UNBIASED AND SKEW-REGULAR HADAMARD MATRICES

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Dedication

To the friends we made along the way.

Abstract

This thesis explores the concept of regularity in Hadamard matrices. By generalizing the definition of regularity, researchers have combined the definitions of skew-type and regular Hadamard matrices to define *skew-regular* Hadamard matrices. In this thesis, we consider skew-regularity in the context of both real and quaternary Hadamard matrices, and showcase some known applications of these objects. The concept of regularity appears in the definition of unbiasedness. Aided by our knowledge of regularity, we then explore unbiasedness for both real and quaternary Hadamard matrices, and classify small orders of mutually unbiased quaternary Hadamard matrices.

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List of Symbols

0_n	Denotes an $n \times n$ all zero matrix
$\bar{z}$	Denotes the complex conjugate of $z \in \mathbb{C}$
$\mathbb{F}_q$	Denotes the finite field of order q
$a \cdot b$	Denotes the inner product of two vectors
A^*	Denotes the conjugate-transpose of the matrix A
A^T	Denotes the transpose of the matrix A
I_n	Denotes the standard identity matrix of order n
J_n	Denotes an $n \times n$ all ones matrix
j_n	Denotes the all ones column vector of length n

Chapter 1

Introduction

This thesis presents a collection of novel mathematical concepts that have been introduced and developed over the last decade. Much of the thesis focuses on skew-regular Hadamard and quaternary Hadamard matrices, as well as the related concept of unbiasedness. The content of the thesis is a combination of published work [24, 27], unpublished work [28], and manuscripts in development.

Hadamard matrices possess remarkable properties that have captivated mathematicians for over a century. Due to their elegant structure, these objects have found applications in many fields outside of mathematics, including telecommunications and quantum computing [43, 48].

Although Hadamard matrices are named after the French mathematician Jacques Hadamard, Hadamard matrices were first studied by James Sylvester [41] in 1867, over 25 years before Jacques Hadamard [12] published his discoveries in 1893. Sylvester focused on a special infinite family of Hadamard matrices by developing a recursive construction using the Kronecker Product. Subsequently, Jacques Hadamard presented two new examples of Hadamard matrices that could not be generated by Sylvester's recursive construction and provided another infinite family of these matrices. Hadamard's work also led to the formulation of the famous *Hadamard conjecture*, which states that there exists a Hadamard matrix of every order that is a multiple of 4. The following century of research has produced many additional examples of Hadamard matrices and advances toward resolving the conjecture. Although experts recognize the conjecture as accurate, the crucial insight required for a complete solution has thus far eluded researchers.

Due to their captivating properties and applications, researchers have explored and classified special types of Hadamard matrices. Moreover, Hadamard matrices have been generalized by relaxing the stringent requirements imposed on their entries. Notable generalizations include quaternary Hadamard matrices, weighing matrices, and orthogonal designs. A detailed treatment of these objects can be found in [40]. In this thesis, we study another special type of Hadamard matrices, skew-regular Hadamard matrices, and extend the definition to the quaternary case. We then provide classifications of these matrices for many small orders and present some applications of these objects.

We then utilize our knowledge of regular (quaternary) Hadamard matrices as a stepping stone to the closely related topic of unbiased (quaternary) Hadamard matrices. These objects generalize the well-known notion of mutually unbiased bases [4]. Similarly, we provide classifications for some small orders and demonstrate the fascinating properties of these objects.

The majority of the thesis is devoted to the discussion of skew-regular and unbiased Hadamard matrices. For each of these discussion topics, we will introduce both the real case and the quaternary case. The differences between the definitions of the real and quaternary cases for these objects are ostensibly minimal. However, because of the broader structure of the quaternary case, the results of the real and quaternary cases employ inherently different approaches in their respective classifications. The reader is encouraged to keep these differences in mind as we explore the intricacies of the classifications of these objects.

1.1 A Note on Matrix Notation

This thesis contains many examples of large matrices, particularly within the appendices. To improve visual clarity and allow us to fit larger matrices within the margins, we denote -1 as $-$ and $-i$ as j. For example, the matrix

$$M = \begin{pmatrix} 1 & i \\ -1 & -i \end{pmatrix}$$

is written as

$$M = \begin{pmatrix} 1 & i \\ - & j \end{pmatrix}.$$

Throughout this work, we will use $\mathrm{Re}(M)$ and $\mathrm{Im}(M)$ to denote the real and imaginary parts of a matrix M. Using the same M as above,

$$\mathrm{Re}(M) = \begin{pmatrix} 1 & 0 \\ - & 0 \end{pmatrix},\ \mathrm{Im}(M) = \begin{pmatrix} 0 & 1 \\ 0 & - \end{pmatrix}.$$

In the same spirit of reducing clutter, the following is a shorthand for diagonal matrices:

$$\mathrm{diag}(1,2,3,4) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

Finally, a note on indexing matrices. If M is an $n \times m$ matrix, unless explicitly stated, the rows and columns of M are naturally indexed by $1, 2, \ldots, n$ and $1, 2, \ldots, m$, respectively.

Chapter 2

Background

This chapter presents the foundations on which we will build throughout the thesis. Sequentially, the following sections will define: Hadamard matrices, quaternary Hadamard matrices, weighing matrices, and end with the introduction of linear error correcting codes.

2.1 Hadamard Matrices

This section presents key definitions and results on Hadamard matrices that will be referenced throughout this thesis. We place particular emphasis on the special types and constructions of these matrices, which recur throughout this work. For a more detailed treatment of these objects, we refer the reader to [7].

We begin by stating the formal definition of a Hadamard matrix.

Definition 2.1. A *Hadamard matrix* of order n is an $n \times n$ matrix H consisting of entries in $\{-1, 1\}$ such that

$$HH^T = nI_n.$$

Put differently, given a Hadamard matrix, H, of order n, the condition $HH^T = nI_n$ is equivalently captured by saying that the row vectors of H are pairwise orthogonal. This condition is also equivalent to stating that the column vectors of H are pairwise orthogonal. This statement is proven in the following proposition.

Proposition 2.2. *If H is a Hadamard matrix of order n, then H^T is a Hadamard matrix of order n.*

Proof. Starting with $HH^T = nI_n$, we multiply on the left by H^{-1} to obtain $H^T = nH^{-1}$. It follows that

$$H^T(H^T)^T = H^T H = nH^{-1}H = nI_n.$$

□

Below are some examples of Hadamard matrices.

Example 2.3. The following are Hadamard matrices of order 1, 2, 4 and 8.

$$H_1 = \begin{pmatrix} 1 \end{pmatrix}, \; H_2 = \begin{pmatrix} 1 & 1 \\ 1 & - \end{pmatrix}, \; H_4 = \begin{pmatrix} - & 1 & 1 & 1 \\ 1 & - & 1 & 1 \\ 1 & 1 & - & 1 \\ 1 & 1 & 1 & - \end{pmatrix}, \text{ and } H_8 = \begin{pmatrix} - & 1 & 1 & 1 & - & 1 & 1 & 1 \\ 1 & - & 1 & 1 & 1 & - & 1 & 1 \\ 1 & 1 & - & 1 & 1 & 1 & - & 1 \\ 1 & 1 & 1 & - & 1 & 1 & 1 & - \\ - & 1 & 1 & 1 & - & - & - & - \\ 1 & - & 1 & 1 & - & 1 & - & - \\ 1 & 1 & - & 1 & - & - & 1 & - \\ 1 & 1 & 1 & - & - & - & - & 1 \end{pmatrix}.$$

A natural question to ask is for which n does there exist a Hadamard matrix of order n? To tackle this question, we must establish the foundations to prove a necessary existence condition for Hadamard matrices.

2.1.1 Equivalence of Hadamard Matrices

Notice that when multiplying any row or column of a Hadamard matrix by -1, the rows and columns remain pairwise orthogonal. Moreover, orthogonality is also retained under permutations of rows and columns. As negating any combination of rows and columns, and permuting the rows and columns of a Hadamard matrix does not change orthogonality, the resulting matrix after performing these operations is still Hadamard. We summarize these observations in the following proposition.

Proposition 2.4. *If H is a Hadamard matrix of order n and P and Q are $n \times n$ signed permutation matrices, then PHQ is a Hadamard matrix of order n.*

Proof. By definition of signed permutation matrices, $PP^T = I_n$ and $QQ^T = I_n$. It follows that

$$(PHQ)(PHQ)^T = PHQQ^T H^T P^T = PHH^T P^T = nPP^T = nI_n.$$

□

This observation that negating or permuting rows and columns of a Hadamard matrix results in another Hadamard matrix naturally gives rise to the following definition of equivalence.

Definition 2.5. Let H and K be two Hadamard matrices of order n. The matrices H and K are said to be *equivalent*, denoted by $H \cong K$, if there exist two $n \times n$ signed permutation matrices P and Q such that $H = PKQ$.

More directly, two Hadamard matrices, H and K, are equivalent if K can be obtained by performing a series of row and column permutations and negations to H. Equivalence is also symmetric, reflexive and transitive, and thus defines an equivalence relation. Thus, we can define an *equivalence class* of a Hadamard matrix to be the set of all Hadamard matrices to which it is equivalent. In Definition 2.5, some authors choose to include the transpose of H, which sometimes cannot be obtained through row/column permutations. However, it is important to note that we do not specifically include H^T in the equivalence class of H.

Similarly, skew-Hadamard equivalence, which closely follows Definition 2.5, but only allows $Q = P^T$. Formally, two Hadamard matrices H and K are said to be SH-equivalent if there exists a signed permutation matrix P such that $H = PKP^T$. Note that if two matrices are SH-inequivalent, this implies there does not exist a signed permutation matrix P such that $H = PKP^T$, but this does not imply that the two matrices are inequivalent. This notion of SH-equivalence is stricter, allowing researchers to tackle problems that would be computationally impractical under the standard definition and is particularly useful for classifying skew-type Hadamard matrices.

The following definition helps establish a standard presentation of Hadamard matrices.

Definition 2.6. A *normalized Hadamard matrix* is a Hadamard matrix in which every entry in the first row and the first column is 1.

In Example 2.3, the matrices H_1, H_2 are normalized Hadamard matrices. Given any Hadamard matrix H, we can multiply rows and columns whose first entries are -1 by -1 to normalize the matrix. This observation is formally summarized below.

Corollary 2.7. *Any Hadamard matrix is equivalent to a normalized Hadamard matrix.*

2.1.2 Existence of Hadamard Matrices

Naturally, one of the first questions to ask when discovering Hadamard matrices is: for which orders does there exist a Hadamard matrix? It is immediately evident that if the order of a Hadamard matrix is greater than one, then the order must be even. However, what is less obvious is that Hadamard matrices of orders $n > 2$ exist only for integers n that are multiples of four. To prove this statement and help answer the question of existence, we present a well-known existence condition for Hadamard matrices.

Proposition 2.8 ([12]). *If H is a Hadamard matrix of order $n > 2$, then n is divisible by four.*

Proof. Corollary 2.7 established that H is equivalent to a normalized Hadamard matrix. Thus, without loss of generality, assume H is normalized. Then the first entry of every column of H will be 1, and for each column j the pair $(h_{2,j}, h_{3,j})$ of possible second and third entries lies in $\{(1,1),(1,-1),(-1,1),(-1,-1)\}$. Next, let a,b,c,d denote the number of columns whose pair of second and third entries is $(1,1),(1,-1),(-1,1)$ and $(-1,-1)$, respectively. By permuting the columns, we have the following visual representation of the normalized Hadamard matrix

$$\left(\begin{array}{cccccccccccccccccccc} 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & - & - & \cdots & - & - & - & - & \cdots & - & - \\ 1 & 1 & \cdots & 1 & 1 & - & - & \cdots & - & - & 1 & 1 & \cdots & 1 & 1 & - & - & \cdots & - & - \\ & & \vdots & & & & & \vdots & & & & & \vdots & & & & & \vdots & & \\ \multicolumn{5}{c}{\underbrace{\hphantom{1\ 1\ \cdots\ 1\ 1}}_{a \text{ columns}}} & \multicolumn{5}{c}{\underbrace{\hphantom{1\ 1\ \cdots\ 1\ 1}}_{b \text{ columns}}} & \multicolumn{5}{c}{\underbrace{\hphantom{1\ 1\ \cdots\ 1\ 1}}_{c \text{ columns}}} & \multicolumn{5}{c}{\underbrace{\hphantom{1\ 1\ \cdots\ 1\ 1}}_{d \text{ columns}}} \end{array}\right).$$

As H is a Hadamard matrix of order n, the rows of H are pairwise orthogonal, and we obtain the following system of equations

$$\begin{aligned} a+b+c+d &= n && \text{since the order of } H \text{ is } n, \\ a+b-c-d &= 0 && \text{as rows one and two are orthogonal,} \\ a-b+c-d &= 0 && \text{as rows one and three are orthogonal,} \\ a-b-c+d &= 0 && \text{as rows two and three are orthogonal.} \end{aligned}$$

This system has the unique solution

$$a = b = c = d = \frac{n}{4}.$$

As each of the variables a, b, c, d is an integer, $n/4$ must be an integer, and so n is divisible by 4.

$\square$

Connecting our previous discussion of equivalence to our current discussion of existence, we present Table 2.1, found in [7], which shows the number of known equivalence classes of Hadamard matrices of order n for $n \leq 40$. As seen in Table 2.1, the number of equivalence classes for larger orders quickly explodes. Therefore, for large orders, determining the number of inequivalent Hadamard matrices is an arduous and computationally intensive task that quickly becomes intractable. However, the classification of Hadamard matrices has progressed greatly over the past several decades.

To give a brief timeline, in 1961, Hall [13] presented the classification of Hadamard matrices of order 16. Just four years later, Hall [14] would also complete the classification of order 20. Nearly 20 years later, in 1981, Ito, Leon, and Longyear [20] published the classification of order 24. This was followed by another decade of research and computational improvements, culminating in Kimura's [29] 1994 classification of order 28. Lastly, it was not until 2013 that Kharaghani and Tayfeh-Rezaie [26] completed the classification of order 32, which remains the largest order classified to date.

Table 2.1: Number of equivalence classes of Hadamard matrices of order $n \leq 40$ [7].

order	inequivalent matrices
1	1
2	1
4	1
8	1
12	1
16	5
20	3
24	60
28	487
32	13710027
36	> 15000000
40	> 366000000000

A natural follow-up question is whether n being a multiple of 4 is not only an existence condition, but also a sufficient condition for the existence of a Hadamard matrix of order n. This question is the subject of a long-standing open problem and is captured by the following famous conjecture.

Conjecture 2.9 (Hadamard Conjecture)**.** There exists a Hadamard matrix of order n if and only if $n = 1,2$ or n is a multiple of four.

Currently, among multiples of 4 less than 1000, the only orders for which no Hadamard matrix is known are 668, 716 and 892. Previously, the smallest unknown order was 428, which was constructed by Kharaghani and Tayfeh-Rezaie [25], making 668 the smallest order for which the Hadamard Conjecture remains unproven.

2.1.3 Special Types of Hadamard Matrices

Having set the stage with equivalence and existence, we now turn to discover some special types of Hadamard matrices. These special families of matrices are of particular interest as their additional structure will be both developed and utilized throughout this work to achieve varying results. In what follows, we accentuate these special types and explore their existence. We begin by introducing the first special type of Hadamard matrix, which is central to our work.

Definition 2.10. H is a *skew-type* Hadamard matrix of order n if $H + H^T = 2I_n$.

Another formulation of the definition states that H is a skew-type Hadamard matrix if $H = A + I_n$, where $A^T = -A$.

Example 2.11. The following are skew-type Hadamard matrices of order 4 and 8.

$$H_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ - & 1 & - & 1 \\ - & 1 & 1 & - \\ - & - & 1 & 1 \end{pmatrix}, \; H_8 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ - & 1 & - & - & 1 & - & 1 & 1 \\ - & 1 & 1 & - & - & 1 & - & 1 \\ - & 1 & 1 & 1 & - & - & 1 & - \\ - & - & 1 & 1 & 1 & - & - & 1 \\ - & 1 & - & 1 & 1 & 1 & - & - \\ - & - & 1 & - & 1 & 1 & 1 & - \\ - & - & - & 1 & - & 1 & 1 & 1 \end{pmatrix}.$$

As with all Hadamard matrices, skew-type Hadamard matrices can only exist when $n = 1, 2$ or n is divisible by four. There are no additional existence conditions that are known, and like ordinary Hadamard matrices, skew-type Hadamard matrices are conjectured to exist for all multiples of four.

Conjecture 2.12. There exists a skew-type Hadamard matrix of order $4k$ for all $k \geq 1$.

Until very recently, Conjecture 2.12 was verified for all $n < 276$. However, in 2024 Đoković [34] found skew-type Hadamard matrices of orders 276, 292, and 372, leaving 21 orders less than 1000 open, with the smallest being 356.

As with Hadamard matrices, significant effort has been put into classifying skew-type Hadamard matrices. Table 2.2 shows the number of known equivalence classes of skew-type Hadamard matrices in terms of our standard definition of equivalence and SH-equivalence. The most recent classification appeared in 2024, which showed that there are at least 157132 SH-inequivalent skew-type Hadamard matrices of order 36 [3].

This classification is helpful for assessing the difficulty of classifying Hadamard matrices of order 36. In 2020, it was shown that there are only 7227 SH-inequivalent skew-type Hadamard matrices of order 32 [16]. Referring back to Table 2.1, there are approximately 1900 times more Hadamard matrices than skew-type Hadamard matrices of order 32. Extrapolating from this ratio, along with the 157132 SH-inequivalent skew-type Hadamard matrices, we estimate the number of inequivalent Hadamard matrices of order 36 is around 200 million.

Table 2.2: Number of equivalence and SH-equivalence classes of skew-type Hadamard matrices [16].

order	# inequivalent matrices	# SH-inequivalent matrices
1	1	1
2	1	1
4	1	1
8	1	1
12	1	1
16	2	2
20	2	2
24	16	16
28	54	64
32	6662	7227
36	–	157132†

†Search is not complete, however, computation suggests that there are no additional equivalence classes [26].

The following result gives a standard presentation of skew-type Hadamard matrices.

Lemma 2.13. *Let $H = (h_{i,j})$ be a skew-type Hadamard matrix of order n. Then H is equivalent to a skew-type Hadamard matrix of the form*

$$\begin{pmatrix} 1 & j_{n-1}^T \\ -j_{n-1} & I_{n-1}+Q \end{pmatrix},$$

where $Q^T = -Q$ and $Qj_{n-1} = 0$.

Proof. Let $D = \operatorname{diag}([1, h_{1,2}, h_{1,3}, \ldots, h_{1,n}])$. Note that negating both the i-th row and i-th column of a skew-type matrix gives another skew-type matrix. Then $K = DHD$ is a skew-type Hadamard matrix of the form

$$K = \begin{pmatrix} 1 & j_{n-1}^T \\ -j_{n-1} & A \end{pmatrix}.$$

Since K is a skew-type Hadamard matrix, this implies that A must have 1s on the diagonal and $A + A^T = 2I_{n-1}$. Therefore, we can write $A = I_{n-1} + Q$, where $Q^T = -Q$ and Q has zero diagonal.

To show that $Qj_{n-1} = 0$, consider the inner product of the first row of K with any other row. Since the rows are orthogonal, we have

$$r_1 \cdot r_i = 1(-1) + \sum_{k=1}^{n-1} (I_{n-1} + Q)_{i-1,k} = 0.$$

The contribution from the identity matrix cancels the -1 from the first column. The remaining terms are the sum of the i-th row of Q, which are zero. □

Next, we introduce the second special type of Hadamard matrix central to our work.

Definition 2.14. H is a *regular* Hadamard matrix of order n if all row sums are constant.

The following is a key property of regular Hadamard matrices.

Proposition 2.15. *If H is a regular Hadamard matrix of order n, then the absolute value of every row and column sum must be $\sqrt{n}$.*

Proof. Let r_i denote the row sum of the i-th row of H. Since j_n is the all-ones column vector, $Hj_n = (r_1, \ldots, r_n)^T$ is the vector of row sums of H. Observe that

$$(Hj_n)^T(Hj_n) = j_n^T H^T H j_n = j_n^T n I_n j_n = n^2.$$

Hence

$$\sum_{i=1}^{n} r_i^2 = n^2,$$

and as H is regular, each $(r_i)^2 = n$, and so $|r_i| = \sqrt{n}$. To see that the column sums of H are also $\pm\sqrt{n}$, repeat the same argument, using H^T in place of H. □

The following is an immediate consequence of Proposition 2.15.

Corollary 2.16. *If there exists a regular Hadamard matrix of order n, then n is a square.*

Proof. By Proposition 2.15, every row sum has absolute value $\sqrt{n}$. Since row sums must be integers, n must be a square. □

Example 2.17. The following is a regular Hadamard matrix of order 16. Note that all row and column sums are four.

$$H = \begin{pmatrix}
1 & - & - & - & - & 1 & 1 & 1 & - & 1 & 1 & 1 & - & 1 & 1 & 1 \\
- & 1 & - & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & 1 & - & 1 & 1 \\
- & - & 1 & - & 1 & 1 & - & 1 & 1 & 1 & - & 1 & 1 & 1 & - & 1 \\
- & - & - & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & - & 1 & 1 & 1 & - \\
- & 1 & 1 & 1 & 1 & - & - & - & - & 1 & 1 & 1 & - & 1 & 1 & 1 \\
1 & - & 1 & 1 & - & 1 & - & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 \\
1 & 1 & - & 1 & - & - & 1 & - & 1 & 1 & - & 1 & 1 & 1 & - & 1 \\
1 & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & - \\
- & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & - & - & - & - & 1 & 1 & 1 \\
1 & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & - & - & 1 & - & 1 & 1 \\
1 & 1 & - & 1 & 1 & 1 & - & 1 & - & - & 1 & - & 1 & 1 & - & 1 \\
1 & 1 & 1 & - & 1 & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & - \\
- & 1 & 1 & 1 & - & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & - & - & - \\
1 & - & 1 & 1 & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & - & - \\
1 & 1 & - & 1 & 1 & 1 & - & 1 & 1 & 1 & - & 1 & - & - & 1 & - \\
1 & 1 & 1 & - & 1 & 1 & 1 & - & 1 & 1 & 1 & - & - & - & - & 1
\end{pmatrix}.$$

Due to the strict requirement of square orders, the orders for which regular Hadamard matrices are believed to exist are much sparser.

Conjecture 2.18 ([45])**.** There exists a regular Hadamard matrix of order $4k^2$ for all $k \geq 1$.

Conjecture 2.18 has been verified for all $k < 103$, with slightly more than 100 values of $k < 1000$ for which the existence of a regular Hadamard matrix of order $4k^2$ is still unknown [2].

The following definition is a crucial precursor to future matrix constructions.

Definition 2.19. A *circulant* matrix is a matrix where each row is cyclically shifted relative to the row directly above. A circulant matrix is fully determined by its first row, as shown below.

$$\text{circ}\begin{pmatrix} 1 & 2 & \cdots & n \end{pmatrix} = \begin{pmatrix} 1 & 2 & \cdots & n \\ n & 1 & \cdots & n-1 \\ \vdots & \vdots & \ddots & \vdots \\ 2 & 3 & \cdots & 1 \end{pmatrix}.$$

The structure of circulant matrices is highly useful and will be explored further in upcoming sections. The definition of circulant matrices allows us to introduce the last special type of Hadamard matrix that is related to our work.

Definition 2.20. A *circulant* Hadamard matrix is a Hadamard matrix such that each row is rotated one element to the right relative to the row directly above.

Circulant Hadamard matrices are very rare. It is conjectured that there are only two examples of circulant Hadamard matrices, which are given below.

$$\text{circ}(1) = (1), \ \text{circ}\begin{pmatrix} - & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} - & 1 & 1 & 1 \\ 1 & - & 1 & 1 \\ 1 & 1 & - & 1 \\ 1 & 1 & 1 & - \end{pmatrix}.$$

Conjecture 2.21 (Ryser's Conjecture, [38])**.** There does not exist a circulant Hadamard matrix of order $n > 4$.

Ryser's conjecture remains unresolved, despite numerous incorrect proofs appearing in the literature within the past few years.

The list of special types of Hadamard matrices discussed here is not exhaustive. Notably, we omit the introductions of Bush-type and symmetric Hadamard matrices. For information about all the different special types of Hadamard matrices and their corresponding existence results, we refer the reader to [7]. Much of our upcoming work deals with skew-type and regular Hadamard matrices. In Chapter 3 we will combine the definitions of skew-type and regular Hadamard matrices to develop a new special type of Hadamard matrices.

2.1.4 Constructions of Hadamard Matrices

In 1867, Sylvester [41] discovered the first construction of Hadamard matrices. We now showcase how Sylvester created the first infinite class of Hadamard matrices.

Theorem 2.22 (Sylvester, [41]). *If H is a Hadamard matrix of order n, then*

$$\begin{pmatrix} H & H \\ H & -H \end{pmatrix}$$

is a Hadamard matrix of order $2n$.

Proof. Consider the following matrix multiplication.

$$\begin{aligned}
\begin{pmatrix} H & H \\ H & -H \end{pmatrix}\begin{pmatrix} H & H \\ H & -H \end{pmatrix}^T &= \begin{pmatrix} H & H \\ H & -H \end{pmatrix}\begin{pmatrix} H^T & H^T \\ H^T & -H^T \end{pmatrix} \\
&= \begin{pmatrix} HH^T + HH^T & HH^T - HH^T \\ HH^T - HH^T & HH^T + HH^T \end{pmatrix} \\
&= \begin{pmatrix} 2nI_n & 0 \\ 0 & 2nI_n \end{pmatrix} \\
&= 2nI_{2n}.
\end{aligned}$$

□

Using Theorem 2.22, Sylvester was able to generate an infinite class of Hadamard matrices. The matrices arising from the following corollary are called the *Sylvester Hadamard matrices*.

Corollary 2.23 (Sylvester, [41])**.** *There exists a Hadamard matrix of order* 2^k *for any integer* $k \geq 0$.

Proof. Recall that $H = \begin{pmatrix} 1 \end{pmatrix}$ is a Hadamard matrix. To construct a Hadamard matrix of order 2^k, iteratively apply Theorem 2.22 k times to H. □

Jacques Hadamard would go on to generalize Sylvester's result in 1893, motivated by solving the maximal determinant problem. The maximal determinant problem commonly refers to finding the largest determinant of a matrix with elements in $\{1, -1\}$. In his 1893 paper, Hadamard [12] presented his determinant bound, commonly referred to as Hadamard's inequality.

Theorem 2.24 (Hadamard's Determinant Inequality, [12])**.** *Let* M *be an* $n \times n$ *matrix with rows* r_i*, then*

$$|\det(M)| \leq \prod_{i=1}^{n} \|r_i\|.$$

The following is an immediate consequence of Hadamard's inequality.

Corollary 2.25 (Hadamard, [12])**.** *Let* M *be an* $n \times n$ *matrix with entries in* ± 1. *Then*

$$|\det(M)| \leq n^{n/2}. \tag{2.1}$$

Moreover, if M *is a matrix whose entries have absolute value at most* B*, then*

$$|\det(M)| \leq B^n n^{n/2}.$$

Hadamard was interested in matrices with complex entries and showed that the Vandermonde matrix of roots of unity achieves the maximum determinant. Hadamard's work showed that (2.1) attains equality if and only if the rows are orthogonal. Matrices achieving this bound are now called Hadamard matrices. The Sylvester Hadamard matrices were the first examples of matrices that attained this bound; however, in the same article, Hadamard also produced Hadamard matrices of order 12 and 20 and generalized Sylvester's construction. To describe Hadamard's generalized construction, we demonstrate the modern formulation of his result using the Kronecker product. In his 1893 article, Hadamard did not present his generalized construction in terms of Kronecker products. However, using the Kronecker product, Hadamard's result can be established more easily, while we also introduce an important tool which will be used throughout this work.

