A RECURSIVE CONSTRUCTION FOR SOME COMBINATORIAL DESIGNS

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Abstract

Following Ionin's modified version of a result of Rajkundlia, generalized Hadamard matrices are applied to recursively construct a class of embeddable quasi-residual balanced incomplete block designs (BIBDs). The construction leads to incidence matrices related to the designs with classical parameters. Further, by a similar method and application of Paley matrices the class of balanced weighing matrices with classical parameters are reconstructed.

Later in the thesis, following an approach by Ionin, the constructed quasi-residual BIBDs are extended to larger quasi-residual designs. Lastly, employing Kharaghani et al. idea a class of orthogonal arrays is used to show that the constructed quasi-residual designs are embeddable and the Ionin's class of symmetric designs are reconstructed.

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Chapter 1

Introduction

This thesis is a collection of different ideas on constructions of various combinatorial structures. The foundation of all these constructions is a special class of balanced incomplete block designs. By applying multiple methods and conditions to their incidence matrices, other combinatorial structures with larger parameters will be obtained.

The origin of incomplete designs goes back to [23], where Yates introduced balanced incomplete block designs (BIBDs). Bose [2, 3, 4] and Fisher [7] also made great contributions to the construction and structure of various incomplete designs. Chapter 2 will discuss the construction of a special family of BIBDs. It will be shown that there are a number of incidence matrices of these designs.

Earlier in 1867, Sylvester [22] introduced Hadamard matrices of order 2^k for any non-negative integer k. Later in 1893, Hadamard [8] constructed Hadamard matrices of orders 12 and 20. In 1933 [19], Paley introduced a construction which shows Hadamard matrices of order $p+1$ when p is a prime number congruent to 3 modulo 4 and Hadamard matrices of order $2(p+1)$ when p is a prime power that is congruent to 1 modulo 4. The smallest order that could not be obtained by Paley's construction was 92. In 1962, a Hadamard matrix of order 92 was constructed using a computer [1]. The Hadamard conjuncture claims that a Hadamard matrix of order $4k$ exists for every positive integer k. Sawade proposed a construction for the Hadamard matrix of order 268 in 1985 [20]. In 2005, Hadi Kharaghani and Behruz Tayfeh-Rezaie [17] constructed a Hadamard matrix of order 428 which left the Hadamard matrix of order 668 the smallest order that is not constructed yet. Many generalizations of Hadamard matrices have led to important structures such as weighing matrices or orthogonal arrays. In [13], the authors used a weighing matrix $\mathrm{W}(n,p)$ and an important family of orthogonal arrays to construct a larger class of weighing matrices

$$\mathrm{W}\left(\frac{p^{m+1}-1}{p-1}(n-1)+1, p^{m+1}\right)$$

where p is a prime power. With a clue from this construction, Chapter 3 will utilize the Paley matrices of order q to reconstruct a family of weighing matrices with parameters $\mathrm{W}(\sum_{i=0}^{m} q^i, q^m)$ where m is an integer and q is a prime power. Additionally, in Chapter 4 some applications of weighing matrices and orthogonal arrays will be elaborated.

In [10], Ionin constructed a family of symmetric BIBDs with the parameters

$$\left(\frac{h((2h-1)^{2m}-1)}{h-1}, h(2h-1)^{2m-1}, h(h-1)(2h-1)^{2m-2}\right)$$

where $h = 3.2^d$ for some integer d and $|2h-1|$ is a prime power, by using weighing matrices and regular Hadamard matrices. In [11], Ionin obtained a quasi-residual design with the parameters

$$\left(\frac{(v-r)(r^{m+1}-1)}{(r-1)}, \frac{(v-1)(r^{m+1}-1)}{(r-1)}, r^{m+1}, (r-\lambda)r^m, \lambda r^m\right)$$

where r is a prime power and m is a positive integer, by applying balanced generalized weighing matrices to a $\mathrm{BIBD}(v-r, v-1, r, r-\lambda, \lambda)$. Chapter 4's foundation is based on the work in [11] and a class of embeddable quasi-residual designs is constructed. In order to prove the embeddablity, the ideas from [14] for the construction of the corresponding symmetric BIBDs will be helpful.

Chapter 2

Preliminaries

In this chapter, the definitions and theorems that are the foundation of this thesis will be represented. The main references for the definitions and terminologies are [12, 21].

Definition 2.1 (Balanced Incomplete Block Design (BIBD)) A BIBD is a pair (X,A) where X is a finite set with v elements (points) and A is a collection of proper k-subsets of X (blocks) such that every point belongs to the same number of blocks (r), and each pair of points belongs to the same number of blocks (λ). The BIBD is described by five parameters (v,b,r,k,λ) where b indicates the number of blocks. In this thesis, a BIBD with these parameters will be shown as BIBD(v,b,r,k,λ).

Designs are useful in many areas, such as coding theory, cryptography, and tournament scheduling. Different applications of designs are stated in chapter 7 of [5].

Example 2.2 A (X,A) design with parameters $(9,12,4,3,1)$ is as following:

$$X = \{1,2,...,9\},$$

$$A = \{123,456,789,147,258,369,159,267,348,168,249,357\}.$$

where a block $\{a,b,c\}$ is denoted abc.

Each BIBD can also be presented as a matrix.

Definition 2.3 (Incidence Matrix) Consider the BIBD (X,A) where $X = \{x_1, ..., x_v\}$ and $A = \{A_1, ..., A_b\}$. Then, the incidence matrix of a BIBD(v,b,r,k,λ) is a $v \times b$, (0,1)-matrix $M = [m_{i,j}]$ such that

$$m_{i,j} = \begin{cases} 1, \text{if } x_i \in A_j \\ 0, \text{otherwise} \end{cases} . \tag{2.1}$$

Example 2.4 Consider the design BIBD$(9,12,4,3,1)$. An incidence matrix of the design is

$$\begin{bmatrix} 1&0&0&1&0&0&1&0&0&1&0&0 \\ 1&0&0&0&1&0&0&1&0&0&1&0 \\ 1&0&0&0&0&1&0&0&1&0&0&1 \\ 0&1&0&1&0&0&0&1&0&0&0&1 \\ 0&1&0&0&1&0&0&0&1&1&0&0 \\ 0&1&0&0&0&1&1&0&0&0&1&0 \\ 0&0&1&1&0&0&0&0&1&0&1&0 \\ 0&0&1&0&1&0&1&0&0&0&0&1 \\ 0&0&1&0&0&1&0&1&0&1&0&0 \end{bmatrix}$$

There are four “1”s in each row and three “1”s in each column. Additionally, the inner product of any two distinct rows of the matrix is 1.

Generally, in an incidence matrix of a BIBD, there are k “1”s in each column, r “1”s in each row and the inner product of any two distinct row is λ.

Each BIBD(v,b,r,k,λ) must satisfy the following necessary conditions:

- $vr = bk$
- $\lambda(v-1) = r(k-1)$
- $b \geq v$

Theorem 2.5 ([21]) Let M be a $v \times b$, (0,1)-matrix and let $2 \leq k < v$. Then, M is the incidence matrix of a BIBD(v,b,r,k,λ) if and only if M is a matrix with r ones in each row and k ones in each column satisfying

$$MM^T = \lambda J + (r-\lambda)I,$$

where I is the $v \times v$ identity matrix, J is the $v \times v$ matrix with all entries equal to 1.

Symmetric BIBDs are an important class of BIBDs. which are defined below. These designs will be mentioned frequently throughout the thesis.

Definition 2.6 (Symmetric BIBDs) A BIBD(v,b,r,k,λ) is called a symmetric BIBD if the number of points is equal to the number of blocks, $v = b$. A symmetric BIBD(v,b,r,k,λ) will be shown as SBIBD(v,k,λ).

In a symmetric design, any two blocks intersect in exactly λ points. If there exists a SBIBD(v,k,λ) with the set of points X and multiset of blocks B, given a block A_0 of this design, the *derived design* (X,D) is defined such that $D = \{A \cap A_0 : A \in B, A \neq A_0\}$. The derived BIBD parameters are $(k, v-1, k-1, \lambda, \lambda-1)$. The *residual design* $(X/A_0, R)$ is defined such that $R = \{A/A_0 : A \in B, A \neq A_0\}$ and has parameters $(v-k, v-1, k, k-\lambda, \lambda)$.

