ASPECTS OF QUANTUM GRAVITY THEORY AND PHENOMENOLOGY

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Abstract

Quantum gravity deals with the formulation of a physical theory consistent with both quantum and gravitational principles. The formulation is based on two main methods of quantisation, the canonical and the covariant one. In the first part of the thesis, the main problems of each method of quantisation are stated. In particular, the problem of time is analysed in the canonical quantisation framework and the conformal sickness problem of the Euclidean quantum gravity is studied with covariant methods.

Quantum gravity phenomenology is studied through two models. The first one is a cosmological model obtained by reduced phase space quantisation. Implications for the early era of the universe as well as how phantom fields might arise are studied. The second one deals with the calculation of the response function of a detector in the presence of Dirac fields in a 2+1 dimensional spacetime. The spectrum detected is expected to invoke the apparent inversion of statistics of a quantum field. This calculation might have potential indications for the actual detection of thermal radiation in a graphene sheet.

Contents

List of Constants

$c = 2.99 \times 10^8 m/s$ . . . . Speed of light

$G = 6.67384 \times 10^{-11} m^3 kg^{-1} s^{-2}$ . . . Gravitational constant

$\hbar = 6.62606957 \times 10^{-34} m^2 kg/s$ . . . Planck constant

$m_P = \frac{\hbar}{l_P c} = \sqrt{\frac{\hbar c}{G}} \approx 2.17 \times 10^{-5}$ . . . Planck mass

$l_P = \sqrt{\frac{\hbar G}{c^3}} \approx 1.62 \times 10^{-33} cm$ . . . . Planck length

$t_P = \frac{l_P}{c} = \sqrt{\frac{\hbar G}{c^5}} \approx 5.4 \times 10^{-44} s$ . . . Planck time

$l_C = \frac{\hbar}{mc}$ . . . . Compton wavelength

$l_S = \frac{2Gm}{c^2}$ . . . . Schwarzschild radius

Λ . . . . cosmological constant

List of Symbols

$\mu, \nu = 0, 1, 2, 3$ . . . . greek indices take spacetime values

$a, b = 1, 2, 3$ . . . . latin indices take spatial values

$\eta_{\mu\nu} = diag(-1, 1, 1, 1)$ . . . . Minkowski metric of flat spacetime

$\gamma_{\mu\nu}$ . . . . 4-dimensional spacetime metric

g_{ab} . . . . 3-dimensional spatial metric

$\Box \equiv \gamma_{\mu\nu}\partial^{\mu}\partial^{\nu}$ . . . . d' Alembertian operator

$\nabla_{\mu} A$ or $A_{;\mu}$ . . . . covariant derivative of A wrt the spacetime metric $\gamma_{\mu\nu}$

D_a or $A_{|a}$ . . . . covariant derivative of A wrt the spatial 3-metric g_{ab}

$\partial_{\mu} A$ or $A_{,\mu}$ . . . . partial derivative

F(x,...f(x,...), g(x,...),...] . . . . function of spacetime points and spacetime functions

$\varphi[f] \equiv \int d^4x f(x)\varphi(x)$ . . . . smeared function

$\frac{\delta}{\delta f}$ . . . . functional derivative wrt the function f

M . . . . manifold

Σ . . . . 3-dimensional spatial surface

M . . . . superspace

$\mathcal{M}$ . . . . phase space

$\bar{\mathcal{M}}$ . . . . constraint surface

$X^\mu(x^\nu)$ . . . . diffeomorphism

Diff(Σ) . . . . diffeomorphisms group of Σ

Diff(M) . . . . diffeomorphisms group of M

ξ^μ . . . . generator of infinitesimal diffeomorphisms

$R_{\mu\nu}$ . . . . Ricci tensor

${}^{(D)}R$ . . . . Ricci scalar in D dimensions

$K_{\mu\nu}$ . . . . extrinsic curvature

$\mathcal{G}_{abcd}$ . . . . DeWitt metric

$[Dx(t)]$ . . . . measure over all paths $x(t)$

$\approx$ . . . . weak equality

$\hat{f}$ . . . . quantum operator

$\{,\}$ . . . . Poisson brackets

$[,]$ . . . . Dirac bracket

e_n . . . . tetrad

γ^μ . . . . Dirac matrices

$\left\langle \vec{a} \middle| \vec{b} \right\rangle$ . . . . Dirac symbol for inner product of the vectors $\vec{a}, \vec{b}$

$\delta^c_{(a}\delta^d_{b)} \equiv \frac{1}{2}(\delta^c_a\delta^d_b + \delta^c_b\delta^d_a)$ . . . . symmetrisation

$\delta^D(x - y) \equiv \delta^D(x, y)$ . . . . Dirac delta continuous function in D dimensions

$T_{\mu\nu}$ . . . . energy-momentum tensor

N . . . . lapse function

N^a . . . . shift vector

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Chapter 1

Introduction

1.1 Quantum gravity

Our present understanding of nature is interpreted in two theoretical frameworks, quantum theory and the general theory of relativity. Quantum theory and its successor, quantum field theory, are the basis of all different kinds of matter and interactions, electrodynamics and the weak and strong interactions. Quantum theory changed our perception of the physical world by claiming that Nature is quantum in principle and that the classical world emerges from the quantum one. As a consequence, the fundamental physical laws should not be formulated to respect the classical concepts, but the quantum ones where the uncertainty principle and the principle of superposition are essential.

General relativity completed the revolution in physics that took place in the beginning of the last century. The special theory of relativity already had given a new insight on how to think about space and time, by unifying the two notions into one, the spacetime continuum. But general relativity, apart from being a theory of gravitational interactions, declared that spacetime interacts with matter and is no longer a background scenery, in which all the action takes place between all kinds of matter. Rather, spacetime is as dynamical as any other physical entity and it deserves to have its own dynamical laws and interactions. It affects the motion of matter and is affected as well by matter. General relativity, however, is a classical theory. If one accepts the quantum nature of our world, then general relativity is not a fundamental theory.

Rather, it is an effective theory valid only in a limited domain and will be superseded by a more fundamental quantum theory of gravity, in the same way general relativity superseded Newtonian theory. In this sense, the revolution that started in the last century is unfinished. A big task for physicists is to understand what a quantum theory of gravitational interactions widely known as quantum gravity would look like. It is estimated, mainly from dimensional arguments, that its domain of applicability is the Planck scale. The Planck scale is defined by the unique way the fundamental constants, the speed of light c, the gravitational constant G and the Planck constant $\hbar$ combine to give units of length, time and mass (see e.g. [1]).

$$l_P = \sqrt{\frac{\hbar G}{c^3}} \approx 1.62 \times 10^{-33}\text{cm} \tag{1.1}$$

$$t_P = \frac{l_P}{c} = \sqrt{\frac{\hbar G}{c^5}} \approx 5.4 \times 10^{-44}\text{s} \tag{1.2}$$

$$m_P = \frac{\hbar}{l_P c} = \sqrt{\frac{\hbar c}{G}} \approx 2.17 \times 10^{-5}\text{g} \approx 1.22 \times 10^{19}\text{GeV} \tag{1.3}$$

That the Planck scale should be the relevant scale for quantum gravity can be seen as follows: At that scale, both the gravitational effects and the quantum effects are non-negligible. Thus, the two characteristic lengths that the two theories define, the Schwarzschild radius $l_S \equiv \frac{2Gm}{c^2}$ and the Compton wavelength $l_C \equiv \frac{\hbar}{mc}$ should be equal and this equality identifies the Planck length[1].

The formulation of a quantum theory of gravity is very important because it is expected to resolve several issues that appear in general relativity and quantum field theory. For example, it is known that general relativity predicts its own breakdown in certain situations as black holes and certain cosmological models, such as the Friedmann–Lemaître–Robertson–Walker (Friedmann-Lemaître-Robertson-Walker (FLRW)) metric, relevant for the description of the present state of the universe and

[1] This can be shown by solving the relation for the Compton length with respect to the mass m and substituting the result in the relation for the Schwarzschild length.

its history. Another equally important open research topic relevant to a quantum gravity theory concerns the quantum origin of the universe and consequently how the classical world emerges from it. The formulation of a consistent theory of quantum cosmology is therefore in order. Such a theory can be part of the full theory of quantum gravity.

However, the formulation of a full theory of quantum gravity is not an easy task and has become a great challenge for theoretical physics for over 50 years, because of both conceptual and practical problems that arise, the most important of which are [2]:

(i) The Planck scale is inapproachable by any contemporary conceivable experiment. As a result, there exist no empirical data that could be used as a guide in this exploration. This fact also makes it difficult to test any proposal for a theory of quantum gravity. However, progress has been made in the last few years in this respect from the quantum gravity phenomenology perspective which will be discussed in the next section.

(ii) The fundamental principles of the two ingredient theories, quantum field theory and general relativity, are so distinct and contradictory[2] that they pose apparently insurmountable obstacles in the road to quantum gravity. Therefore, as Isham and Butterfield comment in [2], we are in front of a paradox: having two extremely successful ingredient theories that cannot be combined to give a consistent physical theory.

[2]Some of the contradicting principles of general relativity and quantum theory are:

- The different nature of time in the two theories: the dynamical time in general relativity vs the fixed static parameter in quantum theories.
- The different principles of a classical theory such as general relativity and the quantum principles. In particular,
  - In classical theories, the physical quantities have definite values.
  - In quantum theories, the physical quantities do not in general have definite values because they are subject to the Heisenberg's uncertainty principle. This leads to their description by probability distributions over many different values.

The conceptual problems appear in two ways. There are problems already existing in the underlying theories and problems that arise when one tries to combine such theories, because of their fundamentally different formulations. The problems of general relativity related to the construction of quantum gravity are to understand the role of diffeomorphisms and how to implement them in a quantum theory, as well as the ontological status of the spacetime points, i.e., if the points have meaning by themselves or are just mathematical representations of the theory. An issue of equal importance is the nature of the gravitational interaction and in particular the equivalence principle[3]. The equivalence principle allows one to represent gravity as a property of spacetime itself, rather than as a field propagating in a passive spacetime background as happens in Newtonian gravity.

On the other hand, questions such as how to quantise a classical theory, what is an observable in quantum gravity, or what is the role of the observer emerge as problems in a theory of quantum gravity because of the fundamental problems of quantum theory. These are the meaning of the probability, the role of measurement, the reduction of the state vector and quantum entanglement. These problems arise specifically in the framework of the conventional Copenhagen interpretation of quantum theory [2, 4]. This interpretation is incompatible with quantum gravity, since all structures related to spacetime would probably have to stay classical because they are thought to be necessary ingredients for the measurement process. For the purposes of quantum gravity, however, such a viewpoint is insufficient and probably inconsistent. The main aim in constructing quantum gravity is to get rid of any external structure and render the theory background independent which is the opposite from what happens in the Copenhagen interpretation. Similar arguments are true for the

[3]The equivalence principle rests on the equality of gravitational and inertial mass. It states that: at every spacetime point in an arbitrary gravitational field it is possible to choose a ‘‘locally inertial coordinate system” such that, within a sufficiently small region of the point in question, the laws of nature take the same form as in special relativity (see e.g. [3]). By small it is meant that the region should be small enough so that the gravitational field stays constant throughout it.

case of quantum field theory due to the fact that its ontological basis is the same as in quantum theory, i.e., the fixed flat background.

From the above short analysis it is understandable that the conflict between general relativity and quantum theory lies in the issue of background independence. This issue is incorporated to some extent in the problem of time in quantum gravity [5, 6]. The problem of time has its origin in the fact that quantum theory is formulated with respect to a classical framework. Thus, time is treated as an external parameter to the system and the system does not interact with space and time (fixed background). On the contrary, in general relativity time has no different status from the other coordinates and in addition, the matter fields interact with the spacetime metric which is the gravitational field itself.

The difficulties mentioned above have naturally led to more than one proposal for a theory of quantum gravity. Therefore, the term quantum gravity is used to describe all the spectrum of approaches that try to unify general relativity and quantum field theory. Three main strategies have been adopted for the resolution of these problems and the construction of a theory of quantum gravity (see e.g.[1, 5, 2]):

1. The first approach is to start with a given classical theory of gravity and apply heuristic quantisation rules. For example,

    - Start with general relativity and quantise it to obtain quantum general relativity or quantum geometrodynamics or,
    - Start with a different classical theory of gravity such as Brans - Dicke theory and apply heuristic quantisation rules.

    There are two different quantum algorithms to apply in the classical theory, the canonical and covariant, which we analyse later in this section. The final theory has to give general relativity and the Standard Model of particle physics in two different low energy limits. The obvious advantage of this approach is that the starting point is already known and this is certainly helpful. However, one does not obtain a unified theory of all physical interactions. Examples of known theories are quantum geometrodynamics, loop quantum gravity (e.g. [7]) and causal dynamical triangulations (CDT) (e.g.[8]).

2. Start from a fundamental quantum theory that is not a quantization of a classical theory and try to derive quantum general relativity and the Standard Model of particle physics as low energy limits. An example of this approach is string theory. Although these approaches yield a unification of all interactions, the starting point is highly speculative.

3. The third approach, which is the most revolutionary, is to start from a set of fundamental principles and formulate a new theory of which the low energy limits are general relativity and quantum field theory. These principles come from the ingredient theories. A major question in this approach is how to actually select the ingredient principles. Experiments might help on this selection since they can verify the validity of certain principles in a variety of energy scales. But still, it is perfectly possible that both of the input theories break down at higher energies. An interesting theory in this category is causal sets theory [9].

Since the language of physics is the technical framework in which it is expressed, it is expected that part of the general problem of how to build a quantum gravity theory is the existent technical framework of the quantisation rules. This framework can be summarised as

(a) Canonical: The canonical framework both enlightens the steps taken towards quantum gravity and generates a variety of problems. One starts with the Hamiltonian formulation of general relativity by casting the spacetime into space and time and by isolating the geometrical entities functioning as the position and conjugate momentum variables, and then quantises them. In gravity, the role of the position variable is played by the gravitational metric. The quantisation can be cast by using two different algorithms. One is the reduced phase space quantisation in which one first constrains the system by eliminating the non-physical degrees of freedom and then quantises the system. The second method is to follow the Dirac algorithm by quantising without singling out at the classical level the non-physical, non-propagating degrees of freedom.

The quantisation procedure brings into play the uncertainty principle and the probabilistic nature of quantum theory. The new quantum picture, therefore, is one of fluctuating geometry of the space, because of the quantisation of the metric. This comes in contradiction to the ordinary quantum theory that presupposes a well-defined classical background against which to define the fluctuations. As a result, in this framework, there appears a difficulty in giving meaning to the quantisation procedure because the commutation relations cannot be defined on a fixed background. This also raises interpretational problems of the theory from which it is not clear how to obtain any physical meaning and predictions.

Even with the above mentioned problems, the Hamiltonian framework is still a very powerful method not least because it heavily lies on the classical general relativity and consequently it should be the correct fundamental classical limit of an underlying quantum theory. Therefore, any quantum theory of gravity constructed starting from general relativity should still be valid as an effective theory in a certain limit, even if this theory is superseded by a more fundamental theory. In addition, a big amount of its mathematical methods are also used in approaches using the covariant framework, such as string theory, which employs the methods of quantisation of constrained systems, and covariant perturbation theory. Finally, an aspect that is very important and is studied in this thesis as well is that many problems of quantum gravity are exhibited more clearly in this formulation. The most important one is the problem of time.

(b) Covariant: It preserves the 4-dimensional covariance at each step. Here, we can also distinguish two methods, the path-integral and the perturbative method [10].

The covariant, or path-integral or sum-over-histories method, starts with a Lagrangian framework. In its early days this approach was not popular because of the severe non renormalisability and the unboundedness of the gravitational action. In later years, though, it became popular in a perturbative framework and is used in the formulation of string theory. However, perturbation methods cannot reveal the full aspects of a new theory of quantum gravity. New approaches and techniques are in order to calculate the gravitational path integral. These involve new techniques but also the construction of new nonperturbative approaches to quantum gravity, such as the causal dynamical triangulation (CDT) and the causal set approach. As for the techniques, an important one is the Euclidean approach to quantum gravity. This approach is motivated by the Wick rotation performed during the path integral calculations of quantum field theory and brings the method into the gravitational theory. However, a problem that arises in gravity is the unboundedness of the gravitational action and it is a severe one. Attempts to resolve it by eliminating the conformal degree of freedom have been reported initially in [11, 12] and progress has been made later in [13, 14, 15]. Euclidean quantum gravity has been used in cosmological models with interesting results so it is required that a resolution of the unboundedness of the action be found in the context of non-perturbative quantum gravity.

1.2 Quantum gravity phenomenology

A theory valid at the Planck scale where both quantum and gravitational effects are strong enough not to be ignored is constructed in a very different way from other physical theories, as already analysed in the previous section. The reason for this different epistemological approach is that there are no direct experimental data from the Planck scale realm to give any insight. quantum gravity theories are constructed by borrowing several assumptions, physical principles and mathematical frameworks from theories already checked for their validity in their domain of applicability. However, it becomes extremely important to have access to experimental and observational data to make the necessary contact of the theoretical framework of quantum gravity with the real world. This can be achieved not by searching directly at the Planck scale, but by looking for indications of Planck scale effects in lower energy scales. Therefore, development of intuition for some quantum gravity effects is necessary so as to know what we are looking for. This intuition can be gained in the context of quantum gravity phenomenology [16, 17, 18], which mainly employs two ways to achieve this. The first is to analyse the structure of the formalisms used in quantum gravity theories and find the low energy limit. Then one compares their predictions at this limit with data from current experiments. A second method is to construct test theories that can be used both to assess the progress of experimental sensitivity and as common language between experimentalists and quantum gravity theorists. Experimentally, the strategy to gain access to the Planck scale realm is to find experimental contexts in which there is effectively a large amplification of some small effects of interest.

Some of the ideas that are used in quantum gravity theories for which experimental data could help clarify their status and validity are: (i) ideas on the discrete or continuous nature of spacetime: whether it is fundamental or it is just a distinction on the level of the description, how discreteness affects the properties of fields and spacetime e.g. the symmetries of the theory, (ii) Lorentz symmetry: does it remain valid, break down or is it deformed at the Planck scale, (iii) dynamical dimensional reduction: is the transition from four to two spacetime dimensions indicated from some quantum gravity theories (e.g. CDT) generic or not, (iv) the problem of the cosmological constant: is it an ultra-violet or infra-red problem, is it time-dependent and what are the experimental bounds?

Experimental and observational procedures can probably test some of the above mentioned ideas. The places to search are astronomical objects such as gamma ray bursts (GRBs), TeV photons and high-energy cosmic rays (threshold anomalies, time delay measurements, polarisation effects) as well as the huge detectors built on Earth. Even though these machines run in much lower energies from the energies we are interested in quantum gravity, it might be possible to test the low energy effects of some quantum gravity theories might be possible to be tested in the future. In addition, in the last few years there has been a considerable progress in the construction of the theoretical framework for experimental tests of analogue gravity in condensed matter systems.

Other possible ways to gain some insight from the theoretical perspective is the semiclassical one, the quantum field theory in curved spacetimes. It is semiclassical because the matter fields are quantised while the gravitational metric is not. It is in this framework that important results such as Hawking radiation, black hole entropy and, in general, black hole thermodynamics laws have been derived [19, 20]. These results serve as a testbed for the quantum gravity approaches. Moreover, they also give motivation for the introduction of the idea of emergent gravity [21]. One of the variants of this idea claims that gravity is to be understood as an emergent, long wavelength phenomenon, which means that, at the macroscopic level, spacetime is smooth and continuous and the field equations have a phenomenological status. At the microscopic level, however, this continuum breaks down and spacetime should be described by different degrees of freedom, not yet known. A classical analogy to this approach is to consider fluid dynamics. At the macroscopic level, the equations of motion of the fluid (Navier-Stokes, continuity equation) are phenomenological and should be replaced at the microscopic limit since the continuum picture is replaced by the molecules of the fluid. In the same sense one can claim that the gravitational equations of motion, i.e., the Einstein equations, are phenomenological and can be obtained in this framework as thermodynamical equations.

1.3 This thesis

The purpose of this thesis is the exposition and study of some aspects of quantum gravity.

- Chapter 2 deals with the essentials of the canonical formulation of general relativity. This formulation is the basis of the canonical approach to quantisation of the gravitational theory studied in chapter 3.
- In chapter 3 the canonical quantisation is studied under the prism of two approaches. The first is the reduced phase space quantisation or choice of time coordinate before quantising the system and the second is the Dirac quantisation or choice of time after quantisation. These are the two prototypes for the canonical quantisation. The problem of time and the implications of each method for the theory of quantum gravity are discussed after the mathematical framework is presented.
- In chapter 4 the alternative approach to the construction of quantum gravity is presented, that is the covariant one. The problems of covariant quantum gravity are stated briefly mainly in relation to Euclidean quantum gravity. The problem of the unboundedness of the Euclidean gravitational action is studied and a possible solution is presented in the context of non perturbative quantum gravity.
- Chapter 5 is devoted to quantum cosmology. Quantum cosmology is the study of the origins of the universe. A quantised gravitational theory is necessary for this study. The main aspects of the field are stated and the minisuperspace models that are mainly used are briefly described. Then, a short reference to classical

cosmology and scalar-field models of dark energy is made. This will be useful in the last section of the chapter where a cosmological model is studied. This model was first reported in [22]. This is a minisuperspace model that has been quantised with the method of selecting time before quantising. The analysis with the help of Euclidean quantum gravity leads to some results regarding phantom energy, a type of dark energy.

