PERRON'S FORMULA AND RESULTING EXPLICIT BOUNDS ON SUMS

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A Thesis
Submitted to the School of Graduate Studies
of the University of Lethbridge
in Partial Fulfillment of the
Requirements for the Degree

MASTER OF SCIENCE

Department of Mathematics and Computer Science
University of Lethbridge
LETHBRIDGE, ALBERTA, CANADA

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Date of Defence: December 14, 2018

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Dedication

To Amir and Tim.

Both very deserving. I hope you can share.

Abstract

By working with *Perron's formula* we prove an *explicit bound* on $\sum_{n\leq x}\frac{a_n}{n^s}$, where $a_n, s \in \mathbb{C}$. We then prove a second explicit bound on this sum for the special case where $s = 0$. These bounds apply to specific sums that are involved in the Prime Number Theorem. Moreover, they are particularly useful in cases where a variant of the Riemann von-Mangoldt explicit formula is not unconditionally available. We choose to implement our bounds on $M(x) = \sum_{n\leq x}\mu(n)$ and $m(x) = \sum_{n\leq x}\frac{\mu(n)}{n}$ (with $\mu(n)$ denoting the Möbius function). This gives constants $C > 0,\ c > 0$ and $x_0 > 0$ for which $|M(x)| \leq Cx\exp\left(-c\sqrt{\log x}\right)$ if $x > x_0$ and a similar kind of bound for $m(x)$. We believe that explicit bounds for $M(x)$ and $m(x)$ like these have never before been published.

Acknowledgments

I have a memory of a question a friend asked me when we were summer scholars in Canberra, during the holiday of 2014–2015. I remember this question as being something like: if you could keep one thing from this experience, what would it be? At the time, I think at least part of my answer was, rather coldly, to do with the knowledge I had gained. There is still a piece of me that will struggle to omit this particularly oblivious response. However, I realise that the people one meets and interacts with along the way, are more important than what one learns academically.

I will begin by thanking Jackie and Andy, who provided me with a home and much appreciated help and company during the first part of my Masters.

My dues also go out to Dennis, for the chance to see (ice) hockey first-hand early in the peace, for the means to a Vegemite fix and, just generally, for the immense kindness he is well known for.

For being my Canadian parents (disguised as landlords), I send the biggest hug to Kathy and George. I hope I never forget the time I have spent in your home.

I would also like to extend my warm gratitude to Barb for reaching out. Keep practicing those actions from the talk!

To my current office mates, my previous office mates, my housemates and just my regular mates, thank you for your friendship! Do feel free to keep in touch.

A specific mention is also much deserved by my friend Quanli. I really appreciate those instances when you helped other friends out in my place, giving me the opportunity to work.

Of course, I would like to send a thank you through the airwaves to Mum, who loves me even when I am at a distance.

With a more formal tone, I gratefully acknowledge Dr. Howard Cheng for answering all of my computer related questions and, more broadly, to my thesis committee as a whole; thank you for your time and valuable input.

My second to last, but definitely vast thank you goes to Tim who was an incredible summer/honours supervisor and mentor. The work I have done over the last (nearly) two years as a Masters student, that has become this thesis, addresses the research problem Tim gifted me with for my honours year. Naturally, he taught me a great deal on this topic, which I am very very grateful for. I hope the dedication makes you feel proud.

My last (but the inverse of least) thank you goes out to my current supervisor Amir for your wisdom, patience, understanding, experience, dedication, kindness, humour …and for ALL of the times you visited my office. (The idea of including Kungfu Panda in the dedication was tempting, but you more than deserve it!)

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Defining the notation and terminology

Notation	*Definition*
$O(\cdot)$	The Big O notation. By writing $f(x) = O(g(x))$ we are expressing that, for some $C > 0$ and $x_0 > 0$, $\lvert f(x)\rvert \le Cg(x)$ if $x > x_0$.
$O^*(\cdot)$	The Big O* notation. By writing $f(x) = O^*(g(x))$ we are expressing that, for some $x_0 > 0$, $\lvert f(x)\rvert \le g(x)$ if $x > x_0$. By writing, for example, $f(x) = h(x) + O^*(g_1(x)) + O^*(g_2(x))$ we are expressing that $f(x) = h(x) + f_1(x) + f_2(x)$, where $f_1(x) = O^*(g_1(x))$ and $f_2(x) = O^*(g_2(x))$.
$\mu(n)$	The Möbius function. It is zero if the square of some prime is a divisor of n, and if not, it is $(-1)^k$, with k recording how many distinct prime divisors n has.
$\Lambda(n)$	The von Mangoldt function. It is $\log p$ if $n = p^m$ for some prime p and $m \ge 1$. If $n \ne p^m$ for any prime p and $m \ge 1$, it is zero.
$\lambda(n)$	The Liouville function. It is $(-1)^k$ and k recording how many prime divisors n has, with the multiplicity of each prime divisor being counted.
$M(x)$	$\sum_{n \le x} \mu(n)$
$m(x)$	$\sum_{n \le x} \frac{\mu(n)}{n}$
$\psi(x)$	$\sum_{n \le x} \Lambda(n)$
$L(x)$	$\sum_{n \le x} \lambda(n)$
$\zeta(w)$	The Riemann zeta-function.
η	A parameter. It is first used in Proposition 2.1.

Notation	*Definition*
A	A constant that depends on η. It is first used in Proposition 2.1.
α	A parameter that belongs to $\mathbb{Z}^{\geq 0}$. It is first used in Proposition 2.1.
$\tilde{A}$	A constant that depends on a sequence (a_n). It is first used in Proposition 2.1.
$\Psi(n)$	A non-decreasing function that depends on a sequence (a_n). It is first used in Proposition 2.1.
$A_1,\ A_2$	Constants that are first used in Proposition 2.1. We state the value we shall use for A_1 in Proposition 2.7 and the value we shall use for A_2 in Proposition 2.8.
N	The integer with the smallest distance to x. If $x+1/2 \in \mathbb{Z}$, then $N := \lfloor x \rfloor$. It is first used in Proposition 2.1.
β	A parameter that is strictly greater than $1/2$ and strictly less than 1. It is first used in Proposition 2.1.
$B_1,\ B_2,\ldots,B_7$	Expressions that are given in Proposition 2.1 or Proposition 2.9.
W	A parameter that is first used in (1.13). We state values for it in Table 3.1.
$\tilde{v}$	A parameter that depends on W. It is first used in (1.13) and we state values for it in Table 3.1.
C_1, C_2	Constants that depend on W and $\tilde{v}$. They are first used in (3.5) (for C_1) and (3.7) (for C_2). We give values for them in Table 3.1.
$\hat{h},\ \tilde{h},\ h,\ \tilde{g},\ g$	Functions that are first used in Proposition 2.14.
$C(x)$	A function that appears in our bounds on $M(x)$ and $m(x)$. We define it differently in Corollaries 3.5, 3.6, 3.9, 3.10, 4.3 and 4.4.

Notation	*Definition*
$C_W,\ c_W,\ C_{W,\varepsilon},\ c_{W,\varepsilon}$	Constants that depend on ε and/or W. We calculate them in order to give explicit bounds on $M(x)$ and $m(x)$. For example, we use these constants as in the bounds $M(x) = O^*\left(C_W x(\log x)\exp(-c_W\sqrt{\log x})\right)$ and $M(x) = O^*\left(C_{W,\varepsilon} x \exp(-c_{W,\varepsilon}\sqrt{\log x})\right)$.
$x_0(W),\ x_0(W,\varepsilon)$	Constants that depend on ε and/or W. They are used as a lower bound on x that specifies when one of the explicit bounds we get on $M(x)$ or $m(x)$ can be used. For example, we use $x_0(W)$ as in $M(x) = O^*(C_W x(\log x)\exp(-c_W\sqrt{\log x}))$, for $x > x_0(W)$.
x_ε	A constant for which we have $0 < \dfrac{\log\log x}{\sqrt{\log x}} < \varepsilon$ whenever $x > x_\varepsilon$.
$C_W^1,\ C_W^2,\ C_W^3$	Constants that represent specific terms of C_W. We define them in Section 3.2.2.
$C_{W,\varepsilon}^1,\ C_{W,\varepsilon}^2,\ C_{W,\varepsilon}^3$	Constants that represent specific terms of $C_{W,\varepsilon}$. We define them in Section 3.2.2.

*   *   *

Terminology	*Definition*
Zero-free region	A region that contains none of the zeros of a function. It is used with respect to $\zeta(w)$.
Bound	By stating that $g(x)$ is a bound on $f(x)$, we mean that $f(x) = O(g(x))$ or $f(x) = O^*(g(x))$.
Explicit Bound	By stating that $g(x)$ is an explicit bound on $f(x)$, we mean that $f(x) = O^*(g(x))$ and we know the value of each constant appearing in $g(x)$.
Implied constant	The constant omitted when using Big O notation.

Chapter 1

Introduction

The Prime Number Theorem (PNT) gives the asymptotic size of the set of primes that are no larger than x. Classically, it is stated in terms of sums $\sum_{n \leq x} a_n$, where the a_n tells us something about the primes. For example, we could have $a_n = \mu(n)$, with $\mu(n)$ denoting the Mobius function, or $a_n = \Lambda(n)$, with $\Lambda(n)$ denoting the von Mangoldt function. When p is a prime and k records how many distinct prime divisors n has, we have

$$\mu(n) := \begin{cases} (-1)^k, & \text{if } n/p^2 \notin \mathbb{Z} \text{ for any } p, \\ 0, & \text{if } n/p^2 \in \mathbb{Z} \text{ for some } p. \end{cases}$$

Furthermore, when p is a prime

$$\Lambda(n) := \begin{cases} \log p & \text{if } n = p^m \text{ for some } p \text{ and } m, \\ 0 & \text{if } n \neq p^m \text{ for any } p \text{ and } m, \end{cases}$$

with $m \geq 1$. Whence, two well-known examples of sums we can use to state the PNT are

$$M(x) := \sum_{n \leq x} \mu(n) \quad \text{and} \quad \psi(x) := \sum_{n \leq x} \Lambda(n).$$

For these sums, the PNT becomes the statements

$$\lim_{x \to \infty} \frac{M(x)}{x} = 0 \quad \text{and} \quad \lim_{x \to \infty} \frac{\psi(x)}{x} = 1 \tag{1.1}$$

(see [21, Exercise 3.4.23 and Theorem 3.3.2]).

Ever since 1896, the idea that the PNT holds has been a fact, due to the proofs of Hadamard and de la Vallée Poussin [29, p. 45]. Their proofs were particularly interesting because they were based upon a subtle link that exists between the primes and the Riemann zeta-function $\zeta(w)$ where $w \in \mathbb{C}\backslash\{1\}$ (henceforth, the zeta-function). The zeta-function is equal to the series $\sum_{n=1}^{\infty} \frac{1}{n^w}$ when $\mathrm{Re}(w) > 1$ and through analytic continuation it has been defined for any $w \in \mathbb{C}\backslash\{1\}$. The proofs of Hadamard and de la Vallée Poussin are certainly not the last instances where an inference was made about the primes using the zeta-function. It turns out that $\zeta(w)$ is at the crux of the PNT. We have Riemann to thank for realizing this and making it known back in 1859 (see [9, p. 299–305] for an English translation of Riemann's paper).

The PNT statements in (1.1) allow one to infer that, for any sufficiently large x, there is some $C > 0$ and $\tilde{C} > 0$, an $f : [1,\infty) \to (0,\infty)$ and an $\tilde{f} : [1,\infty) \to (0,\infty)$, such that

$$|M(x)| \leq Cxf(x) \text{ and } |\psi(x) - x| \leq \tilde{C}x\tilde{f}(x), \tag{1.2}$$

where $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \tilde{f}(x) = 0$. The short, sharp and shiny way to state the bounds in (1.2) is that, for some $x_0 \geq 1$ and any $x > x_0$,

$$M(x) = O(xf(x)) \text{ and } \psi(x) = x + O(x\tilde{f}(x))$$

or (if we want to retain more of the detail) with x satisfying the same condition as above, we can write

$$M(x) = O^*(Cxf(x)) \text{ and } \psi(x) = x + O^*(\tilde{C}x\tilde{f}(x)). \tag{1.3}$$

The uncommon notation O^* is used in [26] and we shall make extensive use of it in this thesis.

Despite the PNT having been proved (numerous times), researchers are still trying to improve the upper bounds in (1.2). Ideally, one prefers to make an improvement on the functions $f(x)$ or $\tilde{f}(x)$. However, a lot of research has also been done on improving C and $\tilde{C}$. This kind of research deals with *explicit bounds*, in which the constants that are involved (such as C and $\tilde{C}$) must all be given.

Explicit bounds on both $M(x)$ and $\psi(x)$ have been extensively studied. One of the older papers on $M(x)$ is [17] and more recent work on this function is given in [24]. An explicit bound on $M(x)$, for $x \in [\exp(13.90), \infty)$, that is presented in [24] is

$$M(x) = O^*\left(\frac{0.013x\log x - 0.118x}{(\log x)^2}\right). \tag{1.4}$$

With respect to research on $\psi(x)$, [27] and [28] started the ball rolling, while more recently [10] and [11] expanded our knowledge. An explicit bound on $\psi(x)$, for $x \in [\exp(20), \infty)$, that is presented in [10, Corollary 1.2] is

$$\psi(x) = x + O^*\left(5.3688 \times 10^{-4}x\right).$$

There is a prominent difference in the way one is able to get explicit bounds on $\psi(x)$ compared to $M(x)$. The main vehicle for working on bounds for $\psi(x)$ is the explicit formula ([21, p. 101])

$$\psi(x) = x - \sum_{\rho}\frac{x^{\rho}}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log\left(1 - \frac{1}{x^2}\right). \tag{1.5}$$

The use of (1.5) continues the trend of involving the zeta-function in the study of the PNT, since the ρ in (1.5) are complex numbers whose real parts are between 0 and 1, and $\zeta(\rho) = 0$.

Explicit formulas, analogous to (1.5), can be proved for functions other than $\psi(x)$. However, some functions, including $M(x)$, do not permit themselves as nicely to this. One big problem is that, for some functions, we have to know the order of the zeros of the zeta-function to get an unconditional explicit formula. This imposes a serious road block to unconditionally using an explicit formula to get explicit bounds in these cases. Fortunately, there is a natural way around this that was pioneered by Landau [16]. Instead of using an explicit formula, we simply start with its precursor, *Perron's formula*.

When $s,\ a_n \in \mathbb{C},\ c > 0$ and $\sum_{n=1}^{\infty} \frac{|a_n|}{n^{\mathrm{Re}(s)+c}}$ converges, Perron's formula [19, Theorem 5.1] is

$$\sum_{n \leq x} \frac{a_n}{n^s} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(\sum_{n=1}^{\infty} \frac{a_n}{n^{s+w}} \right) \frac{x^w}{w} dw. \tag{1.6}$$

Using (1.6) to get a bound on $\sum_{n \leq x} \frac{a_n}{n^s}$ involves a slight twist on what one does to get an explicit formula. To get an explicit formula, using (1.6), we truncate the imaginary part of the endpoints of the integral at a finite $V > 0$ and make up for the loss by including a bound on the error (which we shall call E). This gives a *truncated Perron's formula*

$$\sum_{n \leq x} \frac{a_n}{n^s} = \frac{1}{2\pi} \int_{c-iV}^{c+iV} \left(\sum_{n=1}^{\infty} \frac{a_n}{n^{s+w}} \right) \frac{x^w}{w} dw + E. \tag{1.7}$$

We then work with the integral further by closing the contour[1], so that Cauchy's residue theorem applies. When we are dealing with $a_n = \Lambda(n)$ or $a_n = \mu(n)$, the points at which $\zeta(w) = 0$ end up being on the interior of this closed region. For $\psi(x)$ this works out, but for $M(x)$, having these zeros inside the contour prevents one from moving forward unconditionally after applying Cauchy's residue theorem. To sidestep this problem, all we have to do is close the contour in a way that leaves the zeros of the zeta-function on the exterior of the closed region. Thus, we shall need to use results on the zero-free regions of the Riemann zeta-function.

There are three main types of zero-free regions for the zeta-function in the literature. One of the most commonly used is the classical zero-free region. This region is made up of

[1] Closing a contour involves adding on pieces until the contour ends where it started.

all of those $w = u + iv \in \mathbb{C}$ for which

$$\mathrm{Re}(w) \geq 1 - \frac{1}{W_1 \log|v|} \text{ and } |v| \geq \tilde{v}_1 \tag{1.8}$$

with $W_1,\ \tilde{v}_1 > 0$. In [20] it is proved that we can take $W_1 = 5.573412$ and $\tilde{v}_1 = 2$.

The second type of zero-free region is made up of all of those $w \in \mathbb{C}$ for which

$$0 \leq \mathrm{Re}(w) \leq 1 \text{ and } |v| \leq H \tag{1.9}$$

with $H > 0$. Currently, there are two values for H in play. The smaller of these two values has been more openly accepted. It is $H = 3.06 \times 10^{10}$ (see [22] and [23]). Regarding the larger value of H, we refer to [13].

The third type of zero-free region owes itself to Vinogradov and Korobov (see [31] and [15]) and is made up of all of those $w \in \mathbb{C}$ for which

$$\mathrm{Re}(w) \geq 1 - \frac{1}{W_2(\log|v|)^{2/3}(\log\log|v|)^{1/3}} \text{ and } |v| \geq \tilde{v}_2 \tag{1.10}$$

with $W_2, \tilde{v}_2 > 0$. It is currently known that we may let $W_2 = 57.54$ and $\tilde{v}_2 = 3$ (see [12]).

To guarantee that all points w where $\zeta(w) = 0$ stay on the exterior of the closed region, when closing the contour that appears in the integral of (1.7), one may try to use the zero-free regions (1.8), (1.9) and (1.10) . However, we also need to guarantee that we will have a bound on $\sum_{n=1}^{\infty} \frac{a_n}{n^{s+w}}$ or a bound on its analytic continuation available to use at particular w on the closed contour.

It is true that when $\mathrm{Re}(w) > 1$

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^w} = \frac{1}{\zeta(w)} \tag{1.11}$$

(see [21, Exercise 1.2.2]) and

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^w} = -\frac{\zeta'(w)}{\zeta(w)} \tag{1.12}$$

(see [29, (1.1.8)]). Moreover, when $\text{Re}(w) \leq 1$ (short of those w at which $\zeta(w) = 0$ and also $w = 1$ in the case of (1.12)) the series in (1.11) and (1.12) can be continued analytically. Hence, when we want to get a bound on $\sum_{n \leq x} a_n$ for $a_n = \mu(n)$ and $a_n = \Lambda(n)$, bounds on $1/\zeta(w)$ and $\zeta'(w)/\zeta(w)$ are what we need. Bounds on $1/\zeta(w)$ and $\zeta'(w)/\zeta(w)$ that are relevant if

$$\text{Re}(w) \geq 1 - \frac{1}{W \log v} \text{ and } v \geq \tilde{v}, \tag{1.13}$$

for particular $W \geq 6$ and $\tilde{v} \geq 34$, are proved by Trudgian in [30]. Trudgian hints in [30] that his motivation for proving these bounds was that Perron's formula could then be used to acquire bounds on four particular functions. Three of the functions that he mentions are $\psi(x)$, $m(x) := \sum_{n \leq x} \frac{\mu(n)}{n}$ and $M(x)$.

Even though there is a significant literature devoted to $M(x)$, it appears that none contain an explicit bound acquired using Perron's formula. This is supported by comments of Bordellés and Ramaré. Bordellés [6, p. 2] wrote the following.

> "... [T]he method of contour integration may lead to bounds of the form
>
> $$\sum_{n \leq x} \mu(n) \ll x \exp\left(-c\sqrt{\log x}\right) \ (x \geq 2, c > 0)$$
>
> but no explicit result of this form is known."

Ramaré [25, p. 1359] confirms Bordellés message, but also goes slightly further to give an idea on the quality of the results one may get.

> "No one has yet obtained an explicit error term for the function M from the Mellin transform/Perron formula machinery, though there are no *theoretical* obstructions. The implied constants are, however, expected to be too large for any decent use."

The function $m(x)$ is a natural progression from studying $M(x)$. Moreover, we do not believe Perron's formula has been applied to $m(x)$ either. The literature on getting bounds for this function includes [6] and [25]. An explicit bound on $m(x)$, for $x \in [\exp(13.05), \infty)$, that is presented in [25] is

$$m(x) = O^*\left(\frac{0.0144\log x - 0.1}{(\log x)^2}\right). \tag{1.14}$$

This thesis shall involve proving a bound for a suitable general function $\sum_{n\leq x} \frac{a_n}{n^s}$, where $a_n \in \mathbb{C}$ and $s \in \mathbb{C}$ by means of Perron's formula. We shall also prove a separate bound for the special case $\sum_{n\leq x} a_n$. We complete both of these tasks in Chapter 2. In Chapters 3 and 4, we then demonstrate the use of such results by arriving at explicit bounds on $M(x)$ and $m(x)$.

In terms of software, we calculate our explicit bounds on $M(x)$ and $m(x)$ by means of `Python`® [2] [2] programs that we have written. (See the Appendix for our code.) These programs rely heavily on mpmath [14] so that the precision of the computations can be altered. We also make use of Maple™ [3] [1].

One could also use Perron's formula to get explicit bounds on $\psi(x)$ to see how the results differ to those that have been arrived at via the explicit formula. We shall not do so in this thesis. However, in Chapter 5 we mention this further, in addition to discussing other ways one may use and continue the work we present.

In this thesis, we shall not only present tables of the constants in our explicit bounds on $M(x)$ and $m(x)$, but also the formulas from which we produced these bounds. Our formulas are of great value because they make it easier to re-determine the bounds each time we are provided with improved information.

[2] "Python" is a registered trademark of the Python Software Foundation.
[3] Maple is a trademark of Waterloo Maple Inc.

Chapter 2

Bounds from two explicit truncated Perron's formulae

As the name of this thesis suggests, *Perron's formula* (which was introduced in (1.6)) is the foundation of our work. Put rather humbly, this formula provides a way to transform a discrete sum over the natural numbers into a continuous integral over a contour in the complex plane. The theme of connecting the natural numbers to the complex plane is somewhat at the crux of an analytic number theorist's approach to delving into the intricate behaviour of the primes. Somewhat naturally, there are a number of important sums one can apply Perron's formula to. This is exactly our reason for studying this formula. We shall later apply it to two specific sums, but we want to do so explicitly. This means we first need a formula like (1.7). That is, we need an *explicit truncation of Perron's formula*.

In Section 2.1 we truncate Perron's formula as Titchmarsh does in his Lemma 3.12 [29]. This shall give bounds for any general function $\sum_{n\leq x}\frac{a_n}{n^s}$ $(a_n,\ s\in\mathbb{C})$ of the variable x, that satisfies particular properties. Section 2.2 involves considering two coefficients that we introduced into the bounds in the process of truncating in Section 2.1. In Section 2.3, we then truncate Perron's formula like Arkhipova did in [3]. This truncation just applies to $\sum_{n\leq x}a_n$. In Section 2.4, we bound the integral in the truncated Perrron's formulae we got in Sections 2.1 and 2.3. Section 2.5 is where the reader will find our bounds on $\sum_{n\leq x}\frac{a_n}{n^s}$ and $\sum_{n\leq x}a_n$.

2.1 An explicit Perron's formula from Titchmarsh's Lemma 3.12

In this section our task is to prove Proposition 2.1, which explicitly truncates Perron's formula. The non-explicit form of this proposition and its proof, which our result and proof are based on, is given by Titchmarsh in [29] as Lemma 3.12. This proposition is one of the cornerstones of our work.