Example 2.7 The following is an incidence matrix of SBIBD$(13, 4, 1)$

$$M = \begin{bmatrix} 0&1&0&0&1&0&0&1&0&0&1&0&0 \\ 0&1&0&0&0&1&0&0&1&0&0&1&0 \\ 0&1&0&0&0&0&1&0&0&1&0&0&1 \\ 0&0&1&0&1&0&0&0&1&0&0&0&1 \\ 0&0&1&0&0&1&0&0&0&1&1&0&0 \\ 0&0&1&0&0&0&1&1&0&0&0&1&0 \\ 0&0&0&1&1&0&0&0&0&1&0&1&0 \\ 0&0&0&1&0&1&0&1&0&0&0&0&1 \\ 0&0&0&1&0&0&1&0&1&0&1&0&0 \\ 1&0&0&0&0&0&0&1&1&1&0&0&0 \\ 1&1&1&1&0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&1&1&1&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0&1&1&1 \end{bmatrix} \tag{2.2}$$

From the above incidence matrix the Derived design will be

$$\begin{bmatrix} 0&0&0&0&0&0&1&1&1&0&0&0 \\ 1&1&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&1&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&1&1 \end{bmatrix}$$

and the residual design is

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Any design with parameters (v,b,r,k,λ) such that $v+r=b-1$ and $k+\lambda=r$ is called a *quasi-residual* design and any design such that $\lambda=k-1$ is called a *quasi-derived* design. A quasi-residual design is *embeddable* if it is the residual part of a symmetric design.

Example 2.8 The design BIBD$(9,12,4,3,1)$ in Example 2.4 is a quasi-residual design and is the residual part of the symmetric design BIBD$(13,4,1)$ in Example 2.7. Therefore, BIBD$(9,12,4,3,1)$ is embeddable.

Throughout the rest of this chapter, some other important combinatorial structures and their properties will be introduced.

2.1 Balanced Generalized Weighing Matrices

Definition 2.9 (Weighing Matrix, [13]) A weighing matrix of order v and weight k, denoted as $W(v,k)$, is a $(0,\pm1)$-matrix W of order v such that $WW^T = kI$.

Suppose $W = [w_{i,j}]$ is a $W(n,k)$ and define $W' = [w_{i,j}^2]$. If the equation

$$(W')(W')^T = (k-\lambda)I + \lambda J,$$

is satisfied, where λ is a positive integer then, W is a *balanced* weighing matrix and it is shown as $BW(n,k)$ [18].

Example 2.10 W_1 is a BW$(4,3)$ and W_2 is a BW$(13,9)$

$$W_1 = \begin{bmatrix} 0 & -1 & -1 & -1 \\ 1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix},$$

$$W_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 \\ 0 & 1 & 1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & -1 \\ 1 & 1 & 0 & -1 & -1 & 1 & 0 & -1 & 1 & 0 & -1 & 1 & 0 \\ 1 & 1 & 0 & -1 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & -1 & 1 \\ 1 & -1 & 1 & 0 & 1 & 0 & -1 & -1 & 1 & 0 & 0 & -1 & 1 \\ 1 & -1 & 1 & 0 & -1 & 1 & 0 & 0 & -1 & 1 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 & 0 & -1 & 1 & 1 & 0 & -1 & -1 & 1 & 0 \\ 1 & 0 & -1 & 1 & 1 & 0 & -1 & 0 & -1 & 1 & -1 & 1 & 0 \\ 1 & 0 & -1 & 1 & -1 & 1 & 0 & 1 & 0 & -1 & 0 & -1 & 1 \\ 1 & 0 & -1 & 1 & 0 & -1 & 1 & -1 & 1 & 0 & 1 & 0 & -1 \end{bmatrix}$$

A construction of W_2 will be represented in Chapter 3.

Balanced weighing matrices with classical parameters include $\mathrm{BW}(\frac{q^{n+1}-1}{q-1}, q^n, q^n - q^{n-1})$ for each positive integer n and prime power q [5]. In addition to balanced weighing matrices with classical parameters, Kharaghani, Pender and Suda [14], constructed a family of balanced weighing matrices with parameters $(1+18\frac{9^{m+1}-1}{8}, 9^{m+1}, 4\times 9^m)$ for any nonzero integer m.

The balanced weighing matrices can be generalized by defining them on a finite group.

Definition 2.11 (Balanced Generalized Weighing Matrices (BGW), [5]) Let G be a finite group. A matrix $W = [w_{ij}]$ of order v with entries from $G \cup \{0\}$ is a balanced generalized weighing matrix over G denoted as $\mathrm{BGW}(v,k,\lambda)$ if each row of W contains exactly k nonzero entries and, for any distinct $i, j \in \{1,2,...,v\}$, the multiset $\{w_{hj}^{-1} w_{ij} : w_{hj} \neq 0, w_{ij} \neq 0\}$ contains exactly $\lambda/|G|$ copies of each element in G.

The largest class of known balanced generalized weighing matrix $\mathrm{BGW}(v,k,\lambda)$ over the cyclic group G of order $q-1$, is said to be a BGW matrix with classical parameters, where

$$(v,k,\lambda) = \left(\frac{q^{m+1}-1}{q-1}, q^m, q^m - q^{m-1}\right). \tag{2.3}$$

Example 2.12 The following matrix is a $\mathrm{BGW}(5,4,3)$ over the cyclic multiplication group $G = \{1, a, a^2\}$

$$W = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & a & a^2 \\ 1 & 1 & 0 & a^2 & a \\ 1 & a & a^2 & 0 & 1 \\ 1 & a^2 & a & 1 & 0 \end{bmatrix}$$

The following result provides the existence of the balanced weighing matrices with classical parameters which will be the core of the work in Chapter 4.

Theorem 2.13 ([12]) If q is a prime power, m is a positive integer, and G is a cyclic group whose order divides $q-1$, then there exists a

$$\mathrm{BGW}\left(\frac{q^{m+1}-1}{q-1}, q^m, q^m - q^{m-1}\right) \tag{2.4}$$

over the group G.

2.2 Hadamard Matrices

Definition 2.14 (Hadamard Matrices) A square (-1,1)-matrix of order n for which $HH^T = nI$ is called a Hadamard matrix.

Example 2.15 ([5]) The following matrix is a Hadamard matrix of order 4:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & - & - \\ 1 & - & 1 & - \\ 1 & - & - & 1 \end{bmatrix}. \tag{2.5}$$

A Hadamard matrix with all entries in the first row and the first column equal to 1 is called a *normalized* Hadamard matrix. By multiplying various rows and columns of a Hadamard matrix by -1, a normalized Hadamard matrix can be obtained.

The fundamental concern regarding Hadamard matrices is their existence. The Hadamard conjecture is that there exist Hadamard matrices of order $4n$ for all $n \geq 1$.

One of the construction methods of Hadamard matrices is provided by Kronecker product operations on matrices.

Definition 2.16 (Kronecker Product) The Kronecker product of an $m \times n$ matrix $A = [a_{ij}]$ and an $t \times z$ matrix B is the $(mt) \times (zn)$ block matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{bmatrix}$$

Theorem 2.17 ([12]) If H_1 and H_2 are Hadamard matrices of order n_1 and n_2 then $H_1 \otimes H_2$ is a Hadamard matrix of order $n_1 n_2$.

Example 2.18 Consider the Hadamard matrix of order 4 in Example 2.15 and the Hadamard matrix of order 2, $H_1 = \begin{bmatrix} 1 & 1 \\ 1 & - \end{bmatrix}$, Then,

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & - & - \\ 1 & - & 1 & - \\ 1 & - & - & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ 1 & - \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & 1 & - & 1 & - & 1 & - \\ 1 & 1 & 1 & 1 & - & - & - & - \\ 1 & - & 1 & - & - & 1 & - & 1 \\ 1 & 1 & - & - & 1 & 1 & - & - \\ 1 & - & - & 1 & 1 & - & - & 1 \\ 1 & 1 & - & - & - & - & 1 & 1 \\ 1 & - & - & 1 & - & 1 & 1 & - \end{bmatrix}$$

is a Hadamard matrix of order 8.