- Chapter 6 describes the first steps in an exploration of quantum gravity phenomenology and in particular the description of analogs of quantum gravity effects in a condensed matter system. This system is a 2-dimensional sheet of graphene that resembles the part of a 2+1 dimensional spacetime outside the horizon of a black hole. Since the technical difficulties regarding the construction and stability of a graphene sheet become less trivial due to technological developments, this spacetime might be realisable in the lab. Then, it can be used to test an interesting effect arising in the context of quantum field theory in curved spacetime, the apparent inversion of statistics of a quantum field in spacetimes of odd dimensions. In the chapter, a short explanation of the effect is provided and some new calculations on $2 + 1$-dimensional spacetime are performed for a Dirac field. The results are stated and some conclusions are drawn.

Part I

Quantum Gravity

Chapter 2

Hamiltonian formulation of General Relativity

2.1 Introduction

The Hamiltonian formulation of general relativity [23] plays an important role in the development of numerical relativity and quantum gravity. In general relativity few, relative to other theories, exact solutions to the Einstein equations are known because of the high degree of complexity of the field equations. Numerical relativity is the field of classical general relativity in which one tries to find approximate solutions to the Einstein field equations with the aid of the computers. The Hamiltonian method is used in order to replace the second order differential equations by first order equations, rendering the problem in this way much easier for the computer to tackle. In the field of quantum gravity, the advantage of the Hamiltonian formulation lies on the fact that the quantisation procedure is more clear than other methods and can shed light on a variety of problems regarding the quantisation of a gravitational system. We discuss quantisation matters in the next chapters in more detail, so we do not comment here further.

The advantages of the canonical formulation mentioned above are a result of its possession of some very valuable properties, the most important probably being the fact that it reveals the gauge character of general relativity and its field-theoretic content instead of its geometric properties. General relativity is a theory with constraints, their presence being explicit in its canonical formulation. By imposing the constraints on the gravitational field, it is possible to make the distinction between the physical and unphysical degrees of freedom of the gravitational field clear and determine the minimal number of variables that describe the state of a system. Then, the Poisson bracket relations are formulated straightforwardly and this leads to the replacement of the second order differential equations by the first order ones, a step that makes the technical problems easier to deal with. During the procedure of imposing the constraints, the time variable has been singled out. This gauge choice, however, does not influence the general covariance of the theory.

The physical interpretation of the theory is also straightforward. The canonical variables represent the independent excitations of the gravitational field and thus provide a way of defining gravitational radiation in a coordinate-independent way, while the numerical value of the Hamiltonian of a state of the system is interpreted as the total energy of the system, as in the case of classical mechanics.

In this chapter we describe the Hamiltonian formulation of classical general relativity. This will reveal all the features discussed already and, in addition, it will provide the mathematical tools that we will need in the next chapters for the quantisation of general relativity.

2.2 The 3+1 decomposition

In order to write the Hamiltonian formulation of general relativity, we have to split the 4-dimensional spacetime represented by the manifold and metric $(M, \gamma_{\mu\nu})$ into space and time with topology $M \simeq \mathbb{R} \times \sigma$, where σ can have any arbitrary fixed topology. This 3+1 splitting of M is possible only when M is globally hyperbolic because of a theorem due to Geroch which states that: if the spacetime is globally hyperbolic[4], then it is necessarily of this kind of topology [24]. This topology induces a foliation $\Sigma_t := X_t(\sigma)$ where $X_t : \sigma \to M$ is an embedding defined by $X_t(x) := X(t, x)$.

[4]See in the appendix for a definition.

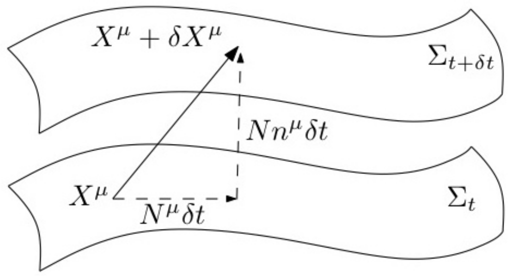

Figure 2.1: Foliation

Similarly, we have the diffeomorphism $X : \mathbb{R}\times\sigma \to M; (t,x) \to X(t,x) := X_t(x)$. Any two foliations $X : M \simeq \mathbb{R}\times\sigma$ and $X' : M \simeq \mathbb{R}\times\sigma'$ are related by a diffeomorphism $\phi \in \mathrm{Diff}(M)$, where $\phi = X' \circ X^{-1}$ and $\mathrm{Diff}(M)$ is the group of all diffeomorphisms on the manifold M. The arbitrariness in the choice of the foliation is equivalent to the invariance of the Einstein-Hilbert action under the action of the group $\mathrm{Diff}(M)$.

One uses the foliations to cast the action in the 3+1 decomposition. The foliation is parametrised by the deformation vector which is [7]

$$T^\mu(X) := \left(\frac{\partial X^\mu(t,x)}{\partial t}\right)_{X=X(x,t)} =: N(X)n^\mu(X) + N^\mu(X) \tag{2.1}$$

where $\mu, \nu = 0,1,2,3$ are the spacetime indices, n^μ is a unit normal vector to Σ_t, i.e., $\gamma_{\mu\nu}n^\mu n^\nu = s$, where $s = -1$ for the Lorentzian case and $s = 1$ for the Euclidean, and N^μ is tangential, i.e., $\gamma_{\mu\nu}n^\mu X^\nu{}_{,a} = 0$. The vector field n^μ is completely determined as a function of the metric and the foliation $\gamma_{\mu\nu}, X$ by these two requirements. The geometrical meaning of X^μ, N, N^μ, as can be deduced from the figure, is the following:

1. The deformation vector $T^\mu(X)$ represents the flow of time throughout spacetime, that is, how the hypersurfaces change with time (therefore it represents the differences between hypersurfaces at different t's). $T^\mu(X)$ is timelike $-N^2 +$

$\gamma_{\mu\nu}N^\mu N^\nu \leq 0$ and N positive everywhere for $T^\mu(X)$ future-directed.

2. The shift vector N^μ describes the spatial diffeomorphisms on the 3-dimensional hypersurface Σ, that is, it describes how to move from one point (t, x) on Σ to one with $(t, x + dx)$.

3. The lapse function N describes the evolution from a point (t, x) on Σ to one with $(t + dt, x)$. It is the function that represents the shift in the orthogonal direction to the hypersurface Σ_t and it indicates the time passed between the surfaces, Σ_{t_1} and Σ_{t_2}.

The freedom to choose the lapse and shift denotes the freedom to choose foliation. It is convenient to parametrise $n^\mu = (1/N, -N^a/N)$, so that $N^\mu = (0, N^a)$. In terms of lapse and shift, the metric tensor can be written as

$$ds^2 = \gamma_{\mu\nu}dx^\mu dx^\nu = (sN^2 + g_{ab}N^a N^b)dt^2 + 2g_{ab}N^b dt dx^a + g_{ab}dx^a dx^b \tag{2.2}$$

where $a = 1, 2, 3$ are spatial indices and are contracted with the 3-dimensional metric g_{ab}. Note that the spatial part g_{ab} is not in general the intrinsic metric on Σ_t which is given by

$$g_{\mu\nu} := \gamma_{\mu\nu} - s n_\mu n_\nu \tag{2.3}$$

and is called first fundamental form. The extrinsic curvature or second fundamental form of Σ_t is

$$K_{\mu\nu} := g^\rho_\mu g^\sigma_\nu \nabla_\rho n_\sigma. \tag{2.4}$$

This tensor is symmetric and it is connected to the Lie derivative of the intrinsic metric $\mathcal{L}_n g_{\mu\nu} = 2K_{\mu\nu}$. It enters the relation between the Ricci scalar of the Σ_t and that of the 4-dimensional manifold M and is known as Codacci equation [7]

$${}^{(3+1)}R = {}^{(3)}R - s[K_{\mu\nu}K^{\mu\nu} - K^2] + 2s\nabla_\mu(n^\nu\nabla_\nu n^\mu - n^\mu\nabla_\nu n^\nu) \tag{2.5}$$

where the notation ${}^{(D)}R$ is used to indicate that the Ricci scalar is of dimension D. We use the Codacci equation (2.5) to write the Einstein-Hilbert action

$$S = \frac{1}{\kappa}\int_M d^4x\sqrt{|\gamma|}\ {}^{(4)}R \tag{2.6}$$

in the 3+1 form with $\kappa = \frac{16\pi G}{c^3}$. Then, the action takes the following form [7]

$$S = \frac{1}{\kappa}\int_{\mathcal{R}} dt \int_\sigma d^3x\sqrt{g}|N|({}^{(3)}R - s[K_{ab}K^{ab} - (K_a^a)^2]). \tag{2.7}$$

The purpose is to write the action in a compact canonical form. In order to do this, we have to perform a Legendre transform from the Lagrangian density to the Hamiltonian. First, we have to find the canonical variables. One can observe that the action depends on the velocities $\dot{g}_{ab}$, but not on the velocities of N, N^a. Therefore, their conjugate momenta are (see e.g. [7, 25] for a derivation of the following relations and of most of the relations in this chapter)

$$p^{ab}(t,x) := \frac{\delta S}{\delta \dot{g}_{ab}(t,x)} = -s\frac{|N|}{N\kappa}\sqrt{g}[K^{ab} - g^{ab}(K_c^c)] \tag{2.8}$$

$$\pi(t,x) := \frac{\delta S}{\delta \dot{N}(t,x)} = 0 \tag{2.9}$$

$$\pi_a(t,x) := \frac{\delta S}{\delta \dot{N}_a(t,x)} = 0 \tag{2.10}$$

The Lagrangian is therefore singular and the system is constrained, since we cannot solve all velocities for momenta. Hence, we have the primary constraints

$$C(t,x) := \pi(t,x) = 0 \qquad C^a(t,x) := \pi^a(t,x) = 0. \tag{2.11}$$

Therefore, one introduces Lagrange multiplier fields $\lambda(x), \lambda_a(x)$ for the primary constraints and then perform the Legendre transform with respect to the remaining velocities $\dot{g}_{ab}, \dot{N}, \dot{N}_a$. The canonical form of the action, after we perform a spatial integration and drop the boundary term is given by

$$S = \int_{\Re} dt \int_{\sigma} d^3x \left(\dot{g}_{ab} p^{ab} + \dot{N}\pi + \dot{N}^a \pi_a - [\lambda C + \lambda^a C_a + N^a H_a + |N| H] \right) \tag{2.12}$$

where

$$H_a := -2 g_a^b D^c p_{bc} \tag{2.13}$$

$$H := -\left(\frac{s\kappa}{\sqrt{g}} [g_{ac} g_{bd} - \frac{1}{2} g_{ab} g_{cd}] p^{ab} p^{cd} + \sqrt{g} R/\kappa \right) \tag{2.14}$$

are called the spatial diffeomorphism and Hamiltonian constraints respectively. The characterisation of these quantities as constraints will be justified briefly in the next section. The quantity in the square brackets

$$\kappa \mathbf{H} := \int_{\sigma} d^3x [\lambda C + \lambda^a C_a + N^a H_a + |N| H] \tag{2.15}$$

$$=: C(\lambda) + \vec{C}(\vec{\lambda}) + \vec{H}(\vec{N}) + H(|N|) \tag{2.16}$$

is the Hamiltonian. The next step is to eliminate the Lagrange multipliers from the action. To this end, we note that because of the relations $\dot{N}^a = \lambda^a, \dot{N} = \lambda$ and because λ^a, λ are completely arbitrary, the lapse function and shift vector are also arbitrary. By treating $\dot{N}^a, \dot{N}$ as Lagrange multipliers and dropping all terms proportional to C_a, C we get the reduced or canonical ADM action [23]

$$S = \frac{1}{\kappa} \int_{\Re} dt \int_{\sigma} d^3x (\dot{g}_{ab} p^{ab} - [N^a H_a + |N| H]) \tag{2.17}$$

which is equivalent to (2.12) insofar as g_{ab}, p^{ab} are concerned.[5] The reduced Hamiltonian is

$$\mathbf{H}_{red} = \frac{1}{\kappa} \int_{\sigma} d^3x [N^a H_a + |N| H] \tag{2.18}$$

[5]What is meant here is that the equivalence is on the surface of constraints which is the subject of the next section.

and is a linear combination of constraints.

2.3 Symplectic structure

At fixed t, the fields $(g_{ab}, N^a, N; p^{ab}, \pi_a, \pi)$ are points in an infinite-dimensional phase space $\mathcal{M}$. The phase space carries the Poisson brackets

$$\{p^{ab}(t,x), g_{cd}(t,y)\} = \frac{\kappa}{2}\delta^a_{(c}\delta^b_{d)}\delta^{(3)}(x,y) \tag{2.19}$$

$$\{\pi(t,x), N(t,y)\} = \frac{\kappa}{2}\delta^{(3)}(x,y) \tag{2.20}$$

$$\{\pi^a(t,x), N_b(t,y)\} = \frac{\kappa}{2}\delta^a_b\delta^{(3)}(x,y) \tag{2.21}$$

and all other vanish. The Poisson bracket is defined to be invariant under diffeomorphisms of σ.[6] From these, the Poisson brackets of any arbitrary functional on the phase space can be defined. Equations (2.20) and (2.21) that contain the redundant variables N, N_b, even though they are often called evolution equations, just describe infinitesimal gauge transformations and they do not correspond to the physical evolution with respect to a physical (gauge invariant) Hamiltonian (that is the reduced ADM Hamiltonian).

That structure given, the evolution of a functional with respect to time is defined by its Poisson bracket with the Hamiltonian. For the fields $\vec{C}(t,x), C(t,x)$ that are the primary constraints, consistency with the equations of motion requires $\dot{\vec{C}}(t,x) := \{\vec{C}(t,x), \mathbf{H}\} = 0$ and $\dot{C}(t,x) := \{C(t,x), \mathbf{H}\} = 0$. Instead, we get for the Poisson brackets of the primary constraints with the diffeomorphism and Hamiltonian constraints

$$\{\vec{C}(t,x), \mathbf{H}\} = \vec{H}(t,x) \qquad \{C(t,x), \mathbf{H}\} = H\left(\frac{N}{|N|}(t,x)\right) \tag{2.22}$$

[6]The variation of the action is performed over the manifold M however.

and therefore we demand the secondary conditions

$$H(x,t) = 0, \qquad H_a(x,t) = 0. \tag{2.23}$$

These are the Hamiltonian and diffeomorphism constraints respectively. Due to the fact that the reduced Hamiltonian (2.18) is a linear combination of primary and secondary constraints, it vanishes. Therefore, general relativity is an example of a constrained system with no true Hamiltonian.

The structure of the phase space determines the evolution equations of the Hamiltonian and diffeomorphism constraints with the Hamiltonian. These are [7]

$$\{\mathbf{H}, \vec{H}(\vec{f})\} = \vec{H}(\mathcal{L}_{\vec{N}}\vec{f}) - H(\mathcal{L}_{\vec{f}}|N|) \tag{2.24}$$

$$\{\mathbf{H}, H(f)\} = H(\mathcal{L}_{\vec{N}}f) + \vec{H}(\vec{N}(|N|, f, g)) \tag{2.25}$$

where $f, \vec{f}$ are any scalar function and vector field on Σ. The above equations are equivalent to the Dirac or hypersurface deformation algebra [5]

$$\{\vec{H}(\vec{f}), \vec{H}(\vec{f}')\} = -\kappa\vec{H}(\mathcal{L}_{\vec{f}}\vec{f}') \tag{2.26}$$

$$\{\vec{H}(\vec{f}), H(f)\} = -\kappa H(\mathcal{L}_{\vec{f}}\vec{f}) \tag{2.27}$$

$$\{H(f), H(f')\} = s\kappa\vec{H}(\vec{N}(f, f', g)) \tag{2.28}$$

or explicitly written

$$\{H_a(x), H_b(y)\} = H_a(y)\partial_b^y\delta(x-y) - H_b(x)\partial_a^y\delta(x-y) \tag{2.29}$$

$$\{H_a(x), H(y)\} = H(x)\partial_a^x\delta(x-y) \tag{2.30}$$

$$\{H(x), H(y)\} = g^{ab}H_a(x)\partial_b^y\delta(x-y) - g^{ab}H_a(y)\partial_b^y\delta(x-y) \tag{2.31}$$

Once we know the Hamiltonian flow of the functions $\vec{H}(\vec{f}), H(f)$ for any $\vec{f}, f$ we can write the equations of motion for any function J on $\mathcal{M}$,

$$\delta_{\vec{f}} J = \{\vec{H}(\vec{f}), J\} \tag{2.32}$$

$$\delta_f J = \{H(f), J\} \tag{2.33}$$

The Dirac algebra is not a Lie algebra. Only in the case that the constraints vanish does it become a Lie algebra with gauge transformation group the Diff(M) group. Therefore the meaning of this algebra is that the constraint surface $\bar{\mathcal{M}}$, which is a submanifold of the phase space $\mathcal{M}$ on which the constraint equations hold, is preserved under the motions generated by the constraints. The motions of the constraints are the following: the Hamiltonian constraint is the generator of time diffeomorphisms (orthogonal to Σ) only when the equations of motion are satisfied, while the diffeomorphisms constraint generates 3-dimensional diffeomorphisms on Σ.

The conservation of the constraint equations under the motions generated by the constraints can be seen by the fact that the right-hand sides vanish on the constraint surface. Constraints of this characteristic are said to be first class, as opposed to second-class constraints whose Poisson brackets do not vanish on-shell. First-class constraints generate gauge transformations on the constraint surface and this is indeed the case for the diffeomorphism and Hamiltonian constraints.

The Poisson structure of the theory helps in the counting of the physical degrees of freedom of a constraint canonical theory by introducing the notion of kinematical phase space. In the case of gravity that we study, the phase space consists of the 3-metric and its conjugate momentum (g_{ab}, p^{ab}) with Poisson brackets as defined above. Its dimension is $(6+6) \times \infty^3 = 12 \times \infty^3$.[7] On this phase space $\mathcal{M}$, the constraint surface $\bar{\mathcal{M}}$ is defined which is the space of (g_{ab}, p^{ab}) satisfying $H^\mu(g,p) \equiv 0$ with

[7] The points in phase space are the 3-metric and its conjugate momentum which are symmetric $2-$tensors. Thus, each has 6 degrees of freedom. Now, there are ∞^3 choices of the metric and respectively of its conjugate momentum at each point of the phase space, hence its dimension is not just 12 but $12 \times \infty^3$ [26].

dimensionality $(12-4)\times\infty^3 = 8\times\infty^3$. However, there is an additional freedom that has to be eliminated related to the gauge invariance of the spacetime, that is the Diff(M) invariance. This freedom is related with the choice of the foliation and the elimination can be done by choosing the lapse function and the shift vector that uniquely define a foliation. Equivalently, we have to choose gauge conditions to impose. These gauge conditions appear in the phase space as orbits on the constraint surface which span a 4-dimensional manifold at each space point. By dividing the constraint surfaces with the orbits gives a submanifold of $(8-4)\times\infty^3 = 4\times\infty^3$ dimension. This is the physical phase space. Its dimension at every space point is 4 and thus the theory has 2 degrees of freedom, in agreement with the result one can find from linearised gravity [26]. A demonstration of the phase space is given in figure 2.2[8].

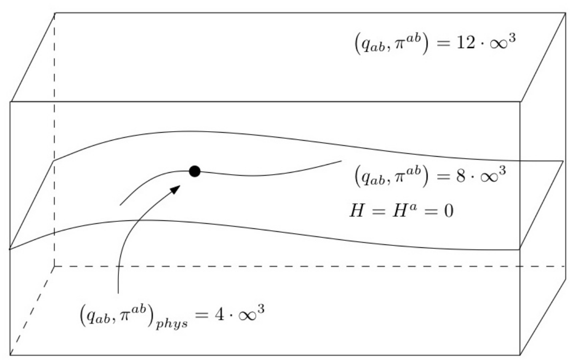

Figure 2.2: Phase space $\mathcal{M}$

In addition to the phase space, the configuration space of pure gravity can also be defined. This is the space of all Riemannian metrics on a 3-dimensional manifold Σ of a fixed but arbitrary topology. This space on which the 3-metrics live is called the superspace. The gauge invariance of gravity, diffeomorphisms, which act on the superspace, leave physical magnitudes such as distances and velocities unchanged.