Definition 2.26. Let $A = (a_{ij})$ be an $n \times m$ matrix and let $B = (b_{ij})$ be a $p \times q$ matrix. The Kronecker product $A \otimes B$ is the $np \times mq$ matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nm}B \end{pmatrix}.$$

The following generalizes Sylvester's result and provides many new orders of Hadamard matrices.

Theorem 2.27 (Hadamard, [12])**.** *If H is a Hadamard matrix of order n and K is a Hadamard matrix of order k, then $H \otimes K$ is a Hadamard matrix of order nk.*

Proof. A careful calculation shows

$$(H \otimes K)(H \otimes K)^T = (HH^T) \otimes (KK^T) = (nI_n) \otimes (kI_k) = nkI_{nk}.$$

□

Example 2.28. In Example 2.3 H_2 is a Sylvester Hadamard matrix of order two. The following demonstrates the construction of a Sylvester Hadamard matrix of order four.

$$H_2 \otimes H_2 = \begin{pmatrix} 1 & 1 \\ 1 & - \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & - \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & - & 1 & - \\ 1 & 1 & - & - \\ 1 & - & - & 1 \end{pmatrix}$$

This construction has an important consequence: once a single Hadamard matrix is known, we can use the Kronecker product with itself and other Hadamard matrices to obtain many further (and potentially new) Hadamard matrices. With this method, given Hadamard matrices of order $4a$ and $4b$, we can form a Hadamard matrix of order $16ab$. The following result alternatively allows us to construct a Hadamard matrix of order $8ab$.

Theorem 2.29 ([1]). *If H is a Hadamard matrix of order $4a$ and K is a Hadamard matrix of order $4b$, then there exists a Hadamard matrix of order $8ab$.*

Proof. We can write H and K as

$$H = \begin{pmatrix} H_1 & H_2 \end{pmatrix} \text{ and } K = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix},$$

where H_1, H_2 are matrices of size $4a \times 2a$ and K_1, K_2 are matrices of size $2b \times 4b$. Then we show

$$\frac{1}{2}(H_1 + H_2) \otimes K_1 + \frac{1}{2}(H_1 - H_2) \otimes K_2 \tag{2.2}$$

is a Hadamard matrix of order $8ab$. First, consider that if an entry in $H_1 + H_2$ is 2 or -2, then the corresponding entry in $H_1 - H_2$ is zero. Similarly, if an entry in $H_1 + H_2$ is zero, then the corresponding entry in $H_1 - H_2$ is 2 or -2. Hence, all entries in the matrix defined in (2.2) are 1 or -1.

Orthogonality follows from the following computation.

$$
\begin{aligned}
&\left(\tfrac{1}{2}(H_1+H_2)\otimes K_1+\tfrac{1}{2}(H_1-H_2)\otimes K_2\right)\left(\tfrac{1}{2}(H_1+H_2)\otimes K_1+\tfrac{1}{2}(H_1-H_2)\otimes K_2\right)^T\\
&=\left(\tfrac{1}{2}(H_1+H_2)\otimes K_1\right)\left(\tfrac{1}{2}(H_1+H_2)\otimes K_1\right)^T+\left(\tfrac{1}{2}(H_1-H_2)\otimes K_2\right)\left(\tfrac{1}{2}(H_1-H_2)\otimes K_2\right)^T\\
&\quad+\left(\tfrac{1}{2}(H_1+H_2)\otimes K_1\right)\left(\tfrac{1}{2}(H_1-H_2)\otimes K_2\right)^T+\left(\tfrac{1}{2}(H_1-H_2)\otimes K_2\right)\left(\tfrac{1}{2}(H_1+H_2)\otimes K_1\right)^T\\
&=\left(\tfrac{1}{4}(H_1+H_2)(H_1+H_2)^T\right)\otimes K_1K_1^T+\left(\tfrac{1}{4}(H_1-H_2)(H_1-H_2)^T\right)\otimes K_2K_2^T\\
&\quad+\left(\tfrac{1}{4}(H_1+H_2)(H_1-H_2)^T\right)\otimes K_1K_2^T+\left(\tfrac{1}{4}(H_1-H_2)(H_1+H_2)^T\right)\otimes K_2K_1^T\\
&=\left(\tfrac{1}{4}(H_1+H_2)(H_1+H_2)^T\right)\otimes 4bI_{2b}+\left(\tfrac{1}{4}(H_1-H_2)(H_1-H_2)^T\right)\otimes 4bI_{2b}\\
&\quad+\left(\tfrac{1}{4}(H_1+H_2)(H_1-H_2)^T\right)\otimes 0_{2b}+\left(\tfrac{1}{4}(H_1-H_2)(H_1+H_2)^T\right)\otimes 0_{2b}\\
&=\left[\left(\tfrac{1}{4}(H_1+H_2)(H_1+H_2)^T\right)+\left(\tfrac{1}{4}(H_1-H_2)(H_1-H_2)^T\right)\right]\otimes 4bI_{2b}\\
&=\left[\tfrac{1}{2}\left(H_1H_1^T+H_2H_2^T\right)\right]\otimes 4bI_{2b}\\
&=\tfrac{1}{2}(4aI_{4a})\otimes 4bI_{2b}\\
&=8abI_{8ab}.
\end{aligned}
$$

□

In 1992, Craigen, Seberry and Zhang [8] published a construction that extends Theorem 2.29 by using four Hadamard matrices. The proof of the construction can be found in [8].

Theorem 2.30 ([8])**.** *If there exist Hadamard matrices of order* $4a$, $4b$, $4c$ *and* $4d$*, then there exists a Hadamard matrix of order* $16abcd$.

Unfortunately, Theorems 2.27, 2.29, and 2.30 have inherent limitations; both theorems are unable to produce Hadamard matrices of order $4n$, where n is odd. In 1944, Williamson introduced a new construction for Hadamard matrices using circulant matrices, producing

Hadamard matrices of order $4n$ for odd and even n.

Theorem 2.31 (Williamson, [46])**.** *Suppose A,B,C and D are symmetric circulant matrices of order n such that*

$$A^2+B^2+C^2+D^2=4nI_n.$$

Then

$$\begin{pmatrix} A & -B & -C & -D \\ B & A & -D & C \\ C & D & A & -B \\ D & -C & B & A \end{pmatrix}$$

is a Hadamard matrix of order $4n$*.*

Proof. The proof follows from multiplying the above matrix by its transpose and utilizing the fact that A,B,C and D are commutative and symmetric. □

Years later, Williamson's array was extended by Goethals and Seidel by removing the requirement that A,B,C, and D had to be symmetric. However, before we showcase this construction, we must first explore some properties of circulant matrices.

Consider the cyclic shift matrix

$$U=\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Any circulant matrix $X=\mathrm{circ}(x_1,x_2,\dots,x_n)$ can be represented as a polynomial in U:

$$X=\sum_{k=1}^{n} x_k U^k.$$

Moreover, since all circulant matrices are generated by U, this implies that any two circulant matrices are commutative. Using the polynomial representation of circulant matrices, we can also prove the following result.

Lemma 2.32. *If X is a circulant matrix of order n and R is the back identity matrix of order n, then $XR = RX^T$.*

Proof. Notice that $UR = RU^T$. Using this, the proof follows from the following calculation.

$$\begin{aligned}
XR &= \left(\sum_{k=1}^{n} x_k U^k\right) R \\
&= \sum_{k=1}^{n} x_k U^k R \\
&= \sum_{k=1}^{n} x_k R (U^k)^T \\
&= R\left(\sum_{k=1}^{n} x_k (U^k)^T\right) \\
&= R\left(\sum_{k=1}^{n} x_k U^k\right)^T \\
&= RX^T.
\end{aligned}$$

□

Theorem 2.33 (Goethals-Seidel, [11])**.** *Suppose A, B, C and D are circulant matrices of order n such that*

$$AA^T + BB^T + CC^T + DD^T = 4nI_n$$

and let R be the back identity matrix of order n. Then

$$H = \begin{pmatrix} A & -BR & -CR & -DR \\ BR & A & -D^T R & C^T R \\ CR & D^T R & A & -B^T R \\ DR & -C^T R & B^T R & A \end{pmatrix}$$

is a Hadamard matrix of order $4n$*. Moreover, if* $A = I_n + Q$ *where* Q *is skew-symmetric, then* H *is a skew-type Hadamard matrix.*

Proof. Note that $R^2 = I_n$, $R^T = R$ and since A, B, C and D are circulant, these matrices commute. Orthogonality follows by applying Lemma 2.32 to simplify the following computation.

$$HH^T = \begin{pmatrix} A & -BR & -CR & -DR \\ BR & A & -D^TR & C^TR \\ CR & D^TR & A & -B^TR \\ DR & -C^TR & B^TR & A \end{pmatrix} \begin{pmatrix} A^T & RB^T & RC^T & RD^T \\ -RB^T & A^T & RD & -RC \\ -RC^T & -RD & A^T & RB \\ -RD^T & RC & -RB & A^T \end{pmatrix}$$

$$= \begin{pmatrix} AA^T + BB^T + CC^T + DD^T & 0 & 0 & 0 \\ 0 & AA^T + BB^T + CC^T + DD^T & 0 & 0 \\ 0 & 0 & AA^T + BB^T + CC^T + DD^T & 0 \\ 0 & 0 & 0 & AA^T + BB^T + CC^T + DD^T \end{pmatrix}$$

$$= \begin{pmatrix} 4nI_n & 0 & 0 & 0 \\ 0 & 4nI_n & 0 & 0 \\ 0 & 0 & 4nI_n & 0 \\ 0 & 0 & 0 & 4nI_n \end{pmatrix}$$

$$= 4nI_{4n}.$$

Finally, since A is skew-type and applying the fact that $XR = RX^T$, it follows that H is a skew-type Hadamard matrix of order $4n$. □

Remark 2.34. The matrices in Theorems 2.31 and 2.33 are sometimes referred to as Williamson and Goethals-Seidel arrays, respectively.

In this section, we have examined constructions that provide Hadamard matrices using existing Hadamard matrices and circulant matrices. In Section 2.4, we will showcase Paley's construction of Hadamard matrices that utilize finite field techniques.

2.1.5 Excess of Hadamard Matrices

We now shift our focus to discover another property that can be used to describe Hadamard matrices.

Definition 2.35. The *excess* of a Hadamard matrix $H = (h_{i,j})$ of order n is defined as

$$\text{excess}(H) = \sum_{i=1}^{n} \sum_{j=1}^{n} h_{i,j}.$$

In plain English, the excess of a Hadamard matrix is the sum of all the entries. Equivalently, excess is equal to the number of 1's minus the number of -1's in a Hadamard matrix. A natural question to ask is: what is the maximum excess of a Hadamard matrix of order n? This question motivates the following definition.

Definition 2.36. Let n be an integer for which a Hadamard matrix of order n exists. Define the *maximum excess* of all Hadamard matrices of order n as

$$\sigma(n) = \max\{\text{excess}(H) : H \text{ is a Hadamard matrix of order } n\}.$$

Apart from considering the maximum excess for a given order for which a Hadamard matrix exists, we can also consider the maximal excess of an equivalence class of a Hadamard matrix.

Definition 2.37. The *maximal excess* of an individual Hadamard matrix H is defined as

$$\sigma(H) = \max\{\text{excess}(K) : K \text{ is equivalent to } H\}.$$

For values of $n < 16$, the maximum excess can be determined using a simple computer program by trying all possible row and column negations and finding the largest possible sum of entries. However, for many orders $n \geq 16$, the problem becomes challenging. The first problem is that naive brute-force computation quickly becomes intractable. The second problem is that as suggested by Definition 2.37, different equivalence classes of Hadamard matrices of a given order may differ in their maximal excess. For orders $n \geq 16$, the number of inequivalent Hadamard matrices rapidly explodes, requiring the computation to be repeated for all inequivalent matrices of that order.

However, there are theoretical results that can help us bound $\sigma(n)$. Before we showcase one of these results, we introduce a famous inequality.

Theorem 2.38 (Cauchy-Schwarz Inequality, [6])**.** *For any real numbers $x_1, \ldots, x_n, y_1, \ldots y_n$ we have*

$$\left(\sum_{k=1}^{n} x_k y_k\right)^2 \leq \left(\sum_{k=1}^{n} x_k^2\right)\left(\sum_{k=1}^{n} y_k^2\right).$$

Moreover, equality occurs if and only if $(x_1, \ldots, x_n) = c(y_1, \ldots, y_n)$ for some nonzero c.

Theorem 2.39 ([5])**.** $\sigma(n) \leq n^{3/2}$.

Proof. Let $H = (h_{i,j})$ be a Hadamard matrix of order n. Let

$$r_i = \sum_{j=1}^{n} h_{i,j}$$

denote the row sum of the i-th row of H. Observe that

$$\text{excess}(H) = \sum_{i=1}^{n}\sum_{j=1}^{n} h_{i,j} = \sum_{i=1}^{n} r_i.$$

Recall from the proof of Proposition 2.15

$$\sum_{i=1}^{n} r_i^2 = n^2. \tag{2.3}$$

Applying the Cauchy-Schwarz Inequality to (2.3) we obtain

$$\frac{\left(\sum_{i=1}^{n} r_i\right)^2}{n} \leq \sum_{i=1}^{n} r_i^2. \tag{2.4}$$

Combining (2.3) and (2.4) we see

$$\frac{\left(\sum_{i=1}^{n} r_i\right)^2}{n} \leq \sum_{i=1}^{n} r_i^2 = n^2.$$

Therefore

$$\sum_{i=1}^{n} r_i \leq n^{3/2}.$$

□

The following theorem shows that Hadamard matrices which attain equality in Theorem 2.39 are equivalent to regular Hadamard matrices.

Theorem 2.40 ([5]). *$\sigma(n) = n^{3/2}$ if and only if there exists a regular Hadamard matrix of order n.*

Proof. Suppose H is a regular Hadamard matrix. Proposition 2.15 shows that the row sums are absolute value $\sqrt{n}$. If the row sums are negative, multiply all rows by -1. With positive row sums, this matrix has excess $n^{3/2}$ as there are n rows with row sum $\sqrt{n}$.

Conversely, suppose H is a Hadamard matrix with excess $n^{3/2}$. This means that (2.4) from Theorem 2.39 attains equality. Hence

$$\left(\sum_{i=1}^{n} r_i\right)^2 = n\sum_{i=1}^{n} r_i^2.$$

Then by Theorem 2.38 it must be the case that $r_1 = r_2 = \cdots = r_n$, and so H is regular. □

For non-regular matrices, finding a theoretical bound for excess is a more challenging problem. Table 2.3 shows the maximum excess achieved, and indicates whether this excess meets the best known theoretical bound. For bounding the maximum excess for non-square orders of Hadamard matrices, we refer the reader to [9].

Table 2.3: Maximum excess for Hadamard matrices of order n for $4 \leq n \leq 56$ [9].

order	4	8	12	16	20	24	28	32	36	40	44	48	52	56
$\sigma(n)$	8	20	36	64	80	112	140	172	216	244	244	324	364	404†

†The theoretical bound is 408 [9]; however, no matrix with that excess is known.

2.2 Linear Error Correcting Codes

In this section, we take a quick detour to review the basic definitions of linear error-correcting codes, followed by a brief discussion of ternary codes, which constitute the only class of codes needed for this work. For a more comprehensive overview, the reader is directed to the standard reference text of Huffman and Pless [18].

Definition 2.41. A *linear code* of *length* n, *dimension* k denoted as an $[n,k]$ code, is a k-dimensional linear subspace C of $\mathbb{F}_q^n$, where the individual vectors in C are called *codewords*. In the case that $q = 2$, the code is called a *binary* code, and if $q = 3$, the code is said to be a *ternary* code.

Linear codes can also be represented using matrices.

Definition 2.42. A *generator matrix* G for an $[n,k]$ linear code C, is a $k \times n$ matrix whose rows are k linearly independent vectors that form a basis for C.

Definition 2.43. Let $x, y \in \mathbb{F}_q^n$. The *Hamming distance* between $x = x_1 \cdots x_n$ and $y = y_1 \cdots y_n$, denoted by $d(x,y)$, is the number of positions i for which $x_i \neq y_i$.

Definition 2.44. The *weight* of a codeword $x \in \mathbb{F}_q^n$, denoted by $\mathrm{wt}(x)$, is the number of non-zero components in x.

Note that we can calculate the weight of a codeword using the fact that $\mathrm{wt}(x) = d(x, \mathbf{0})$, where $\mathbf{0}$ is the all-zero codeword.

Definition 2.45. The *minimum distance* of a linear code C is the smallest Hamming distance between two distinct codewords.

For a linear code C, as $\mathbf{0} \in C$, the set of all possible differences $\{x-y : x, y \in C\}$ generates the same set of vectors as the code C itself. Since the Hamming distance between two codewords x and y is also given by $d(x,y) = \mathrm{wt}(x-y)$, we have an alternate way to calculate the minimum distance of a linear code.

Corollary 2.46. *The minimum distance of a linear code C is also equal to the minimum weight of its non-zero codewords.*

$$d(C) = \min\{wt(c) \mid c \in C, c \neq 0\}.$$

Definition 2.47. The *dual* of a linear code C, denoted by $C^{\perp}$, is defined by

$$C^{\perp} = \{x \in \mathbb{F}_q^n \mid x \cdot c = 0 \text{ for all } c \in C\}.$$

Moreover, a code C is said to be *self-dual* provided that $C = C^{\perp}$.

In other words, $C^{\perp}$ is the orthogonal complement of C with respect to the standard inner product. The following result follows from a well-known theorem from linear algebra.

Theorem 2.48 ([32])**.** *Let q be a prime power and C be an $[n,k]$ linear code over $\mathbb{F}_q$. Then,*

$$\dim(C) + \dim(C^{\perp}) = n.$$

Proof. Let G be a generator matrix for C. By definition, G is a $k \times n$ matrix whose rows form a basis for C. Therefore, the row space of G is exactly the code C, and the rank of G is equal to the dimension of C.

Since the rows of G form a basis for C, a vector x is orthogonal to every vector in C if and only if it is orthogonal to every row of G. This implies that $C^{\perp}$ is exactly the null space (or kernel) of the matrix G.

$$\dim(C^{\perp}) = \text{nullity}(G).$$

By the Rank-Nullity Theorem, for any matrix with n columns, the sum of the rank and the nullity equals the number of columns. Therefore

$$\dim(C) + \dim(C^{\perp}) = \text{rank}(G) + \text{nullity}(G) = n.$$

□

Example 2.49. Consider the following $[3,2]$ linear binary code C with generator matrix

$$G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The code C consists of all possible linear combinations of the two rows of G. Since there are $k = 2$ basis vectors and $q = 2$ coefficients, there are $q^k = 2^2 = 4$ total codewords in C. Thus, the full code is given by $C = \{(0,0,0),(1,0,0),(0,1,0),(1,1,0)\}$. The dual code $C^{\perp}$ consists of all vectors $x = (x_1, x_2, x_3) \in \mathbb{F}_2^3$ that are orthogonal to every codeword in C. Thus, the dual code is given by $C^{\perp} = \{(0,0,0),(0,0,1)\}$.

Next, we introduce the notion of equivalence for linear codes.

Definition 2.50. Two linear codes C_1 and C_2 of length n over $\mathbb{F}_q$ are said to be *equivalent* if there exists an $n \times n$ monomial matrix M over $\mathbb{F}_q$ such that $C_2 = \{cM \mid c \in C_1\}$.

2.2.1 Ternary Codes

We now shift our attention to ternary codes, which arise naturally from the Hadamard matrices introduced earlier in this chapter. Throughout this section, let $C(H)$ denote the ternary code generated by the rows of a Hadamard matrix H, where the entries of the matrix are interpreted as elements of $\mathbb{F}_3$.

Lemma 2.51 ([3]). *Let H be a skew-type Hadamard matrix of order n. If $n \equiv 0 \pmod{12}$, then $C(H)$ is self-dual.*

Proof. Since $n \equiv 0 \pmod{12}$, it follows that $HH^T \equiv 0_n \pmod 3$. This implies $C(H) \subseteq C(H)^{\perp}$. Using Theorem 2.48, we must have $\dim C(H) \leq \frac{n}{2}$. Also, since H is a skew-type Hadamard matrix, we have $H + H^T = 2I_n$. Thus, working over $\mathbb{F}_3$ we obtain

$$n = \text{rank}(2I_n) = \text{rank}(H + H^T) \leq \text{rank}(H) + \text{rank}(H^T) = 2\text{rank}(H).$$

Thus, $\dim C(H) = \text{rank}(H) \geq n/2$, and so $\dim C(H) = \frac{n}{2}$. By Theorem 2.48 $\dim C(H)^{\perp} = \frac{n}{2}$, and so since $C(H) \subseteq C(H)^{\perp}$, we must have $C(H) = C(H)^{\perp}$, which means that $C(H)$ is *self-dual*. □

An interesting problem in coding theory is to find self-dual codes with a large minimum distance d. It was shown in [33] that the minimum weight d of a ternary self-dual code of length n is bounded by

$$d \leq 3\lfloor n/12 \rfloor + 3.$$

The codes that achieve or come close to this minimum distance are given a special name.

Definition 2.52. Let C be an $[n, n/2]$ ternary self-dual code. If $d = 3\lfloor n/12 \rfloor + 3$, then the code C is called *extremal*, and if $d = 3\lfloor n/12 \rfloor$, the code is called *near-extremal*.

2.3 Quaternary Hadamard Matrices

In this section, we will examine our first generalization of Hadamard matrices: quaternary Hadamard matrices. We place particular emphasis on extending definitions from the real to the quaternary case and demonstrating a vital connection between real and quaternary Hadamard matrices.

Hadamard matrices contain entries in $\{\pm 1\}$. The following definition extends this to the set $\{\pm 1, \pm i\}$.

Definition 2.53. A *quaternary Hadamard matrix* of order n is an $n \times n$ matrix H with entries in $\{\pm 1, \pm i\}$ such that $HH^* = nI_n$.

An important distinction between real and quaternary Hadamard matrices is that the rows r_i, r_j are orthogonal with respect to the conjugate inner product: $r_i \cdot \overline{r_j} = 0$.

Example 2.54. The following is a quaternary Hadamard matrix of order 4.

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & - & - \\ 1 & - & i & j \\ 1 & - & j & i \end{pmatrix}.$$

2.3.1 Existence and Equivalence of Quaternary Hadamard Matrices

Many concepts discussed in Section 2.1 can be naturally generalized to quaternary Hadamard matrices. For example, the Kronecker product of two quaternary Hadamard matrices is another quaternary Hadamard matrix.

In addition to permuting rows and columns, multiplying a row or column of a quaternary Hadamard matrix by one of $\{\pm 1, \pm i\}$ results in a quaternary Hadamard matrix. This means that multiplying a quaternary Hadamard matrix by a quaternary permutation matrix (i.e., a square matrix where each row and column has exactly one non-zero entry in $\{\pm 1, \pm i\}$) gives us another quaternary Hadamard matrix. This is the foundation for our definition of equivalence.

Definition 2.55. Let H and K be two quaternary Hadamard matrices of order n. The matrices H and K are said to be *equivalent*, denoted by $H \cong K$, if there exist two $n \times n$ quaternary permutation matrices P and Q such that $H = PKQ$.

Quaternary Hadamard matrices cannot exist for odd orders $n > 1$. However, unlike Hadamard matrices, quaternary Hadamard matrices can exist for all even orders. This has led to the following well-known conjecture.

Conjecture 2.56 (Quaternary Hadamard Conjecture, [44])**.** There exists a quaternary Hadamard matrix of order $2n$ for all integers $n \geq 1$.

Currently, the smallest order for which Conjecture 2.56 is open is $n = 94$ [21].

As with their real counterparts, significant effort has been devoted to classifying quaternary Hadamard matrices. Table 2.4 summarizes the classifications of Quaternary Hadamard matrices. To provide a brief timeline for the data, Szöllősi [42] published an article classifying quaternary Hadamard matrices of orders $n \leq 8$ in 2011. Not long after, Lampio, Östergård, and Szöllősi [30] published classifications of quaternary Hadamard matrices of order 10, 12 and 14. In 2020, the same authors published an article classifying quaternary Hadamard matrices of order 16 [31]. Most recently, in 2021 Östergård and Paavola [35] extended the classification to matrices of order 18. This framework serves as the basis for the investigation presented in Chapter 4.

Table 2.4: Number of equivalence classes of quaternary Hadamard matrices of order $n \leq 18$ [31, 35].

order	inequivalent matrices
1	1
2	1
4	2
6	1
8	15
10	10
12	319
14	752
16	1786763
18	3830723

2.3.2 Special Types of Quaternary Hadamard Matrices

Our discussion will focus on extending the definitions of skew-type and regular Hadamard matrices to the quaternary case. The following closely follows from the real case.

Definition 2.57. H is a *skew-type* quaternary Hadamard matrix of order n if $H + H^* = 2I_n$.

Lemma 2.58. *Let $H = (h_{i,j})$ be a skew-type quaternary Hadamard matrix of order n. Then H is equivalent to a skew-type quaternary Hadamard matrix of the form*

$$\begin{pmatrix} 1 & j_{n-1}^T \\ -j_{n-1} & I_{n-1}+Q \end{pmatrix},$$

where $Q^* = -Q$ and $Qj_{n-1} = 0$.

Proof. Let $D = \text{diag}([1, h_{1,2}, h_{1,3}, \ldots, h_{1,n}])$. Then $K = DHD^*$ is the matrix with the desired form, with the remainder of the proof following Lemma 2.13. □

Definition 2.59. H is a *regular* quaternary Hadamard matrix of order n if all row sums are constant.

The definition of a regular quaternary Hadamard matrix is identical to the definition of a regular Hadamard matrix. Following the proof of Proposition 2.15, we conclude that the square of the magnitude of the row sums must be n. However, since the row sums are of the form $r_i = a + ib$, it must be the case that $a^2 + b^2 = n$. This establishes the following result.

Corollary 2.60. *If there exists a regular quaternary Hadamard matrix of order n, then n is the sum of two integer squares.*

The following establishes that a regular quaternary Hadamard matrix also has constant column sums.

Proposition 2.61. *Let H be a regular quaternary Hadamard matrix of order n with constant row sum s. Then the column sums of H are also constant and equal to s.*

Proof. Since H is regular with row sum s,

$$Hj_n = sj_n.$$

Then, multiplying both sides by H^* we obtain

$$H^*(Hj_n) = H^*(sj_n),$$

and so

$$nj_n = s(H^*j_n).$$

Following Proposition 2.15, the row sum s must satisfy $|s|^2 = s\bar{s} = n$. Thus

$$H^*j_n = \frac{n}{s}j_n = \frac{s\bar{s}}{s}j_n = \bar{s}j_n.$$

This implies that the row sums of H^* are $\bar{s}$, which is equivalent to saying the column sums of H are s. □

2.3.3 A Connection to Hadamard Matrices

The following theorem shows an important connection between real and quaternary Hadamard matrices.