Definition 2.19 (Generalized Hadamard Matrix, [15]) A generalized Hadamard matrix GH(g, λ) over the group G of order g is a $g\lambda \times g\lambda$ matrix such that the entries of the matrix are from the group G and the multiset $\{m_{il} m_{jl}^{-1} : 1 \leq l \leq g\lambda\}$ for each $1 \leq i < j \leq g\lambda$ contains λ copies of each element of the group G.

Example 2.20 Let $G = \{1, a, a^2\}$ be a cyclic group of order 3 such that $a^3 = 1$, then a generalized Hadamard matrix over the group G

$$\mathrm{GH}(3,1) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix}. \tag{2.6}$$

Theorem 2.21 ([6]) If there is a $\mathrm{GH}(g,\lambda_1)$ and $\mathrm{GH}(g,\lambda_2)$ over the group G of order g, then there exists a generalized Hadamard matrix $\mathrm{GH}(g, g\lambda_1\lambda_2)$ over the same group such that

$$\mathrm{GH}(g, g\lambda_1\lambda_2) = \mathrm{GH}(g,\lambda_1) \otimes \mathrm{GH}(g,\lambda_2).$$

Example 2.22 Let H be the $\mathrm{GH}(3,1)$ over the group G in Example 2.20, then

$$H \otimes H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & a & a^2 & 1 & a & a^2 & 1 & a & a^2 \\ 1 & a^2 & a & 1 & a^2 & a & 1 & a^2 & a \\ 1 & 1 & 1 & a & a & a & a^2 & a^2 & a^2 \\ 1 & a & a^2 & a & a^2 & 1 & a^2 & 1 & a \\ 1 & a^2 & a & a & 1 & a^2 & a^2 & a & 1 \\ 1 & 1 & 1 & a^2 & a^2 & a^2 & a & a & a \\ 1 & a & a^2 & a^2 & 1 & a & a & a^2 & 1 \\ 1 & a^2 & a & a^2 & a & 1 & a & 1 & a^2 \end{bmatrix}$$

is a $\mathrm{GH}(3,3)$ over G.

Theorem 2.23 ([6]) Let p be a prime power. There is a generalized Hadamard matrix $\mathrm{GH}(p,1)$ over every finite elementary abelian group of order p.

Proof. Suppose F is the finite field of order p such that G is the additive group of F. The multiplication table for F is a $\mathrm{GH}(p,1)$ over G. □

2.3 Paley Matrices

Definition 2.24 (Quadratic Character) Suppose q is an odd prime power. The quadratic character on the finite field of order q, $\mathrm{GF}(q)$, is a function χ from $\mathrm{GF}(q)$ to $\{-1,0,1\}$ such that

$$\chi(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x \text{ is a nonzero square,} \\ -1 & \text{if } x \text{ is a nonzero. nonsquare.} \end{cases}$$

Example 2.25 Let $GF(5) = \{0,1,2,3,4\}$, then the square elements of $\mathrm{GF}(5)$ are $S = \{1,4\}$. Therefore,

$$\forall a \in S; \chi(a) = 1.$$

Lemma 2.26 ([12]) Suppose q is an odd prime power and $\mathrm{GF}(q)$ is the finite field of order q and let χ be the quadratic character on $\mathrm{GF}(q)$. Then, for any nonzero element of $\mathrm{GF}(q)$, there are exactly $(q-3)/2$ elements x such that $\chi(x+a) = \chi(x)$.

Proof.

$$\sum_{x\in \mathrm{GF}(q)} \chi(x+a)\chi(x) = \sum_{x\in \mathrm{GF}(q)^*} \chi(x+a)\chi(x) = \sum_{x\in \mathrm{GF}(q)^*} \frac{\chi(x+a)}{\chi(x)}$$

$$= \sum_{x\in \mathrm{GF}(q)^*} \chi\left(1+\frac{a}{x}\right) = \sum_{y\in \mathrm{GF}(q)/\{1\}} \chi(y) = -1.$$

where $\mathrm{GF}(q)^*$ is the set of all nonzero elements of $\mathrm{GF}(q)$. Therefore, among the nonzero products $\chi(x+a)\chi(x)$, the number of -1s is one more than the number of 1s. This proves the Lemma. □

Definition 2.27 (Paley Matrix, [12]) Suppose q is an odd prime power and $\mathrm{GF}(q) = \{a_1, a_2, ..., a_q\}$ is the finite field of order q. Let χ be the quadratic character on $\mathrm{GF}(q)$. The matrix $P = [p_{i,j}]$ such that $p_{i,j} = \chi(a_i - a_j)$ is a Paley matrix of order q.

Example 2.28 Let $\mathrm{GF}(7) = \{0, 1, 2, ..., 6\}$, then the square elements of $\mathrm{GH}(7)$ are $\{1, 2, 4\}$. Thus, the following matrix is a Paley matrix of order 7

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & -1 & -1 & 1 & -1 & 1 & 1 \\ 1 & 0 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 0 & -1 & -1 & 1 & -1 \\ -1 & 1 & 1 & 0 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 & 0 & -1 & -1 \\ -1 & 1 & -1 & 1 & 1 & 0 & -1 \\ -1 & -1 & 1 & -1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Lemma 2.29 ([12]) If P is a Paley matrix of order q, then $PP^T = qI - J$ and $PJ = JP = 0$.

Proof. Let $\mathrm{GF}(q)$ be the finite field of order q. Since $\mathrm{GF}(q)$ has equal number of nonzero squares and nonsquares, then $PJ = JP = 0$. Additionally, each element of PP^T is equal to

$$(PP^T)_{a,b} = \sum_{x \in \mathrm{GF}(q)} \chi(a - x)\chi(b - x)$$

If $a = b$, then by Lemma 2.26

$$\sum_{x \in \mathrm{GF}(q)} \chi(a - x)\chi(b - x) = q - 1,$$

and if $a \neq b$, then

$$\sum_{x \in \mathrm{GF}(q)} \chi(a - x)\chi(b - x) = -1.$$

Therefore, $PP^T = qI - J$. □

2.4 Orthogonal Arrays

Orthogonal arrays are one of the most interesting combinatorial structures which can be used for construction of other combinatorial structures such as weighing matrices [13] and ternary codes [16].

Definition 2.30 (Orthogonal Array, [5]) An orthogonal array $\mathrm{OA}(t,\lambda)$ is a $\lambda^2 \times t$ matrix over λ symbols such that in any two columns, each pair of symbols occurs once.

Example 2.31 ([21]) Consider the symbols $\{1,2,3\}$, the following is an $\mathrm{OA}(4,3)$

$$A = \begin{bmatrix} 2 & 3 & 1 & 1 \\ 3 & 2 & 1 & 2 \\ 1 & 1 & 1 & 3 \\ 3 & 1 & 2 & 1 \\ 1 & 3 & 2 & 2 \\ 2 & 2 & 2 & 3 \\ 1 & 3 & 3 & 1 \\ 2 & 1 & 3 & 2 \\ 3 & 3 & 3 & 3 \end{bmatrix}$$

Each two rows of A agree in only one column which is an important property for the class of orthogonal arrays that are introduced in the following.

Proposition 2.32 ([13]) Let p be a prime power and m a positive integer. Then, there exists an orthogonal array of dimension $p^{m+1} \times \frac{p^{m+1}-1}{p-1}$ on p symbols such that every two rows agree in $\frac{p^m-1}{p-1}$ columns.

These orthogonal arrays are the most important combinatorial objects that will be used in Chapter 4. In [13] Kharaghani et al. used the same orthogonal arrays to construct a family of weighing matrices. There are many more classes of orthogonal arrays that are out of scope of this thesis (see [9, 21] for more information on the orthogonal arrays and their constructions).

Chapter 3

A Recursive Construction of a Class of Quasi-Residual BIBDs

In this chapter, firstly, an alteration of the class of generalized Hadamard matrices represented in Theorem 2.23 will be used for the construction of a family of quasi-residual BIBDs with parameters

$$\text{BIBD}\left(p^m, p\frac{p^m-1}{p-1}, \frac{p^m-1}{p-1}, p^{m-1}, \frac{p^{m-1}-1}{p-1}\right)$$

for $m \geq 2$ and p a prime power will be given. Using a similar method of construction and an application of Paley matrices, a family of weighing matrices will be obtained.

