. We will focus on the quadratic term $Y^{(2)}$ below, and treat each of the rank-two simple Lie algebras in turn.

3.2 Examples

3.2.1 A_2

From Chapter 2, subsection 2.2,

$$Y^{(2)} = -L_1 L_2. \quad (3.12)$$

This result agrees with that calculated using (3.9).

For any algebra A_r , $\mathcal{I}_\Lambda = 0$ for all $\Lambda \in F$. Using

$$\begin{aligned} \text{ch}_{\Lambda^1} \text{ch}_{\Lambda^2} &= \text{ch}_{\Lambda^1+\Lambda^2} + 1, \\ (\text{ch}_{\Lambda^1})^2 &= \text{ch}_{2\Lambda^1} + \text{ch}_{\Lambda^2}, \quad (\text{ch}_{\Lambda^2})^2 = \text{ch}_{2\Lambda^2} + \text{ch}_{\Lambda^1}, \\ \text{ch}_{2\Lambda^1} - \mathcal{O}_{2\Lambda^1} &= \text{ch}_{\Lambda^2}, \quad \text{ch}_{2\Lambda^2} - \mathcal{O}_{2\Lambda^2} = \text{ch}_{\Lambda^1}, \end{aligned} \quad (3.13)$$

(3.10) gives the same result.

The interpretation of the result (3.12) is simple. There is one incompatible quadratic product, but it is not uniquely determined. Any of the 3 following possibilities works:

$$(L_1 a_1)(L_2 a_1^{-1}), \quad (L_1 a_1^{-1} a_2)(L_2 a_1 a_2^{-1}), \quad (L_1 a_2^{-1})(L_2 a_2). \quad (3.14)$$

We will see below in Chapter 4, subsection 2.1 that these choices lead to three different, but equivalent, expressions for X .

3.2.2 B_2

To use (3.10), we need

$$\begin{aligned} \mathcal{I}_{\Lambda^1} &= 1, \quad \mathcal{I}_{\Lambda^2} = 0, \\ S_{\Lambda^1, \Lambda^1} &= 1 + \text{ch}_{2\Lambda^2}, \quad S_{\Lambda^1, \Lambda^2} = \text{ch}_{\Lambda^2}, \quad S_{\Lambda^2, \Lambda^2} = 1 + \text{ch}_{\Lambda^1}, \\ \text{ch}_{2\Lambda^1} - \mathcal{O}_{2\Lambda^1} &= \text{ch}_{2\Lambda^2}, \quad \text{ch}_{2\Lambda^2} - \mathcal{O}_{2\Lambda^2} = 1 + \text{ch}_{\Lambda^1}, \end{aligned} \quad (3.15)$$

to find

$$Y^{(2)} = -L_1 L_2 \text{ch}_{\Lambda^2}. \quad (3.16)$$

This is in agreement with the result of Chapter 2, subsection 2.3.

The negative terms in (3.16) reveal incompatibilities between generators. One choice of incompatible products is

$$\begin{aligned} (L_1 a_1)(L_2 a_1^{-1} a_2), \quad (L_1 a_1)(L_2 a_2^{-1}), \\ (L_1 a_1^{-1} a_2^2)(L_2 a_2^{-1}), \quad (L_1)(L_2 a_2^{-1}). \end{aligned} \quad (3.17)$$

The sum of these four terms equals $L_1 L_2 \text{ch}_{\Lambda^2}$, therefore agreeing with (3.16).

In Chapter 4, subsection 2.2 below we will relate this choice of incompatible products to a non-negative expression for X , and an underlying graph.

3.2.3 G_2

From Chapter 2, subsection 2.4,

$$Y^{(2)} = L_1^2 (1 + \text{ch}_{\Lambda^2}) + L_1 L_2 (1 - \text{ch}_{2\Lambda^2}) , \quad (3.18)$$

in accord with that calculated using (3.9).

To verify (3.10),

$$\begin{aligned} \mathcal{I}_{\Lambda^1} &= 1 + \text{ch}_{\Lambda^2} , \quad \mathcal{I}_{\Lambda^2} = \text{ch}_{\Lambda^1} + \text{ch}_{\Lambda^2} , \\ S_{\Lambda^1, \Lambda^2} &= 1 + \text{ch}_{\Lambda^1} + \text{ch}_{2\Lambda^2} + \text{ch}_{3\Lambda^2} , \\ S_{\Lambda^1, \Lambda^2} &= \text{ch}_{\Lambda^2} + \text{ch}_{2\Lambda^2} , \quad S_{\Lambda^2, \Lambda^2} = 1 + \text{ch}_{\Lambda^1} + \text{ch}_{\Lambda^2} , \\ \text{ch}_{2\Lambda^1} - \mathcal{O}_{2\Lambda^1} &= \text{ch}_{3\Lambda^2} - \text{ch}_{\Lambda^2} + 1 , \\ \text{ch}_{2\Lambda^2} - \mathcal{O}_{2\Lambda^2} &= \text{ch}_{\Lambda^1} + \text{ch}_{\Lambda^2} , \end{aligned} \quad (3.19)$$

are useful.

We will verify in Chapter 4, subsection 2.3 below that the expression (3.18) encodes the incompatible products for X . More precisely, we will show that it can be written as a sum of terms

$$Y^{(2)} = - Y_{\text{out, out}}^{(2)} + Y_{\text{in, in}}^{(2)} - Y_{\text{in, out}}^{(2)} . \quad (3.20)$$

The negative terms are incompatible products, either with two outer generators as factors, or one inner and one outer. Since the factor Z^{-1} of X does not involve the inner generators, the allowed products quadratic in the inner generators appear in $Y^{(2)}$; that explains the positive term.

Chapter 4

Gaskell character generators from Demazure character formulas

4.1 Gaskell character generators from Demazure character formulas

In this section, we follow Gaskell [14] and apply the Demazure character formulas to the calculation of character generators. We will be able to interpret our results in terms of certain graphs, as discussed in Chapter 5.

Let us first review the Demazure character formula(s), and set our notation. Demazure [10] introduced the operators \widehat{D}_i , $i = 1, \dots, r$, associated with the simple roots of the Lie algebra X_r , or the corresponding primitive reflections r_i . They are defined by the

action

$$\widehat{D}_i(a^\phi) = \begin{cases} a^\phi + a^{\phi-\alpha_i} + \dots + a^{\phi-\phi_i\alpha_i} & \phi_i \geq 0 ; \\ 0 , & \phi_i = -1 ; \\ -a^{\phi+\alpha_i} - a^{\phi+2\alpha_i} - \dots - a^{\phi+(|\phi_i|-1)\alpha_i} , & \phi_i \leq -2 . \end{cases} \quad (4.1)$$

The number of terms in these expansions is $|\phi_i + 1|$. Alternatively, one can write

$$\widehat{D}_i = (1 - a^{-\alpha_i})^{-1} (1 - a^{-\alpha_i} \hat{r}_i) . \quad (4.2)$$

A unique Demazure operator can be defined for every element of the Weyl group W . Suppose $w \in W$ has a reduced decomposition $w = s_1 s_2 \cdots s_\ell$. Here each $s_j = r_{j'}$ is a primitive reflection, and since the decomposition is reduced, ℓ is the minimum possible length. Then we can define

$$\widehat{D}_w := \widehat{D}_{1'} \widehat{D}_{2'} \cdots \widehat{D}_{\ell'} . \quad (4.3)$$

Reduced decompositions are not unique, however. For example, the longest element w_L of the $su(3)$ Weyl group $W \cong S_3$ has two such decompositions,

$$w_L = r_1 r_2 r_1 = r_2 r_1 r_2 . \quad (4.4)$$

But the braid relation that equates them is also satisfied by the Demazure operators:

$$\widehat{D}_{w_L} = \widehat{D}_1 \widehat{D}_2 \widehat{D}_1 = \widehat{D}_2 \widehat{D}_1 \widehat{D}_2 , \quad (4.5)$$

so that \widehat{D}_{w_L} can be constructed using either of its reduced decompositions. Such braid relations are obeyed for any simple Lie algebra, and the operators \widehat{D}_w are uniquely defined for any $w \in W$. The basic operators are the $\widehat{D}_i := \widehat{D}_{r_i}$.

Notice, however that although the braid relations of the Weyl group are obeyed by the Demazure operators, we have $r_i^2 = 1$, but $\widehat{D}_i^2 \neq 1$. Instead

$$(\widehat{D}_i)^2 = \widehat{D}_i, \quad (4.6)$$

so that the \widehat{D}_i are projection operators. It is also very useful to realize that

$$\widehat{D}_i(1 + \hat{r}_i) = (1 + \hat{r}_i), \quad (4.7)$$

so that \widehat{D}_i does not change expressions that are \hat{r}_i -invariant. Using this fact can reduce computations significantly.

The Demazure character formula can be written simply as

$$\text{ch}_\lambda(a) = \widehat{D}_L(a^\lambda), \quad (4.8)$$

where we have written $\widehat{D}_L := \widehat{D}_{w_L}$ for short. Equivalently, we can write

$$\widehat{\text{ch}} = \widehat{D}_L, \quad (4.9)$$

for the operator $\widehat{\text{ch}}$ introduced in (2.19).

As an example, consider the $su(3)$ representation of highest weight $\lambda = 2\Lambda^1 + \Lambda^2$.

We will use the reduced decomposition $\widehat{D}_L = \widehat{D}_1\widehat{D}_2\widehat{D}_1$. First,

$$\widehat{D}_1 a^{2\Lambda^1 + \Lambda^2} = a^{2\Lambda^1 + \Lambda^2} + a^{2\Lambda^2} + a^{-2\Lambda^1 + 3\Lambda^2}. \quad (4.10)$$

Then

$$\begin{aligned} \widehat{D}_2 \widehat{D}_1 a^{2\Lambda^1 + \Lambda^2} &= (a^{2\Lambda^1 + \Lambda^2} + a^{3\Lambda^1 - \Lambda^2}) \\ &\quad + (a^{2\Lambda^2} + a^{\Lambda^1} + a^{2\Lambda^1 - 2\Lambda^2}) \\ &\quad + (a^{-2\Lambda^1 + 3\Lambda^2} + a^{-\Lambda^1 + \Lambda^2} + a^{-\Lambda^2} + a^{\Lambda^1 - 3\Lambda^2}). \end{aligned} \quad (4.11)$$

To avoid generating terms with negative integer coefficients, that will eventually cancel anyway, we separate out the \hat{r}_1 -invariant part of this result,

$$(a^{2\Lambda^1+\Lambda^2} + a^{2\Lambda^2} + a^{-2\Lambda^1+3\Lambda^2}) + (a^{\Lambda^1} + a^{-\Lambda^1+\Lambda^2}) + (a^{-\Lambda^2}) \quad (4.12)$$

before applying \widehat{D}_1 . By virtue of (4.7), we then need only compute

$$\begin{aligned} \widehat{D}_1 (a^{3\Lambda^1-\Lambda^2} + a^{2\Lambda^1-2\Lambda^2} + a^{\Lambda^1-3\Lambda^2}) = \\ (a^{3\Lambda^1-\Lambda^2} + a^{\Lambda^1} + a^{-\Lambda^1+\Lambda^2} + a^{-3\Lambda^1+2\Lambda^2}) + \\ (a^{2\Lambda^1-2\Lambda^2} + a^{-\Lambda^2} + a^{-2\Lambda^1}) + (a^{\Lambda^1-3\Lambda^2} + a^{-\Lambda^1-2\Lambda^2}). \end{aligned} \quad (4.13)$$

Adding this last result to (4.12) then gives the character $\text{ch}_{2\Lambda^1+\Lambda^2}(a)$, without the need for cancellations between positive and negative terms, as in the Weyl character formula.

Also useful are operators \hat{d}_i , defined by

$$\widehat{D}_i =: 1 + \hat{d}_i, \quad \hat{d}_i := (1 - a^{-\alpha_i})^{-1} a^{-\alpha_i} (1 - \hat{r}_i). \quad (4.14)$$

Their action is

$$\hat{d}_i (a^\phi) = \begin{cases} a^{\phi-\alpha_i} + a^{\phi-2\alpha_i} + \dots + a^{\phi-\phi_i\alpha_i}, & \phi_i \geq 1; \\ 0, & \phi_i = 0; \\ -a^\phi - a^{\phi+\alpha_i} - \dots - a^{\phi+(|\phi_i|-1)\alpha_i}, & \phi_i \leq -1. \end{cases} \quad (4.15)$$

Notice that the number of terms in all three of these last expressions is $|\phi_i|$.

Cartoons of the actions of the Demazure operators are given in Fig. 4.1. They make clear certain relations, such as $\hat{r}_i \widehat{D}_i = \widehat{D}_i$, $\widehat{D}_i = \hat{d}_i + 1$, $\hat{r}_i \hat{d}_i = a^{\alpha_i} \hat{d}_i$, $\hat{d}_i \hat{r}_i = -\hat{d}_i$, etc. The vertical, dashed line in the figure represents the hyperplane in weight space where the

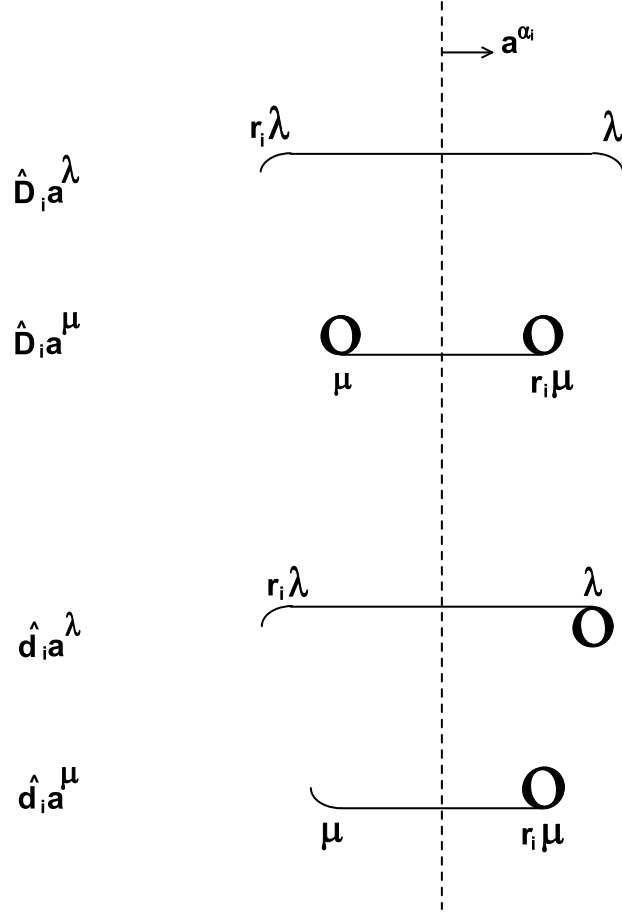


Figure 4.1: Action of the Demazure operators \hat{D}_i and \hat{d}_i . The weight λ has positive, integer Dynkin label λ_i , while μ_i is a negative integer.

i -th Dynkin label vanishes. The actions are indicated both for a weight λ , with positive Dynkin label λ_i , and a weight μ , with $\mu_i < 0$. Raised, horizontal lines represent strings of terms like $a^\lambda + a^{\lambda - \alpha_i} + \dots + a^{r_i \lambda}$, with positive coefficients $+1$. Lowered, horizontal lines correspond to such strings with -1 as their coefficients. The circles, consisting as they do of a raised and a lowered part, contribute 0 , but emphasize that there is no term a^λ in $\hat{d}_i a^\lambda$, e.g.