[8]Note that the symbols in this figure differ from our conventions. The notation q_{ab}, π^{cd} corresponds to the notation g_{ab}, p^{cd} respectively used in this thesis.

These transformations lead to identifying different points (i.e. metrics) on the superspace with the same physical geometry. Therefore two different metrics connected by the action of a class of smooth C^∞ diffeomorphisms are isometrically equivalent and thus considered to be physically indistinguishable. This equivalence class of 3-metrics that are equivalent to one another under coordinate transformations are called 3-geometries.

2.4 Conclusions

In this chapter we discussed the Hamiltonian formulation of general relativity that is essential for the canonical quantisation discussed in the next chapter. The power of this formulation is that it reveals the constraint nature of general relativity and casts it as a classical gauge theory. This helps to make contact with the gauge theories of particle physics even though general relativity differs from them essentially.[9] This chapter is the basis for chapter 3 and gives insight onto the geometrical methods discussed in chapter 4.

[9]General relativity can be cast as a gauge theory, however it is quite different from the gauge theories which describe the other three known interactions, electromagnetism and strong and weak interactions of the Standard Model of particle physics since it is not a pure abelian or a Yang-Mills gauge theory. A way to see this is the fact that the Dirac algebra of deformations is not a pure Lie algebra. In addition, the dynamical nature of the gravitational field inserts an additional freedom non-existent in Yang-Mills theories, which is the freedom to choose a different tetrad basis at each spacetime point. Despite these differences, there are some similarities. For example, the way the connection $\Gamma^{\mu}_{\nu\rho}$, or the spin connection $\omega_{\mu ab}$ of the tetrad formulation, behave, resemble Yang-Mills fields to some respect.

Chapter 3

Canonical Quantisation of General Relativity

3.1 Introduction

One problem of quantum gravity that is of major importance is the problem of time [5, 6] , since it is connected to important conceptual issues. Some of these issues are: the status of the concept of probability and its conservation, the concepts of causality and unitarity[10], the question about the concept of spacetime and the maintenance of classical geometrical concepts in the quantum theory [2, 4, 5]. The problem of time becomes more apparent in the canonical approach of quantum gravity rather than the covariant one, even though it is independent of the quantisation procedure [5, 6]. The source of the problem of time is the different nature of time in Newtonian theory and general relativity. In Newtonian theory, time is a fixed structure, an external system parameter. Time is part of the classical background that is necessary to the construction of a Hilbert space, the definition of an inner scalar product and a consistent interpretation of the quantum theory, in this case the Copenhagen interpretation. The inner product is conserved in time and in this way unitarity and the conservation of the total probability.

On the other hand, general relativity is invariant under the action of the Diff(M) group, the group of diffeomorphisms of the spacetime manifold M, and time does not

[10] Unitarity means conservation of probability which implies that nothing can be created out of nothing or just disappear. Mathematically it is expressed as that the sum of probabilities of all possible outcomes of any event add up to one.

have any special status. Rather, it is just a coordinate on the manifold locally defined and in no way is it an absolute notion. Time here is part of a dynamical background, which means that spacetime influences matter and vice versa.

The question that arises is whether this coordinate nature of time is compatible with the existence of a universal external Newtonian time and how one can formulate a quantum theory of gravity. Note that this problem does not arise in the conventional quantum theory or even in quantum field theories, since they are formulated within the special relativity context which keeps the essence of the Newtonian notion of time. Moreover, if time is just a coordinate with no physical significance,.[11] an important question is how change arises in the context of general relativity. The group Diff(M) is similar to the Yang-Mills group: both are associated with field variables that are non-dynamical and their canonical formalism entails constraints on the canonical variables. However, they are different in the following respect which is quite important: Yang-Mills transformations occur at a fixed spacetime point and is a Lie group, while the group of diffeomorphisms Diff(M), being not a Lie group, moves points around; that is the background is not fixed. This raises the problem of the fundamental ontological significance of the points of the manifold on the conceptual level. At the technical level, though, this creates obstacles. In quantum theory, the metric which is the dynamical background is subject to quantum fluctuations as well. This means that notion, such as causality and spacelike-separated that depend on the metric are dependent on the quantum state. This could lead to the conclusion that time is also state dependent and a microcausality condition[12] cannot be universally

[11]Time parameter in general relativity can be chosen arbitrarily as can be seen by the vanishing of the Hamiltonian constraint, which is also the generator of the time diffeomorphisms.

[12]The microcausality condition is written in quantum field theory as

$$[\hat{\phi}(x), \hat{\phi}(y)] = 0$$

and becomes for the quantised gravitational field

$$[\hat{\gamma}_{\mu\nu}(x), \hat{\gamma}_{\rho\sigma}(y)] = 0$$

defined as in ordinary quantum field theory.

The problem of time is attacked by several ways which are classified according to how one inserts a notion of time into the theory: before or after quantisation. These can be summarised, respectively, as follows:

1. One first reduces the degrees of freedom of the system by imposing the classical constraints of the theory and then quantising the system. This method is also called the reduced phase space quantisation.

2. One quantises the system with its full degrees of freedom using the Dirac algorithm [27]. During this procedure the quantum version of the constraints is imposed and the redundant degrees of freedom are eliminated.

There are also timeless schemes in which time is not a fundamental notion of the theory, but emerges only at the classical or semiclassical level. In this chapter only the first two categories are discussed.

The problems that arise in the context of the canonical quantisation are, in short, the following [1, 5]:

- *The ultra-violet divergence problem*

  The definition at the same point of quantum operators which are the analogues of functions of fields leads to divergences because of the perturbative non-renormalisability of the theory.

- *The operator-ordering problem*

  There is no unique way to perform the replacement of classical variables and functions with quantum operators.

- *The global time problem*

  It is unclear whether the required canonical transformations that untangle the dynamical modes of the gravitational field from non-dynamical ones exist.

- *The multiple choice problem*

  The selection of a time variable is not unique and in general a different selection results in a different quantum theory. It has not been verified whether these different quantum theories are related under a more general covariant scheme or are completely unrelated.

- *The Hilbert space problem*

  Theories belonging to the category "time before quantisation" do not deal with this problem because they can give a natural inner product and lead to a well-defined interpretative framework. However, theories of the category "time after quantisation" lead to the Wheeler-DeWitt equation which is a second-order functional differential equation and leads to problems in the construction of a Hilbert space endowed with a positive-definite inner product on the space of its solutions. Some aspects of this problem are discussed later in this chapter in more detail.

- *The spatial metric reconstruction problem*

  The separation of the canonical variables into physical and non-physical parts is invertible at the classical level. However it is not obvious whether this can happen at the quantum level.

- *The spacetime problem*

  An internal space or time coordinate in a conventional spacetime context should be independent of any background foliation on M, that is, behave as scalar fields on M. In the canonical approach, the functionals are functions of the canonical variables and there is no particular reason for satisfying the condition. This problem consists in finding such functionals.

- *The problem of functional evolution*

  The problem is the appearance of potential anomalies in the algebra of each theory. The result in both is that the consistency of the classical evolution is lost.

The first two problems are not related directly to the problem of time but they arise mainly because of the perturbative non-renormalisability of the classical theory and they do raise questions over some techniques used to attack the problem of time.

3.2 Reduced phase space quantisation/Time before quantisation

The procedure in which a choice of time is made before quantisation is considered to be a conservative approach to the problem of time. However, it was revived recently in the context of loop quantum cosmology [28]. In this case, time is part of the fixed-background structure used in the formulation of the quantum theory. The choice of a time variable before quantisation leads to the construction of the quantum theory in a standard way.

The aim is to isolate time from the true degrees of freedom before we quantise the system. To this end, we formulate the Hamiltonian gravity as described in chapter 2 [1, 5],

(i) Start by introducing a reference foliation $\mathcal{F}^{ref} : \mathcal{R} \times \Sigma \rightarrow M$ in order to define the canonical variables.

(ii) Choose a set of classical functions (embedding variables) $\mathcal{X}^A(\mathcal{F}^{ref}(x,t)) \equiv (T(x; g(t), p(t)], Z^a(x; g(t), p(t)])$, where $g(t)$ and $p(t)$ denote the metric and momentum induced from γ on the hypersurface $\mathcal{F}^{ref}_t(\Sigma)$ of M, to serve as internal time and space coordinates[13].

[13] The notation $F(x; X(y), Y(z)]$ with one parenthesis on the left and a bracket on the right indicates that the object F is both a function of coordinate points x and a function of other functions $X(y), Y(z)$ that are maps between spaces. These objects are called functionals. This notation is extensively used throughout the thesis.

(iii) Perform the canonical transformation

$$(g_{ab}(x), p^{cd}(x)) \to (\mathcal{X}^A(x), \mathcal{P}_B(x), \phi^r(x), p_s(x)) \tag{3.1}$$

where the $\mathcal{X}^A$ specify a particular choice of internal space and time coordinates and $P_B, A, B = 0, 1, 2, 3$ are their conjugate momenta. These represent $8 \times \infty^3$ degrees of freedom. $\phi^r, p_s, s = 1, 2$ represent the true degrees of freedom of the gravitational field. Note that (3.1) is not unique or valid globally because general relativity is not equivalent to a deparametrised theory.

(iv) Then, eliminate $4\times\infty^3$ of the $8\times\infty^3$ embedding variables by writing the classical constraints $H = 0, H_a = 0$ in the form

$$\mathcal{P}_A(x) + h_A(x; \mathcal{X}^B, \phi^r, p_s] \approx 0 \tag{3.2}$$

This procedure is called reduced phase quantisation and it refers to the solving of the constraints on the classical level.

(v) The remaining $4 \times \infty^3$ variables are eliminated by inserting (3.2) for $\mathcal{P}_A$ into the action

$$S = \int dt \int_\Sigma d^3x(\mathcal{P}_a \dot{\mathcal{X}}^A + p_r \dot{\phi}^r - NH - N^a H_a) \tag{3.3}$$

where all fields are functions of x and t and going to the constraint hypersurface, giving

$$S = \int dt \int_\Sigma d^3x(p^r \dot{\phi}_r - h_A(x; \mathcal{X}_t^B, \phi^r, p_s]\dot{\mathcal{X}}_t^A(x)) \tag{3.4}$$

where $\dot{\mathcal{X}}_t^A(x)$ is now a prescribed function of t and x which must not be varied, and

$$H_{true}(t) = \int_\Sigma d^3x h_A(x; \mathcal{X}_t^B, \phi^r, p_s]\dot{\mathcal{X}}_t^A(x) \tag{3.5}$$

is the true, unconstrained Hamiltonian of the system from which the equations of motion for ϕ^r, p_s can be derived. (The variables $X_t^A(x)$ can be interpreted as describing embeddings in a spacetime after these equations, together with the choice of lapse and shift, have been solved.)

(vi) Introduce wave functionals $\Psi[\phi^r(x)]$ defined on Riem(Σ) in order to quantise the constraint (3.2). It becomes

$$i\hbar\frac{\delta\Psi[\phi^r(x)]}{\delta\mathcal{X}^A(x)} = h_A(x;\mathcal{X}^B,\hat{\phi}^r,\hat{p}_s]\Psi[\phi^r(x)] \tag{3.6}$$

where the variables $\mathcal{X}^A$ have not been turned into an operator. Therefore, the quantisation procedure does not involve the configuration variables which stay classical in the same way time as in the Schrödinger equation remains classical. This equation has also the form of a local Schrödinger equation usually called Tomonaga-Schwinger equation. It consists of infinitely many equations with respect to the local bubble time $\mathcal{X}^A(x)$.

The method of quantising after selecting time has some advantages that can be summarised as follows: The time variable is external to the system described by $\hat{\phi}^r, \hat{p}_s$ and the formalism resembles the formalism of ordinary quantum field theory. In consequence, the mathematical structure of quantum field theory, that is a Hilbert-space structure, can be defined and one can recover an inner product, operators and the probability interpretation. Therefore, this approach e relies on the conventional Copenhagen interpretation of quantum theory. In this sense, it is characterised as being a conservative approach to the construction of quantum gravity because, between the Newtonian and the general relativistic approach to the notion of time, it chooses the Newtonian. The fact that this approach relies heavily on the classical notion of time is one of the main reasons for the severe problems that this approach encounters and mentioned in the previous section.

3.3 Dirac quantisation/Time after quantisation

3.3.1 The steps

The Dirac procedure for quantisation of constrained systems goes as follows [1, 27]

1. *The quantisation rule*

   The first step has to do with the identification of configuration variables and their momenta and translation of the Poisson brackets into commutators for the fundamental variables. For a general case this is written as

$$V_3 = \{V_1, V_2\} \to \hat{V}_3 = -\frac{i}{\hbar}[\hat{V}_1, \hat{V}_2] \tag{3.7}$$

   For the case of geometrodynamics,[14] the fundamental variables are the 3-metric and its conjugate momentum (the metric representation of Wheeler), that is $g_{ab}(x), p^{cd}(x)$. Their commutator is

$$[\hat{g}_{ab}, \hat{p}^{cd}] = i\hbar\delta^c_{(a}\delta^d_{b)}\delta(x, y) \tag{3.8}$$

   where δ^b_a and $\delta(x, y)$ are the discretised and continuum Dirac delta functions respectively.

2. *Quantisation of a general variable*

   The second step is to extend the quantisation procedure for the fundamental variables to a general variable $F(x; g, p]$ which is a function of the fundamental ones. However, this cannot be done in the general case. The constraints to be imposed on the wave functionals are highly non-linear functions of the canonical variables g, p and involve non-polynomial products of field operators evaluated at the same point. This results in severe problems of regularisation,

[14] This is the alternative name of the canonical form of general relativity when the variables on the phase space are the metric and its conjugate momentum. In quantum geometrodynamics, the wave functional which is the basic kinematical object, is defined on the space of 3-geometries (i.e. the superspace).

renormalisation, operator ordering and potential anomalies. However, one can employ the ‘‘rule” for the simplest form of the equations and apply the Dirac quantisation. As this intuitive rule is not enough upon which to build a theory, additional criteria must be used to find a proper quantisation procedure, such as the demand for ‘‘Dirac consistency”.[15]

3. *The representation space*

Then one should find an appropriate representation space $\mathcal{F}$ for the dynamical variables on which they should act as operators. One usually employs the functional Schrödinger picture in which operators act on wave functionals defined on an appropriate function space. In the case of gravity the state vectors are taken to be functionals $\Psi[g_{ab}(x)]$ of Riemannian metrics g on the 3-surface Σ

$$\hat{g}_{ab}\Psi[g_{ab}(x)] = g_{ab}(x)\Psi[g_{ab}(x)] \tag{3.11}$$

$$\hat{p}^{cd}\Psi[g_{ab}(x)] = -i\hbar\frac{\delta}{\delta g_{cd}(x)}\Psi[g_{ab}(x)] \tag{3.12}$$

These relations, although widely used in the canonical approach, do not define self-adjoint operators when suitably smeared[16], because there is no Lebesgue

[15]When the classical constraints $\mathcal{C}_a \approx 0$ are quantised according to the Dirac procedure give the restriction $\hat{\mathcal{C}}_a\psi = 0$. Also the requirement that the commutator of two constraints must vanish if applied on wave functions,

$$[\hat{\mathcal{C}}_a, \hat{\mathcal{C}}_b]\psi = 0 \tag{3.9}$$

is called ‘‘Dirac consistency”. This requirement holds only if the commutator has the form

$$[\hat{\mathcal{C}}_a, \hat{\mathcal{C}}_b]\psi = C^c_{ab}(\hat{g}, \hat{p})\hat{\mathcal{C}}_c\psi \tag{3.10}$$

where $C^c_{ab}(\hat{g}, \hat{p})$ are coefficients that must stand to the left of the constraints. If this is not the case, additional terms proportional to $\hbar$ appear called anomalies [1].

[16]Classical and quantum fields dependent on spacetime points are in general more singular than ordinary functions. To resolve this and to be able to define operators in quantum mechanics, one defines the smeared functions

$$\varphi[f] \equiv \int d^4x f(x)\varphi(x) \tag{3.13}$$

are required to yield well-defined operators in quantum mechanics, where $f(x)$ is a test function infinitely differentiable and of compact support defined on spacetime [29]. The test functions encompass and generalise the idea that measurements of local observables usually occur over some area instead of a point of spacetime after some coupling between the field and something else, e.g. a measure on Riem(Σ). The state vectors $\Psi[g]$ define a representation space $\mathcal{F}$ which is only an auxiliary space. There is no need for the auxiliary space to contain only physical states and consequently it is not necessary to demand that it be a Hilbert space or that the operators acting on $\mathcal{F}$ be self-adjoint. The last requirement (i.e. of self-adjointness) might not be necessary even for the constraint operators.

4. *Implementation of the constraints*

As already stated, the formal domain space for the state functionals is RiemΣ. To specify a metric $g_{ab}(x)$ at a point $x \in \Sigma$ requires six numbers, four more than the true degrees of freedom of the gravitational field. Therefore, the implementation of constraints becomes necessary so as to keep only the physical degrees of freedom.

The classical Hamiltonian and diffeomorphism constraints $H \approx 0, H_a \approx 0$ are implemented as operators on the state vectors (see e.g. [1])

$$\hat{H}\Psi \equiv \Big(-16\pi G\hbar^2 \mathcal{G}_{abcd}\frac{\delta^2}{\delta g_{ab}\delta g_{cd}} - \frac{\sqrt{g}}{16\pi G}({}^{(3)}R - 2\Lambda)\Big)\Psi = 0 \tag{3.14}$$

$$\hat{H}_a\Psi \equiv -2(D_b g_{ac})\frac{\hbar}{i}\frac{\delta\Psi}{\delta g_{bc}} = 0 \tag{3.15}$$

where D_a is the covariant derivative with respect the 3-metric g_{ab}. Equation (3.14) is called the Wheeler-DeWitt equation and there are infinitely many of these(one at each point), while equation (3.15) is called the diffeomorphism or momentum constraint. We discuss further these equations and the problems associated to them in the next subsection. Only solutions of these equations can be regarded as candidates for physical states. The solution space is denoted by $\mathcal{F}_o$. The solution space however is still large and does not coincide with the physical space $\mathcal{F}_{phys}$. Additional conditions on the wave functionals should be

measurement device. Such couplings disturb the field values near the area of measurement.

imposed, such as normalisability, to ensure the probability interpretation, even though its meaning is not clear in quantum gravity. Therefore we have the following scheme

$$\mathcal{F}_{phys} \subset \mathcal{F}_o \subset \mathcal{F} \tag{3.16}$$

5. *The role of the observables*

   In the classical theory, an observable is any physical quantity that has a weakly vanishing Poisson bracket with the Hamiltonian, $\{A, H[f, \vec{f}]\} \approx 0$. Note that the weak equality means that this relation is valid only on the subspace of the phase space on which the constraints $H_a = 0, H = 0$ hold. Equivalently, one can postulate in the quantum theory

$$[\hat{A}, \hat{H}[f, \vec{f}]] = 0 \tag{3.17}$$

   for all test functions $f, \vec{f}$ and holding on the subspace of the solutions to the Dirac constraints $\hat{H}\Psi = 0, \hat{H}_a\Psi = 0$.

6. *The role of the physical Hilbert space*

   It is not certain whether the observables have to be represented by a Hilbert space and if yes which one is the proper one. It cannot be the auxiliary space $\mathcal{F}$, since it contains states that are not solutions of the constraints, but it is unclear whether it is $\mathcal{F}_o$ or only $\mathcal{F}_{phys} \subset \mathcal{F}$.

3.3.2 Discussion on the constraints

The diffeomorphism constraint

The classical functions $\vec{H}[\vec{f}]$ are the infinitesimal generators of the diffeomorphism group of Σ and the same might be expected to apply here. This can be the case only if the Dirac algebra (2.26) can be freed from the operator-ordering problems for the $\vec{H}(\vec{f})$ and be preserved at the quantum level. Indeed, the problem of operator-ordering can be solved if we impose the condition that $\vec{H}[\vec{f}]$ form a self-adjoint representation of the Lie algebra of Diff(Σ).

The implementation of the diffeomorphism constraint is that the group Diff(Σ) acts as a group of transformations on the space RiemΣ of Riemannian metrics on Σ. This leads to a picture in which RiemΣ is fibered by the orbits of the Diff(Σ) action. Then the $\hat{H}_a \Psi = 0$ implies that the state functional Ψ is constant on the orbits of Diff(Σ) and therefore implies that the wave function is a function of diffeomorhism-invariant quantities.