Proposition 2.1. *Assume that, for some* σ *that is fixed,* $c>0,\ \eta>0,\ A:=A(\eta)>0$, *and* $\alpha\in\mathbb{R}$, *we have that*

$$\sum_{n=1}^{\infty}\frac{|a_n|}{n^{\sigma+c}}=O^*\left(\frac{A}{(\sigma+c-1)^{\alpha}}\right), \tag{2.1}$$

whenever $1<\sigma+c<1+\eta$. *Also assume that for some* $\tilde{A}>0, n\in\mathbb{N}$ *and a non-decreasing function* $\Psi(n)$, *we have that*

$$a_n=O^*\left(\tilde{A}\Psi(n)\right). \tag{2.2}$$

Furthermore, assume that for some $A_1>0$, *and* $x>1$,

$$\sum_{1\le r\le x+1/2}\frac{1}{r}=O^*\left(A_1\log x\right) \tag{2.3}$$

and for some $A_2>0$ *and* $x>1$,

$$\sum_{1\le r\le \frac{x+1}{2}}\frac{1}{r}=O^*\left(A_2\log x\right). \tag{2.4}$$

Let $V>0,\ 1/2<\beta<1$, *and* $s:=\sigma+it$. *On denoting the integer with the smallest distance to* x *by* N *(if* $x+1/2\in\mathbb{Z}$, *then denote* $N:=\lfloor x\rfloor$*), and defining* x *to be non-integer, for* $x>\max\{3/2,1/(4\beta-2)\}$ *we have that*

$$\begin{aligned}\sum_{n\le x}\frac{a_n}{n^s} &= \frac{1}{2\pi i}\int_{c-iV}^{c+iV}\left(\sum_{n=1}^{\infty}\frac{a_n}{n^{s+w}}\right)\frac{x^w}{w}dw+O^*\left(B_1\frac{x^c}{V(\sigma+c-1)^{\alpha}}\right)\\ &\quad +O^*\left((B_2\Psi(2x)+B_3\Psi(N))\frac{x^{1-\sigma}\log x}{V}\right)+O^*\left(B_4\frac{\Psi(N)x^{1-\sigma}}{V|x-N|}\right)\end{aligned} \tag{2.5}$$

with

$$B_1 := \frac{A}{\pi \log 2}\left(2+\mathrm{Arctan}\frac{c}{V}+\frac{\pi}{2}+\left|\mathrm{Arctan}\frac{c}{V}-\frac{\pi}{2}\right|\right), \tag{2.6}$$

$$B_2 := \frac{8\tilde{A}A_1}{\pi}\left(1+\left|\mathrm{Arctan}\frac{c}{V}-\frac{\pi}{2}\right|\right), \tag{2.7}$$

$$B_3 := \frac{3(2^{\sigma+c-1})\tilde{A}A_2}{\pi(1-\beta)}\left(1+\mathrm{Arctan}\frac{c}{V}+\frac{\pi}{2}\right), \textit{ and} \tag{2.8}$$

$$B_4 := \frac{2^{\sigma+c}\tilde{A}}{\pi}\left(1+\max\left\{\mathrm{Arctan}\frac{c}{V}+\frac{\pi}{2},\left|\mathrm{Arctan}\frac{c}{V}-\frac{\pi}{2}\right|\right\}\right). \tag{2.9}$$

We shall break the proof of this proposition into five lemmas. When proving each lemma, we assume the appropriate conditions from Proposition 2.1 without giving them specifically in the statement of the lemma.

Lemma 2.2. *We have*

$$\begin{aligned}\sum_{n<x}\frac{a_n}{n^s} &= \frac{1}{2\pi i}\int_{c-iV}^{c+iV}\left(\sum_{n=1}^{\infty}\frac{a_n}{n^{s+w}}\right)\frac{x^w}{w}dw\\ &+O^*\left(\frac{1}{\pi}\left(1+\mathrm{Arctan}\frac{c}{V}+\frac{\pi}{2}\right)\sum_{n<x}\frac{|a_n|}{n^\sigma}\frac{(x/n)^c}{V\left|\log\frac{x}{n}\right|}\right)\\ &+O^*\left(\frac{1}{\pi}\left(1+\left|\mathrm{Arctan}\frac{c}{V}-\frac{\pi}{2}\right|\right)\sum_{n>x}\frac{|a_n|}{n^\sigma}\frac{(x/n)^c}{V\left|\log\frac{x}{n}\right|}\right).\end{aligned} \tag{2.10}$$

Proof. If we can prove that

$$\frac{1}{2\pi i}\int_{c-iV}^{c+iV}\left(\frac{x}{n}\right)^w\frac{dw}{w} = \begin{cases}1+O^*\left(\frac{1}{\pi}\left(1+\mathrm{Arctan}\frac{c}{V}+\frac{\pi}{2}\right)\frac{(x/n)^c}{V\left|\log\frac{x}{n}\right|}\right) & \text{for } n<x,\\ O^*\left(\frac{1}{\pi}\left(1+\left|\mathrm{Arctan}\frac{c}{V}-\frac{\pi}{2}\right|\right)\frac{(x/n)^c}{V\left|\log\frac{x}{n}\right|}\right) & \text{for } n>x,\end{cases} \tag{2.11}$$

we would have that

$$\begin{aligned}\frac{1}{2\pi i}\sum_{n=1}^{\infty}\frac{a_n}{n^s}\int_{c-iV}^{c+iV}\left(\frac{x}{n}\right)^w\frac{dw}{w} &= \sum_{n<x}\frac{a_n}{n^s}+O^*\left(\frac{1}{\pi}\left(1+\mathrm{Arctan}\frac{c}{V}+\frac{\pi}{2}\right)\sum_{n<x}\frac{|a_n|}{n^\sigma}\frac{(x/n)^c}{V\left|\log\frac{x}{n}\right|}\right)\\ &+O^*\left(\frac{1}{\pi}\left(1+\left|\mathrm{Arctan}\frac{c}{V}-\frac{\pi}{2}\right|\right)\sum_{n>x}\frac{|a_n|}{n^\sigma}\frac{(x/n)^c}{V\left|\log\frac{x}{n}\right|}\right).\end{aligned}$$

We may then interchange the series and the integral on the left-hand side of the equation above. To do so, we use the absolute convergence of $\sum_{n=1}^{\infty} \frac{a_n}{n^{s+c+iv}}$ (when $\mathrm{Re}(s)+c>1$ and $v \in [-V,V]$), which is a consequence of (2.1), and the Weierstrass M-test to justify having uniform convergence of $\sum_{n=1}^{\infty} \frac{a_n}{n^{s+c+iv}}$ (when $\mathrm{Re}(s)+c>1$ and $v \in [-V,V]$). This would then give the lemma. Hence, all we really need to do is prove (2.11). We do this separately for $n<x$ and $n>x$.

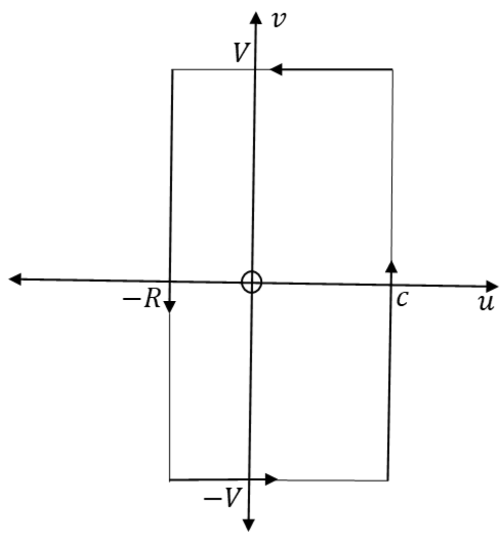

Figure 2.1: A contour needed in Lemma 2.2 for $n<x$.

For $n<x$, we close the contour using three line segments so that it becomes a rectangle that captures the pole $1/w$ has at $w=0$. This rectangle is shown in Figure 2.1. Then, with $R>0$, Cauchy's residue theorem tells us that

$$\frac{1}{2\pi i}\int_{c-iV}^{c+iV}\left(\frac{x}{n}\right)^w \frac{dw}{w}-1=-\frac{1}{2\pi i}\left(\int_{c+iV}^{-R+iV}+\int_{-R+iV}^{-R-iV}+\int_{-R-iV}^{c-iV}\right)\left(\frac{x}{n}\right)^w \frac{dw}{w}.$$

From this, one can use the triangle inequality to get that

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_{c-iV}^{c+iV}\left(\frac{x}{n}\right)^w \frac{dw}{w}-1\right| \quad \leq \quad & \lim_{R\to\infty}\frac{1}{2\pi}\left(\left|\int_{-R-iV}^{c-iV}\left(\frac{x}{n}\right)^w \frac{dw}{w}\right|+\left|\int_{c+iV}^{-R+iV}\left(\frac{x}{n}\right)^w \frac{dw}{w}\right|\right.\\ & \left.+\left|\int_{-R+iV}^{-R-iV}\left(\frac{x}{n}\right)^w \frac{dw}{w}\right|\right). \qquad (2.12)\end{aligned}$$

Now, set $w = \mathrm{Re}(w) + i\mathrm{Im}(w) := u + iv$. Since

$$\begin{aligned}\lim_{R\to\infty}\left|\frac{1}{2\pi i}\int_{-R+iV}^{-R-iV}\left(\frac{x}{n}\right)^w\frac{dw}{w}\right| &\leq \lim_{R\to\infty}\frac{1}{2\pi}\int_{-V}^{V}\left(\frac{x}{n}\right)^{-R}\frac{dv}{|-R+iv|}\\ &\leq \lim_{R\to\infty}\frac{1}{2\pi}\left(\frac{x}{n}\right)^{-R}\frac{1}{R}\int_{-V}^{V}dv\\ &= 0\end{aligned}$$

(because $n/x < 1$), only the first and second terms on the right-hand side of (2.12) contribute in (2.11).

For the first term in (2.12), we use integration by parts and the triangle inequality to obtain

$$\begin{aligned}\lim_{R\to\infty}\frac{1}{2\pi}\left|\int_{-R-iV}^{c-iV}\left(\frac{x}{n}\right)^w\frac{dw}{w}\right| \quad &\leq \quad \frac{1}{2\pi}\left|\frac{(x/n)^{c-iV}}{(c-iV)\log\frac{x}{n}}\right|\\ &\quad +\frac{1}{2\pi}\lim_{R\to\infty}\left(\left|\frac{(x/n)^{-R-iV}}{(-R-iV)\log\frac{x}{n}}\right| + \left|\frac{1}{\log\frac{x}{n}}\int_{-R-iV}^{c-iV}\left(\frac{x}{n}\right)^w\frac{dw}{w^2}\right|\right)\\ &\leq \quad \frac{1}{2\pi}\frac{(x/n)^c}{V\log\frac{x}{n}} + \frac{1}{2\pi\log\frac{x}{n}}\int_{-\infty}^{c}\left(\frac{x}{n}\right)^u\frac{du}{u^2+V^2}.\end{aligned} \tag{2.13}$$

Then, after a change of variables $u \mapsto V\tan\theta$, we get that

$$\frac{1}{2\pi}\frac{(x/n)^c}{V\log\frac{x}{n}} + \frac{1}{2\pi\log\frac{x}{n}}\int_{-\infty}^{c}\left(\frac{x}{n}\right)^u\frac{du}{u^2+V^2} \leq \frac{1}{2\pi}\frac{(x/n)^c}{V\log\frac{x}{n}} + \frac{\left(\mathrm{Arctan}\frac{c}{V}+\frac{\pi}{2}\right)(x/n)^c}{2\pi V\log\frac{x}{n}}. \tag{2.14}$$

The second term on the right-hand side of (2.12) is also bounded by (2.14). This can be shown as we have just done for the first term in (2.12). Hence, we have the result for $n < x$.

For $n > x$, we again close the contour so that it becomes a rectangle. However, this time we do not let the rectangle capture the pole that $1/w$ has at $w = 0$. This rectangle is shown

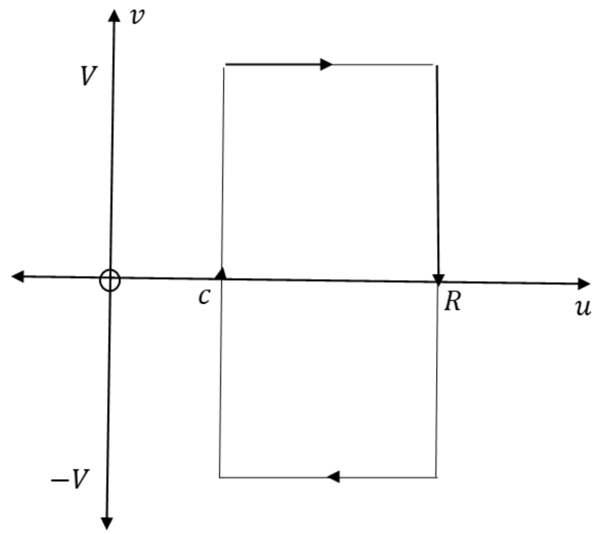

Figure 2.2: A contour needed in Lemma 2.2 for $n > x$.

in Figure 2.2. With $R > 0$, we get the following,

$$\left|\frac{1}{2\pi i}\int_{c-iV}^{c+iV}\left(\frac{x}{n}\right)^w \frac{dw}{w}\right| \leq \lim_{R\to\infty}\frac{1}{2\pi}\left(\left|\int_{R-iV}^{c-iV}\left(\frac{x}{n}\right)^w \frac{dw}{w}\right| + \left|\int_{c+iV}^{R+iV}\left(\frac{x}{n}\right)^w \frac{dw}{w}\right| + \left|\int_{R+iV}^{R-iV}\left(\frac{x}{n}\right)^w \frac{dw}{w}\right|\right).$$

The proof is completed using almost identical reasoning as we used above in the $n < x$ case. □

We next work with the bounds that appear on the right-hand side of (2.10) in Lemma 2.2 over several intervals. We begin with the intervals $n > 2x$ and $n < x/2$.

Lemma 2.3. *We have*

$$\sum_{\substack{n>2x \\ or \\ n<x/2}} \frac{|a_n|}{n^{\sigma+c}} \frac{x^c}{V|\log\frac{x}{n}|} = O^*\left(\frac{A}{\log 2}\frac{x^c}{V(\sigma+c-1)^\alpha}\right).$$

Proof. This follows from condition (2.1) along with the fact that, for $n < x/2$,

$$|\log(x/n)| > \log 2$$

and the fact that, for $n > 2x$,

$$|\log(x/n)| = |\log(n/x)| > \log 2.$$

□

The next bound we shall work with is for $n = N$. Back in Proposition 2.1, N was introduced as the integer with the smallest distance to x. This dictates that

$$N - 1/2 \leq x \leq N + \frac{1}{2}. \tag{2.15}$$

We shall make use of this condition.

Lemma 2.4. *When $x > 1$, we have*

$$\frac{|a_N|}{N^{\sigma+c}} \frac{x^c}{V|\log\frac{x}{N}|} = O^*\left(2^{\sigma+c}\tilde{A}\frac{\Psi(N)x^{1-\sigma}}{V|x-N|}\right).$$

Proof. Firstly, from condition (2.2) in Proposition 2.1, we have

$$\frac{|a_N|}{N^{\sigma+c}} \frac{x^c}{V|\log\frac{x}{N}|} = O^*\left(\tilde{A}\frac{\Psi(N)x^c}{N^{\sigma+c}V|\log\frac{x}{N}|}\right). \tag{2.16}$$

We next work with[4]

$$\log\frac{x}{N} = \log\left(1 + \frac{(x-N)}{N}\right) = \frac{(x-N)}{N} - \frac{(x-N)^2}{2N^2} + \frac{(x-N)^3}{3N^3} - \cdots. \tag{2.18}$$

Unless $x + 1/2 \in \mathbb{Z}$, we are unable to tell if $x - N < 0$ or $x - N > 0$ so we must consider both cases.

[4]The ratio test confirms that absolute convergence of (2.18) occurs when

$$\left|\frac{x-N}{N}\right| < 1. \tag{2.17}$$

Furthermore, as $N - 1/2 \leq x \leq N + 1/2$, we know that we have (2.17) since $\frac{1}{2N} < 1$.

For $x-N<0$, we consider the expression

$$-\frac{(x-N)}{N}+\frac{(x-N)^2}{2N^2}-\frac{(x-N)^3}{3N^3}+\cdots,$$

which is the absolute value of (2.18). From this we get that

$$\left|\log\frac{x}{N}\right| \geq -\frac{(x-N)}{N}=\frac{|x-N|}{N}. \tag{2.19}$$

For $x-N>0$ we may ignore all terms, but the first two in (2.18). This is due to the fact that, since $x\leq N+1/2$ as in (2.15), we have that

$$\frac{(x-N)^k}{kN^k}-\frac{(x-N)^{k+1}}{(k+1)N^{k+1}} \geq \frac{(x-N)^k}{N^{k+1}k(k+1)}\left((k+1)N-k/2\right), \text{ for } k\geq 3. \tag{2.20}$$

Coupled with the restrictions $x-N>0$ and $x>1$ (making $N\geq 1$), (2.20) implies that

$$\frac{(x-N)^k}{kN^k}-\frac{(x-N)^{k+1}}{(k+1)N^{k+1}}>0, \text{ for } k\geq 3. \tag{2.21}$$

which allows one to ignore the terms in question.

Furthermore, since $x-N>0$, we have $|\log(x/N)|=\log(x/N)$. This fact, (2.18) and (2.21) give

$$\left|\log\frac{x}{N}\right| \geq \frac{(x-N)}{N}-\frac{(x-N)^2}{2N^2}. \tag{2.22}$$

We also get that $(x-N)/N<1$ because $x\leq N+1/2$ and $N\geq 1$, so $x<2N$. Thus, from (2.22) we have that

$$\frac{(x-N)}{N}-\frac{(x-N)^2}{2N^2}>\frac{(x-N)}{N}-\frac{(x-N)}{2N}=\frac{|x-N|}{2N}. \tag{2.23}$$

With (2.16), (2.19) and (2.23) we get that

$$\frac{|a_N|}{N^{\sigma+c}}\frac{x^c}{V\left|\log\frac{x}{n}\right|}=O^*\left(2\tilde{A}\frac{\Psi(N)x^cN^{1-(\sigma+c)}}{V|x-N|}\right).$$

Furthermore, since $x > 1$ and $N \geq x - 1/2$, we get that $N > x - x/2$. Whence,

$$N^{1-(\sigma+c)} < 2^{(\sigma+c)-1} x^{1-(\sigma+c)}$$

because $\sigma + c > 1$. We now have the lemma. □

The final two intervals we have to deal with are $N < n \leq 2x$ and $x/2 \leq x < N$.

Lemma 2.5. *When $x > 3/2$, we have*

$$\sum_{N<n\leq 2x} \frac{|a_n|}{n^{\sigma+c}} \frac{x^c}{V\left|\log\frac{x}{n}\right|} = O^*\left(8\tilde{A}A_1 \frac{\Psi(2x)x^{1-\sigma}\log x}{V}\right).$$

Proof. We must have $n > x$ because $N < n \leq 2x$. (If alternatively we had that $n < x$, n and N would both have the smallest distance to x, which is impossible because $N < n$.) From this fact, along with (2.2) in Proposition 2.1 and the condition $\sigma + c > 1$, we have that

$$\sum_{N<n\leq 2x} \frac{|a_n|}{n^{\sigma+c}} \frac{x^c}{V\left|\log\frac{x}{n}\right|} = O^*\left(\tilde{A}\frac{\Psi(2x)x^{-\sigma}}{V} \sum_{N<n\leq 2x} \frac{1}{\left|\log\frac{x}{n}\right|}\right). \tag{2.24}$$

We then set $n = N + r$. Using this definition for n as well as the restrictions $n > x$ and $x \leq N + 1/2$, we get that

$$\left|\log\frac{x}{n}\right| = \log\frac{n}{x} \geq \log\frac{N+r}{N+1/2}. \tag{2.25}$$

Now, along with the definition of n, the condition $N < n \leq 2x$ gives the restriction $0 < r \leq 2x - N$. Since $N < n$ and $x \leq N + 1/2$, this restriction on r becomes $1 \leq r \leq N + 1$. We shall deal with $1 \leq r \leq N$ separately from $r = N + 1$.

If $1 \le r \le N$, (2.25) is [5]

$$\left(\frac{r}{N}-\frac{r^2}{2N^2}+\frac{r^3}{3N^3}-\frac{r^4}{4N^4}+\cdots\right)+\left(-\frac{1}{2N}+\frac{1}{2(2N)^2}-\frac{1}{3(2N)^3}+\frac{1}{4(2N)^4}-\cdots\right) \tag{2.26}$$

$$\ge \qquad \frac{r}{N}\left(1-\frac{r}{2N}\right)-\frac{1}{2N} \tag{2.27}$$

because $\dfrac{1}{k(2N)^k}-\dfrac{1}{(k+1)(2N)^{k+1}}>0,$ for $k\ge 2$, and $\dfrac{r^k}{kN^k}-\dfrac{r^{k+1}}{(k+1)N^{k+1}}>0,$ for $k\ge 3$.

We next simplify (2.27) in two cases, $r=1$ and $r\ge 2$.

On letting $r=1$,

$$\frac{r}{N}\left(1-\frac{r}{2N}\right)-\frac{1}{2N}=\frac{1}{2N}\left(1-\frac{1}{N}\right). \tag{2.28}$$

Then, on imposing the restriction $x>3/2$, we use the fact that $N\ge 2$ to get

$$\frac{1}{2N}\left(1-\frac{1}{N}\right)\ge\frac{1}{4N}=\frac{r}{4N}. \tag{2.29}$$

If we instead let $r\ge 2$ (which means $r-1\ge r/2$) and use the fact that $r\le N$, we get that

$$\frac{r}{N}\left(1-\frac{r}{2N}\right)-\frac{1}{2N}\ge(r-1)\frac{1}{2N}\ge\frac{r}{2}\frac{1}{2N}. \tag{2.30}$$

Hence, if $1\le r\le N$, from (2.25), (2.27), (2.28), (2.29) and (2.30) we have that

$$\left|\log\frac{x}{n}\right|\ge\frac{r}{4N} \tag{2.31}$$

Furthermore, since $N\le 2x$, we have

$$\frac{r}{4N}\ge\frac{r}{8x}. \tag{2.32}$$

We now consider what we can conclude if $r=N+1$.

[5]The ratio test confirms that absolute convergence of (2.26) occurs if $r<N$, while the alternating series test confirms that convergence of (2.26) occurs in such a case that $r=N$.

If we substitute $r = N+1$ into the expression $\log \frac{N+r}{N+1/2}$ from (2.25) we get $\log 2$. Thus, (2.31) is also valid for $r = N+1$ if

$$\log 2 \geq \frac{r}{4N} = \frac{N+1}{4N}.$$

That is, for $N > 1/(4\log 2 - 1)$. This does not cause any great concern since $N \geq 2$ by the assumption $x > 3/2$.

Considering (2.24) and (2.32) together gives

$$\sum_{N<n\leq 2x} \frac{|a_n|}{n^{\sigma+c}} \frac{x^c}{V\left|\log \frac{x}{n}\right|} = O^*\left(8\tilde{A}\frac{\Psi(2x)x^{1-\sigma}}{V} \sum_{1\leq r\leq x+1/2} \frac{1}{r}\right). \tag{2.33}$$

(Note that the upper restriction on r in (2.33) comes from using the definition $n = N+r$, with $n \leq 2x$ to get $r \leq 2x - N$, then applying the fact that $N \geq x - 1/2$.)

We have the lemma on applying (2.3) in (2.33). □

Lemma 2.6. *When $x > \max\{1, 1/(4\beta - 2)\}$, for β as in Proposition 2.1, we have*

$$\sum_{x/2\leq n<N} \frac{|a_n|}{n^{\sigma+c}} \frac{x^c}{V\left|\log \frac{x}{n}\right|} = O^*\left(\frac{3\left(2^{\sigma+c-1}\right)\tilde{A}A_2}{1-\beta} \frac{x^{1-\sigma}\Psi(N)\log x}{V}\right).$$

Proof. From the conditions $x/2 \leq n$ and $\sigma + c > 0$, (2.2) in Proposition 2.1, we have that

$$\sum_{x/2\leq n<N} \frac{|a_n|}{n^{\sigma+c}} \frac{x^c}{V\left|\log \frac{x}{n}\right|} = O^*\left(2^{\sigma+c}\tilde{A}\Psi(N)\frac{x^{-\sigma}}{V} \sum_{x/2\leq n<N} \frac{1}{\left|\log \frac{x}{n}\right|}\right). \tag{2.34}$$

We then set $n = N - r$. Along with $n < N$ ($n,\ N \in \mathbb{N}$) this means that $r \geq 1$. Then, since $N - r = n < x$ (the alternative would mean that $n = N$, which is not true) and $x \geq N - 1/2$, we get that

$$\left|\log \frac{x}{n}\right| = \log \frac{x}{n} \geq \log \frac{N-1/2}{N-r}. \tag{2.35}$$

Moreover, if we let $x > 1$ (forcing $N \geq 1$ to guarantee convergence)[6], (2.35) is

$$\left(\frac{r}{N} + \frac{r^2}{2N^2} + \frac{r^3}{3N^3} + \frac{r^4}{4N^4} + \cdots\right) - \left(\frac{1}{2N} + \frac{1}{2(2N)^2} + \frac{1}{3(2N)^3} + \frac{1}{4(2N)^4} + \cdots\right) \quad (2.37)$$

$$\geq \frac{r}{N}\left(1 - \frac{N}{r}\frac{1}{2N-1}\right). \quad (2.38)$$

(This last line was a consequence of the inequalities

$$\frac{r^2}{2N^2} + \frac{r^3}{3N^3} + \frac{r^4}{4N^4} + \cdots > 0$$

and

$$\frac{1}{2N} + \frac{1}{2(2N)^2} + \frac{1}{3(2N)^3} + \frac{1}{4(2N)^4} + \cdots \leq \frac{1}{2N}\frac{1}{1-\frac{1}{2N}} = \frac{1}{2N-1}.)$$

We now simplify (2.38) when $r = 1$ and $r \geq 2$.