Before the presentation of the constructions, the GH matrices in Theorem 2.23 will be modified to a class of (0,1)-matrices with some particular properties. For a better understanding of these matrices, the following example is stated.

Example 3.1 Suppose H is the generalized Hadamard matrix $\text{GH}(3,1)$ over the cyclic group of order 3, $G = \{1, a, a^2\}$ in Example 2.20. Let G_1 be the group generated by $U_3 = \text{circ}(0,1,0)$. By replacing each element of the group $G = \{1, a, a^2\}$ by the elements of the group $G_1 = \{I_3, U_3, U_3^2\}$, respectively, the following block matrix will be obtained

$$H = \begin{bmatrix} I_3 & I_3 & I_3 \\ I_3 & U_3 & U_3^2 \\ I_3 & U_3^2 & U_3 \end{bmatrix}, \tag{3.1}$$

By considering H as a (0,1)-matrix

$$H = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Proposition 3.2 Suppose F is the cyclic group generated by $U_p = \text{circ}(0,1,0,...,0)$ where p is a prime number. Define $G_m = \{g_1 \otimes g_2 \otimes ... \otimes g_m; g_i \in F\}$. Then, G_m with the multiplication is an elementary abelian group of order p^m for any positive integer m.

Proof. The identity element of G_m is the element $I \otimes I \otimes ... \otimes I$ and the inverse of $\tau \in G_m$ is equal to the transpose of τ in terms of matrices since $\tau(\tau)^T = I \otimes I \otimes ... \otimes I$. Additionally, according to the definition of Kronecker product and its properties, G_m satisfies the closure and the associativity properties.

To show that G_m is abelian, suppose $\tau, \theta \in G_m$, then it will be proven that $\tau\theta = \theta\tau$. Let $\tau = g_1 \otimes g_2 \otimes ... \otimes g_m$ and $\theta = h_1 \otimes h_2 \otimes ... \otimes h_m$, then

$$\tau\theta = (g_1 \otimes g_2 \otimes ... \otimes g_m)(h_1 \otimes h_2 \otimes ... \otimes h_m) = g_1h_1 \otimes g_2h_2 \otimes ... \otimes g_mh_m,$$

since h_i and g_i are the elements of the cyclic group generated by U_p for any $i \in \{1,2,...,m\}$, then $g_i h_i = h_i g_i$. Therefore,

$$\tau\theta = h_1 g_1 \otimes h_2 g_2 \otimes ... \otimes h_m g_m = \theta\tau.$$

Thus, G_m is an abelian group.

Lastly, it has to be shown that the order of each element of G_m is p. In order to show that, it will be proven that for an arbitrary element τ in G_m, $\tau^p = I \otimes I \otimes ... \otimes I$ and p is the smallest integer that satisfies the equation $\tau^x = I \otimes I \otimes ... \otimes I$. Then, by the definition of elementary abelian group, G_m is an elementary abelian group of order p^m.

Let $\tau \in G_m$ and $\tau = g_1 \otimes g_2 \otimes ... \otimes g_m$ and suppose for some integer x, $\tau^x = I \otimes I \otimes ... \otimes I$. Since g_i is an element of the group generated by U_p for each $i \in \{1, 2, ..., m\}$, then

$$\tau = U_p^{n_1} \otimes U_p^{n_2} \otimes ... \otimes U_p^{n_m},$$

for some integers $n_1, n_2, ..., n_m \in \{0, 1, ..., p-1\}$. Thus,

$$\tau^x = U_p^{n_1 x} \otimes U_p^{n_2 x} \otimes ... \otimes U_p^{n_m x} = I \otimes I \otimes ... \otimes I.$$

Thus, for each i, $n_i x \equiv 0 (\text{mod } p)$. Since $n_i \nmid p$, then $x = p$.

Therefore, $ord(\tau)|p$ but p is a prime number and τ is chosen arbitrarily. Thus, $ord(\tau) = p$ which proves that G_m is an elementary abelian group. □

Construction 3.3 By Theorem 2.23, for any prime number p and positive integer m there exists a GH$(p^m, 1)$ over an elementary abelian group of order p^m, namely G. Let G_m be the elementary Abelian group in Proposition 3.2. By defining an isomorphism between the group elements of G and G_m, a (0,1)-matrix will be obtained. These matrices will be referred to as *realized GH matrices* over the group G_m.

Example 3.4 Suppose G is the cyclic group generated by U_3, and let H be a realized GH$(3,1)$ over G (See Example 3.1). Then,

$$H_2 = H \otimes H = \begin{bmatrix} H & H & H \\ H & N_1 & N_2 \\ H & N_2 & N_1 \end{bmatrix}, \tag{3.2}$$

where

$$N_1 = \begin{bmatrix} U_3 & U_3 & U_3 \\ U_3 & U_3^2 & I_3 \\ U_3 & I_3 & U_3^2 \end{bmatrix},$$

$$N_2 = \begin{bmatrix} U_3^2 & U_3^2 & U_3^2 \\ U_3^2 & I_3 & U_3 \\ U_3^2 & U_3 & I_3 \end{bmatrix}.$$

By considering $H \otimes H$ as a $(0,1)$-matrix, $H \otimes H$ is equal to

$$
\left[\begin{array}{ccccccccc|ccccccccc|ccccccccc}
1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0\\
0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0\\
0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1\\
1&0&0&0&1&0&0&0&1&1&0&0&0&1&0&0&0&1&1&0&0&0&1&0&0&0&1\\
0&1&0&0&0&1&1&0&0&0&1&0&0&0&1&1&0&0&0&1&0&0&0&1&1&0&0\\
0&0&1&1&0&0&0&1&0&0&0&1&1&0&0&0&1&0&0&0&1&1&0&0&0&1&0\\
1&0&0&0&0&1&0&1&0&1&0&0&0&0&1&0&1&0&1&0&0&0&0&1&0&1&0\\
0&1&0&1&0&0&0&0&1&0&1&0&1&0&0&0&0&1&0&1&0&1&0&0&0&0&1\\
0&0&1&0&1&0&1&0&0&0&0&1&0&1&0&1&0&0&0&0&1&0&1&0&1&0&0\\
\hline
1&0&0&1&0&0&1&0&0&0&1&0&0&1&0&0&1&0&0&0&1&0&0&1&0&0&1\\
0&1&0&0&1&0&0&1&0&0&0&1&0&0&1&0&0&1&1&0&0&1&0&0&1&0&0\\
0&0&1&0&0&1&0&0&1&1&0&0&1&0&0&1&0&0&0&1&0&0&1&0&0&1&0\\
1&0&0&0&1&0&0&0&1&0&1&0&0&0&1&1&0&0&0&0&1&1&0&0&0&1&0\\
0&1&0&0&0&1&1&0&0&0&0&1&1&0&0&0&1&0&1&0&0&0&1&0&0&0&1\\
0&0&1&1&0&0&0&1&0&1&0&0&0&1&0&0&0&1&0&1&0&0&0&1&1&0&0\\
1&0&0&0&0&1&0&1&0&0&1&0&1&0&0&0&0&1&0&0&1&0&1&0&1&0&0\\
0&1&0&1&0&0&0&0&1&0&0&1&0&1&0&1&0&0&1&0&0&0&0&1&0&1&0\\
0&0&1&0&1&0&1&0&0&1&0&0&0&0&1&0&1&0&0&1&0&1&0&0&0&0&1\\
\hline
1&0&0&1&0&0&1&0&0&0&0&1&0&0&1&0&0&1&0&1&0&0&1&0&0&1&0\\
0&1&0&0&1&0&0&1&0&1&0&0&1&0&0&1&0&0&0&0&1&0&0&1&0&0&1\\
0&0&1&0&0&1&0&0&1&0&1&0&0&1&0&0&1&0&1&0&0&1&0&0&1&0&0\\
1&0&0&0&1&0&0&0&1&0&0&1&1&0&0&0&1&0&0&1&0&0&0&1&1&0&0\\
0&1&0&0&0&1&1&0&0&1&0&0&0&1&0&0&0&1&0&0&1&1&0&0&0&1&0\\
0&0&1&1&0&0&0&1&0&0&1&0&0&0&1&1&0&0&1&0&0&0&1&0&0&0&1\\
1&0&0&0&0&1&0&1&0&0&0&1&0&1&0&1&0&0&0&1&0&1&0&0&0&0&1\\
0&1&0&1&0&0&0&0&1&1&0&0&0&0&1&0&1&0&0&0&1&0&1&0&1&0&0\\
0&0&1&0&1&0&1&0&0&0&1&0&1&0&0&0&0&1&1&0&0&0&0&1&0&1&0
\end{array}\right]
$$

Then, it can be verified that

$$H_2 H_2^T = 3(J_{27} - I_9 \otimes J_3) + 9I_{27}.$$

Lemma 3.5 Let q be a prime power such that $q = p^n$ for some prime number p and some positive integer n. Let G_n be the group of order p^n in Proposition 3.2. Suppose H is the realized generalized Hadamard matrix $\mathrm{GH}(q,1)$ over G_n. Let

$$H_m = \underbrace{H \otimes H \otimes \ldots \otimes H}_{m \text{ times}}.$$

Thus,

$$H_m H_m^T = q^{m-1}(J_{q^{m+1}} - I_{q^m} \otimes J_q) + q^m I_{q^{m+1}}.$$

Proof. The lemma can be proven by mathematical induction on $m \geq 1$.