The Wheeler-DeWitt equation

The Wheeler-DeWitt equation in contrast to the diffeomorphism constraint has no simple group-theoretic interpretation because of the presence of the $g^{ab}(x)$ factor in the algebra of the constraints which means that it is not a genuine Lie algebra. This is the reason why the operator-ordering problem becomes more difficult. For a choice of ordering in which all the p^{cd} variables are placed to the right of the g_{ab} variables the constraint becomes [25]

$$- \hbar^2 \kappa^2 \mathcal{G}_{abcd}(x, g] \frac{\delta^2 \Psi[g]}{\delta g_{ab}(x) \delta g_{cd}(x)} - \frac{\sqrt{|g(x)|}}{\kappa^2} R(x, g] \Psi[g] = 0 \tag{3.18}$$

where $\mathcal{G}_{abcd}$ is the DeWitt metric and is defined as where

$$\mathcal{G}^{\mu\nu\rho\sigma} = \frac{1}{2}(g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho} + C g^{\mu\nu} g^{\rho\sigma}) \tag{3.19}$$

The Wheeler-DeWitt equation is the most important aspect of the Dirac constraint quantisation approach of quantum gravity and everything must be extracted from it.

However, this equation is subject to the problems discussed in the introduction of this chapter. The ones on which we focus our discussion are:

(i) The ordering chosen in the above version of the Wheeler-DeWitt equation is not necessarily the correct one, even though it is the simplest one.

(ii) The Wheeler-DeWitt equation contains products of functional differential operators evaluated at the same spatial point which lead to divergences. A regularisation procedure might therefore be needed to be employed.

(iii) An important question is how to solve the Wheeler-DeWitt equation. A possible way is to try to solve it as a functional differential equation. However, this approach needs additional boundary conditions to be imposed on the wave functional if the eigenvalue 0 is to be included as is suggested from the form of the equation, and the theory provides no clue on how to select the boundary conditions. In practice, checking that 0 is a genuine eigenvalue is usually ignored, a potential source of misleading results.

(iv) How the solutions of the Wheeler-DeWitt equation are to be interpreted. This raises two related questions:

- The Wheeler-DeWitt equation does not have the structure of a local Schrödinger equation,[17] thus a choice of Hilbert space is not clear a priori. This is the Hilbert space problem. So, a relevant question concerns what kind of inner product is to be satisfied by the solutions.
- Since the notion of time has not been singled out, it has to be extracted from the Wheeler-DeWitt equation as an internal property of the system. How can this happen?

3.3.3 Semiclassical approximation

The WKB approximation to pure quantum gravity

In the semiclassical approximation to quantum gravity, one starts with the Wheeler-DeWitt equation plus matter degrees of freedom. The WKB ansatz for a solution $\Psi[g]$

[17] A local Schrödinger equation consists of infinitely many equations with respect to the local "bubble time" $\mathcal{X}^A(x)$ [1].

of this equation is [25, 1]

$$\Psi[g] = A[g]e^{iS[g]/\hbar\kappa^2} \tag{3.20}$$

where $S[g]$ is real and rapidly varying phase and $A[g]$ is a positive, real function of the metric that is slowly varying in the sense that

$$\hbar\kappa^2 \left| \frac{\delta A[g]}{\delta g_{ab}} \right| \ll \left| A[g] \frac{\delta S[g]}{\delta g_{ab}} \right| \tag{3.21}$$

Inserting (3.20) and (3.21) in the Wheeler-DeWitt equation, we obtain the result that, to the lowest order of the expansion of powers $\hbar\kappa^2 \backsimeq l_P^2, l_P \equiv \sqrt{\frac{G\hbar}{c^3}}$, the phase S satisfies the Hamilton-Jacobi equation of classical general relativity

$$\mathcal{G}_{abcd}(x, g) \frac{\delta S[g]}{\delta g_{ab}(x)} \frac{\delta S[g]}{\delta g_{cd}(x)} - |g|^{1/2}(x) R(x, g) = 0, \tag{3.22}$$

while the momentum constraints imply that the wave function is a function of invariant variables independent of the scheme. The amplitude factor A obeys the conservation law

$$\mathcal{G}_{abcd}(x, g) \frac{\delta}{\delta g_{ab}(x)} \left(A^2[g] \frac{\delta S[g]}{\delta g_{cd}(x)} \right) = 0 \tag{3.23}$$

The precise form of (3.22) and (3.23) depends on the choice of operator ordering in the Wheeler-DeWitt equation. Here, the simplest one is assumed, in which the functional derivatives stand to the right of the DeWitt metric.

Some notes on the method are:

1. It is only the gravitational field that is considered semiclassically. The matter fields are fully quantised.

2. Since the starting point is the Wheeler-DeWitt equation, the related problems are present. That is, the factor ordering problem, the singular operator products and the question of the boundary conditions in RiemΣ imposed for the solution

of the Wheeler-DeWitt equation.

3. The simple WKB approximation breaks down at the turning points of S and particular treatment is needed in these regions.

4. The WKB ansatz is one of the many possible types of solution to the Wheeler-DeWitt equation. There is no preferred reason why this ansatz is correct. More specifically, one cannot exclude in principle superpositions of WKB solutions. However, in such a scheme each term would define its own notion of time and it is difficult to see what this would imply. The situation resembles the Schrödinger cat problem of conventional quantum theory where a single value of a classical property must be extracted from a quantum state that is a linear superposition of eigenstates. This problem is relevant to quantum cosmology where a wave function of the form $e^{iS[g]}+e^{-S[g]}$ is a natural semiclassical solution to the Hartle-Hawking ansatz. We discuss more on quantum cosmology in a later section.

Chapter 4

Covariant Quantum Gravity

4.1 The path integral in quantum field theory and the Wick rotation

The path integral in quantum mechanics is defined as the propagator for a particle to go from a point (x', t') to a point (x'', t'') and is expressed as a formal sum over all possible continuous paths connecting these points

$$Z = \langle x'', t'' | x', t' \rangle = \int [Dx(t)] e^{iS[x,t]/\hbar} \tag{4.1}$$

The symbol $[Dx(t)]$ is a formal notation for the limiting process taken. The expression (4.1) obeys the Schrödinger equation for $t > 0$ and the composition law

$$\langle x'', t'' | x', t' \rangle = \int_{-\infty}^{\infty} du \, \langle x'', t'' | u, t \rangle \, \langle u, t | x', t' \rangle \tag{4.2}$$

This law holds because the propagator is a propagator in external time. For this reason it will not hold in quantum gravity where there is no external time parameter.

In quantum field theory, the path integral for a scalar field ϕ is defined as

$$Z[\phi] = \int [D\phi] e^{iS[\phi]} \tag{4.3}$$

In contrast to the quantum mechanical path integral that is well-defined, in quantum field theory its definition is not mathematically rigorous, since it lacks a measure-theoretic foundation. Still, though, the path integral plays an important role in quantum field theory, especially in perturbative calculations and the derivation of the Feynman rules or in gauge theories. In the latter case, the Faddeev-Popov method[18] is employed to eliminate the non-dynamical degrees of freedom. However, ambiguities from operator ordering are also present and are reflected in the integration measure. A way out of this mathematical ambiguity is to perform a rotation of time to the 4-dimensional space via the Wick rotation $t \to -i\tau$. By flipping the time, the trajectories are in imaginary time $x(\tau)$ and the metric $\eta^{\mu\nu}$ has changed from Lorentzian $(-+++)$ to Euclidean $(++++)$. The advantages of the Euclidean path integral in quantum field theory can be summarised to be:

1. Before the Wick rotation, the rapidly oscillating terms far from the classical trajectory can cause convergence problems, because of the i factor in front of the action. By turning to Euclidean time, the Euclidean action S_E is bounded from below and the convergence properties of the path integral are improved since it turns the fast oscillating integral to one with decaying exponentials (damped oscillation).

2. The extremisation procedure of the action between two instants of time is improved. In Lorentzian time, one has to deal with hyperbolic equations with initial and final values and this does not constitute a well-posed boundary problem, since there might be either no solution or infinitely many of them. By Wick rotating, this problem changes to one with elliptic equations with given boundary values which is well-posed.

In short, the idea is to perform all path integrals on the Euclidean section and then analytically continue the results anticlockwise in the complex-t plane back to the Lorentzian or Minkowski section. This rotation back is guaranteed, because the theory has a solid Hamiltonian-theoretic basis and therefore the unitary time evolution,

[18] See the Appendix for a short discussion of the Faddeev-Popov method.

causality and the existence of a Hilbert space upon which a Fock space can be defined is guaranteed.

4.2 Euclidean quantum gravity

4.2.1 The gravitational path integral

The quantisation with the path integral method can be applied to gravity as well. The gravitational path integral, or gravitational partition function, is defined as the amplitude to go from a state with metric g_1 and matter fields ϕ_1 on a surface S_1 to a state with g_2, ϕ_2 on S_2 and it is the sum over all field configurations γ and ϕ which take the given values on the initial and final surfaces S_1 and S_2 [11]

$$\langle \gamma_2, \phi_2, S_2 | \gamma_1, \phi_1, S_1 \rangle \equiv \int [D\gamma][D\phi] e^{iS[\gamma,\phi]/\hbar}. \tag{4.4}$$

The assumption here is that the surfaces S_1 and S_2 and the region between them are compact (a ‘‘closed” universe).[19] The pure gravitational partition function is

$$Z[\gamma_{\mu\nu}] = \int_M [D\gamma_{\mu\nu}] e^{iS[\gamma_{\mu\nu}]} \tag{4.5}$$

where the sum is over all metrics on a 4-dimensional manifold M divided by the diffeomorphisms group Diff(M). The gravitational action in (4.5) is

$$S[\gamma_{\mu\nu}] = \frac{1}{16\pi G} \int_M d^4x \sqrt{-\gamma}(R - 2\Lambda) - \frac{1}{8\pi G} \int_{\partial M} d^3x \sqrt{-g} K \tag{4.6}$$

where R is the Ricci scalar, Λ is the cosmological constant[20] and K is the extrinsic curvature. The surface term is added in order to ensure the multiplication of the amplitude of the path integral [30] as well as the continuity by dividing the integral

[19] The alternative assumption one considers in the case of asymptotically flat space is that the gravitational and matter fields die off in some suitable way at spatial infinity.

[20] The cosmological constant is the present value of the energy density of the vacuum space in the universe.

at intermediate surfaces.

The diffeomorphism invariance in the canonical quantisation is expressed in the Wheeler-DeWitt equation and the momentum constraints. An analogy should also exist in the case of the path integral. Indeed, the wave functions generated by the path integral satisfy the Wheeler-DeWitt equation and the momentum constraints if and only if the path integral is constructed in an invariant way. This means that the action, measure and class of paths summed over should be invariant under diffeomorphisms [31, 32]. In the canonical quantisation, in order to solve the Wheeler-DeWitt equation one needs to impose boundary conditions. The question of boundary conditions on the wave function appears in the path integral as a question on the choice of the contour of integration and on the choice of the paths that are summed over [32]. The problem of the boundary conditions becomes important in quantum cosmology and it will shortly be discussed in chapter 5.

The application of the path integral quantisation method in the case of gravity has similar problems as the one of ordinary quantum field theory such as the rapidly oscillating terms and the resulting convergence problems, however it has several differences and more difficulties, the most important of which are the following:

- The quantum field theory path integral is formulated on a fixed Minkowskian background, while in quantum gravity the metric, and consequently time, is a dynamical variable.
- The gravitational path integral contains integration over the whole 4-metric, that is, including the time variable (in the form of the lapse function). Due to the integration over time, no composition law holds[21], contrary to the quantum mechanical path integral.
- It is perturbatively nonrenormalisable and the perturbation series around the

[21] See the appendix for a short review of the basic laws of quantum theory.

fixed Minkowski metric is not appropriate for a fundamental definition. So, other methods are in order for the construction of the functional integral.

- Conceptually, an interpretation of the sum over all metrics is yet to be found.

The above remarks indicate that the covariant quantisation of the gravitational field generates too many problems. However, the power of this quantisation approach lies in the fact that, contrary to the canonical quantisation, it allows a change of topology, a fact that cannot be excluded in principle in quantum gravity. Therefore, we discuss this method in the subsequent sections and, in particular, the possible resolution of the problems that the Euclidean path integral presents.

4.2.2 Euclidean gravitational path integral

To avoid problems arising in ordinary quantum field theory, one uses the Wick rotation for the definition of the Euclidean gravitational path integral,

$$Z = \int_M [D\gamma] e^{-S_E[\gamma]} \tag{4.7}$$

where M denotes all Euclidean signature, real Riemannian metrics with a given topology. The Euclidean gravitational path integral is obtained by changing the configuration space of the theory from Lorentzian spacetime metrics $\gamma_{\mu\nu}$ with signature $(-+++)$ to Euclidean metrics $\gamma^E_{\mu\nu}$ with signature $(++++)$[22] and simultaneously replacing the complex amplitudes $e^{iS[\gamma_{\mu\nu}]}$ by real Boltzmann weights $e^{-S[\gamma^E_{\mu\nu}]}$. The definition (4.7) is a natural one if the Wick rotation $t \to -i\tau$ is applicable, as in any field theory in a fixed (Minkowskian) spacetime background where its causal structure is manifest.

The Euclidean gravitational action is obtained by Wick rotating the boundary, so that its induced metric g becomes positive definite everywhere and then, integrating

[22]In this case the configuration space or superspace according to the definition of the previous chapters contains the 4-dimensional metrics not the 3-dimensional ones.

over all positive definite metrics γ which induce the given metric g on the boundary[23]

$$S_E[\gamma] = -\frac{1}{16\pi G}\int_M d^4x(\gamma)^{1/2}(R-2\Lambda) - \frac{1}{8\pi G}\int_{\partial M} d^3x(g)^{1/2}K \qquad (4.8)$$

The resulting path integral, however, has several new problems not present in the ordinary Euclidean quantum field theory. For example,

- The change of the configuration space of the theory has no particular physical justification. Even though in ordinary quantum field theory Wick rotation renders the calculations easier, in the case of gravity one physical problem is replaced by another one even more severe as it is revealed below.

- The spacetime is not the fixed background Minkowskian spacetime. Instead, the time variable is now dynamical. It is not clear, therefore, that a Wick rotation can be defined at all and even if it does, it does not preserve the Hamiltonian structure of the theory.

- In addition, there is no a priori reason for which the two problems are related or equivalent. Not every Euclidean metric possesses a Lorentzian sector, that is, leads to a signature $(-+++)$ after a rotation $\tau \to it$.[24] Such a sector exists only for metrics with special symmetries such as homogeneity or spherical symmetry.

- The Euclidean gravitational action is not bounded from below, because of the conformal-factor problem of the Euclidean gravitational action which we discuss below. The Euclidean path integral therefore is not guaranteed to converge and a serious unsolved problem still persists after the Wick rotation.

One way to see the unboundedness of the action is to perform a conformal transformation $\bar{\gamma}_{\mu\nu} = \Omega^2\gamma_{\mu\nu}$, where Ω is a positive function which is equal to one on the

[23]The direction of the Wick rotation should be chosen to be consistent with that for the matter fields.

[24]This is the rotation to get back the initial Lorentzian spacetime which was the starting point before the Wick rotation $t \to -i\tau$.

boundary ∂M. This decomposition is fixed by requiring $\gamma_{\mu\nu}$ to satisfy a coordinate invariant condition of the form

$$R(\bar{\gamma}) = 0 \tag{4.9}$$

and fixing the boundary conditions on Ω such as $\Omega = 1$ on ∂M. The Ricci scalar and external curvature are transformed under conformal transformations as [11]

$$\bar{R} = \Omega^{-2} R - 6\Omega^{-3} \Box \Omega \tag{4.10}$$

$$\bar{K} = \Omega^{-1} K + 3\Omega^{-2} \Omega_{;a} n^a \tag{4.11}$$

where $\Box \equiv \partial_\mu \partial^\mu$ is the D' Alembertian operator, n^a is the unit outward normal to the boundary ∂M. Thus the action is written as [11]

$$S_E[\bar{\gamma}] = -\frac{1}{16\pi G} \int_M d^4x (\gamma)^{1/2} (\Omega^2 R + 6\Omega_{;\mu} \Omega_{;\nu} \gamma^{\mu\nu} - 2\Lambda \Omega^4) - \frac{1}{8\pi G} \int_{\partial M} d^3x (g)^{1/2} \Omega^2 K \tag{4.12}$$

and it becomes clear that the action can be made arbitrarily negative by choosing a rapidly varying conformal factor Ω. The presence of such metrics in the path integral leads to manifest divergence. This is the conformal-factor problem. This divergence appears in the kinematical formulation of the theory and it is distinct from the ultraviolet divergences of the Einstein theory. In order to perform regularisation and renormalisation techniques, one first has to take care of the divergence due to the appearance of metrics belonging to the same conformal class.

The first observation of the conformal-factor problem and a suggestion to solve it was initially given in [11]. The proposal was that the construction of a convergent Euclidean gravitational path integral can be manipulated by a conformal rotation. This involves a change of the variables of integration from $\gamma_{\mu\nu}$ to Ω and $\bar{\gamma}_{\mu\nu}$ which satisfy (4.9) and a distortion of the contour of the Ω-integration to complex values, i.e $\Omega \to i\Omega$ with the new Ω real. The action then becomes positive definite and the resulting integral converges. This proposal, though, is ad hoc (conjectured) and its physical meaning unclear, because it has not been derived from a canonical Hamiltonian formulation. As a result, there are no implications whether a definition of a unitary time evolution over a basis of states built upon a stable vacuum is possible in quantum gravity and any effort to investigate this becomes even harder, because of the dynamical nature of time variable in gravity. These properties are important in a theory, because they guarantee the boundedness of the Euclidean action. In addition, it starts with a divergent integral, a quantity that does not really exist, and after manipulation produces a convergent one. Considering that such a manipulation is not needed for gauge theories such as electromagnetism and Yang-Mills, a more natural resolution of this problem would be desirable. In the next section we briefly discuss later proposals to solve the conformal-factor problem.

4.3 Functional integration and conformal sickness of Euclidean quantum gravity

4.3.1 Introduction

In order to understand the conformal factor problem in its essence, it is better to work in a Lorentzian framework with real time and then perform an analytic continuation to determine the correct Euclidean form. In this way, one avoids the need for a conjectured definition of the Euclidean action similar to the one in [11] because the existence of a firm canonical Hamiltonian formulation of the theory is guaranteed and consequently the Hilbert space structure as well. An additional asset of the Lorentzian approach is that one can gain insight from the results obtained in the Hamiltonian framework and try to rederive them in the covariant, since the physical results are independent of the theoretical framework considered. The important result of the Hamiltonian analysis is in our case the fact that the conformal mode is constrained by the on-shell field equations [33]. This observation indicates that the kinetic term of the conformal mode must be cancelled, since it is not a true degree of freedom and cannot propagate. Therefore it cannot be responsible for physical instabilities as the one appearing in the Euclidean gravitational action. So the purpose is to derive the result that the conformal mode is not propagating in the context of the covariant framework. In this way additional features of the conformal factor problem might be revealed.

The observation that the conformal mode is a non-propagating degree of freedom can be used in the construction of the correct path integral measure $[Dg]$. In particular, all gravitational degrees of freedom are expected to contribute to the measure. However, as it is already mentioned, the gravitational field contains non-physical degrees of freedom that should not be taken into account during the calculation of physical quantities. This is what happens in this case for the conformal mode. It should be eliminated as non-physical. The elimination of the redundant degrees of freedom is performed via a procedure that resembles the Faddeev-Popov procedure of quantum field theory[25] [34]. The Faddeev-Popov procedure factors out the gauge group of the theory and one integrates over the equivalence classes of fields with respect to this group.

The calculation of the gravitational path integral via functional integration methods has been attempted in both perturbative [35, 33, 36] and non-perturbative context [13, 15, 14]. Our interest is for a result coming from the non-perturbative calculation that will be used in chapter 5. However, the discussion of the calculation here starts with the perturbative one since the steps are similar and can be demonstrated more clearly in the perturbative approach. Then, we briefly discuss the non-perturbative technique.

The technical issue we confront here is to specify correctly the functional measure on the coset space of spacetime manifolds modulo coordinate reparametrisations. This

[25] The Faddeev-Popov procedure is briefly described in the appendix.

will be recognised by using perturbative methods built in references [35, 36, 33]. Then this formalism is generalised to the non-perturbative case [13]. This calculation is performed by choosing a specific gauge, the proper time gauge that fits properly to the case we consider. The non-perturbative method has further been developed in [15]. There, a specific gauge is not necessary for the calculation of the measure. The advantage of this approach is that a calculation can be performed in a more detailed and explicit way and there is no need to take approximations of the result to make some conclusions. The difference between [15] and [13] is that they obtain the physical measure with a different decomposition. While in [15] the measure was decomposed into the group of diffeomorphisms to obtain the gauge fixed metrics and then the diffeomorphism orbits were further decomposed into a trace and a traceless part. Then, the conformal mode transforms due to the trace part of diffeomorphisms parametrised by a scalar field.