Theorem 2.62. *If H is a Quaternary Hadamard matrix of order n and $R = Re(H)$ and $I = Im(H)$, then*

$$K = \begin{pmatrix} -R+I & R+I \\ R+I & R-I \end{pmatrix}$$

is a Hadamard matrix of order $2n$.

Proof. It is clear that K has entries in $\{\pm 1\}$. By expanding $HH^* = nI_n$ we obtain

$$HH^* = (R+iI)(R^T - iI^T) = (RR^T + II^T) + i(IR^T - RI^T) = nI_n,$$

so

$$RR^T + II^T = nI_n \quad \text{and} \quad RI^T = IR^T.$$

Then

$$\begin{aligned}
KK^T &= \begin{pmatrix} -R+I & R+I \\ R+I & R-I \end{pmatrix} \begin{pmatrix} -R^T+I^T & R^T+I^T \\ R^T+I^T & R^T-I^T \end{pmatrix} \\
&= \begin{pmatrix} 2(RR^T+II^T) & -2(RI^T-IR^T) \\ 2(RI^T-IR^T) & 2(RR^T+II^T) \end{pmatrix} \\
&= \begin{pmatrix} 2nI_n & 0 \\ 0 & 2nI_n \end{pmatrix} \\
&= 2nI_{2n}.
\end{aligned}$$

□

Note that we can express the construction in Theorem 2.62 using Kronecker products as follows.

$$K = \begin{pmatrix} - & 1 \\ 1 & 1 \end{pmatrix} \otimes \text{Re}(H) + \begin{pmatrix} 1 & 1 \\ 1 & - \end{pmatrix} \otimes \text{Im}(H).$$

2.4 Weighing Matrices

In this section, we introduce another generalization of Hadamard matrices. Retaining their orthogonality, Hadamard matrices can be generalized to include zeroes. The following definition formalizes this extension by allowing entries in $\{0, \pm 1\}$.

Definition 2.63. A *weighing matrix* of order n and weight k is an $n \times n$ square matrix W with entries in $\{0, 1, -1\}$, such that $WW^T = kI_n$, denoted $W(n,k)$.

If W is a $W(n,k)$, then from straightforward matrix multiplication, we can verify that $W^TW = kI_n$. Therefore, there are k non-zero entries in every row and column of W.

As mentioned, weighing matrices are a generalization of Hadamard matrices. If W is a $W(n,n)$, then $WW^T = nI_n$, and so W is a Hadamard matrix. Another special case of weighing matrices is when $k = n-1$. A weighing matrix $W(n, n-1)$ is called a *conference matrix* of order n.

Remark 2.64. If $H = A + I_n$, then H is a skew-type Hadamard matrix of order n if and only if A is a skew-symmetric conference matrix of order n.

Example 2.65. Consider the matrices

$$W_1 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & - & - & 1 \\ 1 & 1 & 0 & 1 & - & - \\ 1 & - & 1 & 0 & 1 & - \\ 1 & - & - & 1 & 0 & 1 \\ 1 & 1 & - & - & 1 & 0 \end{pmatrix}, \; W_2 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ - & 0 & - & - & 1 & - & 1 & 1 \\ - & 1 & 0 & - & - & 1 & - & 1 \\ - & 1 & 1 & 0 & - & - & 1 & - \\ - & - & 1 & 1 & 0 & - & - & 1 \\ - & 1 & - & 1 & 1 & 0 & - & - \\ - & - & 1 & - & 1 & 1 & 0 & - \\ - & - & - & 1 & - & 1 & 1 & 0 \end{pmatrix}.$$

W_1 is a symmetric conference matrix $W(6,5)$ and W_2 is a skew conference matrix $W(8,7)$.

Unlike Hadamard matrices, the restrictions on the orders of weighing matrices are less strict. For example, the matrices from Example 2.65 are of order 6 and 7. The following is a conjecture about the existence of weighing matrices for specific orders.

Conjecture 2.66 ([7])**.** If n is a multiple of four, then there exists a weighing matrix $W(n,k)$ for all $k \leq n$.

In the case that $k = n$, Conjecture 2.66 is the Hadamard conjecture. This conjecture naturally generalizes the Hadamard conjecture, extending the statement to assert that a weighing matrix of any order that is a multiple of four can exist for any weight less than or equal to the order.

2.4.1 Paley Construction

In Section 2.1, we showcased Sylvester's construction for Hadamard matrices of order 2^n, and extensions of Sylvester's construction using the Kronecker product. The weakness of these constructions is that they only give Hadamard matrices of order $4n$ for even n. At the end of Section 2.1, we also covered constructions using the Williamson and Goethals-Seidel arrays, which are able to produce Hadamard matrices of order $4n$ for odd n. However, the Hadamard matrices coming from these arrays require us to search for four matrices that can be used within the arrays. But, in 1933, seven years before Williamson published his construction, English mathematician Raymond Paley [36] discovered a connection between field theory and Hadamard matrices. This connection to field theory allowed Paley to construct an infinite class of Hadamard matrices of order $4n$ for odd n. We begin by presenting preliminary results from field theory, followed by Paley's construction.

Definition 2.67. Suppose q is an odd prime power. Let $\mathbb{F}_q$ denote the finite field of order q, and suppose that $x \in \mathbb{F}_q \setminus \{0\}$. We say that x is a *quadratic residue* if there exists an $a \in \mathbb{F}_q$ such that $a \equiv x^2 \pmod{q}$, and x is a *quadratic non-residue* if no such a exists. Finally, define $\mathrm{QR}(q)$ to be the set containing the quadratic residues of $\mathbb{F}_q$ and $\mathrm{QN}(q)$ to be the set containing quadratic non-residues of $\mathbb{F}_q$.

In other words, $x \in \mathbb{F}_q \setminus \{0\}$ is a quadratic residue if it is a square in $\mathbb{F}_q$. Note that the element 0 is neither a quadratic residue nor a quadratic non-residue.

Definition 2.68. Let q be an odd prime power and let $\mathbb{F}_q$ denote the finite field of order q. The *quadratic character* is a function $\chi_q : \mathbb{F}_q \to \{-1, 0, 1\}$ defined by

$$\chi_q(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \in \mathrm{QR}(q) \\ -1 & \text{if } x \in \mathrm{QN}(q) \end{cases}$$

for $x \in \mathbb{F}_q$.

The next proposition unveils the number of elements in $\mathrm{QR}(q)$ and $\mathrm{QN}(q)$.

Proposition 2.69. *Let q be an odd prime power. Then there are $(q-1)/2$ quadratic residues and $(q-1)/2$ quadratic non-residues in $\mathbb{F}_q$.*

Proof. Define the map $\phi : \mathbb{F}_q \setminus \{0\} \mapsto \mathbb{F}_q \setminus \{0\}$ defined by $\phi(x) = x^2$ for $x \in \mathbb{F}_q \setminus \{0\}$. Note that $\phi(\mathbb{F}_q \setminus \{0\}) = \mathrm{QR}(q)$. Observe $\phi(x)$ is a two-to-one mapping since $(x)^2 = (-x)^2$ for any $x \in \mathbb{F}_q$. Thus, ϕ can be shown to be a group homomorphism with $|\ker\phi| = 2$. By the First Isomorphism Theorem we have

$$\phi(\mathbb{F}_q \setminus \{0\}) \cong (\mathbb{F}_q \setminus \{0\})/\ker\phi,$$

so

$$|\phi(\mathbb{F}_q \setminus \{0\})| = \frac{|\mathbb{F}_q \setminus \{0\}|}{|\ker\phi|} = \frac{q-1}{2}.$$

Hence $|\mathrm{QR}(q)| = (q-1)/2$. Since there are $q-1$ elements in $\mathbb{F}_q \setminus \{0\}$, and every element is either a quadratic residue or a quadratic non-residue, it follows that $|\mathrm{QN}(q)| = (q-1)/2$. □

We will make use of the well-known result from field theory that the multiplicative group $(\mathbb{F}_q \setminus \{0\}, \times)$ is a cyclic group. A generator of this group is said to be a *primitive element* of the field $\mathbb{F}_q$.

The following lemma shows two useful properties of the quadratic character.

Lemma 2.70. *Let q be an odd prime power. Then*

$$\sum_{x \in \mathbb{F}_q} \chi_q(x) = 0.$$

Proof. By Proposition 2.69

$$\sum_{x \in \mathbb{F}_q} \chi_q(x) = |\mathrm{QR}(q)| - |\mathrm{QN}(q)| + \chi_q(0) = \frac{q-1}{2} - \frac{q-1}{2} + 0 = 0.$$

□

Lemma 2.71. *Let q be an odd prime power. If $y \neq 0$ then*

$$\sum_{x \in \mathbb{F}_q} \chi_q(x)\chi_q(x+y) = -1.$$

Proof. Notice that if $x = 0$ then $\chi_q(0)\chi_q(0+y) = 0$. If $x \neq 0$, since $\mathbb{F}_q$ is a field, there exists a unique $z \in \mathbb{F}_q$ such that $x + y = xz$, and since $y \neq 0$, it follows that $z \neq 1$. As x takes on all values in $\mathbb{F}_q$ except for 0, then z will take on all values in $\mathbb{F}_q$ except for 1 since

$$z = \frac{x+y}{x} = 1 + \frac{y}{x} \text{ and } y \neq 0.$$

Then using Lemma 2.70

$$\begin{aligned}
\sum_{x \in \mathbb{F}_q} \chi_q(x)\chi_q(x+y) &= \sum_{x \in \mathbb{F}_q} \chi_q(x)\chi_q(xz) \\
&= \sum_{x \in \mathbb{F}_q} (\chi_q(x))^2 \chi_q(z) \\
&= \sum_{z \in \mathbb{F}_q \setminus \{1\}} \chi_q(z) \\
&= 0 - \chi_q(1) \\
&= -1.
\end{aligned}$$

□

Definition 2.72. Let q be an odd prime power. Define the *Paley matrix* $P = (p_{i,j})$ to be the $q \times q$ matrix in which the rows and columns are indexed by $\mathbb{F}_q$, such that

$$p_{i,j} = \chi_q(i-j)$$

for $i, j \in \mathbb{F}_q$.

The following is an immediate consequence of the previous two lemmas.

Corollary 2.73 (Paley, [36]). *If P is a Paley matrix of order q, then*

1. $PJ_q = J_qP = 0$, *and*

2. $PP^T = qI_q - J_q$.

Proof. Part (1) follows from Lemma 2.70. To verify part (2), observe that the i,j entry of PP^T is given by

$$\sum_{k\in\mathbb{F}_q} \chi_q(i-k)\chi_q(j-k).$$

If $i = j$ then

$$\sum_{k\in\mathbb{F}_q} \chi_q(i-k)\chi_q(i-k) = \sum_{k\in\mathbb{F}_q} (\chi_q(i-k))^2 = q-1.$$

If $i \neq j$, let $x = j-k$ and $y = i-j$. Since $y \neq 0$ we can apply Lemma 2.71 to obtain

$$\begin{aligned}\sum_{k\in\mathbb{F}_q} \chi_q(i-k)\chi_q(j-k) &= \sum_{x\in\mathbb{F}_q} \chi_q(i-j+x)\chi_q(x)\\ &= \sum_{x\in\mathbb{F}_q} \chi_q(y+x)\chi_q(x)\\ &= -1.\end{aligned}$$

□

Corollary 2.74 (Paley, [36]). *Let q be an odd prime power and let P be a Paley matrix of order q. If $q \equiv 1 \pmod 4$ then P is symmetric, and if $q \equiv -1 \pmod 4$ then P is skew-symmetric.*

Proof. Notice that

$$\chi_q(i-j) = \chi_q(-1)\chi_q(j-i).$$

Suppose $q \equiv 1 \pmod 4$. For a primitive element α we have

$$-1 = \alpha^{(q-1)/2} = \alpha^{(4k+1-1)/2} = (\alpha^k)^2$$

and so $\chi_q(-1) = 1$. Similarly, if $q \equiv -1 \pmod 4$ then $\chi_q(-1) = -1$. □

We now present Paley's method of constructing Hadamard matrices using conference matrices.

Theorem 2.75 (Paley, [36]). *Let q be an odd prime power. If $q \equiv 1 \pmod 4$, then there exists a symmetric conference matrix of order $q+1$. If $q \equiv -1 \pmod 4$, then there exists a skew-symmetric conference matrix of order $q+1$.*

Proof. If $q \equiv 1 \pmod 4$, define

$$W_1 = \begin{pmatrix} 0 & j_q^T \\ j_q & P \end{pmatrix}.$$

If $q \equiv -1 \pmod 4$, define

$$W_2 = \begin{pmatrix} 0 & j_q^T \\ -j_q & P \end{pmatrix}.$$

By the definition of the Paley matrix, it follows that W_1 and W_2 have weight q. By Corollary 2.74, W_1 is symmetric and W_2 is skew-symmetric. By part (2) of Corollary 2.73, the rows of W_1 and W_2 are orthogonal. □

Theorem 2.76 (Paley, [36]). *Let q be a prime power.*

1. *If $q \equiv 1 \pmod 4$ then there is a symmetric Hadamard matrix of order $2q+2$.*

2. *If $q \equiv -1 \pmod 4$, then there is a skew-type Hadamard matrix of order $q+1$.*

Proof. (1) If $q \equiv 1 \pmod 4$ then by Theorem 2.75, we can construct a symmetric conference matrix of order $q+1$, say W_1. Then define

$$H = \begin{pmatrix} W_1 + I_{q+1} & W_1 - I_{q+1} \\ W_1 - I_{q+1} & -W_1 - I_{q+1} \end{pmatrix}.$$

Since W_1 has a zero diagonal, H has entries in $\{\pm 1\}$. Similarly, since W_1 is symmetric, it follows that H is symmetric. To show the multiplication of HH^T, we make some simplifications. First, observe that

$$\begin{aligned}(W_1+I_{q+1})^2+(W_1-I_{q+1})^2 &= 2W_1^2+2I_{q+1}^2\\ &= 2qI_{q+1}+2I_{q+1}\\ &= (2q+2)I_{q+1}.\end{aligned}$$

Similarly, since $(W_1+I_{q+1})^2=(-W_1-I_{q+1})^2$ we have $(-W_1-I_{q+1})^2+(W_1-I_{q+1})^2 = 2(q+1)I_{q+1}$. Using these identities, we have

$$\begin{aligned}HH^T &= \begin{pmatrix} W_1+I_{q+1} & W_1-I_{q+1}\\ W_1-I_{q+1} & -W_1-I_{q+1}\end{pmatrix}\begin{pmatrix} W_1+I_{q+1} & W_1-I_{q+1}\\ W_1-I_{q+1} & -W_1-I_{q+1}\end{pmatrix}\\ &= \begin{pmatrix}(2q+2)I_{q+1} & 0\\ 0 & (2q+2)I_{q+1}\end{pmatrix}\\ &= (2q+2)I_{2q+2}.\end{aligned}$$

(2) If $q \equiv -1 \pmod 4$ then by Theorem 2.75, we can construct a skew-symmetric conference matrix of order $q+1$, say W_2. We know $W_2W_2^T = qI_{q+1}$. As W_2 is skew-symmetric with a zero diagonal, it follows that W_2+I_{q+1} is the desired Hadamard matrix of order $q+1$. □

Remark 2.77. Another way to construct the matrix from part (1) of Theorem 2.76 is by using Kronecker products. For a prime power $q \equiv 1 \pmod 4$, a straightforward calculation shows that if W_1 is a conference matrix of order $q+1$

$$H = \begin{pmatrix}1 & 1\\ 1 & -\end{pmatrix}\otimes W_1 + \begin{pmatrix}1 & -\\ - & -\end{pmatrix}\otimes I_{q+1}$$

is the same Hadamard matrix of order $2q+2$ as seen in the proof of Theorem 2.76.

Example 2.78. Example 2.65 showed two conference matrices, W_1 of order 6 and W_2 of order 8, constructed using Theorem 2.75. Using W_1 as outlined in Theorem 2.76, we construct a symmetric Hadamard matrix of order 12.

$$\begin{pmatrix} W_1+I_6 & W_1-I_6 \\ W_1-I_6 & -W_1-I_6 \end{pmatrix} = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & - & - & 1 & 1 & - & 1 & - & - & 1 \\
1 & 1 & 1 & 1 & - & - & 1 & 1 & - & 1 & - & - \\
1 & - & 1 & 1 & 1 & - & 1 & - & 1 & - & 1 & - \\
1 & - & - & 1 & 1 & 1 & 1 & - & - & 1 & - & 1 \\
1 & 1 & - & - & 1 & 1 & 1 & 1 & - & - & 1 & - \\
- & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - \\
1 & - & 1 & - & - & 1 & - & - & - & 1 & 1 & - \\
1 & 1 & - & 1 & - & - & - & - & - & - & 1 & 1 \\
1 & - & 1 & - & 1 & - & - & 1 & - & - & - & 1 \\
1 & - & - & 1 & - & 1 & - & 1 & 1 & - & - & - \\
1 & 1 & - & - & 1 & - & - & - & 1 & 1 & - & -
\end{pmatrix}.$$

Using W_2, we obtain the following skew-type Hadamard matrix of order 8.

$$W_2+I_8 = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
- & 1 & - & - & 1 & - & 1 & 1 \\
- & 1 & 1 & - & - & 1 & - & 1 \\
- & 1 & 1 & 1 & - & - & 1 & - \\
- & - & 1 & 1 & 1 & - & - & 1 \\
- & 1 & - & 1 & 1 & 1 & - & - \\
- & - & 1 & - & 1 & 1 & 1 & - \\
- & - & - & 1 & - & 1 & 1 & 1
\end{pmatrix}.$$

The following is an immediate consequence that comes from applying Theorem 2.27 to known orders of Hadamard matrices and the matrices constructed using Theorem 2.76.

Corollary 2.79. *Let q be a prime power and let n be the order of a Hadamard matrix. If $q \equiv 1 \pmod 4$ then there is a Hadamard matrix of order $2n(q+1)$. If $q \equiv -1 \pmod 4$,* *then there is a Hadamard matrix of order $n(q+1)$.*

Corollary 2.79 allows us to use known Hadamard matrices alongside the matrices from Theorem 2.76 to construct many new orders of Hadamard matrices. However, for the case of $q \equiv 1 \pmod 4$, the construction doubles the order of the conference matrix, resulting in Hadamard matrices of order $2n(q+1)$. However, it is possible to remove the doubling condition in two ways. The first is by applying Theorem 2.29. The second involves modifying the construction of Theorem 2.76.

Lemma 2.80 (Hall, [15])**.** *Let W be a matrix of order m such that $W^T = \sigma W$, $\sigma = \pm 1$ and $WW^T = (m-1)I_m$. Let A and B be matrices of order n such that $AA^T = BB^T = nI_n$ and $AB^T = -\sigma BA^T$. Define $K = A \otimes I_m + B \otimes W$. Then $KK^T = nmI_{nm}$.*

Proof. Notice that $\sigma^2 = 1$. A straightforward computation shows that

$$\begin{aligned}
KK^T &= (A \otimes I_m + B \otimes W)(A \otimes I_m + B \otimes W)^T \\
&= (A \otimes I_m + B \otimes W)(A^T \otimes I_m + B^T \otimes W^T) \\
&= (AA^T \otimes I_m) + (AB^T \otimes W^T) + (BA^T \otimes W) + (BB^T \otimes WW^T) \\
&= (nI_n \otimes I_m) + (-\sigma BA^T \otimes \sigma W) + (BA^T \otimes W) + (nI_n \otimes (m-1)I_m) \\
&= (nI_n \otimes I_m) - \sigma^2 (BA^T \otimes W) + (BA^T \otimes W) + (nI_n \otimes (m-1)I_m) \\
&= nI_{nm} + n(m-1)I_{nm} \\
&= nmI_{nm}.
\end{aligned}$$

□

Corollary 2.81 (Hall, [15])**.** *If q is a prime power $q \equiv 1 \pmod 4$, and H is a Hadamard matrix of order n, then there exists a Hadamard matrix of order $n(q+1)$.*

Proof. Let W be a symmetric conference matrix of order $q+1$ constructed in Theorem 2.75.

Define

$$M = I_{n/2} \otimes \begin{pmatrix} 0 & 1 \\ - & 0 \end{pmatrix},$$

and let $B = MH$. Notice that $M^T = -M$ and $MM^T = I_n$. Then, note that

$$BB^T = (MH)(MH)^T = MHH^TM^T = nMM^T = nI_n.$$

Next, observe that

$$HB^T = H(MH)^T = HH^TM^T = -nM.$$

Since W is symmetric, the matrices W, H and B satisfy the conditions of Lemma 2.80. Therefore $K = H \otimes I_{q+1} + B \otimes W$ is the desired Hadamard matrix of order $n(q+1)$. □

Using Theorem 2.75, it is also possible to construct an infinite family of quaternary Hadamard matrices. This family of quaternary Hadamard matrices will be used extensively in Chapter 3.

Theorem 2.82. *Let $q \equiv 1 \pmod 4$ be an odd prime power. Then there exists a skew-type quaternary Hadamard matrix of order $q+1$.*

Proof. Let $W = W(q+1, q)$ be the symmetric conference matrix constructed in Theorem 2.75. Then define

$$H = I_{q+1} - iW.$$

From the construction of W, it is clear that H has entries in $\{\pm 1, \pm i\}$. Then, as W is symmetric, observe that

$$H + H^* = (I_{q+1} - iW) + (I_{q+1} - iW)^* = I_{q+1} - iW + I_{q+1} + iW = 2I_{q+1},$$

and

$$\begin{aligned} HH^* &= (I_{q+1} - iW)(I_{q+1} - iW)^* \\ &= (I_{q+1} - iW)(I_{q+1} + iW) \\ &= I_{q+1} + W^2 \\ &= I_{q+1} + qI_{q+1} \\ &= (q+1)I_{q+1}. \end{aligned}$$

□

Remark 2.83. We can also prove part (1) of Theorem 2.76 by applying Theorem 2.62 to Theorem 2.82 to construct a Hadamard matrix of order $2(q+1)$.

2.4.2 Orthogonal Designs

Originally inspired by the Williamson array, we now study orthogonal matrices with entries that are commuting variables.

Definition 2.84. Let $x_1, x_2, \dots, x_u$ be commuting variables. An *orthogonal design* of order n and type $(s_1, \dots, s_u)$ is an $n \times n$ matrix D, with entries from the set $\{0, \pm x_1, \dots, \pm x_u\}$, such that $DD^T = sI_n$, where

$$s = \sum_{i=1}^{u} s_i x_i^2.$$

We denote D by $\mathrm{OD}(n; s_1, s_2, \dots, s_u)$. Moreover, the design is said to be *full* if

$$\sum_{i=1}^{u} s_i = n,$$

or equivalently, if D has no zero entries.

To help break down the definition, let D be an $\mathrm{OD}(n; s_1, \dots, s_u)$. Then D has the following properties, which are proven in [10].

1. In each row of D, the variable x_i appears exactly s_i times with coefficients ± 1.

2. We can write

$$D = x_1 W_1 + x_2 W_2 + \cdots + x_u W_u,$$

   where $W_i = W(n, s_i)$, W_i and W_j are disjoint and $W_i W_j^T = -W_j W_i^T$ for $i \neq j$.

3. D^T is also an orthogonal design of order n and type $(s_1, \ldots, s_u)$.

Remark 2.85. An orthogonal design of order n over just one variable x_1 (i.e., an orthogonal design of type (s_1)) is essentially a weighing matrix multiplied by the indeterminate x_1. In general, replacing each variable x_i in an OD$(n; s_1, \ldots, s_u)$ by a value in $\{0, \pm 1\}$, gives a weighing matrix $W(n, k)$, where k is the sum of those s_i for which x_i was replaced by 1 or -1.

Example 2.86. The Williamson array from Theorem 2.31 is an OD$(4; 1, 1, 1, 1)$.

It is also possible to generalize orthogonal designs to have variables multiplied by a fourth root of unity.

Definition 2.87. Let $x_1, x_2, \ldots, x_k$ be real variables. Then a *quaternary orthogonal design* of type $(s_1, s_2, \ldots, s_k)$ is a square matrix D of order n, with entries from the set $\{0, \pm\varepsilon_1 x_1, \ldots, \pm\varepsilon_k x_k\}$, such that $DD^* = sI_n$, where

$$s = \sum_{i=1}^{k} s_i x_i^2,$$

and $\varepsilon_j \in \{1, i\}$. We say that D is a COD$(n; s_1, s_2, \ldots, s_k)$.

If $\sum_{i=1}^{k} s_i = n$, by replacing each indeterminate with 1 or -1, we obtain a quaternary Hadamard matrix of order n.

Constructions of orthogonal designs and their quaternary counterparts will be detailed in Sections 3.1 and 3.2, respectively. For further insights into weighing matrices, orthogonal designs, and the various possible generalizations of Hadamard matrices, we refer the reader to [40].

Chapter 3

Skew-Regularity

(This Chapter is based on co-authored published work [24] and work in forthcoming publications [28])

In this chapter, we introduce the idea of skew-regular Hadamard matrices. This concept uses the regular and skew-type Hadamard matrices introduced in the previous chapter. Although formally defined only recently [24, 28], the underlying structure of skew-regular Hadamard matrices was utilized as early as 2005 to construct an infinite family of symmetric designs [19]. Even earlier, Kharaghani and Seberry [22] utilized skew-type quaternary Hadamard matrices with two distinct row sums to construct Hadamard matrices of maximal excess, which closely resembles our approach of utilizing skew-regular quaternary Hadamard matrices in Section 3.2.

3.1 Skew-Regular Hadamard matrices

In this section, we introduce the notion of skew-regularity for Hadamard matrices and their quaternary counterparts. Skew-regular matrices combine aspects of both the regular and skew-type properties of Hadamard matrices. We start by highlighting an inherent restriction that emerges from combining these two properties.

Lemma 3.1. *A Hadamard matrix H of order $n > 1$ cannot be both skew-type and regular.*

Proof. Let H be a regular Hadamard matrix of order n. Without loss of generality, assume H has constant row sum $\sqrt{n}$.

Assume H is skew-type, satisfying $H + H^T = 2I_n$. Then

$$(H + H^T)j_n = 2j_n,$$

which implies

$$Hj_n + H^T j_n = 2j_n.$$

But since H is regular, $Hj_n = (\sqrt{n})j_n$, so

$$(\sqrt{n})j_n + H^T j_n = 2j_n.$$

Then

$$H^T j_n = (2 - \sqrt{n})j_n,$$

so by Proposition 2.15 we must have $2 - \sqrt{n} = \sqrt{n}$, which implies $n = 1$. Thus, no such matrix exists for $n > 1$. □

This restriction motivates the following definition.

Definition 3.2. *H* is a *semi-regular* Hadamard matrix if all of its row sums are constant in absolute value.

The following result establishes an important property of semi-regular Hadamard matrices.

Lemma 3.3 ([5]). *Let H be a Hadamard matrix of order n. The absolute values of row sums are all equal if and only if the row sums are all equal to* $\pm\sqrt{n}$.

Proof. If the row sums are all equal to $\pm\sqrt{n}$, then it is clear that the absolute values of row sums are all the same. Conversely, if the absolute values of all row sums are equal, then from the proof of Proposition 2.15 we know for row sums $(r_1, r_2, \ldots, r_n)$ we have

$$\sum_{i=1}^{n}(r_i)^2 = n^2.$$

Hence $(r_i)^2 = n$ and so the row sums are equal to $\pm\sqrt{n}$. □

With this relaxed condition of regularity, we can now combine the definitions of regularity and skew-type Hadamard matrices.