A unique operator \hat{d}_w can again be defined for any $w \in W$, using reduced decom-

positions of w , if we set

$$\hat{d}_{id} \left(a^\lambda \right) = a^\lambda . \quad (4.16)$$

In agreement with (4.6), we have

$$\hat{d}_i^2 + \hat{d}_i = 0 , \quad (4.17)$$

so that the Demazure character formula (4.8) can be rewritten as

$$\text{ch}_\lambda(a) = \sum_{w \in W} \hat{d}_w \left(a^\lambda \right) , \quad (4.18)$$

or

$$\widehat{\text{ch}} = \sum_{w \in W} \hat{d}_w . \quad (4.19)$$

We will now apply the Demazure character formulas to the calculation of character generators, following Gaskell [14]. In a remarkable paper, Gaskell discovered some of the Demazure results on characters independently, and applied them to character generators. The motivation was to find formulas that did not involve negative terms and cancellations, such as the general one (2.10) due to Patera and Sharp [23]. The relevant minus signs can be traced to the $\det w$ factor in the Weyl character formula (2.8). As illustrated by the $A_2 \cong su(3)$ example above, however, the Demazure character formula can avoid such negative terms, and so can lead to more useful formulas for X .

To save writing, let us introduce the notation

$$[x] := (1 - x)^{-1} = \sum_{n=0}^{\infty} x^n . \quad (4.20)$$

The generating function for highest-weights can be written as

$$\text{H}(L, a) := \prod_{\Lambda \in F} (1 - L^\Lambda a^\Lambda)^{-1} = \prod_{\Lambda \in F} [L^\Lambda a^\Lambda] , \quad (4.21)$$

and the generating function of interest is then

$$X = \widehat{\text{ch}} \left(\prod_{\Lambda \in F} (1 - L^\Lambda a^\Lambda)^{-1} \right) = \widehat{\text{ch}}(\mathbf{H}) = \widehat{D}_L(\mathbf{H}) . \quad (4.22)$$

Choosing a reduced decomposition of \widehat{D}_L , X can be calculated by successive applications of the basic Demazure operators \widehat{D}_i .

To proceed, Gaskell [14] derived the product rule

$$\widehat{D}_i(FG) = (\widehat{D}_i F)G + (\widehat{r}_i F)(\widehat{d}_i G) . \quad (4.23)$$

This also implies

$$\widehat{D}_i(FG) = F(\widehat{D}_i G) + (\widehat{d}_i F)(\widehat{r}_i G) . \quad (4.24)$$

In terms of the operators \widehat{d}_i , these identities read as

$$\widehat{d}_i(FG) = (\widehat{d}_i F)G + (\widehat{r}_i F)(\widehat{d}_i G) \quad (4.25)$$

and

$$\widehat{d}_i(FG) = F(\widehat{d}_i G) + (\widehat{d}_i F)(\widehat{r}_i G) . \quad (4.26)$$

For us, the most useful of these product rules will be (4.24).

To apply the Demazure operators on individual factors of $\mathbf{H}(L, a)$ and the results, we need

$$\widehat{d}_i[F] = [F] \widehat{d}_i F[\widehat{r}_i F] \quad (4.27)$$

or

$$\widehat{D}_i[F] = [F] + [F] \widehat{d}_i F[\widehat{r}_i F] . \quad (4.28)$$

Now \widehat{D}_i always acts to produce an r_i -invariant expression. The right-hand-side of the last result is therefore r_i -invariant, although it is not obvious. A manifestly invariant formula

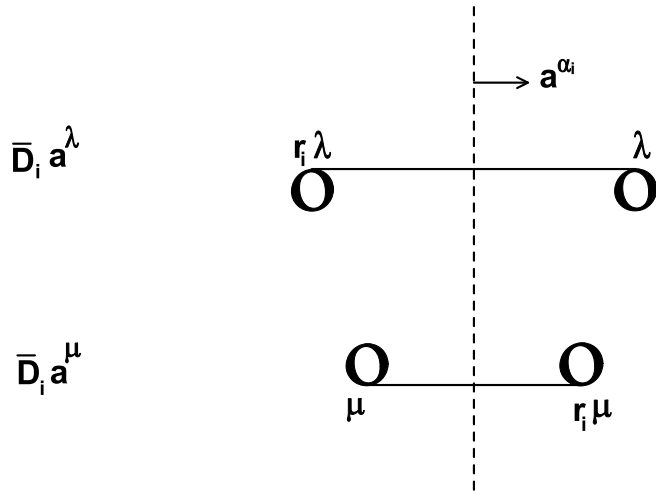


Figure 4.2: The action of the operator $\bar{D}_i = \hat{D}_i a^{-\alpha_i}$.

can be written, however, as

$$\hat{D}_i [F] = [F] (1 + \bar{D}_i F) [\hat{r}_i F] . \quad (4.29)$$

Since we will need to write it often, we have defined

$$\bar{D}_i := \hat{d}_i - \hat{r}_i . \quad (4.30)$$

From the expression (4.2), we can show that

$$\bar{D}_i = \hat{D}_i a^{-\alpha_i} , \quad (4.31)$$

demonstrating that \bar{D}_i , too, generates r_i -invariant expressions. The action of \bar{D}_i is depicted in Fig. 4.2.

4.2 Rank-Two Simple Lie Algebras

The calculations quickly become unwieldy with increasing rank. For simplicity, therefore, we will restrict to consideration of the complex, simple Lie algebras of rank two. We believe our results are indicative of general properties of the character generators of the simple Lie algebras, however.

We should point out that we make certain choices in how we perform our Demazure calculations, such as the order of factors in the highest-weight generating function H , the reduced decomposition of w_L , and which of the product rules (4.23-4.26) we apply. Of course, none of these choices changes the final result, but they can simplify the calculations substantially, and affect the way the final answer is expressed. It will become clear in Chapter 5 why we make the choices we do, when a connection with graphs is established.

4.2.1 A_2

Choosing the reduced decomposition $w_L = r_2 r_1 r_2$, we need to calculate

$$X = \widehat{D}_L H = \widehat{D}_2 \widehat{D}_1 \widehat{D}_2 ([L_2 a_2] [L_1 a_1]). \quad (4.32)$$

First consider the application of \widehat{D}_2 . Since $[L_1 a_1]$ is \hat{r}_2 -invariant, it is unaffected. So,

$$\begin{aligned} \widehat{D}_2 ([L_2 a_2] [L_1 a_1]) &= [L_2 a_2] (1 + \bar{D}_2 L_2 a_2) [L_2 a_1 a_2^{-1}] [L_1 a_1] \\ &= [L_2 a_2] [L_2 a_1 a_2^{-1}] [L_1 a_1]. \end{aligned} \quad (4.33)$$

where we used (4.29), and $\bar{D}_2 L_2 a_2 = 0$.

When we apply \widehat{D}_1 , $[L_2 a_2]$ is untouched: $\widehat{D}_1 [L_2 a_2] = [L_2 a_2]$. Using the product

rule (4.24), we get

$$\begin{aligned} \widehat{D}_1 \left(\widehat{D}_2 ([L_2 a_2] [L_1 a_1]) \right) &= \underline{[L_2 a_2]} [L_2 a_1 a_2^{-1}] [L_1 a_1] (1 + \bar{D}_1 L_1 a_1) [L_1 a_1^{-1} a_2] \\ &\quad + \underline{[L_2 a_2]} [L_2 a_1 a_2^{-1}] \hat{d}_1 L_2 a_1 a_2^{-1} [L_2 a_1^{-1}] [L_1 a_1^{-1} a_2] , \end{aligned} \quad (4.34)$$

which simplifies since $\bar{D}_1 L_1 a_1 = 0$. At this point, we can save effort by anticipating the application of \widehat{D}_2 , and rewriting the result as

$$\begin{aligned} \widehat{D}_1 \left(\widehat{D}_2 ([L_2 a_2] [L_1 a_1]) \right) &= \widehat{D}_2 (\underline{[L_2 a_2]} [L_1 a_1]) [L_1 a_1^{-1} a_2] \\ &\quad + \underline{[L_2 a_2]} [L_2 a_1 a_2^{-1}] L_2 a_1^{-1} [L_2 a_1^{-1}] [L_1 a_1^{-1} a_2] . \end{aligned} \quad (4.35)$$

where $\hat{d}_1 L_2 a_1 a_2^{-1} = L_2 a_1^{-1}$. In this last expression, terms that are \hat{r}_2 -invariant are made plain by underlines, and all such terms are annihilated by \widehat{D}_2 . This procedure will save considerable work in more complicated cases.

For A_2 , we therefore find

$$\begin{aligned} &\widehat{D}_2 \left(\widehat{D}_1 \left(\widehat{D}_2 ([L_2 a_2] [L_1 a_1]) \right) \right) \\ &= \widehat{D}_2 (\underline{[L_2 a_2]} [L_1 a_1]) [L_1 a_1^{-1} a_2] (1 + \bar{D}_2 L_1 a_1^{-1} a_2) [L_1 a_2^{-1}] \\ &\quad + \underline{[L_2 a_2]} [L_2 a_1 a_2^{-1}] L_2 a_1^{-1} [L_2 a_1^{-1}] [L_1 a_1^{-1} a_2] \\ &\quad \quad \quad \times (1 + \bar{D}_2 L_1 a_1^{-1} a_2) [L_1 a_2^{-1}] . \end{aligned} \quad (4.36)$$

Since $\bar{D}_2 L_1 a_1^{-1} a_2$ vanishes, the final result is

$$\begin{aligned} X &= \widehat{D}_2 \widehat{D}_1 \widehat{D}_2 ([L_2 a_2] [L_1 a_1]) \\ &= [L_2 a_2] [L_2 a_1 a_2^{-1}] [L_1 a_1] [L_1 a_1^{-1} a_2] [L_1 a_2^{-1}] \\ &\quad + [L_2 a_2] [L_2 a_1 a_2^{-1}] L_2 a_1^{-1} [L_2 a_1^{-1}] [L_1 a_1^{-1} a_2] [L_1 a_2^{-1}] . \end{aligned} \quad (4.37)$$

Incidentally, the same result can be found by

$$\begin{aligned}
Y &= \widehat{D}_L \mathcal{Y} = \widehat{D}_1 \widehat{D}_2 \widehat{D}_1 \mathcal{Y} \\
&= \widehat{D}_1 \widehat{D}_2 \{ (1 - L_2 a_1 a_2^{-1}) (1 - L_2 a_1^{-1}) [\widehat{D}_1 (1 - L_1 a_1^{-1} a_2)] (1 - L_2 a_2^{-1}) \} \\
&= \widehat{D}_1 \widehat{D}_2 \{ (1 - L_2 a_1 a_2^{-1}) (1 - L_2 a_1^{-1}) (1 - L_2 a_2^{-1}) \} \\
&= \widehat{D}_1 \{ (1 - L_2 a_1^{-1}) (1 - L_1 a_2)^{-1} + (1 - L_2 a_2) (1 - L_2 a_1^{-1}) L_1 a_2^{-1} \} \\
&= (1 - L_1 a_2^{-1}) + (1 - L_2 a_2) L_1 a_2^{-1}, \tag{4.38}
\end{aligned}$$

for example.

The structure of the generating functions is more easily seen if we write

$$\begin{aligned}
A &= L_2 a_2, \quad B = L_2 a_1 a_2^{-1}, \quad C = L_2 a_1^{-1}, \\
D &= L_1 a_1, \quad E = L_1 a_1^{-1} a_2, \quad F = L_1 a_2^{-1}, \tag{4.39}
\end{aligned}$$

so that

$$X = [A] [B] \left([D] + C [C] \right) [E] [F]. \tag{4.40}$$

First, reconsider the 3 choices of incompatible product displayed in (3.14). They are DC , EB and FA , respectively. Expanding the expression of (4.40) using (4.20) results in no terms involving the product DC . Choosing DC as the sole incompatible product for the A_2 case therefore leads to (4.40). Rewriting that expression as $Z^{-1} (1 - DC)$, where

$$Z^{-1} = [A] [B] [C] [D] [E] [F], \tag{4.41}$$

makes clear that the incompatibility DC is related to (4.40). Similarly, we can write

$$\begin{aligned}
X &= [C] [A] \left([E] + B [B] \right) [F] [D] = Z^{-1} (1 - EB) \\
&= [B] [C] \left([F] + A [A] \right) [D] [E] = Z^{-1} (1 - FA), \tag{4.42}
\end{aligned}$$

corresponding to the other 2 choices EB and FA , respectively, for the incompatible product.