On letting $r = 1$,

$$\frac{r}{N}\left(1 - \frac{N}{r}\frac{1}{2N-1}\right) = \frac{1}{N}\left(1 - \frac{N}{2N-1}\right).$$

Then, on imposing the restriction $x > \beta/(2\beta-1) - 1/2 = 1/(4\beta-2)$ for some $1/2 < \beta < 1$,[7] we use the restriction $N \geq \beta/(2\beta-1)$ to get

$$\frac{1}{N}\left(1 - \frac{N}{2N-1}\right) \geq (1-\beta)\frac{r}{N}. \quad (2.39)$$

On letting $r \geq 2$, we use the inequality $2N-1 \geq N$ (which we have because $N \geq 1$), to

[6] The ratio test confirms that absolute convergence of (2.37) occurs if $r/N < 1$. To work towards this end, we get that

$$\frac{r}{N} \leq \frac{1}{2} + \frac{1}{4N} \quad (2.36)$$

by combining the definition $n = N - r$ with the restriction $n \geq x/2$, then using the fact that $x \geq N - 1/2$. Since we want $r/N < 1$ we only need to restrict N so that $\frac{1}{2} + \frac{1}{4N} < 1$.

[7] We have $1/2 < \beta$ because $\beta \geq N/(2N-1)$ due to the restriction on N and $N/(2N-1) > 1/2$. Moreover, we set $\beta < 1$ to stop the right-hand side of (2.39) from becoming non-positive.

get that

$$\frac{r}{N}\left(1-\frac{N}{r}\frac{1}{2N-1}\right) \geq (r-1)\frac{1}{N} \geq \frac{r}{2N}.$$

Hence,

$$\left|\log\frac{x}{n}\right| \geq \min\{1/2, 1-\beta\}\frac{r}{N}. \tag{2.40}$$

Then, since $N \leq x+1/2$, and $x>1$, we have $N<3x/2$. Thus, with $1/2<\beta<1$, we find that

$$\min\{1/2, 1-\beta\}\frac{r}{N} > \frac{2}{3}(1-\beta)\frac{r}{x}. \tag{2.41}$$

Considering (2.34), (2.40) and (2.41) gives

$$\sum_{x/2\leq n<N}\frac{|a_n|}{n^{\sigma+c}}\frac{x^c}{V\left|\log\frac{x}{n}\right|} = O^*\left(\frac{3(2^{\sigma+c-1})\tilde{A}}{1-\beta}\frac{x^{1-\sigma}\Psi(N)}{V}\sum_{1\leq r\leq\frac{x+1}{2}}\frac{1}{r}\right). \tag{2.42}$$

(Note that the upper restriction on r in (2.42) comes from using the definition $n=N-r$, with $n\geq x/2$, to get that $r\leq N-x/2$, then applying the fact that $N\leq x+1/2$.)

We have the lemma on applying condition (2.4) in (2.42). □

Proof of Proposition 2.1. Proposition 2.1 is now the result of collecting together Lemmas 2.2 to 2.6. □

2.2 The coefficients A_1 and A_2

Before we move on to the next section and derive an explicit truncated Perron's formula from Theorem 1 in [3], we shall detour to provide expressions for the implied constants A_1 and A_2. These constants showed up in the truncated Perron's formula we gave in Proposition 2.1. Their values are independent of the particular sum $\sum_{n\leq x}\frac{a_n}{n^s}$ that we wish to bound.

Hypothesis (2.3) in Proposition 2.1 states that for some $A_1>0$ and $x>1$,

$$\sum_{1\leq r\leq x+\frac{1}{2}}\frac{1}{r} \leq A_1\log x.$$

Proposition 2.7 below gives the expression we shall use for A_1.

Proposition 2.7. *Suppose $x \geq x_0$ for some fixed $x_0 > 1$. We may then let*

$$A_1 = \frac{1 + \log\left(1 + \frac{1}{2x_0}\right)}{\log x_0} + 1. \tag{2.43}$$

Proof. We know that,

$$\sum_{1 \leq r \leq x + \frac{1}{2}} \frac{1}{r} \leq 1 + \int_1^{x+\frac{1}{2}} \frac{1}{u} du.$$

On evaluating the integral and rearranging, we get that

$$1 + \int_1^{x+\frac{1}{2}} \frac{1}{u} du = \left(\frac{1 + \log\left(1 + \frac{1}{2x}\right)}{\log x} + 1 \right) \log x.$$

We then impose the condition $x \geq x_0$ to get the result. □

Hypothesis (2.4) in Proposition 2.1 expresses that for some $A_2 > 0$ and $x > 1$,

$$\sum_{1 \leq r \leq \frac{x+1}{2}} \frac{1}{r} \leq A_2 \log x.$$

Proposition 2.8 below gives the expression we shall use for A_2.

Proposition 2.8. *Suppose $x \geq x_0$ for some fixed $x_0 > 1$. We may let*

$$A_2 = \frac{1 + \log\left(1 + \frac{1}{x_0}\right) - \log 2}{\log x_0} + 1.$$

Proof. We know that,

$$\sum_{1 \leq r \leq \frac{x+1}{2}} \frac{1}{r} \leq 1 + \int_1^{\frac{x+1}{2}} \frac{1}{u} du,$$

which is, in turn, bounded by

$$1 + \int_1^{\frac{x+1}{2}} \frac{1}{u} du = \left(\frac{1 + \log\left(1 + \frac{1}{x}\right) - \log 2}{\log x} + 1 \right) \log x.$$

We then impose the condition $x \geq x_0$ to get the result. □

2.3 An explicit Perron's formula from Arkhipova's theorem

As in Section 2.1, our task in this section is to prove an explicit truncated Perron's formula. However, our approach in this section is based on Theorem 1 and its proof by Arkhipova in [3]. Our contribution is in making Arkhipova's theorem explicit. Unlike Proposition 2.1 of the last section, we only write this result for the special case $\sum_{n\leq x} a_n$ as was done in [3]. However, the proposition we give is just as much a cornerstone of what is to come, as is Proposition 2.1.

Proposition 2.9. *Assume that for some* $\eta > 0$, $A = A(\eta) > 0$ *and* $\alpha \in \mathbb{R}$, *we have that*

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n^c} = O^*\left(\frac{A}{(c-1)^{\alpha}}\right), \tag{2.44}$$

where $1 < c < 1+\eta$. *Also assume that for some* $\tilde{A} > 0, n \in \mathbb{N}$, *and a non-decreasing function* $\Psi(n)$ *we have that*

$$a_n = O^*\left(\tilde{A}\Psi(n)\right). \tag{2.45}$$

Let $2c < V < x$. *On defining* x *to be non-integer, we have that*

$$\begin{aligned} \sum_{n\leq x} a_n \quad = \quad & \frac{1}{2\pi i}\int_{c-iV}^{c+iV}\left(\sum_{n=1}^{\infty}\frac{a_n}{n^w}\right)\frac{x^w}{w}dw + O^*\left(B_5\frac{x^c}{V(c-1)^{\alpha}}\right) \\ & + O^*\left(B_6\frac{\Psi(x)x\log V}{V}\right) + O^*\left(B_7\frac{\Psi(2x)x\log V}{V}\right) \end{aligned} \tag{2.46}$$

with

$$B_5 := \frac{2A}{\pi\log 2} \tag{2.47}$$

$$B_6 := \frac{2^c\tilde{A}(\pi\log 2c + \log\frac{\pi}{2} + 1)}{\pi\log 2c}, \text{ and} \tag{2.48}$$

$$B_7 := \frac{\tilde{A}(4+\pi+4\log 2c + 4\log\frac{\pi}{4})}{2\pi\log 2c}. \tag{2.49}$$

We shall break the proof of this theorem into four lemmas. In each lemma we assume the appropriate conditions from Proposition 2.9 without giving them in the statement of the lemma.

Lemma 2.10. *We have that*

$$\begin{aligned}\sum_{n<x} a_n &= \frac{1}{2\pi i}\int_{c-iV}^{c+iV}\left(\sum_{n=1}^{\infty}\frac{a_n}{n^w}\right)\frac{x^w}{w}dw \\ &\quad +O^*\left(\sum_{n<x}|a_n|\left(\frac{x}{n}\right)^c \min\left\{1,\frac{1}{\pi V\left|\log\frac{x}{n}\right|}\right\}\right) \\ &\quad +O^*\left(\sum_{n>x}|a_n|\left(\frac{x}{n}\right)^c \min\left\{\frac{1}{2},\frac{1}{\pi V\left|\log\frac{x}{n}\right|}\right\}\right). \end{aligned} \tag{2.50}$$

Proof. If we can prove that

$$\frac{1}{2\pi i}\int_{c-iV}^{c+iV}\left(\frac{x}{n}\right)^w\frac{dw}{w} = \begin{cases} 1+O^*\left(\left(\frac{x}{n}\right)^c\min\left\{1,\frac{1}{\pi V\left|\log\frac{x}{n}\right|}\right\}\right) & \text{for } n<x, \\ O^*\left(\left(\frac{x}{n}\right)^c\min\left\{\frac{1}{2},\frac{1}{\pi V\left|\log\frac{x}{n}\right|}\right\}\right) & \text{for } n>x, \end{cases} \tag{2.51}$$

we would have that

$$\begin{aligned}\frac{1}{2\pi i}\sum_{n=1}^{\infty}a_n\int_{c-iV}^{c+iV}\left(\frac{x}{n}\right)^w\frac{dw}{w} &= \sum_{n<x}a_n + O^*\left(\sum_{n<x}|a_n|\left(\frac{x}{n}\right)^c\min\left\{1,\frac{1}{\pi V\left|\log\frac{x}{n}\right|}\right\}\right) \\ &\quad +O^*\left(\sum_{n>x}|a_n|\left(\frac{x}{n}\right)^c\min\left\{\frac{1}{2},\frac{1}{\pi V\left|\log\frac{x}{n}\right|}\right\}\right).\end{aligned}$$

For reasons analogous to those given when we proved Lemma 2.2, we may bring the series inside the integral on the left-hand side of the equation above.

The proof of (2.51) is done in, for example, [8, pp. 105–106].[8] We shall not give the proof herein, but with such a proof, one has the lemma. □

We next work with the bounds on the right-hand side of (2.50) over several intervals. The first two intervals we consider are $n \geq 2x$ and $n \leq x/2$. However, we have already dealt

[8]The bound given in [8] is slightly weaker than what we have given.

with bounds over these intervals in Lemma 2.3. For completeness, we include the result below as Lemma 2.11. (Note that even though we are now including the equality in both intervals it makes no substantial differences in the proof.)

Lemma 2.11. *We have*

$$\sum_{\substack{n\geq 2x \\ or \\ n\leq x/2}} |a_n| \left(\frac{x}{n}\right)^c \min\left\{1, \frac{1}{\pi V \left|\log\frac{x}{n}\right|}\right\} = O^*\left(B_5 \frac{x^c}{V(c-1)^\alpha}\right)$$

for B_5 as in Proposition 2.9.

The other two intervals we consider are $x/2 < n < x$ and $x < n < 2x$. For these intervals we have Lemmas 2.12 and 2.13.

Lemma 2.12. *We have*

$$\sum_{\frac{x}{2}<n<x} |a_n| \left(\frac{x}{n}\right)^c \min\left\{1, \frac{1}{\pi V \left|\log\frac{x}{n}\right|}\right\} = O^*\left(B_6 \frac{\Psi(x) x \log V}{V}\right)$$

for B_6 as in Proposition 2.9.

Proof. From hypothesis (2.45) in Proposition 2.9 and the restriction $x/2 < n < x$, we have that

$$|a_n| \left(\frac{x}{n}\right)^c = O^*\left(\tilde{A}\Psi(x)2^c\right).$$

We are now left with the job of bounding the sum

$$\sum_{\frac{x}{2}<n<x} \min\left\{1, \frac{1}{V\pi \left|\log\frac{x}{n}\right|}\right\}.$$

We shall begin by considering [9]

$$\log\frac{n}{x} = \log\left(1+\frac{(n-x)}{x}\right) = \frac{(n-x)}{x} - \frac{(n-x)^2}{2x^2} + \frac{(n-x)^3}{3x^3} - \cdots. \tag{2.52}$$

Since $n-x<0$, the absolute value of (2.52) is

$$-\frac{(n-x)}{x} + \frac{(n-x)^2}{2x^2} - \frac{(n-x)^3}{3x^3} + \cdots,$$

but $\dfrac{(n-x)^k}{kx^k} - \dfrac{(n-x)^{k+1}}{(k+1)x^{k+1}} > 0$, for even $k \geq 2$, so

$$\log\frac{x}{n} = \left|\log\frac{n}{x}\right| \geq -\frac{(n-x)}{x} = \frac{|n-x|}{x}.$$

Thus,

$$\sum_{\frac{x}{2}<n<x} \min\left\{1, \frac{1}{V\pi\log\frac{x}{n}}\right\} = O^*\left(\sum_{\frac{x}{2}<n<x} \min\left\{1, \frac{x}{V\pi|n-x|}\right\}\right).$$

Now, we will let $m := n-x$, which transforms the sum on the right-hand side into

$$\sum_{-\frac{x}{2}<m<0} \min\left\{1, \frac{x}{V\pi|m|}\right\}. \tag{2.53}$$

Furthermore, if we consider when the summand in (2.53) goes from being $\dfrac{x}{V\pi|m|}$ to being 1, we find that we may write (2.53) as

$$\frac{1}{\pi}\sum_{-\frac{x}{2}<m\leq-\frac{x}{\pi V}} \frac{x}{V|m|} + \sum_{-\frac{x}{\pi V}<m<0} 1.$$

[9]The ratio test confirms that absolute convergence of (2.52) occurs in such a case that $|(n-x)/x| < 1$. This condition is satisfied by all relevant x and all $x/2 < n < x$. More specifically, from the fact that $x/2 < n < x$ we know that $-1/2 < (n-x)/x < 0$.

Then, due to the fact that,

$$\sum_{-\frac{x}{2}<m\leq -\frac{x}{\pi V}} \frac{x}{V|m|} \leq \pi + \frac{x}{V}\int_{\frac{x}{\pi V}}^{\frac{x}{2}} \frac{dt}{t}$$

and

$$\sum_{-\frac{x}{\pi V}<m<0} 1 \leq \frac{x}{\pi V},$$

we get that

$$\sum_{-\frac{x}{2}<m<0} \min\left\{1, \frac{x}{\pi V|m|}\right\} \leq \frac{x}{V}\log V\left(\frac{V}{x\log V} + \frac{1}{\pi} + \frac{\log\frac{\pi}{2}}{\pi\log V} + \frac{1}{\pi\log V}\right).$$

Under the assumption $2c < V < x$, the last bound becomes

$$\frac{x}{V}\log V \frac{(\pi + \log 2c + \log\frac{\pi}{2} + 1)}{\pi\log 2c}.$$

Thus, we are done. □

Lemma 2.13. *We have*

$$\sum_{x<n<2x} |a_n|\left(\frac{x}{n}\right)^c \min\left\{\frac{1}{2}, \frac{1}{\pi V\left|\log\frac{x}{n}\right|}\right\} = O^*\left(B_7\frac{\Psi(2x)x\log V}{V}\right)$$

for B_7 as in Proposition 2.9.

Proof. From hypothesis (2.45) in Theorem 2.9 and the restriction $x < n < 2x$, we have that

$$|a_n|\left(\frac{x}{n}\right)^c = O^*\left(\tilde{A}\Psi(2x)\right). \tag{2.54}$$

Thus, we now only need to get a bound on the sum

$$\sum_{x<n<2x} \min\left\{\frac{1}{2}, \frac{1}{\pi V\left|\log\frac{x}{n}\right|}\right\}.$$

We shall begin by considering [10]

$$\log\frac{n}{x} = \log\left(1+\frac{(n-x)}{x}\right) = \frac{(n-x)}{x} - \frac{(n-x)^2}{2x^2} + \frac{(n-x)^3}{3x^3} - \cdots. \tag{2.55}$$

Here, $n < 2x,\ n-x > 0$ and $x > 0$, so we may ignore all terms, but the first two in (2.55) because

$$\frac{(n-x)^k}{kx^k} - \frac{(n-x)^{k+1}}{(k+1)x^{k+1}} \geq \frac{(n-x)^k}{x^{k+1}k(k+1)}x > 0$$

for $k \geq 3$. Moreover, we have $n > x$. This means $|\log(n/x)| = \log(n/x)$. Hence,

$$\left|\log\frac{n}{x}\right| \geq \frac{(n-x)}{x} - \frac{(n-x)^2}{2x^2}.$$

We also notice that

$$\frac{(n-x)}{2x} > \frac{(n-x)^2}{2x^2}$$

because $n < 2x$, from which we get that $(n-x)/x < 1$. Hence,

$$\frac{(n-x)}{x} - \frac{(n-x)^2}{2x^2} > \frac{(n-x)}{x} - \frac{(n-x)}{2x} = \frac{|n-x|}{2x}$$

and so,

$$\left|\log\frac{x}{n}\right| = \left|\log\frac{n}{x}\right| > \frac{|n-x|}{2x}.$$

Thus,

$$\sum_{x<n<2x} \min\left\{\frac{1}{2}, \frac{1}{\pi}\frac{1}{V\left|\log\frac{x}{n}\right|}\right\} = O^*\left(\sum_{x<n<2x} \min\left\{\frac{1}{2}, \frac{2x}{V\pi|n-x|}\right\}\right).$$

Now, we will define $m := n-x$, which transforms the above sum into

$$\sum_{0<m<x} \min\left\{\frac{1}{2}, \frac{2x}{V\pi|m|}\right\}. \tag{2.56}$$

[10] The ratio test confirms that absolute convergence of (2.55) occurs in such a case that $|(n-x)/x| < 1$. This condition is satisfied by all relevant x and all $x < n < 2x$. More specifically, from the fact that $x < n < 2x$ we know that $0 < (n-x)/x < 1$.

The summand in (2.56) is $1/2$ only until $x = \frac{4x}{\pi V}$, so we replace (2.56) with

$$\frac{1}{2}\sum_{0<m<\frac{4x}{\pi V}} 1 + \frac{2}{\pi}\sum_{\frac{4x}{\pi V}\leq m<x} \frac{x}{V|m|}.$$

Then, using the upper bounds

$$\sum_{0<m<\frac{4x}{\pi V}} 1 \leq \frac{4x}{\pi V}$$

and

$$\sum_{\frac{4x}{\pi V}\leq m<x} \frac{x}{V|m|} \leq \frac{\pi}{4} + \frac{x}{V}\int_{\frac{4x}{\pi V}}^{x} \frac{dt}{t},$$

we get that

$$\sum_{0<m<x} \min\left\{\frac{1}{2}, \frac{2x}{V\pi|m|}\right\} \leq \frac{x\log V}{V}\left(\frac{2}{\pi\log V} + \frac{V}{2x\log V} + \frac{2}{\pi} + \frac{2\log\frac{\pi}{4}}{\pi\log V}\right). \tag{2.57}$$

Under the assumption $2c < V < x$, (2.57) becomes

$$\frac{x}{V}\log V\frac{(4+\pi+4\log 2c+4\log\frac{\pi}{4})}{2\pi\log 2c}.$$

Therefore, we are done. □

Proof of Proposition 2.9. Proposition 2.9 is now the result of collecting together Lemmas 2.10 to 2.13. □

2.4 Working with the integrals

We next prove a bound on the integrals that appears in (2.5) and (2.46).

Herein, $f(w)$, with $w \in \mathbb{C}$, should be taken to be either the series $\sum_{n=1}^{\infty} \frac{a_n}{n^{s+w}}$ with $a_n \in \mathbb{C}$, the series $\sum_{n=1}^{\infty} \frac{a_n}{n^w}$ or the meromorphic continuation of whichever of these two series are of interest.

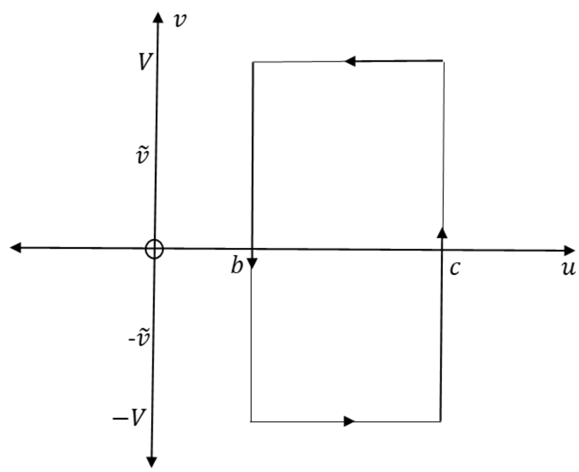

Figure 2.3: A contour needed in Proposition 2.14.

Proposition 2.14. *Let* $V,\ x>1$ *and let* $\tilde{v}, c>0$*, with* $\tilde{v}<V$*. Also let* $s=\sigma+it$ *and* $w=u+iv$*. Assume that, for some* $b<c$ *and all* $u\in[b,c]$*, there exists functions* $g(b,c,V)$ *and* $\tilde{g}(b,c,V)$ *such that*

$$f((\sigma+u)+i(t+V))=O^{*}\left(g(b,c,V)\right) \tag{2.58}$$

and

$$f((\sigma+u)+i(t-V))=O^{*}\left(\tilde{g}(b,c,V)\right). \tag{2.59}$$

Also, assume that for all $v\in[\tilde{v},V]$ *there exists a function* $h(b,V)$ *such that*

$$f((\sigma+b)+i(t+v))=O^{*}\left(h(b,V)\right), \tag{2.60}$$

for all $v\in[-\tilde{v},-V]$ *there exists a function* $\tilde{h}(b,V)$ *such that*

$$f((\sigma+b)+i(t+v))=O^{*}\left(\tilde{h}(b,V)\right), \tag{2.61}$$

and for all $v\in[-\tilde{v},\tilde{v}]$ *there exists a function* $\hat{h}(b,\tilde{v})$ *such that*

$$f((\sigma+b)+i(t+v))=O^{*}\left(\hat{h}(b,\tilde{v})\right). \tag{2.62}$$

If we restrict our attention to $f(w)$ with an analytic continuation to points w that are on the contour shown in Figure 2.3[11]*, a meromorphic continuation to points w that are inside this contour and for which the number of poles is finite, we have*

$$\begin{aligned}\frac{1}{2\pi i}\int_{c-iV}^{c+iV} f(s+w)\frac{x^w}{w}dw &= R+O^*\left(\frac{1}{2\pi}\left(g(b,c,V)+\tilde{g}(b,c,V)\right)\ \frac{x^c}{V\log x}\right) \\ &\quad +O^*\left(\frac{1}{2\pi}\left(h(b,V)+\tilde{h}(b,V)\right)x^b\log V\right) \\ &\quad +O^*\left(\frac{\tilde{v}}{|b|\pi}\hat{h}(b,\tilde{v})x^b\right), \qquad (2.63)\end{aligned}$$

with R being the sum of the residues of $f(s+w)\frac{x^w}{w}$ with respect to the poles of this function that are located within the contour of Figure 2.3.

We shall break the proof of this theorem into four lemmas.

Lemma 2.15. *Let $V,\ c>0$ and $b<c$. Consider the contour in Figure 2.3. Then, with R defined as in Proposition 2.14, we have that*

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_{c-iV}^{c+iV} f(s+w)\frac{x^w}{w}dw - R\right| &\le \frac{1}{2\pi}\left(\left|\int_{c+iV}^{b+iV} f(s+w)\frac{x^w}{w}dw\right|\right. \\ &\quad +\left|\int_{b+iV}^{b-iV} f(s+w)\frac{x^w}{w}dw\right| \\ &\quad \left.+\left|\int_{b-iV}^{c-iV} f(s+w)\frac{x^w}{w}dw\right|\right). \qquad (2.64)\end{aligned}$$

Proof. We apply Cauchy's Theorem to the integral on the right-hand side of (2.63) with respect to the contour in Figure 2.3 and invoke the triangle inequality. □

Lemma 2.16. *Let $V,c>0,\ b<c$, and $x>1$. Furthermore, assume that one has* (2.58). *Then,*

$$\int_{c+iV}^{b+iV} f(s+w)\frac{x^w}{w}dw = O^*\left(g(b,c,V)\frac{x^c}{V\log x}\right). \qquad (2.65)$$

[11] Although b is depicted as being greater than zero in Figure 2.3, this is not necessary. In Chapter 4 we shall in fact use a $b<0$.

Proof. After re-writing the integral as

$$\int_c^b f((\sigma+u)+i(t+V))\frac{x^{u+iV}}{u+iV}du$$

we can proceed by getting a bound on

$$\int_b^c |f((\sigma+u)+i(t+V))|\frac{x^u}{V}du.$$

Making use of the bound on $f((\sigma+u)+i(t+V))$ given in (2.58) finishes the proof. □

Lemma 2.16 also provides one with a bound on

$$\int_{b-iV}^{c-iV} f(s+w)\frac{x^w}{w}dw$$

if we write $\tilde{g}(b,c,V)$ instead of $g(b,c,V)$ so we shall move on.