Let $m = 1$, then $H = [H_{i,j}]$ is a block matrix consisting of blocks $H_{i,j}$ of size q such that each block is equal to one of the group elements of G_n. Thus, $HH^T = [(HH^T)_{i,j}]$ is also a block matrix such that each block $(HH^T)_{i,j}$ is equal to $\sum_{z=1}^{q} H_{i,z}H_{z,j}^T$. Therefore, the diagonal blocks of HH^T are equal to

$$\sum_{z=1}^{q} H_{i,z}H_{z,i}^T = \sum_{z=1}^{q} H_{i,z}(H_{i,z})^T = qI_q,$$

Now the off-diagonal blocks are equal to

$$\sum_{z=1}^{q} H_{i,z}H_{z,j}^T = \sum_{z=1}^{q} H_{i,z}(H_{j,z})^T$$

According to the definition of generalized Hadamard matrix GH$(q,1)$ and the construction 3.3, each element of G_n appears only once in the multiset $\{H_{i,z}(H_{j,z})^T : 1 \leq z \leq q\}$. Therefore,

$$\sum_{z=1}^{q} H_{i,z}(H_{j,z})^T = \sum_{g \in G_n} g = J_q.$$

Suppose the statement is true for $m-1$, it will be proven that it is also true for m.

$H_m = H \otimes H_{m-1}$ thus, each block of the block matrix H_m, $(H_m)_{i,j}$, is equal to $((H)_{i,j})H_{m-1}$ of size q^m. Therefore,

$$(H_m H_m^T)_{i,j} = \sum_{z=1}^{q} (H_m)_{i,z}[(H_m)_{j,z}]^T = \sum_{z=1}^{q} (H_{i,z}H_{m-1})((H_{j,z})^T H_{m-1}^T) = (HH^T)_{i,j}(H_{m-1}H_{m-1}^T) \quad (3.3)$$

If $i \neq j$, then equation (3.3) is equal to $(J_q)(H_{m-1}H_{m-1}^T)$ and if $i = j$, then it is equal to $(qI_q)(H_{m-1}H_{m-1}^T)$. Therefore, by the induction hypothesis,

$$H_m H_m^T = q^{m-1}(J_{q^{m+1}} - I_{q^m} \otimes J_q) + q^m I_{q^{m+1}}.$$

□

The next example shows that there are three disjoint incidence matrices of BIBD$(9,12,4,3,1)$ that can be constructed using the above Lemma.

Example 3.6 Let $U_3 = \text{circ}(0,1,0)$, then $U_3^2 = \text{circ}(0,0,1)$ and let

$$\text{diag}(U^i) = \begin{bmatrix} U_3^i & 0 & 0 \\ 0 & U_3^i & 0 \\ 0 & 0 & U_3^i \end{bmatrix},$$

for $i \in \{0,1,2\}$

$$H = \begin{bmatrix} I_3 & I_3 & I_3 \\ I_3 & U_3 & U_3^2 \\ I_3 & U_3^2 & U_3 \end{bmatrix}.$$

Consider the matrix M_0 as following

$$M_0 = \left[\begin{array}{c|c} M_{0_1} & M_{0_2} \end{array} \right]$$

where

$$M_{0_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

$$M_{0_2} = \begin{bmatrix} I_3 & I_3 & I_3 \\ I_3 & U_3 & U_3^2 \\ I_3 & U_3^2 & U_3 \end{bmatrix}.$$

$$M_0 = \left[\begin{array}{ccc|ccc|ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{array}\right],$$

There are four and three nonzero entries in each row and column of M_0, respectively and the inner product of any two distinct rows is 1. Thus, the matrix M_0 is an incidence matrix of BIBD$(9,12,4,3,1)$.

Similarly, two more incidence matrices can be obtained as

$$M_1 = \left[\begin{array}{c|c} M_{1_1} & M_{1_2} \end{array}\right]$$

where

$$M_{1_1} = U_3 \otimes \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

$$M_{1_2} = \begin{bmatrix} I_3 & I_3 & I_3 \\ I_3 & U_3 & U_3^2 \\ I_3 & U_3^2 & U_3 \end{bmatrix} \begin{bmatrix} U_3 & 0 & 0 \\ 0 & U_3 & 0 \\ 0 & 0 & U_3 \end{bmatrix} = \begin{bmatrix} U_3 & U_3 & U_3 \\ U_3 & U_3^2 & I_3 \\ U_3 & I_3 & U_3^2 \end{bmatrix},$$

$$M_1 = \left[\begin{array}{ccc|ccc|ccc|ccc} 0&1&0&0&1&0&0&1&0&0&1&0\\ 0&1&0&0&0&1&0&0&1&0&0&1\\ 0&1&0&1&0&0&1&0&0&1&0&0\\ \hline 0&0&1&0&1&0&0&0&1&1&0&0\\ 0&0&1&0&0&1&1&0&0&0&1&0\\ 0&0&1&1&0&0&0&1&0&0&0&1\\ \hline 1&0&0&0&1&0&1&0&0&0&0&1\\ 1&0&0&0&0&1&0&1&0&1&0&0\\ 1&0&0&1&0&0&0&0&1&0&1&0 \end{array}\right].$$

$$M_2 = \left[\begin{array}{c|c} M_{2_1} & M_{2_2} \end{array}\right],$$

where

$$M_{2_1} = U_3^2 \otimes \begin{bmatrix} 1\\ 1\\ 1 \end{bmatrix},$$

$$M_{2_2} = \begin{bmatrix} I_3 & I_3 & I_3\\ I_3 & U_3 & U_3^2\\ I_3 & U_3^2 & U_3 \end{bmatrix} \begin{bmatrix} U_3^2 & 0 & 0\\ 0 & U_3^2 & 0\\ 0 & 0 & U_3^2 \end{bmatrix} = \begin{bmatrix} U_3^2 & U_3^2 & U_3^2\\ U_3^2 & I_3 & U_3\\ U_3^2 & U_3 & I_3 \end{bmatrix},$$

$$M_2 = \left[\begin{array}{ccc|ccc|ccc|ccc} 0&0&1&0&0&1&0&0&1&0&0&1\\ 0&0&1&1&0&0&1&0&0&1&0&0\\ 0&0&1&0&1&0&0&1&0&0&1&0\\ \hline 1&0&0&0&0&1&1&0&0&0&1&0\\ 1&0&0&1&0&0&0&1&0&0&0&1\\ 1&0&0&0&1&0&0&0&1&1&0&0\\ \hline 0&1&0&0&0&1&0&1&0&1&0&0\\ 0&1&0&1&0&0&0&0&1&0&1&0\\ 0&1&0&0&1&0&1&0&0&0&0&1 \end{array}\right].$$

The sum of the above three matrices is

$$M_0 + M_1 + M_2 = \left[\begin{array}{ccc|ccc|ccc|ccc} 1&1&1&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&1&1&1\\ \hline 1&1&1&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&1&1&1\\ \hline 1&1&1&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&1&1&1 \end{array}\right].$$

The above matrix shows that M_0, M_1 and M_2 are disjoint and they provide a decomposition for the matrix $J_{9\times 12}$.