Definition 3.4. A *skew-regular* Hadamard matrix is a skew-type semi-regular Hadamard matrix.

Example 3.5. The following is a skew-regular Hadamard matrix of order 4.

$$H = \begin{pmatrix} 1 & 1 & - & 1 \\ - & 1 & 1 & 1 \\ 1 & - & 1 & 1 \\ - & - & - & 1 \end{pmatrix}.$$

This simple example enabled Ionin and Kharaghani [19] to construct an infinite class of new symmetric designs. It also motivated the search for skew-regular Hadamard matrices for other orders, which are discussed later in this section.

Lemma 3.6 ([24])**.** *If H is a skew-regular Hadamard matrix of order $4n^2$, then H has $2n^2+n$ rows with row sums $2n$ and $2n^2-n$ rows with row sums $-2n$.*

Proof. Suppose H has k rows with row sum $2n$. Since H is also skew-type, and since $H+H^T = 2I_{4n^2}$, the sum of the entries in H is $4n^2$. Since each row sum is $\pm 2n$, we have $k(2n)+(4n^2-k)(-2n) = 4n^2$. Thus $k = 2n^2+n$, and since there are $4n^2$ rows precisely $2n^2-n$ have row sums $-2n$. □

Theorem 3.7 ([24])**.** *Let H be a Hadamard matrix of order $4n^2$. H is equivalent to a skew-regular Hadamard matrix if and only if it is equivalent to a regular Hadamard matrix and a skew-type Hadamard matrix.*

Proof. Assume H is equivalent to a skew-regular Hadamard matrix K. Then there exist signed permutation matrices P,Q such that $PHQ = K$. Since K is skew-regular, H is equivalent to a skew-type Hadamard matrix. By Lemma 3.3, all row sums must be $\pm 2n$. Negating all the rows of K with negative sums shows that K is equivalent to a regular Hadamard matrix with row sums $2n$. Since equivalence is transitive, H is then also equivalent to a regular Hadamard matrix.

Conversely, assume H is equivalent to a skew-type Hadamard matrix S and a regular Hadamard matrix R. Then by the transitivity of equivalence, S and R are equivalent, and so there exist signed permutation matrices P,Q such that $S = PRQ$. Then observe that the matrix QSQ^T is skew-type, and QPR has row sums $\pm 2n$. Therefore, $QSQ^T = QPR$ is skew-regular. □

Theorem 3.7 restricts the existence of skew-regular Hadamard matrices to orders for which regular Hadamard matrices exist. As shown in Corollary 2.16, every regular Hadamard matrix is of order $4n^2$. The following result restricts the existence of skew-regular Hadamard matrices of order $4n^2$ to only odd values of n.

Theorem 3.8 ([24])**.** *If there exists a skew-regular Hadamard matrix of order $4n^2$, then n is odd.*

Proof. Let H be a skew-regular Hadamard matrix of order $4n^2$. Then $H + H^T = 2I_{4n^2}$ and so $H^T = 2I_{4n^2} - H$. From this, we can see that the sum of each column of H is $2n+2 \pmod 4$. By Lemma 3.6 H has $2n^2 - n$ rows with row sum $-2n$. Negating a row of H will change every column sum by exactly $2 \pmod 4$, and so negating each of the $2n^2 - n$ negative rows, the column sums of H are changed by $2(2n^2 - n) \pmod 4$. As each column sum of H was $2n+2 \pmod 4$, after negation the column sums are all

$$(2n+2) + 2(2n^2 - n) \equiv 2 \pmod 4.$$

But after negating the row sums, the new matrix is regular, and so the column sums must be of absolute value $2n$. If n were even, this would contradict that the column sums are 2 (mod 4), and so n must be odd. □

The following are two immediate consequences of combining Theorems 3.7 and 3.8.

Corollary 3.9. *If H is a skew-type Hadamard matrix of order* $16n^2$*, then H is not equivalent to a regular Hadamard matrix.*

Corollary 3.10. *If H is a regular Hadamard matrix of order* $16n^2$*, then H is not equivalent to a skew-type Hadamard matrix.*

3.1.1 Classification of Skew-Type Order 36

So far, we have only seen one example of a skew-regular Hadamard matrix. In this section, we find many more examples of skew-regular Hadamard matrices and highlight some of their applications. The next smallest order for which skew-regular matrices can exist is 36. To find examples of order 36, we look to the classification of SH-inequivalent skew-type Hadamard matrices of order 36 [3]. This classification found that there are at least 157132 SH-inequivalent skew-type Hadamard matrices of order 36. Using this classification, we try to find how many of these matrices are SH-equivalent to a skew-regular Hadamard matrix.

Since row and column permutations merely reorder the respective sums, preserving their values, it suffices to restrict our attention to signed permutation matrices that have entries only on the diagonal. Note that we only consider SH-equivalence as the matrices are already skew-type, and so to retain this property, whenever we negate the *i*-th row of a skew-type Hadamard matrix, we must also negate the *i*-th column of the matrix.

A systematic search of all signed permutation matrices of order 36 is computationally expensive, as for each of the 157132 skew-type Hadamard matrices, there are 2^{36} possible signed permutation matrices to check. A preliminary benchmark indicates that computing just one of the 157132 matrices would take approximately two minutes on a single-core 3.2 GHz CPU.

However, this computation can be performed much faster. First, note that we are looking for a signed permutation matrix P with nonzero entries only on the main diagonal such that $PHP^T = PHP$ (note that $P = P^T$ since P is a diagonal matrix) has row sums ± 6. Since negating rows does not change the absolute value of the row sums for skew-regular matrices, we are in essence searching for P such that $H(Pj_{36})$ is a column vector with entries in $\{\pm 6\}$, which means that $H(Pj_{36}) = 0 \pmod 3$. In other words, Pj_{36}, which is a ± 1 column vector, is contained in the null space of H over $\mathbb{F}_3$. Note that as an immediate consequence of Lemma 2.51, the dimension of the null space of each H over $\mathbb{F}_3$ is 18. We then apply the following algorithm, first introduced in [24], to each skew-type Hadamard matrix H.

The Skew-Regular Algorithm:

1. Compute the basis $\mathcal{B} = \{u_1, \ldots, u_{18}\}$ for the null space of H over $\mathbb{F}_3$.

2. Generate all 2^{18} candidate vectors v formed by the linear combinations $v = \sum_{k=1}^{18} c_k u_k$, where the coefficients $c_k \in \{\pm 1\}$.

3. Filter the candidates to retain only ± 1 vectors v, which are possible valid Pj_{36} vectors.

4. For each remaining candidate v, construct the corresponding diagonal matrix P and verify if PHP has row sums ± 6. Successful vectors are said to *regularize* H, and are called *regularizing vectors*.

Using this algorithm, instead of 2^{36} candidates for Pj_{36}, we only have 2^{18} candidates coming from the null space of H over $\mathbb{F}_3$, dramatically reducing running time for each computation. Using this approach, a single-core 3.2 GHz CPU took approximately 15 minutes to examine all 157132 matrices. Table A.1 in Appendix A.1 shows the number of valid vectors identified in step four that successfully yield a skew-regular Hadamard matrix. This computation shows that every one of the 157132 matrices from the classification is SH-equivalent to a skew-regular Hadamard matrix.

Theorem 3.11 ([24]). *There are at least* 157132 *skew-regular Hadamard matrices of order* 36 *up to SH-equivalence.*

Example 3.12. The following is a skew-regular Hadamard matrix of order 36.

$$H=\left(\begin{array}{cccccccccccccccccccccccccccccccccccc}
1&1&-&-&1&1&1&-&1&1&-&-&-&-&-&1&1&-&1&1&-&-&1&-&-&1&1&-&1&1&1&-&1&1&1&1\\
-&1&1&1&-&1&-&1&-&-&-&1&-&-&-&1&-&-&1&1&-&1&1&-&1&-&1&-&-&-&1&-&-&-&-&1\\
1&-&1&-&1&-&-&-&1&-&1&1&-&-&-&-&-&1&1&1&1&1&1&1&-&-&1&1&1&-&1&1&1&1&-&1\\
1&-&1&1&1&-&-&1&1&-&-&-&1&1&-&1&-&-&-&1&-&-&1&1&1&1&-&1&-&1&1&1&1&-&1&1\\
-&1&-&-&1&1&1&-&1&-&-&-&1&-&-&-&-&1&1&1&-&1&-&1&1&1&-&1&-&-&-&-&1&-&-&-\\
-&-&1&1&-&1&-&1&-&-&-&-&-&-&1&-&1&1&1&1&1&-&-&1&1&1&1&1&1&1&1&-&1&1&1&-\\
-&1&1&1&-&1&1&1&-&1&1&-&1&-&-&-&-&-&-&1&1&1&1&-&-&1&-&1&1&1&-&1&1&1&-&1\\
1&-&1&-&1&-&-&1&1&1&-&-&-&-&1&-&-&-&1&-&1&1&1&-&1&1&-&-&-&1&-&-&-&1&-&-\\
-&1&-&-&-&1&1&-&1&-&-&1&-&1&1&-&-&-&-&-&1&1&1&1&1&1&1&1&-&1&1&1&-&1&1&1\\
-&1&1&1&1&1&-&-&1&1&1&-&-&1&1&1&-&1&1&-&-&-&-&-&1&1&-&1&1&-&1&1&-&1&-&1\\
1&1&-&1&1&1&-&1&1&-&1&1&1&-&1&-&1&-&-&1&-&-&1&1&1&1&1&-&1&-&-&1&-&1&-&-\\
1&-&-&1&1&1&1&1&-&1&-&1&-&1&-&-&-&1&1&1&-&1&-&1&1&-&-&-&1&1&-&1&-&1&1&1\\
1&1&1&-&-&1&-&1&1&1&-&1&1&1&-&-&1&1&-&-&-&1&1&-&1&-&-&1&1&-&1&-&1&1&1&-\\
1&1&1&-&1&1&1&1&-&-&1&-&-&1&1&1&-&1&-&1&1&1&1&1&-&1&-&-&1&-&1&-&-&-&1&-\\
1&1&1&1&1&-&1&-&-&-&-&1&1&-&1&-&1&-&1&-&1&1&-&-&1&1&-&-&1&-&1&1&1&-&1&1\\
-&-&1&-&1&1&1&1&1&-&1&1&1&-&1&1&1&-&1&1&-&1&-&-&-&-&-&1&-&1&1&1&-&1&1&-\\
-&1&1&1&1&-&1&1&1&1&-&1&-&1&-&-&1&1&1&1&1&-&1&-&-&1&1&1&-&-&-&1&-&-&1&-\\
1&1&-&1&-&-&1&1&1&-&1&-&-&-&1&1&-&1&1&-&-&1&1&-&1&-&1&1&1&1&-&1&1&-&1&-\\
-&-&-&1&-&-&1&-&1&-&1&-&1&1&-&-&-&-&1&1&1&-&1&-&1&-&-&-&1&-&1&-&-&1&1&-\\
-&-&-&-&-&-&-&1&1&1&-&-&1&-&1&-&-&1&-&1&-&1&-&-&-&1&1&-&1&-&1&1&-&-&1&1\\
1&1&-&1&1&-&-&-&-&1&1&1&1&-&-&1&-&1&-&1&1&1&-&-&1&1&1&1&-&1&1&-&-&1&1&-\\
1&-&-&1&-&1&-&-&-&1&1&-&-&-&-&-&1&-&1&-&-&1&1&1&-&1&-&1&-&-&1&1&-&-&1&-\\
-&-&-&-&1&1&-&-&-&1&-&1&-&-&1&1&-&-&-&1&1&-&1&-&1&-&-&1&1&-&-&1&1&-&1&-\\
1&1&-&-&-&-&1&1&-&1&-&-&1&-&1&1&1&1&1&1&1&-&1&1&1&-&-&1&-&-&1&1&-&1&-&1\\
1&-&1&-&-&-&1&-&-&-&-&-&-&1&-&1&1&-&-&1&-&1&-&-&1&1&1&1&1&-&-&1&-&1&-&-\\
-&1&1&-&-&-&-&-&-&-&-&1&1&-&-&1&-&1&1&-&-&-&1&1&-&1&-&-&1&1&-&1&-&1&1&-\\
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-&1&-&1&1&-&-&1&1&-&-&-&-&-&-&1&1&-&-&-&1&1&-&1&-&-&-&1&1&-&-&-&-&1&1&1\\
-&1&1&-&1&-&-&-&-&1&1&-&1&1&1&-&1&-&1&1&-&1&1&1&1&-&1&1&1&1&-&-&-&-&1&1\\
-&-&-&-&1&-&1&1&-&-&1&1&-&-&-&-&1&1&-&-&-&-&1&-&1&1&-&1&1&1&1&-&-&-&-&1\\
1&1&-&-&1&1&-&1&-&-&-&-&1&1&-&-&-&-&1&-&1&-&-&-&-&-&1&1&1&1&1&1&-&-&-&-\\
-&1&-&-&-&-&-&1&1&1&1&1&-&1&-&1&1&-&1&1&1&1&-&1&1&1&-&-&1&1&1&1&1&-&-&-\\
-&1&-&1&1&-&-&-&-&-&-&-&-&1&1&-&1&1&-&1&-&1&1&-&-&-&-&-&-&1&1&1&1&1&-&-\\
-&1&1&-&1&-&1&1&-&1&1&-&-&-&-&-&-&-&-&-&-&-&-&1&1&-&1&-&-&-&1&1&1&1&1&-\\
-&-&-&-&1&1&-&1&-&-&1&-&1&1&-&1&1&1&1&-&1&1&1&-&1&1&1&-&-&-&-&1&1&1&1&1
\end{array}\right).$$

3.1.2 Existence of Order 100

The next order for which a skew-regular Hadamard matrix can exist is of order 100. As in order 36, we are attempting to use skew-type Hadamard matrices to identify those equivalent to skew-regular Hadamard matrices. Using a similar process as before, the computation can be reduced to checking 2^{50} permutation matrices, this time over $\mathbb{F}_5$.

The skew-type Hadamard matrices of order 100 were formed using the Goethals-Seidel array from Theorem 2.33. Unlike order 36, where every known skew-type is equivalent to a regular one, five of the six skew-type Hadamard matrices of order 100 tested were found to be inequivalent to a regular Hadamard matrix [24].

The successful candidate, which was found to be equivalent to two skew-regular Hadamard matrices, was found by taking

$$\begin{aligned}
A &= \text{circ}\left(1\ 1\ 1\ 1\ -\ 1\ -\ -\ 1\ 1\ 1\ -\ -\ 1\ 1\ -\ -\ -\ 1\ 1\ -\ 1\ -\ -\ -\right), \\
B &= \text{circ}\left(1\ 1\ 1\ 1\ 1\ -\ 1\ 1\ 1\ -\ 1\ 1\ -\ 1\ -\ -\ 1\ -\ -\ -\ 1\ -\ -\ -\ -\right), \\
C &= \text{circ}\left(1\ -\ -\ 1\ -\ -\ -\ 1\ -\ -\ -\ 1\ 1\ 1\ 1\ -\ -\ -\ 1\ -\ -\ -\ 1\ -\ -\right), \text{ and} \\
D &= \text{circ}\left(1\ 1\ -\ 1\ -\ -\ -\ -\ -\ 1\ -\ 1\ -\ -\ 1\ -\ 1\ -\ -\ -\ -\ -\ 1\ -\ 1\right).
\end{aligned}$$

The two skew-regular Hadamard matrices of order 100, as well as the vectors that were used to form the permutation matrices, can be found in [24].

We now know that skew-regular Hadamard matrices exist for the three smallest permissible orders: 4, 36 and 100. Motivated by these findings, we make the following conjecture.

Conjecture 3.13. There exists a skew-regular Hadamard matrix of order $4n^2$ if and only if n is odd.

3.1.3 Applications of Skew-Regular Hadamard Matrices

We now discuss applications of skew-regular Hadamard matrices, including the construction of orthogonal designs and regular Hadamard matrices.

The first application of skew-regular Hadamard matrices is found in [19]. In this article, the structure of the skew-regular Hadamard matrix of order 4 allowed the authors to construct regular Hadamard matrices, which proved instrumental in developing an infinite class of symmetric designs with new parameters.

The discovery of skew-regular Hadamard matrices of orders 36 and 100 enables us to construct two new classes of regular Hadamard matrices, which hold significant potential for generating symmetric designs with new parameters, likely by extending the methodology presented in [19]. The proof of our main result in this section relies on the use of orthogonal designs.

Lemma 3.14 ([24]). *Let H be a skew-type Hadamard matrix of order n. Then, for every integer $k \geq 0$, there exists an*

$$OD\left(n(n-1)^k;(n-1)^k,(n-1)^{k+1}\right).$$

Proof. By Lemma 2.13 we can assume

$$H=\begin{pmatrix} 1 & j_{n-1}^T \\ -j_{n-1} & I_{n-1}+Q_{n-1} \end{pmatrix},$$

where $Q_{n-1}j_{n-1}=0$. Let $D_0=aI_n+b(H-I_n)$, then D_0 is an $\mathrm{OD}(n;1,n-1)$ in variables a and b. Then define $A=aJ_{n-1}$ and $B=bI_{n-1}+aQ_{n-1}$. Let D_1 be the matrix obtained from D_0 by substituting A for a and B for b.

First, notice that $Q_{n-1}^T=-Q_{n-1}$ and $Q_{n-1}J_{n-1}=0$. Since $HH^T=nI_n$, we have

$$(I_{n-1}+Q_{n-1})(I_{n-1}+Q_{n-1})^T=(I_{n-1}+Q_{n-1})(I_{n-1}-Q_{n-1})=nI_{n-1}-J_{n-1}$$

which implies $Q_{n-1}^2 = J_{n-1} - (n-1)I_{n-1}$.

Since D_0 is an $\mathrm{OD}(n;1,n-1)$, block-rows of the resulting matrix D_1 will be pairwise orthogonal since A and B commute (which can be shown using $QJ = JQ = 0$). Moreover, expanding the diagonal of $D_1 D_1^T$, we have

$$\begin{aligned}
AA^T + (n-1)BB^T &= (aJ)(aJ)^T + (n-1)(bI+aQ)(bI+aQ)^T \\
&= a^2J^2 + (n-1)(bI+aQ)(bI-aQ) \\
&= a^2(n-1)J + (n-1)(b^2I - a^2Q^2) \\
&= a^2(n-1)J + (n-1)b^2I - a^2(n-1)(J-(n-1)I) \\
&= a^2(n-1)J + (n-1)b^2I - a^2(n-1)J + a^2(n-1)^2I \\
&= \left((n-1)b^2 + (n-1)^2a^2\right) I_{n-1}.
\end{aligned}$$

Thus, D_1 is an orthogonal design with parameters $\mathrm{OD}(n(n-1);n-1,(n-1)^2)$. Continuing the recursive process, for $k \geq 1$, we construct D_k, an OD $\left(n(n-1)^k;(n-1)^k,(n-1)^{k+1}\right)$ in variables a,b, obtained from D_{k-1} by relabeling a to be the variable that appears $(n-1)^{k-1}$, and b to be the variable that appears $(n-1)^k$ times in D_{k-1} and then substituting A for a and B for b. □

The following result follows by setting $a = b = 1$ from the orthogonal designs obtained from Lemma 3.14.

Corollary 3.15 ([24])**.** *Let H be a skew-type Hadamard matrix of order n. Then there exists a Hadamard matrix of order $n(n-1)^k$ for all integers $k \geq 0$.*

The main result of this section shows that when H is skew-regular, the Hadamard matrices in Corollary 3.15 corresponding to even k are regular.

Theorem 3.16 ([24])**.** *Let H be a skew-regular Hadamard matrix of square order $n = m^2$. Then, there are regular Hadamard matrices of order $n(n-1)^{2k}$ for all integers $k \geq 0$.*

Proof. We follow the construction of the orthogonal designs from Lemma 3.14, and carefully track the row sums at each step of the recursion. Firstly, it follows from Lemma 3.6 that there are

(i) $A = \frac{m^2+m}{2}$ rows in D_0 with sum $a+(m-1)b$, and

(ii) $B = \frac{m^2-m}{2}$ rows in D_0 with sum $a-(m+1)b$.

For $k \geq 1$, D_k is obtained from D_{k-1} as described in Lemma 3.14. The counting process is divided into three steps, depending on the value of k in the construction.

1. $k = 1$. There are

   (i) $A_1 = A(m^2-1)$ rows with sum $(m^2-1)a+(m-1)b$, and

   (ii) $B_1 = B(m^2-1)$ rows with sum $(m^2-1)a-(m+1)b$.

2. $k = 2$. There are

   (i) $A_2 = A(m^2-1)^2$ rows with sum $(m^2-1)b+(m-1)(m^2-1)a$, and

   (ii) $B_2 = B(m^2-1)^2$ rows with sum $(m^2-1)b-(m+1)(m^2-1)a$.

3. Continuing with the recursive construction, we see that there are

   (i) $A_{2k-1} = A(m^2-1)^{2k-1}$ rows with sum $(m^2-1)^k a+(m-1)(m^2-1)^{k-1}b$,

   (ii) $B_{2k-1} = B(m^2-1)^{2k-1}$ rows with sum $(m^2-1)^k a-(m+1)(m^2-1)^{k-1}b$,

   (iii) $A_{2k} = A(m^2-1)^{2k}$ rows with sum $(m^2-1)^k b+(m-1)(m^2-1)^k a$, and

   (iv) $B_{2k} = B(m^2-1)^{2k}$ rows with sum $(m^2-1)^k b-(m+1)(m^2-1)^k a$.

   Replacing $a = b = 1$ in (iii) and (iv), we see there are

   (i) $A_{2k} = \frac{(m^2+m)(m^2-1)^{2k}}{2}$ rows with sum $m(m^2-1)^k$, and

   (ii) $B_{2k} = \frac{(m^2-m)(m^2-1)^{2k}}{2}$ rows with sum $-m(m^2-1)^k$.

This shows that the constructed Hadamard matrices of order $m^2(m^2-1)^{2k}$ have constant row sums of absolute value $m(m^2-1)^k$. After negating the rows with negative row sums, we observe that the resulting matrix is regular for each nonnegative integer k.

□

Using the skew-regular Hadamard matrices of order 36 and 100, we can construct two new infinite families of regular Hadamard matrices.

Corollary 3.17. *There are regular Hadamard matrices of order* $36(35)^{2k}$ *and* $100(99)^{2k}$ *for every integer* $k \geq 0$.

3.2 Skew-Regular Quaternary Hadamard matrices

In this section, we extend the definition of skew-regular Hadamard matrices to the quaternary case. We begin with an example.

Example 3.18. The following is a quaternary Hadamard matrix of order 10.

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 & j & j & j & - & - & - \\ - & 1 & j & j & 1 & - & 1 & j & j & i \\ - & j & 1 & j & 1 & 1 & - & i & j & j \\ - & j & j & 1 & - & 1 & 1 & j & i & j \\ j & - & - & 1 & 1 & j & j & 1 & - & 1 \\ j & 1 & - & - & j & 1 & j & 1 & 1 & - \\ j & - & 1 & - & j & j & 1 & - & 1 & 1 \\ 1 & j & i & j & - & - & 1 & 1 & j & j \\ 1 & j & j & i & 1 & - & - & j & 1 & j \\ 1 & i & j & j & - & 1 & - & j & j & 1 \end{pmatrix}.$$

Observe that H is skew-type as $H+H^* = 2I_{10}$. Moreover,

$$Hj_{10} = \begin{pmatrix} 1-3i \\ 1-3i \\ \vdots \\ 1-3i \end{pmatrix},$$

so H is also regular.

Unlike the case of real Hadamard matrices, quaternary Hadamard matrices can be both skew-type and regular, as shown in Example 3.18. This allows for a simpler definition of skew-regularity in the quaternary case.

Definition 3.19. A *skew-regular* quaternary Hadamard matrix is a regular skew-type quaternary Hadamard matrix.

Since skew-regular quaternary Hadamard matrices are regular, the order n must be a sum of two squares. However, the following result implies that one of the squares must be one.

Lemma 3.20. *Let H be a skew-regular quaternary Hadamard matrix of order n. Then the row sum of H is of the form $1+ki$.*

Proof. Let $s = a+ib$ be the row sum of H. Then

$$Hj_n = sj_n,$$

which means j_n is an eigenvector of H, with eigenvalue s. But since H is skew-type,

$$H+H^* = 2I_n.$$

This means that for any eigenvalue λ and corresponding eigenvector $v \neq 0$ we have

$$\begin{aligned} v^*(H+H^*)v &= v^*(2I_n)v \\ v^*Hv + v^*H^*v &= 2v^*v \\ v^*(\lambda v) + (Hv)^*v &= 2v^*v \\ \lambda v^*v + \overline{\lambda} v^*v &= 2v^*v \\ \lambda + \overline{\lambda} &= 2. \end{aligned}$$

Since s is an eigenvalue, the real part of the row sum s must be 1. □

The following is an immediate consequence of Lemma 3.20.

Corollary 3.21. *Let H be a skew-regular quaternary Hadamard matrix of order n. Then $n = 1 + k^2$ for an odd integer k.*

Proof. By Lemma 3.20, we know the row sum of H must be of the form $1 + ki$, where $1 + k^2 = n$. Since n is even, k must be odd. □

Analagous to the real case, the following definition generalizes regularity and skew-regularity for quaternary Hadamard matrices.

Definition 3.22. A quaternary Hadamard matrix is *semi-regular* if the absolute value of all row sums is constant. Moreover, a *skew-semi-regular* quaternary Hadamard matrix is a semi-regular skew-type quaternary Hadamard matrix.

In other words, a quaternary Hadamard matrix of order $n = a^2 + b^2$ is called semi-regular if it has row sums belonging to the set $\{\pm a \pm bi, \pm b \pm ai\}$.

Constructions of these objects will be given after the following construction of skew-regular quaternary Hadamard matrices.

3.2.1 An Infinite Family of Skew-Regular Quaternary Hadamard Matrices

We now present a construction for the only known infinite family of skew-regular quaternary Hadamard matrices, first introduced in [28]. However, we must first elaborate on the properties of the quadratic character, which was first introduced in Section 2.4.

Throughout this section, let p be an odd prime, $q = p^2$. Fix an element $\gamma \in \mathbb{F}_q \setminus \mathbb{F}_p$. For $i \in \{0, \dots, p-1\}$, define the set

$$C_i = i\gamma + \mathbb{F}_p.$$

Note that $C_0 = \mathbb{F}_p$ and $C_0, C_1, \dots, C_{p-1}$ are arbitrary additive cosets that partition $\mathbb{F}_q$. We begin by proving that every nonzero element of $\mathbb{F}_p$ is a square in $\mathbb{F}_q$.