Lemma 2.17. *Let* V, $x>1$ *and* $\tilde{v}>0$, *where* $\tilde{v}<V$. *Also let* $s=\sigma+it$. *Furthermore, assume hypotheses* (2.60), (2.61), *and* (2.62). *Then,*

$$\begin{aligned}\int_{b-iV}^{b+iV} f(s+w)\frac{x^w}{w}dw &= O^*\left(\left(h(b,V)+\tilde{h}(b,V)\right)x^b\log V\right)\\ &\quad +O^*\left(\frac{2\tilde{v}}{|b|}\hat{h}(b,\tilde{v})x^b\right). \qquad (2.66)\end{aligned}$$

Proof. After re-writing the integral as

$$\int_{-V}^{-\tilde{v}}+\int_{-\tilde{v}}^{\tilde{v}}+\int_{\tilde{v}}^{V}\left(f((\sigma+b)+i(t-v))\frac{x^{b+iv}}{b+iv}i\right)dv$$

we see (by the triangle inequality) that a bound on the integral is given by a bound on

$$\int_{-V}^{-\tilde{v}}+\int_{\tilde{v}}^{V}\left(|f((\sigma+b)+i(t+v))|\frac{x^b}{|v|}\right)dv+\int_{-\tilde{v}}^{\tilde{v}}\left(|f((\sigma+b)+i(t+v))|\frac{x^b}{|b|}\right)dv.$$

Thus, all we require to finish the proof are the bounds given in (2.60), (2.61), and (2.62). □

Proof of Proposition 2.14. Proposition 2.14 is now the result of collecting together Lemmas 2.15 to 2.17. □

2.5 The Titchmarsh and Arkhipova bounds

We may now give the two penultimate theorems of this chapter. These theorems provide bounds on $\sum_{n\le x}\frac{a_n}{n^s}$ or $\sum_{n\le x} a_n$. The first theorem corresponds to the truncated Perron's formula we got in Section 2.1 and the second theorem comes from the truncated Perron's formula we proved in Section 2.3.

Theorem 2.18. *Under the assumptions of Propositions 2.1 and 2.14,*

$$\begin{aligned}
\sum_{n\le x}\frac{a_n}{n^s} \;=\; & R+O^*\left(\frac{1}{2\pi}\exp(\log g(b,c,V)+\log(x^c)-\log V-\log\log x)\right)\\
&+O^*\left(\frac{1}{2\pi}\exp(\log \tilde{g}(b,c,V)+\log(x^c)-\log V-\log\log x)\right)\\
&+O^*\left(\frac{1}{2\pi}\exp(\log h(b,V)+\log(x^b)+\log\log V)\right)\\
&+O^*\left(\frac{1}{2\pi}\exp(\log \tilde{h}(b,V)+\log(x^b)+\log\log V)\right)\\
&+O^*\left(\frac{\tilde{v}}{\pi}\exp(\log \hat{h}(b,\tilde{v})+\log(x^b)-\log|b|)\right)\\
&+O^*\left(B_1\exp(\log(x^c)-\log V-\alpha\log(\sigma+c-1))\right)\\
&+O^*\left(B_2 x\exp(\log\Psi(2x)-\log(x^\sigma)+\log\log x-\log V)\right)\\
&+O^*\left(B_3 x\exp(\log\Psi(N)-\log(x^\sigma)+\log\log x-\log V)\right)\\
&+O^*\left(B_4 x\exp(\log\Psi(N)-\log(x^\sigma)-\log V-\log|x-N|)\right). \qquad (2.67)
\end{aligned}$$

Proof. Combine the result of Proposition 2.1 and Proposition 2.14, then re-write the bound so that each term is made up of a coefficient, a factor of x and a power of an exponential. □

Theorem 2.19. *Under the assumptions of Propositions 2.9 and 2.14,*

$$\begin{aligned}
\sum_{n\leq x} a_n &= R + O^*\left(\frac{1}{2\pi}\exp(\log g(b,c,V) + \log(x^c) - \log V - \log\log x)\right) \\
&\quad + O^*\left(\frac{1}{2\pi}\exp(\log \tilde{g}(b,c,V) + \log(x^c) - \log V - \log\log x)\right) \\
&\quad + O^*\left(\frac{1}{2\pi}\exp(\log h(b,V) + \log(x^b) + \log\log V)\right) \\
&\quad + O^*\left(\frac{1}{2\pi}\exp(\log \tilde{h}(b,V) + \log(x^b) + \log\log V)\right) \\
&\quad + O^*\left(\frac{\tilde{v}}{\pi}\exp(\log \hat{h}(b,\tilde{v}) + \log(x^b) - \log|b|)\right) \\
&\quad + O^*\left(B_5 \exp\left(\log\left(x^c\right) - \log V - \alpha\log(c-1)\right)\right) \\
&\quad + O^*\left(B_6 x \exp\left(\log \Psi(x) + \log\log V - \log V\right)\right) \\
&\quad + O^*\left(B_7 x \exp\left(\log \Psi(2x) + \log\log V - \log V\right)\right). \qquad (2.68)
\end{aligned}$$

Proof. In our proof of Theorem 2.18, use Proposition 2.9 rather than Proposition 2.1. Everything else is the same. □

We now carry Theorems 2.18 and 2.19 forward with us into the next chapter. In Chapter 3, we shall be switching gears from working with $\sum_{n\leq x} \frac{a_n}{n^s}$ or $\sum_{n\leq x} a_n$ where (a_n) has not been specified, to specifying (a_n) and working with the corresponding sum $\sum_{n\leq x} a_n$.

Chapter 3

The function $M(x)$

Now that we have bounds on $\sum_{n\leq x} \frac{a_n}{n^s}$, with $a_n, s \in \mathbb{C}$, we can apply these bounds to examples of sums of this type. One such sum is

$$M(x) = \sum_{n\leq x} \mu(n).$$

The aim of this chapter is to arrive at explicit bounds on $M(x)$ via the results we proved in Chapter 2. In particular, we will show that there is an $x_0 > 0$ such that when $x \in (x_0, \infty)$ we have

$$M(x) = O^*\left(Cx(\log x)\exp\left(-c\sqrt{\log x}\right)\right). \tag{3.1}$$

We will also show that there is another $x_0 > 0$ such that when $x \in (x_0, \infty)$ we have

$$M(x) = O^*\left(Cx\exp\left(-c\sqrt{\log x}\right)\right) \tag{3.2}$$

and we will give explicit values for the x_0, C and c in (3.1) and (3.2).

One of the strongest bounds we are aware of is

$$M(x) = O^*\left(\frac{0.013x\log x - 0.118x}{(\log x)^2}\right) \tag{3.3}$$

from [24]. This bounds can be used when $x \in [\exp(13.90), \infty)$, but we will see that the bounds we shall get in this chapter will be stronger than (3.3) if x is sufficiently large. Hence, for $M(x)$, (3.2) is the tightest type of explicit bound that is known unconditionally for x in an unbounded interval.

We shall get explicit bounds on $M(x)$ in Section 3.1. We will then analyze these bounds in Section 3.2. This analysis shall include determining how large x needs to be for one of our bounds to become stronger than (3.3).

3.1 Arriving at explicit bounds on $M(x)$

Chapter 2 contains two theorems that we shall now apply to arrive at explicit bounds on $M(x)$. Section 3.1.1 shall rely on Theorem 2.18, and Section 3.1.2 shall rely on Theorem 2.19.

The two approaches we shall consider produce bounds that only apply at x in a subset of (x_0,∞). We want our bounds to work for x in the whole interval (x_0,∞). Hence, we will have to transform the bounds on $M(x)$ that we get, into bounds that apply at any $x \in (x_0,\infty)$. We do this in Section 3.1.3.

3.1.1 The approach from Titchmarsh's Lemma 3.12

Theorem 2.18 is our ticket to proving explicit bounds on $M(x)$. The hypotheses of Theorem 2.18 are exactly the hypotheses of Propositions 2.1 and 2.14. Hence, we need to make sure $M(x)$ satisfies the hypotheses of Propositions 2.1 and 2.14. We shall first demonstrate that these hypotheses are satisfied. Then, we shall build on Theorem 2.18 until we have explicit bounds on $M(x)$.

Proposition 2.1 gave an explicit truncated Perron's formula. There were four hypotheses involved in proving this proposition. In this context, for some $A = A(\eta)$, $\eta > 0$, $1 < c < 1+\eta$ and $\alpha \in \mathbb{Z}^{\geq 0}$, the first hypothesis is

$$\sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^c} = O^*\left(\frac{A}{(c-1)^{\alpha}}\right).$$

This hypothesis is satisfied by the following Proposition 3.1. We prove Proposition 3.1 by referring to the proof of Lemma 3.1 in [12].

Proposition 3.1. *We get that for* $0 < \eta \le 1/\log(1.5)$ *and* $1 < c < 1+\eta$,

$$\sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^c} < \frac{\left(1-\log(1.5)+\frac{1}{2}\eta\log^2(1.5)\right)\eta+1}{(c-1)}.$$

Proof. The proof of Lemma 3.1 in [12] gives that

$$\zeta(c) \le \frac{\left(1-\log(1.5)+\frac{1}{2}(c-1)\log^2(1.5)\right)(c-1)+1}{(c-1)},$$

for $1 < c \le 1+1/\log(1.5)$. Moreover, we know that

$$\sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^c} \le \zeta(c).$$

Hence, the proposition holds if $0 < c-1 < \eta \le 1/\log(1.5)$. □

Thus, we can take $\alpha = 1$ and $A = \left(1-\log(1.5)+\frac{1}{2}\eta\log^2(1.5)\right)\eta+1$ for $c-1 \le \eta \le 1/\log(1.5)$.

In this context, the second hypothesis in Proposition 2.1 is that $|\mu(n)| \le \tilde{A}\Psi(n)$. We have $|\mu(n)| \le 1$, so we can take $\tilde{A} = 1$ and $\Psi(n) = 1$.

The third and fourth hypotheses were addressed in Propositions 2.7 and 2.8.

We can now move onto the hypotheses of Proposition 2.14, which was all about the integral

$$\frac{1}{2\pi i}\int_{c-iV}^{c+iV} f(w)\frac{x^w}{w}dw.$$

In this case

$$f(w) = \frac{1}{\zeta(w)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^w} \tag{3.4}$$

because we will have $\mathrm{Re}(w) = c > 1$. (See [21, p. 208] for a proof of (3.4)). However, in what follows, we will not always have $\mathrm{Re}(w) > 1$; sometimes $\mathrm{Re}(w) \le 1$. In these instances, we define $f(w) = 1/\zeta(w)$ to be the analytic continuation of $\sum_{n=1}^{\infty} \frac{\mu(n)}{n^w}$. Such an analytic continuation exists with respect to all of the w we are interested in because $\zeta(w) \ne 0$ in the region we shall consider.

There were six hypotheses involved in Proposition 2.14. In this context, five of the hypotheses require a bound on $1/\zeta(w)$. These bounds each correspond to one of the five dark lines in Figure 3.1.

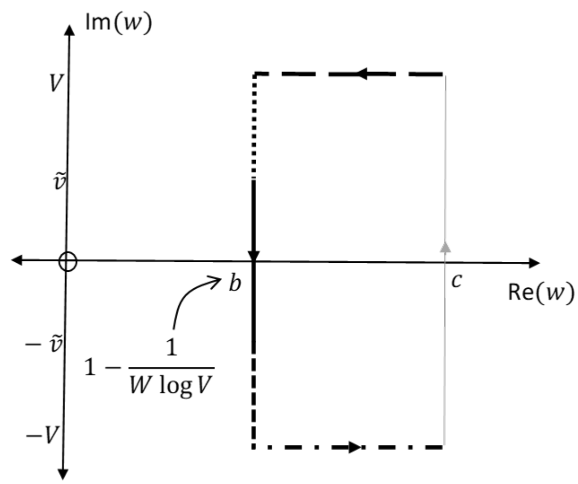

Figure 3.1: The (dark) lines corresponding to five of the hypotheses of Proposition 2.14. (A different line-style indicates correspondence to a different hypotheses.)

The four bounds on $1/\zeta(w)$ that each correspond to one of the non-solid lines in Figure 3.1 were dealt with by Trudgian in [30]. These bounds look like

$$\frac{1}{\zeta(w)} = O^*\left(C_1 \log V\right). \tag{3.5}$$

We shall need (3.5) for $w \in \mathbb{C}$ with $\mathrm{Re}(w) \geq 1 - \frac{1}{W \log V}$ and $\tilde{v} \leq |\mathrm{Im}(w)| \leq T$. Hence, we can take $g(b,c,V) = \tilde{g}(b,c,V) = h(b,V) = \tilde{h}(b,V) = C_1 \log V$. To use these bounds we shall set

$$b := 1 - \frac{1}{W \log V}. \tag{3.6}$$

Furthermore, the C_1, W and $\tilde{v}$ that we need will come from [30, Table 2]. We have included each set of W, $\tilde{v}$ and C_1 in Table 3.1, alongside another constant C_2 that we shall consider next.

Table 3.1: The W, $\tilde{v}$, C_1 and C_2 we will use to get bounds on $M(x)$. (The first three columns of this table are from [30, Table 2]).

W	$\tilde{v}$	C_1	C_2
6	34.00	3.2×10^{30}	3.33
7	34.00	1.3×10^{10}	3.29
8	50.28	3.1×10^{6}	3.24
9	70.59	9.6×10^{4}	3.21
10	90.87	1.5×10^{4}	3.19
11	111.12	4.4×10^{3}	3.17
12	132.16	1.9×10^{3}	3.16

The other hypothesis in Proposition 2.14 that involves a bound on $1/\zeta(w)$ corresponds to the dark solid line in Figure 3.1. For this line, $\text{Re}(w) = b$ and $-\tilde{v} \leq \text{Im}(w) \leq \tilde{v}$. In Proposition 2.14, V can be any number larger than $\tilde{v}$. Moreover, as V increases, b shifts closer and closer to $\text{Im}(w) = 1$, according to (3.6). As a consequence, an appropriate bound for $1/\zeta(w)$ on the dark solid line in Figure 3.1 needs to be a bound over the whole closed-shaded rectangle shown in Figure 3.2(b). Fortunately, if we use the Maximum Modulus Theorem, which is,

> "Let f be analytic in a bounded region D, continuous in $\overline{D}$ (the closure of D), and let $|f(z)| \leq M$ for $z \in \partial D$ (the boundary of D). Then $|f(z)| \leq M$ in D; moreover, if $|f(z)| = M$ for some $z \in D$, then f is constant in D.",

as stated in [5, p. 132], we do not have to be concerned with the values of $1/|\zeta(w)|$ inside the rectangle in Figure 3.2(b). We need only consider values of $1/|\zeta(w)|$ on the boundary of the rectangle. To this end, we have consulted graphs from Maple that show $1/|\zeta(w)|$ on each of the four boundary lines for the various W and $\tilde{v}$ in Table 3.1. We determine a bound by finding the graph with the highest point and re-producing the particular section of that graph containing the maximum so that we can read what the maximum is from the graph. Figure 3.2(a), (c) and (d) show the graphs of $1/|\zeta(w)|$ on the rectangle's boundary lines for $W = 6$ and $\tilde{v} = 34$. Figure 3.3 shows the close-up of the maximum in Figure 3.2(d).

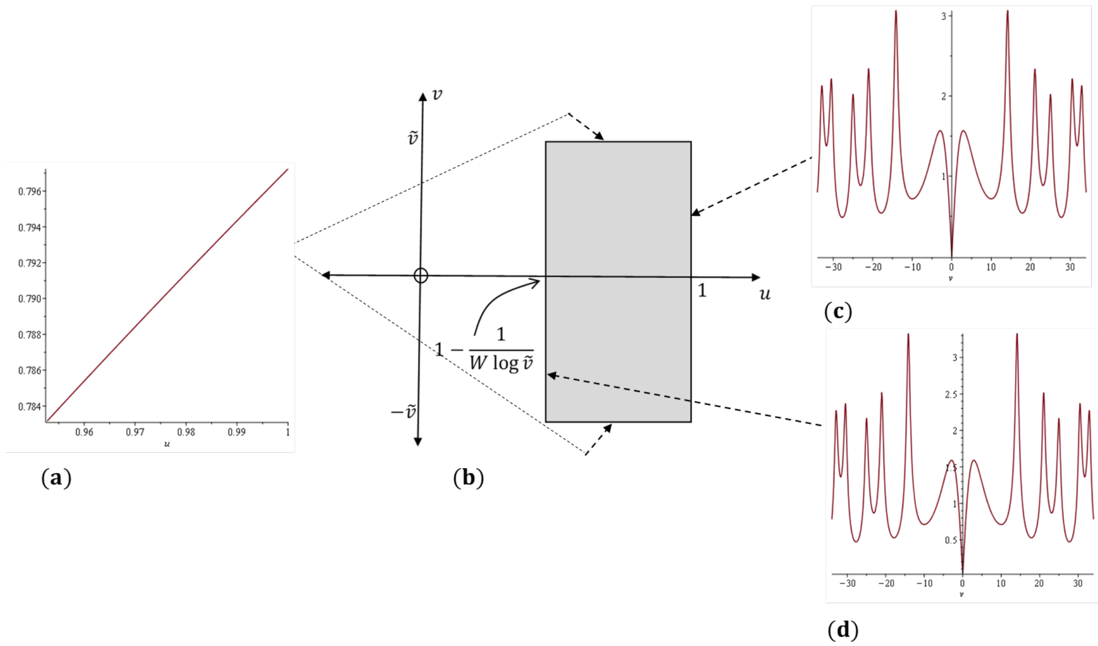

Figure 3.2: [(a), (c) and (d)] Graphs of $1/|\zeta(w)|$ on the rectangle's four boundary lines for $W=6$ and $\tilde{v}=34$. [(b)] The closed-shaded rectangle in which $w=u+iv$ that satisfy (3.7) are found.

From this method, we get the bounds

$$\frac{1}{\zeta(w)}=O^*\left(C_2\right). \tag{3.7}$$

That is, we can take $\hat{h}(b,\tilde{v})=C_2$. The constants we obtain for C_2 with respect to each W and $\tilde{v}$ are included in Table 3.1.

We now have just one hypothesis of Proposition 2.14 left to examine. Proposition 3.2 below fills this void.

Proposition 3.2. *If W and $\tilde{v}$ are as defined in Table 3.1 and $V>\tilde{v}$ then $1/\zeta(w)$ is holomorphic whenever $Re(w)\geq 1-\dfrac{1}{W\log V}$ and $Im(w)\leq V$.*

Proof. The entries in Table 3.1 for W that we shall use are no smaller than 6 and the entries for $\tilde{v}$ that we will use are no smaller than 34. Hence, we know that any w satisfying $\mathrm{Re}(w)\geq 1-\dfrac{1}{W\log V}$ and $\mathrm{Im}(w)\leq V$ will not make $\zeta(w)=0$.[12] □

[12] As per the zero-free region (1.8) that is stated in Chapter 1.

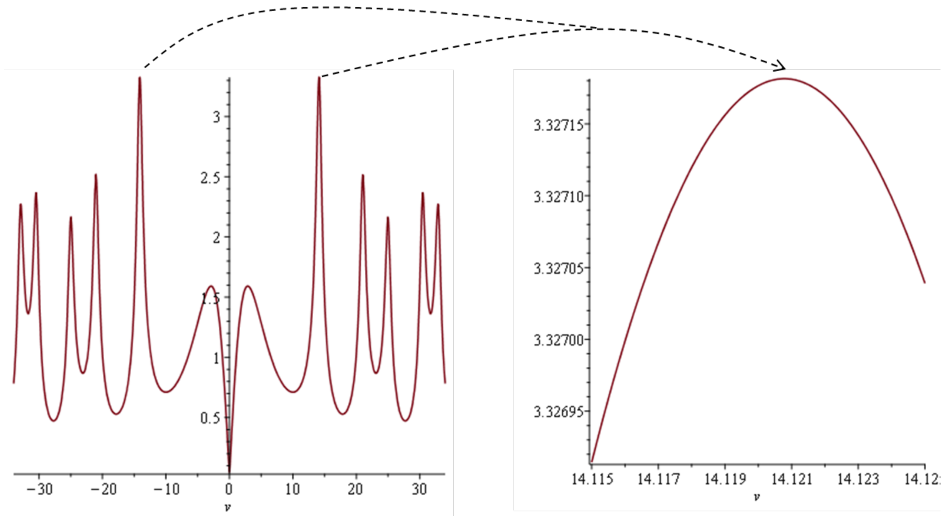

Figure 3.3: A close-up of the maximum of $1/|\zeta(w)|$ in Figure 3.2(d).

Due to Proposition 3.2, the value of R in Theorem 2.18 is zero for $M(x)$.

We now know that $M(x)$ meets the requirements to use Theorem 2.18. Hence, we have the green light to give the following corollary of Theorem 2.18.

Corollary 3.3. *For x, N, B_1, B_2, B_3 and B_4 defined in Propositions 2.1, V restricted as it was in Proposition 2.14, C_1, C_2, W and $\tilde{v}$ described in Table 3.1, $1 < c < 1+\eta$ for $0 < \eta \leq 1/\log(1.5)$ and $b = 1 - 1/(W \log V)$, we have that*

$$
\begin{aligned}
M(x) \quad = \quad & O^*\left(\frac{C_1}{\pi}\exp\left(\log\log V + \log(x^c) - \log V - \log\log x\right)\right) \\
& + O^*\left(\frac{C_1}{\pi}\exp\left(\log(x^b) + 2\log\log V\right)\right) \\
& + O^*\left(\frac{\tilde{v}C_2}{\pi}\exp\left(\log(x^b) - \log|b|\right)\right) \\
& + O^*\left(B_1 \exp\left(\log(x^c) - \log V - \log(c-1)\right)\right) \\
& + O^*\left((B_2 + B_3)x\exp\left(\log\log x - \log V\right)\right) \\
& + O^*\left(B_4 x \exp\left(-\log V - \log|x - N|\right)\right). \qquad (3.8)
\end{aligned}
$$

We now want to remove the dependence our bound has on the parameters b, c and V. We stated our definition for b in (3.6). For the definitions of c and V, we allow Landau's work [16] to guide us and set

$$c := 1 + \frac{1}{\log x} \tag{3.9}$$

and

$$V := \exp\left(\sqrt{\frac{1}{W}\log x}\right). \tag{3.10}$$

With these definitions, we can now state a more explicit version of Corollary 3.3 as the following theorem, which is the main theoretical result of this section.