In general there are p disjoint incidence matrices of BIBD$(p^2, p^2+p, p+1, p, 1)$, where p is a prime power and they result in a decomposition for the matrix $J_{p^2\times(p^2+p)}$.

Lemma 3.7 Suppose $p = q^n$ where q is a prime number and n is a positive integer. Let

$$M_i = \left[\begin{array}{c|c} U_p^i \otimes e_p & H\mathrm{diag}(U_p^i) \end{array}\right], \tag{3.4}$$

for $i = 0, .., p-1$, such that U_p is the shift of order p, $\mathrm{diag}(U_p^i) = I_p \otimes U_p^i$, and H is the realized GH$(p, 1)$ over the group $G_n = \{g_1 \otimes g_2 \otimes \ldots \otimes g_n | g_i \in < U_p >\}$ in Proposition 3.2. Then, M_i is an incidence matrix of

$$\mathrm{BIBD}\left(p^2, p^2+p, p+1, p, 1\right)$$

for each i.

Proof. It will be shown that $M_iM_i^T = J_{p^2} + pI_{p^2}$ for any $i \in \{0,1,...,p-1\}$.

$$M_iM_i^T = (U_p^i \otimes e_p)((U_p^i)^T \otimes e_p^T) + H\text{diag}(U_p^i)\text{diag}((U_p^i)^T)H^T$$

$$= (U_p^i(U_p^i)^T) \otimes (e_pe_p^T) + H\text{diag}(U_p^i)\text{diag}((U_p^i)^T)H^T,$$

Since U_p is the shift of order p, then $U_p^i(U_p^i)^T = I_p$ and $e_pe_p^T = J_p$,

$$M_iM_i^T = I_p \otimes J_p + HH^T,$$

By Lemma 3.5, $HH^T = J_{p^2} - I_p \otimes J_p + pI_{p^2}$. Thus,

$$M_iM_i^T = J_{p^2} + pI_{p^2},$$

Therefore, for each $i \in \{0,1,...,p-1\}$, M_i is the incidence matrix of BIBD$\left(p^2, p^2+p, p+1, p, 1\right)$.

□

Lemma 3.8 Let $\{I_p, U_p, ..., U_p^{p-1}\}$ be the group generated by U_p. Then,

$$\sum_{i=0}^{p-1} U_p^i = J_p.$$

Proof. Suppose x_i is a vector of order p such that the ith coordinate is one and the rest of the coordinates are zeros. Then, $U_p = \text{circ}(x_2)$ consequently, $U_p^i = \text{circ}(x_{i+1})$ for $i \in \{0,1,..,p-1\}$. Therefore,

$$\sum_{i=0}^{p-1} U_p^i = \sum_{i=1}^{p} \text{circ}(x_i) = \text{circ}(j)$$

where j is a vector of order p with all coordinates equal to one. Thus,

$$\sum_{i=0}^{p-1} U_p^i = J_p.$$

□

Corollary 3.9 The matrices of equation (3.4) are disjoint.

In the following another property of these incidence matrices will be discussed.

Lemma 3.10 Let $S = \{I_p, U_p, U_p^2, ..., U_p^{p-1}\}$ be the cyclic group generated by the shift of order p. Suppose $G =< \sigma >$ is a cyclic group of order p such that $\sigma : S \to S$; $\sigma(U_p^i) = U_p^{i+1}$ for any $i \in \{0, 1, ..., p-2\}$ and $\sigma(U_p^{p-1}) = I_p$. Then, for $\sigma^r \in G$

$$\sigma^r(U_p^i)(\sigma^r(U_p^j))^T = (U_p^i)(U_p^j)^T$$

for any $i, j, r \in \{0, 1, ..., p-1\}$.

Proof. By the definition of the bijection σ, $\sigma^r(U_p^i) = U_p^{i+r}$, $\sigma^r(U_p^j) = U_p^{j+r}$. Therefore,

$$(\sigma^r(U_p^i))(\sigma^r(U_p^j))^T = (U_p^{i+r})(U_p^{j+r})^T$$

$$= U_p^i U_p^r (U_p^r)^T (U_p^j)^T = U_p^i (U_p^j)^T.$$

□

Proposition 3.11 Let $M = \{M_0, M_1, ..., M_{p-1}\}$ be the set of incidence matrices constructed in Lemma 3.7. Suppose $G =< \sigma >$ where σ is the bijection from M to itself such that $\sigma(M_i) = M_{i+1}$ for $i = 0, .., p-2$ and $\sigma(M_{p-1}) = M_0$. Then, for any $\sigma^r \in G$

$$(\sigma^r(M_i))(\sigma^r(M_j))^T = M_i M_j^T,$$

where $i, j, r \in \{0, 1, ..., p-1\}$.

Proof. By the definition of G

$$(\sigma^r(M_i))(\sigma^r(M_j))^T = (M_\theta)(M_\beta)^T,$$

for some $\theta, \beta \in \{0, 1, ..., p-1\}$ such that $\theta = i + r$ and $\beta = j + r$. Thus, by equation (3.4) in Lemma 3.7 and Lemma 3.10

$$(M_\theta)(M_\beta)^T = \left[\begin{array}{c|c} (U_p^\theta(U_p^\beta)^T)\otimes(e_pe_p^T) & HH^T\mathrm{diag}(U_p^\theta(U_p^\beta)^T) \end{array}\right]$$

$$= \left[\begin{array}{c|c} (U_p^i(U_p^j)^T)\otimes(e_pe_p^T) & HH^T\mathrm{diag}(U_p^i(U_p^j)^T) \end{array}\right] = M_iM_j^T.$$

□

Theorem 3.12 (Main Theorem) Suppose p be a prime power. For $m \geq 3$ let

$$M_{m_i} = \left[\begin{array}{c|c} M_{m-1_i}\otimes e_p & H_{m-1}\mathrm{diag}(U_p^i) \end{array}\right], i=0,\dots,p-1 \tag{3.5}$$

and for $m=2$ let

$$M_{2_i} = \left[\begin{array}{c|c} U_p^i\otimes e_p & H_1\mathrm{diag}(U_p^i) \end{array}\right], \tag{3.6}$$

such that $\mathrm{diag}(U_p^i) = I_{p^{m-1}}\otimes U_p^i$, and H_m is the matrix constructed in Lemma 3.5. Then, for each i, M_{m_i} is the incidence matrix of

$$\mathrm{BIBD}\left(p^m, p\frac{p^m-1}{p-1}, \frac{p^m-1}{p-1}, p^{m-1}, \frac{p^{m-1}-1}{p-1}\right)$$

for $m \geq 2$.

Proof. By mathematical induction on m it will be shown

$$M_{m_i}M_{m_i}^T = \frac{p^{m-1}-1}{p-1}J_{p^m} + p^{m-1}I_{p^m}.$$

By Lemma 3.7, there are p $\mathrm{BIBD}(p^2, p^2+p, p+1, p, 1)$ with disjoint incidence matrices namely

$$\{M_{2_0}, M_{2_1}, \dots, M_{2_{p-1}}\}.$$

Suppose the statement is true for $m-1$, then there are p

$$\text{BIBD}\left(p^{m-1}, p\frac{p^{m-1}-1}{p-1}, \frac{p^{m-1}-1}{p-1}, p^{m-2}, \frac{p^{m-2}-1}{p-1}\right)$$

with incidence matrices $\{M_{m-1_0}, M_{m-1_1}, ..., M_{m-1_{p-1}}\}$. Thus,

$$M_{m_i}M_{m_i}^T = (M_{m-1_i} \otimes e_p)(M_{m-1_i}^T \otimes e_p^T) + H_{m-1}\text{diag}(U_p^i)\text{diag}(U_p^i)^T H_{m-1}^T$$

$$= (M_{m-1_i}M_{m-1_i}^T) \otimes (e_p e_p^T) + H_{m-1}H_{m-1}^T,$$

Since

$$M_{m-1_i}M_{m-1_i}^T = \frac{p^{m-2}-1}{p-1}J_{p^{m-1}} + \left(\frac{p^{m-1}-1}{p-1} - \frac{p^{m-2}-1}{p-1}\right)I_{p^{m-1}},$$

therefore,

$$M_{m_i}M_{m_i}^T = \left(\frac{p^{m-2}-1}{p-1}J_{p^{m-1}} + \left(\frac{p^{m-1}-1}{p-1} - \frac{p^{m-2}-1}{p-1}\right)I_{p^{m-1}}\right) \otimes J_p$$

$$+p^{m-2}J_{p^m} - p^{m-2}(I_{p^{m-1}} \otimes J_p) + p^{m-1}I_{p^m}$$

$$= \frac{p^{m-2}-1}{p-1}J_{p^m} + \left(\frac{p^{m-1}-1}{p-1} - \frac{p^{m-2}-1}{p-1}\right)(I_{p^{m-1}} \otimes J_p) + p^{m-2}J_{p^m} - p^{m-2}(I_{p^{m-1}} \otimes J_p) + p^{m-1}I_{p^m}$$

$$= \left(\frac{p^{m-2}-1}{p-1} + p^{m-2}\right)J_{p^m} + p^{m-1}I_{p^m}$$

$$= \frac{p^{m-1}-1}{p-1}J_{p^m} + p^{m-1}I_{p^m}.$$

Therefore, M_{m_i} is the desired incidence matrix. □

The incidence matrices in Theorem 3.12 have the following property.