Lemma 3.23. *Let p be an odd prime and $q = p^2$. If $x \in \mathbb{F}_p \setminus \{0\}$, then $\chi_q(x) = 1$.*

Proof. Let α be a primitive element of $\mathbb{F}_q$. Then $x = \alpha^k$ for some integer $0 \le k < q-1$. First, note that since $x \in \mathbb{F}_p \setminus \{0\}$, we have $x^{p-1} = 1$. Then

$$x^{\frac{q-1}{2}} = x^{\frac{p^2-1}{2}} = x^{\frac{(p-1)(p+1)}{2}} = (x^{p-1})^{\frac{p+1}{2}} = (1)^{\frac{p+1}{2}} = 1.$$

Hence

$$1 = x^{\frac{q-1}{2}} = (\alpha^k)^{\frac{q-1}{2}} = \alpha^{\frac{k(q-1)}{2}},$$

and since α generates $\mathbb{F}_q \setminus \{0\}$, it must be that $q-1$ divides $k(q-1)/2$. Then there exists an integer n such that $n(q-1) = k(q-1)/2$, which implies that $n = k/2$, so $k = 2n$ is an even integer. Therefore,

$$x = \alpha^k = \alpha^{2n} = (\alpha^n)^2,$$

and so x is a square in $\mathbb{F}_q$. □

Next, we show that quadratic character sums over $C_1, \dots, C_{p-1}$ are constant.

Lemma 3.24. *For* $1 \leq n, m \leq p-1$ *we have*

$$\sum_{x \in C_n} \chi_q(x) = \sum_{x \in C_m} \chi_q(x).$$

Proof. By the definition of the cosets $C_1, \ldots, C_{p-1}$, there exists a scalar $k = nm^{-1} \in \mathbb{F}_p \setminus \{0\}$ such that $C_n = \{kx \mid x \in C_m\}$. Since $k \in \mathbb{F}_p \setminus \{0\}$, by Lemma 3.23 $\chi_q(k) = 1$, which shows

$$\sum_{y \in C_n} \chi_q(y) = \sum_{x \in C_m} \chi_q(kx) = \chi_q(k) \sum_{x \in C_m} \chi_q(x) = \sum_{y \in C_m} \chi_q(y).$$

□

Our last step before presenting the main construction is to evaluate the quadratic character sums over cosets of $\mathbb{F}_p$.

Lemma 3.25 ([28])**.** *Let* p *be an odd prime and set* $q = p^2$. *For* $t \in \mathbb{F}_q$ *define*

$$S(t) = \sum_{x \in t + \mathbb{F}_p} \chi_q(x).$$

Then

$$S(t) = \begin{cases} p-1, & t \in \mathbb{F}_p, \\ -1, & t \notin \mathbb{F}_p. \end{cases}$$

Proof. By Lemma 3.23, every nonzero element of $\mathbb{F}_p$ is a square in $\mathbb{F}_q$. Thus for $t \in \mathbb{F}_p$

$$S(t) = \sum_{x \in t + \mathbb{F}_p} \chi_q(x) = \sum_{x \in \mathbb{F}_p} \chi_q(x) = p-1.$$

Next, suppose $t \in \mathbb{F}_q \setminus \mathbb{F}_p$. Recall from Lemma 2.70

$$\sum_{x \in \mathbb{F}_q} \chi_q(x) = 0.$$

Since $C_0, C_1, \ldots, C_{p-1}$ partition $\mathbb{F}_q$ we have

$$0 = \sum_{x \in \mathbb{F}_q} \chi_q(x) = \sum_{x \in \mathbb{F}_p} \chi_q(x) + \sum_{x \in \cup_{i=1}^{p-1} C_i} \chi_q(x).$$

Then by Lemma 3.24 the character sums over $C_1, \ldots, C_{p-1}$ are constant, so

$$p - 1 = \sum_{x \in \mathbb{F}_p} \chi_q(x) = - \sum_{x \in \cup_{i=1}^{p-1} C_i} \chi_q(x) = -(p-1) \sum_{x \in C_1} \chi_q(x).$$

Therefore

$$\sum_{x \in t + \mathbb{F}_p} \chi_q(x) = \sum_{x \in C_1} \chi_q(x) = -1,$$

which implies $S(t) = -1$.

□

Theorem 3.26 ([28])**.** *Let p be an odd prime. Then there exists a skew-regular quaternary Hadamard matrix of order $p^2 + 1$ with row sum $1 - pi$.*

Proof. Let $q = p^2$. Since p is odd, $q = p^2 = (2k+1)^2 \equiv 1 \pmod 4$. Then let $H = I_{q+1} - iW$ be the skew-type quaternary Hadamard matrix of order $q+1$ constructed in Theorem 2.82, where $W = W(q+1, q)$ is the symmetric conference matrix from Theorem 2.75. Index the first row and column of H by the symbol ∞, and the remaining rows and columns by elements of $\mathbb{F}_q$. Then, using the additive cosets $C_0, C_1, C_2, \ldots, C_{p-1}$, we partition the index set $\{\infty\} \cup \mathbb{F}_q$ into four parts:

$$\{\infty\}, \quad \mathbb{F}_p = C_0, \quad \mathcal{H}_1 = C_1 \cup \cdots \cup C_{(p-1)/2}, \quad \mathcal{H}_2 = C_{(p-1)/2+1} \cup \cdots \cup C_{p-1}.$$

Next, define the vector $v = (v_\alpha)$ by

$$v_\infty = 1, \qquad v_x = \begin{cases} 1, & \text{if } x \in \mathbb{F}_p, \\ -i, & \text{if } x \in \mathcal{H}_1, \text{ and} \\ +i, & \text{if } x \in \mathcal{H}_2. \end{cases}$$

Finally, define

$$M = \mathrm{diag}(v), \text{ and } S = MHM^*.$$

We will show that S is the desired skew-regular quaternary Hadamard matrix. First, notice that

$$S+S^* = (MHM^*)+(MHM^*)^* = MHM^* + MH^*M^* = M(H+H^*)M^* = 2MM^* = 2I_{q+1},$$

so S is a skew-type quaternary Hadamard matrix. Next, we must show S is regular. First, notice that the (α,β) entry of S is given by

$$S_{\alpha,\beta} = v_\alpha H_{\alpha,\beta}\overline{v_\beta}.$$

Then, define

$$R_\alpha = \sum_{\beta\in\mathbb{F}_q\cup\{\infty\}} H_{\alpha,\beta}\overline{v_\beta}.$$

Thus, the row sum of the row indexed by α is given by

$$\sum_{\beta\in\mathbb{F}_q\cup\{\infty\}} S_{\alpha,\beta} = \sum_{\beta\in\mathbb{F}_q\cup\{\infty\}} v_\alpha H_{\alpha,\beta}\overline{v_\beta} = v_\alpha R_\alpha.$$

It suffices to show that for every $\alpha \in \mathbb{F}_q \cup \{\infty\}$, $v_\alpha R_\alpha = 1 - pi$. We consider four cases for the row–index α.

Case 1: $\alpha = \infty$. By definition

$$H_{\infty,\infty} = 1, \qquad H_{\infty,x} = -i \ \ (\forall x \in \mathbb{F}_q).$$

Hence

$$v_\alpha R_\alpha = 1\overline{v_\infty} + \sum_{x\in\mathbb{F}_q} (-i)\ \overline{v_x} = 1 + \left[p\cdot(-i) + \tfrac{q-p}{2}\cdot(-1) + \tfrac{q-p}{2}\cdot(1)\right] = 1 - pi.$$

Case 2: $\alpha \in \mathbb{F}_p$. Split the sum $R_\alpha = \sum_\beta H_{\alpha,\beta}\,\overline{v_\beta}$ into four parts:

1. $\beta = \infty$: We have
$$v_\alpha H_{\alpha,\infty}\overline{v_\infty} = -i.$$

2. $\beta = \alpha$: We have
$$v_\alpha H_{\alpha,\alpha}\overline{v_\alpha} = 1.$$

3. $\beta \in \mathbb{F}_p \setminus \{\alpha\}$: since $\alpha, \beta \in \mathbb{F}_p$, we have $\chi_q(\alpha - \beta) = 1$, which gives
$$v_\alpha \cdot \sum_{\beta\in\mathbb{F}_p\setminus\{\alpha\}} H_{\alpha,\beta} = 1 \cdot \sum_{\beta\in\mathbb{F}_p\setminus\{\alpha\}} \chi_q(\alpha-\beta)\cdot(-i) = -i(p-1).$$

4. $\beta \in \mathbb{F}_q \setminus \mathbb{F}_p$: Using Lemma 3.25
$$\begin{aligned} v_\alpha \cdot \sum_{\beta\in\mathbb{F}_q\setminus\mathbb{F}_p} (-i)\cdot\chi_q(\alpha-\beta)\overline{v_\beta} &= -i\cdot \sum_{\beta\in\cup_{i=1}^{p-1} C_i} \chi_q(\alpha-\beta)\overline{v_\beta} \\ &= -i\cdot\sum_{\beta\in\mathcal{H}_1}\chi_q(\alpha-\beta)\cdot i \ -i\cdot\sum_{\beta\in\mathcal{H}_2}\chi_q(\alpha-\beta)\cdot(-i) \\ &= -\frac{p-1}{2} + \frac{p-1}{2} \\ &= 0. \end{aligned}$$

Adding the four parts gives

$$v_\alpha R_\alpha = -i + 1 - i(p-1) + 0 = 1 - pi.$$

Case 3: $\alpha \in \mathcal{H}_1$ We split R_α into five parts. Assume $\alpha \in C_k$ for $k \in \{1, \ldots, (p-1)/2\}$.

1. $\beta = \infty$: We have

$$v_\alpha H_{\alpha,\infty}\overline{v_\infty} = -i \cdot -i \cdot 1 = -1.$$

2. $\beta = \alpha$: We have

$$v_\alpha H_{\alpha,\alpha}\overline{v_\alpha} = -i \cdot 1 \cdot i = 1.$$

3. $\beta \in \mathbb{F}_p$: Since $\alpha \notin \mathbb{F}_p$, we apply Lemma 3.25 to obtain

$$v_\alpha \sum_{\beta \in \mathbb{F}_p} -i \cdot \chi_q(\alpha - \beta)\overline{v_\beta} = -1 \cdot \sum_{\beta \in \mathbb{F}_p} \chi_q(\alpha - \beta) \cdot 1 = -1 \cdot (-1) = 1.$$

4. $\beta \in C_k$: In this case, α and β are in the same coset, so $\alpha - \beta \in \mathbb{F}_p$. Hence

$$v_\alpha \sum_{\beta \in C_k} -i \cdot \chi_q(\alpha - \beta)\overline{v_\beta} = -1 \cdot \sum_{\beta \in C_k} \chi_q(\alpha - \beta) \cdot i = -i \cdot \sum_{x \in \mathbb{F}_1} \chi_q(x) = -i \cdot (p-1).$$

5. $\beta \in \mathbb{F}_q \setminus (\mathbb{F}_p \cup C_k)$: We have

$$\begin{aligned} v_\alpha \sum_{\beta \in \mathbb{F}_q \setminus (\mathbb{F}_p \cup C_k)} -i \cdot \chi_q(\alpha - \beta)\overline{v_\beta} &= -\sum_{\beta \in \mathcal{H}_1 \setminus C_k} \left[\chi_q(\alpha - \beta) \cdot i\right] - \sum_{\beta \in \mathcal{H}_2} \left[\chi_q(\alpha - \beta) \cdot (-i)\right] \\ &= -i \cdot \left[\frac{p-1}{2} + 1\right] + i \cdot \left[\frac{p-1}{2}\right] \\ &= -i. \end{aligned}$$

Adding everything,

$$v_\alpha R_\alpha = -1 + 1 + 1 - i \cdot (p-1) - i = 1 - pi.$$

Case 4: $\alpha \in \mathcal{H}_2$ We split R_α into five parts as above. Parts (1-4) contribute 1, 1, -1 and $-i \cdot (p-1)$, respectively. The final part also follows closely. Assume $\alpha \in C_k$ for some $k \in \{(p-1)/2+1, \dots, p-1\}$.

5. $\beta \in \mathbb{F}_q \setminus (\mathbb{F}_p \cup C_k)$: We have

$$\begin{aligned} v_\alpha \sum_{\beta \in \mathbb{F}_q \setminus (\mathbb{F}_p \cup C_k)} -i \cdot \chi_q(\alpha - \beta)\overline{v_\beta} &= i \cdot \sum_{\beta \in \mathcal{H}_1} \chi_q(\alpha-\beta) \; + i \cdot \sum_{\beta \in \mathcal{H}_2 \setminus C_k} \left[\chi_q(\alpha-\beta) \cdot (-1)\right] \\ &= i \cdot \left[\frac{p-1}{2}\right] - i \cdot \left[\frac{p-1}{2} + 1\right] \\ &= -i. \end{aligned}$$

Adding everything,

$$v_\alpha R_\alpha = 1 + 1 - 1 - i \cdot (p-1) - i = 1 - pi.$$

Therefore, for any $\alpha \in \mathbb{F}_q \cup \{\infty\}$ the row sum $R_\alpha = 1 - pi$. Hence, S is regular. □

Remark 3.27. The infinite family of skew-regular quaternary Hadamard matrices coming from Theorem 3.26 constitute the first known examples of these objects.

An example of the construction can be shown in Example A.1 found in Appendix A.2.

3.2.2 Applications of Skew-Regular Quaternary Hadamard Matrices

We now show how skew-regular quaternary Hadamard matrices can be used to construct quaternary orthogonal designs, as well as both their regular and skew-semi-regular counterparts.

The following results closely follow the constructions using the skew-regular Hadamard matrices from Section 3.1.3.

Lemma 3.28 ([28])**.** *There is a* $COD(1+p^2; 1, p^2)$ *for each odd prime* p.

Proof. Let H be the skew-regular quaternary Hadamard matrices constructed in Theorem 3.26. Write $H = I_{p^2+1} + S$. Since $S^* = -S$, the matrix $aI_{p^2+1} + bS$ is a COD$(1+p^2; 1, p^2)$ in variables a and b with constant row sum $a - pbi$. $\square$

Lemma 3.29 ([28]). *There is a*

$$COD\left((1+p^2)p^{2k}; p^{2k}, p^{2k+2}\right)$$

for each odd prime p and non-negative integer k.

Proof. Let H be the skew-regular quaternary Hadamard matrix of order p^2+2 constructed in Theorem 3.26. By Lemma 2.58, H is equivalent to a skew-type quaternary Hadamard matrix of the form

$$\begin{pmatrix} 1 & j_{p^2}^T \\ -j_{p^2} & I_{p^2}+Q \end{pmatrix},$$

where $Qj_{p^2} = 0$ and $Q^* = -Q$.

Assume that D_0 is the COD$(1+p^2; 1, p^2)$ constructed in Lemma 3.28 in variables a, b, with a repeated one time. Then define $A = aJ_{p^2}$ and $B = bI_{p^2} + aQ$. Following the proof of Lemma 3.14, $Q^2 = J_{p^2} - p^2 I_{p^2}$. We then replace a with A and b with B to obtain a matrix D_1, which is an COD$((1+p^2)p^2; p^2, p^4)$, where a is repeated p^4 times and b is repeated p^2 times. Continuing the recursive process, we construct D_k from D_{k-1} by relabeling a to be the variable that appears p^{2k-2} times, and b to be the variable that appears p^{2k} times, and then replacing a with A, and b with B. $\square$

Lemma 3.30 ([28]). *Let p be an odd prime. Then there is a regular quaternary Hadamard matrix of order $p^{2k}(1+p^2)$.*

Proof. We follow the construction in Lemma 3.29, setting $a = b = 1$, and carefully track the changes in row sums for each step of the recursive process.

(i) For $k = 0$, the order is $1 + p^2$ and the row sum is $1 - pi$,

(ii) for $k = 1$, the order is $p^2(1+p^2)$ and the row sum is $p^2 - pi$, and

(iii) for $k = 2$, the order is $p^4(1+p^2)$ and the row sum is $p^2 - p^3 i$.

Inductively,

(iv) for even $k \geq 0$, the order is $p^{2k}(1+p^2)$ and the row sum is $p^k - (p^{k+1})i$, and

(v) for odd $k \geq 0$, the order is $p^{2k}(1+p^2)$ and the row sum is $p^{k+1} - (p^k)i$.

□

Next, we show how to construct families of regular and skew-type quaternary Hadamard matrices of twice the order of a skew-regular quaternary Hadamard matrix.

Lemma 3.31 ([28])**.** *Let H be a skew-regular quaternary Hadamard matrix of order $n = 1+p^2$ with the row sum $1-pi$. Then the matrix*

$$K = \begin{pmatrix} H & iH \\ iH^* & H^* \end{pmatrix}$$

is a skew-type quaternary Hadamard matrix of order $2+2p^2$ with row sums in $\{1+p+(1-p)i, 1-p+(1+p)i\}$.

Proof. Recall from Proposition 2.61, since H is regular, it has constant row and column sums. Thus, the row sum of the first $1+p^2$ rows is

$$1 - pi + i(1-pi) = 1+p+(1-p)i,$$

and the sum of the remaining rows is

$$(i-p)+(1+pi) = 1-p+(1+p)i.$$

Also

$$K+K^* = \begin{pmatrix} H & iH \\ iH^* & H^* \end{pmatrix} + \begin{pmatrix} H^* & -iH \\ -iH^* & H \end{pmatrix} = 2I_{2+2p^2}.$$

□

Lemma 3.32. *Let H be a regular quaternary Hadamard matrix of order $n = 1+p^2$ with the row sum $1-pi$. Then the matrix*

$$K = \begin{pmatrix} H & iH \\ iH & H \end{pmatrix}$$

is a regular quaternary Hadamard matrix of order $2+2p^2$ with row sums $1+p+(1-p)i$.

Proof. The proof follows from Lemma 3.31. □

Remark 3.33. Note that in Lemma 3.32, H does not need to be skew-regular, while in Lemma 3.31, H being skew-regular is a necessary condition.

The following are immediate consequences of applying Theorem 3.26 to Lemmas 3.31 and 3.32, respectively.

Corollary 3.34. *There is a skew-type quaternary Hadamard matrix of order $2+2p^2$ for each odd prime number p.*

Corollary 3.35. *There is a regular quaternary Hadamard matrix of order $2+2p^2$ for each odd prime number p.*

As an application of skew-regular Hadamard matrices constructed in Theorem 3.26, we can also introduce a new infinite class of Hadamard matrices with maximal excess. However, we first need to introduce a result about the excess of weighing matrices. The proof of the following lemma is omitted as it closely follows the proof of Theorem 2.39.

Lemma 3.36. *Let W be a weighing matrix $W(n,k^2)$. Then the excess of W is at most nk, and the excess is equal to nk if and only if all row sums of the matrix are equal to k.*

Theorem 3.37 ([28])**.** *For each odd prime p, there is a Hadamard matrix of order* $4+4p^2$ *with maximal excess* $8p(1+p^2)$.

Proof. Let $H = I + Q$ be the skew-regular quaternary matrix constructed in Theorem 3.26. Consider the three weighing matrices

$$Q_1 = \begin{pmatrix} H & iH \\ iH & H \end{pmatrix},$$

$$Q_2 = \begin{pmatrix} Q & iQ \\ iQ & Q \end{pmatrix},$$

and

$$Q_3 = \begin{pmatrix} I & iI \\ iI & I \end{pmatrix}.$$

Next, for $j = 1,2,3$ write $Q_j = A_j + iB_j$, where A_j, B_j are matrices with entries in $\{0, \pm 1\}$. Next, to convert the three quaternary matrices to real weighing matrices, for $j = 1,2,3$, define

$$W_j = A_j \otimes \begin{pmatrix} 1 & 1 \\ 1 & - \end{pmatrix} + B_j \otimes \begin{pmatrix} - & 1 \\ 1 & 1 \end{pmatrix}.$$

Then W_1 is a weighing matrix $W(4+4p^2, 4+4p^2)$ (i.e., a Hadamard matrix), W_2 is a weighing matrix $W(4+4p^2, 4p^2)$, and W_3 is a weighing matrix $W(4+4p^2, 4)$, where $W_1 = W_2 + W_3$. Then define $S_{j,k}$, for $j = 1,2,3$, and $k = 1,2,\ldots,4+4p^2$, to be the k-th row sum of the j-th weighing matrix. From the construction, we know

- $S_{1,k}$ alternates between $2+2p$ and $2-2p$, respectively,
- $S_{2,k}$ alternates between $2p$ and $-2p$, respectively, and
- $S_{3,k} = 2$, for $k = 1,2,\cdots,4+4p^2$.

Negating the rows with negative row sums in W_1 increases the row sums of W_1 and W_2 by $4p$ for each negation, and decreases the row sums of W_3 by 4. After negating all the negative row sums in W_1, the total sum of W_3 is zero, and by Lemma 3.36, W_2 achieves its maximum excess of $2p(4+4p^2)$. Ignoring the zero entries, after negations of the rows of W_1, we can assume that W_3 consists of blocks of the form

$$B = \begin{pmatrix} 1 & 1 & - & 1 \\ - & 1 & - & - \\ - & 1 & 1 & 1 \\ - & - & - & 1 \end{pmatrix}.$$

Since both the row sums and column sums of W_2 are all $2p$, negating columns and rows of B does not increase the total sum of W_1. Since no row and column negations can increase the excess, it follows that the maximum excess of the Hadamard matrix W_1 of order $4+4p^2$ is $8p(1+p^2)$. □

Remark 3.38. There are two known classes of Hadamard matrices of order 40, one with a maximum excess of $240 = 40\sqrt{40-4}$ [9], and the second with a maximum excess of 244. For $p = 3$, Theorem 3.37 gives a matrix with maximum excess $8(3)(1+3^2) = 240$.

Chapter 4

Unbiased Hadamard Matrices

(The work in this chapter is based on co-authored published articles [24, 27])

In this chapter, we introduce the idea of unbiased Hadamard matrices and their quaternary counterparts. As we will discover, the notion of unbiasedness is closely connected to the concept of regularity of Hadamard matrices, which was explored in the previous chapters.

Unlike the novel concept of skew-regularity, the topic of unbiasedness has been studied for over half a century, with its foundations rooted in quantum physics [39].

We begin with the definition of unbiased Hadamard matrices.

Definition 4.1. Two Hadamard matrices H and K of order n are said to be *unbiased* if

$$HK^T = \sqrt{n}L,$$

where L is a Hadamard matrix of order n. If H and K are unbiased, K is said to be an *unbiased mate* of H, and vice versa.

The definition of unbiased Hadamard matrices is closely tied to the concept of regularity. Firstly, since $HK^T = \sqrt{n}L$, pairs of unbiased Hadamard matrices can only exist for square orders, just like regular Hadamard matrices. Additionally, given a pair of unbiased Hadamard matrices H and K, we can use the rows of one matrix to make the other regular. To streamline this process, we have the following definition.

Definition 4.2. Let H be a (quaternary) Hadamard matrix of order n. Then the vector v is called a *regularizing vector* if $H\,\mathrm{diag}(v)$ is a regular (quaternary) Hadamard matrix. Such a vector v is said to *regularize* the matrix H.

In other words, v is a regularizing vector of H if the inner product of v with every row of H is $\pm\sqrt{n}$ [27]. From the definition of unbiased matrices, it is clear that since $HK^T = \sqrt{n}L$, every row of K regularizes H and vice versa. Moreover, if H and K are an unbiased pair of Hadamard matrices, we can think of K being made up of n mutually orthogonal regularizing vectors of H. This notion will serve as the foundation for many of our computer searches and will also enable us to connect the search for regularizing vectors in Section 3.1.1 to the concept of unbiasedness.

The definition of unbiasedness and our discussion of regularizing vectors can be easily extended to the quaternary case.

Definition 4.3. Two quaternary Hadamard matrices H and K of order n are said to be *unbiased* if

$$HK^* = (a+bi)L,$$

where L is a quaternary Hadamard matrix of order n and $a^2+b^2 = n$.

Gathering a collection of quaternary Hadamard matrices that are pairwise unbiased, we have the following definition.

Definition 4.4. Let $U = \{H_1, H_2, \ldots, H_k\}$ be a set of quaternary Hadamard matrices of order n. The set U is called a *mutually unbiased* set of quaternary Hadamard matrices if for all $1 \leq i, j \leq k$, and $i \neq j$, H_i and H_j are unbiased. These sets are often abbreviated as **"MUQHs"**, and if the matrices are all real, these are called **"MUHs"**.

An interesting problem in the study of unbiased quaternary Hadamard matrices is determining the largest set of mutually unbiased quaternary Hadamard matrices. We denote the upper bound for the largest number of mutually unbiased quaternary Hadamard matrices of order n by $|\mathbf{MUQH}(n)|$, and the upper bound for the largest number of mutually unbiased Hadamard matrices of order n by $|\mathbf{MUH}(n)|$.

We will now present bounds for $|\mathbf{MUQH}(n)|$ and $|\mathbf{MUH}(n)|$.

Lemma 4.5 ([4])**.** *If H and K are unbiased quaternary Hadamard matrices of order $2n$ and n is odd, then at least one of the matrices is not semi-regular.*

Proof. Suppose that H and K are semi-regular quaternary Hadamard matrices of order $2n$, where n is odd and $HK^* = (a+bi)L$ where $a^2+b^2 = 2n$. Consider the matrix

$$M = \frac{1}{1+i}(H+J_{2n})\frac{1}{1+i}(K^*+J_{2n}).$$

Since the entries of H and K are in $\{\pm 1, \pm i\}$, the entries of $H+J_{2n}$ and K^*+J_{2n} belong to the set $\{0,2,1+i,1-i\}$, which are all divisible by $1+i$. Thus, M is a matrix with Gaussian integer entries. Expanding the product and simplifying, we have

$$M = \frac{1}{2i}(HK^*+HJ_{2n}+J_{2n}K^*+J_{2n}J_{2n}^*) = \frac{-i}{2}(HK^*+HJ_{2n}+J_{2n}K^*+2nJ_{2n}).$$

Then, for M to be a Gaussian integer matrix, the entries of $HK^*+HJ_{2n}+J_{2n}K^*+2nJ_{2n}$ must be divisible by 2. However, $HK^* = (a+bi)L$, where a and b are both odd. Moreover, since H and K are both semi-regular, the entries of HJ_{2n} and $J_{2n}K^*$ are in the form $x+iy$, where x and y are both odd. Finally, since the entries of $2nJ_{2n}$ are even, the entries of $HK^*+HJ_{2n}+J_{2n}K^*+2nJ_{2n}$ cannot be divisible by 2, and so H and K cannot both be semi-regular. □

Theorem 4.6 ([4])**.** *For every odd integer n, $|\boldsymbol{MUQH}(2n)| \leq 2$.*

Proof. Suppose that $\{H,K,L\}$ are mutually unbiased quaternary Hadamard matrices of order $2n$. Let

$$P = \text{diag}([H_{1,1}, H_{1,2}, \ldots, H_{1,2n}]).$$

Notice that $\{HP^*, KP^*, LP^*\}$ is still a set of mutually unbiased quaternary Hadamard matrices, and the first row of HP^* is all one. This implies that KP^* and LP^* must be semi-regular, contradicting Lemma 4.5. □

Following a similar process, we can obtain an equivalent bound for $|\mathbf{MUH}(n)|$.

Lemma 4.7 ([17]). *If H and K are unbiased Hadamard matrices of order $4n^2$ and n is odd, then at least one of the matrices is not semi-regular.*

Proof. The proof closely follows Lemma 4.5. Suppose that H and K are semi-regular Hadamard matrices of order $4n^2$, where n is odd. Consider the matrix

$$M = \frac{1}{2}(H + J_{4n^2})\frac{1}{2}(K^T + J_{4n^2}).$$

M is an integer matrix. However, expanding the product, we see

$$M = \frac{1}{4}(HK^T + HJ_{4n^2} + J_{4n^2}K^T + 4n^2 J_{4n^2}),$$

which is not an integer matrix since the entries of HK^T, HJ_{4n^2} and $J_{4n^2}K^T$ are all $2 \pmod 4$. □

Theorem 4.8 ([17]). *For every odd integer n, $|\boldsymbol{MUH}(4n^2)| \leq 2$.*

Proof. Suppose that $\{H, K, L\}$ are mutually unbiased Hadamard matrices of order $4n^2$ and let

$$P = \mathrm{diag}([H_{1,1}, H_{1,2}, \ldots, H_{1,2n}]).$$

Then $\{HP, KP, LP\}$ is a set of mutually unbiased matrices, which implies that KP and LP must be semi-regular, contradicting Lemma 4.7. □

We will now work towards a result that addresses even values of n not included in Theorem 4.6 and Theorem 4.8.