Theorem 3.4. *For N defined in Proposition 2.1, x defined in Proposition 2.1 that also satisfy*

$$x > \max\left\{\exp\left(\frac{1}{\eta}\right), \exp\left(\frac{4}{W}\right), \exp(1), \exp\left(W(\log\tilde{v})^2\right)\right\},$$

with respect to an $0 < \eta \leq 1/\log(1.5)$, *and* W *and* $\tilde{v}$ *as described in Table 3.1, we have that*

$$\begin{aligned} M(x) &= O^*\left(\frac{C_1 e}{\pi\sqrt{W\log x}} x\exp\left(-\sqrt{\frac{1}{W}\log x}\right)\right) \\ &\quad + O^*\left(\frac{C_1}{\pi W} x\exp\left(-\sqrt{\frac{1}{W}\log x} + \log\log x\right)\right) \\ &\quad + O^*\left(\frac{2\tilde{v}C_2}{\pi} x\exp\left(-\sqrt{\frac{1}{W}\log x}\right)\right) \\ &\quad + O^*\left((B_1 e + B_2 + B_3) x\exp\left(-\sqrt{\frac{1}{W}\log x} + \log\log x\right)\right) \\ &\quad + O^*\left(\frac{B_4}{|x-N|} x\exp\left(-\sqrt{\frac{1}{W}\log x}\right)\right) \end{aligned} \tag{3.11}$$

with C_1 *and* C_2 *described in Table 3.1. Furthermore, if we define* A_1 *and* A_2 *as in Proposi-*

tion 2.7 and 2.8 and have $1/2 < \beta < 1$, *we get*

$$B_1 \leq \frac{\left(1-\log(1.5)+\frac{1}{2}\eta\log^2(1.5)\right)\eta+1}{\pi\log 2} \left(2+\operatorname{Arctan}\left(\frac{1}{\exp\left(\sqrt{\frac{1}{W}\log x}\right)}+\frac{1}{(\log x)\exp\left(\sqrt{\frac{1}{W}\log x}\right)}\right)+\pi\right), \tag{3.12}$$

$$B_2 \leq \frac{8A_1}{\pi}\left(1+\pi/2\right), \tag{3.13}$$

$$B_3 = \frac{3(2^{\frac{1}{\log x}})A_2}{\pi(1-\beta)} \left(1+\operatorname{Arctan}\left(\frac{1}{\exp\left(\sqrt{\frac{1}{W}\log x}\right)}+\frac{1}{(\log x)\exp\left(\sqrt{\frac{1}{W}\log x}\right)}\right)+\pi/2\right), \tag{3.14}$$

and

$$B_4 = \frac{2^{1+\frac{1}{\log x}}}{\pi} \left(1+\operatorname{Arctan}\left(\frac{1}{\exp\left(\sqrt{\frac{1}{W}\log x}\right)}+\frac{1}{(\log x)\exp\left(\sqrt{\frac{1}{W}\log x}\right)}\right)+\pi/2\right). \tag{3.15}$$

Proof. Continuing from (3.8), if we replace c, b and V using definitions (3.6), (3.9) and (3.10), we get that

$$\begin{aligned} M(x) &= O^*\left(\frac{C_1}{\pi}x\exp\left(\log\left(\sqrt{\frac{1}{W}\log x}\right)+1-\sqrt{\frac{1}{W}\log x}-\log\log x\right)\right) \\ &+O^*\left(\frac{C_1}{\pi}x\exp\left(-\sqrt{\frac{1}{W}\log x}\right)+\log\left(\frac{1}{W}\log x\right)\right) \\ &+O^*\left(\frac{\tilde{\nu}C_2}{\pi}x\exp\left(-\sqrt{\frac{1}{W}\log x}-\log\left|\frac{\sqrt{W\log x}-1}{\sqrt{W\log x}}\right|\right)\right) \\ &+O^*\left(B_1x\exp\left(1-\sqrt{\frac{1}{W}\log x}+\log\log x\right)\right) \end{aligned}$$

$$+O^*\left((B_2+B_3)x\exp\left(\log\log x-\sqrt{\frac{1}{W}\log x}\right)\right)$$
$$+O^*\left(B_4 x\exp\left(-\sqrt{\frac{1}{W}\log x}-\log|x-N|\right)\right).$$

Now, if we suppose that $x>\exp(4/W)$, so that

$$\frac{\sqrt{W\log x}-1}{\sqrt{W\log x}}>\frac{1}{2}$$

and suppose that $x\geq e$, so that $\log\log x\geq 0$, the bound on $M(x)$ in the proposition is proved on rearranging the terms.

We should also note that in Corollary 3.3, $c<1+\eta$ for $\eta\leq 1/\log(1.5)$, and so $\log x\geq 1/\eta$. Furthermore, in Proposition 2.14, $V>\tilde{v}$, and thus, $\exp\left(\sqrt{\frac{1}{W}\log x}\right)>\tilde{v}$. These conditions, together with those we previously mentioned, necessitate having

$$x>\max\left\{\exp\left(\frac{1}{\eta}\right),\exp\left(\frac{4}{W}\right),\exp(1),\exp\left(W(\log\tilde{v})^2\right)\right\}.$$

Moreover, in B_1 B_2, B_3 and B_4, the variables c and V appear in the expressions

$$\left|\operatorname{Arctan}\frac{c}{V}-\frac{\pi}{2}\right|,\ \left(\operatorname{Arctan}\frac{c}{V}+\frac{\pi}{2}\right),\ 2^{c-1},\ \text{and } 2^c.$$

When we use the definitions of c and V, the last three expressions are equal to

$$\operatorname{Arctan}\left(\frac{1}{\exp\sqrt{\frac{1}{W}\log x}}+\frac{1}{(\log x)\exp\sqrt{\frac{1}{W}\log x}}\right)+\frac{\pi}{2},2^{\frac{1}{\log x}}\ \text{and}\ 2^{1+\frac{1}{\log x}}.$$

Furthermore, the assumptions $c>0$ and $V>0$, allow one to infer that

$$\left|\operatorname{Arctan}\frac{c}{V}-\frac{\pi}{2}\right|\leq\frac{\pi}{2}.$$

□

We now give two corollaries of Theorem 3.4 that we shall use to produce our explicit bounds. These two corollaries differ in the way they treat the $\log\log x$ factors in (3.11).

Corollary 3.5. *With all notation and conditions as in Theorem 3.4, for*

$$C(x) := \frac{C_1 e}{\pi\sqrt{W}(\log x)^{3/2}} + \frac{C_1}{\pi W} + \frac{2\tilde{v}C_2}{\pi \log x} + B_1 e + B_2 + B_3 + \frac{B_4}{|x-N|\log x},$$

we find that

$$M(x) = O^*\left(C(x)x(\log x)\exp\left(-\sqrt{\frac{1}{W}\log x}\right)\right).$$

Proof. Factor $x(\log x)\exp\left(-\sqrt{\frac{1}{W}\log x}\right)$ from (3.11) in Theorem 3.4. □

For convenience, from now on we denote the lower bound on x in Corollary 3.5 by

$$x_0(W) := \max\left\{\exp\left(\frac{1}{\eta}\right), \exp\left(\frac{4}{W}\right), \exp(1), \exp\left(W(\log\tilde{v})^2\right)\right\}. \tag{3.16}$$

Corollary 3.6. *Let* $0 < \varepsilon < \sqrt{\frac{1}{W}}$ *and let* x *as defined in Proposition 2.1 be such that*

$$x > \max\left\{\exp\left(\frac{1}{\eta}\right), \exp\left(\frac{4}{W}\right), \exp(1), \exp\left(W(\log\tilde{v})^2\right)\right\}$$

and

$$0 < \frac{\log\log x}{\sqrt{\log x}} < \varepsilon. \tag{3.17}$$

With all other notation and conditions as in Theorem 3.4 and

$$C(x) := \frac{C_1}{\pi W} + B_1 e + B_2 + B_3 + \left(\frac{C_1 e}{\pi\sqrt{W\log x}} + \frac{2\tilde{v}C_2}{\pi} + \frac{B_4}{|x-N|}\right)\exp(-\varepsilon\sqrt{\log x}),$$

we find that

$$M(x) = O^*\left(C(x)x\exp\left(\left(-\sqrt{\frac{1}{W}} + \varepsilon\right)\sqrt{\log x}\right)\right).$$

Proof. We use the hypothesis $\frac{\log\log x}{\sqrt{\log x}} < \varepsilon$ to infer that

$$\exp\left(-\sqrt{\frac{1}{W}\log x} + \log\log x\right) < \exp\left(\left(-\sqrt{\frac{1}{W}} + \varepsilon\right)\sqrt{\log x}\right),$$

which allows one to further bound (3.11) in Theorem 3.4. We then finish by factoring

$$x\exp\left(\left(-\sqrt{\frac{1}{W}} + \varepsilon\right)\sqrt{\log x}\right)$$

from the bound on $M(x)$. □

We denote the lower bound on x in Corollary 3.6 by

$$x_0(W,\varepsilon) := \max\left\{\exp\left(\frac{1}{\eta}\right), \exp\left(\frac{4}{W}\right), \exp(1), \exp\left(W(\log\tilde{v})^2\right), x_\varepsilon\right\}, \tag{3.18}$$

where

$$0 < \frac{\log\log x}{\sqrt{\log x}} < \varepsilon.$$

for $x > x_\varepsilon$. One ideally wants x_ε to be as small as possible.

We will soon use Corollaries 3.5 and 3.6 to get bounds on $M(x)$, but before doing so, we shall set β appropriately. It is possible for the parameters β to cause an increase in $x_0(W)$ or $x_0(W,\varepsilon)$. We can prevent this by making sure $1/(4\beta-2) \le x_0$ with $x_0 = x_0(W)$ for Corollary 3.5 and $x_0 = x_0(W,\varepsilon)$ for Corollary 3.6. Thus, we want to set

$$\beta := \frac{1}{2} + \frac{1}{4x_0}. \tag{3.19}$$

When we calculate our bounds on $M(x)$ we will have $|x-N| = 0.5$.[13] With this choice, the $C(x)$ in Corollary 3.5 decreases as x increases. Hence, for $x > x_0(W)$ we are able to bound $C(x)$ by $C_W := C(x_0(W))$. In the same way, for Corollary 3.6, we are able to bound $C(x)$ by

[13] Since $x-N$ appears in a denominator of one of the terms in the $C(x)$ of Corollary 3.5, it makes sense that we restrict it to the largest value it can take, which is 0.5.

$C_{W,\varepsilon} := C(x_0(W,\varepsilon))$.

We are now ready to get bounds on $M(x)$.

The explicit bounds on $M(x)$ that we shall get from Corollary 3.5 will have the form

$$M(x) = O^*\left(C_W(\log x)x\exp\left(-c_W\sqrt{\log x}\right)\right),$$

for $x > x_0(W)$, where $c_W = \sqrt{1/W}$, $x_0(W)$ was given in (3.16) and $C_W = C(x_0(W))$, with the $C(x)$ of Corollary 3.5. The bounds on $M(x)$ we shall get, using Corollary 3.5, will be calculated using W from 6 to 12 (and $|x-N| = 0.5$ as previously mentioned). We shall also use $\eta = \dfrac{1}{W(\log\tilde{v})^2}$.[14]

The Python code that we have written to generate our bounds on $M(x)$, according to Corollary 3.5, can be found in Appendix A.1. When run, this program will prompt for a decision as per what precision it should use to evaluate the bounds. It will also prompt for a decision as per what $x-N$ and W should be. It then does the necessary calculations and presents the bound it gets on the screen. The program works for all $0 < |x-N| \le 1/2$, but only $W = 6,\ 7,\ 8,\ 9,\ 10,\ 11$ and 12 can be used. As shown in Table 3.1, $\tilde{v}$, C_1 and C_2 are determined according to what W is. Appropriately, when running our code no decision has to be made by the user about $\tilde{v}$, C_1 or C_2. The appropriate choices for these parameters are written into the program. One bound on $M(x)$ is calculated each time the code is run. We present the bounds on $M(x)$ that we get using this code in Table 3.2. For Table 3.2, and each of the analogous tables presenting bounds that are to come later, we rounded the output of our program up when recording $\log x_0(W)$ and C_W. When recording c_W we rounded the output of our program down.

[14] Since η appears in the numerator of B_1, but x has to be bigger than both $\exp(1/\eta)$ and $\exp\left(W(\log\tilde{v})^2\right)$, choosing $\eta = \dfrac{1}{W(\log\tilde{v})^2}$ will decrease the bound the most, while not forcing x to be larger than $\exp\left(W(\log\tilde{v})^2\right)$. For this η, $\exp(1/\eta) = \exp\left(W(\log\tilde{v})^2\right)$.

Table 3.2: The bounds we get via Corollary 3.5 and the code in Appendix A.1.

W	$\log x_0(W)$	C_W	c_W
12	286.25	7.4×10^1	0.288
11	244.09	1.6×10^2	0.301
10	203.35	5.1×10^2	0.316
9	163.09	3.5×10^3	0.333
8	122.79	1.3×10^5	0.353
7	87.05	6.0×10^8	0.377
6	74.62	1.8×10^{29}	0.408

The explicit bounds on $M(x)$ that we shall get from Corollary 3.6 will have the form

$$M(x) = O^*\left(C_{W,\varepsilon} x \exp\left(-c_{W,\varepsilon}\sqrt{\log x}\right)\right),$$

for $x > x_0(W,\varepsilon)$, where $c_{W,\varepsilon} = \sqrt{1/W} - \varepsilon$, $x_0(W,\varepsilon)$ can be found back in (3.18). Furthermore, $C_{W,\varepsilon} = C(x_0(W,\varepsilon))$, with the $C(x)$ from Corollary 3.6. The `Python` code for these bounds is given in Appendix A.2. This code operates in much the same way as the code we used to implement Corollary 3.5. However, when run, it will prompt for a decision on what ε and x_ε should be, in addition to the other information requested by our Corollary 3.5 program.

We shall get one bound on $M(x)$ for each of $W = 6,\ 7,\ 8,\ 9,\ 10,\ 11$ and 12. We write ε_W for the value of ε that we shall use with a particular W when calculating a bound. We know that we can use any $0 < \varepsilon_W < \sqrt{1/W}$. However, we have opted to use a strategic approach to making the decision on how to set each ε_W. We begin by selecting ε_{12}. We shall have $\varepsilon_{12} = 0.28$, but one does not guarantee that this ε_{12} cannot be improved upon. However, since ε_{12} is very close to $\sqrt{1/12} \approx 0.289$, the bound we get will be valid for a larger set of x than it would be if we had $\varepsilon_{12} < 0.28$. We get the first bound by feeding $W = 12$ and $\varepsilon_{12} = 0.28$ into our program. We then want to find a suitable value of ε_{11}. To do so, we select an $\varepsilon_{11} < 0.28$. At this stage, it does not matter exactly what value less than 0.28 we assign to ε_{11}. With this ε_{11} and $W = 11$ we then get a bound

$$M(x) \leq C_{11,\varepsilon_{11}} \exp(-c_{11,\varepsilon_{11}} \sqrt{\log x}). \tag{3.20}$$

This bound comes with a condition on x, say $x > x_0(11, \varepsilon_{11})$. We check to see how close this $x_0(11, \varepsilon_{11})$ is to the point where (3.20) improves on the $W = 12$ bound. We are interested in the smallest $x \geq x_0(12, 0.28)$ that satisfies

$$C_{11,\varepsilon_{11}} \exp\left(-c_{11,\varepsilon_{11}} \sqrt{\log x}\right) \leq C_{12,0.28} \exp\left(-c_{12,0.28} \sqrt{\log x}\right).$$

Namely, we are intersested in the point

$$\tilde{x}_0(11, \varepsilon_{11}) = \exp\left(\left(\frac{1}{-c_{12,0.28} + c_{11,\varepsilon_{11}}} \log\left(\frac{C_{11,\varepsilon_{11}}}{C_{12,0.28}}\right)\right)^2\right). \tag{3.21}$$

We want $\tilde{x}_0(11, \varepsilon_{11})$ to be just a little bit smaller than $x_0(11, \varepsilon_{11})$. If this is not the case, we alter ε_{11} and repeat the process to adjust $\tilde{x}_0(11, \varepsilon_{11})$ and $x_0(11, \varepsilon_{11})$. We continue in this way until we do get the kind of $\tilde{x}_0(11, \varepsilon_{11})$ we desire. After settling on an ε_{11}, we do the same for $W = 10$. However, this time we have the task of checking $x_0(10, \varepsilon_{10})$ to the smallest point $\tilde{x}_0(10, \varepsilon_{10}) > x_0(11, \varepsilon_{11})$ where the $W = 10$ bound improves over the $W = 11$ bound. We follow the same process for $W = 9,\ 8,\ 7$ and 6 (in this order). The bounds we get are given in Table 3.3. Altogether we will have seven bounds. For Table 3.3 and each of the analogous tables presenting bounds that are to come later, we rounded the output of our program up when recording $\log x_0(W, \varepsilon)$ and $C_{W,\varepsilon}$. When recording $c_{W,\varepsilon}$ we rounded the output of our program down.

Table 3.3: The bounds we get via Corollary 3.6 and the code in Appendix A.2.

W	ε	$\log x_0(W,\varepsilon)$	$C_{W,\varepsilon}$	$c_{W,\varepsilon}$
12	0.28	489.15	7.3×10^{1}	0.008
11	0.26	607.78	1.5×10^{2}	0.041
10	0.23	864.36	5.0×10^{2}	0.086
9	0.19	1474.63	3.5×10^{3}	0.143
8	0.14	3364.98	1.3×10^{5}	0.213
7	0.08	14305.32	6.0×10^{8}	0.297
6	0.04	79589.39	1.7×10^{29}	0.368

3.1.2 The approach from Arkhipova's theorem

The work that is required to get bounds on $M(x)$ from Theorem 2.19 is almost no different to what we did for Theorem 2.18. Furthermore, we do not need to bother with checking any of the hypotheses of Theorem 2.19 because it was all done in the last section. Thus, we can begin by stating the relevant corollary of Theorem 2.18.

Corollary 3.7. *For x, B_5, B_6 and B_7 defined in Propositions 2.9, $2c < V < x$ defined in Proposition 2.14, C_1, C_2, W and $\tilde{v}$ described in Table 3.1, $1 < c < 1+\eta$ for $0 < \eta \le 1/\log(1.5)$ and $b = 1 - 1/(W\log V)$, we have*

$$\begin{aligned} M(x) &= O^*\left(\frac{C_1}{\pi}x\exp\left(\log\log V + \log(x^{c-1}) - \log V - \log\log x\right)\right) \\ &\quad + O^*\left(\frac{C_1}{\pi}x\exp\left(\log(x^{b-1}) + 2\log\log V\right)\right) \\ &\quad + O^*\left(\frac{\tilde{v}C_2}{\pi}x\exp\left(\log(x^{b-1}) - \log|b|\right)\right) \\ &\quad + O^*\left(B_5 x\exp\left(\log(x^{c-1}) - \log V - \log(c-1)\right)\right) \\ &\quad + O^*\left((B_6+B_7)x\exp\left(\log\log V - \log V\right)\right). \end{aligned} \tag{3.22}$$

We keep the same definitions for c, b and V as in Section 3.1.1. Hence, we have the following, which is the main theoretical result of this section.

Theorem 3.8. *For x defined in Proposition 2.9 that also satisfy*

$$x > \max\left\{\exp\left(\frac{1}{\eta}\right), \exp\left(\frac{4}{W}\right), \exp(1), \exp\left(W(\log\tilde{v})^2\right), \exp\left(W(\log 4)^2\right)\right\},$$

with respect to an $0 < \eta \leq 1/\log(1.5)$, W *and* $\tilde{v}$ *described in Table 3.1, we have*

$$\begin{aligned} M(x) &= O^*\left(\frac{C_1 e}{\pi\sqrt{W\log x}} x \exp\left(-\sqrt{\frac{1}{W}\log x}\right)\right) \\ &\quad + O^*\left(\frac{C_1}{\pi W} x \exp\left(-\sqrt{\frac{1}{W}\log x + \log\log x}\right)\right) \\ &\quad + O^*\left(\frac{2\tilde{v}C_2}{\pi} x \exp\left(-\sqrt{\frac{1}{W}\log x}\right)\right) \\ &\quad + O^*\left(eB_5 x \exp\left(-\sqrt{\frac{1}{W}\log x + \log\log x}\right)\right) \\ &\quad + O^*\left(\frac{(B_6+B_7)}{\sqrt{W}} x \exp\left(-\sqrt{\frac{1}{W}\log x + \frac{1}{2}\log\log x}\right)\right), \end{aligned} \tag{3.23}$$

where C_1 *and* C_2 *described in Table 3.1 and*

$$B_5 = \frac{2\left(1 - \log(1.5) + \frac{1}{2}\eta\log^2(1.5)\right)\eta + 1}{\pi\log 2} \tag{3.24}$$

$$B_6 \leq \frac{2^{1+\frac{1}{\log x}}\left(\pi\log\left(2 + \frac{2}{\log x}\right) + \log\frac{\pi}{2} + 1\right)}{\pi\log 2} \tag{3.25}$$

$$B_7 \leq \frac{4 + \pi + 4\log\frac{\pi}{4} + 4\log\left(2 + \frac{2}{\log x}\right)}{2\pi\log 2}. \tag{3.26}$$

Proof. We follow the same lines of reasoning as in the proof of Theorem 3.4. The biggest difference is that we now must have $x > \exp\left(W(\log 4)^2\right)$ because in Proposition 2.9 we needed $V > 2c$. This translates to

$$\exp\left(\sqrt{\frac{1}{W}\log x}\right) > 2 + \frac{2}{\exp(\log\log x)}. \tag{3.27}$$

which holds if $\exp\left(\sqrt{\frac{1}{W}\log x}\right) > 4$. □

Next, we provide two corollaries of Theorem 3.8 that we will use to get explicit bounds on $M(x)$.

Corollary 3.9. *With all notation and conditions as in Theorem 3.8 and with*

$$C(x) := \frac{C_1 e}{\pi\sqrt{W}(\log x)^{3/2}} + \frac{C_1}{\pi W} + \frac{2\tilde{v}C_2}{\pi \log x} + B_5 e + \frac{B_6 + B_7}{\sqrt{W \log x}}$$

we find that

$$M(x) = O^*\left(C(x)x(\log x)\exp\left(-\sqrt{\frac{1}{W}\log x}\right)\right).$$

Corollary 3.10. *Let* $0 < \varepsilon < \sqrt{1/W}$*, and let* x *defined in Proposition 2.9 be such that*

$$x > \max\left\{\exp\left(\frac{1}{\eta}\right), \exp\left(\frac{4}{W}\right), \exp(1), \exp\left(W(\log \tilde{v})^2\right), \exp\left(W(\log 4)^2\right)\right\}$$

and

$$0 < \frac{\log\log x}{\sqrt{\log x}} < \varepsilon.$$

With all notation and conditions as in Theorem 3.8 and

$$C(x) := \frac{C_1}{\pi W} + B_5 e + \left(\frac{C_1 e}{\pi\sqrt{W\log x}} + \frac{2\tilde{v}C_2}{\pi}\right)\exp(-\varepsilon\sqrt{\log x}) + \frac{B_6 + B_7}{\sqrt{W}}\exp\left(-\frac{\varepsilon}{2}\sqrt{\log x}\right),$$

we find that

$$M(x) = O^*\left(C(x)x\exp\left(\left(-\sqrt{\frac{1}{W}} + \varepsilon\right)\sqrt{\log x}\right)\right).$$

Corollaries 3.9 and 3.10 can be proved by following the reasoning we used in the proof of Corollaries 3.5 and 3.6 respectively.

We now have everything we need to get bounds for $M(x)$. The only parameters we set differently here than in Section 3.1.1 are ε and x_ε. We use the same approach for determining ε that we used in Section 3.1.1. We also re-use the notation C_W, c_W, $x_0(W)$, $C_{W,\varepsilon}$, $c_{W,\varepsilon}$ and

$x_0(W,\varepsilon)$ from Section 3.1.1. However, one should keep in mind that (when appropriate) we define these notations differently in this section, compared to when we used them in Section 3.1.1.

The code in which we have implemented Corollary 3.9 is given in Appendix A.3 and the bounds that we get from this code are in Table 3.4. The code implementing Corollary 3.10 is in Appendix A.4 and the corresponding bounds are presented in Table 3.5.

Table 3.4: The bounds we get via Corollary 3.9 and the code in Appendix A.3.

W	$\log x_0(W)$	C_W	c_W
12	286.25	5.6×10^1	0.288
11	244.09	1.4×10^2	0.301
10	203.35	4.9×10^2	0.316
9	163.09	3.5×10^3	0.333
8	122.79	1.3×10^5	0.353
7	87.05	6.0×10^8	0.377
6	74.62	1.8×10^{29}	0.408

Table 3.5: The bounds we get via Corollary 3.10 and the code in Appendix A.4.

W	ε	$\log x_0(W,\varepsilon)$	$C_{W,\varepsilon}$	$c_{W,\varepsilon}$
12	0.28	489.15	5.5×10^1	0.008
11	0.25	680.87	1.4×10^2	0.051
10	0.22	980.19	4.9×10^2	0.096
9	0.18	1710.54	3.4×10^3	0.153
8	0.13	4093.11	1.3×10^5	0.223
7	0.08	14305.32	6.0×10^8	0.297
6	0.03	418968.18	1.7×10^{29}	0.388

3.1.3 Bounds that work for any value of x

In Section 3.1.1 the bounds we got on $M(x)$ were applicable only to $x = \lfloor x \rfloor + 1/2$. In Section 3.1.2 the bounds we got hold only for non-integer values of x. We shall now transform our bounds into ones that we know are valid for all $x \in (x_0, \infty)$ where $x_0 = x_0(W)$, if we are considering Corollaries 3.5 or 3.9, and $x_0 = x_0(W,\varepsilon)$, if we are considering Corollaries 3.6 or 3.10. We do so using the following propositions. These propositions are similar to Corollary 4 of [7]. The proofs we provide are like the one given in [7].

Firstly, Proposition 3.11 transforms the bounds in Table 3.2 into bounds on $M(x)$ for any $x > x_0(W)$.