Proposition 3.13 Let $M = \{M_{m_0}, M_{m_1}, ..., M_{m_{p-1}}\}$ be the set of incidence matrices constructed in Theorem 3.12. Suppose $G =< \sigma >$ where σ is the bijection from M to itself such that $\sigma(M_{m_i}) = M_{m_{i+1}}$ for $i = 0, .., p-2$ and $\sigma(M_{m_{p-1}}) = M_{m_0}$. Then, for any $\sigma^r \in G$

$$(\sigma^r(M_{m_i}))(\sigma^r(M_{m_j}))^T = M_{m_i} M_{m_j}^T,$$

where $i, j, r \in \{0, 1, ..., p-1\}$.

Proof. The notation in Theorem 3.12 will be used during the proof. The statement will be proven by mathematical induction on m. Let $m = 2$, then by Proposition 3.11 the statement is true.

By equation (3.5) and the definition of G

$$(\sigma^r(M_{m_i}))(\sigma^r(M_{m_j}))^T$$

$$= \left[\begin{array}{c|c} [(M_{m-1_{r+i}})(M_{m-1_{r+j}})^T] \otimes (e_p e_p^T) & H_{m-1} H_{m-1}^T diag(U_p^{i+r} (U_p^{r+j})^T) \end{array} \right]$$

for some $r \in \{0, 1, ..., p-1\}$. By Lemma 3.10 and the induction hypothesis

$$(\sigma^r(M_{m_i}))(\sigma^r(M_{m_j}))^T = M_{m_i} M_{m_j}.$$

□

3.1 Embeddablity of the Quasi-Residual Designs

This section shows that the constructed quasi-residual designs in Theorem 3.12 are embeddable. The corresponding SBIBDs will be obtained. The symmetric BIBDs will be also used in Chapter 4 and larger quasi-residual designs will be constructed.

Example 3.14 Let $\{M_0, M_1, M_2\}$ be the three disjoint incidence matrices of BIBD$(9,12,4,3,1)$. Then, each of the following matrices are the incidence matrices of SBIBD$(13,4,1)$

$$S_i = \begin{bmatrix} 0_9 & M_i \\ e_4 & D \end{bmatrix}, \tag{3.7}$$

where $D = U_4^j \otimes e_3^T$ for some $0 \leq j \leq 2$. For instance, suppose $j = 0, i = 0$,

$$D = I_4 \otimes e_3^T = \left[\begin{array}{ccc|ccc|ccc|ccc} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right].$$

It will be proven that $S_0 S_0^T = J_{13} + 3I_{13}$.

$$S_0 S_0^T = \begin{bmatrix} M_0 M_0^T & M_0 D^T \\ D M_0^T & DD^T + e_4 e_4^T \end{bmatrix}$$

Since the matrix M_0 is the incidence matrix of BIBD$(9,12,4,3,1)$, then $M_0 M_0^T = J_9 + 3I_9$, and $DD^T = 3I_4$. Additionally, by Example 3.6

$$M_0D^T = \left[\begin{array}{ccc|ccc|ccc|ccc}
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
\hline
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
\hline
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0
\end{array}\right]
\left[\begin{array}{ccc|ccc|ccc|ccc}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1
\end{array}\right]^T$$

$$= \left[\begin{array}{c|c|c|c}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
\hline
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
\hline
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{array}\right].$$

Thus, $M_0D^T = J_{9\times 4}$. Consequently, $S_0S_0^T = J_9 + 3I_9$ which shows S_0 is the incidence matrix of SBIBD$(13,4,1)$ therefore, M_0 is embeddable. In a similar way, M_1 and M_2 are also embeddable.

During the rest of this section, a class of quasi-derived designs will be obtained which leads to the proof of the embeddablity of the quasi-residual designs in Theorem 3.12.

3.1.1 Construction of the Quasi-Derived Designs

Lemma 3.15 Suppose p is a prime power, then the following matrices are the incidence matrices of the quasi-derived BIBD$(p+1,p^2+p,p,1,0)$

$$D_i = U_{p+1}^i \otimes e_p^T$$

for any $i \in \{0,1,2,...,p-1\}$.

Proof. This follows by Theorem 2.5, since

$$D_i D_i^T = (U_{p+1}^i (U_{p+1}^i)^T) \otimes (e_p^T e_p) = pI_{p+1}.$$

□

Lemma 3.16 Suppose p is a prime power and S is an incidence matrix of

$$\text{SBIBD}\left(\frac{p^m-1}{p-1}, \frac{p^{m-1}-1}{p-1}, \frac{p^{m-2}-1}{p-1}\right),$$

then, $S \otimes e_p^T$ is an incidence matrix of the quasi-derived

$$\text{BIBD}\left(\frac{p^m-1}{p-1}, p\frac{p^m-1}{p-1}, p\frac{p^{m-1}-1}{p-1}, \frac{p^{m-1}-1}{p-1}, p\frac{p^{m-2}-1}{p-1}\right).$$

Proof. Using Theorem 2.5

$$(S \otimes e_p^T)(S \otimes e_p^T)^T = (SS^T) \otimes (e_p^T e_p) = \left(\frac{p^{m-2}-1}{p-1} J_{\frac{p^m-1}{p-1}} + p^{m-2} I_{\frac{p^m-1}{p-1}}\right) \otimes (p)$$

$$= p\frac{p^{m-2}-1}{p-1} J_{\frac{p^m-1}{p-1}} + p^{m-1} I_{\frac{p^m-1}{p-1}}$$

□

3.1.2 Construction of the Symmetric Designs

Lemma 3.17 Suppose p is a prime power. Let M_i be one of the p disjoint incidence matrices of BIBD$(p^2, p^2+p, p+1, p, 1)$ that are defined in Lemma 3.7. Then M_i is embeddable.

Proof. It will be proven that the following matrix is an incidence matrix of SBIBD$(p^2+p+1, p+1, 1)$

$$S_i = \begin{bmatrix} 0_{p^2} & M_i \\ e_{p+1} & D \end{bmatrix} \tag{3.8}$$

where D is one of the incidence matrices of the quasi-derived BIBD$(p+1, p^2+p, p, 1, 0)$ defined in Lemma 3.15.

It will be shown that $S_iS_i^T = J_{p^2+p+1} + pI_{p^2+p+1}$.

$$S_iS_i^T = \begin{bmatrix} M_iM_i^T & M_iD^T \\ DM_i^T & DD^T + e_{p+1}e_{p+1}^T \end{bmatrix}$$

By hypothesis, $M_iM_i^T = J_{p^2} + pI_{p^2}$ and by Lemma 3.15 $DD^T = pI_{p+1}$.