4.3.2 Functional integration method

The space we consider is the space of all 4-metrics $\gamma_{\mu\nu}(x)$ or else superspace denoted by M.[26] The points on this space are the 4-metrics obviously. The infinitesimal one-form $\delta\gamma_{\mu\nu}(x) \equiv h_{\mu\nu}(x)$ lies in the cotangent space at the point $\gamma_{\mu\nu}$. The inner product on the cotangent space is

$$\langle h|h\rangle_T \equiv \int d^4x\sqrt{-\gamma}h_{\mu\nu}\mathcal{G}^{\mu\nu\rho\sigma}h_{\rho\sigma}(x) \tag{4.13}$$

where $\mathcal{G}^{\mu\nu\rho\sigma}$ is the supermetric. The symmetries of the supermetric are found by the demand that the measure remains invariant under the transformations that leave the metric invariant. These transformations are the general coordinate transformations

$$x^\mu \to x^\mu + \xi^\mu(x) \tag{4.14}$$

[26] This is exactly the definition given in chapter 2 with the difference that now the points are not 3-metrics but rather 4-metrics

An additional requirement is that the functional measure be invariant under the transformations that leave invariant the scalar $ds^2 = \gamma_{\mu\nu} dx^\mu dx^\nu$. These are the passive relabeling of the coordinates. The corresponding transformation of the metric on the spacetime manifold is

$$h_{\mu\nu} \to h_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \tag{4.15}$$

This can be seen as a relabeling of coordinates on M that leaves the point $\gamma_{\mu\nu}$, that is the geometry corresponding to this metric, unchanged.

The supermetric $\mathcal{G}^{\mu\nu\rho\sigma}(\gamma)$ must transform as a contravariant 4-tensor because $\gamma_{\mu\nu}(x)$ transforms covariantly as a symmetric tensor under (4.14). Therefore the supermetric has the following symmetry properties:

- It is symmetric under the interchange of its first or last two indices

$$\mathcal{G}^{\mu\nu\rho\sigma} = \mathcal{G}^{\nu\mu\rho\sigma} = \mathcal{G}^{\mu\nu\sigma\rho} \tag{4.16}$$

- It is symmetric under interchange of its first two with the last two indices

$$\mathcal{G}^{\mu\nu\rho\sigma} = \mathcal{G}^{\rho\sigma\mu\nu} \tag{4.17}$$

$\mathcal{G}^{\mu\nu\rho\sigma}(\gamma)$ must be a purely local function of the coordinates of the superspace. This means that it must not contain derivatives of $\gamma_{\mu\nu}(x)$. The demand that the supermetric has these symmetries, in addition to the demand for ultralocality[27] leads to the

[27]This is the demand for ultralocality of the supermetric and is expressed in its definition by not inserting derivatives of the metric tensor. The metric tensor is the fundamental field coordinate of the theory. Inserting its derivatives in the definition of the supermetric introduces spurious dynamics in the kinematical definition of the inner product [33].

assumption that the supermetric must contain only terms of the form

$$\frac{1}{2}(\gamma^{\mu\rho}\gamma^{\nu\sigma} + \gamma^{\mu\sigma}\gamma^{\nu\rho}), \quad C\gamma^{\mu\nu}\gamma^{\rho\sigma} \tag{4.18}$$

Restricting the metric on M to be covariant and ultralocal determines it (up to an overall irrelevant normalisation) to be

$$\mathcal{G}^{\mu\nu\rho\sigma}(\gamma) = \frac{1}{2}(\gamma^{\mu\rho}\gamma^{\nu\sigma} + \gamma^{\mu\sigma}\gamma^{\nu\rho} + C\gamma^{\mu\nu}\gamma^{\rho\sigma}) \tag{4.19}$$

where C is an undetermined constant. The supermetric is independent of the overall constant in front of the parenthesis but the constant C is not irrelevant. The signature of the supermetric depends on it. This can be seen by decomposing the metric into traceless and trace parts

$$h_{\mu\nu} = h^{TF}_{\mu\nu} + \frac{h\gamma_{\mu\nu}}{4} \tag{4.20}$$

The traceless part is independent of C, however this is not the case for the trace part. The eigenvalue of $\mathcal{G}$ on the scalar trace mode is $1 + \frac{CD}{2}$, where D is the spacetime dimension, and thus it is dependent on the constant C. The dependence comes as follows:

- for $C > \frac{D}{2}$ the signature is positive
- for $C < \frac{D}{2}$ the signature is negative
- for $C = \frac{D}{2}$ the signature is indefinite (non-invertible)

The functional measure of the integration is defined by the Gaussian normalisation condition

$$\int [Dh_{\mu\nu}] e^{\frac{i}{2}\langle h|h\rangle_T} = 1 \tag{4.21}$$

and is invariant under the infinitesimal general coordinate transformations.

We next continue to extract the infinite gauge orbit volume in an invariant way via a procedure that resembles the Faddeev-Popov. To this end, one first introduces a change of coordinates in the tangent space of M at $\gamma_{\mu\nu}$ (this is the York decomposition)

$$h_{\mu\nu} = h^{\perp}_{\mu\nu} + (L\xi)_{\mu\nu} + (2\sigma + \frac{2}{D}\nabla_\lambda \xi^\lambda)\gamma_{\mu\nu} \tag{4.22}$$

where L is the conformal Killing form that maps vectors into traceless symmetric tensors

$$(L\xi)_{\mu\nu} \equiv \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu - \frac{2}{D}(\nabla_\lambda \xi^\lambda)\gamma_{\mu\nu} \tag{4.23}$$

The scalar σ is the gauge invariant piece of the trace, while $h^{\perp}_{\mu\nu}$ is the gauge invariant piece of the traceless part of $h_{\mu\nu}$. The choice of orthogonal coordinates on the tangent space of M is not necessary and $h^{\perp}_{\mu\nu}$ may be required to satisfy an arbitrary coordinate (gauge) condition

$$(F \cdot h^{\perp})_\mu \equiv F^\nu h^{\perp}_{\mu\nu} = 0 \tag{4.24}$$

with the only condition on F being that the operator $F \circ L$ be locally invertible so that (4.22) can be solved uniquely for ξ

$$\xi_\mu = (F \circ L)^{-1\nu}_{\mu} (F \circ h^{TF})_\nu \tag{4.25}$$

Otherwise the local chart (4.22) is singular at the point $\gamma_{\mu\nu}$. To extract the infinite gauge orbit volume generated by the gauge direction ξ_μ, we must find the Jacobian of the transformation to the new field coordinates $(h^{\perp}_{\mu\nu}, \xi_\mu, \sigma)$

$$[Dh_{\mu\nu}] = J[Dh^{\perp}_{\mu\nu}][D\sigma][D\xi_\mu] \tag{4.26}$$

This can be done by substituting the decomposition (4.22) into the inner product (4.13), completing the square of the term quadratic in ξ_μ and computing the Gaussian integrals over each of the components. This procedure will result in the following form for the Jacobian [33]

$$J = [\det_V(F \circ F^\dagger)]^{-\frac{1}{2}} \det_V(F \circ L) \tag{4.27}$$

The first factor is a normalisation factor independent of the $h_{\mu\nu}$ that makes no contribution to the Feynman rules. The second factor is recognised as the Faddeev-Popov determinant for the gauge condition (4.24) on the traceless components of $h_{\mu\nu}$. The result (4.27) is valid to all orders of perturbation theory even though it has been derived by tangent space methods involving only Gaussian integrals [33].

The Jacobian (4.27) helps us to factor out the infinite diffeomorphism gauge group volume out of the covariant quantum measure (4.26) in a manifestly covariant way. If the action is independent of the vector gauge orbit parameter ξ_μ, integration over ξ_μ would yield the infinite volume of the diffeomorphism group, so

$$\{\mathrm{Vol}(\mathcal{G})\}^{-1} \int [Dh_{\mu\nu}] = \{\mathrm{Vol}(\mathcal{G})\}^{-1} \int [D\xi_\mu] \int J[Dh^\perp][D\sigma] = \int J[Dh^\perp_{\mu\nu}][D\sigma] \tag{4.28}$$

when integrated over functions independent of ξ_μ. The final step is to extend the integration measure defined on the tangent space to a measure on the full metric. This is done by extending the coordinates on the tangent space (4.22) to coordinates on M. This is straightforward at least locally by writing

$$\gamma_{\mu\nu}(x) = \frac{\partial X^\rho}{\partial x^\mu} \frac{\partial X^\sigma}{\partial x^\nu} e^{2\sigma(X)} \gamma^\perp_{\rho\sigma}(X) \tag{4.29}$$

$$(F \cdot \gamma^\perp)_\mu = 0 \tag{4.30}$$

and σ may be fixed by the requirement that $\gamma^\perp$ has constant scalar curvature, that is the Yamabe condition:

$$R[\gamma_{\mu\nu}] = 6e^{-3\sigma} \Delta_0^\perp e^\sigma + e^{2\sigma} R^\perp \tag{4.31}$$

and $\Delta_0^\perp$ is the scalar Laplace-Beltrami operator with respect to the metric $\gamma_\perp$. The vacuum amplitude for quantum gravity can then be written in covariant form:

$$Z = \{\mathrm{Vol}(\mathcal{G})\}^{-1}\int [D\gamma_{\mu\nu}]\exp(iS_{inv}[\gamma]) \tag{4.32}$$

$$= \int J[D\gamma^{\perp}_{\mu\nu}][D\sigma]\exp\big(iS_{inv}[e^{2\sigma}\gamma^{\perp}]\big) \tag{4.33}$$

$$= [\det_V(F\circ F^{\dagger})]^{\frac{1}{2}}\int [D\sigma][D\gamma^{\perp}_{\mu\nu}]\det_V(F\circ L)|_{\gamma=e^{2\sigma}\gamma^{\perp}}\exp\big(iS_{inv}[e^{2\sigma}\gamma^{\perp}]\big) \tag{4.34}$$

and this is the final result for the invariant functional integral over geometries.

4.3.3 Non-perturbative calculation of path integral

The above mentioned method is one that can shed light on the non-perturbative calculation of the confomal part of the gravitational path integral. In this section we proceed to the identification of the physical variables by decomposing the diffeomorphisms as a traceless and a trace part and then the Faddeev-Popov procedure is employed for the conformal sector of the metric. The Jacobian of the pure scale transformations is a scalar determinant and makes the classical negative action positive [15]. In the following, for convenience, the notation for the Euclidean action is changed only for this section and is denoted as $\gamma_{\mu\nu}$. The calculation starts with the Euclidean gravitational path integral

$$Z = \int [D\gamma_{\mu\nu}]e^{-S_E} \tag{4.35}$$

that can be decomposed in terms of a conformal mode and a set of conformal equivalence class of metrics $\gamma_{\mu\nu} = e^{2\phi}\bar{\gamma}_{\mu\nu}$ The path integral can be written as

$$Z = \int [D\phi][D\bar{\gamma}_{\mu\nu}]\exp\left\{\frac{1}{16\pi G}\int d^4x e^{2\phi}\sqrt{\bar{\gamma}}[\bar{R}+6(\bar{\nabla}\phi)^2]\right\} \tag{4.36}$$

where the action is written with respect to the conformal field ϕ. It is clear that the kinetic term of the conformal mode of this action is positive and this renders the action unbounded from below. The approach to solve this problem considered here, as already discussed, is to assume that this divergence comes from the presence of non-dynamical degrees of freedom. This choice was triggered by results obtained in the Hamiltonian framework stating that the conformal mode is constrained, as well as the positive results from the covariant framework presented in the previous sections which use the same assumption. In order to eliminate the unphysical degree of freedom, one has to identify the metrics related with a gauge on the DeWitt space and eliminate them from the measure of the path integral. The outcome will be a new effective action that contains only the physical degrees of freedom.

The decomposition of the metric field is the same as in equation (4.22) using the fact that the metric transformations are given by (4.29). The coordinate transformation (4.22) generates a Faddeev-Popov determinant that contributes in the measure of the integral. The elimination of the conformal part of the Faddeev-Popov determinant of diffeomorphisms is done by a parametrisation of the trace by a scalar field. In order to do this, one separates the vector generator of the diffeomorphisms ξ^μ, into a divergenceless part represented by a vector $\hat{\xi}^\mu$ and a divergence part represented by a scalar σ

$$\xi_\mu = \hat{\xi}_\mu + \nabla_\mu \sigma \tag{4.37}$$

$$\nabla^\mu \xi_\mu = \nabla^\mu \nabla_\mu \sigma \tag{4.38}$$

After the coordinate transformation (4.22), the path integral (4.36) acquires a Jacobian determinant in the measure $\det M$

$$Z = \int [D\phi][D\gamma^{\perp}_{\mu\nu}][D\hat{\xi}^\mu][D\sigma] \det M \exp\left\{ \frac{1}{16\pi G} \int d^4x e^{2\phi} \sqrt{\bar{\gamma}} [\bar{R} + 6(\bar{\nabla}\phi)^2] \right\} \tag{4.39}$$

where

$$\det M = \det_S[8(1+2C)(-2(\nabla)^4+4\nabla_\mu\nabla^2\nabla^\mu+4\nabla_\mu\nabla_\rho\nabla^\mu\nabla^\rho)]^{1/2}\det_V\tilde{V}\det_T\tilde{T} \quad (4.40)$$

The tensor and vector parts of $\det M$ are irrelevant for the analysis presented here and they are excluded from the measure. The calculation of the scalar part of the determinant $\det M$ contributes an exponent to the action $S_{trace} = \frac{1}{2}\zeta'(0)$ where $\zeta(\lambda) = \sum_{n=1}^{\infty}\lambda^{-n}$ is the zeta function that is used in regularisation of the path integrals in curved space [37] and $\zeta'(0)$ denotes the derivative with respect to the eigenvalues λ of the matrix M when $\lambda = 0$. In the weak gravity regime, $\nabla^\mu R_{\mu\nu} \approx 0$, the scalar operator in the determinant

$$-2(\nabla)^4 + 4\nabla_\mu\nabla^2\nabla^\mu + 4\nabla_\mu\nabla_\rho\nabla^\mu\nabla^\rho \quad (4.41)$$

is approximated by

$$6\nabla^4(1 + \frac{4}{3}(\nabla^4)^{-1}\nabla_\mu R^{\mu\nu}\nabla_\nu) \quad (4.42)$$

Therefore, the determinant becomes

$$\det_S[8(1+2C)6\nabla^4] \quad (4.43)$$

After explicit calculation of the $\det_S\nabla^4$ [15] [14], the action term takes the form

$$S_{eff} = [8(1+2C)6]\frac{1}{2}\zeta'(0) + S_{class} \quad (4.44)$$

$$= -\frac{1}{64\pi^2 G}\frac{R}{6}[8(1+2C)6] + S_{class} \quad (4.45)$$

$$= -\frac{1}{64\pi^2 G}\frac{R}{6}[8(1+2C)6] - \frac{1}{16\pi G}R \quad (4.46)$$

and the final result is that the action becomes

$$S_{eff} = -\frac{R}{16\pi G}\left[1 + \frac{2(1+2C)}{\pi}\right] \tag{4.47}$$

For the case $C = -2$ that represents the Einstein's action, the Euclidean Einstein gravity has a positive definite effective action. The effect of this calculation is the change of an overall sign in the action by a minus sign. This change has also been observed in the continuum limit of the discrete calculation of the path integral [38]. In conclusion, the term that causes the conformal divergence is cancelled by the contribution of the Faddeev-Popov determinant and in particular of its conformal mode part.

Part II

Applications of Quantum Gravity

Chapter 5

Cosmology

5.1 Introduction

Cosmology is the field that studies the large-scale structure of the universe and its dynamics. It deals with questions such as the formation and evolution of the universe. The theoretical framework on which the cosmological models are built is general relativity. The present-day universe is described by a cosmological model known as Friedmann-Lemaître-Robertson-Walker (FLRW) metric

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right) \tag{5.1}$$

where $a(t)$ is the scale factor and measures the expansion and contraction of the spatial geometry. The values $k = 1, 0, -1$ correspond respectively to open, flat or closed spatial geometry. This metric is homogeneous and isotropic at large scales and indeed fits the present observations of the universe.

Another indication from the cosmological observations are that the universe is under an accelerating expansion [39, 40]. However, the current known forms of matter cannot explain exactly this phenomenon and new forms of matter have been suggested to explain it. A new suggested form of energy is dark energy which is supposed to permeate all space. Dark energy can have a non-dynamical and a dynamical form [41]. Its non-dynamical form is known as the cosmological constant Λ. The cosmological constant is a constant energy density of the order of 10^{-12} GeV/m that fills the space homogeneously.[28] The inclusion of the cosmological constant in the FLRW model leads to the ΛCDM model, the "standard model" of cosmology nowadays. The dynamical form of the dark energy consists of scalar fields with varying energy density in space and time known as quintessence and phantom energy. In particular, the phantom energy is exotic since its energy density increases with time and it violates the dominant energy condition [43].[29]

The simplest way to obtain a model with a phantom energy component is to consider a homogeneous scalar field ϕ with negative kinetic energy. The action of an ordinary scalar field minimally coupled to gravity is

$$S_\phi = \int d^4x\sqrt{-\gamma}[-\frac{1}{2}(\nabla\phi)^2 - V(\phi)] \tag{5.2}$$

This field is also called quintessence. The energy-momentum tensor field obtained by varying the scalar field action with respect to the metric is

$$T_{\mu\nu} = -\frac{2}{\sqrt{-\gamma}}\frac{\delta S_\phi}{\delta\gamma^{\mu\nu}} \tag{5.3}$$

or, explicitly written,

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \gamma_{\mu\nu}\left[\frac{1}{2}\gamma^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + V(\phi)\right] \tag{5.4}$$

The components of the energy-momentum tensor for the case of the flat FLRW background ($k = 0$) that give the energy and pressure density respectively are

$$\rho = -T^0_0 = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{5.5}$$

[28]The above mentioned value is the observational one. The estimated theoretical value is of the order of 10^{108} GeV/m. The two values have a discrepancy of the order of 10^{120} orders of magnitude. This consists the cosmological problem (see e.g [42] for more details).

[29]The dominant energy condition is interpreted as that the speed of energy flow of matter is always less than the speed of light (see e.g. [44]).

and

$$p = T_i^i = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{5.6}$$

The equation-of-state parameter is given by $w = \frac{p}{\rho}$, therefore

$$w = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)} \tag{5.7}$$

A vanishing potential $V = 0$ (free scalar field) has $w = 1$, while a vanishing kinetic energy is equivalent to a cosmological constant $w = -1$. Anything between these values but not crossing the value $w = -1$ is achievable. The phantom field corresponds to a negative kinetic energy and corresponds to a value of $w < -1$. This happens for $\dot{\phi}^2/2 < V(\phi)$. The action of the phantom field is

$$S_{phantom} = \int d^4x\sqrt{-\gamma}[\frac{1}{2}(\nabla\phi)^2 - V(\phi)] \tag{5.8}$$

which has an opposite sign to the kinetic term compared to the action (5.2). This leads to the following values for the stress-energy tensor

$$\rho = -T_0^0 = -\frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{5.9}$$

and

$$p = T_i^i = -\frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{5.10}$$

finally giving

$$w = \frac{-\frac{1}{2}\dot{\phi}^2 - V(\phi)}{-\frac{1}{2}\dot{\phi}^2 + V(\phi)} \tag{5.11}$$

The FLRW model describes the current universe very well, so it is a good candidate cosmological model from which we can extract the history of the universe as well as its fate. The FLRW metric shows that at the beginning of space and time, the universe was extremely hot and dense, confined to an infinitesimal spacetime volume. This is the well-known classical singularity called the Big Bang from which all known forms of matter and energy have originated. In order to study the evolution of the universe at this state, it is necessary to include both gravitational and quantum effects. Therefore a proper description of the origins of the universe should be done through a theory of quantum gravity applied to cosmological models. The field that studies the origins of the universe is called quantum cosmology. Quantum cosmology is motivated by the question of the initial state of the universe, meaning the conditions out of which the universe could have arisen. It is true that the Big Bang model explains a lot of observational features, however it fails to specify a certain initial quantum state out of which the universe evolved [32, 45]. The problem of the initial conditions in quantum cosmology is a very important one. It arises by the fact that the dynamical equation of the theory has an infinite set of solutions and there is no way to single out one of them. In order to specify a solution, initial and/or boundary conditions should be imposed, as in the case of quantum theory. In ordinary quantum theory, the initial conditions are specified by the conditions determined by the environment of the quantum system. However, in cosmology no such external system exists to the universe and there is no indication of a proper boundary condition. The initial conditions are thus elevated to a physical law in quantum cosmology.

The main mathematical object of quantum cosmology is the wave function of the universe [32, 45]

$$\Psi[g_{ij}(x), \phi(x), B] \tag{5.12}$$

which gives the amplitude that the universe contains a 3-surface B on which the 3-metric is $g_{ij}(x)$ and the matter field configuration is $\phi(x)$. In order to give meaning to this, three elements are necessary: (i) dynamics of the theory, in this case the functional Schrödinger equation or the Wheeler-DeWitt equation that arises after the quantisation of the cosmological model, (ii) initial and/or boundary conditions necessary to solve the dynamical equation which, in the case of quantum cosmology are elevated to a physical law, and (iii) an interpretation scheme for the wave function.