Proposition 3.11. *If for $x = \lfloor x \rfloor + 1/2 > x_0(W)$, we have*

$$M(x) = O^*\left(Cx(\log x)\exp\left(-c\sqrt{\log x}\right)\right),$$

then for $\lfloor x \rfloor - \nu > \tilde{x}_0 = \max\left\{x_0(W) - (1/2+\nu), \exp\left(c^2\right)\right\}$ where $0 < \nu \leq 1$, we have

$$\begin{aligned} M(\lfloor x \rfloor - \nu) \quad = \quad & O^*\left(\left(\left(C\left(1+\frac{3}{2\tilde{x}_0}\right)\left(1+\frac{\log\left(1+\frac{3}{2\tilde{x}_0}\right)}{\log \tilde{x}_0}\right)\right.\right.\right. \\ & \left.\left. +\frac{\exp\left(c\sqrt{\log \tilde{x}_0}\right)}{\tilde{x}_0(\log \tilde{x}_0)}\right)(\lfloor x \rfloor - \nu)(\log(\lfloor x \rfloor - \nu))\exp\left(-c\sqrt{\log(\lfloor x \rfloor - \nu)}\right)\right). \end{aligned}$$

Proof. Firstly, since

$$|M(\lfloor x \rfloor - \nu)| = |M(\lfloor x \rfloor + 1/2) - \mu(\lfloor x \rfloor)|,$$

the triangle inequality allows one to infer that

$$|M(\lfloor x \rfloor - \nu)| \leq |M(\lfloor x \rfloor + 1/2)| + 1.$$

From this, we get that

$$M(\lfloor x \rfloor - \nu) = O^*\left(C\left(\lfloor x \rfloor + \frac{1}{2}\right)\log\left(\lfloor x \rfloor + \frac{1}{2}\right)\exp\left(-c\sqrt{\log(\lfloor x \rfloor + 1/2)}\right)\right) + 1. \quad (3.28)$$

Moreover,

$$C\left(\lfloor x\rfloor+\frac{1}{2}\right)\log\left(\lfloor x\rfloor+\frac{1}{2}\right)\exp\left(-c\sqrt{\log\left(\lfloor x\rfloor+1/2\right)}\right)+1$$
$$\leq C\left(\lfloor x\rfloor-\nu+\nu+\frac{1}{2}\right)\log\left(\lfloor x\rfloor-\nu+\nu+\frac{1}{2}\right)\exp\left(-c\sqrt{\log\left(\lfloor x\rfloor-\nu\right)}\right)+1. \qquad (3.29)$$

Thus, on using the fact that $0<\nu\leq 1$ as well as (3.28) and (3.29) together, we get that

$$M(\lfloor x\rfloor-\nu)=O^{*}\left(\left(C\left(1+\frac{3}{2(\lfloor x\rfloor-\nu)}\right)\left(1+\frac{\log\left(1+\frac{3}{2(\lfloor x\rfloor-\nu)}\right)}{\log(\lfloor x\rfloor-\nu)}\right)\right.\right.$$
$$\left.\left.+\frac{\exp(c\sqrt{\log(\lfloor x\rfloor-\nu)})}{(\lfloor x\rfloor-\nu)\log(\lfloor x\rfloor-\nu)}\right)(\lfloor x\rfloor-\nu)(\log(\lfloor x\rfloor-\nu))\exp\left(-c\sqrt{\log(\lfloor x\rfloor-\nu)}\right)\right).$$

We now bound all, but the $(\lfloor x\rfloor-\nu)(\log(\lfloor x\rfloor-\nu))\exp\left(-c\sqrt{\log(\lfloor x\rfloor-\nu)}\right)$ factor. For this to be possible, we have to make sure $\frac{\exp(c\sqrt{\log(\lfloor x\rfloor-\nu)})}{(\lfloor x\rfloor-\nu)\log(\lfloor x\rfloor-\nu)}$ never increases as x increases. Having $\lfloor x\rfloor-\nu>\exp(c^2)$ does the job. □

From Proposition 3.11, we get Table 3.6.

Table 3.6: The bounds we get via Proposition 3.11.

W	$\log x_0(W)$	C_W	c_W
12	286.25	7.5×10^{1}	0.288
11	244.09	1.7×10^{2}	0.301
10	203.35	5.2×10^{2}	0.316
9	163.09	3.6×10^{3}	0.333
8	122.79	1.4×10^{5}	0.353
7	87.05	6.1×10^{8}	0.377
6	74.62	1.9×10^{29}	0.408

Secondly, Proposition 3.12 will transform the bounds in Table 3.3 into bounds on $M(x)$ for any $x>x_0(W,\varepsilon)$.

Proposition 3.12. *If for* $x = \lfloor x \rfloor + 1/2 > x_0(W,\varepsilon)$, *we have*

$$M(x) = O^*\left(Cx\exp\left(-c\sqrt{\log x}\right)\right)$$

then, for $\lfloor x \rfloor - \nu > \tilde{x}_0 = \max\left\{x_0(W,\varepsilon) - (1/2+\nu), \exp\left(c^2\right)\right\}$ *where* $0 < \nu \leq 1$, *we have*

$$M(\lfloor x \rfloor - \nu) = O^*\left(\left[C\left(1+\frac{3}{2\tilde{x}_0}\right) + \frac{\exp\left(c\sqrt{\log \tilde{x}_0}\right)}{\tilde{x}_0}\right](\lfloor x \rfloor - \nu)\exp\left(-c\sqrt{\log(\lfloor x \rfloor - \nu)}\right)\right).$$

The proof of Proposition 3.12 is similar to the proof of Proposition 3.11.

From Proposition 3.12 we get Table 3.7.

Table 3.7: The bounds we get via Proposition 3.12.

W	ε	$\log x_0(W,\varepsilon)$	$C_{W,\varepsilon}$	$c_{W,\varepsilon}$
12	0.28	489.15	7.4×10^1	0.008
11	0.26	607.78	1.6×10^2	0.034
10	0.23	864.36	5.1×10^2	0.086
9	0.19	1474.63	3.6×10^3	0.143
8	0.14	3364.98	1.4×10^5	0.213
7	0.08	14305.32	6.1×10^8	0.297
6	0.04	79589.39	1.8×10^{29}	0.368

Thirdly, we prove Proposition 3.13, which we can apply to Table 3.4 to produce bounds that are valid for all integer values of $x \in (x_0(W), \infty)$.

Proposition 3.13. *If for* $x = \lfloor x \rfloor + \nu > x_0(W)$ *where* $0 < \nu < 1$, *we have*

$$M(x) = O^*\left(Cx(\log x)\exp(-c\sqrt{\log x})\right),$$

then we have

$$M(\lfloor x\rfloor) = O^*\left(C\left(1+\log\left(1+\frac{1}{x_0(W)-1}\right)\frac{1}{\log(x_0(W)-1)}\right)\right.$$
$$\left.\left(1+\frac{1}{x_0(W)-1}\right)\lfloor x\rfloor(\log\lfloor x\rfloor)\exp(-c\sqrt{\log\lfloor x\rfloor})\right).$$

Proof. Firstly, the triangle inequality allows one to infer that

$$|M(\lfloor x\rfloor)| \leq |M(\lfloor x\rfloor) - M(\lfloor x\rfloor+\nu)| + |M(\lfloor x\rfloor+\nu)|.$$

From this, we get that

$$M(\lfloor x\rfloor) = O^*\left(C(\lfloor x\rfloor+\nu)(\log(\lfloor x\rfloor+\nu))\exp\left(-c\sqrt{\log(\lfloor x\rfloor+\nu)}\right)\right). \tag{3.30}$$

Thus, on using the fact that $\nu < 1$ along with (3.30), we get that

$$M(\lfloor x\rfloor) = O^*\left(C\left(1+\frac{1}{\lfloor x\rfloor}\right)\left(1+\log\left(1+\frac{1}{\lfloor x\rfloor}\right)\frac{1}{\log\lfloor x\rfloor}\right)\right.$$
$$\left.\lfloor x\rfloor(\log\lfloor x\rfloor)\exp\left(-c\sqrt{\log\lfloor x\rfloor}\right)\right).$$

Furthermore, $\lfloor x\rfloor > x_0(W)-1$ since $\nu < 1$, thus,

$$M(\lfloor x\rfloor) = O^*\left(C\left(1+\frac{1}{x_0(W)-1}\right)\left(1+\log\left(1+\frac{1}{x_0(W)-1}\right)\frac{1}{\log(x_0(W)-1)}\right)\right.$$
$$\left.\lfloor x\rfloor(\log\lfloor x\rfloor)\exp\left(-c\sqrt{\log\lfloor x\rfloor}\right)\right).$$

□

With Proposition 3.13 we can now put together Table 3.8 that complements Table 3.4. The bounds on $M(x)$ given in Table 3.8 are valid for all integer values of $x \in (x_0(W), \infty)$.

Table 3.8: The bounds we get via Proposition 3.13.

W	$\log x_0(W)$	C_W	c_W
12	286.25	5.7×10^1	0.288
11	244.09	1.5×10^2	0.301
10	203.35	5.0×10^2	0.316
9	163.09	3.6×10^3	0.333
8	122.79	1.4×10^5	0.353
7	87.05	6.1×10^8	0.377
6	74.62	1.9×10^{29}	0.408

Lastly, we give Proposition 3.14, which we can apply to Table 3.5. To prove this proposition, one just needs to adapt what we did to prove Proposition 3.13.

Proposition 3.14. *If for* $x=\lfloor x\rfloor+\nu>x_0(W,\varepsilon)$ *where* $0<\nu<1$, *and*

$$M(x)=O^*\left(Cx\exp(-c\sqrt{\log x})\right),$$

then we have

$$M(\lfloor x\rfloor)=O^*\left(C\left(1+\frac{1}{x_0(W,\varepsilon)-1}\right)\lfloor x\rfloor\exp(-c\sqrt{\log\lfloor x\rfloor})\right).$$

With Proposition 3.14 we can now put together Table 3.9 that complements Table 3.5.

Table 3.9: The bounds we get via Proposition 3.14.

W	ε	$\log x_0(W,\varepsilon)$	$C_{W,\varepsilon}$	$c_{W,\varepsilon}$
12	0.28	489.15	5.6×10^1	0.008
11	0.25	680.87	1.5×10^2	0.051
10	0.22	980.19	5.0×10^2	0.096
9	0.18	1710.54	3.5×10^3	0.153
8	0.13	4093.11	1.4×10^5	0.223
7	0.08	14305.32	6.1×10^8	0.297
6	0.03	418968.18	1.8×10^{29}	0.388

At the moment we have Table 3.4 and Table 3.5 for non-integers and Table 3.8 and Table 3.9 for integers. However, the bounds given in the latter two tables are upper bounds for the bounds given in the former two tables. Hence, we have no further need for Tables 3.4 and 3.5. Tables 3.8 an 3.9 can be used for whatever $x \in (x_0, \infty)$ we choose.

We now have quite a few bounds, do we really need this many? This must be considered for Tables 3.6 and 3.8 separately from Tables 3.7 and 3.9.

The values of $x_0(W)$ and c_W in Tables 3.6 and 3.8 are identical. Hence, we can easily pair up the bounds in these two tables and look at the values of C_W to consider whether the bound in Table 3.6 or the bound in Table 3.8 is sharper for each $x_0(W)$. On doing so, one discovers that in all cases the results in Table 3.8 are at least as good as those in Table 3.6. In the cases where the values of C_W are not the same in the two tables, Tables 3.8 contains the smaller value of C_W.

Tables 3.7 and 3.9 record different values for $x_0(W)$, $C_{W,\varepsilon}$ and $c_{W,\varepsilon}$ for some of the values of W. However, the first bound in Tables 3.9 is clearly sharper than the first bound in Table 3.7. If we begin with this first bound in Table 3.7, then consider each of the other bounds in Tables 3.7 and 3.9 in turn to determine the sharpest bound for each x, we find that the sharper bounds are the ones in Table 3.9.

3.2 Analyzing the bounds

Our task in the previous sections was to get explicit bounds on $M(x)$. We successfully did this, creating Tables 3.8 and 3.9. However, the bounds in these tables should certainly not be taken as the be all and end all. They represent just a sample of what can be produced using Corollaries 3.9 and 3.10 together with Proposition 3.13 and 3.14. For example, if we were interested in considering alternative values for ε or if C_1 was updated in the literature, we could return to these corollaries and re-do the bounds. This warrants spending time considering what influence the parameters W and ε have on the bounds and what the reasons are for the size of C_W and x_0. We shall consider what influence W and ε have in Sections 3.2.1. The reasons for the size of C_W and x_0 will be considered in Section 3.2.2.

With respect to the tables of bounds on $M(x)$ given in the previous section, Tables 3.8

and 3.9 were the sharpest. To get these two tables we applied Propositions 3.13 and 3.14 to the two tables that we got using Corollaries 3.9 and 3.10. Hence, for the analysis in the following sections, we have opted to focus our attention only on Corollaries 3.9 and 3.10.

Before we consider Corollaries 3.9 and 3.10 in the way we have mentioned, we would like to know at which $x \in (x_0(W,\varepsilon),\infty)$ our bounds in Table 3.9 will be sharper than

$$M(x) = O^*\left(\frac{0.013x\log x - 0.118x}{(\log x)^2}\right) \tag{3.31}$$

from [24]. It turns out that this will be the case when $x \in (e^{13267.27},\infty)$.

3.2.1 Analyzing the influence of W and ε

We have the ability to alter the bounds we get on $M(x)$ using Corollaries 3.9 and 3.10, by adjusting the parameters W and ε. Thus, it is important to know how these parameters affect the bounds.

In Corollary 3.9, C_W, c_W and $x_0(W)$ are the variables that determine the bounds

$$M(x) = O^*\left(C_W x(\log x)\exp\left(-c_W\sqrt{\log x}\right)\right)$$

over $x \in (x_0(W),\infty)$. Hence, we need to analyze the influence of W on these variables. We begin our analysis by giving a table of examples. Table 3.10 illustrates the influence W has on the bounds we get by employing Corollary 3.9 for a selection of values of W.

Table 3.10: The influence of W on the bounds in Corollary 3.9.

W	$\log x_0(W)$	C_W	c_W
10	203.35	4.9×10^2	0.316
9	163.09	3.5×10^3	0.333
8	122.79	1.3×10^5	0.353

The patterns we have observed when we decrease W in Table 3.10 are summarized in Table 3.11. In Table 3.11 if the parameter W influences one of the variables c_W, $x_0(W)$

or C_W, we record the nature of that influence (that is, whether the variable increases or decreases) in the cell corresponding to the variable. For example, the table shows that a decrease in W causes an increase in c_W. Moreover, we use *italics* in the table to indicate that a decrease in W influences the relevant variable in a way that improves the bound. For example, an increase in c_W improves the bound, so the word *Increases* next to c_W in Table 3.11 is in italics.

Table 3.11: The influence of a decrease in W on C_W, c_W and $x_0(W)$ in Corollary 3.9. (We use italics as an indication that decreasing W improves the bound via its influence on the relevant variable.)

Variable	W
c_W	*Increases*
C_W	Increases
$x_0(W)$	*Decreases*

At the moment, the patterns we summarize in Table 3.11 are only supported by the examples we give in Table 3.10. However, the patterns one observes in these examples are visible in the formulas given in Corollary 3.9.

Firstly, in Corollary 3.9 we have $c_W = \sqrt{1/W}$. Clearly, our conclusion about c_W in Table 3.11 is right; c_W increases if W decreases.

Secondly, the terms in $C_W = C(x_0(W))$ that involve W are

$$\frac{C_1}{\pi W}, \frac{C_1 e}{\pi\sqrt{W}(\log x)^{3/2}} \text{ and } \frac{B_6+B_7}{\sqrt{W\log x}}. \tag{3.32}$$

Moreover, there is also an increase in C_1 as W decreases. Hence, the expressions in (3.32) definitely increase whenever W decreases. This confirms what Table 3.11 shows.

Likewise, Table 3.11 also agrees with the formulas in the case of $x_0(W)$. Although $x_0(W)$ has a somewhat more complicated formula, taking into account the $x_0(W)$ for the bounds we calculated, we can roughly consider $x_0(W) = \exp\left(W(\log\tilde{v})^2\right)$. Whence, a decrease in W corresponds to a decrease in $\tilde{v}$, and consequently, we get a decrease in $x_0(W)$.

We next explore the influence of W and ε on the bounds given in Corollary 3.10. In Corollary 3.10, $C_{W,\varepsilon}$, $c_{W,\varepsilon}$ and $x_0(W,\varepsilon)$ are the variables that determine the bounds

$$M(x) = O^*\left(C_{W,\varepsilon}x\exp\left(-c_{W,\varepsilon}\sqrt{\log x}\right)\right)$$

over $x \in (x_0(W,\varepsilon),\infty)$. Thus, it is the effect of W and ε on these variables that we need to analyze. This time we consider two examples. Table 3.12 shows the changes to the variables $C_{W,\varepsilon}$, $c_{W,\varepsilon}$ and $x_0(W,\varepsilon)$ coresponding to particular changes in the value of W (and $\varepsilon = 0.2$). Table 3.13 does the same with respect to a change in the value of ε (and $W = 6$).

Table 3.12: The effect of W on the bounds in Corollary 3.10 for $\varepsilon = 0.2$.

W	$\log x_0(W,\varepsilon)$	$C_{W,\varepsilon}$	$c_{W,\varepsilon}$
10	1279.61	4.9×10^{2}	0.11
9	1279.61	3.5×10^{3}	0.13
8	1279.61	1.3×10^{5}	0.15

Table 3.13: The effect of ε on the bounds in Corollary 3.10 for $W = 6$.

ε	$\log x_0(W,\varepsilon)$	$C_{W,\varepsilon}$	$c_{W,\varepsilon}$
0.3	398.30	1.7×10^{29}	0.10
0.2	1279.61	1.7×10^{29}	0.20
0.1	8099.12	1.7×10^{29}	0.30

A summary of the patterns that we can see in Tables 3.12 and 3.13 are summarized in Table 3.14.

Table 3.14: The influence of a decrease in W or ε on C_W, $c_{W,\varepsilon}$ and $x_0(W,\varepsilon)$ in Corollary 3.10.

	Parameter	
Variable	W	ε
$c_{W,\varepsilon}$	*Increases*	*Increases*
$x_0(W,\varepsilon)$	Not Applicable	Increases
$C_{W,\varepsilon}$	Increases	Not Applicable

Column one shows the patterns visible in Table 3.12, while column two shows the patterns visible in Table 3.13. When we record Not Applicable in a cell, it means the value of the corresponding variable did not change when we decreased the parameter (either ε or W) in question.

Now we refer to the formulas given in Corollary 3.10 to see whether our summary in Table 3.14 makes sense. Firstly, we shall consider $x_0(W,\varepsilon) = x_\varepsilon$, where $x_\varepsilon > 0$ is such that for $x > x_\varepsilon$ we have

$$0 < \frac{\log\log x}{\sqrt{\log x}} < \varepsilon, \tag{3.33}$$

since this was the case when we calculated our bounds. Hence, W will not directly influence $x_0(W,\varepsilon)$ (confirming our Not Applicable record in Table 3.14), but if ε decreases, $x_0(W,\varepsilon)$ will increase, as we observed in our examples and noted in Table 3.14. Secondly, $c_{w,\varepsilon} = \sqrt{1/W} - \varepsilon$, which confirms the observation we recorded in Table 3.14. A decrease in W will increase $c_{W,\varepsilon}$ and a decrease in ε will increase $c_{W,\varepsilon.}$ Lastly, W shows up in the formulas for $C_{W,\varepsilon}$ in the terms

$$\frac{C_1}{\pi W}, \; \frac{C_1 e}{\pi\sqrt{W\log x}}\exp\left(-\varepsilon\sqrt{\log x}\right) \text{ and } \frac{B_6+B_7}{\sqrt{W}}\exp\left(-\frac{\varepsilon}{2}\sqrt{\log x}\right).$$

These terms will all increase if we were to decrease W. Hence, our record in Table 3.14 is accurate. Furthermore, in the formula for $C_{W,\varepsilon}$ given in Corollary 3.10 there is a factor of $\exp(-\varepsilon\sqrt{\log x})$ and a factor of $\exp\left(-\frac{\varepsilon}{2}\sqrt{\log x}\right)$. This hints that decreasing ε should increases $C_{W,\varepsilon}$. However, decreasing ε will also increase $x_0(W,\varepsilon)$. An increase in $x_0(W,\varepsilon)$ should help to decrease $C_{W,\varepsilon}$ through the terms

$$\frac{C_1 e}{\pi\sqrt{W\log x}}\exp(-\varepsilon\sqrt{\log x}) \text{ and } \frac{B_6+B_7}{\sqrt{W}}\exp\left(-\frac{\varepsilon}{2}\sqrt{\log x}\right).$$

Hence, there is a pull and tug going on as to whether $C_{W,\varepsilon}$ should increase or decrease when ε decreases. It could be this pull and tug or simply the fact that the ε does not influence

$C_{W,\varepsilon}$ in a strong way for our choice of parameters, but the influence of ε on $C_{W,\varepsilon}$ is not clear. Hence, the record of Not applicable in Table 3.14 is suitable.

3.2.2 Analyzing the size of C_W, $x_0(W)$, $C_{W,\varepsilon}$ and $x_0(W,\varepsilon)$

Our bounds generally involve large values for C_W, $x_0(W)$, $C_{W,\varepsilon}$ and $x_0(W,\varepsilon)$. Hence, one might naturally ask: why is this? In the present section, we shall address this question.

Firstly, we consider the $C_W = C(x_0(W))$ we get from Corollary 3.9. To narrow down our search for why this variable is generally large, we evaluate the terms of C_W separately for each different value of W, with some of the terms grouped together. Namely, we shall evaluate

$$C_W^1 := \frac{C_1 e}{\pi\sqrt{W}(\log x_0(W))^{3/2}},$$

$$C_W^2 := \frac{C_1}{\pi W} \tag{3.34}$$

and

$$C_W^3 := \frac{2\tilde{\nu}C_2}{\pi \log x_0(W)} + B_5 e + \frac{B_6 + B_7}{\sqrt{W \log x_0(W)}}.$$

Table 3.15 gives the values for C_W^1, C_W^2 and C_W^3 for W = 6 to 12.

Table 3.15: The terms of C_W in Corollary 3.9 for $W = 6$ to 12.

W	$\log x_0(W)$	C_W	C_W^1	C_W^2	C_W^3
12	286.25	5.7×10^{1}	1.0×10^{-1}	5.1×10^{1}	2.3×10^{0}
11	244.09	1.5×10^{2}	3.1×10^{-1}	1.3×10^{2}	2.3×10^{0}
10	203.25	5.0×10^{2}	1.5×10^{0}	4.8×10^{2}	2.3×10^{0}
9	163.09	3.6×10^{3}	1.4×10^{1}	3.4×10^{3}	2.3×10^{0}
8	122.79	1.4×10^{5}	7.0×10^{2}	1.3×10^{5}	2.3×10^{0}
7	87.05	6.1×10^{8}	5.3×10^{6}	6.0×10^{8}	2.4×10^{0}
6	74.62	1.9×10^{29}	1.8×10^{27}	1.7×10^{29}	2.5×10^{0}

Table 3.15 makes it clear that C_W^2 is the term that most effects C_W. If we look back at the formula for C_W^2 in (3.34), we find that it can only be C_1 that is causing this. Moreover, as W and x_0 shrink, C_W^1 contributes more and more. This is further evidence of the influence

of C_1. Therefore, we can conclude that, with respect to Corollary 3.9, C_W is large because of the size of C_1.

If we do the same analysis on the $C_{W,\varepsilon} = C(x_0(W,\varepsilon))$ that comes from Corollary 3.10, with

$$C^1_{W,\varepsilon} := \frac{C_1 e}{\pi\sqrt{W \log x_0(W,\varepsilon)}} \exp\left(-\varepsilon\sqrt{\log x_0(W,\varepsilon)}\right),$$

$$C^2_{W,\varepsilon} := \frac{C_1}{\pi W}$$

and

$$C^3_{W,\varepsilon} := B_5 e + \frac{2\tilde{v}C_2}{\pi} \exp\left(-\varepsilon\sqrt{\log x_0(W,\varepsilon)}\right) + \frac{B_6 + B_7}{\sqrt{W}} \exp\left(-\frac{\varepsilon}{2}\sqrt{\log x_0(W,\varepsilon)}\right),$$

we get Table 3.16.

Table 3.16: The terms of $C_{W,\varepsilon}$ in Corollary 3.10 for $W = 6$ to 12.

W	ε	$\log x_0(W,\varepsilon)$	$C_{W,\varepsilon}$	$C_{W,\varepsilon,1}$	$C_{W,\varepsilon,2}$	$C^3_{W,\varepsilon}$
12	0.28	489.15	5.5×10^{1}	4.4×10^{-2}	5.1×10^{1}	1.9×10^{0}
11	0.25	680.87	1.4×10^{2}	6.5×10^{-2}	1.3×10^{2}	1.7×10^{0}
10	0.22	980.87	4.9×10^{2}	1.4×10^{-1}	4.8×10^{2}	1.5×10^{0}
9	0.18	1710.54	3.4×10^{3}	4.0×10^{-1}	3.4×10^{3}	1.4×10^{0}
8	0.13	4093.11	1.3×10^{5}	3.7×10^{0}	1.3×10^{5}	1.4×10^{0}
7	0.08	14305.32	6.0×10^{8}	2.5×10^{3}	6.0×10^{8}	1.3×10^{0}
6	0.03	418968.18	1.7×10^{29}	6.5×10^{18}	1.7×10^{29}	1.3×10^{0}

This table leads one to the same kind of conclusion we drew from Table 3.15 for C_W. That is, $C_{W,\varepsilon}$ is large because of C_1.