4.2.2 B_2

The longest element of the B_2 Weyl group has the reduced decompositions $w_L = r_1 r_2 r_1 r_2 = r_2 r_1 r_2 r_1$. Choosing the first, we write

$$X = \widehat{D}_L H = \widehat{D}_1 \widehat{D}_2 \widehat{D}_1 \widehat{D}_2 ([L_2 a_2] [L_1 a_1]) . \quad (4.43)$$

Applying (4.29), we get

$$\widehat{D}_2 ([L_2 a_2] [L_1 a_1]) = [L_2 a_2] [L_2 a_1 a_2^{-1}] [L_1 a_1] , \quad (4.44)$$

so that

$$\begin{aligned} \widehat{D}_1 \widehat{D}_2 ([L_2 a_2] [L_1 a_1]) &= \\ & [L_2 a_2] [L_2 a_1 a_2^{-1}] [L_1 a_1] (1 + \bar{D}_1 L_1 a_1) [L_1 a_1^{-1} a_2^2] \\ & + [L_2 a_2] [L_2 a_1 a_2^{-1}] \hat{d}_1 L_2 a_1 a_2^{-1} [L_2 a_1^{-1} a_2] [L_1 a_1^{-1} a_2^2] , \end{aligned} \quad (4.45)$$

where (4.24) was used. This simplifies because $\bar{D}_1 L_1 a_1 = 0$. Here again we can rewrite for manifest r_2 -invariance, before applying \widehat{D}_2 :

$$\begin{aligned} \widehat{D}_1 \widehat{D}_2 ([L_2 a_2] [L_1 a_1]) &= \underline{\widehat{D}_2 ([L_2 a_2] [L_1 a_1])} [L_1 a_1^{-1} a_2^2] \\ & + \underline{[L_2 a_2] [L_2 a_1 a_2^{-1}]} L_2 a_1^{-1} a_2 [L_2 a_1^{-1} a_2] [L_1 a_1^{-1} a_2^2] , \end{aligned} \quad (4.46)$$

where we have substituted $\hat{d}_1 L_2 a_1 a_2^{-1} = L_2 a_1^{-1} a_2$. Then using $\hat{d}_2 \hat{D}_2 = 0$, and that \hat{d}_2 annihilates all r_2 -invariant terms, we get

$$\begin{aligned}
& \hat{D}_2 \hat{D}_1 \hat{D}_2 ([L_2 a_2] [L_1 a_1]) \\
&= \underline{\hat{D}_2 ([L_2 a_2] [L_1 a_1])} [L_1 a_1^{-1} a_2^2] (1 + \bar{D}_2 L_1 a_1^{-1} a_2^2) [L_1 a_1 a_2^{-2}] \\
&+ \underline{[L_2 a_2] [L_2 a_1 a_2^{-1}]} L_2 a_1^{-1} a_2 [L_2 a_1^{-1} a_2] [L_1 a_1^{-1} a_2^2] \\
&\quad \times (1 + \bar{D}_2 L_1 a_1^{-1} a_2^2) [L_1 a_1 a_2^{-2}] \\
&+ \underline{[L_2 a_2] [L_2 a_1 a_2^{-1}]} L_2 a_1^{-1} a_2 [L_2 a_1^{-1} a_2] \hat{d}_2 L_2 a_1^{-1} a_2 \\
&\quad \times [L_2 a_2^{-1}] [L_1 a_1 a_2^{-2}] \\
&+ \underline{[L_2 a_2] [L_2 a_1 a_2^{-1}]} \hat{d}_2 L_2 a_1^{-1} a_2 [L_2 a_2^{-1}] [L_1 a_1 a_2^{-2}]. \tag{4.47}
\end{aligned}$$

Using (4.46), this can be rewritten as

$$\begin{aligned}
& \underline{\hat{D}_1 \hat{D}_2 ([L_2 a_2] [L_1 a_1])} (1 + \bar{D}_2 L_1 a_1^{-1} a_2^2) [L_1 a_1 a_2^{-2}] \\
&+ \underline{[L_2 a_2] [L_2 a_1 a_2^{-1}] [L_2 a_1^{-1} a_2] \hat{d}_2 L_2 a_1^{-1} a_2 [L_2 a_2^{-1}] [L_1 a_1 a_2^{-2}]} . \tag{4.48}
\end{aligned}$$

Notice that $\bar{D}_2 L_1 a_1^{-1} a_2^2 = L_1$. Taking account of the r_1 -invariant terms, it is now simple to derive

$$\begin{aligned}
& \hat{D}_1 \hat{D}_2 \hat{D}_1 \hat{D}_2 ([L_2 a_2] [L_1 a_1]) \\
&= \underline{\hat{D}_1 \hat{D}_2 ([L_2 a_2] [L_1 a_1])} (1 + \bar{D}_2 L_1 a_1^{-1} a_2^2) [L_1 a_1 a_2^{-2}] [L_1 a_1^{-1}] \\
&+ \underline{[L_2 a_2] [L_2 a_1 a_2^{-1}] [L_2 a_1^{-1} a_2] \hat{d}_2 L_2 a_1^{-1} a_2 [L_2 a_2^{-1}]} \\
&\quad \times [L_1 a_1 a_2^{-2}] [L_1 a_1^{-1}]. \tag{4.49}
\end{aligned}$$

Finally, substituting for $\widehat{D}_1\widehat{D}_2 ([L_2a_2] [L_1a_1])$ from above, we get

$$\begin{aligned}
& \widehat{D}_1\widehat{D}_2\widehat{D}_1\widehat{D}_2 ([L_2a_2] [L_1a_1]) \\
&= [L_2a_2] [L_2a_1a_2^{-1}] [L_1a_1] [L_1a_1^{-1}a_2^2] \\
&\quad \times (1 + \bar{D}_2L_1a_1^{-1}a_2^2) [L_1a_1a_2^{-2}] [L_1a_1^{-1}] \\
&+ [L_2a_2] [L_2a_1a_2^{-1}] L_2a_1^{-1}a_2 [L_2a_1^{-1}a_2] [L_1a_1^{-1}a_2^2] \\
&\quad \times (1 + \bar{D}_2L_1a_1^{-1}a_2^2) [L_1a_1a_2^{-2}] [L_1a_1^{-1}] \\
&+ [L_2a_2] [L_2a_1a_2^{-1}] [L_2a_1^{-1}a_2] L_2a_2^{-1} [L_2a_2^{-1}] \\
&\quad \times [L_1a_1a_2^{-2}] [L_1a_1^{-1}] . \tag{4.50}
\end{aligned}$$

Using the notation

$$\begin{aligned}
A &= L_2a_2, \quad B = L_2a_1a_2^{-1}, \quad C = L_2a_1^{-1}a_2, \quad D = L_2a_2^{-1}, \\
E &= L_1a_1, \quad F = L_1a_1^{-1}a_2^2, \quad G = L_1a_1a_2^{-2}, \quad H = L_1a_1^{-1}, \tag{4.51}
\end{aligned}$$

the B_2 character generator takes a compact form. Defining

$$[AB] := [A][B] , \tag{4.52}$$

and similarly for more than two factors, we can write

$$\begin{aligned}
X &= [ABC] D [DGH] \\
&+ [AB] C [CF] (1+z) [GH] \\
&+ [ABEF] (1+z) [GH] . \tag{4.53}
\end{aligned}$$

Here we have also defined $z := \bar{D}_2L_1a_1^{-1}a_2^2 = L_1$ for the sole inside generator required for the B_2 generating function.

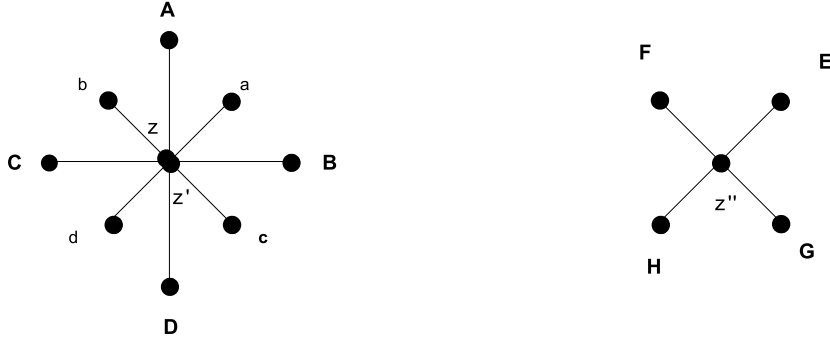


Figure 4.3: The B_2 fundamental weight diagrams. The weights are labelled by the corresponding elements of the integrity basis I_X .

The weights of the generators $A - H$ and z are depicted in Fig. 4.3. For all character generators, the generator weights fill out the r fundamental weight diagrams of the relevant rank- r simple Lie algebra. The two fundamental weight diagrams of B_2 are shown in the Figure.

The form of the character generator can be related to the set of incompatible products obtained above. In terms of the integrity basis elements, the choice (3.17) gives $\{EC, ED, FD, zD\}$ as the set of incompatible products. It is easily seen that the expression (4.53) does not contain these products, but does contain all other products quadratic in the elements of $\{A, B, \dots, F, z\}$.

4.2.3 G_2

For G_2 , $w_L = r_1 r_2 r_1 r_2 r_1 r_2 = r_2 r_1 r_2 r_1 r_2 r_1$ are the two reduced decompositions of the longest element of the Weyl group. We will calculate

$$X = \widehat{D}_L H = \widehat{D}_1 \widehat{D}_2 \widehat{D}_1 \widehat{D}_2 \widehat{D}_1 \widehat{D}_2 ([L_2 a_2] [L_1 a_1]). \quad (4.54)$$

First, since $\bar{D}_2 L_2 a_2 = 0$,

$$\widehat{D}_2 ([L_2 a_2] [L_1 a_1]) = [L_2 a_2] [L_2 a_1 a_2^{-1}] [L_1 a_1] , \quad (4.55)$$

using (4.29). Since $[L_2 a_2]$ is r_1 -invariant, applying \widehat{D}_1 to this last result gives

$$\begin{aligned} & \widehat{D}_1 \widehat{D}_2 ([L_2 a_2] [L_1 a_1]) \\ &= [L_2 a_2] [L_2 a_1 a_2^{-1}] [L_1 a_1] [L_1 a_1^{-1} a_2^3] \\ &+ [L_2 a_2] [L_2 a_1 a_2^{-1}] \hat{d}_1 L_2 a_1 a_2^{-1} [L_2 a_1^{-1} a_2^2] [L_1 a_1^{-1} a_2^3] , \end{aligned} \quad (4.56)$$

using the product rule (4.24), as usual. $\hat{d}_1 L_2 a_1 a_2^{-1}$ is just $L_2 a_1^{-1} a_2^2$.

We will try to keep track of the invariant terms as we proceed. This will simplify the calculations greatly. We rewrite the above equation as

$$\begin{aligned} & \widehat{D}_1 \widehat{D}_2 ([L_2 a_2] [L_1 a_1]) \\ &= \underline{\widehat{D}_2 ([L_2 a_2] [L_1 a_1])} [L_1 a_1^{-1} a_2^3] \\ &+ \underline{[L_2 a_2] [L_2 a_1 a_2^{-1}]} L_2 a_1^{-1} a_2^2 [L_2 a_1^{-1} a_2^2] [L_1 a_1^{-1} a_2^3] , \end{aligned} \quad (4.57)$$

where the underlines indicate r_2 -invariant factors. Applying \widehat{D}_2 then gives

$$\begin{aligned} & \widehat{D}_2 \widehat{D}_1 \widehat{D}_2 ([L_2 a_2] [L_1 a_1]) \\ &= \underline{\widehat{D}_2 ([L_2 a_2] [L_1 a_1])} [L_1 a_1^{-1} a_2^3] (1 + \bar{D}_2 L_1 a_1^{-1} a_2^3) [L_1 a_1^2 a_2^{-3}] \\ &+ \underline{[L_2 a_2] [L_2 a_1 a_2^{-1}]} L_2 a_1^{-1} a_2^2 [L_2 a_1^{-1} a_2^2] [L_1 a_1^{-1} a_2^3] \\ &\quad \times (1 + \bar{D}_2 L_1 a_1^{-1} a_2^3) [L_1 a_1^2 a_2^{-3}] \\ &+ \underline{[L_2 a_2] [L_2 a_1 a_2^{-1}]} [L_2 a_1^{-1} a_2^2] \hat{d}_2 L_2 a_1^{-1} a_2^2 [L_2 a_1 a_2^{-2}] [L_1 a_1^2 a_2^{-3}] . \end{aligned} \quad (4.58)$$

Using (4.57), this expression can be simplified to

$$\begin{aligned}
& \widehat{D}_2 \widehat{D}_1 \widehat{D}_2 ([L_2 a_2] [L_1 a_1]) \\
&= \widehat{D}_1 \widehat{D}_2 ([L_2 a_2] [L_1 a_1]) (1 + \bar{D}_2 L_1 a_1^{-1} a_2^3) [L_1 a_1^2 a_2^{-3}] \\
&+ \underline{[L_2 a_2] [L_2 a_1 a_2^{-1}] [L_2 a_1^{-1} a_2^2]} \hat{d}_2 L_2 a_1^{-1} a_2^2 [L_2 a_1 a_2^{-2}] [L_1 a_1^2 a_2^{-3}] , \quad (4.59)
\end{aligned}$$

where the underlines now indicate r_1 -invariant factors, in preparation for the application of \widehat{D}_1 .