Of course, the most important demand from the theory so as to be characterised as physical, is to be able to explain the observations for the state of the universe today. One of the most important observations that a quantum cosmological model should be able to explain is the classicality of spacetime when the universe is large. The requirements imposed on a system so as to be characterised as classical constrain the form of the wave function of the universe. The requirements for a spacetime to be classical can be summarised as follows:

(i) The wave function must predict that the canonical variables are strongly correlated according to classical laws. This means that the wave function or some distribution constructed from it must be peaked about one or more classical configurations.

(ii) The quantum mechanical interference between distinct configurations should be negligible.

The form of the wave functions that commonly arise in quantum cosmology are not of wave packet form $e^{ikx-i\omega t}$ as in ordinary quantum theory, but of WKB form. They can be classified as oscillatory, having the form e^{iS}, or exponential, of the form e^{-S_E}. By using a distribution function of coordinates and momenta such as the Wigner function, one can identify the peaks of the wave function. One finds that (i) a wave function of the form e^{-S_E} predicts no correlation between q and p and thus cannot correspond to classical behaviour, (ii) a wave function of the form e^{iS} predicts a strong correlation between q and p and, as a result, it is peaked about not a single classical solution, but about a set of solutions to the field equations [32]. Thus, only the oscillatory solutions correspond to classical spacetime. These observations about the allowed form of the wave function are useful in this chapter since we study the WKB solutions of a minisuperspace model.

Finally, an interesting aspect is that, although the original wave function does not carry a particular variable playing the role of time, a notion of time may emerge for certain types of wave functions. The affine parameter along the histories about which the wave function is peaked is the time variable. So time and spacetime are only derived concepts appropriate only to certain regions of minisuperspace and dependent on initial conditions.

The present chapter is devoted to an application in the field of cosmology. Starting from a quantised FLRW model and using results coming from the non-perturbative gravity mentioned in chapter 4, we find implications about the behaviour of the universe in its early stages. Our study focuses on the one hand on the application of the tunneling method on this cosmological model and on the other hand on some implications regarding phantom cosmology.

5.2 Minisuperspace models

The configuration space of quantum cosmology is the infinite-dimensional superspace of the 3-geometries already mentioned. However, the infinite dimension sets a huge obstacle in dealing with the full formalism of quantum cosmology. One resolution to this problem is to restrict the configuration space to a finite dimension. This can be achieved by taking advantage of several symmetries of the spacetime, such as homogeneity and isotropy, to reduce the degrees of freedom of the theory. The resulting configuration space is called minisuperspace.

The minisuperspace approximation has several advantages. At a technical level, one deals with a finite-dimensional problem in the classical theory that is more easily quantised and leads to a problem of the ordinary quantum theory instead of the more complicated quantum field theory. In addition, this approach can help in the understanding several conceptual problems of quantum cosmology and quantum gravity such as the problem of time, the interpretation of the wave function of the universe and the boundary conditions.

On the other hand, minisuperspace models are not free of problems since setting most of the field modes and their conjugate momenta equal to zero is equivalent to a violation of the uncertainty principle. However, the most severe problem is that it is not even known to be part of a valid approximation to the full theory. In this respect, one can adopt the point of view that the minisuperspace models are just toy models that keep some features of the full theory in isolation from others and having nothing to do with it in the end. However, there is the opposite point of view arguing that the minisuperspace models do actually have a connection with the full theory [32]. In this thesis we do not adopt any particular point of view about the validity of the minisuperspace approximation.

We now proceed to study the mathematical form of the minisuperspace models. Typically, the general form of the 3+1 decomposed metric in the minisuperspace models is obtained by choosing a homogeneous lapse function, $N = N(t)$ and zero shift vector $\dot{N}^i = 0$

$$ds^2 = -N(t)^2 dt^2 + g_{ij}(\vec{x}, t) dx^i dx^j \tag{5.13}$$

The 3-metric g_{ij} is restricted to be homogeneous and is described by a finite number of functions of t, $q^\alpha(t)$ where $\alpha = 0, 1, ..., (n-1)$. Then the general form of the action obtained by inserting the 3+1 metric with the above restrictions is

$$S[q^\alpha(t), N(t)] = \int dt N \left[\frac{1}{2N^2} f_{\alpha\beta}(q) \dot{q}^\alpha \dot{q}^\beta + U(q) \right] \equiv \int dt L \tag{5.14}$$

The function $f_{\alpha\beta}(q)$ is the reduced DeWitt metric with indefinite signature $(-+++...)$. The form of the action is that of a relativistic particle in a curved spacetime of n dimensions with a potential. By varying the action, one obtains the dynamical and constraint equations respectively:

$$\frac{1}{N}\frac{d}{dt}\left(\frac{\dot{q}^\alpha}{N}\right) + \frac{1}{N^2}\Gamma^\alpha_{\beta\gamma}\dot{q}^\beta\dot{q}^\gamma + f^{\alpha\beta}\frac{\partial U}{\partial q^\beta} = 0 \tag{5.15}$$

$$\frac{1}{2N^2}f_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta + U(q) = 0 \tag{5.16}$$

In order to ensure consistency, the equations resulting from the variation of the action must be equivalent to those obtained if one inserts the decomposed metric into the Einstein equations. However, this is not always the case and should be checked. The Hamiltonian construction then goes as follows. The conjugate momenta are

$$p_\alpha = \frac{\partial L}{\partial \dot{q}^\alpha} = f_{\alpha\beta}\frac{\dot{q}^\beta}{N} \tag{5.17}$$

and the canonical Hamiltonian is

$$H_c = p_\alpha \dot{q}^\alpha - L = N\left[\frac{1}{2}f^{\alpha\beta}p_\alpha p_\beta + U(q)\right] \equiv NH \tag{5.18}$$

The Hamiltonian form of the action is

$$S = \int dt[p_\alpha\dot{q}^\alpha - NH] \tag{5.19}$$

from which one can see that it is a constrained system with constraint

$$H(q^\alpha, p_\alpha) = \frac{1}{2}f^{\alpha\beta}p_\alpha p_\beta + U(q) = 0 \tag{5.20}$$

The next step is to proceed to the quantisation of the model. The quantisation of the model can be done by both canonical and functional methods. The quantisation procedure follows the rules of canonical and covariant quantisation schemes discussed in the previous chapters, so we do not enter into the details here. After the quantisation, boundary conditions should be imposed on the dynamical equations in order to find a unique solution. The problem of the boundary conditions has been discussed briefly in the introduction and will not concern us further here.

5.3 Application: Signs and cosmology

We now proceed to the discussion of an application of quantum cosmology first reported in [22]. We use the formalism of the reduced phase space quantisation, of the Euclidean path integral and of the minisuperspace models to built this section. As discussed in chapter 4, the Euclidean path integral diverges because the gravitational action is unbounded from below. After the first plausible proposal to solve this problem in [11] by rotating the conformal factor in the complex plane, more rigorous treatments followed with more successful results [35, 36, 38, 13]. The observation was that the conformal mode of the gravitational field is not a dynamical degree of freedom, therefore the kinetic term related to it has to be cancelled. This can happen by including a contribution from the measure of the functional integral. In a similar way, as in the gauge theories where the Faddeev-Popov procedure is employed in order to eliminate the physically irrelevant degrees of freedom, a term in the measure appears that can be cancelled out with the terms arising from the conformal part of the metric. This procedure was explained in chapter 4. It has been shown that this cancellation of the terms related to the conformal degree of freedom can happen under the assumption for the value of the constant appearing in the DeWitt metric that determines its signature is $C < -\frac{2}{D}$. In the case of 4-dimensional Einstein gravity the value of C is $C = -2$ which indeed lies in this range.

The Euclidean action, without the boundary term, obtained from an initial Wick rotation $t \to -i\tau$ of a Lorentzian metric S_L is defined to be, as shown from eq. (4.47)

$$S_E = -\frac{1}{16\pi G}\Big[1 + \frac{2(1+2C)}{\pi}\Big] \int d^4x_E \sqrt{\gamma_E} R \tag{5.21}$$

where C is the constant in the DeWitt metric. For the case of Einstein gravity that is $C = -2$, the factor in front of the integral is positive. For convenience we denote it by $b^2 \equiv -\frac{1}{16\pi G}[1 - \frac{6}{\pi}]$ in the rest of the section. A Wick rotation back by applying $t \to i\tau$ to the above Euclidean action results in a new Lorentzian action S'_L

$$S'_L = -iS'_E = -ib^2 \int d^4x\sqrt{\gamma}R = -b^2 \int d^4x\sqrt{\gamma}R \tag{5.22}$$

where now the Euclidean action S'_E has been obtained by a Wick rotation $t \to -i\tau$ of a Lorentzian action S_L. The relation between S_L and S'_L is

$$S'_L = -\frac{b^2}{\kappa} S_L \tag{5.23}$$

where $\kappa = 1/16\pi G$. We study the possible implications of the above result in the following minisuperspace metric for cosmology

$$ds^2 = -N^2 dt^2 + a^2(t)\left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right) \tag{5.24}$$

which is similar to the metric (5.1), but now the lapse function is taken to be arbitrary. The classical Lagrangian for this metric is derived from a Legendre transform

$$S_L = p_a \dot{a} - N\left[-\frac{p_a^2}{24a} - 6ka\right] \tag{5.25}$$

where p_a is the conjugate momentum to $a(t)$ and the canonical form of the action is

$$S[a, p_a, \phi, p_\phi, N] = \int dt(p_a\dot{a} + p_\phi\dot{\phi} - NH) \tag{5.26}$$

where the total Hamiltonian is [5]

$$H = -\frac{p_a^2}{24a} - 6ka + \frac{p_\phi^2}{2a^3} + a^2 V(\phi) \tag{5.27}$$

The rescaled Lagrangian produces a rescaled conjugate momentum, that is

$$p'_a = \frac{\partial S'_L}{\partial \dot{a}} = -\frac{b^2}{\kappa} p_a \tag{5.28}$$

which results in a modified total Hamiltonian as well

$$H' = p'_a \dot{a} - S'_L = -\frac{b^2}{\kappa} H \tag{5.29}$$

The dynamics of this system is usually studied as coupled to a scalar field. The Hamiltonian of the gravitational system plus the matter Hamiltonian H_m is

$$H_{tot} = -\frac{b^2}{\kappa} H + H_m \tag{5.30}$$

However our treatment of the Euclidean-Lorentzian correspondence after Wick rotation is relevant only for the pure gravitational part, not the matter Hamiltonian. In the following, therefore, we assume that the matter Hamiltonian does not scale. Under this assumption, the total Hamiltonian seems to differ non-trivially from the original one H related to the S_L action. In the following subsections we study how this modification in the Hamiltonian affects the physics in two different cases. The first one is a study on the changes of the WKB wave function and how these changes could affect the tunneling in quantum cosmology. The second is to understand the role of the rescaling in the definition of phantom scalar fields.

Tunneling and quantum cosmology

We start by assuming an analysis of the type of reduced phase space quantisation (constrain the system and then quantise it) or choice of time before quantising, for a minisuperspace FLRW cosmological model in the presence of a scalar field. After selecting a time variable, we perform the quantisation procedure and obtain a Schrödinger equation that we solve in the lowest-order approximation with the WKB approximation similar to the solution of the Wheeler-DeWitt equation with the WKB approximation mentioned in section (3.3.3).

In the Hamiltonian (5.27) we set $k = 1$ and $V(\phi) = 0$, that is the quantum field is massless and confined in a spacetime with positive spatial curvature. Choosing as time coordinate the scale factor, $t = a(t)$, the Hamiltonian constraint takes the form

$$-\frac{p_a^2}{24t} - 6t + \frac{p_\phi^2}{2t^3} = 0 \tag{5.31}$$

Solving for the conjugate momentum of a we find for the squared conjugate momentum

$$p_a^2 = 24t\left[-6t + \frac{p_\phi^2}{2t^3}\right] \tag{5.32}$$

The squared true Hamiltonian is given by

$$h_a^2 \equiv p_a^2 = -144t^2 + \frac{12p_\phi^2}{t^2} \tag{5.33}$$

Equation (5.33) is a function of time and has 4 roots, two real and two imaginary,

$$t_{1,2} = \pm\frac{\sqrt{p_\phi}}{12^{1/4}} \tag{5.34}$$

$$t_{3,4} = \pm i\frac{\sqrt{p_\phi}}{12^{1/4}} \tag{5.35}$$

Analysis of the sign shows that in the interval between the two real roots, (t_1, t_2) we have a positive squared Hamiltonian, $h_a^2 > 0$, and thus it is a classically allowed region, while in the intervals from $(-\infty, t_1)$ and $(t_2, +\infty)$, we have $h_a^2 < 0$. Then, the Hamiltonian h_a becomes imaginary and quantum tunneling can be possible. This becomes more obvious by using the WKB approximation method to solve the time dependent Schrödinger equation

$$p_a\psi(t) = i\hbar\frac{\partial}{\partial t}\psi(t) \tag{5.36}$$

that resulted from the quantisation of the system by the reduced phase space approach and the procedure is the same as in ordinary quantum mechanics. Therefore, we derive the wave function in three regions. For $p_a^2 > 0$ they have the form

$$\psi(t) = \frac{c}{\sqrt{p_a}}e^{\pm\frac{i}{\hbar}\int_{t_1}^{t_2} dt p_a(t)} \tag{5.37}$$

where the positive sign denotes the expanding case and the negative the contracting one. p_a^2 takes negative values in two different regions, therefore the wave functions will be

$$\psi(t) = \frac{d}{\sqrt{|p_a|}}e^{\pm\frac{1}{\hbar}\int_{-\infty}^{t_1} dt|p_a(t)|} \tag{5.38}$$

and

$$\psi(t) = \frac{d}{\sqrt{|p_a|}}e^{\pm\frac{1}{\hbar}\int_{t_2}^{\infty} dt|p_a(t)|} \tag{5.39}$$

The constants c, d in equations (5.37), (5.39) are to be defined by the imposition of proper boundary conditions. In this section we will not be concerned with any choice of boundary conditions. We only consider the general form of the solutions and their behaviour in the intervals. In the case that the cosmological part of the Hamiltonian is multiplied by a factor $\mu = -b^2/\kappa$ as in (5.30), the Hamiltonian becomes

$$H = \mu\Big(-\frac{p_a^2}{24a} - 6ka\Big) + \frac{p_\phi^2}{2a^3} + a^2V(\phi) \tag{5.40}$$

which we allow to be either positive or negative according to the sign of C in (5.21). For the same values as before, $k = 1, V(\phi) = 0$ and choice of coordinate time the scale factor $a(t) = t$, the true squared Hamiltonian is

$$h_a^2 \equiv p_a^2 = -144a^2 + \frac{12p_\phi}{2\mu a^2} \tag{5.41}$$

The roots of this Hamiltonian have the form

$$t_{1,2} = \pm\frac{\sqrt{p_\phi}}{(12\mu)^{1/4}} \tag{5.42}$$

$$t_{3,4} = \pm i\frac{\sqrt{p_\phi}}{(12\mu)^{1/4}} \tag{5.43}$$

of which there are again two real and two imaginary roots. In the case $\mu > 0$, the extended Hamiltonian is positive in the interval between the real roots and negative otherwise, and for the imaginary time, in the interval between the two roots it is negative and positive otherwise. Conversely, when $\mu < 0$, as in the case for Einstein Gravity which has $C = -2$ the real and imaginary roots switch. The form of the solutions to the Schrödinger equation in the WKB approximation will again have the same form as previously, but now the wave functions depend on the parameter μ. The general solution for all the regions, classically allowed and not allowed, is

$$\psi(t) = \frac{c}{\sqrt{|p_a(t,\mu)|}} e^{\pm\frac{i}{\hbar}\int_{t_1}^{t_2} dt p_a(t,\mu)} \tag{5.44}$$

Depending on whether μ is positive or negative we recover the solution (5.44) in each interval. The dependence on the parameter μ does not have any impact on the nature of the WKB wave functions nor the tunneling behaviour. Therefore, even though the "Euclidean effective action" is no longer unbounded, the physics remains unchanged. In the case that the potential $V(\phi)$ is non-zero, the roots are given in the appendix C. In this case, here will be a distinct change in the turning points due to a change in the sign of the action. Although this is a quantitative change in the location of the turning points or the tunneling process, the system is not really affected qualitatively.

Phantom Cosmology

The above analysis shows that the standard FLRW cosmology is qualitatively unchanged, at least for the case of the closed universe, $k = 1$. However, there is one aspect where this simple change in sign can bring new physics, which is when gravity is coupled to phantom fields. As already mentioned, these scalar fields are usually taken to be those which have a negative kinetic term in the Lagrangian and give rise to negative pressure systems ([46] and references therein). They can be under certain assumptions viable candidates for dark energy. We begin by observing that the typical gravity and matter coupled system has the action

$$S = \int d^4x\sqrt{-\gamma}\left[\kappa R - \frac{1}{2}\partial_\mu\phi\gamma^{\mu\nu}\partial_\nu\phi - V(\phi)\right] \tag{5.45}$$

where we have set $\kappa = 1/16\pi G$. If we introduce the scaling due to the quantum corrections in the gravity sector, the action becomes

$$S = \int d^4x\sqrt{-\gamma}\left[-b^2 R - \frac{1}{2}\partial_\mu\phi\gamma^{\mu\nu}\partial_\nu\phi - V(\phi)\right] \tag{5.46}$$

We factor out the minus sign as an overall signature of the system

$$S = -\int d^4x\sqrt{-\gamma}\left[b^2 R + \frac{1}{2}\partial_\mu\phi\gamma^{\mu\nu}\partial_\nu\phi + V(\phi)\right] \tag{5.47}$$

and analytically continue in the scalar field $\phi' = i\phi$, assuming that $V(\phi) = \phi^4/4!$. The action takes the form

$$S = -\int d^4x\sqrt{-\gamma}\left[b^2 R - \frac{1}{2}\partial_\mu\phi'\gamma^{\mu\nu}\partial_\nu\phi' + V(\phi')\right] \tag{5.48}$$

This is slightly different from the usual phantom fields where the kinetic term changes sign, the potential remains as it is. What we have achieved here is a relative change of sign between the kinetic term of the potential and the other terms in the coupled

Lagrangian. In the case of the flat universe, that is $\kappa = 0$ one can compute the density and the pressure of this scalar field as

$$\rho = -\frac{\dot{\phi}'^2}{2} + \frac{V(\phi')}{2} \tag{5.49}$$

$$p = -\frac{1}{2}\left[\dot{\phi}'^2 + V(\phi')\right] \tag{5.50}$$

and find that $w = p/\rho < -1$ for this redefined cosmology [46]. The negative w indicates the existence of phantom fields. We conclude therefore that at least in the case of the flat FLRW universe, a change in the sign can lead to the existence of phantom fields. This model can be further investigated in various ways. For example, one can choose the case for a closed or an open universe, or select a different internal time, for example the field ϕ and then quantise the system.

Conclusion

It turns out that for Euclidean gravity the reversal of the sign of the action due to quantum correction makes an important difference in the path integral computation which becomes a convergent integral. However it does not affect the 'classical' or the semiclassical description of the system per se. Quantum corrections which produce higher curvature terms also found in [15] in the strong gravity limit are expected to affect the dynamics of the system. In this section, the WKB approximation for the tunneling wave function in quantum cosmology was used. The results remained qualitatively unchanged from those predicted using the original Lagrangian. It is remarkable though that it is able to interpret the change in sign in the gravitational sector of the theory to provide a plausible origin of phantom negative pressure. This method of creating negative pressure is confined to scalar potentials which are mapped to themselves under such an analytic continuation. Further analysis is necessary to check the validity for other forms of potentials.

Chapter 6

Detection of thermal radiation in 2+1 dimensional spacetime

6.1 Introduction

The discovery of Hawking radiation from black holes [47] is a particularly important result for theoretical physics. It implies that in the semiclassical picture of a background gravitational field on which the quantum fields propagate, matter and energy can escape from the black holes, which, when considered as objects of the classical gravitational theory, do not permit even light to escape. Since then, it has also been proved that thermalisation radiation can also be produced in Rindler spacetime[30], also known as the Unruh effect [48]. Hawking and Unruh effects are obtained in the framework of quantum field theory in curved spacetime, that is the approximation in which the spacetime is considered as fixed curved background, and are usually thought to be thermal in their nature because of their dependence on temperature.

In a more axiomatic treatment of quantum field theory in curved spacetime known as axiomatic quantum field theory, an interesting result is the thermalisation theorem that contains the mathematical structure for Hawking-Unruh effect. Briefly, the thermalisation theorem states that (see e.g. [49])

Thermalisation Theorem. *The pure state which is the vacuum from the point of view of an inertial observer is a canonical ensemble from the point of view of*

[30]The spacetime corresponding to an accelerated observer in a flat spacetime. See the appendix for a short introduction.

a uniformly accelerated observer. The temperature characterising the ensemble is proportional to the magnitude of the acceleration of the observer.