Next we consider the size of $x_0(W)$ and $x_0(W,\varepsilon)$.

In Corollary 3.9

$$x_0(W) = \max\left\{\exp\left(\frac{1}{\eta}\right), \exp\left(\frac{4}{W}\right), \exp(1), \exp\left(W(\log\tilde{v})^2\right), \exp\left(W(\log\tilde{4})^2\right)\right\}$$

For the bounds we calculated using Corollary 3.9 we ended up with $x_0(W) = \exp(W(\log\tilde{v})^2)$.

Whence, it is the size of W and $\tilde{v}$ that determines the size of $x_0(W)$.

In Corollary 3.10

$$x_0(W,\varepsilon) = \max\left\{\exp\left(\frac{1}{\eta}\right), \exp\left(\frac{4}{W}\right), \exp(1), \exp\left(W(\log\tilde{v})^2\right), \exp\left(W(\log 4)^2\right), x_\varepsilon\right\}. \tag{3.35}$$

We got $x_0(W,\varepsilon) = x_\varepsilon$ for all of the bounds we calculated using Corollary 3.10. Thus, $x_0(W,\varepsilon)$ increases as ε decreases.

Chapter 4

The function $m(x)$

The aim of this chapter is to arrive at explicit bounds on $m(x)$ via the results we proved in Chapter 2, as we did with respect to $M(x)$ in Chapter 3. We shall end this chapter by determining at which $x \in (x_0, \infty)$ our bounds will be sharper than

$$m(x) = O^*\left(\frac{0.0144\log x - 0.1}{(\log x)^2}\right) \tag{4.1}$$

from [25], which can be used when $x \in [\exp(13.05), \infty)$.

4.1 Arriving at explicit bounds on $m(x)$

The end result of this section will be explicit bounds on $m(x)$. To get to this result, we first use our work from Chapter 2 in Section 4.1.1. This shall give bounds for $x = \lfloor x \rfloor + 1/2$ with $x \in (x_0, \infty)$ for some x_0. In Section 4.1.2, we then alter the bounds we have so that they apply to all sufficiently large values of x.

4.1.1 The approach from Titchmarsh's Lemma 3.12

The results we shall give in this section are analogous to those we gave in Section 3.1.1. The essence of the proofs required in this section are identical to those in Section 3.1.1. Hence, we opt not to include them.

We first consider Theorem 2.18. In this case, $a_n = \mu(n)$ and $s = 1$. Naturally, we have to explain that the hypotheses of this theorem are satisfied for these values of a_n and s. The work we did to justify having the hypotheses of Theorem 2.18 in Section 3.1.1 essentially applies again here. However, the fact that $s = 1$, not 0, will slightly change Proposition 3.1. Here, we need Proposition 4.1.

Proposition 4.1. *We get that for* $0 < \eta \leq 1/\log(1.5)$ *and* $0 < c \leq \eta$,

$$\sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^{1+c}} \leq \frac{\left(1 - \log(1.5) + \frac{1}{2}\eta\log^2(1.5)\right)\eta + 1}{c}.$$

One might have noticed that we need $c > 0$ in Proposition 4.1, instead of requiring $c > 1$, as in Proposition 3.1. Whence, we will have

$$c := \frac{1}{\log x}. \tag{4.2}$$

Furthermore, we shall have

$$b := -\frac{1}{W \log V}. \tag{4.3}$$

However, we keep the definition for V as it was in Section 3.1.1. Namely,

$$V := \exp\left(\sqrt{\frac{1}{W}\log x}\right).$$

We have the value of α, A, $\tilde{A}$, $\Psi(n)$, $g(b,c,V)$, $\tilde{g}(b,c,V)$, $h(b,V)$,$\tilde{h}(b,V)$, $\hat{h}(b,\tilde{v})$, C_1, C_2, $\tilde{v}$, W and R exactly as in Section 3.1.1.

Theorem 4.2 below is what we get on applying Theorem 2.18 to $m(x)$.

Theorem 4.2. *For N defined in Proposition 2.1, x defined in Proposition 2.1 that also satisfy*

$$x > \max\left\{\exp\left(\frac{1}{\eta}\right), \exp(1), \exp\left(W(\log\tilde{v})^2\right)\right\},$$

with respect to an $0 < \eta \leq 1/\log(1.5)$, *and* W *and* $\tilde{v}$ *are listed in Table 3.1, we have that*

$$\begin{aligned} m(x) &= O^*\left(\frac{C_1 e}{\pi\sqrt{W \log x}} \exp\left(-\sqrt{\frac{1}{W}\log x}\right)\right) \\ &\quad + O^*\left(\frac{C_1}{\pi W} \exp\left(-\sqrt{\frac{1}{W}\log x} + \log\log x\right)\right) \\ &\quad + O^*\left(\frac{\tilde{v} C_2 \sqrt{W}}{\pi} \exp\left(-\sqrt{\frac{1}{W}\log x} + \frac{1}{2}\log\log x\right)\right) \\ &\quad + O^*\left((B_1 e + B_2 + B_3) \exp\left(-\sqrt{\frac{1}{W}\log x} + \log\log x\right)\right) \\ &\quad + O^*\left(\frac{B_4}{|x-N|} \exp\left(-\sqrt{\frac{1}{W}\log x}\right)\right), \end{aligned} \tag{4.4}$$

where C_1 *and* C_2 *can also be found in Tables 3.1 and*

$$B_1 = \frac{\left(1 - \log(1.5) + \frac{1}{2}\eta\log^2(1.5)\right)\eta + 1}{\pi \log 2} \left(2 + \operatorname{Arctan}\left(\frac{1}{(\log x)\exp\left(\sqrt{\frac{1}{W}\log x}\right)}\right) + \pi\right), \tag{4.5}$$

$$B_2 = \frac{8A_1}{\pi}(1 + \pi/2), \tag{4.6}$$

$$B_3 = \frac{3(2^{\frac{1}{\log x}})A_2}{\pi(1-\beta)} \left(1 + \operatorname{Arctan}\left(\frac{1}{(\log x)\exp\left(\sqrt{\frac{1}{W}\log x}\right)}\right) + \frac{\pi}{2}\right), \tag{4.7}$$

$(1/2 < \beta < 1)$, *and*

$$B_4 = \frac{2^{1+\frac{1}{\log x}}}{\pi} \left(1 + \operatorname{Arctan}\left(\frac{1}{(\log x)\exp\left(\sqrt{\frac{1}{W}\log x}\right)}\right) + \frac{\pi}{2}\right). \tag{4.8}$$

We next give two corollaries and use each of them to put together a table of bounds for $m(x)$.

Corollary 4.3. *With all notation and conditions as in Theorem 4.2, for*

$$C(x) := \frac{C_1 e}{\pi\sqrt{W}(\log x)^{3/2}} + \frac{C_1}{\pi W} + \frac{\tilde{v}C_2\sqrt{W}}{\pi\sqrt{\log x}} + B_1 e + B_2 + B_3 + \frac{B_4}{|x-N|\log x},$$

we find that

$$m(x) = O^*\left(C(x)(\log x)\exp\left(-\sqrt{\frac{1}{W}\log x}\right)\right).$$

From now on, we denote the lower bound on x in Corollary 4.3 by

$$x_0(W) := \max\left\{\exp\left(\frac{1}{\eta}\right), \exp(1), \exp\left(W(\log\tilde{v})^2\right)\right\}.$$

Table 4.1 was constructed using the code that applies Corollary 4.3, which is in Appendix A.5.

Table 4.1: The bounds we get via Corollary 4.3 and the code in Appendix A.5.

W	$\log x_0(W)$	C_W	c_W
12	286.25	1.0×10^2	0.288
11	244.09	1.8×10^2	0.301
10	203.35	5.3×10^2	0.316
9	163.09	3.5×10^3	0.333
8	122.79	1.3×10^5	0.353
7	87.05	6.0×10^8	0.377
6	74.62	1.8×10^{29}	0.408

Corollary 4.4. *For $0 < \varepsilon < \sqrt{1/W}$ and x defined in Proposition 2.1 that also satisfy*

$$x > \max\left\{\exp\left(\frac{1}{\eta}\right), \exp(1), \exp\left(W(\log\tilde{v})^2\right)\right\}$$

and

$$0 < \frac{\log\log x}{\sqrt{\log x}} < \varepsilon,$$

with all other notation and conditions as in Theorem 4.2 and

$$\begin{aligned} C(x) \;&:=\; \frac{C_1}{\pi W} + B_1 e + B_2 + B_3 + \left(\frac{C_1 e}{\pi\sqrt{W\log x}} + \frac{B_4}{|x-N|} \right) \exp(-\varepsilon\sqrt{\log x}) \\ &+ \frac{\tilde{\nu} C_2 \sqrt{W}}{\pi} \exp(-\frac{\varepsilon}{2}\sqrt{\log x}), \end{aligned}$$

we find that

$$m(x) = O^*\left(C(x) \exp\left(\left(-\sqrt{\frac{1}{W}} + \varepsilon \right) \sqrt{\log x} \right) \right).$$

We denote the lower bound on x in Corollary 4.4 by

$$x_0(W,\varepsilon) := \max\left\{ \exp\left(\frac{1}{\eta}\right), \exp(1), \exp\left(W(\log\tilde{\nu})^2\right), x_\varepsilon \right\}$$

where

$$0 < \frac{\log\log x}{\sqrt{\log x}} < \varepsilon$$

for $x > x_\varepsilon$.

Table 4.2 comes about on using the code in Appendix A.6 that applies Corollary 4.3.

Table 4.2: The bounds we get via Corollary 4.4 and the code in Appendix A.6.

W	ε	$\log x_0(W,\varepsilon)$	$C_{W,\varepsilon}$	$c_{W,\varepsilon}$
12	0.28	489.15	7.4×10^1	0.008
11	0.25	680.88	1.7×10^2	0.051
10	0.22	980.19	5.1×10^2	0.096
9	0.19	1474.63	3.5×10^3	0.143
8	0.14	3364.98	1.3×10^5	0.213
7	0.08	14305.32	6.0×10^8	0.297
6	0.02	418968.18	1.7×10^{29}	0.388

4.1.2 Bounds that work for any value of x

The bounds in Table 4.1 and 4.2 are deduced for x satisfying $x = \lfloor x \rfloor + 1/2 \in (x_0, \infty)$ for some x_0. Transforming these bounds into ones that are valid at any $x \in (x_0, \infty)$ is what we shall achieve in this section. We shall give two propositions that we can apply to Tables 4.1 and 4.2 to produce additional tables that display such bounds.

We first give Proposition 4.5, which we shall use to transform Table 4.1.

Proposition 4.5. *If for* $x = \lfloor x \rfloor + 1/2 > x_0(W)$, *we have*

$$m(x) = O^*\left(C(\log x)\exp\left(-c\sqrt{\log x}\right)\right),$$

then, when $\lfloor x \rfloor - \nu > \tilde{x}_0 = \max\left\{x_0(W) - (1/2+\nu), \exp(c^2)\right\}$ *with* $0 < \nu \leq 1$, *we have*

$$\begin{aligned} m(\lfloor x \rfloor - \nu) &= \left[\frac{\exp\left(-\log \tilde{x}_0 + c\sqrt{\log \tilde{x}_0}\right)}{\log \tilde{x}_0}\right. \\ &\quad \left. + C\left(1 + \frac{\log\left(1 + \frac{3}{2\tilde{x}_0}\right)}{\log(\tilde{x}_0)}\right)\right](\log(\lfloor x \rfloor - \nu))\exp\left(-c\sqrt{\log(\lfloor x \rfloor - \nu)}\right). \end{aligned}$$

Proof. Firstly, since

$$|m(\lfloor x \rfloor - \nu)| = \left| m(\lfloor x \rfloor + 1/2) - \frac{\mu(\lfloor x \rfloor)}{\lfloor x \rfloor} \right|,$$

the triangle inequality allows one to infer that

$$|m(\lfloor x \rfloor - \nu)| \leq |m(\lfloor x \rfloor + 1/2)| + \frac{1}{\lfloor x \rfloor}.$$

From this we get that

$$m(\lfloor x \rfloor - \nu) = O^*\left(C\log\left(\lfloor x \rfloor + \frac{1}{2}\right)\exp\left(-c\sqrt{\log(\lfloor x \rfloor + 1/2)}\right) + \frac{1}{\lfloor x \rfloor}\right). \tag{4.9}$$

Moreover,

$$
\begin{aligned}
&C\log\left(\lfloor x\rfloor+\frac{1}{2}\right)\exp\left(-c\sqrt{\log\left(\lfloor x\rfloor+1/2\right)}\right)+\frac{1}{\lfloor x\rfloor}\\
&\quad=C\log\left(\lfloor x\rfloor-\nu+\nu+\frac{1}{2}\right)\exp\left(-c\sqrt{\log\left(\lfloor x\rfloor-\nu+\nu+1/2\right)}\right)+\frac{1}{\lfloor x\rfloor-\nu+\nu}. \qquad (4.10)
\end{aligned}
$$

Thus, on using the fact that $0<\nu\leq 1$ as well as (4.9) and (4.10) together, we get that

$$
\begin{aligned}
m(\lfloor x\rfloor-\nu)=&O^*\left(C\left(1+\frac{\log\left(1+\frac{3}{2(\lfloor x\rfloor-\nu)}\right)}{\log(\lfloor x\rfloor-\nu)}\right)(\log(\lfloor x\rfloor-\nu))\exp\left(-c\sqrt{\log(\lfloor x\rfloor-\nu)}\right)\right.\\
&\left.+\exp\left(-\log(\lfloor x\rfloor-\nu)\right)\right).
\end{aligned}
$$

We now factor $(\log(\lfloor x\rfloor-\nu))\exp(-c\sqrt{\log(\lfloor x\rfloor-\nu)})$, then bound what remains. This is possible if we make sure that $\dfrac{\exp(-\log(\lfloor x\rfloor-\nu)+c\sqrt{\log(\lfloor x\rfloor-\nu)})}{\log(\lfloor x\rfloor-\nu)}$ never increases as x increases. We can do so by setting $\lfloor x\rfloor-\nu>\exp(c^2)$. □

The results that culminate from Propositions 4.5 are presented in Table 4.3.

Table 4.3: The bounds we get via Proposition 4.5.

W	$\log x_0(W)$	C_W	c_W
12	286.25	1.1×10^{2}	0.288
11	244.09	1.9×10^{2}	0.301
10	203.35	5.4×10^{2}	0.316
9	163.09	3.6×10^{3}	0.333
8	122.79	1.4×10^{5}	0.353
7	87.05	6.1×10^{8}	0.377
6	74.62	1.9×10^{29}	0.408

We now give Proposition 4.6, which we will apply on Table 4.2.

Proposition 4.6. *If for* $x=\lfloor x\rfloor+1/2>x_0(W,\varepsilon)$, *we have*

$$m(x)=O^*\left(C\exp\left(-c\sqrt{\log x}\right)\right),$$

then, when $\lfloor x\rfloor - \nu > \tilde{x}_0 = \max\left\{x_0(W,\varepsilon) - (1/2+\nu), \exp(c^2)\right\}$ *with* $0 < \nu \leq 1$*, we have*

$$m(\lfloor x\rfloor - \nu) = \left[\exp\left(-\log\tilde{x}_0 + c\sqrt{\log\tilde{x}_0}\right) + C\right]\exp\left(-c\sqrt{\log(\lfloor x\rfloor - \nu)}\right).$$

One can prove Proposition 4.6 by following the kind of ideas we demonstrated in proving Proposition 4.5.

Proposition 4.6 gives us Table 4.4.

Table 4.4: The bounds we get via Proposition 4.6.

W	ε	$\log x_0(W,\varepsilon)$	$C_{W,\varepsilon}$	$c_{W,\varepsilon}$
12	0.28	489.15	7.5×10^{1}	0.008
11	0.25	680.88	1.8×10^{2}	0.051
10	0.22	980.19	5.2×10^{2}	0.096
9	0.19	1474.63	3.6×10^{3}	0.143
8	0.14	3364.98	1.4×10^{5}	0.213
7	0.08	14305.32	6.1×10^{8}	0.297
6	0.02	418968.18	1.8×10^{29}	0.388

Table 4.3 and 4.4 are both applicable at any value of $x \in (x_0(W,\varepsilon),\infty)$.

To end this chapter, we point out that the bounds in Table 4.4 are sharper than (4.1) when $x \in (e^{1405.32},\infty)$.

Chapter 5

Concluding Remarks

We have proven bounds on $\sum_{n\leq x} \frac{a_n}{n^s}$ and $\sum_{n\leq x} a_n$ and have demonstrated how we use these bounds by applying them to $M(x)$ and $m(x)$. We shall now tidy up a bit by providing details on which roads are still open for exploration in terms of continuing this research. We shall sort our discussions into three categories. In Section 5.1 we shall discuss parts of our work that warrant *revision* or further consideration. In Section 5.2, we mention *applications* of our work beyond what we have presented. In Section 5.3, we outline ways in which our work could be *improved*. We then finish with a conclusion that summarizes the thesis.

5.1 Revising our work

We shall point out two ways our work would benefit from revision.

In order to get bounds on $M(x)$ in Chapter 3 and $m(x)$ in Chapter 4, we needed to know values for parameters C_1, W and $\tilde{v}$, as introduced in these chapters. Instances of these C_1, W and $\tilde{v}$ are available in [30]. However, as we explained with respect to $M(x)$ in Section 3.2.1 and 3.2.2, C_1, W and $\tilde{v}$ play an important role in determining how strong our bounds are. Thus, if improved values for C_1, W and $\tilde{v}$ were to appear in the literature, it would be interesting to feed them into our Corollaries 3.9, 3.10, 4.3 and 4.4 to get new bounds on $M(x)$ and $m(x)$. We emphasise that this would involve just a re-writing of part of our code, which we have included as appendices. We wrote our code only for the C_1, W and $\tilde{v}$ that we had available.

In addition to C_1, W and $\tilde{v}$, we also needed to know values for a constant C_2. In Chapter 3, we provided values for this constant (in Table 3.1), in addition to details on how we came about these values. The important thing to note now is that the way in which we found these values for C_2 lacked rigour. Hence, this aspect of our work could be revised with the aim of making it rigorous. It is possible that interval arithmetic could be of use in this respect.

5.2 Applying our work

There are two ways that we foresee our work being applied further. We detail the first application in Section 5.2.1 and the second application in Section 5.2.2.

5.2.1 The functions $\psi(x)$, $L(x)$ and ...

Our Theorem 2.18 that gives a bound on $\sum_{n\leq x} \frac{a_n}{n^s}$, and our Theorem 2.19 that gives a bound on $\sum_{n\leq x} a_n$, are general. We may apply them to any function, as long as the function meets the requirements of the hypotheses of these theorems. Hence, the first application of our work that we suggest is to apply our bounds to different functions.

Two functions that were suggested in [30] as applications of the work in that paper are $\psi(x)$, which we met in the introduction, and $L(x) := \sum_{n\leq x} \lambda(n)$ where $\lambda(n) := (-1)^k$. In this case, k records how many prime divisors n has, with the multiplicity of each prime divisor being counted.

One would be particularly interested to see what bounds would result for $\psi(x)$ from our work. Having such bounds would give one an idea on how much is gained by working with the explicit formula rather than Perron's formula.

In the case of $\psi(x)$, rather than $1/\zeta(w)$, the analysis would involve $-\zeta'(w)/\zeta(w)$. Bounds on $\zeta'(w)/\zeta(w)$ are given in [30, Table 2]. In the case of $L(x)$ the analysis would involve $\zeta(2w)/\zeta(w)$. Furthermore, we would consider $1/2 < \mathrm{Re}(w)$. Hence, we can get bounds on $\zeta(2w)/\zeta(w)$ by using the bounds on $1/\zeta(w)$ in [30], and the fact that $\zeta(2w)$ is bounded by a finite number as long as w stays far enough away from the line $\mathrm{Re}(w) = 1/2$.

If one wishes to look for different functions to apply our work to, other than $\psi(x)$ and $L(x)$, one good resource for doing so would be [21].

5.2.2 Another route to bounds on $M(x)$ or $m(x)$

In Chapters 3 and 4, we employed our work on Lemma 3.12 of Titchmarsh's book [29] and Theorem 1 of Arkhipova [3] to bound $M(x)$ and $m(x)$. However, there is another way that we could have achieved our aim of getting explicit bounds for these functions. Namely, it is possible to turn our $m(x)$ bounds (from Chapter 4) into $M(x)$ bounds or alternatively to turn our $M(x)$ bounds (from Chapter 3) into $m(x)$ bounds. We shall now discuss the two possible approaches to this.

In [30, p. 2] Trudgian makes the statement that "One can recover explicit bounds for $M(x)$ using (1.1) and partial summation." The (1.1) he refers to is a bound on $m(x)$. In this case, the partial summation formula and the triangle inequality give the bound

$$|M(x)| = O^*\left(x|m(x)|\right) + O^*\left(\int_1^x |m(t)|dt\right). \tag{5.1}$$

To turn this into a bound that looks like the ones we have calculated in this thesis for $M(x)$, we can apply one of our bounds on $m(x)$ to the term $O^*\left(x|m(x)|\right)$ of (5.1) and do a bit of work on the second term, $O^*\left(\int_1^x |m(t)|dt\right)$.

If we wanted to instead turn our $M(x)$ bound into $m(x)$ bounds, partial summation itself is not worth pursuing. This is due to the fact that partial summation will give a bound on $m(x)$ that does not have the property $\lim_{x\to\infty} m(x) = 0$. The bounds we proved in this thesis all demonstrate this property. Hence, we shall need an improvement on the inequality derived by partial summation between these two functions if we hope to get a non-trivial bound for $m(x)$. Fortunately, there are in fact several such improvements out there. There is a paper of Balazard [4] in which two such improvements on partial summation are provided. One of the improvements is

$$|m(x)| \leq \frac{|M(x)|}{x} + \frac{1}{x^2}\int_1^x |M(t)|dt + \frac{8}{3x}. \tag{5.2}$$

There is another improvement on partial summation that belongs to El Marraki [18, (1.1)]. It is

$$|m(x)| \leq \frac{|M(x)|}{x} + \frac{1}{x}\int_1^x \frac{|M(t)|}{t}dt + \frac{\log x}{x}.$$

If one was to bound $M(x)$ or $m(x)$ in these alternative ways, checking whether they give anything sharper than the bounds we have arrived at is certainly called for.

5.3 Improving our work

We shall explain an alternative choice to one that was made in this thesis. We believe this choice might give results that complement those we have presented herein.

In Proposition 2.14 we employed the assumption $V > \tilde{v}$. This assumption ended up becoming the condition

$$x > \exp\left(W(\log\tilde{v})^2\right). \tag{5.3}$$

For the bounds we got out of Corollaries 3.9 and 4.3, $x_0(W)$ was set as $\exp\left(W(\log\tilde{v})^2\right)$ because of (5.3). Since the large size of $x_0(W)$ is one of the drawbacks of our work, it is natural to ask oneself: what would we get if we were to instead assume that $V \leq \tilde{v}$? The aim in assuming this would be to get bounds on $M(x)$ and $m(x)$ when $x \leq \exp\left(W(\log\tilde{v})^2\right)$. In fact, we would be looking to get a result for $v_0 < V < \tilde{v}$, for a suitably chosen $v_0 > 0$. If we keep the same definition for V, this gives

$$\exp\left(W(\log v_0)^2\right) < x < \exp\left(W(\log\tilde{v})^2\right) = x_0(W),$$

hopefully with a sufficiently small v_0. If v_0 is sufficiently small, we could then use a computer to get a result for $1 < x \leq \exp\left(W(\log v_0)^2\right)$.

5.4 Conclusion

In the previous chapters, we have proved an *explicit bound* on a general function $\sum_{n \le x} \frac{a_n}{n^s}$, in addition to a second bound that is relevant in the special case when $s = 0$. These bounds are particularly useful with respect to sums for which there is no unconditional version of the explicit formula available.

To prove our bounds on $\sum_{n \le x} \frac{a_n}{n^s}$, in Chapter 2 we took a few steps back from the explicit formula and used its precursor, *Perron's formula*. The two theorems (Theorems 2.18 and 2.19) we proved were derived using Lemma 3.12 of [29] and Theorem 1 of [3] respectively, but the general approach we used is that of Landau [16].