When \widehat{D}_1 is applied, we get

$$\begin{aligned}
& \widehat{D}_1 \widehat{D}_2 \widehat{D}_1 \widehat{D}_2 ([L_2 a_2] [L_1 a_1]) \\
&= \widehat{D}_1 \widehat{D}_2 ([L_2 a_2] [L_1 a_1]) (1 + \bar{D}_2 L_1 a_1^{-1} a_2^3) \widehat{D}_1 [L_1 a_1^2 a_2^{-3}] \\
&+ \widehat{D}_1 \widehat{D}_2 ([L_2 a_2] [L_1 a_1]) \hat{d}_1 \bar{D}_2 (L_1 a_1^{-1} a_2^3) [L_1 a_1^{-2} a_2^3] \\
&+ \underline{[L_2 a_2] [L_2 a_1 a_2^{-1}] [L_2 a_1^{-1} a_2^2]} \hat{d}_2 L_2 a_1^{-1} a_2^2 [L_2 a_1 a_2^{-2}] \\
&\hspace{20em} \times \widehat{D}_1 [L_1 a_1^2 a_2^{-3}] \\
&+ \underline{[L_2 a_2] [L_2 a_1 a_2^{-1}] [L_2 a_1^{-1} a_2^2]} \hat{d}_2 L_2 a_1^{-1} a_2^2 [L_2 a_1 a_2^{-2}] \\
&\hspace{10em} \times \hat{d}_1 L_2 a_1 a_2^{-2} [L_2 a_1^{-1} a_2] [L_1 a_1^{-2} a_2^3] \\
&+ \underline{[L_2 a_2] [L_2 a_1 a_2^{-1}] [L_2 a_1^{-1} a_2^2]} \hat{d}_1 \hat{d}_2 L_2 a_1^{-1} a_2^2 \\
&\hspace{10em} \times [L_2 a_1^{-1} a_2] [L_1 a_1^{-2} a_2^3] . \quad (4.60)
\end{aligned}$$

Using the results above, this becomes

$$\begin{aligned}
& \widehat{D}_1 \widehat{D}_2 \widehat{D}_1 \widehat{D}_2 ([L_2 a_2] [L_1 a_1]) \\
&= \underline{\widehat{D}_2 \widehat{D}_1 \widehat{D}_2 ([L_2 a_2] [L_1 a_1])} (1 + \bar{D}_1 L_1 a_1^2 a_2^{-3}) [L_1 a_1^{-2} a_2^3] \\
&+ \left(\widehat{D}_1 \widehat{D}_2 ([L_2 a_2] [L_1 a_1]) \right) \hat{d}_1 \bar{D}_2 (L_1 a_1^{-1} a_2^3) [L_1 a_1^{-2} a_2^3] \\
&+ \underline{[L_2 a_2] [L_2 a_1 a_2^{-1}] [L_2 a_1^{-1} a_2^2]} (1 + \widehat{D}_2 L_2 a_1^{-1} a_2^2) [L_2 a_1 a_2^{-2}] \\
&\quad \times L_2 a_1^{-1} a_2 [L_2 a_1^{-1} a_2] [L_1 a_1^{-2} a_2^3] , \tag{4.61}
\end{aligned}$$

where now the underlines indicate r_2 -invariant factors.

Letting \widehat{D}_2 act, we obtain

$$\begin{aligned}
& \widehat{D}_2 \widehat{D}_1 \widehat{D}_2 \widehat{D}_1 \widehat{D}_2 ([L_2 a_2] [L_1 a_1]) \\
&= \underline{\widehat{D}_2 \widehat{D}_1 \widehat{D}_2 ([L_2 a_2] [L_1 a_1])} (1 + \bar{D}_1 L_1 a_1^2 a_2^{-3}) \widehat{D}_2 [L_1 a_1^{-2} a_2^3] \\
&+ \underline{\widehat{D}_2 \widehat{D}_1 \widehat{D}_2 ([L_2 a_2] [L_1 a_1])} \hat{d}_2 \bar{D}_1 (L_1 a_1^2 a_2^{-3}) [L_1 a_1 a_2^{-3}] \\
&+ \left(\widehat{D}_1 \widehat{D}_2 ([L_2 a_2] [L_1 a_1]) \right) \hat{d}_1 \bar{D}_2 (L_1 a_1^{-1} a_2^3) \widehat{D}_2 [L_1 a_1^{-2} a_2^3] \\
&+ \left(\widehat{D}_1 \widehat{D}_2 ([L_2 a_2] [L_1 a_1]) \right) \bar{D}_2 \hat{d}_1 \bar{D}_2 (L_1 a_1^{-1} a_2^3) [L_1 a_1 a_2^{-3}] \\
&+ \left(\widehat{D}_2 \widehat{D}_1 \widehat{D}_2 ([L_2 a_2] [L_1 a_1]) \right) \hat{r}_2 \hat{d}_1 \bar{D}_2 (L_1 a_1^{-1} a_2^3) [L_1 a_1 a_2^{-3}] \\
&+ \underline{[L_2 a_2] [L_2 a_1 a_2^{-1}] [L_2 a_1^{-1} a_2^2]} (1 + \bar{D}_2 L_2 a_1^{-1} a_2^2) [L_2 a_1 a_2^{-2}] \\
&\quad \times L_2 a_1^{-1} a_2 [L_2 a_1^{-1} a_2] \widehat{D}_2 [L_1 a_1^{-2} a_2^3] \\
&+ \underline{[L_2 a_2] [L_2 a_1 a_2^{-1}] [L_2 a_1^{-1} a_2^2]} (1 + \bar{D}_2 L_2 a_1^{-1} a_2^2) [L_2 a_1 a_2^{-2}] \\
&\quad \times [L_2 a_1^{-1} a_2] \hat{d}_2 L_2 a_1^{-1} a_2 [L_2 a_2^{-1}] [L_1 a_1 a_2^{-3}] . \tag{4.62}
\end{aligned}$$

Here we have used our usual product rule (4.24), but also the identity

$$F(\hat{d}_i G) + (\hat{d}_i F)(\hat{r}_i G) = F(\bar{D}_i G) + (\widehat{D}_i F)(\hat{r}_i G) , \tag{4.63}$$

so that the $(\widehat{D}_2\widehat{D}_1\widehat{D}_2([L_2a_2] [L_1a_1]))$ term results.

This expression can be simplified. For example, $\hat{d}_2\bar{D}_1(L_1a_1^2a_2^{-3}) = 0$. Completing the calculation, and referring to (4.61), yields

$$\begin{aligned}
& \widehat{D}_2\widehat{D}_1\widehat{D}_2\widehat{D}_1\widehat{D}_2 [L_2a_2] [L_1a_1] \\
&= \underline{\widehat{D}_1\widehat{D}_2\widehat{D}_1\widehat{D}_2 [L_2a_2] [L_1a_1]}(1 + \bar{D}_2L_1a_1^{-2}a_2^3)[L_1a_1a_2^{-3}] \\
&+ \widehat{D}_1\widehat{D}_2 [L_2a_2] [L_1a_1]\underline{\bar{D}_2\hat{d}_1\bar{D}_2(L_1a_1^{-1}a_2^3)[L_1a_1a_2^{-3}]} \\
&+ \left(\widehat{D}_2\widehat{D}_1\widehat{D}_2([L_2a_2] [L_1a_1])\right)\hat{r}_2\hat{d}_1\bar{D}_2(L_1a_1^{-1}a_2^3)[L_1a_1a_2^{-3}] \\
&+ \underline{[L_2a_2] [L_2a_1a_2^{-1}] [L_2a_1^{-1}a_2^2]}(1 + \bar{D}_2L_2a_1^{-1}a_2^2)[L_2a_1a_2^{-2}] \\
&\quad \times \underline{[L_2a_1^{-1}a_2] L_2a_2^{-1}[L_2a_2^{-1}][L_1a_1a_2^{-3}]} , \tag{4.64}
\end{aligned}$$

where the r_1 -invariant terms have been underlined.

Finally, applying \widehat{D}_1 we obtain

$$\begin{aligned}
& \widehat{D}_1\widehat{D}_2\widehat{D}_1\widehat{D}_2\widehat{D}_1\widehat{D}_2 ([L_2a_2] [L_1a_1]) \\
&= \underline{\widehat{D}_1\widehat{D}_2\widehat{D}_1\widehat{D}_2([L_2a_2] [L_1a_1])}(1 + \bar{D}_2L_1a_1^{-2}a_2^3)\widehat{D}_1[L_1a_1a_2^{-3}] \\
&\quad + \underline{\widehat{D}_1\widehat{D}_2\widehat{D}_1\widehat{D}_2([L_2a_2] [L_1a_1])}\hat{d}_1\bar{D}_2(L_1a_1^{-2}a_2^3)[L_1a_1^{-1}] \\
&+ \widehat{D}_1\widehat{D}_2([L_2a_2] [L_1a_1])\underline{\bar{D}_2\hat{d}_1\bar{D}_2(L_1a_1^{-1}a_2^3)}\widehat{D}_1[L_1a_1a_2^{-3}] \\
&\quad + \underline{\widehat{D}_1\widehat{D}_2([L_2a_2] [L_1a_1])}\hat{d}_1\bar{D}_2\hat{d}_1\bar{D}_2(L_1a_1^{-1}a_2^3)[L_1a_1^{-1}] \\
&+ \left(\widehat{D}_2\widehat{D}_1\widehat{D}_2([L_2a_2] [L_1a_1])\right)\hat{r}_2\hat{d}_1\bar{D}_2(L_1a_1^{-1}a_2^3)\widehat{D}_1[L_1a_1a_2^{-3}] \\
&\quad + \left(\widehat{D}_2\widehat{D}_1\widehat{D}_2([L_2a_2] [L_1a_1])\right)\bar{D}_1\hat{r}_2\hat{d}_1\bar{D}_2(L_1a_1^{-1}a_2^3)[L_1a_1^{-1}] \\
&+ \left(\bar{D}_1(\widehat{D}_2\widehat{D}_1\widehat{D}_2 [L_2a_2] [L_1a_1])\right)\hat{r}_1\hat{r}_2\hat{d}_1\bar{D}_2(L_1a_1^{-1}a_2^3)[L_1a_1^{-1}] \\
&\quad + \underline{[L_2a_2] [L_2a_1a_2^{-1}] [L_2a_1^{-1}a_2^2]}(1 + \widehat{D}_2L_2a_1^{-1}a_2^2)[L_2a_1a_2^{-2}], \\
&\quad \times \underline{[L_1a_1^{-1}a_2]\hat{d}_2L_2a_1^{-1}a_2[L_2a_2^{-1}]\widehat{D}_1[L_1a_1a_2^{-3}]} . \tag{4.65}
\end{aligned}$$

In this expression, both $\hat{d}_1\bar{D}_2\hat{d}_1\bar{D}_2(L_1a_1^{-1}a_2^3)$ and $\bar{D}_1\hat{r}_2\hat{d}_1\bar{D}_2(L_1a_1^{-1}a_2^3)$ vanish. Also, the second and seventh summands on the right hand side cancel, because $\hat{d}_1\bar{D}_2(L_1a_1^{-2}a_2^3) = -\hat{r}_1\hat{r}_2\hat{d}_1\bar{D}_2(L_1a_1^{-1}a_2^3)$. Therefore, the above result reduces to

$$\begin{aligned}
& \underline{\hat{D}_1\hat{D}_2\hat{D}_1\hat{D}_2}([\underline{L_2a_2}] [\underline{L_1a_1}])(1 + \bar{D}_2L_1a_1^{-2}a_2^3) \hat{D}_1[L_1a_1a_2^{-3}] \\
& + \underline{\hat{D}_1\hat{D}_2}([\underline{L_2a_2}] [\underline{L_1a_1}])\bar{D}_2\hat{d}_1\bar{D}_2(L_1a_1^{-1}a_2^3)\hat{D}_1[L_1a_1a_2^{-3}] \\
& + \left(\hat{D}_2\hat{D}_1\hat{D}_2([\underline{L_2a_2}] [\underline{L_1a_1}])\right)\hat{r}_2\hat{d}_1\bar{D}_2(L_1a_1^{-1}a_2^3) \hat{D}_1[L_1a_1a_2^{-3}] \\
& + \underline{[\underline{L_2a_2}] [\underline{L_2a_1a_2^{-1}}] [\underline{L_2a_1^{-1}a_2^2}]}(1 + \hat{D}_2L_2a_1^{-1}a_2^2)[\underline{L_2a_1a_2^{-2}}] \\
& \quad \times \underline{[\underline{L_1a_1^{-1}a_2}] L_2a_2^{-1}[\underline{L_2a_2^{-1}}]} \hat{D}_1[L_1a_1a_2^{-3}] , \tag{4.66}
\end{aligned}$$

where $\hat{D}_1[L_1a_1a_2^{-3}] = [L_1a_1a_2^{-3}] [L_1a_1^{-1}]$.

Substituting for $\hat{D}_1\hat{D}_2\hat{D}_1\hat{D}_2([\underline{L_2a_2}] [\underline{L_1a_1}])$, $\hat{D}_1[L_1a_1a_2^{-3}]$, $\hat{D}_1\hat{D}_2([\underline{L_2a_2}] [\underline{L_1a_1}])$, and $\hat{D}_2\hat{D}_1\hat{D}_2([\underline{L_2a_2}] [\underline{L_1a_1}])$ from above, and defining

$$\begin{aligned}
A &= L_2a_2, \quad B = L_2a_1a_2^{-1}, \quad C = L_2a_1^{-1}a_2^2, \\
D &= L_2a_1a_2^{-2}, \quad E = L_2a_1^{-1}a_2, \quad F = L_2a_2^{-1}, \\
G &= L_1a_1, \quad H = L_1a_1^{-1}a_2^3, \quad I = L_1a_1^2a_2^{-3}, \\
J &= L_1a_1^{-2}a_2^3, \quad K = L_1a_1a_2^{-3}, \quad L = L_1a_1^{-1}, \tag{4.67}
\end{aligned}$$

we can write the character generator for G_2 as

$$\begin{aligned}
X &= \widehat{D}_1 \widehat{D}_2 \widehat{D}_1 \widehat{D}_2 \widehat{D}_1 \widehat{D}_2 ([A] [G]) \\
&= [A] [B] [G] [H] (1 + \bar{D}_2 H) [I] (1 + \bar{D}_1 I) [J] (1 + \bar{D}_2 J) [K] [L] \\
&\quad + [A] [B] C [C] [H] (1 + \bar{D}_2 H) [I] (1 + \bar{D}_1 I) [J] (1 + \bar{D}_2 J) [K] [L] \\
&\quad + [A] [B] [C] \hat{d}_2 C [D] [I] (1 + \bar{D}_1 I) [J] (1 + \bar{D}_2 J) [K] [L] \\
&\quad + [A] [B] [G] [H] \hat{d}_1 \bar{D}_2 H [J] (1 + \bar{D}_2 J) [K] [L] \\
&\quad + [A] [B] C [C] [H] \hat{d}_1 \bar{D}_2 H [J] (1 + \bar{D}_2 J) [K] [L] \\
&\quad + [A] [B] [C] (1 + \bar{D}_2 C) [D] E [E] [J] (1 + \bar{D}_2 J) [K] [L] \\
&\quad + [A] [B] [G] [H] \bar{D}_2 \hat{d}_1 \bar{D}_2 H [K] [L] \\
&\quad + [A] [B] C [C] [H] \bar{D}_2 \hat{d}_1 \bar{D}_2 H [K] [L] \\
&\quad + [A] [B] [G] [H] (1 + \bar{D}_2 H) [I] \hat{r}_2 \hat{d}_1 \bar{D}_2 H [K] [L] \\
&\quad + [A] [B] C [C] [H] (1 + \bar{D}_2 H) [I] \hat{r}_2 \hat{d}_1 \bar{D}_2 H [K] [L] \\
&\quad + [A] [B] [C] \hat{d}_2 C [D] [I] \hat{r}_2 \hat{d}_1 \bar{D}_2 H [K] [L] \\
&\quad + [A] [B] [C] (1 + \bar{D}_2 C) [D] [E] F [F] [K] [L] . \tag{4.68}
\end{aligned}$$

The weights of these outside elements (see (2.14)) of the integrity basis are depicted in Fig. 4.4, where the weight diagrams of the fundamental representations of G_2 are drawn.