Therefore, the thermalisation theorem leads to the Unruh effect. A common misconception is that if the thermalisation theorem holds, then a uniformly accelerated detector[31] observes the Planckian spectrum.[32] However, these statements are in general non-equivalent

1. The thermalisation theorem holds (Unruh effect)

2. A uniformly accelerated detector observes the Planckian spectrum

If the two above statements were equivalent, one would expect that the Bose-Einstein spectrum[33] should be observed for integral spin particles and the Fermi-Dirac for half-integral spin particles. In fact, this is only true in even dimensions of spacetime, but not in odd ones. What a detector actually measures in odd dimensions is the spectrum of Fermi-Dirac for scalar fields and the Bose-Einstein spectrum for the half-integral spin fields. This inversion is apparent, because there is no underlying change on the Dirac bracket relations that the fields satisfy. This effect is summarised as follows:

Apparent Statistics Inversion. *A uniformly accelerated detector coupled to a quantised scalar field observes the Bose-Einstein (but not always exactly Planckian) spectrum if the dimension of the spacetime is even, while it observes the Fermi-Dirac spectrum if the dimension is odd.*

[31]We define rigorously what one means by a detector below.

[32]Planck's law is the spectral distribution for electromagnetic radiation in thermodynamic equilibrium, when there is no net flow of matter or energy

$$E_\lambda(T) = \frac{2hc^2}{\lambda^5}\frac{1}{e^{hc/\lambda k_B T} - 1} \tag{6.1}$$

[33]The Bose-Einstein distribution is the spectral distribution for integral spin particles, i.e., bosons, in thermodynamic equilibrium. At the massless limit, it reduces to the Planckian. The equivalent of Bose-Einstein distribution for half-spin particles is the Fermi-Dirac distribution.

The effect depends on the detection of particles from a ‘‘particle detector”. In particular, we are concerned with a DeWitt pointlike detector of monopole type, referred to as a DeWitt detector [50]. It is defined as an object endowed with an internal structure characterised by two internal energy levels labeled as $|g\rangle$ and $|e\rangle$ for the ground and excited state respectively and coupled linearly to a quantum field via a ‘‘monopole moment” [50, 51]. It is also assumed that the interaction with the field takes place at a point and thus the detector is an idealised point without size.

The interesting quantity is the transition rate, that is, the probability per unit proper time for the detector to make a transition from one energy eigenstate to another while the field undergoes a transition from the initial Minkowski vacuum to arbitrary excited states. The transition rate is an important quantity because it gives the notion of what is meant by a ‘‘particle” in the context of quantum field theory in curved spacetimes. In particular, when a transition happens, the detector clicks and we can say that a particle has been observed in this context. The final state of the particle is actually not observed.

An explanation of the apparent inversion of statistics was given in [49, 52]. It is actually a consequence of the global nature of the fields that cannot be eliminated on the surface on which the detector is confined. Therefore, the contribution of an additional factor related to the density of states per unit volume modulo the surface has to be taken into account. This term leads to a not perfectly Planckian spectrum of the detector and consequently to the deviation from the thermalization theorem.

It is quite interesting that it might be possible to detect the apparent inversion of statistics phenomenon in the lab in a 2-dimensional graphene sheet [53]. Graphene is a 2-dimensional allotrope of carbon with the carbon atoms arranged on a hexagonal honeycomb structure [54]. Even though graphene can be relatively easy to make, it is very difficult to be found in the free state. Therefore, it came as a surprise that it was produced in the lab [55]. However, even before that, its properties were well-

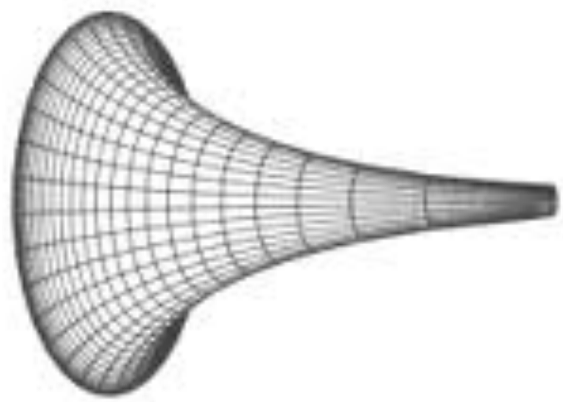

Figure 6.1: Beltrami spacetime

known and theoretically studied. The particular interest for field theorists comes from the fact that its low-energy excitations are massless, Dirac fermions and as a result, the physics of quantum electrodynamics (QED) can be mimicked on the graphene sheet. In this case though the propagating velocity equals the Fermi velocity u_F which is 300 times smaller than c. Therefore, many of the unusual properties of QED appear at much smaller speeds, a fact that makes much easier the realisation of quantum field theoretic experiments. In addition, it behaves in a metallic way so that the Dirac fermions propagate on a locally curved space, hence our interest for the study of quantum gravity effects. As it was demonstrated in [53, 56], the graphene excitations can propagate on a $2 + 1$ dimensional spacetime with a black hole if the sheet is curved such that it resembles the Beltrami spacetime of figure 6.1. Then, if one places a detector in this graphene sheet, it may be possible to detect thermal radiation.

In this chapter we deal with the attempt to theoretically extract the apparent statistics inversion in a $2+1$ dimensional spacetime with a black hole in the presence of a Dirac field by examining the response function of the detector. This will be useful to a possible future realisation of the Beltrami spacetime in the lab. We first introduce the mathematical definitions for the DeWitt detector, then we describe the $2 + 1$ dimensional spacetime and the solution of the Dirac equation we will need in order to evaluate the response function of the detector. Our results mainly concern the related Wightman function which is the necessary two-point function for the calculation of the response function.

6.2 Unruh-DeWitt detector

We proceed to review the method of calculating the response function for a Dirac field in the Rindler spacetime. This will be useful for the study of the response function of a Dirac field in a 2+1-dimensional spacetime with black hole, the so-called Banados-Teitelboim-Zanelli (BTZ) spacetime [57]. We therefore consider a point-like detector with two internal energy levels, labeled as $|g\rangle$ and $|e\rangle$ for the ground and excited state respectively. The detector couples to a Dirac field via a monopole moment. The Hamiltonian of the interaction is given by

$$H_{int} = c\chi(\tau)\mu(\tau)\psi(x(\tau)) \tag{6.2}$$

where c is a small coupling constant, μ is the monopole moment of the detector, χ is the switching function, which is positive during the interaction and vanishing elsewhere and τ is the trajectory of the detector. The spectrum of the interaction Hamiltonian is discrete. The initial and final states of the system are described by $|\tau_i\rangle = |g\rangle \otimes |\psi_i\rangle$ and $|\tau_f\rangle = |e\rangle \otimes |\psi_f\rangle$ and the transition rate is given by

$$P(E) = c^2 |\langle e| \mu(0) |g\rangle|^2 \mathcal{F}(\omega) \tag{6.3}$$

where the function

$$\mathcal{F}(\omega) = \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' e^{-i\omega(\tau-\tau')} S^{+}(x(\tau), x'(\tau')) \tag{6.4}$$

is the detector response function or power spectrum. The response function is independent of the details of the detector and is determined by the positive Wightman function $S^+(x(\tau), x'(\tau'))$, while the $|\langle e| \mu(0) |g\rangle|^2$ is the part which depends on the details of the detector.

Some notes are in order now to clarify a few details of the process.

- The mathematical analysis is performed up to first order perturbation theory.
- The adiabatic switching factor has only been introduced in order to suppress spurious transient effects. Its precise form is not essential.
- The interaction is kept switched on for a duration of proper time with order of magnitude much less than the spacing of detector's energy levels so as not to have any effect on the process.
- The transition rate formula shows that the transition rate of the DeWitt detector is proportional to the "response function" which depends only on the field but not on the structure of the detector.

Without entering to the details of the calculation [49], a scalar quantum field in the Rindler vacuum[34] turns out to have a power spectrum of the form

$$\mathcal{F}_n(\omega) = \frac{\pi}{\omega} \frac{D_n^R(\omega)}{e^{\omega/T} - 1} \tag{6.5}$$

The appearance of the Bose distribution factor $\frac{1}{e^{\omega/T}-1}$ is a consequence of the thermalisation theorem and this is its sole contribution to the response function. The theorem does not give any information on the multiplying factor $D_n^R(\omega)$ which depends on the details of the Rindler mode function. $D_n^R(\omega)$ is interpreted as the density of states at energy ω per unit volume on the $(n-1)$ hyperboloid. It turns out that the Rindler noise is identical to the thermal noise in even dimension spacetime for

[34] See the appendix for a definition.

massless scalar field because in the massless case the contribution of $D_n^R(\omega)$ multiplied with the Bose-Einstein distribution does not change the distribution for even spacetime dimensions while it does affect the spectrum in odd dimensions and turns it to a Fermi-Dirac distribution. This reveals that the expectation that the addition of one or more spacetime dimensions will not affect the power spectrum is wrong. The reason is that even though one considers the world line of the uniformly accelerated observer lying entirely in a two-dimensional plane, this does not happen for the quantum field. Quantum fields cannot be confined to this plane but they extend over the full spacetime. Therefore, as not initially expected, the presence of extra dimensions or mass does make a difference, and an important one.

We now turn in the case of the Dirac field with which we are concerned. The DeWitt detector is coupled linearly to the scalar density of the Dirac field via the Lagrangian

$$L_{int} = \mu(\tau)\psi(x(\tau)) \tag{6.6}$$

where μ is the hermitian "monopole" operator of the detector and the transition rate

$$P(E) = |\langle e|\,\mu(0)\,|g\rangle|^2 \mathcal{F}_n(\omega) \tag{6.7}$$

where $\mathcal{F}_n(\omega) = \int_{-\infty}^{\infty} d\tau e^{-i\omega\tau} S_n(\tau)$ and

$$S_n(\tau - \tau') \equiv \langle 0_M|\,\bar{\psi}(x(\tau))\psi(x(\tau))\bar{\psi}(x(\tau'))\psi(x(\tau'))\,|0_M\rangle \tag{6.8}$$

is the Wightman function for the Dirac field. By using Wick's theorem to factor this expression and by disregarding an infinite constant which does not contribute to the response function if ω is non-vanishing, we get

$$S_n(\tau - \tau') = \mathrm{tr}\, S_n^+(x(\tau), x(\tau'))S_n^-(x(\tau'), x(\tau)) \tag{6.9}$$

where $S_n^{\pm}$ are the positive and negative frequency Wightman functions respectively. This is the expression we will later need to compute the Wightman function in the $2+1$ dimensional spacetime.

6.3 Dirac field in 2+1 black hole spacetime

We give a short account of the $2+1$-dimensional spacetime with a black hole (BTZ spacetime) [57, 58]. Detailed reviews on $2+1$ dimensional gravity and black holes can also be found in [59, 60]. In the following we give a solution to the Dirac equation in this spacetime first presented in [61]. We will use the solutions of this equation in order to calculate the Wightman function and the response function of a detector confined in the $2+1$ dimensional black hole spacetime. The action in the vacuum is

$$S = \frac{1}{2\pi}\int d^2x\ dt\sqrt{-\gamma}[R + 2l^{-2}] + B \tag{6.10}$$

where B is a surface term and l is related to the cosmological constant by $-\Lambda = l^{-2}$. The equations of motion derived by variation of the action with respect to the metric are

$$R_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}R = \frac{1}{l^2}\gamma_{\mu\nu} \tag{6.11}$$

The metric of this spacetime is

$$ds^2 = -(N^{\perp})^2 dt^2 + f^{-2}dr^2 + r^2(d\phi + N^{\phi}dt)^2 \tag{6.12}$$

where the squared lapse $(N^{\perp})^2(r)$ and the angular shift N^{ϕ} are given by

$$N^{\perp} = f = (-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2})^{1/2}, \quad N^{\phi} = -\frac{J}{2r^2} \quad (|J| \leq Ml) \tag{6.13}$$

The coordinates take values in the ranges $-\infty < t < \infty$, $0 < r < \infty$ and $0 \leq \phi \leq 2\pi$. The lapse function vanishes for values of r equal to

$$r_{\pm}^2 = \frac{Ml^2}{2}\left\{1 \pm \left[1 - \left(\frac{J}{Ml}\right)^2\right]^{1/2}\right\} \tag{6.14}$$

and this designates the horizons of the black hole. In particular, the external horizon of the black hole is located at r_+. The horizon exists only if

$$M > 0, \quad |J| \leq Ml \tag{6.15}$$

where M is the mass and J is the angular momentum of the black hole. We can write the lapse, mass and angular momentum as functions of the outer and inner horizons

$$N^{\perp} = f = \left(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}\right)^{1/2} = \left[\frac{(r^2 - r_-^2)(r^2 - r_+^2)}{r^2 l^2}\right]^{1/2} \tag{6.16}$$

and

$$M = \frac{r_+^2 + r_-^2}{l^2} \quad J = \frac{2r_+ r_-}{l} \tag{6.17}$$

Then, the metric can be written in the form

$$ds^2 = -\frac{(r^2 - r_-^2)(r^2 - r_+^2)}{r^2 l^2}dt^2 + \frac{l^2 r^2}{(r^2 - r_-^2)(r^2 - r_+^2)}dr^2 + r^2\left(d\phi - \frac{r_+ r_-}{lr^2}dt\right)^2 \tag{6.18}$$

After a change of the coordinate r to $r^2 = r_+^2 \cosh^2\mu - r_-^2 \sinh^2\mu$, the metric changes to

$$ds^2 = -\sinh^2\mu\left(r_+\frac{dt}{l} - r_- d\phi\right)^2 + l^2 d\mu^2 + \cosh^2\mu\left(-r_-\frac{dt}{l} + r_+ d\phi\right)^2 \tag{6.19}$$

or, if we define the Killing directions

$$x^+ = \frac{r_+ t}{l} - r_-\phi, \quad x^- = \frac{-r_- t}{l} + r_+\phi \tag{6.20}$$

the metric takes the final form with respect to x^+, x^-

$$ds^2 = -\sinh^2\mu(dx^+)^2 + l^2 d\mu^2 + \cosh^2\mu(dx^-)^2 \tag{6.21}$$

The next step is to enter this metric in the Dirac equation and solve it for non-minimally coupled fermions

$$\gamma^1 l^{-1}\Big(\partial_\mu + \frac{\sinh\mu}{2\cosh\mu} + \frac{\cosh\mu}{2\sinh\mu}\Big)\psi + \gamma^0\frac{\partial_{x^+}\psi}{\sinh\mu} + \gamma^2\frac{\partial_{x^-}\psi}{\cosh\mu} + \frac{1}{2l}\psi = 0 \tag{6.22}$$

We assume that the solution has the form

$$\psi = \frac{e^{-i(k^+x^+ - k^-x^-)}}{\sqrt{\sinh\mu\cosh\mu}}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

where we have defined

$$k^+ = \frac{\omega - m\Omega}{2\pi l T_H} = \frac{\omega r_+ - m\Omega r_+}{2\pi l r_+ T_H} \tag{6.23}$$

$$k^- = \frac{\omega r_- - (r_+ m)/l}{2\pi l r_+ T_H} \tag{6.24}$$

with $\Omega = \frac{J}{2r_+^2}$ is the time eigenfrequency, m is the azimuthal quantum number and $T_H = \frac{r_+^2 - r_-^2}{2\pi l^2 r_+}$ the Hawking temperature. The equations then are written as

$$\Big(\frac{d}{d\mu} - \frac{ilk^+}{\sinh\mu}\Big)\psi_2 = -\Big(\frac{1}{2} - \frac{ilk^-}{\cosh\mu}\Big)\psi_1 \tag{6.25}$$

$$\Big(\frac{d}{d\mu} + \frac{ilk^+}{\sinh\mu}\Big)\psi_1 = -\Big(\frac{1}{2} + \frac{ilk^-}{\cosh\mu}\Big)\psi_2 \tag{6.26}$$

The solutions to the above equations are

$$\frac{\psi_1}{\sqrt{\sinh\mu\cosh\mu}} = \Big[\Big(\frac{1+\sqrt{z}}{\sinh\mu}\Big)^{1/2} + \Big(\frac{1-\sqrt{z}}{\sinh\mu}\Big)^{1/2}\Big] z^{(1+ilk^+)/2}(1-z)^{-1/2}\frac{\Gamma(1-\alpha)\lambda\Gamma(\lambda)}{\beta\Gamma(\beta)}P_{-\alpha}^{(\lambda,-1)}(1-2z) \tag{6.27}$$

and

$$\frac{\psi_2}{\sqrt{\sinh\mu\cosh\mu}} = \Big[\big(\tfrac{1+\sqrt{z}}{\sinh\mu}\big)^{1/2} - \big(\tfrac{1-\sqrt{z}}{\sinh\mu}\big)^{1/2}\Big] z^{ilk^+/2}(1-z)^{-1/2}\big(\tfrac{-\lambda}{\alpha}\big)\tfrac{(1-\alpha)\Gamma(1-\alpha)\Gamma(\lambda)}{\beta\Gamma(\beta)} P_{-\alpha}^{(\lambda,-1)}(1-2z) \tag{6.28}$$

where we have defined

$$z = \tanh^2\mu \tag{6.29}$$

$$\alpha = \frac{ilk^+}{2} + \frac{ilk^-}{2} + \frac{1}{2} \tag{6.30}$$

$$\beta = \frac{ilk^+}{2} - \frac{ilk^-}{2} \tag{6.31}$$

$$\lambda = \alpha + \beta = ilk^+ + \frac{1}{2} \tag{6.32}$$

and $P_b^a(z)$ are the Jacobi polynomials. This result will be useful in the calculation of the Wightman function

$$S^+(x,x') =< 0|\psi(x)\bar{\psi}(x')|0> . \tag{6.33}$$

A preliminary result shows that the Wightman function is of the form of a matrix times a thermal distribution:

$$\psi\bar{\psi} = \begin{pmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{pmatrix} \times \frac{2i}{\cosh^2[n(k^+ + k^-)](e^{\omega/2T_R}+1)(e^{\omega/2T_L}-1)} \tag{6.34}$$

where Ψ_{ij} are yet to be specified. In order to find this result we have set $m = 0$. These results indicate that the spectrum has the form of a product of a right and a left Planckian terms with right and left temperatures T_R and T_L respectively that are defined through the relations $T_L^{-1} = \frac{1}{T_H}(1 - \frac{r_-}{r_+})$ and $T_R^{-1} = \frac{1}{T_H}(1 + \frac{r_-}{r_+})$. This form of the spectrum is relevant to the conformal field theory close to the horizon. A more complete calculation of the response function however might give the full picture on the inversion of statistics. It is expected that the contribution of the Ψ_{ij} terms will be nontrivial and therefore it will affect the spectrum.

Chapter 7

Conclusions

Quantum gravity is an open research field that explores the old fundamental principles of physics and looks for new ones. In this thesis, we reviewed the already known quantisation methods of the classical gravitational theory, general relativity and discussed their implementations for the formulation of quantum gravity. As already stated in the introduction, quantum gravity lacks an experimental basis because it is expected that quantum gravity effects are only strong enough at the Planck scale. Other methods therefore are in order so as to establish contact with the physical world. These are briefly described as the field of quantum gravity phenomenology. The applications in this respect are (i) a cosmological model of the early universe and (ii) the detection of quantum fields with half-integer spin in a spacetime with odd spacetime dimensions.

The cosmological model employs the minisuperspace approach, that is the infinite degrees of freedom from superspace are eliminated and we are left with a finite number. The configuration space of the finite number of variables left is called minisuperspace. The method of quantisation that is used is the reduced phase space, or, the elimination of the time coordinate before the quantisation of the system. Even though this approach generates problems in the road for rendering gravity quantised, it does have some interesting advantages as it was discussed in chapter 3. The Hamiltonian derived in this way was then used in the context of Euclidean quantum gravity. When one tries to solve the conformal factor problem of Euclidean general relativity by non-perturbative methods as discussed in chapter 4, an interesting result appears. A factor containing a constant depending on the supermetric multiplies the action. The results of the study of the model first appeared in [22] and showed that the tunneling of the universe will not be affected by this constant. On the other hand, under the presence of a scalar field when one attempts to perform a Wick rotation, a change of sign appears in the kinetic term of the field that could be interpreted as phantom energy. The inclusion of such fields could be useful in modern cosmology in order to explain the accelerating expansion of the present universe as well as the very small value of the cosmological constant Λ which has been one of the most important unsolved problems of cosmology nowadays.