In Chapter 3, we arrived at bounds on $M(x)$. Firstly, we got the bounds of the form

$$M(x) = O^*\left(C_W x (\log x) \exp\left(-c_W \sqrt{\log x}\right)\right) \tag{5.4}$$

when $x > x_0(W)$. We gave particular examples for C_W, c_W and $x_0(W)$ in Table 3.8. Two examples of the bounds we computed are, for $x > e^{74.62}$,

$$M(x) = O^*\left(1.9 \times 10^{29} x (\log x) \exp\left(-0.408 \sqrt{\log x}\right)\right)$$

and for $x > e^{286.25}$,

$$M(x) = O^*\left(5.7 \times 10 x (\log x) \exp\left(-0.288 \sqrt{\log x}\right)\right).$$

Secondly, we got bounds of the shape

$$M(x) = O^*\left(C_W x \exp\left(-c_{W,\varepsilon} \sqrt{\log x}\right)\right) \tag{5.5}$$

for $x > x_0(W, \varepsilon)$. We gave particular examples for C_W, $c_{W,\varepsilon}$ and $x_0(W, \varepsilon)$ in Table 3.9. Two examples are, for $x > e^{418968.18}$,

$$M(x) = O^*\left(1.8 \times 10^{29} x \exp\left(-0.388\sqrt{\log x}\right)\right)$$

and for $x > e^{489.15}$,

$$M(x) = O^*\left(5.6 \times 10 x \exp\left(-0.008\sqrt{\log x}\right)\right).$$

In Chapter 4, we did the same for $m(x)$. We arrived at bounds of the same shape as (5.4) and (5.5), without the factor of x that appears before the exponential. We gave these bounds in Tables 4.3 and 4.4.

Regarding both $M(x)$ and $m(x)$, we found that the C_W, $x_0(W)$ and $x_0(W,\varepsilon)$ in our bounds are rather large. This was not surprising. Ramaré expressed that this would likely be the case in [25, p. 1359]. However, in the case of $M(x)$ we also demonstrated that if one was to decrease C_1, our C_W and $C_{W,\varepsilon}$ would improve. On the other hand, $x_0(W)$ is, in the case of (5.4), at the mercy of W and $\tilde{v}$. Smaller values for W and smaller values for $\tilde{v}$ are preferable. Where (5.5) is concerned, $x_0(W,\varepsilon)$ is determined by our decision on ε as per (3.17). The larger the value of ε, the smaller the value of $x_0(W,\varepsilon)$, but the larger the value of $c_{W,\varepsilon}$. Hence, a large ε is better for $x_0(W,\varepsilon)$, but not for $c_{W,\varepsilon}$. Nevertheless, the bounds we got demonstrated the usefulness of our theorems from Chapter 2, since explicit bounds like (5.4) and (5.5) have never before found their way into the literature. Moreover, our bounds are inevitably sharper than the bounds that have, thus far, been available when x is sufficiently large.

We do not know of any published formulas that provide explicit bounds on $\sum_{n\leq x}\frac{a_n}{n^s}$ exactly like the ones we have presented, nor have we come across explicit bounds on $M(x)$ and $m(x)$ as sharp as those we have provided. Moreover, by presenting the formulas we developed, along with our explicit bounds, we have made it possible for our work to be used to prove even stronger explicit bounds in the future. With there being much one can still do in this line of research, we hope that this work ignites further exploration.

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Appendix A

Code

This Appendix contains the `Python` code we have used to get the explicit bounds in this thesis. The code is divided into six sections, each of which corresponds to one of Corollaries 3.5, 3.6, 3.9, 3.10, 4.3 or 4.4. In some of the cases, where there is repetition between code for different corollaries, we have omitted parts of the code so as to reduce repetition. In these cases, we indicate where the missing code can be found. Furthermore, in Sections A.1 and A.2 we include comments in the code (preceded by a #) to explain what each part does. The code in the other four sections use the same structure as the code in these two sections. Hence, we shall include no comments in these later sections.

In Table A.1, we list the names we have used in our code for each of the relevant parameters, variables and functions of this thesis.

Table A.1: The names used in our code for the parameters, variables and functions of this thesis.

Name of variable or function in the code	Corresponding variable/parameter /function in thesis	Name of variable or function in the code	Corresponding variable/parameter /function in thesis
`xMinusN`	$\|x-N\|$	`B2(x)`	B_2
`W`	W	`B3(x)`	B_3
`C1`	C_1	`B4(x)`	B_4
`vTilde`	$\tilde{v}$	`C(x)`	$C(x)$
`C2`	C_2	`B5(x)`	B_5
`eta`	η	`B6(x)`	B_6
`A1(x)`	A_1	`B7(x)`	B_7
`A2(x)`	A_2	`Epsilon`	ε
`Beta(x)`	β	`x0`	$\log x_0(W)$ or $\log x_0(W,\varepsilon)$
`B1(x)`	B_1	`xEpsilon`	x_ε

A.1 Code that implements Corollary 3.5

```
# Loading everything that is provided by mpmath that shall be used
#  in the proceeding code
from mpmath import exp, log, sqrt, atan, fabs, pi, mpf, mp
```

```
# Asking for a number greater than 0 that shall be used to change the
# precision.
mp.dps = input('What will you have the precision > 0 as?')

# Asking for a number strictly greater than 0 and strictly less than 1
# that shall then be used as the value of xMinusN.
xMinusN = 1
while xMinusN == 1:
    xMinusN = mpf(input('What will you have 0 < x - N < 1 as?'))
    if xMinusN >= 1:
        print('x - N cannot be what you entered.')
        xMinusN = 1
    elif xMinusN <= 0:
        print('x - N cannot be what you entered.')
        xMinusN = 1

# Asking for either the numbers 6, 7, 8, 9, 10, 11, or 12 that shall
# then be used as the value of W. Once a suitable number is decided
# on for W, the program defines C1, vTilde and C2 according to the
# values given earlier in this thesis
W = 1
while W == 1:
    W = mpf(input('What will you have W as, W = 6, 7 , 8 , 9, 10,
    11 or 12?'))
    if W == 6:
      C1 = mpf('3.2')*mpf(10)**mpf(30)
      vTilde = mpf('34.0')
      C2 = mpf('3.33')
    elif W == 7:
      C1 = mpf('1.3')*mpf(10)**mpf(10)
      vTilde = mpf('34.0')
      C2 = mpf('3.29')
    elif W == 8:
      C1 = mpf('3.1')*mpf(10)**mpf(6)
      vTilde = mpf('50.28')
      C2 = mpf('3.24')
    elif W == 9:
      C1 = mpf('9.6')*mpf(10)**mpf(4)
      vTilde = mpf('70.59')
      C2 = mpf('3.21')
    elif W == 10:
      C1 = mpf('1.5')*mpf(10)**mpf(4)
      vTilde = mpf('90.87')
      C2 = mpf('3.19')
```

```
    elif W == 11:
      C1 = mpf('4.4')*mpf(10)**mpf(3)
      vTilde = mpf('111.12')
      C2 = mpf('3.17')
    elif W == 12:
      C1 = mpf('1.9')*mpf(10)**mpf(3)
      vTilde = mpf('132.16')
      C2 = mpf('3.16')
    else:
      print('W cannot be', W)
      W = 1

# Evaluating an expression that shall give the value of eta.
eta = mpf(1)/(W*(log(vTilde))**mpf(2))

# Relaying the current value of xMinusN, W, C1, vTilde, C2 and eta.
print('You have xMinusN as', xMinusN, ', W as', W, ', C1 as', C1, ',
vTilde as', vTilde, ', C2 as', C2, 'and eta as', eta)

# Evaluating the maximum of several quantities that shall then be used
# as the value of x0.
x0 = max(mpf(1)/eta, mpf(4)/W, mpf(1), W*(log(vTilde))**mpf(2))

# Defining x0, A1(x), A2(x), B1(x), B2(x), B3(x), B4(x) and C(x).
def A1(x):
    A1 = (mpf(1)+ log(mpf(1)+mpf(1)/(mpf(2)*x)))/log(x)+mpf(1)
    return A1

def A2(x):
    A2 =(mpf(1) + log(mpf(1)+mpf(1)/x)-log(mpf(2)))/log(x) + mpf(1)
    return A2

def Beta(x):
    Beta = mpf(1)/mpf(2)+mpf(1)/(mpf(4)*x)
    return Beta

def B1(x):
    B1 =(mpf(1)-log(mpf('1.5'))+mpf('0.5')*eta*(log(mpf('1.5'))
   **mpf(2))*eta+mpf(1))/(pi*log(mpf(2)))*(mpf(2)+
   atan(mpf(1)/exp(sqrt(mpf(1)/W*log(x)))+mpf(1)/(log(x)
   *exp(sqrt(mpf(1)/W*log(x)))))+ pi)
    return B1

def B2(x):
    B2 = mpf(8)*A1(x)/pi*(mpf(1)+ pi/mpf(2))
```

```
    return B2

def B3(x):
    B3 = (mpf(3)*mpf(2)**(mpf(1)/log(x))*A2(x))/(pi*(mpf(1)-Beta(x)))
   *(mpf(1)+ atan(mpf(1)/exp(sqrt(mpf(1)/W*log(x)))+mpf(1)/(log(x)
   *exp(sqrt(mpf(1)/W*log(x)))))+ pi/mpf(2))
    return B3

def B4(x):
    B4 = (mpf(2)**(mpf(1)+ mpf(1)/log(x)))/pi*(mpf(1)+ atan(mpf(1)
   /exp(sqrt(mpf(1)/W*log(x)))+ mpf(1)/(log(x)*exp(sqrt(mpf(1)/W
   *log(x)))))+ pi/mpf(2))
    return B4

def C(x):
    C = (C1*exp(mpf(1)))/(pi*sqrt(W)*(log(x))**mpf('1.5')) + C1/(pi*W)
   + (mpf(2)*vTilde*C2)/(pi*log(x))+B1(x)*exp(mpf(1))+B2(x)+B3(x)
   +B4(x)/(xMinusN*log(x))
    return C

# Evaluating the constants needed to express the bound and relaying
#  the bound.
print('x > exp(', x0,'), M(x)=O*(',C(exp(x0)),'x(logx)exp(',
-sqrt(mpf(1)/(W)),'sqrt(log(x))))')
```

A.2 Code that implements Corollary 3.6

We have omitted some parts of the code below. All of this omitted code appears in Section A.1. We indicate what we have omitted by replacing it with a ⋮ . To locate the omitted code in Section A.1, one should look for the code in Section A.1 that corresponds to the code on either side of the ⋮ in this section. Everything between these pieces of code in Section A.1 is exactly what we omitted.

```
from mpmath import exp, log, sqrt, atan, fabs, pi, mpf, mp

                                    ⋮

eta = mpf(1)/(W*(log(vTilde))**mpf(2))

# Asking for a number greater than 0 and strictly less than the
# square root of 1/W that shall then be used as the value of Epsilon.
Epsilon = 1
while Epsilon ==1:
    Epsilon = mpf(input('What will you have 0 <= Epsilon
   < sqrt(1/W) as?'))
```

```
    if Epsilon >= sqrt(mpf(1)/W):
        print('Epsilon cannot be what you entered.')
        Epsilon = 1
    elif Epsilon <=0:
        print('Epsilon cannot be what you entered.')
        Epsilon = 1

# Asking for a suitable number that shall be used as the value of
# EpsilonX
EpsilonX = 8
while log(log(EpsilonX))/sqrt(log(EpsilonX))>Epsilon:
    EpsilonX = mpf(input('What will you have EpsilonX as?'))
    if EpsilonX <= 1:
        print('EpsilonX cannot be what you entered.')
        EpsilonX = 8
    elif log(log(EpsilonX))/sqrt(log(EpsilonX))>Epsilon:
        print('EpsilonX cannot be what you entered.')

print('You have xMinusN as', xMinusN, ', W as', W, ', C1 as', C1, ',
vTilde as', vTilde, ', C2 as', C2, 'eta as', eta, 'Epsilon as',
 Epsilon, 'and EpsilonX as', EpsilonX)

x0 = max(mpf(1)/eta, mpf(4)/W, mpf(1), W*(log(vTilde))
**mpf(2), log(EpsilonX))
```

⋮

```
def C(x):
    C = C1/(pi*W)+B1(x)*exp(mpf(1))+B2(x)+B3(x)+((C1*exp(mpf(1)))
   /(pi*sqrt(W*log(x))) + (mpf(2)*vTilde*C2)/pi +B4(x)/(xMinusN))
   *exp(-Epsilon*sqrt(log(x)))
    return C

print('x > exp(', x0,'), M(x)=O*(',C(exp(x0)),'xexp(',
-sqrt(mpf(1)/(W))+ Epsilon,'sqrt(log(x))))')
```

A.3 Code that implements Corollary 3.9

```
from mpmath import exp, log, sqrt, pi, mpf, mp

mp.dps = input('What will you have the precision > 0 as?')

W = 1
while W == 1:
    W = mpf(input('What will you have W as, W = 6, 7 , 8 , 9, 10,
```

```
   11 or 12?’))
    if W == 6:
      C1 = mpf(’3.2’)*mpf(10)**mpf(30)
      vTilde = mpf(’34.0’)
      C2 = mpf(’3.33’)
    elif W == 7:
      C1 = mpf(’1.3’)*mpf(10)**mpf(10)
      vTilde = mpf(’34.0’)
      C2 = mpf(’3.29’)
    elif W == 8:
      C1 = mpf(’3.1’)*mpf(10)**mpf(6)
      vTilde = mpf(’50.28’)
      C2 = mpf(’3.24’)
    elif W == 9:
      C1 = mpf(’9.6’)*mpf(10)**mpf(4)
      vTilde = mpf(’70.59’)
      C2 = mpf(’3.21’)
    elif W == 10:
      C1 = mpf(’1.5’)*mpf(10)**mpf(4)
      vTilde = mpf(’90.87’)
      C2 = mpf(’3.19’)
    elif W == 11:
      C1 = mpf(’4.4’)*mpf(10)**mpf(3)
      vTilde = mpf(’111.12’)
      C2 = mpf(’3.17’)
    elif W == 12:
      C1 = mpf(’1.9’)*mpf(10)**mpf(3)
      vTilde = mpf(’132.16’)
      C2 = mpf(’3.16’)
    else:
      print(’W cannot be’, W)
      W = 1

eta = mpf(1)/(W*(log(vTilde))**mpf(2))

print(’You have W as’, W, ’, C1 as’, C1, ’, vTilde as’, vTilde, ’,
C2 as’, C2, ’and eta as’, eta)

x0 = max(mpf(1)/eta, mpf(4)/W, mpf(1), W*(log(vTilde))
**mpf(2),W*(log(mpf(4)))**mpf(2))

def A1(x):
    A1 = (mpf(1)+ log(mpf(1)+mpf(1)/(mpf(2)*x)))/log(x)+mpf(1)
    return A1
```

```
def A2(x):
    A2 =(mpf(1) + log(mpf(1)+mpf(1)/x)-log(mpf(2)))/log(x) + mpf(1)
    return A2

def Beta(x):
    Beta = mpf(1)/mpf(2)+mpf(1)/(mpf(4)*x)
    return Beta

def B5(x):
    B5 = mpf(2)*(mpf(1)-log(mpf('1.5'))+mpf('0.5')*eta
   *(log(mpf('1.5'))**mpf(2))*eta+mpf(1))/(pi*log(mpf(2)))
    return B5

def B6(x):
    B6 = (mpf(2)**(mpf(1)/log(x))*(pi*log(mpf(2)+mpf(2)/log(x))
   + log(pi/mpf(2))+mpf(1)))/(pi*mpf(2))
    return B6

def B7(x):
    B7 = (mpf(4) + pi + mpf(4)*log(pi/mpf(4))+mpf(4)
   *log(mpf(2)+mpf(2)/log(x)))/(pi*log(mpf(2)))
    return B7

def C(x):
    C = (C1*exp(mpf(1)))/(pi*sqrt(W)*(log(x))
   **mpf('1.5')) + C1/(pi*W) + (mpf(2)*vTilde*C2)
   /(pi*log(x))+ B5(x)*exp(mpf(1)) + (B6(x)+B7(x))
   /(sqrt(W*log(x)))
    return C

print('x > exp(', x0,'), M(x)=O*(',C(exp(x0)),'x(logx)exp(',
-sqrt(mpf(1)/(W)),'sqrt(log(x))))')
```

A.4 Code that implements Corollary 3.10

We have omitted some parts of the code below. All of this omitted code appears in Section A.3. We indicate what we have omitted by replacing it with a $\vdots$. To locate the omitted code in Section A.3, one should look for the code in Section A.3 that corresponds to the code on either side of the $\vdots$ in this section. Everything between these pieces of code in Section A.3 is exactly what we omitted.

```
from mpmath import exp, log, sqrt, pi, mpf, mp
```

$\vdots$

```
eta = mpf(1)/(W*(log(vTilde))**mpf(2))

Epsilon = 1
while Epsilon == 1:
    Epsilon = mpf(input('What will you have 0 <= Epsilon
   < sqrt(1/W)as?'))
    if Epsilon >= sqrt(mpf(1)/W):
        print('Epsilon cannot be what you entered.')
        Epsilon = 1
    elif Epsilon <=0:
        print('Epsilon cannot be what you entered.')
        Epsilon = 1

EpsilonX = 8
while log(log(EpsilonX))/sqrt(log(EpsilonX)) > Epsilon:
    EpsilonX = mpf(input('What will you have EpsilonX as?'))
    if EpsilonX <=1:
        print('EpsilonX cannot be what you entered.')
        EpsilonX = 8
    elif log(log(EpsilonX))/sqrt(log(EpsilonX)) > Epsilon:
        print('Epsilon cannot be what you entered.')

print('You have W as', W, ', C1 as', C1, ', vTilde as', vTilde, ',
C2 as',  C2, 'eta as', eta, 'Epsilon as', Epsilon, 'and EpsilonX as',
EpsilonX)

x0 = max(mpf(1)/eta, mpf(4)/W, mpf(1), W*(log(vTilde))
**mpf(2),W*(log(mpf(4)))**mpf(2),log(EpsilonX))

                                    ⋮

def C(x):
    C = C1/(pi*W)+ B5(x)*exp(mpf(1)) + ((C1*exp(mpf(1)))
   /(pi*sqrt(W*log(x))) + (mpf(2)*vTilde*C2)/pi+(B6(x)+B7(x))
   /(sqrt(W)))*exp(-Epsilon*sqrt(log(x)))
    return C

print('x > exp(', x0,'), M(x)=O*(',C(exp(x0)),'xexp(',
-sqrt(mpf(1)/(W))+Epsilon,'sqrt(log(x))))')
```

A.5 Code that implements Corollary 4.3

```
from mpmath import exp, log, sqrt, atan, fabs, pi, mpf, mp

mp.dps = input('What will you have the precision > 0 as?')
```

```
xMinusN = 1
while xMinusN == 1:
    xMinusN = mpf(input('What will you have 0 < x - N < 1 as?'))
    if xMinusN >= 1:
        print('x - N cannot be what you entered.')
        xMinusN = 1
    elif xMinusN <= 0:
        print('x - N cannot be what you entered.')
        xMinusN = 1

W = 1
while W == 1:
    W = mpf(input('What will you have W as, W = 6, 7 , 8 , 9, 10,
   11 or 12?'))
    if W == 6:
      C1 = mpf('3.2')*mpf(10)**mpf(30)
      vTilde = mpf('34.0')
      C2 = mpf('3.33')
    elif W == 7:
      C1 = mpf('1.3')*mpf(10)**mpf(10)
      vTilde = mpf('34.0')
      C2 = mpf('3.29')
    elif W == 8:
      C1 = mpf('3.1')*mpf(10)**mpf(6)
      vTilde = mpf('50.28')
      C2 = mpf('3.24')
    elif W == 9:
      C1 = mpf('9.6')*mpf(10)**mpf(4)
      vTilde = mpf('70.59')
      C2 = mpf('3.21')
    elif W == 10:
      C1 = mpf('1.5')*mpf(10)**mpf(4)
      vTilde = mpf('90.87')
      C2 = mpf('3.19')
    elif W == 11:
      C1 = mpf('4.4')*mpf(10)**mpf(3)
      vTilde = mpf('111.12')
      C2 = mpf('3.17')
    elif W == 12:
      C1 = mpf('1.9')*mpf(10)**mpf(3)
      vTilde = mpf('132.16')
      C2 = mpf('3.16')
    else:
      print('W cannot be', W)
```

```
      W = 1

eta = mpf(1)/(W*(log(vTilde))**mpf(2))

print('You have xMinusN as', xMinusN, ', W as', W, ', C1 as', C1, ',
tTilde as', vTilde, ', C2 as', C2, 'and eta as', eta)

x0 = max(mpf(1)/eta, mpf(1), W*(log(vTilde))**mpf(2))

def A1(x):
    A1 = (mpf(1)+ log(mpf(1)+mpf(1)/(mpf(2)*x)))/log(x)+mpf(1)
    return A1

def A2(x):
    A2 =(mpf(1) + log(mpf(1)+mpf(1)/x)-log(mpf(2)))/log(x) + mpf(1)
    return A2

def Beta(x):
    Beta = mpf(1)/mpf(2)+mpf(1)/(mpf(4)*x)
    return Beta

def B1(x):
    B1 =(mpf(1)-log(mpf('1.5'))+mpf('0.5')*eta*(log(mpf('1.5'))
   **mpf(2))*eta+mpf(1))/(pi*log(mpf(2)))*(mpf(2)+ atan(mpf(1)
   /(log(x)*exp(sqrt(mpf(1)/W*log(x)))))+ pi)
    return B1

def B2(x):
    B2 = mpf(8)*A1(x)/pi*(mpf(1)+ pi/mpf(2))
    return B2

def B3(x):
    B3 = (mpf(3)*mpf(2)**(mpf(1)/log(x))*A2(x))/(pi*(mpf(1)-Beta(x)))
   *(mpf(1)+ atan(mpf(1)/(log(x)*exp(sqrt(mpf(1)/W*log(x)))))
   + pi/mpf(2))
    return B3

def B4(x):
    B4 = (mpf(2)**(mpf(1)+ mpf(1)/log(x)))/pi*(mpf(1)+ atan(mpf(1)
   /(log(x)*exp(sqrt(mpf(1)/W*log(x)))))+ pi/mpf(2))
    return B4

def C(x):
    C = (C1*exp(mpf(1)))/(pi*sqrt(W)*(log(x))**mpf('1.5'))
   + C1/(pi*W) + (vTilde*C2*sqrt(W))/(pi*sqrt(log(x)))
```

```
   +B1(x)*exp(mpf(1))+B2(x)+B3(x)+B4(x)/(xMinusN*log(x))
    return C

print('x > exp(', x0,'), m(x)=O*(',C(exp(x0)),'(logx)exp(',
-sqrt(mpf(1)/(W)),'sqrt(log(x))))')
```

A.6 Code that implements Corollary 4.4

We have omitted some parts of the code below. All of this omitted code appears in Section A.5. We indicate what we have omitted by replacing it with a ⋮. To locate the omitted code in Section A.5, one should look for the code in Section A.5 that corresponds to the code on either side of the ⋮ in this section. Everything between these pieces of code in Section A.5 is exactly what we omitted.

```
from mpmath import exp, log, sqrt, atan, fabs, pi, mpf, mp

                                    ⋮

eta = mpf(1)/(W*(log(vTilde))**mpf(2))

Epsilon = 1
while Epsilon ==1:
    Epsilon = mpf(input('What will you have 0 <= Epsilon
   < sqrt(1/W) as?'))
    if Epsilon >= sqrt(mpf(1)/W):
        print('Epsilon cannot be what you entered.')
        Epsilon = 1
    elif Epsilon <=0:
        print('Epsilon cannot be what you entered.')
        Epsilon = 1

EpsilonX = 8
while log(log(EpsilonX))/sqrt(log(EpsilonX))>Epsilon:
    EpsilonX = mpf(input('What will you have EpsilonX as?'))
    if EpsilonX <= 1:
        print('EpsilonX cannot be what you entered.')
        EpsilonX = 8
    elif log(log(EpsilonX))/sqrt(log(EpsilonX))>Epsilon:
        print('EpsilonX cannot be what you entered.')

print('You have xMinusN as', xMinusN, ', W as', W, ', C1 as', C1, ',
vTilde as', vTilde, ', C2 as', C2, 'eta as', eta, 'Epsilon as',
 Epsilon, 'and EpsilonX as', EpsilonX)

x0 = max(mpf(1)/eta, mpf(1), W*(log(vTilde))**mpf(2),
log(EpsilonX))
```

⋮

```
def C(x):
    C = C1/(pi*W)+B1(x)*exp(mpf(1))+B2(x)+B3(x)+((C1*exp(mpf(1)))
   /(pi*sqrt(W*log(x)))+B4(x)/(xMinusN))*exp(-Epsilon*sqrt(log(x)))
   + ((vTilde*C2*sqrt(W))/pi)*exp(-(Epsilon/2)*sqrt(log(x)))
    return C

print('x > exp(', x0,'), m(x)=O*(',C(exp(x0)),'exp(',
-sqrt(mpf(1)/(W))+ Epsilon,'sqrt(log(x))))')
```