The other weights are labelled by our notation for the corresponding inside generators.

The inside generators, the elements of I_{in} , make their appearance when we calculate

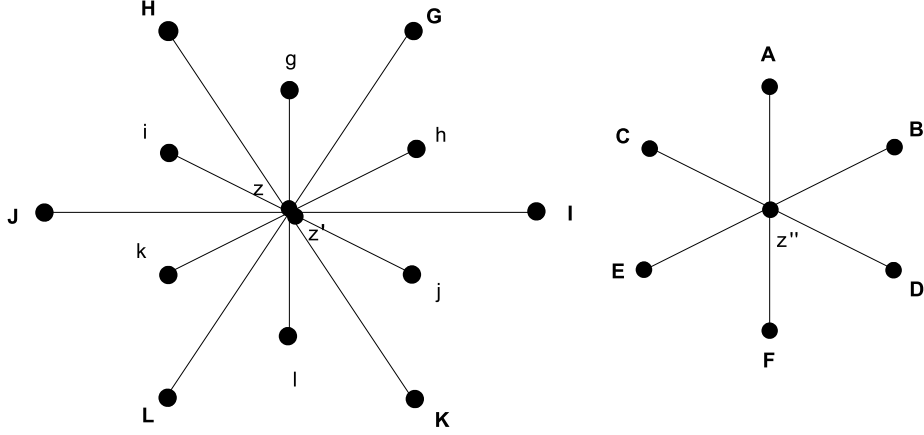


Figure 4.4: The G_2 fundamental weight diagrams. The weights are indicated by the elements of I_X .

numerator factors, such as $\hat{r}_2 \hat{d}_1 \bar{D}_2 H = j$. The final result is therefore

$$\begin{aligned}
X = & [ABGH](1+g+h)[I](1+z')[J](1+k+\ell)[KL] \\
& + [AB]C[CH](1+g+h)[I](1+z')[J](1+k+\ell)[KL] \\
& + [ABC](z''+D)[DI](1+z')[J](1+k+\ell)[KL] \\
& \quad + [ABGH]i[J](1+k+\ell)[KL] \\
& \quad + [AB]C[CH]i[J](1+k+\ell)[KL] \\
& + [ABC](1+z'')[D]E[EJ](1+k+\ell)[KL] \\
& \quad + [ABGH]z[KL] + [AB]C[CH]z[KL] \\
& \quad + [ABGH](1+g+h)[I]j[KL] \\
& \quad + [AB]C[CH](1+g+h)[I]j[KL] \\
& + [ABC](z''+D)[DI]j[KL] \\
& \quad + [ABC](1+z'')[DE]F[FKL]. \tag{4.69}
\end{aligned}$$

Here we have again shortened by using $[A][B] =: [AB]$, etc.

Now consider the choice of incompatible products underlying this expression for X written in terms of integrity basis elements. By inspecting (4.69), we can find the incompatibilities between outer generators:

$$\{GC, GD, GE, GF, HD, HE, HF, IE, IF, JF\} . \quad (4.70)$$

None of these products appears in the expansion of any of the terms of (4.69). Therefore, we write

$$Y_{\text{out,out}}^{(2)} = G(C + D + E + F) + H(D + E + F) + I(E + F) + JF . \quad (4.71)$$

Similarly, for inner-outer incompatible products, we find

$$\begin{aligned} Y_{\text{in,out}}^{(2)} &= z''(G + H) + (z + g + h + i)(D + E + F) + z'(E + F) \\ &\quad + j(E + F) + (k + \ell)F + z(I + J) + iI + jJ . \end{aligned} \quad (4.72)$$

Compatible inner-inner products give

$$Y_{\text{in,in}}^{(2)} = (z'' + g + h)(j + k + \ell + z') + i(k + \ell + z') + z'(k + \ell) . \quad (4.73)$$

Substituting these last 3 results into (3.20) verifies the result (3.18) derived from the general formula (2.21).

The same results, (4.40), (4.53), and (4.69), can be obtained by making the choices $\widehat{D}_1 \widehat{D}_2 \widehat{D}_1$ ($[D] [A]$), $\widehat{D}_2 \widehat{D}_1 \widehat{D}_2 \widehat{D}_1$ ($[E] [A]$), and $\widehat{D}_2 \widehat{D}_1 \widehat{D}_2 \widehat{D}_1 \widehat{D}_2 \widehat{D}_1$ ($[G] [A]$) for A_2 , B_2 and

G_2 respectively. It is easy to show that for A_2 and B_2 . For G_2 we find

$$\begin{aligned}
X = & [GH](1+g+h)[I](1+z')[J](1+k+l)[K]L[LEF] \\
& + [GH](1+g+h)[I](1+z')[J](k+l+K)[KDEF] \\
& + [GH](1+g+h)[I](z'+J)[JC](1+z'')[DEF] \\
& + [GH](g+h+I)[IBC](1+z'')[DEF] \\
& + [GHABC](1+z'')[DEF] \\
& + [GH]i[JC](1+z'')[DEF] \\
& + [GH]i[J](k+l+K)[KDEF] \\
& + [GH]i[J](1+k+l)[K]L[LEF] \\
& + [GH](1+g+h)[I]j[KDEF] \\
& + [GH](1+g+h)[I]j[K]L[LEF] \\
& + [GH]z[KDEF] + [GH]z[K]L[LEF]
\end{aligned} \tag{4.74}$$

We have checked that this expression equals the result (4.69).

Chapter 5

Character generators, semi-standard tableaux, posets and graphs

5.1 Character generators, semi-standard tableaux, posets and graphs

By (2.2), the character generator is the generating function of the multiplicities $\text{mult}_\lambda(\sigma)$:

$$X = \sum_{\lambda \in P_{\geq}} \sum_{\sigma \in P} L^\lambda a^\sigma \text{mult}_\lambda(\sigma) . \quad (5.1)$$

Many combinatorial ways of calculating such multiplicities are known, including those involving Young tableaux and variants. This “microscopic” point of view leads to an improved understanding of the structure of the character generators. The first to exploit this fact was Stanley [24], for the algebras $A_r \cong su(r+1)$. King [18] extended Stanley’s work to include

the algebras $C_r \cong sp(2r)$.

Most relevant to us, however, was the connection made explicit by Baclawski [6] to certain partially-ordered sets, or posets, related to tableaux. A poset is a set, together with a binary operation (partial order) \geq , satisfying reflexivity ($x \geq x$, $\forall x \in P$), antisymmetry (if $x \geq y$ and $y \geq x$, then $x = y$), and transitivity (if $x \geq y$ and $y \geq z$, then $x \geq z$). It is a partial order because two elements $x, y \in P$ can be incomparable, i.e. neither $x \geq y$ nor $y \geq x$ is true.

The connection with posets was already made in [24], but much less directly than in [6]. Baclawski emphasized its importance and wrote explicit formulas in terms of poset objects. Later these considerations were generalized to all the classical Lie algebras A_r, B_r, C_r, D_r (or all $su(N), so(N), sp(2N)$) in [19, 20], using generalized Young tableaux.* At about the same time, Baclawski and Towber [7] treated the exceptional G_2 algebra by introducing a generalization of a poset.

In the remainder of this section, we will treat the algebras A_r, B_2 and G_2 in turn. This first case is the simplest, and best understood.

5.1.1 $A_r \cong su(r + 1)$

Certain posets are encoded in the structure of Young tableaux, and related objects. For example, consider the algebras $A_r \cong su(r + 1)$. Their multiplicities $\text{mult}_\lambda(\sigma)$ equal the number of semi-standard Young tableaux of shape λ and weight σ (see [11], e.g.). These Young tableaux can be constructed by joining together the semi-standard tableaux of the fundamental representation, and these fundamental tableaux become the columns of the full semi-standard tableaux. The only complication is that they must be placed in a certain

*For a discussion of character generators and tableaux methods with a different emphasis, see [9].

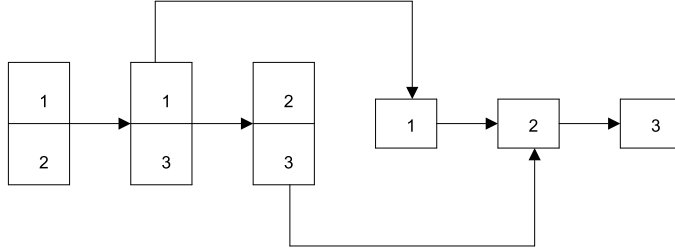


Figure 5.1: $su(3)$ Hasse diagram with the fundamental semi-standard tableaux.

order.

More precisely, the columns of the semi-standard tableaux, the fundamental tableaux, are the elements of a poset P . The partial order can be encoded in a so-called Hasse diagram, a graph whose vertices are the elements of the poset, and whose edges indicate the order (see [25], e.g.). The poset P is locally finite, meaning it has an order that is completely determined by its cover relations. $x > y$ is a cover relation if no poset element z exists such that $x > z > y$. To every cover relation $x > y$ of the poset P , there is an edge $\{x, y\}$ in its Hasse diagram $\mathcal{H}(P)$.

The Hasse diagram relevant to $su(3)$ semi-standard tableaux is drawn in Figure 5.1, with the fundamental semi-standard tableaux drawn where the corresponding vertices would be. They are lined up horizontally, to make obvious the connection with the semi-standard tableaux for $su(3)$. Any number of copies of the fundamental tableaux of each kind can be used to build a valid semi-standard tableaux, as long as the partial order is respected.

Recall that the weight of an integrity basis element $L^\Lambda a^\mu$, is μ (while the fundamental weight Λ is its shape). The weights of the fundamental semi-standard tableaux are the weights of the integrity basis elements (4.39) for the character generator. The Hasse

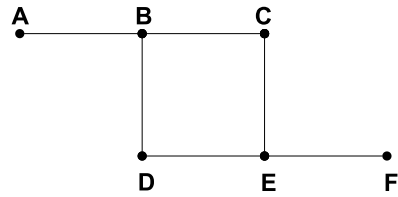


Figure 5.2: Hasse diagram of the $su(3)$ fundamental poset.

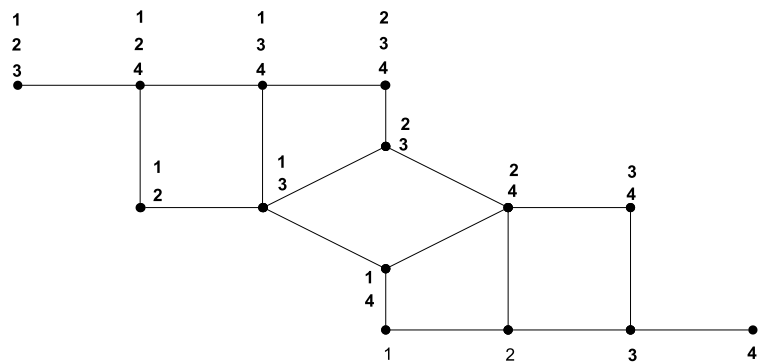


Figure 5.3: Hasse diagram of the $su(4)$ fundamental poset, with the fundamental tableaux indicated next to the corresponding vertices.

diagram can be labelled by those basis elements, and then the diagram provides a method of constructing the generating function directly. For $su(3)$, the resulting Hasse diagram is the first one drawn in Figure 5.2. The corresponding $su(4) \cong A_3$ Hasse diagram is shown in Fig. 5.3.

Consider the $su(3)$ character generator (4.40). The two terms are easily seen to correspond to the two longest paths (or walks) $ABDEF$ and $ABCEF$ on the Hasse diagram from “the beginning” A , or the greatest element, to “the end”, or least element, F . These two paths correspond to the two maximal chains, or totally ordered sets, $A \geq B \geq D \geq$

$E \geq F$ and $A \geq B \geq C \geq E \geq F$ in the corresponding poset. The two maximal chains are treated in equal fashion, since

$$[D] + C [C] = 1 + D [D] + C [C] . \quad (5.2)$$

From this last expression, one can see that the extra factor of C in (4.40) is necessary to avoid over-counting.

The point of view just explained was discovered by Baclawski [6], and applied to the simple algebras A_r and C_r (as well as to $U(N)$). Using generalized tableaux, character generators for all classical algebras (A_r, B_r, C_r , and D_r) were studied in [19, 20].