The second model is a study on the response function of the detector confined in a $2 + 1$ spacetime with black hole in the presence of a Dirac field. The theoretical calculation of the probability rate of a detector in an odd-dimensional spacetime under the presence of a Dirac field has become interesting during the last few years after the production of graphene sheets in the lab [62]. The quantum field theoretic effects in graphene resemble the quantum field theoretic effects on a $2+1$ dimensional spacetime [54, 63, 64]. Therefore, an open possibility is to be able to simulate these effects in the lab [53, 56]. The realisation of experiments like this is certain to open new horizons in quantum gravity phenomenology and consequently quantum gravity. Therefore both theoretical as well as experimental preparation is necessary for the conduct of these experiments.

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Appendix A

Aspects of General Relativity

A.1 Causal structure of spacetime

In this section, we follow the definitions in [44]. In order to define what a globally hyperbolic manifold is, a few necessary definitions are in order.

- *Causal curve* is a curve whose tangent vector is timelike or null at all points in the curve.
- *The future domain of dependence* $D^+(\Sigma)$ *of a set* Σ is defined as follows:

$$D^+(\Sigma) = \Big\{ p \in M \Big| \text{every past inextendible causal curve through p intersects } \Sigma \Big\} \tag{A.1}$$

  and equivalently for the past domain of dependence of Σ.

- *Domain of dependence* of Σ is therefore the union of the past and future domains of dependence

$$D(\Sigma) = D^+(\Sigma) \cup D^-(\Sigma) \tag{A.2}$$

- A subset $S \subset M$ is said to be *achronal* if there do not exist $p, q \in S$ such that q belongs to the chronological (i.e., timelike) future of p.
- A closed achronal set Σ for which $D(\Sigma) = M$ is called a Cauchy surface.
- A spacetime $(M, \gamma_{\mu\nu})$ which possesses a Cauchy surface Σ is said to be globally hyperbolic.

Appendix B

Aspects of Quantum Field Theory

B.1 The path integral in quantum theory

Here we follow [65]. The path integral is defined independent of the limiting procedure chosen as

$$K(x_N, x_0) = \langle x_N | x_0 \rangle = \int_{x_0}^{x_N} [Dx(t)] e^{iS[x_N, x_0]/\hbar} \tag{B.1}$$

where x_0 is the initial spacetime point and x_N the final one and $[Dx(t)]$ is a formal notation for the following limiting process

$$\langle x_N | x_0 \rangle = \lim_{N \to \infty} \int dx_1 dx_2 ... dx_{N-1} \left(\frac{mN}{2\pi i t \hbar} \right)^{N/2} \prod_{j=0}^{N-1} \exp\left\{ -\frac{m(x_{j+1} - x_j)^2 N}{2it\hbar} - \frac{itV(x_j)}{\hbar N} \right\} \tag{B.2}$$

where m is the mass of the particle, V the potential and $t = t_N - t_0$.
The rules for the path integral are

- Amplitudes for events occurring in succession in time multiply

$$K(b, a) = \int [Dx(t)] e^{i(S[b,c] + S[c,a])/\hbar} = \int_{x_c} dx_c K(b, c) K(c, a) \tag{B.3}$$

- For more events in succession, the composition law holds. This law holds because the propagator is a propagator in external time.

$$K(b, a) = \int_{x_c} \int_{x_d} dx_c dx_d K(b, c) K(c, d) K(d, a) \tag{B.4}$$

Postulates

Postulate I. *(probability-theoretical principle)*
The probability (amplitude) for a system to go from state A to state B equals the sum of the probabilities (probability amplitudes) of going from state A to state B extending over all possible paths that connect A and B.

Postulate II. *(complex probability amplitude and introduction of $\hbar$) The probability* *amplitude for every path has the same magnitude and a phase given by the classical action for that path measured in units of $\hbar$.*

B.2 Time in conventional quantum theory and quantum field theory

The following observations about the nature of time in conventional quantum theory are reported in [5]. The conventional quantum theory is formulated with a Newtonian notion of time, that is time is an external parameter to the system. This fact is manifest in the formulation of the theory in the following ways:

1. Time is not a physical observable since it is not represented by an operator, but rather it is treated as an external parameter. This is reflected in the time-dependent Schrödinger equation

$$i\hbar \frac{d\psi_t}{dt} = \hat{H}\psi_t \tag{B.5}$$

2. It is difficult to describe quantum mechanically a truly closed system. Since the ultimate closed system is the universe, this explains the difficulty to formulate a quantum theory of the universe in the context of conventional quantum theory.

3. The idea that the events happen at a single time plays crucial role in conventional quantum theory. In particular, the notion of measurement and consequently the value of an observable measured are defined at a specific moment of time in the context of Copenhagen interpretation. In addition, the conservation of the scalar product on a Hilbert space of states under time evolution and the unitarity requirement that probabilities sum to one are connected with the notion of an external time. Finally, the commutation relations are defined for a specific value of the time parameter and this feature is essential to the selection of the observables from which the Hilbert space of a quantum system is constructed.

All the above observations hold in the quantum field theory which is formulated based on special relativity. However, the status of time in SR is not fundamentally different from the conventional quantum theory.

B.3 Statistical mechanics and Euclidean field theory

Here we follow the treatment in [66]. To define the path integral more rigorously, one performs a Wick rotation $t \to -i\tau$. In the following presentation we study how this can be achieved for the simple case of a scalar field theory. The Minkowskian path integral is

$$Z = \int D\phi e^{(i/\hbar)\int d^4x[\frac{1}{2}(\partial\phi)^2 - V(\phi)]} \equiv \int D\phi e^{iS[\phi]/\hbar} \tag{B.6}$$

from which the Euclidean functional integral is defined after the Wick rotation has been performed

$$Z = \int D\phi e^{-(1/\hbar)\int d_E^3 x[\frac{1}{2}(\partial\phi)^2 + V(\phi)]} \equiv \int D\phi e^{-S_E[\phi]/\hbar} \tag{B.7}$$

where $d^4x = -id_E^4x$ and $d_E^4x \equiv d\tau d^3x$. In (B.6) $(\partial\phi)^2 = (\frac{\partial\phi}{\partial t})^2 - (\vec{\nabla}\phi)^2$ while in (B.7) $(\partial\phi)^2 = (\frac{\partial\phi}{\partial\tau})^2 + (\vec{\nabla}\phi)^2$. The unit $(\vec{\nabla}\phi)^2 + V(\phi)$ does not change under a Wick rotation.

The Euclidean action may be regarded as a static energy functional integral of the field $\phi(x)$. Thus, given a configuration $\phi(x)$ in a 4-dimensional space, the more it varies the less probable it is to contribute to the Euclidean functional integral. The Euclidean functional integral has similarities with the partition function used in statistical mechanics. Indeed, the partition function of a quantum scalar field with Hamiltonian H is given by

$$Z = \mathrm{tr}\, e^{-\beta H} = \sum_n < n|e^{-\beta H}|n >= \int D\phi e^{-\int_0^\beta d\tau \int d^3x L(\phi)} \tag{B.8}$$

with the integral evaluated over all paths $\phi(\vec{x}, \tau)$ such that the periodic boundary condition holds $\phi(\vec{x}, 0) = \phi(\vec{x}, \beta)$. As a result, one can study a field theory at finite temperature by rotating to Euclidean space and imposing the periodic boundary condition.

The connection between temperature and cyclic imaginary time comes from the fact that e^{-iHT} of quantum physics and $e^{-\beta H}$ of thermal physics are formally related by analytic continuation.

B.4 Faddeev-Popov procedure

We assume that the path integral has the form

$$Z \equiv \int [Df_{\mu\nu}] e^{iS[f_{\mu\nu}]} \tag{B.9}$$

where $f_{\mu\nu}$ is invariant under the gauge transformations $f_{\mu\nu} \to f_{\mu\nu} - \partial_\mu\epsilon_\nu - \partial_\nu\epsilon_\mu \equiv f^\epsilon_{\mu\nu}$. This gauge invariance makes the integral formally infinite because the integration is performed over gauge orbits. In order to get rid of these, we apply the Faddeev-Popov procedure by following the steps

1. First choose a gauge constraint $G_a[f]$, where a is the number of gauge conditions, in order to fix the gauge. Then the integration in the path integral is performed over the constraint subspace $G_a[f]$.

2. To implement this, one defines a functional $\Delta_G[f]$ through

$$\Delta_G[f] \int [D\epsilon] \prod_a \delta(G_a[f]) = 1 \tag{B.10}$$

The integration measure is a formal integration over the gauge group and is invariant. Using this invariance of the measure, one can show that $\Delta_G[f]$ is gauge invariant, that is $\Delta_G[f] = \Delta_G[f^\epsilon]$.

3. Introduce "1" into the path integral

$$Z = \int [D\epsilon] \int [Df] \Delta_G[f] \prod_a \delta(G_a[f^\epsilon]) e^{iS[f]} \tag{B.11}$$

make the substitution $f^\epsilon \to f$ and use the gauge invariance of the determinant we have

$$Z = \int [D\epsilon] \int [Df] \Delta_G[f] \prod_a \delta(G_a[f]) e^{iS[f]} \tag{B.12}$$

Now the infinite term $\int [D\epsilon]$ that comes from the integration over the volume of the gauge orbits can be omitted. The path integral then can be defined as

$$Z \equiv \int [Df] \Delta_G[f] \prod_a \delta(G_a[f]) e^{iS[f]} \tag{B.13}$$

and depends on the gauge G only formally, that is the results are independent of the choice of gauge.

B.5 Quantum field theory in flat spacetime

B.5.1 Scalar field in Minkowski space

We follow the treatment of [67]. The Klein-Gordon equation for a scalar field is

$$(\Box + m^2)\phi = 0 \tag{B.14}$$

where $\Box \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$ and $\eta_{\mu\nu} \equiv \text{diag}(-1, 1, 1, 1)$ is the Minkowski metric. The solutions of (B.14) are of the form of the plane waves

$$u_k(t, \mathbf{x}) \propto e^{i\mathbf{k}\mathbf{x} - i\omega t} \tag{B.15}$$

where

$$\omega \equiv (k^2 + m^2)^{1/2}, \quad k = |\mathbf{k}| = (\sum_{i=1}^{n-1} k_i^2)^{1/2} \tag{B.16}$$

The solutions of positive frequency modes with respect to t are

$$\frac{\partial}{\partial t} u_{\mathbf{k}}(t, \mathbf{x}) = -i\omega u_k(t, \mathbf{x}), \quad \omega > 0 \tag{B.17}$$

We define the scalar product of two scalar fields on a spacelike hypersurface of simultaneity at instant t as

$$(\phi_1, \phi_2) = -i \int d^{n-1}x \{\phi_1(x)\partial_t\phi_2^*(x) - [\partial_t\phi_1(x)]\phi_2^*(x)\} \tag{B.18}$$

and the orthogonality condition holds

$$(u_{\mathbf{k}}, u_{\mathbf{k}'}) = 0, \quad \mathbf{k} \neq \mathbf{k}' \tag{B.19}$$

We choose the normalisation constant to be

$$u_{\mathbf{k}} = [2\omega(2\pi)^{n-1}]^{-1/2} e^{i\mathbf{k}\mathbf{x}-i\omega t} \tag{B.20}$$

Then the functions $u_{\mathbf{k}}$ are normalised with respect to the scalar product

$$(u_{\mathbf{k}}, u_{\mathbf{k}'}) = \delta^{D-1}(\mathbf{k} - \mathbf{k}') \tag{B.21}$$

where $\delta^n(\mathbf{x} - \mathbf{x}')$ denotes the n-dimensional continuous Dirac delta function. The quantisation of the system follows by replacing the Poisson with Dirac brackets

$$[\phi(t, \mathbf{x}), \phi(t, \mathbf{x}')] = 0 \tag{B.22}$$

$$[\pi(t, \mathbf{x}), \pi(t, \mathbf{x}')] = 0 \tag{B.23}$$

$$[\phi(t, \mathbf{x}), \pi(t, \mathbf{x}')] = i\delta^{D-1}(\mathbf{x} - \mathbf{x}') \tag{B.24}$$

where π is the canonically conjugate variable to ϕ. The field modes (B.20) and their conjugate variables form a complete orthonormal basis so the ϕ may be expanded as

$$\phi(t, \mathbf{x}) = \sum_{\mathbf{k}} [a_{\mathbf{k}} u_{\mathbf{k}}(t, \mathbf{x}) + a^{\dagger}_{\mathbf{k}} u^*_{\mathbf{k}}(t, \mathbf{x})] \tag{B.25}$$

Then the Poisson brackets for ϕ, π are equivalent to

$$[a_{\mathbf{k}}, a_{\mathbf{k}'}] = 0 \tag{B.26}$$

$$[a^{\dagger}_{\mathbf{k}}, a^{\dagger}_{\mathbf{k}'}] = 0 \tag{B.27}$$

$$[a_{\mathbf{k}}, a^{\dagger}_{\mathbf{k}'}] = \delta_{\mathbf{k}\mathbf{k}'} \tag{B.28}$$

The operators $a_{\mathbf{k}}, a^{\dagger}_{\mathbf{k}}$ are the annihilation and creation operators defined as

$$a_{\mathbf{k}} |0_M\rangle = 0, \ \forall \mathbf{k} \tag{B.29}$$

$$a^{\dagger}_{\mathbf{k}} |0_M\rangle = |1_{\mathbf{k}}\rangle, \ \forall \mathbf{k} \tag{B.30}$$

where $|0_M\rangle$ is the vacuum or no-particle state with respect to a static observer in the Minkowski spacetime.

B.5.2 Scalar field in Rindler space

The Rindler spacetime results from a change of coordinates in the Minkowski spacetime. If we consider the 2-dimensional Minkowski spacetime,

$$ds^2 = -dt^2 + dx^2 \tag{B.31}$$

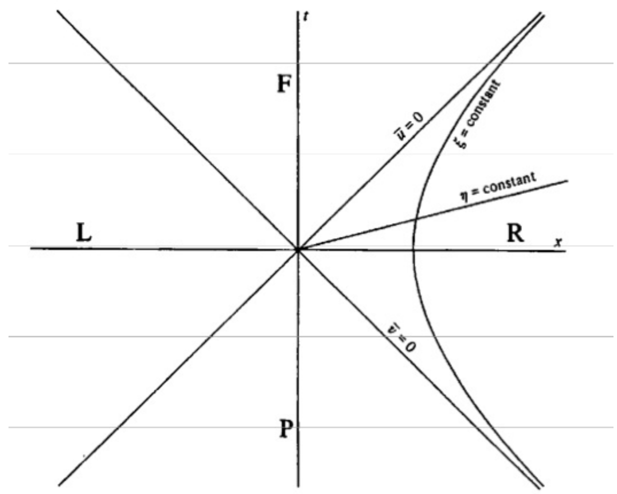

Figure B.1: Rindler diagram

a change in the (t, x) coordinates such that

$$t = a^{-1}e^{a\xi}\sinh a\eta \tag{B.32}$$

$$x = a^{-1}e^{a\xi}\cosh a\eta \tag{B.33}$$

where $a = const > 0$ and $-\infty < \eta, \xi < \infty$ gives the metric

$$ds^2 = e^{2a\xi}(-d\eta^2 + d\xi^2) \tag{B.34}$$

The coordinates (η, ξ) cover only the $x > |t|$ portion of Minkowski space. The lines $\eta = const$ are straight lines while the lines $\xi = const$ are hyperbolae

$$ae^{-a\xi} = \alpha^{-1} = \text{proper acceleration} \tag{B.35}$$

where the proper acceleration is the acceleration of the reference system of the accelerated observer. Therefore the hyperbolae represent accelerated observers. Diagram B.1 shows the null structure of the Rindler spacetime.[35] The wave equation in this spacetime is

$$e^{2a\xi}\Box\phi = \Big(-\frac{\partial^2}{\partial\eta^2} + \frac{\partial^2}{\partial\xi^2}\Big)\phi = 0 \tag{B.36}$$

with solutions

$$u_{\mathbf{k}} = (a\pi\omega)^{-1/2}e^{ik\xi\pm i\omega\eta},\ \omega = |k| > 0,\ -\infty < k < \infty \tag{B.37}$$

The upper sign applies in region L while the lower in the region R. These modes have positive frequency with respect to the timelike Killing vector $+\partial_\eta$ in R and $-\partial_\eta$ in L and satisfy

$$\mathcal{L}_{\pm\partial_\eta}u_k = -i\omega u_k \tag{B.38}$$

[35] This figure is from the book [67].

In order to expand the field ϕ with respect to these modes we have to define a complete set. This can be done by defining the following complete sets:

$$^{R}u_k = \begin{cases} (4\pi\omega)^{-1/2} e^{ik\xi - i\omega\eta} & \text{in R} \\ 0 & \text{in L} \end{cases}$$

in the R region and

$$^{L}u_k = \begin{cases} (4\pi\omega)^{-1/2} e^{ik\xi + i\omega\eta} & \text{in L} \\ 0 & \text{in R} \end{cases}$$

in the L region. Then the field may be expanded as

$$\phi = \sum_{k=-\infty}^{\infty} (b_k^{(1)L} u_k + b_k^{(1)\dagger L} u_k^* + b^{(2)R} u_k + b^{(2)\dagger R} u_k^*) \tag{B.39}$$

Then the procedure to define the Rindler vacuum is the same as in the Minkowski space and we get

$$b_k^{(1)} |0_R\rangle = b_k^{(2)} |0_R\rangle = 0 \tag{B.40}$$

B.6 Dirac equation in flat and curved space

In Minkowski space and in cartesian coordinates, the Dirac equation is

$$(i\gamma^\mu + m)\,\psi(x) = 0 \tag{B.41}$$

where γ^μ are 4×4 matrices called Dirac matrices that satisfy the relation

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu} \tag{B.42}$$

and $\psi(x)$ is the Dirac spinor. In order to go to curved space, the metric formalism is not enough. So one has to introduce an anholonomic basis, that is, non coordinate. This is required because the wave components of the spinors transform with respect to a two-values representation of the Lorentz group and therefore one needs to introduce a local Lorentz group and local orthonormal frames.

This can be achieved by introducing the tetrad formalism in which a basis $e_n = (e_0, e_1, e_2, e_3)$ is chosen at each point. The tetrads are related to the tangent vectors along coordinates lines by

$$e_n = e_\nu^\mu \partial_\mu \tag{B.43}$$

The usual choice is that the tetrads are orthonormal

$$e_n e_m \equiv g_{\mu\nu} e_m^\mu e_n^\nu = \eta_{\mu\nu} \tag{B.44}$$

The anholonomic Dirac matrices can be defined by

$$\gamma^n \equiv e_\nu^n \gamma^\nu \tag{B.45}$$

where $e^{\nu}_{n} e^{m}_{\nu} = \delta^{m}_{n}$. It can be shown that they satisfy the relation

$$\{\gamma^n, \gamma^m\} = 2\eta^{nm} \tag{B.46}$$

that can be proved from the following one

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{B.47}$$

This last relation is true because of the equivalence principle. The equivalence principle also demands the replacement of the partial derivative in the Dirac equation with the covariant one

$$\partial_\mu \to D_\mu \equiv \partial_\mu + \frac{i}{4}\sigma^{mk}\omega_{\mu mk} \tag{B.48}$$

where $\sigma^{mk} = i[\gamma^m, \gamma^n]/2$ is the generator of the Lorentz group and $\omega_{\mu mk}$ denotes the components of the connection. The Dirac equation in the curved space is therefore written

$$\left(i\gamma^n D_n + m\right)\psi(x) = 0 \tag{B.49}$$

Appendix C

Miscellaneous

C.1 Non-degenerate turning points

This is the treatment reported in [22]. General form of the solutions for

$$p_a^2 = 24(-6t^2 + \frac{p_\phi^2}{2t\mu} + \frac{t^4 V(\phi)}{\mu}) \tag{C.1}$$

$$t_{1,2} = \pm\frac{\sqrt{2}}{2V(\phi)M^{1/3}}[V(\phi)M^{1/3}K]^{1/2} \tag{C.2}$$

$$t_{2,3} = \pm\frac{1}{2V(\phi)M^{1/3}}[V(\phi)M^{1/3}(-K + 12\mu M^{1/3} + i\sqrt{3}(M^{2/3} - 16\mu^2))]^{1/2} \tag{C.3}$$

$$t_{5,6} = \pm\frac{1}{2V(\phi)M^{1/3}}[V(\phi)M^{1/3}(-K + 12\mu M^{1/3} - i\sqrt{3}(M^{2/3} - 16\mu^2))]^{1/2} \tag{C.4}$$

where

$$M = -2p_\phi^2 V^2(\phi) \pm 64\mu^3 + 2p_\phi V(\phi)\sqrt{p_\phi^2 V^2(\phi) \mp 64\mu^3} \tag{C.5}$$

$$K = M^{2/3} + 16\mu^2 \pm 4\mu M^{1/3} \tag{C.6}$$

and the upper sign corresponds to $\mu > 0$, while the lower sign to $\mu < 0$. Thus a rescaling of the Lagrangian due to quantum corrections does change the turning points, though qualitatively the system remains the same.