Definition 4.9. Given a quaternary Hadamard matrix H of order n, label the rows as $r_1, r_2, \ldots, r_n$. Then the n *auxiliary matrices* associated with H are given by $c_i = r_i^* r_i$.

Lemma 4.10 ([23]). *The auxiliary matrices $c_1, \ldots, c_n$ associated to a quaternary Hadamard matrix H of order n satisfy*

(1) $c_i^* = c_i$,

(2) $c_i c_j = 0$ *if* $i \neq j$,

(3) $c_i c_j = nc_i$ *if* $i = j$, *and*

(4) $\sum_{i=1}^{n} c_i = nI_n$.

Proof. Property (1) follows from the properties of the conjugate transpose

$$c_i^* = (r_i^* r_i)^* = r_i^* (r_i^*)^* = r_i^* r_i = c_i.$$

To verify properties (2) and (3), note that

$$c_i c_j = (r_i^* r_i)(r_j^* r_j) = r_i^* (r_i r_j^*) r_j.$$

Since H is a quaternary Hadamard matrix, if $i \neq j$ then $(r_i r_j^*) = 0$. Otherwise if $i = j$ then $r_i r_j^* = n$ and $nr_i^* r_i = nc_i$. Lastly, since H^* is a quaternary Hadamard matrix

$$\sum_{i=1}^{n} c_i = \sum_{i=1}^{n} r_i^* r_i = H^* H = nI_n.$$

□

Lemma 4.11. *Let H be a quaternary Hadamard matrix of order n. Then the set of matrices*

$$\{c_i - I_n : 2 \leq i \leq n\}$$

is linearly independent.

Proof. Assume there exists a linear combination over $\mathbb{R}$ of these matrices that sums to zero

$$\sum_{i=2}^{n} \alpha_i (c_i - I_n) = 0.$$

Then

$$\sum_{i=2}^{n} \alpha_i c_i - \left(\sum_{i=2}^{n} \alpha_i\right) I_n = 0.$$

Multiplying by an arbitrary auxiliary matrix c_j, we obtain

$$\left(\sum_{i=2}^{n} \alpha_i c_i\right) c_j = \left(\sum_{i=2}^{n} \alpha_i\right) I_n c_j.$$

Then by properties (2) and (3) of Lemma 4.10

$$\left(\sum_{i=2}^{n} \alpha_i c_i\right) c_j = \alpha_j c_j^2 = \alpha_j n c_j.$$

Hence

$$\alpha_j n c_j = \left(\sum_{i=2}^{n} \alpha_i\right) c_j$$

and since $c_j \neq 0$, this implies

$$\alpha_j n = \sum_{i=2}^{n} \alpha_i.$$

As c_j was arbitrary, this means each coefficient α_i is equal to some constant α, and so

$$\alpha n = \sum_{i=2}^{n} \alpha = \alpha(n-1),$$

which implies $\alpha = 0$ and so every coefficent α_i must be 0. □

Lemma 4.12. *Let $\{H_1, H_2, \ldots, H_k\}$ be a set of mutually unbiased quaternary Hadamard matrices of order n and define $S_{i,j} = r_{i,j}^* r_{i,j} - I_n$ where $r_{i,j}$ is the j-th row of H_i. Then each $S_{i,j}$ is Hermitian. Moreover*

$$\left\{S_{i,j} : 1 \leq i \leq k,\ 2 \leq j \leq n\right\},$$

is linearly independent.

Proof. By property (1) of Lemma 4.10, each $S_{i,j}$ is Hermitian. Next, consider that

$$\langle S_{i,j}, S_{p,q} \rangle = \operatorname{tr}(S_{i,j} S_{p,q}^*) = \operatorname{tr}(S_{i,j} S_{p,q}) = \operatorname{tr}((r_{i,j}^* r_{i,j} - I_n)(r_{p,q}^* r_{p,q} - I_n)).$$

Then to simplify the product, since $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ we have

$$-\operatorname{tr}(r_{i,j}^* r_{i,j}) - \operatorname{tr}(r_{p,q}^* r_{p,q}) + \operatorname{tr}(I_n) = -\operatorname{tr}(r_{i,j} r_{i,j}^*) - \operatorname{tr}(r_{p,q} r_{p,q}^*) + n = -n,$$

and

$$\operatorname{tr}(r_{i,j}^* r_{i,j} r_{p,q}^* r_{p,q}) = \langle r_{i,j}, r_{p,q} \rangle \cdot \operatorname{tr}(r_{i,j}^* r_{p,q}) = \langle r_{i,j}, r_{p,q} \rangle \cdot \operatorname{tr}(r_{p,q} r_{i,j}^*) = |\langle r_{i,j}, r_{p,q} \rangle|^2.$$

Hence

$$\operatorname{tr}(S_{i,j} S_{p,q}^*) = |\langle r_{i,j}, r_{p,q} \rangle|^2 - n.$$

Notice if $i \neq p$, since H_i and H_p are unbiased

$$|\langle r_{i,j}, r_{p,q} \rangle|^2 - n = (\sqrt{n})^2 - n = 0,$$

and so

$$\langle S_{i,j}, S_{p,q} \rangle = 0 \quad \text{for } i \neq p.$$

Hence, $S_{i,j}$ and $S_{p,q}$ corresponding H_i and H_p are orthogonal whenever $i \neq p$. Moreover, by Lemma 4.11 the matrices $S_{i,j}$ are linearly independent for a single matrix H_i, which completes the proof. □

Theorem 4.13 ([47])**.** *For every integer n for which there exists a quaternary Hadamard matrix of order n,* $|\boldsymbol{MUQH}(n)| \leq n$.

Proof. Let $\{H_1, H_2, \ldots, H_k\}$ be a set of mutually unbiased quaternary Hadamard matrices of order n. Then, following Lemma 4.12 define $S_{i,j} = r_{i,j}^* r_{i,j} - I_n$. We have shown that each

$S_{i,j}$ is Hermitian and $S = \{S_{i,j} : 1 \le i \le k,\ 2 \le j \le n\}$ is linearly independent. Moreover, $S_{i,j}$ has a zero diagonal, and so span$\{S_{i,j}\}$ is a subspace of all Hermitian matrices with a zero diagonal. Since the set of Hermitian matrices of order n with zero diagonal has dimension $n(n-1)$, we have

$$k(n-1) = \dim(S) \le 2 \times \frac{n(n-1)}{2} = n(n-1),$$

and so $k \le n$ as desired. □

Theorem 4.14 ([17])**.** *For every integer n for which a Hadamard matrix exists,*

$$|\boldsymbol{MUH}(n)| \le n/2.$$

Proof. Let $\{H_1, H_2, \ldots, H_k\}$ be a set of mutually unbiased Hadamard matrices of order n. Following Theorem 4.13, the set S is linearly independent, and each $S_{i,j}$ is now symmetric. Thus, span$\{S_{i,j}\}$ is a subspace of the space of all symmetric matrices with a zero diagonal, which has dimension $n(n-1)/2$. Therefore

$$k(n-1) \le \frac{n(n-1)}{2},$$

so $k \le n/2$. □

4.1 Unbiased Skew-Type Hadamard Matrices of Order 36

We will once again utilize the classification of SH-inequivalent skew-type Hadamard matrices of order 36 [3]. In Section 3.1.1, we found that up to SH-equivalence, each one of the 157132 skew-type Hadamard matrices of order 36 was equivalent to a regular (specifically a skew-regular) Hadamard matrix.

The Algorithm Section 3.1.1, originating from [24], worked by finding vectors v and then using the permutation matrix $P = \text{diag}(v)$ to turn each skew-type Hadamard matrix H

into a skew-regular Hadamard matrix $K = PHP$. Essentially, the algorithm looked for all vectors v such that the inner product of v and each row of H was ± 6 and then used these vectors to regularize and preserve the skew-type property of each matrix simultaneously. The number of regularizing vectors v found for each matrix is detailed in Table A.1.

According to our previous discussion, to find how many of these skew-type Hadamard matrices of order 36 have unbiased mates, we need to see how many of these matrices have a set of 36 mutually orthogonal regularizing vectors. By extending the algorithm for skew-regularity, this question can be answered easily. For each skew-type Hadamard matrix H, we employ the following algorithm.

The Updated Skew-Regular Algorithm:

1-4. These steps follow the same as the search for regularizing vectors, detailed in Section 3.1.1. Let $\mathcal{R}$ be the set of all regularizing vectors for H.

5. Exhaustively search for sets of 36 mutually orthogonal vectors in $\mathcal{R}$.

Each set of 36 mutually orthogonal vectors will then correspond to an unbiased mate of H. Applying the updated algorithm to the set of 157132 skew-type Hadamard matrices of order 36, we have the following.

Theorem 4.15. *There are at least two SH-inequivalent skew-type Hadamard matrices of order* 36 *that have an unbiased mate.*

For each matrix that forms an unbiased pair, the unbiased mate is unique up to equivalence. The two unbiased pairs can be found in Appendix B.1.

Remark 4.16. The reason Theorem 4.15 gives a lower bound for the number of matrices with unbiased pairs is that the search for SH-inequivalent skew-type Hadamard matrices is not complete. However, computation suggests that there are likely no additional equivalence classes [3].

4.2 Search for Unbiased Quaternary Hadamard Matrices

In studying unbiased quaternary Hadamard matrices, we have two main goals: to find the largest set of mutually unbiased quaternary Hadamard matrices that contains each inequivalent quaternary Hadamard matrix of order n, and to find examples that attain these bounds.

This section will utilize the classifications of quaternary Hadamard matrices by Lampio [30] for orders $n \leq 10$, and the classification of order 18 by Östergård [35]. Using these classifications, we detail the exhaustive search for every equivalence class in each classification, first published in [27]. As a byproduct of this search, the algorithm will also classify all regular quaternary Hadamard matrices of small orders.

Throughout this section, we will also provide the number of inequivalent regularizing vectors for each equivalence class of quaternary Hadamard matrices. In this context, we define two vectors as equivalent if and only if they are scalar multiples of one another. For classifying orders $2,4,8$, and 10, a straightforward computer search is used. For each quaternary Hadamard matrix H of order n, apply the following algorithm.

Unbiased Classification Algorithm:

1. Find all inequivalent regularizing vectors of H.

2. From the set of inequivalent regularizing vectors, find all possible sets of n mutually orthogonal regularizing vectors. These sets of n mutually orthogonal regularizing vectors form the unbiased mates of H.

3. After finding all unbiased mates of H, determine the largest set of mutually unbiased matrices by checking all possible combinations of matrices.

By Theorem 4.8, the maximum size of a set of mutually unbiased quaternary Hadamard matrices is two for orders 2, 10, and 18. This means that step 3 of the classification algorithm can be ignored for these orders, as all unbiased mates for a particular matrix are found in step 2.

4.2.1 Unbiased Quaternary Hadamard Matrices Orders 2 and 4

The two smallest orders for which non-singleton sets of mutually unbiased quaternary Hadamard matrices exist are orders 2 and 4. There is one equivalence class of quaternary Hadamard matrices of order 2, and two equivalence classes of order 4.

The quaternary Hadamard matrix of order 2 only has two inequivalent regularizing vectors, which form an unbiased mate. The following is a set of mutually unbiased quaternary Hadamard matrices of order 2, meeting the bound of Theorem 4.8.

$$\left\{ \begin{pmatrix} 1 & 1 \\ 1 & - \end{pmatrix}, \begin{pmatrix} 1 & j \\ j & 1 \end{pmatrix} \right\}.$$

The computations for order 4 are summarized in Table 4.1. Note that the largest set of MUQH includes the original matrix. The following are the largest sets of mutually unbiased quaternary Hadamard matrices of order 4 for each equivalence class. The first element in the set is the representative for the particular equivalence class.

$$\left\{ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & - & - \\ 1 & - & 1 & - \\ 1 & - & - & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & i & j \\ 1 & 1 & j & i \\ i & j & 1 & 1 \\ j & i & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & i & 1 & j \\ 1 & j & 1 & i \\ i & 1 & j & 1 \\ j & 1 & i & 1 \end{pmatrix}, \begin{pmatrix} 1 & i & j & 1 \\ 1 & j & i & 1 \\ i & 1 & 1 & j \\ j & 1 & 1 & i \end{pmatrix} \right\},$$

$$\left\{ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & - & - \\ 1 & - & i & j \\ 1 & - & j & i \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & - \\ 1 & 1 & - & 1 \\ 1 & - & 1 & 1 \\ - & 1 & 1 & 1 \end{pmatrix} \right\}.$$

Our computations establish the following results.

Lemma 4.17. *All quaternary Hadamard matrices of order $n \leq 4$ are equivalent to a regular*

Table 4.1: Classification of Unbiased Quaternary Hadamard Matrices of Order 4.

Matrix #	**# Reg. Vectors**	**# Unbiased Mates**	**Largest Set of MUQH**
1	16	10	4
2	8	4	2

quaternary Hadamard matrix.

Lemma 4.18. *Only one equivalence class of quaternary Hadamard matrices of order* 4 *attain the upper bound of* $|\boldsymbol{MUQH(4)}| = 4$.

4.2.2 Unbiased Quaternary Hadamard Matrices of Order 8

There are 15 equivalence classes of unbiased quaternary Hadamard matrices of order 8 [30]. Running the same computation as for orders 2 and 4, we find:

- One equivalence class forms a set of 8 mutually unbiased quaternary Hadamard matrices.
- Three equivalence classes form a set of 4 mutually unbiased quaternary Hadamard matrices.
- Seven equivalence classes form a pair of mutually unbiased quaternary Hadamard matrices.
- Four equivalence classes are not regular, and cannot have any unbiased mates.

The computations are summarized in detail in Table 4.2.

Theorem 4.19. *There are* 11 *equivalence classes of quaternary Hadamard matrices of order* 8 *that are equivalent to a regular quaternary Hadamard matrix, and* 4 *equivalence classes that are not.*

Theorem 4.20. *Only one equivalence class of quaternary Hadamard matrices of order* 8 *attain the upper bound of* $|\boldsymbol{MUQH(8)}| = 8$.

Table 4.2: Classification of Unbiased Quaternary Hadamard Matrices of Order 8.

Matrix #	**# Reg. Vectors**	**# Unbiased Mates**	**Largest Set of MUQH**
1	224	3248	8
2	96	288	4
3	96	288	4
4	32	64	2
5	224	912	4
6	96	112	2
7	32	24	2
8	96	112	2
9	32	24	2
10	0	0	1
11	0	0	1
12	32	64	2
13	0	0	1
14	32	100	2
15	0	0	1

The theoretically maximal set of mutually unbiased quaternary Hadamard matrices is found in Appendix B.2.1.

Remark 4.21. Notably, for orders 2, 4, and 8, it is the real Hadamard matrices that form the theoretical maximal set of mutually unbiased quaternary Hadamard matrices, which attain equality in Theorem 4.13.

4.2.3 Unbiased Quaternary Hadamard Matrices of Order 10

There are 10 equivalence classes of unbiased quaternary Hadamard matrices of order 10 [30]. The computation for order 10 found:

- Two equivalence classes form a pair of mutually unbiased quaternary Hadamard matrices, which meet the upper bound of Theorem 4.8.
- Eight equivalence classes are not regular, and cannot have any unbiased mates.

The computations are summarized in detail in Table 4.3. The unbiased pair corresponding to matrix 9 can be found in Appendix B.2.2. The example corresponding to matrix #2 can be found in [4].

Table 4.3: Classification of Unbiased Quaternary Hadamard Matrices of Order 10 [27].

Matrix #	# Reg. Vectors	# Unbiased Mates	Largest Set of MUQHs
1	176	0	1
2	304	0	1
3	432	36	2
4	152	0	1
5	304	0	1
6	232	0	1
7	132	0	1
8	232	0	1
9	112	1	2
10	80	0	1

Theorem 4.22 ([27]). *There are* 2 *equivalence classes of quaternary Hadamard matrices of order* 10 *that are regular, and* 8 *equivalence classes that are not.*

Theorem 4.23 ([27]). *There are two equivalence classes of quaternary Hadamard matrices of order* 10 *that attain the upper bound of* $|\textbf{\textit{MUQH(10)}}| = 2$.

4.2.4 Unbiased Quaternary Hadamard Matrices of Order 18

In a recent classification, Östergård and Paavola found 3,830,723 inequivalent quaternary Hadamard matrices of order 18 [35]. However, the authors released a database of only 1,955,625 matrices, representing the equivalence classes obtained by including complex conjugation as an equivalence operation. Consequently, our computations for this section are restricted to the 1,955,625 available matrices, requiring us to adopt the broader definition of equivalence that includes conjugate transposes. The unbiased classification algorithm detailed at the beginning of Section 4.2 takes approximately three minutes to run on a single-core 3.2 GHz CPU for a single quaternary Hadamard matrix of order 18. Because there are more than 1.9 million matrices, the previous algorithm is impractical for classifying order 18.

We present an updated algorithm, designed specifically for order 18. For each normalized quaternary Hadamard matrix H, we implement the following algorithm, first introduced in [27].

Unbiased Classification Algorithm for Order 18:

1. Split $H = [H_1|H_2]$, where H_1 and H_2 each have 9 columns.
2. Compute $H_1 v$ and $H_2 v$ for all vectors v having entries in $\{1, -1, i, -i\}$.
3. Find all vector pairs (v, w) such that $H_1 v + H_2 w \equiv 0 \pmod 3$.
4. Concatenate each pair (v, w), and define $\mathcal{R}_H$ to be the set of concatenated vectors v, w such that the sum of the entries is $3 + 3i$. Note that every regularizing vector of H is equivalent to a vector in $\mathcal{R}_H$.
5. Exhaustively search for sets of 18 mutually orthogonal vectors in $\mathcal{R}_H$ to find an unbiased mate of H.

Our search establishes the following result.

Theorem 4.24 ([27])**.** *There are* 184 *equivalence classes of quaternary Hadamard matrices of order* 18 *that attain the upper bound of* $|\textbf{\textit{MUQH(18)}}| = 2$.

Moreover, the search revealed that most quaternary Hadamard matrices of order 18 are regular.

Theorem 4.25 ([27])**.** *Out of the 1,955,625 inequivalent quaternary Hadamard matrices of order 18:*

- $\mathcal{R}_H = \emptyset$ *for 28 matrices, meaning that there are* 28 *non-regular equivalence classes.*
- *1,955,597 equivalence classes are regular.*

Example 4.26. Below is a non-regular quaternary Hadamard matrix of order 18.

$$
\begin{pmatrix}
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&i&i&i&-&-&-&-&-&-&j&j&j\\
1&1&1&-&-&-&1&j&j&i&i&-&-&-&j&1&1&i\\
1&1&-&1&-&j&i&-&j&i&j&1&i&j&-&i&-&1\\
1&1&i&-&1&j&j&-&i&1&1&j&-&1&-&-&i&-\\
1&1&j&j&i&i&-&i&j&j&i&-&1&1&j&i&-&-\\
1&i&-&-&j&1&1&i&-&1&j&-&1&-&1&-&j&i\\
1&i&1&j&i&-&j&j&i&-&j&i&1&j&i&j&-&i\\
1&i&-&-&1&j&-&j&j&j&-&i&i&i&i&1&1&j\\
1&-&1&i&-&1&1&-&j&j&i&i&j&j&i&-&i&j\\
1&-&i&1&1&-&-&1&j&1&i&1&j&-&-&-&j&i\\
1&-&j&-&-&j&-&i&i&i&i&j&j&1&i&1&j&1\\
1&-&i&-&j&i&j&i&1&-&1&1&i&-&j&1&-&j\\
1&-&j&i&1&-&1&1&-&-&-&j&i&1&j&-&i&1\\
1&-&j&1&j&-&i&-&i&1&j&i&j&i&j&1&i&-\\
1&j&i&j&-&i&-&1&-&i&j&i&-&1&1&-&1&j\\
1&j&-&i&i&i&1&j&1&-&j&j&j&i&i&i&j&-\\
1&j&-&1&-&1&-&j&i&j&i&j&i&-&1&j&i&i
\end{pmatrix}.
$$

4.2.5 Non-Regular Hadamard Matrices of Order 36

Recall from Section 2.3.3 that a quaternary Hadamard matrix of order $2n$ can be converted to a Hadamard matrix.

Lemma 4.27 ([27]). *Let H be a quaternary Hadamard matrix of order 18. Split $H = R + iI$, where $R = Re(H)$ and $I = Im(H)$. Then*

$$K = R \otimes C + I \otimes D$$

is a Hadamard matrix of order 36, where

$$C = \begin{pmatrix} - & 1 \\ 1 & 1 \end{pmatrix}, \text{ and } D = \begin{pmatrix} 1 & 1 \\ 1 & - \end{pmatrix}.$$

Furthermore, H is equivalent to a regular quaternary Hadamard matrix if and only if K is equivalent to a regular Hadamard matrix.

Proof. Theorem 2.62 shows K is a Hadamard matrix. Moreover, without loss of generality, assume H is regular with constant row sum $3+3i$. Then consider any row of H. When we convert this row to two real rows, the first row will have a sum of 6 because there are 3 more i entries than $-i$ entries in H. Similarly, the second row will have a sum of 6 because there are 3 more 1 entries than -1 entries in H.

Conversely, suppose K is equivalent to a regular matrix. Since permuting rows and columns does not change row sums, there exist signed diagonal matrices P and Q such that PKQ is regular. Without loss of generality, assume PKQ has a constant row sum of 6.

- If P negates both rows coming from a single row of H, this is the same as negating a row of H.

- If Q negates both columns coming from a single column of H, this is the same as negating a column of H.

- If P negates the first of two rows coming from a single row of H, after swapping the order of the two rows corresponding to a single row of H in K, this is equivalent to multiplying a column of H by i.

- If P negates the second of two rows coming from a single row of H, after swapping the order of the two rows corresponding to a single row of H in K, this is equivalent to multiplying a column of H by $-i$.

- If Q negates the first of two columns coming from a single column of H, after swapping the order of the two columns corresponding to a single column of H in K, this is equivalent to multiplying a column of H by i.

- If Q negates the second of two columns coming from a single column of H, after swapping the order of the two columns corresponding to the single column of H in K, this is equivalent to multiplying a column of H by $-i$.

This process shows PKQ is equivalent to a regular matrix made up of 2×2 blocks $C, -C, D$ and $-D$. Since the matrix will have a constant row sum of 6, there must be 3 more C blocks than $-C$ blocks, and 3 more D blocks than $-D$ blocks in every block row and column. Thus, performing the operations described above on H yields an equivalent quaternary Hadamard matrix with constant row and column sums of $3+3i$. □

Converting the 28 quaternary Hadamard matrices of order 18 to Hadamard matrices, there are 16 equivalence classes of non-regular Hadamard matrices of order 36 [27]. The fact that only a tiny fraction of quaternary Hadamard matrices of order 18 are non-regular is an indication that non-regular Hadamard matrices of order 36 are quite rare.

From these 16 equivalence classes of non-regular Hadamard matrices, we classified them up to maximum excess classes as follows.

- 8 equivalence classes where each matrix had a maximum excess of 204, and
- 8 equivalence classes where each matrix had a maximum excess of 208.

Two examples of Hadamard matrices of order 36 with maximum excess 204 and 208 are shown in Appendix B.3.1.

4.2.6 An Application to Extremal Codes

Converting each of the 184 Quaternary Hadamard matrices with unbiased pairs to real Hadamard matrices, we obtained 6 equivalent extremal codes with 3-rank 18 and minimum distance 12 [27]. This code, discovered by Pless [37] in 1972, is known as the Pless symmetry code. The six matrices leading to the Pless symmetry code belong to two equivalence classes of Hadamard matrices, with each equivalence class containing 3 Hadamard matrices.

Table 4.4 shows the parameters of the codes from the 184 Quaternary Hadamard matrices with unbiased pairs. In total, there were 6 matrices leading to extremal codes, 68 matrices leading to near-extremal codes and 39 leading to self-dual codes with minimum distance 6 [27]. The remaining codes were not self-dual.

Table 4.4: Count of (3-rank, minimum distance) pairs [27].

(3-rank, Minimum Distance)	Count
(14, 9)	9
(14, 12)	3
(16, 6)	18
(16, 9)	40
(16, 12)	1
(18, 6)	39
(18, 9)	68
(18, 12)	6

Remark 4.28. For codes of length 36, the Pless symmetry code, first appearing in a 1972 article by Vera Pless [37], is currently the only known extremal ternary self-dual code.

Appendix B.3.2 provides an example of a Hadamard matrix that generates an extremal ternary code, along with a summary of the equivalence of the objects discussed in this section.

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Appendix A

Detailed Results from Chapter 3

A.1 Skew-Regular Hadamard Matrices of Order 36

Table A.1 shows the number of valid vectors identified in step four of the skew-regular algorithm that successfully yield a skew-regular Hadamard matrix.

Table A.1: Sorting the skew-type Hadamard matrices by the number of regularizing vectors.