Let us write Baclawski's results, concentrating on the case of A_r . Denote by P the poset with fundamental tableaux as elements, the so-called fundamental poset. We can label the elements of the poset with the corresponding elements of the integrity basis I_X , as in the first diagram of Fig. 5.2 for $su(3)$. The result is called a labelling of the poset P , since the labels can be added and multiplied. A multi-chain is a chain with repeated elements, such as $m = A \geq A \geq B \geq D \geq D \geq E \geq F \geq F \geq F$, where, by abuse of notation, we use the labels to denote the poset elements. The label of such a multi-chain is easily obtained:

$$\ell(m) = \ell(A \geq A \geq B \geq D \geq D \geq E \geq F \geq F \geq F) = A^2 B D^2 F^3 . \quad (5.3)$$

The first result is simply written as

$$X = \sum_{m \in M(P)} \ell(m) , \quad (5.4)$$

where $M(P)$ denotes the set of multi-chains of P .

As pointed out above, the relevance to X of maximal chains is immediately obvious. To write the formula [6] that makes the connection explicit, consider the poset \hat{P} , the

extended fundamental poset, obtained by adjoining two new elements, $\hat{0}$ and $\hat{1}$, to the poset P . The element $\hat{0}$ satisfies $x \geq \hat{0}$, and $\hat{1}$ obeys $\hat{1} \geq x$, both for all $x \in \hat{P}$. The labelling of \hat{P} that we use is simply obtained by adjoining the labels $\ell(\hat{0}) = \ell(\hat{1}) = 1$ to the labelling of P .

The links of a poset are relevant here. A chain C of a poset P is called saturated if no $z \in P \setminus C$ exists such that $x \geq z \geq y$ for $x, y \in C$, such that $C \cup \{z\}$ is a chain. Roughly speaking, there are no gaps in a saturated chain. A cover relation is a two-element saturated chain, and a link is a saturated chain with three-elements.

Let $\text{Link}(\hat{P})$ denote the set of links of \hat{P} . For the $su(3)$ case, we have

$$\begin{aligned} \text{Link}(\hat{P}) = \{ & \hat{1} > A > B, A > B > C, A > B > D, B > C > E, \\ & B > D > E, C > E > F, D > E > F, E > F > \hat{0} \}. \end{aligned} \quad (5.5)$$

A linking of a poset P is a partition of $\text{Link}(\hat{P})$ into two disjoint subsets $\text{Link}^\pm(\hat{P})$, such that, for every pair $x > y$ in \hat{P} , there exists a unique saturated chain $x = x_0 > x_1 > \cdots > x_{n-1} > x_n = y$, every link of which is in $\text{Link}^+(\hat{P})$. For the A_2 example, one linking of the poset P is specified by the choice

$$\text{Link}^-(\hat{P}) = \{ B > C > E \}. \quad (5.6)$$

Then $\text{Link}^+(\hat{P}) = \text{Link}(\hat{P}) \setminus \text{Link}^-(\hat{P})$.

Another concept required for the formula is that of a descent set $\mathcal{DS}(m)$, of a maximal chain $m = x_0 > x_1 > \cdots > x_n$ of \hat{P} :

$$\mathcal{DS}(m) := \{ x_i \mid 0 < i < n \text{ and } (x_{i-1} > x_i > x_{i+1}) \in \text{Link}^-(\hat{P}) \}. \quad (5.7)$$

Its label is therefore

$$\ell(\mathcal{DS}(m)) = \prod_{x \in \mathcal{DS}(m)} \ell(x). \quad (5.8)$$

Let $\text{Max}(\hat{P})$ denote the set of maximal chains in \hat{P} . Baclawski [6] proved

$$X = \sum_{m \in \text{Max}(\hat{P})} [\ell(m)] \ell(\mathcal{DS}(m)) . \quad (5.9)$$

For the extended poset \hat{P} relevant to A_2 , $\text{Max}(\hat{P})$ contains two chains, $\hat{1} > A > B > D > E > F > \hat{0}$ and $\hat{1} > A > B > C > E > F > \hat{0}$. For the first, the linking specified by (5.6) gives a null descent set, while for the second maximal chain, the descent set is $\{C\}$. The formula (5.9) therefore immediately reproduces the result (4.40).

With the alternate choice $\text{Link}^-(\hat{P}) = \{B > D > E\}$, Baclawski's formula (5.9) yields $X = [ABCEF] + [AB]D[DEF]$, an equivalent form.

Notice that for both linkings of the poset P , the product CD is incompatible. Incompatibilities are fixed by the poset itself, rather than by a choice of linking of P . CD is an incompatible product because C and D are incomparable in the poset, as is clear from its Hasse diagram in Fig. 5.2.

We should mention that Baclawski [6] also derived formulas of a recursive nature, that lead to nested expressions for X . Considerably shortened expressions can result this way, since the sub-poset structure is taken into account. Since our goal is an understanding of the full character generator and corresponding (generalized) posets, however, we will not study those formulas here.

In Fig. 5.3 the Hasse diagram of the fundamental poset for $A_3 \cong su(4)$ is depicted, with the vertices labelled by the corresponding fundamental semi-standard tableaux. Using (5.4) or (5.9) on this diagram yields the A_3 character generator in straightforward fashion. Higher ranks involve larger fundamental posets and Hasse diagrams, but do not require any new important complications.

Clearly, the fundamental poset P encodes the essence of the character generator

X, for the algebras A_r . This poset can also be constructed without reference to semi-standard tableaux. The alternative construction uses the Weyl group and its Bruhat order (see [17], e.g.). That means it is more easily adaptable to general Lie algebras than are the semi-standard tableaux relevant to $A_r \cong su(r+1)$.

The elements of the poset are in one-to-one correspondence with the weights of the r fundamental representations of A_r . First consider the weights of a fixed fundamental representation $R(\Lambda)$, $\Lambda \in F$. Every such weight $\mu = w\Lambda$, for some $w \in W \cong S_{r+1}$, the Weyl group of A_r . For all weights in a fixed fundamental representation $R(\Lambda)$, the Bruhat order \prec on W determines the required partial order: $L^\Lambda a^{w\Lambda} \geq L^\Lambda a^{v\Lambda}$ for $w \prec v$.

To define the Bruhat order, recall that the length $\ell(w)$ of $w \in W$ is the number of primitive reflections in a reduced decomposition of w . Given w_1 and w_2 in W , write $w_2 \rightarrow w_1$ iff both $w_2 = r_\alpha w_1$ for some positive root α and $\ell(w_2) = \ell(w_1) + 1$. The Bruhat order \prec is given by putting $w \succ v$ iff there exist Weyl group elements w_1, w_2, \dots, w_n such that $w \rightarrow w_n \rightarrow \dots w_1 \rightarrow v$.

What about relations between generators with weights in different fundamental representations? They can be stated simply if the vertex corresponding to the weight μ in $R(\Lambda^j)$ is indicated by the triple $[\Lambda^j, \mu; w]$, with a fixed $w \in W$ adjoined obeying $\mu = w\Lambda^j$. Of course, there is an ambiguity in the choice of w for a fixed weight μ of $R(\Lambda^j)$. Consider, however, the reduced decomposition of w_L used in the Demazure calculation of the character generator. For $su(r+1)$, we can use

$$w_L = (r_r r_{r-1} \cdots r_1)(r_r r_{r-1} \cdots r_2)(r_r r_{r-1} \cdots r_3) \cdots (r_r r_{r-1})(r_r). \quad (5.10)$$

This expression motivates the label $[\Lambda^r, \Lambda^r; \text{id}]$ and, more generally,

$$[\Lambda^j, \Lambda^j; (r_r r_{r-1} \cdots r_{j-1}) \cdots (r_r r_{r-1})(r_r)] \quad (5.11)$$

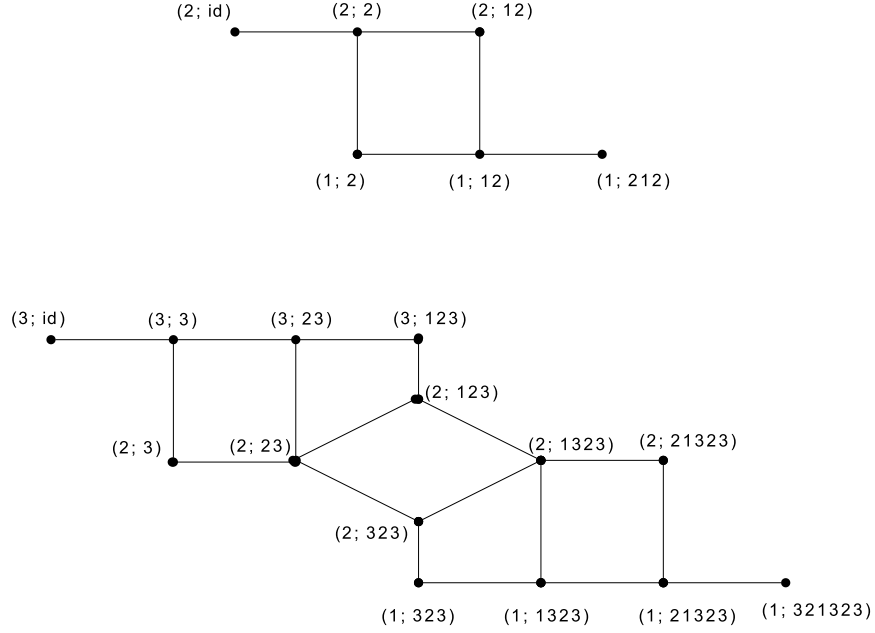


Figure 5.4: Hasse diagrams for the $su(3)$ and $su(4)$ fundamental posets. Next to the vertices, the corresponding integrity basis elements are indicated, by their shapes and a Weyl group element that maps their shape to their weight. For example, $(2; 123)$ indicates shape Λ^2 and weight $r_1 r_2 r_3 \Lambda^2 = -\Lambda^1 + \Lambda^3$, i.e., $L_2 a_1^{-1} a_3$.

for all the highest-weight vertices. Notice that the length of these Weyl elements increases as j decreases in Λ^j . That is, our choice of reduced decomposition for w_L induces a total order on the set F of fundamental weights.

Once the Weyl elements are fixed for the highest-weight vertices, the Bruhat order can then be used to assign Weyl elements to all the vertices of the required Hasse diagram. The edges of the Hasse diagram are then determined by the cover relations

$$[\Lambda^j, \mu; w] > [\Lambda^i, \nu; v], \quad \text{if } i = j \text{ and } w \prec v, \\ \text{or if } j = i + 1 \text{ and } w = v. \quad (5.12)$$

See Fig. 5.4 for illustrations of the cases A_2 and A_3 .

Before leaving the A_r algebras, we should mention that the semi-standard tableaux have significance for the vectors (states) in representations, not just their multiplicities. As is well known, a vector in an arbitrary irreducible representation of A_r can be constructed from the vectors of the fundamental representations, which are in turn constructible from the vectors of the first fundamental (basic) representation. The latter can be labelled by single boxes, numbered from 1 to $r + 1$. Totally antisymmetric j -fold tensor products of the basic vectors yield the vectors of the fundamental representation of highest weight Λ^j . Symmetrizing these, according to the rows of a fixed Young tableau, produces the vectors of the representation of highest weight equal to the tableau shape.

Consequently, the generating function X and the related fundamental poset P encode something of this construction.[†] Conversely, knowing how to construct the vectors from those of the fundamental representations, can tell us about the character generator X . As we will discuss below, the G_2 character generator was found this way in [7].

5.1.2 B_2

The A_r case is simple. All fundamental representations are minuscule, i.e., their weights form a single Weyl orbit $W\Lambda^j$. This means, in particular, that there are no inside generators in the case of A_r .

For the algebra B_2 , however, there is one inside generator, $z := \bar{D}_2 L_1 a_1^{-1} a_2^2 = L_1$. From the expression (4.53), it is clear that this inside generator z is not treated in the same way as the outer generators A, \dots, H . While X is linear in z , it contains arbitrarily high powers of each of the outside generators.

[†]Arguably, the most important use of the character generator is to tell us about this method of building vectors of highest-weight representations.

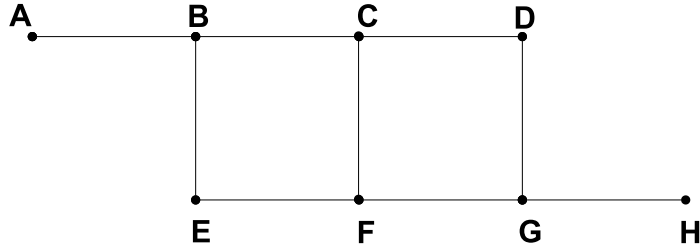


Figure 5.5: Hasse diagram of the fundamental-orbit poset of B_2 .

As in the A_2 case, however, the result (4.53) can be understood in terms of a graph related to a poset. We can use the same construction as for A_2 , including the cover relations (5.12), as long as only the elements of the Weyl orbits $W\Lambda^j$ are included in the poset. For the B_2 case, the Hasse diagram of this poset is drawn in Fig. 5.5. We will call the poset so constructed the fundamental-orbit poset P_o .

The connection of (4.53) to P_o is clear. If we modify the labelling of the maximal chains of P_o so that

$$\ell(\dots > F > G > \dots) = \dots F(1+z)G \dots, \quad (5.13)$$

then Baclawski's formula (5.9) still works. That is, in maximal chains, we introduce a labelling of the edge $\{F, G\}$ of the Hasse diagram connecting vertices F and G , that corresponds to the cover relation $F > G$.

The extra labelling has a Demazure interpretation: $z = \bar{D}_2F$ and

$$\widehat{D}_2[F] = [F] (1 + \bar{D}_2F) [\hat{r}_i F] = [F] (1 + z) [G]. \quad (5.14)$$

by eqn. (4.29).

The latter result shows that only the $\{F, G\}$ edge needs this extra factor, because no other generator other than F has a weight with a Dynkin label greater than 1. If

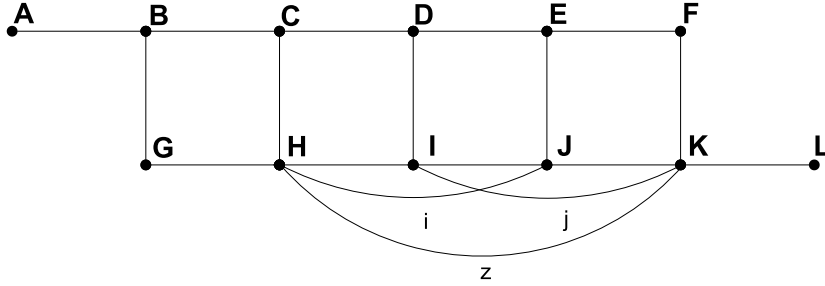


Figure 5.6: G_2 graph.

generator V has i -th Dynkin label equal to 1, then $\bar{D}_i V = 0$. We can, therefore extend the labelling to include all the edges of the Hasse diagram between vertices with weights in the same fundamental representation. Label with $(1 + \bar{D}_j V)$ the edge $\{V, \hat{r}_j V\}$. For all cases considered so far, except $V = F$, this label is just 1.

5.1.3 G_2

For G_2 , $\|I_{\text{in}}\| = 9$, so the situation becomes more complicated. That is made plain by looking at the final expression for X . The fundamental-orbit poset P_o has $\|I_{\text{out}}\| = 12$ elements, and is easily constructed. If, as for B_2 , we continue to label with $(1 + \bar{D}_j V)$ the edge $\{V, \hat{r}_j V\}$, there are many terms recognizable in (4.69) as coming from the maximal chains of P_o . However, there are many more terms in (4.69) than there are maximal chains in P_o .

To proceed, we introduce new edges to the Hasse diagram of P_o , and label the new edges as needed to produce the expression (4.69). The result is the graph of Figure

5.6, where the labels of the edges are indicated. Three new edges are required: the $\{H, J\}$ edge with label $\hat{d}_1 \bar{D}_2 H = i$, $\{I, K\}$ with label $\hat{r}_2 \hat{d}_1 \bar{D}_2 H = j$, and $\{H, K\}$ labelled by $\bar{D}_2 \hat{d}_1 \bar{D}_2 H = z$. Consider now maximal paths (or walks) on this graph, from the beginning vertex G to the end F . The terms of (4.69) can be put in one-to-one correspondence with the maximal paths, so that a formula of the Baclawski type can still be written, as long as the edge labels are included as factors.

The final element required is an explanation of the edge labels. They are sums of inside generators, but which ones? Their individual expressions in terms of Demazure operators are not particularly illuminating.

However, notice that the new edges only relate outside generators with weights from the same fundamental representation, say $R(\Lambda)$, for some $\Lambda \in F$. Focus on a vertex $L^\Lambda a^{v\Lambda} = \hat{v}(L^\Lambda a^\Lambda)$, where $v \in W$. It is clear that any inside generators with weights in $\hat{d}_v a^\Lambda$ must appear as labels of edges ending on that vertex.

Consider the label $\hat{d}_1 \bar{D}_2 H = i$ of the new $\{H, J\}$ edge. Since $J = \hat{r}_1 \hat{r}_2 \hat{r}_1 G$, we calculate

$$\hat{d}_{r_1 r_2 r_1} G = \hat{d}_1 \hat{d}_2 \hat{d}_1 G =: \hat{d}_{121} G = i + z' + J. \quad (5.15)$$

The inside generator i labels one edge ending on J , while $(1 + z')$ is the label of the other edge ending there.

This way, the labels of edges ending on a graph vertex can be found. To determine where the edges should begin, we simply reverse the process. Start with the outside generator of lowest weight, $L^\Lambda a^{wL\Lambda} = \hat{w}_L(L^\Lambda a^\Lambda)$, $\Lambda \in F$. To work backwards, we need to consider Demazure operators like the \hat{d}_m , but where the role of the simple root α_m is taken by its negative $-\alpha_m$. We denote such a Demazure operator by $\hat{d}_{\underline{m}}$, and also use the

convention that $\hat{d}_{\underline{\ell m}} := \hat{d}_{\underline{\ell}} \hat{d}_{\underline{m}}$, etc.

For the $\{H, J\}$ edge, the generator of lowest weight is L . We calculate

$$\hat{d}_{\underline{2121}} L = \hat{d}_{\underline{2}} \hat{d}_{\underline{1}} \hat{d}_{\underline{2}} \hat{d}_{\underline{1}} L = z + i + h + g + H . \quad (5.16)$$

Comparing (5.16) and (5.15), we see that only i is common, and so i will label the edge beginning at H and ending on J .

This procedure works for all the new edges. It is easy to find

$$\begin{aligned} \hat{d}_{2121} G &= z + j + k + \ell + K , \\ \hat{d}_{\underline{121}} L &= j + z' + I, \quad \hat{d}_{\underline{2121}} L = z + i + h + g + H . \end{aligned} \quad (5.17)$$

The first two of these results confirms that the $\{I, K\}$ edge is labelled by j ; the first and third give z as the $\{H, K\}$ label.

The nontrivial $\bar{D}_i V$ part of the labels $(1 + \bar{D}_i V)$ for the edges $\{V, \hat{r}_i V\}$ of the Hasse diagram of P_o that is contained in our graph, can also be obtained this way.

Our graph, as shown in Fig. 5.6, has an obvious resemblance to that devised long ago by Baclawski and Towber [7], depicted in Fig. 5.7. Every element of the integrity basis (both inside and outside generators) is represented by a vertex in that graph. The Baclawski-Towber graph is not the Hasse diagram of a poset, however, but rather its generalization for a generalized poset. The generalization is necessary because an inner generator ι appears at most linearly in X . That means ι^2 is an incompatible product. Incompatibilities correspond to incomparable elements of a poset, however, and the poset partial order \geq obeys the reflexivity property: $x \geq x$, for all x in a poset P . For x in a poset, then, x is always comparable to x .

If the partial order \geq is replaced by a binary relation \gg without reflexivity, how-

ever, a generator ι can be incomparable to itself. All the inner generators ι do not obey $\iota \gg \iota$, and so ι^2 can be an incompatible product. In the generalized poset, the inner generators are incompatible with themselves, while the outer generators are not. As a consequence, the vertices of the corresponding graph are not all treated on an equal footing. With this modification, however, formulas like the poset ones written by Baclawski [6] can also be written for graphs of the type in Fig. 5.7.

In contrast, we prefer to work with a graph more closely related to the Hasse diagram of the fundamental-orbit poset P_o . We do not increase the number of vertices by introducing new ones for every inner generator, i.e., for every integrity basis element with a weight not an element of a fundamental Weyl orbit $W\Lambda$, $\Lambda \in F$. Instead, we introduce edge labels involving the inner generators for the edges of the Hasse diagram of P_o , and add new edges (only) with such labels. In our opinion, the resulting graph is simpler than that of ref. [7]; compare of Figs. 5.6 and 5.7. We will call our graph and its generalization to other simple Lie algebras the character-generator graph, and denote it \mathcal{G}_X .

We should point out, however, that just as Baclawski and Towber treat the vertices for inner and outer generators differently, we do not treat all the edges of \mathcal{G}_X on equal footing. An edge between two outer generators related by a primitive reflection r_j gets special treatment. The 1 of the labels $(1 + \bar{D}_i V)$ of the edges $\{V, \hat{r}_i V\}$ must be added.

If we focus on the character generator of the algebra G_2 only, however, our result only amounts to a slight simplification of that of [7]. On the other hand, an important difference is revealed if we compare the methods used.

As was discussed above, semi-standard tableaux reveal the poset structure underlying the $su(r + 1) \cong A_r$ character generator. They also encode a construction of the

vectors of an irreducible highest-weight representation, using as a basis the vectors of the fundamental representations. While writing down the vectors of a fixed representation is much more involved than finding its weights and multiplicities, doing the former does tell us about the latter, and so about the character of the representation. A general construction, for all highest weight representations can, therefore, reveal the structure of the character generator.

This construction of vectors is possible for any simple Lie algebra, providing a way to the character generator of that algebra. In [7], the authors defined what they called a shape algebra, which is useful for such constructions, but is framed in a more general context. More importantly for us, they constructed the required basis for G_2 explicitly, and were consequently able to draw the generalized poset graph of Fig. 5.7, and write the character generator X . Their work used the special relation of G_2 to the octonions, however. It was therefore not able to yield results in a general form, useful for any simple Lie algebra. Their G_2 results were not derived or written in terms of objects common to all simple Lie groups and/or their algebras, like the Weyl group, for example.

Our method, however, is essentially that of Gaskell [13], taking into account the poset structure of Baclawski [6]. As such it uses only general methods, involving Weyl groups and their Bruhat order, Demazure operators, and a total ordering of fundamental weights induced by a reduced decomposition of w_L . It therefore leads to results that we believe indicate the general form of the character generator for all simple Lie algebras.

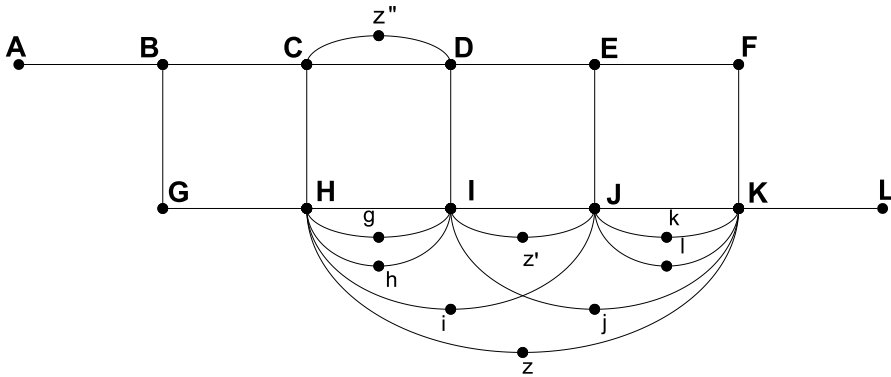


Figure 5.7: The G_2 generalized-poset graph of Baclawski and Towber [7].

5.1.4 General simple Lie algebras - possible universal picture

Let us now sketch the construction of the character-generator graph in terms that apply to an arbitrary simple Lie algebra X_r .

We must emphasize that the construction outlined below has not been proven correct. All we can say at this point is that it works for the algebras we considered, and is expressed in universal terms. It therefore has a hope of applying to all simple Lie algebras.

First, construct the fundamental-orbit poset P_o . Its elements are in one-to-one correspondence with the outside generators of the integrity basis (2.14) for X , and so with pairs (Λ, μ) , where $\Lambda \in F$ and $\mu \in P_\Lambda$, the set of weights (of non-zero multiplicity) in the fundamental representation $R(\Lambda)$. We write

$$P_o = \{ [\Lambda, \mu; w] \mid \Lambda \in F, \mu \in P_\Lambda; w \in W \text{ such that } \mu = w\Lambda \} . \quad (5.18)$$

Notice that only one Weyl group element is associated with each pair (Λ, μ) , i.e. each element of P_o . The choice of these Weyl elements is not unique; the different possible

choices allow the same character generator X to be described by different posets.

To make one such choice, fix a reduced decomposition of w_L , and write it as

$$w_L := s_L s_{L-1} \cdots s_1 , \quad (5.19)$$

where each s_a is a primitive reflection of W , so that $L = \ell(w_L)$, the length of w_L . More generally, we will use

$$w_{L,a} := s_a s_{a-1} \cdots s_1 . \quad (5.20)$$

Set the highest-weight elements of P_o to be $[\Lambda^j, \Lambda^j; w_{\max}^{(j)}]$, where $w_{\max}^{(j)}$ is the longest of the Weyl group elements $w_{L,a}$ fixing Λ^j : $w_{\max}^{(j)} \Lambda^j = \Lambda^j$. Then the remaining elements can be assigned Weyl group elements using the Bruhat order: $[\Lambda, \mu; w] > [\Lambda, \nu; v]$ if $w \prec v$.

The reduced decomposition of w_L selected also induces a total order \geq on the fundamental weights of F . Let \Rightarrow denote its cover relations. We put $\Lambda^j > \Lambda^i$ if $\ell(w_{\max}^{(j)}) < \ell(w_{\max}^{(i)})$.

The partial order of P_o can then finally be fully defined as

$$\begin{aligned} [\Lambda, \mu; w] > [\Lambda', \nu; v], & \quad \text{if } \Lambda = \Lambda' \text{ and } w \prec v, \\ & \quad \text{or if } \Lambda \Rightarrow \Lambda' \text{ and } w = v . \end{aligned} \quad (5.21)$$

Let $E(\mathcal{G})$ and $V(\mathcal{G})$ indicate the edge set and the vertex set, respectively, of a graph \mathcal{G} . The character-generator graph \mathcal{G}_X is built on the skeleton $\mathcal{H}(P_o)$. More precisely,

$$V(\mathcal{G}_X) = V(\mathcal{H}(P_o)) , \quad E(\mathcal{G}_X) \supset E(\mathcal{H}(P_o)) . \quad (5.22)$$

Label the vertices of the Hasse diagram $\mathcal{H}(P_o)$ of P_o using

$$\ell([\Lambda, \mu; w]) = L^\Lambda a^\mu . \quad (5.23)$$

We will also label the edges of the resulting character-generator graph, using Demazure objects. First, all edges of the P_o Hasse diagram are labelled by 1. Additional labels are introduced as follows, and they will add to the 1s already present, or label new edges of $\mathcal{G}_X \supset \mathcal{H}(P_o)$, when they do not vanish.

The “new” edges, the elements of $E(\mathcal{G}_X) \setminus E(\mathcal{H}(P_o))$, are found using Demazure calculations. Suppose that $T = L^\Lambda$ and $B = \hat{w}_L T$ indicate the top and bottom vertices of the same shape $\Lambda \in F$. Consider the vertices V_1 and V_2 , with $V_1 > V_2$ in $\mathcal{H}(P_o)$. Suppose further that $V_1 = \hat{b}_1 B$, and $V_2 = \hat{t}_2 T$, with $b_1, t_2 \in W$. Calculate $\hat{d}_{t_2} T$ and $\hat{d}_{\underline{b}_1} B$. Denote the sum of terms common to both as

$$d(V_1, V_2) := \hat{d}_{\underline{b}_1} B \cap \hat{d}_{t_2} T . \quad (5.24)$$

If $d(V_1, V_2) \neq 0$, then $\{V_1, V_2\}$ will belong to $E(\mathcal{G}_X)$.