# Vectors	# Matrices	# Vectors	# Matrices
160	2	88	12958
152	10	84	15900
148	2	80	18204
144	8	76	18242
140	14	72	18094
136	46	68	15104
132	52	64	12022
128	122	60	7986
124	214	56	4612
120	416	52	2414
116	662	48	1192
112	1122	44	424
108	1838	40	146
104	3112	36	50
100	4822	32	20
96	7176	28	12
92	10128	24	6

A.2 Example of Construction Using Theorem 3.26

Example A.1. Suppose $p = 5$. We have

$$H = \left(\begin{array}{cccccccccccccccccccccccccc}
1 & j & j & j & j & j & j & j & j & j & j & j & j & j & j & j & j & j & j & j & j & j & j & j & j & j \\
j & 1 & j & j & j & j & j & i & i & i & j & j & i & i & j & i & j & i & j & i & i & j & j & i & i & i \\
j & j & 1 & j & j & j & j & j & i & i & i & i & j & i & i & j & i & j & i & j & i & i & j & j & i & i \\
j & j & j & 1 & j & j & i & j & j & i & i & j & i & j & i & i & i & i & j & i & j & i & i & j & j & i \\
j & j & j & j & 1 & j & i & i & j & j & i & i & j & i & j & i & j & i & i & j & i & i & i & i & j & j \\
j & j & j & j & j & 1 & i & i & i & j & j & i & i & j & i & j & i & j & i & i & j & j & i & i & i & j \\
j & j & j & i & i & i & 1 & j & j & j & j & j & i & i & i & j & j & i & i & j & i & j & i & j & i & i \\
j & i & j & j & i & i & j & 1 & j & j & j & j & j & i & i & i & i & j & i & i & j & i & j & i & j & i \\
j & i & i & j & j & i & j & j & 1 & j & j & i & j & j & i & i & j & i & j & i & i & i & i & j & i & j \\
j & i & i & i & j & j & j & j & j & 1 & j & i & i & j & j & i & i & j & i & j & i & j & i & i & j & i \\
j & j & i & i & i & j & j & j & j & j & 1 & i & i & i & j & j & i & i & j & i & j & i & j & i & i & j \\
j & j & i & j & i & i & j & j & i & i & i & 1 & j & j & j & j & j & i & i & i & j & j & i & i & j & i \\
j & i & j & i & j & i & i & j & j & i & i & j & 1 & j & j & j & j & j & i & i & i & i & j & i & i & j \\
j & i & i & j & i & j & i & i & j & j & i & j & j & 1 & j & j & i & j & j & i & i & j & i & j & i & i \\
j & j & i & i & j & i & i & i & i & j & j & j & j & j & 1 & j & i & i & j & j & i & i & j & i & j & i \\
j & i & j & i & i & j & j & i & i & i & j & j & j & j & j & 1 & i & i & i & j & j & i & i & j & i & j \\
j & j & i & i & j & i & j & i & j & i & i & j & j & i & i & i & 1 & j & j & j & j & j & i & i & i & j \\
j & i & j & i & i & j & i & j & i & j & i & i & j & j & i & i & j & 1 & j & j & j & j & j & i & i & i \\
j & j & i & j & i & i & i & i & j & i & j & i & i & j & j & i & j & j & 1 & j & j & i & j & j & i & i \\
j & i & j & i & j & i & j & i & i & j & i & i & i & i & j & j & j & j & j & 1 & j & i & i & j & j & i \\
j & i & i & j & i & j & i & j & i & i & j & j & i & i & i & j & j & j & j & j & 1 & i & i & i & j & j \\
j & j & i & i & i & j & j & i & i & j & i & j & i & j & i & i & j & j & i & i & i & 1 & j & j & j & j \\
j & j & j & i & i & i & i & j & i & i & j & i & j & i & j & i & i & j & j & i & i & j & 1 & j & j & j \\
j & i & j & j & i & i & j & i & j & i & i & i & i & j & i & j & i & i & j & j & i & j & j & 1 & j & j \\
j & i & i & j & j & i & i & j & i & j & i & j & i & i & j & i & i & i & i & j & j & j & j & j & 1 & j \\
j & i & i & i & j & j & i & i & j & i & j & i & j & i & i & j & j & i & i & i & j & j & j & j & j & 1
\end{array}\right).$$

Then, using the additive cosets $C_0, C_1, C_2, \ldots, C_{p-1}$, as described in Theorem 3.26, define

$$v = \Big[\underbrace{1}_{\infty} \; \underbrace{1\;1\;1\;1\;1}_{\mathbb{F}_p = C_0} \; \underbrace{j\;j\;j\;j\;j}_{C_1} \; \underbrace{j\;j\;j\;j\;j}_{C_2} \; \underbrace{i\;i\;i\;i\;i}_{C_3} \; \underbrace{i\;i\;i\;i\;i}_{C_4} \Big].$$

Defining $M = diag(v)$, we have

$$S = MHM^* = \left(\begin{array}{cccccccccccccccccccccccccc}
1 & j & j & j & j & j & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - & - & - & - & - \\
j & 1 & j & j & j & j & 1 & - & - & - & 1 & 1 & - & - & 1 & - & - & 1 & - & 1 & 1 & - & - & 1 & 1 & 1 \\
j & j & 1 & j & j & j & 1 & 1 & - & - & - & - & 1 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & - & - & 1 & 1 \\
j & j & j & 1 & j & j & - & 1 & 1 & - & - & 1 & - & 1 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & - & - & 1 \\
j & j & j & j & 1 & j & - & - & 1 & 1 & - & - & 1 & - & 1 & - & - & 1 & 1 & - & 1 & 1 & 1 & 1 & - & - \\
j & j & j & j & j & 1 & - & - & - & 1 & 1 & - & - & 1 & - & 1 & 1 & - & 1 & 1 & - & - & 1 & 1 & 1 & - \\
- & - & - & 1 & 1 & 1 & 1 & j & j & j & j & j & i & i & i & j & i & j & j & i & j & i & j & i & j & j \\
- & 1 & - & - & 1 & 1 & j & 1 & j & j & j & j & j & i & i & i & j & i & j & j & i & j & i & j & i & j \\
- & 1 & 1 & - & - & 1 & j & j & 1 & j & j & i & j & j & i & i & i & j & i & j & j & j & j & i & j & i \\
- & 1 & 1 & 1 & - & - & j & j & j & 1 & j & i & i & j & j & i & j & i & j & i & j & i & j & j & i & j \\
- & - & 1 & 1 & 1 & - & j & j & j & j & 1 & i & i & i & j & j & j & j & i & j & i & j & i & j & j & i \\
- & - & 1 & - & 1 & 1 & j & j & i & i & i & 1 & j & j & j & j & i & j & j & j & i & i & j & j & i & j \\
- & 1 & - & 1 & - & 1 & i & j & j & i & i & j & 1 & j & j & j & i & i & j & j & j & j & i & j & j & i \\
- & 1 & 1 & - & 1 & - & i & i & j & j & i & j & j & 1 & j & j & j & i & i & j & j & i & j & i & j & j \\
- & - & 1 & 1 & - & 1 & i & i & i & j & j & j & j & j & 1 & j & j & j & i & i & j & j & i & j & i & j \\
- & 1 & - & 1 & 1 & - & j & i & i & i & j & j & j & j & j & 1 & j & j & j & i & i & j & j & i & j & i \\
1 & 1 & - & - & 1 & - & i & j & i & j & j & i & i & j & j & j & 1 & j & j & j & j & j & i & i & i & j \\
1 & - & 1 & - & - & 1 & j & i & j & i & j & j & i & i & j & j & j & 1 & j & j & j & j & j & i & i & i \\
1 & 1 & - & 1 & - & - & j & j & i & j & i & j & j & i & i & j & j & j & 1 & j & j & i & j & j & i & i \\
1 & - & 1 & - & 1 & - & i & j & j & i & j & j & j & j & i & i & j & j & j & 1 & j & i & i & j & j & i \\
1 & - & - & 1 & - & 1 & j & i & j & j & i & i & j & j & j & i & j & j & j & j & 1 & i & i & i & j & j \\
1 & 1 & - & - & - & 1 & i & j & j & i & j & i & j & i & j & j & j & j & i & i & i & 1 & j & j & j & j \\
1 & 1 & 1 & - & - & - & j & i & j & j & i & j & i & j & i & j & i & j & j & i & i & j & 1 & j & j & j \\
1 & - & 1 & 1 & - & - & i & j & i & j & j & j & j & i & j & i & i & i & j & j & i & j & j & 1 & j & j \\
1 & - & - & 1 & 1 & - & j & i & j & i & j & i & j & j & i & j & i & i & i & j & j & j & j & j & 1 & j \\
1 & - & - & - & 1 & 1 & j & j & i & j & i & j & i & j & j & i & j & i & i & i & j & j & j & j & j & 1
\end{array}\right)$$

which is a skew-regular quaternary Hadamard matrix of order 26 with constant row sum $1-5i$.

Appendix B

Detailed Results from Chapter 4

B.1 Unbiased Skew-type Hadamard Matrices of Order 36

The number of regularizing vectors for each matrix can be found in Table A.1. Both matrices with unbiased mates had 144 regularizing vectors. Below are the two inequivalent unbiased pairs (H_i, K_i) where H_i is the skew-type Hadamard matrix.

$$H_1 = \begin{pmatrix}
1&- \\
1&1&-&-&-&-&-&-&-&-&-&-&-&-&-&-&-&-&-&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\
1&1&1&-&-&-&-&-&-&-&-&1&1&1&1&1&1&1&1&-&-&-&-&-&-&-&-&-&1&1&1&1&1&1&1&1 \\
1&1&1&1&-&-&-&-&-&-&1&-&-&1&1&1&1&1&1&-&-&1&1&1&1&1&1&1&-&-&-&-&-&-&-&1 \\
1&1&1&1&1&-&-&-&-&1&1&1&1&-&-&-&-&1&1&1&1&-&-&-&-&1&1&1&-&-&-&-&1&1&1&- \\
1&1&1&1&1&1&-&-&1&-&-&1&1&-&-&1&1&-&-&-&1&-&1&1&1&-&-&1&-&-&1&1&-&-&1&- \\
1&1&1&1&1&1&1&-&-&-&1&-&1&1&1&-&-&-&-&1&-&1&-&-&1&-&1&-&-&1&1&1&-&1&-&- \\
1&1&1&1&1&1&1&1&-&-&-&-&-&-&1&-&1&-&1&1&1&-&-&1&-&1&-&-&1&1&-&-&-&-&1&1 \\
1&1&1&1&1&-&1&1&1&1&-&-&-&1&-&-&-&1&-&-&-&-&1&1&-&1&-&-&-&-&1&1&1&1&-&1 \\
1&1&1&1&-&1&1&1&-&1&-&1&-&-&-&1&-&-&1&-&-&1&1&-&-&-&1&1&1&1&-&1&1&-&-&- \\
1&1&1&-&-&1&-&1&1&1&1&-&-&1&-&1&-&1&-&1&-&-&-&1&1&-&-&1&1&1&-&-&-&1&1&- \\
1&1&-&1&-&-&1&1&1&-&1&1&-&1&1&-&1&-&-&-&1&-&1&-&1&-&1&-&1&-&-&-&1&1&1&- \\
1&1&-&1&-&-&-&1&1&1&1&1&1&-&1&-&1&-&-&1&-&1&-&1&-&-&-&1&-&1&1&-&1&-&-&1 \\
1&1&-&-&1&1&-&1&-&1&-&-&1&1&-&-&1&1&1&-&1&1&-&1&1&-&1&-&1&-&1&-&1&-&-&- \\
1&1&-&-&1&1&-&-&1&1&1&-&-&1&1&1&-&-&1&1&1&1&1&-&-&-&-&-&-&-&-&1&1&-&1&1 \\
1&1&-&-&1&-&1&1&1&-&-&1&1&1&-&1&-&-&1&-&1&1&-&-&1&1&-&1&-&1&-&-&-&1&-&1 \\
1&1&-&-&1&-&1&-&1&1&1&-&-&-&1&1&1&1&-&-&1&-&-&-&-&1&1&1&1&1&1&1&-&-&-&- \\
1&1&-&-&-&1&1&1&-&1&-&1&1&-&1&1&-&1&-&1&-&-&1&-&1&1&1&-&-&-&1&-&-&-&1&1 \\
1&1&-&-&-&1&1&-&1&-&1&1&1&-&-&-&1&1&1&1&-&1&1&1&-&1&-&-&1&-&-&1&-&1&-&- \\
1&-&1&1&-&1&-&-&1&1&-&1&-&1&-&1&1&-&-&1&1&1&-&-&-&1&1&-&1&-&1&-&-&1&-&1 \\
1&-&1&1&-&-&1&-&1&1&1&-&1&-&-&-&-&1&1&-&1&1&1&-&1&-&-&-&1&1&1&-&-&-&1&1 \\
1&-&1&-&1&1&-&1&1&-&1&1&-&-&-&-&1&1&-&-&-&1&-&-&1&1&1&-&-&1&-&1&1&-&1&1 \\
1&-&1&-&1&-&1&1&-&-&1&-&1&1&-&1&1&-&-&1&-&1&1&-&-&1&-&1&1&-&1&-&1&-&1&- \\
1&-&1&-&1&-&1&-&-&1&-&1&-&-&1&1&1&1&-&1&1&1&1&1&1&-&-&-&-&1&-&-&1&1&-&- \\
1&-&1&-&1&-&-&1&1&1&-&-&1&-&1&-&1&-&1&1&-&-&1&-&1&-&1&1&1&-&-&1&-&1&-&1 \\
1&-&1&-&-&1&1&-&-&1&1&1&1&1&1&-&-&-&-&-&1&-&-&1&1&1&-&1&1&-&-&1&1&-&-&1 \\
1&-&1&-&-&1&-&1&1&-&1&-&1&-&1&1&-&-&1&-&1&-&1&1&-&1&1&-&-&1&1&-&1&1&-&- \\
1&-&1&-&-&-&1&1&1&-&-&1&-&1&1&-&-&1&1&1&1&1&-&1&-&-&1&1&-&-&1&1&-&-&1&- \\
1&-&-&1&1&1&1&-&1&-&-&-&1&-&1&1&-&1&-&-&-&1&-&1&-&-&1&1&1&-&-&-&1&1&1&1 \\
1&-&-&1&1&1&-&-&1&-&-&1&-&1&1&-&-&1&1&1&-&-&1&-&1&1&-&1&1&1&1&-&1&-&-&- \\
1&-&-&1&1&-&-&1&-&1&1&1&-&-&1&1&-&-&1&-&-&1&-&1&1&1&-&-&1&-&1&1&-&1&1&- \\
1&-&-&1&1&-&-&1&-&-&1&1&1&1&-&1&-&1&-&1&1&-&1&1&-&-&1&-&1&1&-&1&-&-&-&1 \\
1&-&-&1&-&1&1&1&-&-&1&-&-&-&-&1&1&1&1&1&1&-&-&-&1&-&-&1&-&-&1&1&1&1&-&1 \\
1&-&-&1&-&1&-&1&-&1&-&-&1&1&1&-&1&1&-&-&1&1&1&-&-&1&-&1&-&1&-&1&-&1&1&- \\
1&-&-&1&-&-&1&-&1&1&-&-&1&1&-&1&1&-&1&1&-&-&-&1&1&1&1&-&-&1&-&1&1&-&1&- \\
1&-&-&-&1&1&1&-&-&1&1&1&-&1&-&-&1&-&1&-&-&-&1&1&-&-&1&1&-&1&1&-&-&1&1&1
\end{pmatrix},$$

$$K_1 = \begin{pmatrix}
- & - & - & - & 1 & - & - & 1 & - & - & 1 & 1 & 1 & - & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - & - & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & - & 1 & - & 1 & 1 & - & - & - & 1 & - & - & 1 & 1 & 1 & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & 1 & 1 & 1 & 1 & 1 & 1 \\
- & - & - & - & 1 & 1 & 1 & - & 1 & - & 1 & 1 & - & - & 1 & - & - & - & 1 & - & 1 & 1 & 1 & 1 & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 \\
- & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 & - & - & 1 & - & - & 1 & 1 & - & 1 & - & - & 1 & 1 & 1 & 1 & 1 & - & - & 1 & - & 1 & - & 1 & 1 & 1 \\
1 & - & 1 & 1 & 1 & - & 1 & 1 & - & - & - & 1 & 1 & - & 1 & - & - & 1 & - & - & 1 & - & - & - & 1 & - & - & - & - & - & - & 1 & - & 1 & 1 & 1 \\
- & - & 1 & 1 & 1 & 1 & 1 & - & 1 & - & - & - & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & - & - & - & 1 & - & - & 1 & 1 & 1 & 1 & - & - & 1 & 1 & 1 \\
1 & - & - & - & - & - & - & 1 & - & 1 & 1 & - & - & - & 1 & 1 & 1 & - & - & 1 & 1 & - & 1 & - & - & - & - & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 \\
1 & 1 & 1 & - & - & - & 1 & - & 1 & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - & - & 1 & - & 1 & 1 & - & 1 & 1 & 1 & 1 & - & 1 & - & - & 1 & 1 & 1 \\
1 & - & 1 & 1 & - & - & - & 1 & 1 & 1 & - & - & - & - & - & - & - & - & 1 & - & - & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & - & - & - & 1 & 1 & 1 \\
1 & 1 & - & 1 & 1 & - & 1 & 1 & 1 & - & - & 1 & - & 1 & 1 & 1 & - & - & - & 1 & - & 1 & - & - & - & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & - & 1 & 1 \\
- & - & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & 1 & - & - & 1 & - & - & - & 1 & 1 & 1 & - & - & - & - & 1 & - & - & 1 & 1 & 1 & - & 1 & 1 \\
- & 1 & 1 & - & - & - & 1 & - & - & 1 & - & - & 1 & - & - & - & - & - & 1 & 1 & - & - & 1 & - & - & 1 & - & - & 1 & - & - & 1 & 1 & - & 1 & 1 \\
1 & 1 & 1 & - & - & 1 & - & 1 & - & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & 1 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 \\
1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & - & - & - & - & 1 & - & 1 & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & - & 1 & - & 1 & - & 1 & 1 \\
1 & - & - & 1 & - & 1 & - & - & - & 1 & 1 & 1 & - & 1 & - & 1 & - & 1 & 1 & - & - & - & - & - & 1 & - & 1 & - & 1 & - & - & - & 1 & - & 1 & 1 \\
- & 1 & - & - & 1 & 1 & - & 1 & 1 & 1 & - & - & 1 & - & - & 1 & - & 1 & - & - & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & 1 & - & 1 & - & - & 1 & 1 \\
- & 1 & - & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & - & 1 & 1 \\
- & 1 & 1 & - & - & 1 & - & - & 1 & - & - & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - & 1 & - & - & - & - & - & - & - & - & - & - & 1 & 1 \\
- & 1 & 1 & 1 & - & - & 1 & - & - & 1 & 1 & - & - & - & 1 & 1 & - & 1 & - & - & - & 1 & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & 1 \\
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1 & 1 & 1 & 1 & - & - & - & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - & - & 1 & - & 1 & 1 & 1 & - & 1 \\
- & 1 & 1 & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - & - & - & 1 & 1 & 1 & 1 & - & 1 & - & - & - & - & 1 & 1 & - & - & 1 & 1 & - & 1 & 1 & - & 1 \\
- & 1 & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & - & - & - & - & 1 & 1 & - & - & 1 & - & - & 1 & - & 1 & - & - & - & 1 & 1 & - & 1 \\
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1 & 1 & - & - & - & 1 & 1 & - & - & - & - & - & - & 1 & 1 & - & 1 & - & 1 & - & 1 & 1 & - & - & - & - & 1 & 1 & 1 & - & - & 1 & - & 1 & - & 1 \\
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- & - & - & - & - & - & 1 & 1 & 1 & - & 1 & - & 1 & - & 1 & 1 & 1 & - & - & - & - & - & - & - & 1 & 1 & 1 & 1 & - & - & - & - & 1 & - & - & 1 \\
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1 & - & 1 & - & 1 & - & 1 & 1 & 1 & 1 & 1 & - & - & 1 & - & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & - & - & 1 & - & - & 1 & - & - & - & 1 \\
- & - & - & - & - & - & 1 & - & - & 1 & - & 1 & 1 & 1 & - & - & - & - & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & - & - & 1 & 1 & - & - & - & - & 1 \\
1 & 1 & - & 1 & 1 & 1 & 1 & 1 & - & - & 1 & - & 1 & - & - & 1 & - & - & 1 & - & - & 1 & 1 & 1 & - & - & - & - & - & - & 1 & - & - & - & - & 1 \\
1 & - & 1 & 1 & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & 1
\end{pmatrix},$$

$$H_2 = \begin{pmatrix}
1 & - \\
1 & 1 & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & - & - & - & - & - & - & - & - & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - & - & - & - & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & - & - & - & - & - & - & 1 & - & - & 1 & 1 & 1 & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - & - & 1 \\
1 & 1 & 1 & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & - & - & - & - & 1 & 1 & 1 & - & - & - & - & 1 & 1 & 1 & - \\
1 & 1 & 1 & 1 & 1 & 1 & - & - & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - & - & 1 & - & 1 & 1 & 1 & - & - & 1 & - & - & 1 & 1 & - & - & 1 & - \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & 1 & - & - & 1 & 1 & - & 1 & - & - & 1 & 1 & - & - & - & - & 1 & 1 & - & 1 & 1 & 1 & 1 & - & - & - & - \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & 1 & - & 1 & - & - & - & 1 & 1 & - & 1 & - & 1 & 1 & - & - & - & - & 1 & - & - & - & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & - & - & - & - & - & - & 1 & 1 & - & - & - & - & 1 & - & 1 & - & 1 & - & 1 & - & - & 1 & 1 & 1 & - & 1 \\
1 & 1 & 1 & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & - & - & - & - & 1 & 1 & - & - & 1 & - & 1 & - & 1 & 1 & - & 1 & 1 & - & - \\
1 & 1 & 1 & - & - & 1 & - & 1 & 1 & 1 & 1 & - & - & - & 1 & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & - & 1 & - & - & - & 1 & 1 & 1 & - & 1 & - \\
1 & 1 & - & 1 & - & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & - & - & 1 & - & 1 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & - & - & 1 & - & 1 & - \\
1 & 1 & - & 1 & - & - & 1 & - & 1 & 1 & 1 & 1 & 1 & - & - & - & 1 & 1 & - & 1 & - & 1 & - & 1 & - & - & - & 1 & 1 & 1 & 1 & - & - & - & - & 1 \\
1 & 1 & - & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & - & - & 1 & - & 1 & - & - & 1 & - & 1 & - & 1 & - & 1 & 1 & - & - & 1 & 1 & 1 & - & - \\
1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 & - & - & - & 1 & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & - & - & - & 1 & - \\
1 & 1 & - & - & 1 & - & 1 & 1 & 1 & - & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 & - & - & - & 1 & 1 & - & 1 & - & - & 1 & - & 1 & - & - & 1 \\
1 & 1 & - & - & 1 & - & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & - & 1 & - & - & - & 1 & - & 1 & - & - & 1 & - & - \\
1 & 1 & - & - & - & 1 & 1 & 1 & - & 1 & - & 1 & - & 1 & - & - & 1 & 1 & 1 & 1 & 1 & - & 1 & 1 & - & 1 & - & - & - & - & - & 1 & - & - & 1 & 1 \\
1 & 1 & - & - & - & 1 & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & 1 & - & - & 1 & - & 1 & - & 1 & - & - & - & 1 & 1 & - & 1 & - & 1 & - & 1 & - & 1 \\
1 & - & 1 & 1 & - & 1 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & - & - & - & 1 & 1 & 1 & 1 & - & - & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 \\
1 & - & 1 & 1 & - & - & - & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & - & - & - & - & - & 1 & - & 1 & 1 & - & 1 & - & - & 1 & - & 1 & - & - & 1 & 1 & 1 \\
1 & - & 1 & - & 1 & 1 & 1 & - & 1 & - & - & 1 & - & - & - & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & - & - & 1 & - & 1 \\
1 & - & 1 & - & 1 & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & - & - & 1 & - & 1 & 1 & - & - & - & 1 & 1 & 1 & - & - & 1 & - & - & 1 & 1 \\
1 & - & 1 & - & 1 & - & 1 & - & 1 & 1 & - & - & - & - & 1 & 1 & 1 & - & 1 & 1 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 & - & 1 & - \\
1 & - & 1 & - & 1 & - & 1 & - & - & 1 & 1 & - & 1 & 1 & - & - & - & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & - & - & - \\
1 & - & 1 & - & - & 1 & - & 1 & 1 & - & 1 & 1 & 1 & - & - & - & 1 & - & 1 & 1 & - & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & - & 1 & - & - & 1 \\
1 & - & 1 & - & - & 1 & - & 1 & - & 1 & - & - & 1 & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & - & 1 & 1 & - & 1 & 1 & 1 & 1 & - & - & - & 1 & - & - \\
1 & - & 1 & - & - & - & 1 & 1 & 1 & - & 1 & 1 & - & - & 1 & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & 1 & - & 1 & - & 1 & - & 1 & 1 & - \\
1 & - & - & 1 & 1 & 1 & - & 1 & - & 1 & 1 & - & - & - & - & 1 & - & 1 & 1 & - & - & - & - & - & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & - & - & 1 & 1 \\
1 & - & - & 1 & 1 & 1 & - & - & 1 & - & 1 & - & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & - & - & - & - & - & - & 1 & - & - & 1 & 1 & 1 & 1 \\
1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & - & 1 & 1 & - & - & 1 & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & 1 & 1 & 1 & - & 1 & - & - \\
1 & - & - & 1 & 1 & - & - & 1 & - & 1 & - & 1 & 1 & - & 1 & 1 & 1 & - & - & - & 1 & 1 & - & 1 & - & 1 & 1 & - & - & 1 & - & 1 & 1 & - & - & 1 \\
1 & - & - & 1 & - & 1 & 1 & 1 & - & - & - & - & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & 1 & - & - & - & 1 & 1 & 1 & - & 1 & - & 1 & - & 1 & - \\
1 & - & - & 1 & - & 1 & 1 & - & - & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & - & - & 1 & 1 & 1 & 1 & - & - & 1 & - & - & 1 & 1 & 1 & - & - \\
1 & - & - & 1 & - & - & 1 & - & 1 & 1 & - & - & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 & - & - & 1 & 1 & 1 & - & - & - & 1 & 1 & - & 1 & 1 & - \\
1 & - & - & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & - & - & - & - & - & - & 1 & 1 & - & 1 & 1 & - & - & 1 & - & 1 & 1 & 1 & 1
\end{pmatrix},$$