The labels of the edges of \mathcal{G}_X are given by

$$\tilde{\ell}(\{V_1, V_2\}) = \begin{cases} 1 + d(V_1, V_2), & \text{if } \{V_1, V_2\} \in E(\mathcal{H}(P_o)) ; \\ d(V_1, V_2) , & \text{if } \{V_1, V_2\} \notin E(\mathcal{H}(P_o)) . \end{cases} \quad (5.25)$$

We can write a formula analogous to (5.9) for the general case if we consider \mathcal{G}_X the Hasse diagram of a poset P_X . The new poset P_X has the same elements as the fundamental-orbit poset P_o , but its cover relations are those of P_o augmented by those encoded in the new edges of \mathcal{G}_X .

Maximal chains in P_X are relevant here, but their labels must include the edge labels as factors, along with those of the vertices. We define

$$\tilde{\ell}(\cdots V_1 > V_2 \cdots) := \cdots \tilde{\ell}(V_1) \tilde{\ell}(\{V_1, V_2\}) \tilde{\ell}(V_2) \cdots . \quad (5.26)$$

The symbol $\tilde{\ell}$ indicates the labelling of \mathcal{G}_X , to distinguish it from the labelling ℓ of P_o .

We need the extended poset \hat{P}_X , and its labelling. But its labelling is trivially different from that of P_X : vertices $\hat{1}$ and $\hat{0}$, and the two extra edges involving them, are all assigned 1 as labels. We will also use $\tilde{\ell}$ for the labels of \hat{P}_X .

Incompatible products are treated in (5.9) using linkings of the extended poset \hat{P} and the resulting descent sets, defined in (5.7). In the general case, only incompatibilities between two outside generators need to be handled this way. Therefore, it is the linking of \hat{P}_o that is relevant. Suppose m is a maximal chain in \hat{P}_X . Then we define

$$\mathcal{DS}_o(m) := \{x_i \mid 0 < i < n \text{ and } (x_{i-1} > x_i > x_{i+1}) \in \text{Link}^-(\hat{P}_o)\} . \quad (5.27)$$

Finally, we are able to write

$$X = \sum_{m \in \text{Max}(\hat{P}_X)} [\tilde{\ell}(m)] \ell(\mathcal{DS}_o(m)) . \quad (5.28)$$

Here the shorthand notation of (4.20) and (4.52) only applies to the vertex factors:

$$[\tilde{\ell}(\cdots V_1 > V_2 \cdots)] := \cdots [\tilde{\ell}(V_1)] \tilde{\ell}(\{V_1, V_2\}) [\tilde{\ell}(V_2)] \cdots . \quad (5.29)$$

The formula (5.28) for the character generator X is one of our main results. Hopefully, our conjecture generalizes Baclawski's formula (5.9) so that it can be applied to any simple Lie algebra.

Chapter 6

Conclusion

In this thesis, two main results were obtained:

- a new general formula, (2.27), was derived for the character generating functions of simple Lie algebras. This result followed from the Patera-Sharp formula [23] which in turn was proved starting from the Weyl character formula.
- taking a lead from the work of Gaskell [14], we applied the Demazure character formula, (4.8), to calculate the character generators, and our results were interpreted in terms of posets and graphs, following Baclawski [6], King and collaborators [19, 20], and Baclawski and Towber [7].

Our motivation for both projects in this thesis is centered on finding general character-generator formulas. That is, formulas that are valid for all simple Lie algebras and expressed in terms of structures common to them, such as the fundamental-orbit posets and quantities expressed only in terms of the Demazure operators.

Much work has been done to calculate character generators for simple Lie algebras. Patera and Sharp [23] derived a nice formula for the character generating function, (2.10),

from the the well-known Weyl character formula, (2.8). But calculating character generators using this formula becomes more complicated, and lengthy as we go to higher rank algebras. In the first of two parts of this project, we studied the formula due to Patera and Sharp, and developed a new general formula, (2.27). The most important features of the result are simplicity and universality. First, there is no explicit sum over the Weyl group. Hence our formula provides a less daunting approach to achieving these calculations. This is evident from the examples in Chapter 2, section 2 for the algebras A_1 , A_2 , B_2 , and G_2 . Our formula, like Patera and Sharp's, is a general and universal result because it applies to all simple Lie algebras. It also makes clear the distinction between outside and inside elements of the integrity basis, and helps determine quadratic incompatibilities (see Chapter 3). The results we obtained in examples (see Chapter 2, Sect. 2.2) are consistent with known results [15,23].

In the second part of our project, we followed Gaskell's work [14] and applied Demazure character formulas, (4.8), to calculate the character generators. This method involves the use of Demazure operators, defined and discussed in Chapter 4, Sect. 1.

Stanley [24] and Baclawski [6] pioneered the use of the posets underlying character generators. Baclawski, using some non-universal methods, found fundamental posets: partially-ordered sets, whose elements are elements of the integrity basis I_X . But in their later work, Baclawski and Towber [7] generalized the notion of posets and derived the G_2 generalized poset graph, using a non-universal method. A more general approach which applies only to classical Lie algebras was developed by King and collaborators [19,20]. Though an important advance, their results do not apply to all simple Lie algebras.

The works just mentioned lack universality. Gaskell [14] calculated character generators for simple Lie algebras using Demazure operators. This turned out to be an inter-

esting and convenient way of doing the calculations. Though his method does not address the issue of the underlying graphs, it provides a very useful general approach.

Knowing about the posets and generalized posets, we applied Gaskell's approach and arrived at a conjecture for a universal method of constructing character generators for simple Lie algebras. Our calculations reveal the underlying generalized-poset structure of the character generator. The properties of the Demazure operators used are valid for any simple Lie algebra. We applied this method to derive A_2 , B_2 , and G_2 graphs and character generators (see Chapters 4 and 5). Our method, however, provides Demazure expressions for the inside generator contributions, and hence a general and simple way of finding the graphs and character generators.

In addition, we construct graphs that are simpler than the ones found by Baclawski and Towber. Compare the G_2 graphs of Fig. 5.7, and Fig. 5.6. We believe our approach indicates a general way of calculating character generators and graphs for higher rank simple Lie algebras. We made this explicit at the end of Chapter 5.

Possible future work includes:

- using the Demazure character formula, (4.8), to extend our result (5.28) to other generating functions $X_w := \widehat{D}_w H$, for $w \in W$. Notice that $X = X_{w_L}$.
- making the second formula (5.28) rigorous by proving, possibly by induction, that the character generator formula applies to all general simple Lie algebras. We can also check that the formula works for other specific algebras.
- attempting to extract a new character formula from our expression for the character generating function. We can also study the formulas to see if there are interesting character identities to be realized.

- using the character generating formula as a guide to the construction of a basis of states in the highest weight representations of simple Lie algebras.

Finally, let us mention what was the original motivation for this work. In conformal field theory, the fusion rules are important data [27]. They are rules counting the couplings of primary fields in a conformal field theory. In Wess-Zumino-Witten conformal field theories, the fusion rules are truncated versions of tensor product decompositions for a fixed Lie algebra. Remarkably, the fusion eigenvalues of Wess-Zumino-Witten conformal field theories are characters of simple Lie algebras [26]. As a possible application, therefore, our results and methods can be applied to find generating functions for fusion eigenvalues of Wess-Zumino-Witten models.

Appendix A

Alternate form of the Weyl character formula and an identity

The identity (3.6) can be seen most easily using a different form of the Weyl character formula:

$$\text{ch}_\lambda = \sum_{w \in W} a^{w\lambda} \prod_{\alpha \in \Delta_+} (1 - a^{-w\alpha})^{-1}. \quad (\text{A.1})$$

The usual formula (2.8) is recovered from this as follows. Each Weyl element $w \in W$ separates the positive roots into two disjoint sets:

$$\begin{aligned} \Delta_+^w &:= \{\alpha \in \Delta_+ \mid w\alpha \in \Delta_+\}, & \Delta_-^w &:= \{\alpha \in \Delta_+ \mid w\alpha \in \Delta_-\}, \\ \Delta_+^w \cup \Delta_-^w &= \Delta_+, & \Delta_+^w \cap \Delta_-^w &= \{\}, \\ w\Delta_+^w &= \Delta_+^w, & w\Delta_-^w &= -\Delta_-^w. \end{aligned} \quad (\text{A.2})$$

It can be shown that $\det w = (-1)^{\|\Delta_-^w\|}$, and

$$-w\rho + \rho = \sum_{\beta \in \Delta_-^w} \beta. \quad (\text{A.3})$$

Using these results, (A.1) becomes

$$\begin{aligned}
\text{ch}_\lambda &= \sum_{w \in W} a^{w\lambda} \prod_{\beta \in \Delta_-^w} (-a^{w\beta})(1 - a^{w\beta})^{-1} \prod_{\alpha \in \Delta_+^w} (1 - a^{-w\alpha})^{-1} \\
&= \sum_{w \in W} (\det w) a^{w\lambda - w \sum_{\gamma \in \Delta_-^w} \gamma} \prod_{\beta \in \Delta_-^w} (1 - a^{w\beta})^{-1} \prod_{\alpha \in \Delta_+^w} (1 - a^{-w\alpha})^{-1}, \quad (\text{A.4})
\end{aligned}$$

so that (2.8) results.

Now, (A.1) gives

$$\widehat{\text{ch}} = \sum_{w \in W} \hat{w} \prod_{\alpha \in \Delta_+} (1 - a^{-\alpha})^{-1}. \quad (\text{A.5})$$

Therefore,

$$\widehat{\text{ch}} (a^\mu \mathcal{O}_\lambda(a)) = c_\lambda \sum_{w \in W} a^{w\mu} \prod_{\alpha \in \Delta_+} (1 - a^{-w\alpha})^{-1} \sum_{u \in W} a^{wu\lambda}. \quad (\text{A.6})$$

After a simple change of summation variables, we find (3.6).

Appendix B

Maple program for \mathcal{Y} , rank 2

```
restart;

 $\mathcal{Y} := \text{proc}(c_{12}, c_{21}, L_1, L_2)$ 

local  $yy, m_1, m_2, \text{term}_1, Q_1, Y, ww, n_1, n_2, \text{term}_2, Q_2$ ;

yy:=1;

 $Q_1 := a_1^1 a_2^0$ ;

 $\text{term}_1 := \text{op}(Q_1)$ ;

 $m_1 := \text{degree}(\text{term}_1, a_1)$ ;

 $m_2 := \text{degree}(\text{term}_1, a_2)$ ;

ww:=1;

 $Q_2 := a_1^0 a_2^1$ ;

 $\text{term}_2 := \text{op}(Q_2)$ ;

 $n_1 := \text{degree}(\text{term}_2, a_1)$ ;

 $n_2 := \text{degree}(\text{term}_2, a_2)$ ;

while  $(m_1^3 + m_2^3) <> -1$  do
```

```

if  $(m_1^3 + m_2^3) <> -1$  then
   $term_1 := term_1((a_1^2) * (a_2^{c_{12}}))^{(-m_1)}$ ;
   $m_1 := degree(term_1, a_1)$ ;
   $m_2 := degree(term_1, a_2)$ ;
   $yy := yy(1 - L_1 term_1)$ ;
else
   $yy := yy$ ;
fi;
if  $(m_1^3 + m_2^3) <> -1$  then
   $term_1 := term_1(a_1^{c_{21}} a_2^2)^{-m_2}$ ;
   $m_1 := degree(term_1, a_1)$ ;
   $m_2 := degree(term_1, a_2)$ ;
   $yy := yy(1 - L_1 term_1)$ ;
else
   $yy := yy$ ;
fi;
od;
while  $(n_2^3 + n_1^3) <> -1$  do
  if  $(n_2^3 + n_1^3) <> -1$  then
     $term_2 := term_2(a_1^{c_{21}} a_2^2)^{-n_2}$ ;
     $n_1 := degree(term_2, a_1)$ ;
     $n_2 := degree(term_2, a_2)$ ;
     $ww := ww(1 - L_2 term_2)$ ;

```

```

else

ww:=ww;

fi;

if  $(n_2^3 + n_1^3) < -1$  then

 $term_2 := term_2(a_1^2 a_2^{c_{12}})^{-n_1}$ ;

 $n_1 := degree(term_2, a_1)$ ;

 $n_2 := degree(term_2, a_2)$ ;

 $ww := ww(1 - L_2 term_2)$ ;

else

ww:=ww;

fi;

od;

Y:=(yy) (ww);

end;

```

Appendix C

Maple program for G_2 Weyl reflections

```
sweyl $\mathcal{Y} := \text{proc}(\mathcal{Y}, a_1, a_2)$   
  
local  $Y, \text{terms}, n, \text{term}, m_1, m_2, i;$   
  
 $Y := \mathcal{Y};$   
  
 $\text{terms} := \text{op}(\mathcal{Y});$   
  
 $n := \text{nops}(\mathcal{Y});$   
  
for  $i$  from 1 to  $n$  do  
  
term:=terms[i];  
  
 $m_1 := \text{degree}(\text{term}, a_1);$   
  
 $m_2 := \text{degree}(\text{term}, a_2);$   
  
while  $m_1 < -1$  or  $m_2 < -1$  do  
  
if  $m_1 < -1$  then  
  
term := -term *  $(a_1^{-2} a_2^3)^{m_1+1};$ 
```



```

 $m_1 := \text{degree}(\text{term}, a_1);$ 
 $m_2 := \text{degree}(\text{term}, a_2);$ 
fi;
if  $m_2 < -1$  then
 $\text{term} := -\text{term} * (a_1 a_2^{-2})^{m_2+1};$ 
 $m_1 := \text{degree}(\text{term}, a_1);$ 
 $m_2 := \text{degree}(\text{term}, a_2);$ 
fi;
od;
if  $m_1 = -1$  or  $m_2 = -1$  then
 $Y := Y\text{-terms}[i];$ 
else
 $Y := Y\text{-terms}[i] + \text{term};$ 
fi;
od;
Y;
end;
```

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