$$K_2 = \begin{pmatrix}
1 & - & - & 1 & 1 & 1 & 1 & 1 & - & - & 1 & 1 & - & - & - & - & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 & 1 & 1 & - & 1 & - & - & 1 & 1 & 1 & 1 & 1 \\
1 & - & - & - & 1 & - & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & 1 & - & 1 & - & - & 1 & 1 & 1 & - & - & 1 & - & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 \\
- & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & - & - & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & 1 & 1 & - & 1 & - & - & 1 & 1 & 1 & 1 \\
- & 1 & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 & 1 & - & - & 1 & - & - & - & 1 & - & - & 1 & 1 & - & 1 & - & - & - & - & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & - & - & - & 1 & 1 & - & 1 & 1 & 1 & - & - & 1 & - & - & 1 & - & - & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & 1 & - & 1 & 1 & 1 \\
- & - & - & 1 & 1 & - & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & 1 & - & - & - & - & - & 1 & - & - & - & - & - & - & - & 1 & 1 & - & 1 & 1 & 1 \\
1 & 1 & 1 & - & 1 & 1 & - & - & - & 1 & - & 1 & - & - & 1 & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & - & 1 & 1 & 1 & - & - & 1 & - & 1 & 1 & 1 \\
- & - & 1 & 1 & - & 1 & 1 & - & - & - & 1 & - & 1 & 1 & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - & 1 & - & - & 1 & - & 1 & - & 1 & - & - & 1 & 1 & 1 \\
- & 1 & 1 & - & - & - & - & - & - & - & 1 & - & - & - & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & - & - & 1 & - & - & - & 1 & - & - & - & 1 & 1 & 1 \\
1 & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & - & - & 1 & - & 1 & - & 1 & 1 & - & 1 & - & - & 1 & 1 & - & 1 & - & - & 1 & 1 & 1 & 1 & 1 & - & 1 & 1 \\
1 & 1 & 1 & - & - & 1 & 1 & 1 & - & - & 1 & - & - & 1 & - & 1 & 1 & 1 & - & - & 1 & - & 1 & - & 1 & - & 1 & 1 & - & 1 & 1 & 1 & 1 & - & 1 & 1 \\
- & - & - & - & - & 1 & - & - & - & 1 & - & - & 1 & - & 1 & - & - & - & 1 & - & - & 1 & - & - & 1 & - & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & 1 \\
- & - & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & - & 1 & - & - & - & - & - & 1 & - & 1 & 1 & - & 1 & 1 \\
1 & 1 & 1 & - & - & - & 1 & - & 1 & 1 & - & 1 & - & 1 & - & - & - & - & - & 1 & - & - & - & - & - & 1 & 1 & - & 1 & - & 1 & - & 1 & - & 1 & 1 \\
- & - & 1 & 1 & - & - & 1 & - & 1 & 1 & - & - & - & - & - & - & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & - & 1 & 1 \\
- & - & - & 1 & - & 1 & - & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & 1 & 1 & 1 & 1 & - & 1 & - & - & - & 1 & 1 & 1 & 1 & - & - & - & - & 1 & 1 \\
1 & 1 & - & - & - & - & - & 1 & - & - & - & 1 & 1 & 1 & 1 & - & 1 & - & - & - & 1 & 1 & - & 1 & 1 & - & - & 1 & 1 & - & - & - & - & - & 1 & 1 \\
1 & - & - & - & 1 & 1 & 1 & - & 1 & - & 1 & - & 1 & 1 & - & 1 & - & - & 1 & - & - & - & 1 & 1 & 1 & 1 & - & - & - & - & - & - & - & - & 1 & 1 \\
1 & 1 & - & 1 & - & - & 1 & - & - & - & - & - & 1 & 1 & - & - & - & 1 & 1 & 1 & - & 1 & 1 & - & - & - & - & 1 & - & - & - & 1 & 1 & 1 & - & 1 \\
1 & - & 1 & - & - & 1 & - & - & 1 & 1 & 1 & 1 & 1 & - & - & - & 1 & 1 & - & - & 1 & - & - & 1 & - & - & - & - & - & - & - & 1 & 1 & 1 & - & 1 \\
- & 1 & - & - & 1 & - & - & - & 1 & - & 1 & 1 & - & 1 & - & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 \\
- & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 & 1 & - & 1 & 1 & - & - & 1 & - & 1 & - & 1 & - & 1 & - & - & - & 1 & - & 1 & - & 1 & 1 & - & 1 \\
1 & - & - & - & - & 1 & - & 1 & 1 & 1 & - & - & - & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & - & - & 1 & - & 1 & 1 & - & 1 \\
1 & - & 1 & - & 1 & - & 1 & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & 1 & - & - & - & - & 1 & - & - & 1 & 1 & 1 & 1 & 1 & - & 1 & - & 1 \\
1 & - & - & 1 & - & - & - & - & 1 & - & - & - & - & 1 & 1 & 1 & - & 1 & - & - & 1 & - & - & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & - & 1 \\
- & 1 & 1 & - & 1 & 1 & 1 & 1 & - & 1 & - & - & 1 & 1 & - & - & - & - & - & - & 1 & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & - & 1 \\
- & - & - & - & - & 1 & 1 & 1 & - & - & - & 1 & - & - & - & - & - & 1 & - & 1 & - & - & - & 1 & 1 & 1 & - & 1 & - & 1 & 1 & - & - & 1 & - & 1 \\
- & - & 1 & 1 & 1 & - & - & - & - & - & 1 & 1 & 1 & 1 & 1 & - & - & - & - & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & 1 & - & - & 1 & 1 & 1 & - & - & 1 \\
- & 1 & - & 1 & - & 1 & 1 & - & - & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & - & 1 & - & - & - & - & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & - & 1 \\
1 & - & 1 & 1 & - & - & - & 1 & - & 1 & 1 & - & - & - & - & 1 & - & - & - & 1 & - & 1 & 1 & 1 & 1 & - & - & - & 1 & 1 & - & - & 1 & - & - & 1 \\
- & - & 1 & - & 1 & - & - & 1 & - & - & - & - & 1 & - & - & 1 & 1 & 1 & 1 & - & - & - & - & - & - & 1 & 1 & 1 & 1 & - & - & - & 1 & - & - & 1 \\
- & 1 & - & - & 1 & 1 & - & - & 1 & - & 1 & - & - & - & - & 1 & - & - & - & 1 & 1 & 1 & - & - & - & - & - & 1 & 1 & - & 1 & 1 & - & - & - & 1 \\
- & 1 & - & - & - & - & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 & 1 & 1 & - & - & - & - & 1 & - & - & - & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & - & - & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & 1 & 1 & - & - & - & 1 & 1 & - & - & - & - & 1 \\
1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & - & - & 1 & - & - & 1 & 1 & - & 1 & - & - & 1 & - & - & 1 & - & - & - & 1 & - & - & - & - & 1 \\
1 & - & 1 & 1 & 1 & 1 & 1 & - & 1 & - & - & 1 & - & - & 1 & - & 1 & - & - & - & - & 1 & 1 & - & - & - & 1 & 1 & - & 1 & - & - & - & - & - & 1
\end{pmatrix}.$$

B.2 Sets of Mutually Unbiased Quaternary Hadamard Matrices

This section gives some examples of maximal sets of $\mathbf{MUQH}(n)$.

B.2.1 Maximal Set of Order 8

Table B.1: A set of 8 mutually unbiased quaternary Hadamard matrices of order 8.

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & - & - & - & - \\
1 & 1 & - & - & 1 & 1 & - & - \\
1 & 1 & - & - & - & - & 1 & 1 \\
1 & - & 1 & - & 1 & - & 1 & - \\
1 & - & 1 & - & - & 1 & - & 1 \\
1 & - & - & 1 & 1 & - & - & 1 \\
1 & - & - & 1 & - & 1 & 1 & -
\end{pmatrix}
\quad
\begin{pmatrix}
1 & 1 & 1 & - & i & j & j & j \\
1 & 1 & - & 1 & j & i & j & j \\
1 & - & 1 & 1 & j & j & i & j \\
i & j & j & j & 1 & 1 & 1 & - \\
- & 1 & 1 & 1 & j & j & j & i \\
j & i & j & j & 1 & 1 & - & 1 \\
j & j & i & j & 1 & - & 1 & 1 \\
j & j & j & i & - & 1 & 1 & 1
\end{pmatrix}
$$

$$
\begin{pmatrix}
1 & 1 & i & j & j & j & 1 & - \\
1 & 1 & j & i & j & j & - & 1 \\
1 & - & j & j & i & j & 1 & 1 \\
i & j & 1 & 1 & 1 & - & j & j \\
- & 1 & j & j & j & i & 1 & 1 \\
j & i & 1 & 1 & - & 1 & j & j \\
j & j & 1 & - & 1 & 1 & i & j \\
j & j & - & 1 & 1 & 1 & j & i
\end{pmatrix}
\quad
\begin{pmatrix}
1 & 1 & j & j & 1 & - & i & j \\
1 & 1 & j & j & - & 1 & j & i \\
1 & - & i & j & 1 & 1 & j & j \\
i & j & 1 & - & j & j & 1 & 1 \\
- & 1 & j & i & 1 & 1 & j & j \\
j & i & - & 1 & j & j & 1 & 1 \\
j & j & 1 & 1 & i & j & 1 & - \\
j & j & 1 & 1 & j & i & - & 1
\end{pmatrix}
$$

$$
\begin{pmatrix}
1 & i & 1 & j & j & - & j & 1 \\
1 & j & 1 & i & j & 1 & j & - \\
1 & j & - & j & j & 1 & i & 1 \\
i & 1 & j & 1 & - & j & 1 & j \\
- & j & 1 & j & i & 1 & j & 1 \\
j & 1 & i & 1 & 1 & j & - & j \\
j & 1 & j & - & 1 & j & 1 & i \\
j & - & j & 1 & 1 & i & 1 & j
\end{pmatrix}
\quad
\begin{pmatrix}
1 & i & j & 1 & 1 & j & j & - \\
1 & j & i & 1 & - & j & j & 1 \\
1 & j & j & - & 1 & i & j & 1 \\
i & 1 & 1 & j & j & 1 & - & j \\
- & j & j & 1 & 1 & j & i & 1 \\
j & 1 & 1 & i & j & - & 1 & j \\
j & 1 & - & j & i & 1 & 1 & j \\
j & - & 1 & j & j & 1 & 1 & i
\end{pmatrix}
$$

$$
\begin{pmatrix}
1 & i & j & - & j & 1 & 1 & j \\
1 & j & j & 1 & i & 1 & - & j \\
1 & j & j & 1 & j & - & 1 & i \\
i & 1 & - & j & 1 & j & j & 1 \\
- & j & i & 1 & j & 1 & 1 & j \\
j & 1 & 1 & j & 1 & i & j & - \\
j & 1 & 1 & j & - & j & i & 1 \\
j & - & 1 & i & 1 & j & j & 1
\end{pmatrix}
\quad
\begin{pmatrix}
1 & j & 1 & j & 1 & j & - & i \\
1 & j & 1 & j & - & i & 1 & j \\
1 & j & - & i & 1 & j & 1 & j \\
i & - & j & 1 & j & 1 & j & 1 \\
- & i & 1 & j & 1 & j & 1 & j \\
j & 1 & i & - & j & 1 & j & 1 \\
j & 1 & j & 1 & i & - & j & 1 \\
j & 1 & j & 1 & j & 1 & i & -
\end{pmatrix}
$$

B.2.2 Maximal Set of Order 10

Below is the unbiased pair corresponding to matrix #9 from Table 4.3.

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & i & i & - & - & - & j & j \\
1 & 1 & i & - & j & 1 & i & j & - & - \\
1 & 1 & - & j & i & j & j & i & i & - \\
1 & i & j & i & j & - & j & 1 & - & i \\
1 & - & 1 & j & - & - & i & 1 & i & j \\
1 & - & i & 1 & j & i & j & - & 1 & - \\
1 & - & - & 1 & i & j & i & j & j & i \\
1 & - & j & - & 1 & 1 & - & - & i & 1 \\
1 & j & - & - & - & i & 1 & i & j & 1
\end{pmatrix}, \quad
\begin{pmatrix}
1 & 1 & 1 & j & i & 1 & - & j & 1 & - \\
1 & j & - & 1 & 1 & i & j & 1 & j & i \\
1 & i & j & 1 & 1 & j & i & 1 & - & j \\
j & 1 & 1 & 1 & j & - & - & i & 1 & 1 \\
j & 1 & 1 & i & i & i & 1 & j & - & 1 \\
j & - & 1 & i & j & 1 & 1 & 1 & 1 & - \\
j & i & - & j & i & 1 & 1 & i & 1 & 1 \\
i & 1 & 1 & j & j & 1 & 1 & i & - & i \\
i & 1 & - & i & j & 1 & i & j & 1 & 1 \\
i & - & 1 & j & 1 & - & 1 & j & 1 & 1
\end{pmatrix}.$$

B.3 Applications of Unbiased Hadamard Matrices of Order 36

B.3.1 Relation to Non-Regularizable Hadamard Matrices

Below are two examples of non-regularizable Hadamard matrices of order 36 with maximum excess 204 and 208, respectively.

$$H_{204} = \begin{pmatrix}
- & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
- & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & 1 & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & - & - & - & - \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & 1 & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & 1 & - & 1 \\
- & 1 & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - & - & - & - & 1 & 1 & 1 & - & 1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & - & - & - & - & 1 & 1 & - \\
1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - & - & - & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - & - & 1 & 1 & 1 & - & - \\
- & 1 & - & 1 & 1 & - & - & 1 & - & 1 & 1 & - & 1 & - & - & - & 1 & 1 & 1 & - & 1 & - & - & - & - & 1 & 1 & 1 & - & - & - & 1 & 1 & 1 & 1 & - \\
1 & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & - & - & - & - & 1 & 1 & - & - & - & - & - & - & 1 & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & - & - \\
- & 1 & - & 1 & - & 1 & 1 & - & 1 & 1 & - & - & 1 & - & 1 & - & - & - & 1 & 1 & 1 & 1 & - & 1 & - & - & 1 & 1 & - & - & 1 & - & 1 & - & - & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & - & - & 1 & - & - & 1 & - & - & - & - & - & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 & 1 & - & - & 1 & - & - & - & - & 1 & 1 \\
- & 1 & - & 1 & 1 & - & 1 & - & - & - & 1 & 1 & - & 1 & 1 & - & - & - & 1 & 1 & - & - & - & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & - & 1 \\
1 & 1 & 1 & 1 & - & - & - & - & - & 1 & 1 & - & 1 & 1 & - & - & - & 1 & 1 & - & - & 1 & - & 1 & 1 & - & - & - & 1 & - & 1 & 1 & - & - & 1 & 1 \\
- & 1 & - & 1 & - & - & - & 1 & 1 & - & 1 & 1 & 1 & - & 1 & - & - & - & - & 1 & 1 & - & 1 & 1 & 1 & - & - & - & - & 1 & 1 & 1 & - & 1 & 1 & - \\
1 & 1 & 1 & 1 & - & 1 & 1 & 1 & - & - & 1 & - & - & - & - & - & - & 1 & 1 & 1 & - & - & 1 & - & - & - & - & 1 & 1 & 1 & 1 & - & 1 & 1 & - & - \\
- & 1 & - & 1 & 1 & 1 & - & - & 1 & - & - & - & 1 & 1 & - & 1 & 1 & - & - & - & - & - & 1 & - & - & - & - & 1 & 1 & 1 & 1 & 1 & 1 & - & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & - & - & 1 & - & - & - & 1 & 1 & - & 1 & 1 & - & - & - & 1 & - & 1 & - & - & - & 1 & 1 & 1 & 1 & - & 1 & - & - & - & 1 & - \\
- & 1 & 1 & 1 & - & - & 1 & - & - & - & 1 & - & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & - & - & - & - & - & 1 \\
1 & 1 & 1 & - & - & 1 & - & - & - & 1 & - & - & - & - & 1 & - & 1 & - & - & 1 & 1 & - & - & - & 1 & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 \\
- & 1 & 1 & - & - & 1 & 1 & 1 & - & - & - & 1 & 1 & - & - & 1 & - & - & 1 & - & - & 1 & 1 & - & 1 & - & 1 & - & - & - & - & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & - & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & - & - & 1 & 1 & - & 1 & - & - & 1 & 1 & - & - & - & - & - & - & - & 1 & 1 & 1 & 1 & - & 1 & - \\
- & 1 & 1 & - & - & - & - & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & - & 1 & - & 1 & - & - & 1 & - & - & 1 & 1 & - & - & - & 1 & - & 1 & 1 & 1 & 1 \\
1 & 1 & - & - & - & 1 & - & 1 & 1 & - & - & - & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & - & 1 & - & - & 1 & 1 & - & - & - & 1 & - & - & 1 & - & 1 & - \\
- & 1 & 1 & - & 1 & - & 1 & 1 & - & - & - & 1 & - & 1 & - & - & 1 & 1 & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & - & - & 1 & - & 1 & - & 1 & 1 \\
1 & 1 & - & - & - & - & 1 & - & - & 1 & 1 & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & 1 & - & - & - & - & 1 & - \\
- & 1 & 1 & - & 1 & - & - & - & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & - & - & - & 1 & 1 & - & - & 1 & 1 & 1 & - & - & 1 & 1 & - & - \\
1 & 1 & - & - & - & - & - & 1 & 1 & - & 1 & 1 & - & - & 1 & - & 1 & 1 & - & - & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & - & - & 1 & 1 & - & - & 1 \\
- & 1 & 1 & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & - & - & - & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 & - & - & 1 & 1 & - & - & - & - \\
1 & 1 & - & - & 1 & - & - & 1 & 1 & - & 1 & - & - & 1 & - & 1 & - & - & 1 & - & 1 & 1 & - & - & 1 & - & 1 & 1 & - & 1 & 1 & - & - & 1 & - & 1 \\
- & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & - & 1 & 1 & 1 & - & - & 1 & - & 1 & - & - & 1 & - & 1 & - & 1 & 1 & - & - & 1 & - & - & 1 & - \\
1 & 1 & - & - & 1 & 1 & 1 & - & - & - & - & - & 1 & 1 & 1 & - & - & 1 & - & - & - & - & 1 & 1 & 1 & 1 & 1 & 1 & - & - & 1 & 1 & - & 1 & - & - \\
- & 1 & 1 & - & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 & 1 & - & - & - & 1 & - & 1 & 1 & 1 & 1 & - & - & - & 1 & - & 1 & 1 & - & 1 & - & - & - & - \\
1 & 1 & - & - & - & - & 1 & - & 1 & - & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & 1 & - & - & 1 & - & 1 & 1 & - & 1 & - & 1 \\
- & 1 & 1 & - & - & 1 & - & - & - & - & 1 & 1 & 1 & 1 & - & - & 1 & - & - & - & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & - \\
1 & 1 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 & - & - & 1 & - & - & - & 1 & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 & - & - & - & 1 & - & - & 1 \\
- & 1 & - & - & 1 & - & 1 & 1 & - & 1 & - & - & - & - & 1 & - & 1 & - & 1 & - & - & 1 & 1 & 1 & - & 1 & - & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 \\
1 & 1 & - & 1 & - & - & 1 & - & 1 & 1 & - & 1 & - & 1 & - & - & - & - & - & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & - & - & 1 & 1 & 1 & -
\end{pmatrix},$$

$$H_{208} = \begin{pmatrix}
- & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
- & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & - & - \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & 1 \\
- & 1 & - & 1 & 1 & 1 & 1 & 1 & 1 & - & 1 & - & - & - & - & - & 1 & 1 & - & 1 & - & 1 & - & 1 & 1 & 1 & 1 & - & 1 & - & - & - & - & - & 1 & - \\
1 & 1 & 1 & 1 & 1 & - & 1 & - & - & - & - & - & - & 1 & - & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - & 1 & - & 1 & - & - \\
- & 1 & - & 1 & 1 & 1 & - & - & 1 & - & - & - & 1 & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & - & - & - & 1 & 1 & 1 & - & - & 1 & 1 & - & - & 1 \\
1 & 1 & 1 & 1 & 1 & - & - & 1 & - & - & - & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & - & - & - & - & 1 & 1 & - & - & - & 1 & 1 & - & - & 1 & 1 \\
- & 1 & - & 1 & - & - & 1 & 1 & - & 1 & 1 & 1 & 1 & - & 1 & - & - & - & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & 1 & 1 & - & - & - & 1 & 1 & - \\
1 & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & 1 & - & - & - & - & - & - & 1 & 1 & - & 1 & - & - & - & - & 1 & - & 1 & 1 & - & - & 1 & 1 & 1 & - & - \\
- & 1 & - & 1 & 1 & - & 1 & - & 1 & - & - & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & - & - & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 & 1 & - & - & - & - \\
1 & 1 & 1 & 1 & - & - & - & - & - & - & 1 & 1 & - & - & 1 & - & 1 & 1 & - & - & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & - & 1 & - & - & 1 & - & 1 \\
- & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & - & - & - & - & - & 1 & 1 & - & - & - & 1 & - & 1 & - & - & 1 & 1 & 1 & - & 1 & 1 & - & - & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & - & - & - & - & 1 & - & - & 1 & - & 1 & 1 & 1 & - & - & - & 1 & - & - & - & - & 1 & 1 & 1 & - & 1 & 1 & - & - & 1 & 1 & 1 & - \\
- & 1 & - & 1 & - & 1 & - & - & - & - & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & - & - & 1 & - & - & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - & - & - & - & 1 & - & - & - & - & - & 1 & 1 & - & 1 & 1 & - & - & 1 & - & 1 & 1 & - & 1 & - & 1 & - \\
- & 1 & 1 & 1 & - & - & 1 & 1 & - & - & 1 & - & 1 & 1 & - & - & - & - & 1 & - & - & - & 1 & 1 & 1 & 1 & 1 & 1 & - & 1 & - & - & 1 & - & - & 1 \\
1 & 1 & 1 & - & - & 1 & 1 & - & - & 1 & - & - & 1 & - & - & 1 & - & 1 & - & - & - & 1 & 1 & - & 1 & - & 1 & - & 1 & 1 & - & 1 & - & - & 1 & 1 \\
- & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & - & - & - & 1 & 1 & - & - & - & 1 & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 & - & 1 & 1 & - & - & - & 1 \\
1 & 1 & - & - & - & 1 & 1 & 1 & 1 & - & - & - & - & 1 & 1 & - & - & 1 & 1 & 1 & - & - & 1 & - & 1 & - & - & 1 & - & - & 1 & - & - & 1 & 1 & 1 \\
- & 1 & 1 & - & - & - & - & 1 & 1 & - & 1 & - & - & 1 & 1 & 1 & 1 & 1 & 1 & - & 1 & 1 & - & - & - & - & 1 & 1 & - & 1 & - & 1 & - & - & 1 & - \\
1 & 1 & - & - & - & 1 & 1 & 1 & - & - & - & - & 1 & 1 & 1 & - & 1 & - & - & - & 1 & - & - & 1 & - & 1 & 1 & - & 1 & 1 & 1 & 1 & - & 1 & - & - \\
- & 1 & 1 & - & 1 & - & 1 & - & - & 1 & - & - & - & 1 & 1 & - & 1 & 1 & 1 & 1 & - & 1 & - & - & - & 1 & - & - & 1 & 1 & 1 & - & 1 & - & - & 1 \\
1 & 1 & - & - & - & - & - & - & 1 & 1 & - & 1 & 1 & 1 & - & - & 1 & - & 1 & - & 1 & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & - & - & - & - & 1 & 1 \\
- & 1 & 1 & - & 1 & 1 & - & - & - & 1 & 1 & - & 1 & 1 & - & 1 & - & - & 1 & 1 & - & - & - & 1 & - & - & 1 & 1 & 1 & - & 1 & - & - & 1 & 1 & - \\
1 & 1 & - & - & 1 & - & - & 1 & 1 & 1 & - & - & 1 & - & 1 & 1 & - & 1 & 1 & - & - & 1 & 1 & 1 & - & 1 & 1 & - & - & - & - & - & 1 & 1 & - & - \\
- & 1 & 1 & - & - & 1 & 1 & - & - & - & - & 1 & 1 & - & 1 & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & - & - & - & - & 1 & - & 1 & - \\
1 & 1 & - & - & 1 & 1 & - & - & - & 1 & 1 & 1 & - & - & 1 & - & 1 & - & 1 & 1 & - & - & 1 & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - & - & - & - \\
- & 1 & 1 & - & - & - & 1 & 1 & 1 & 1 & - & 1 & 1 & - & - & - & 1 & - & - & 1 & - & 1 & - & - & 1 & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & - \\
1 & 1 & - & - & - & 1 & 1 & - & 1 & - & 1 & 1 & - & - & - & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & - & 1 & - & - & 1 & 1 & - & 1 & - & - & 1 \\
- & 1 & 1 & - & 1 & 1 & - & - & 1 & - & 1 & 1 & 1 & 1 & - & - & - & 1 & - & - & 1 & 1 & 1 & - & 1 & 1 & - & - & - & 1 & 1 & - & - & 1 & - & - \\
1 & 1 & - & - & 1 & - & - & 1 & - & - & 1 & - & 1 & - & - & 1 & 1 & 1 & - & 1 & 1 & - & - & - & 1 & - & - & 1 & 1 & 1 & - & - & 1 & 1 & - & 1 \\
- & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & - & 1 & - & - & - & - & 1 & - & 1 & - & - & - & 1 & 1 & - & - & - & - & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & - & - & 1 & - & 1 & 1 & - & - & 1 & 1 & - & 1 & - & 1 & - & - & - & - & - & 1 & 1 & - & - & 1 & - & 1 & 1 & - & 1 & 1 & 1 & - & 1 & - \\
- & 1 & - & - & 1 & - & 1 & - & - & 1 & 1 & 1 & - & 1 & 1 & - & - & 1 & - & - & 1 & - & - & 1 & 1 & - & 1 & - & - & - & - & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & - & 1 & - & - & - & - & 1 & 1 & 1 & - & 1 & 1 & - & - & 1 & 1 & - & 1 & - & - & 1 & 1 & - & - & - & - & - & 1 & 1 & 1 & 1 & - & 1 & -
\end{pmatrix}.$$

B.3.2 Relation to Linear Codes

Below is some additional information about the six pairs of unbiased Hadamard matrices of order 36 that were found to generate the Pless symmetry code. Starting from the six inequivalent quaternary Hadamard matrices of order 18, $Q_1, \ldots, Q_6$ and converting these matrices to their real counterparts $H_1, \ldots, H_6$ using Theorem 2.62, we have the following equivalences.

- H_1, H_2 and H_3 are equivalent.
- H_4, H_5 and H_6 are equivalent.
- $H_1, \ldots, H_6$ generate equivalent codes with 3-rank 18 and minimum distance 12.

The following is an example of an unbiased pair that generates the extremal code with 3-rank 18 and minimum distance 12. Note that both H and K generate the same code.

$$H = \left(\begin{array}{cccccccccccccccccccccccccccccccccccc}
- & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
- & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & - & - \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & 1 \\
- & 1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & - & 1 & - & - & - & 1 & 1 & - & 1 & - & 1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & - & 1 & - & - & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - & - & - & - & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - & - & - & 1 & 1 \\
- & 1 & - & 1 & 1 & - & - & 1 & 1 & - & 1 & - & 1 & - & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & - & - & - & - & 1 & - & 1 & - & 1 & 1 & - & 1 & - \\
1 & 1 & 1 & 1 & - & - & 1 & 1 & - & - & - & - & - & - & 1 & 1 & 1 & - & 1 & 1 & - & - & - & - & - & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - \\
- & 1 & - & 1 & 1 & - & 1 & - & 1 & - & - & 1 & - & 1 & 1 & - & 1 & 1 & 1 & - & 1 & - & - & 1 & - & 1 & - & - & 1 & - & - & 1 & - & 1 & 1 & - \\
1 & 1 & 1 & 1 & - & - & - & - & - & - & 1 & 1 & 1 & 1 & - & - & 1 & - & - & - & - & - & 1 & 1 & 1 & 1 & - & 1 & - & - & 1 & 1 & 1 & 1 & - & - \\
- & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - & - & 1 & - & - & 1 & 1 & - & - & 1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & - \\
1 & 1 & 1 & 1 & 1 & - & - & - & 1 & 1 & - & - & 1 & 1 & - & - & - & 1 & - & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - \\
- & 1 & - & 1 & 1 & - & - & 1 & - & 1 & 1 & - & 1 & - & 1 & - & - & - & 1 & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & 1 & - & - & 1 & - & 1 \\
1 & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & - & - & - & - & - & - & 1 & 1 & - & - & - & 1 & 1 & - & - & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 \\
- & 1 & - & 1 & 1 & - & 1 & - & 1 & - & - & 1 & 1 & - & - & 1 & - & - & 1 & - & - & 1 & 1 & - & 1 & - & 1 & - & - & 1 & 1 & 1 & - & 1 & - & 1 \\
1 & 1 & 1 & 1 & - & - & - & - & - & - & 1 & 1 & - & - & 1 & 1 & - & 1 & - & - & 1 & 1 & - & - & - & - & - & - & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 \\
- & 1 & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 & - & - & 1 & 1 & - & - & - & - & 1 & - & 1 & 1 & - & 1 & - & 1 & - & 1 & - & - & 1 & - & 1 & 1 & - \\
1 & 1 & 1 & - & 1 & 1 & 1 & 1 & - & - & - & - & 1 & 1 & - & - & - & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - & - & - & 1 & 1 & 1 & 1 & - & - \\
- & 1 & 1 & - & 1 & - & - & 1 & 1 & - & - & 1 & - & 1 & 1 & - & - & - & - & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & - & 1 & 1 & - & 1 & - & - & 1 \\
1 & 1 & - & - & - & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & - & - & 1 & 1 & 1 & - & - & 1 & - & 1 & 1 & - & - & 1 & 1 & - & - & - & - & 1 & 1 \\
- & 1 & 1 & - & - & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & - & 1 & - & - & - & 1 & 1 & - & 1 & - & - & 1 & - & 1 & 1 & - & 1 & - & - & 1 & 1 & - \\
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