Also, $M_i = [(M_i)_{x,y}]$ is a block matrix consisting of blocks $(M_i)_{x,y}$ of size $p \times p$ such that each row of $(M_i)_{x,y}$ has exactly one nonzero entry. Similarly, $D = [D_{x,y}]$ is consisting of blocks $D_{x,y}$ of dimension $1 \times p$ and each block is either equal to e_p^T or 0_p^T. Consequently, M_iD^T is also a block matrix such that each block $(M_iD^T)_{x,y}$ is of size $p \times 1$ and is equal to

$$\sum_{z=1}^{p+1} (M_i)_{x,z}(D_{y,z})^T \tag{3.9}$$

since there is exactly one block $D_{y,z}$ equal to e_p^T for a fixed y and all $1 \leq z \leq p+1$ and there is exactly one nonzero entry in each row of $(M_i)_{x,z}$, thus

$$\sum_{z=1}^{p+1} (M_i)_{x,z}(D_{y,z})^T = \sum_{z=1}^{p+1} (M_i)_{x,z}(e_p) = J_{p\times 1},$$

thus $M_iD^T = J_{p^2\times(p+1)}$. Therefore, $S_iS_i^T = J_{p^2+p+1} + pI_{p^2+p+1}$ and S_i is the desired incidence matrix. □

Theorem 3.18 Suppose p is a prime power and M_{m_i} is one of the incidence matrices of the quasi-residual

$$\text{BIBD}\left(p^m, p\frac{p^m-1}{p-1}, \frac{p^m-1}{p-1}, p^{m-1}, \frac{p^{m-1}-1}{p-1}\right)$$

that are constructed in Theorem 3.12. Then, M_{m_i} is embeddable for any $m \geq 2$.

Proof. Proof by mathematical induction on m. Let $m = 2$, then by Lemma 3.17, M_{2_i} is embeddable. Now suppose the statement is true for $m-1$ and let S be the incidence matrix of

$$\text{SBIBD}\left(\frac{p^m-1}{p-1}, \frac{p^{m-1}-1}{p-1}, \frac{p^{m-2}-1}{p-1}\right),$$

and

$$S_i = \begin{bmatrix} 0_{p^m} & M_{m_i} \\ e_{\frac{p^m-1}{p-1}} & D \end{bmatrix} \tag{3.10}$$

where $D = S \otimes e_p^T$. It will be shown that

$$S_i S_i^T = \frac{p^{m-1}-1}{p-1} J_{\frac{p^{m+1}-1}{p-1}} + p^{m-1} I_{\frac{p^{m+1}-1}{p-1}},$$

and

$$S_i S_i^T = \begin{bmatrix} M_{m_i} M_{m_i}^T & M_{m_i} D^T \\ D M_{m_i}^T & DD^T + e_{\frac{p^m-1}{p-1}} e^T_{\frac{p^m-1}{p-1}} \end{bmatrix}$$

$$M_{m_i} M_{m_i}^T = \frac{p^{m-1}-1}{p-1} J_{p^m} + p^{m-1} I_{p^m}$$

since M_{m_i} is the incidence matrix of BIBD $\left(p^m, p\frac{p^m-1}{p-1}, \frac{p^m-1}{p-1}, p^{m-1}, \frac{p^{m-1}-1}{p-1}\right)$ and

$$DD^T = p\left(\frac{p^{m-2}-1}{p-1} J_{\frac{p^m-1}{p-1}} + p^{m-2} I_{\frac{p^m-1}{p-1}}\right).$$

$M_{m_i} = [(M_{m_i})_{x,y}]$ is a block matrix consisting of blocks $(M_{m_i})_{x,y}$ of size $p \times p$ and each row of $(M_{m_i})_{x,y}$ has exactly one nonzero entry. Also, $D = [D_{x,y}]$ is a block matrix consisting of blocks $D_{x,y}$ of dimension $1 \times p$ which each $D_{x,y}$ is either equal to 0_p^T or e_p^T. Thus, the matrix $M_{m_i}D^T = [(M_{m_i}D^T)_{x,y}]$ is also a block matrix consisting of blocks of size $p \times 1$ and

$$(M_{m_i}D^T)_{x,y} = \sum_{z=1}^{\frac{p^m-1}{p-1}} (M_{m_i})_{x,z}(D_{y,z})^T \tag{3.11}$$

Since $D = S \otimes e_p^T$ and S has $\frac{p^{m-1}-1}{p-1}$ nonzero entries in each row, then there are $\frac{p^{m-1}-1}{p-1}$ blocks $D_{y,z}$ equal to e_p^T for a fixed y and $z \in \{1, 2, ..., \frac{p^m-1}{p-1}\}$. Thus, equation (3.11) is equal to

$$\frac{p^{m-1}-1}{p-1} e_p,$$

so

$$M_{m_i}D_i^T = \frac{p^{m-1}-1}{p-1} J_{p^m \times \frac{p^m-1}{p-1}},$$

Therefore,

$$S_i S_i^T = \frac{p^{m-1}-1}{p-1} J_{\frac{p^{m+1}-1}{p-1}} + p^{m-1} I_{\frac{p^{m+1}-1}{p-1}},$$

and S_i is the incidence matrix of

$$\text{SBIBD}\left(\frac{p^{m+1}-1}{p-1}, \frac{p^m-1}{p-1}, \frac{p^{m-1}-1}{p-1}\right).$$

□

In conclusion, it has been shown that the constructed quasi-residual BIBDs in Theorem 3.12 are embeddable for any $m \geq 2$ using a recursive construction.

3.2 A Recursive Construction of Weighing Matrices

In this section the class of weighing matrices with classical parameters will be reconstructed. The use of Paley matrices is essential in the construction.

Example 3.19 Suppose $\{M_0, M_1, M_2\}$ are the three disjoint incidence matrices of BIBD$(9,12,4,3,1)$ in Example 3.6. Consider $0M_0 + 1M_1 + (-1)M_2$

$$C = M_1 - M_2 = \begin{bmatrix} 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 1 & -1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 & 1 & -1 & -1 & 0 & 1 & 1 & -1 & 0 \\ -1 & 0 & 1 & -1 & 0 & 1 & 1 & -1 & 0 & 0 & 1 & -1 \\ -1 & 0 & 1 & 1 & -1 & 0 & 0 & 1 & -1 & -1 & 0 & 1 \\ 1 & -1 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 & 0 & 1 & 0 & 1 & -1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 & -1 & 0 & -1 & 0 & 1 & 0 & 1 & -1 \end{bmatrix}$$

Thus,

$$CC^T = 9I_9 - J_9,$$

which indicates that all the off-diagonal entries are equal to -1 and all the diagonal entries are equal to 8.

Lemma 3.20 Let p be an odd prime power and $P = [p_{i,j}]$ be a Paley matrix of order p, then

$$C_1 C_1^T = p^2 I_{p^2} - J_{p^2}$$

where $C_1 = \left[\begin{array}{c|c} P \otimes e_p & H_1(I_p \otimes P) \end{array} \right]$ and H_1 is the realized GH$(p,1)$ in Lemma 3.5.

Proof. Since

$$H_1 H_1^T = J_{p^2} - I_p \otimes J_p + pI_{p^2},$$

and

$$PP^T = pI_p - J_p,$$

then

$$C_1C_1^T = PP^T \otimes J_p + H_1H_1^T(I_p \otimes PP^T)$$

$$= (pI_p \otimes J_p - J_{p^2}) + (J_{p^2} - I_p \otimes J_p + pI_{p^2})(pI_{p^2} - I_p \otimes J_p)$$

$$= p^2 I_{p^2} - J_{p^2}.$$

In particular, all the off-diagonal entries of $C_1C_1^T$ are -1. □

Theorem 3.21 Let p be an odd prime power. Suppose

$$C_m = \left[\begin{array}{c|c} C_{m-1} \otimes e_p & H_m(I_{p^m} \otimes P) \end{array} \right]$$

for $m \geq 2$ where $P = [p_{i,j}]$ is the Paley matrix of order p and

$$C_1 = \left[\begin{array}{c|c} P \otimes e_p & H_1(I_p \otimes P) \end{array} \right]$$

and H_m is the (0,1)-matrix in Lemma 3.5, then

$$C_mC_m^T = p^{m+1} I_{p^{m+1}} - J_{p^{m+1}}$$

for $m \geq 1$.

Proof. Mathematical induction on m will be used to show

$$C_mC_m^T = p^{m+1} I_{p^{m+1}} - J_{p^{m+1}}$$

for $m \geq 1$.

The $m = 1$ case is proven by Lemma 3.20. Now suppose the statement is true for $m - 1$.

Then

$$C_m C_m^T = ((C_{m-1}C_{m-1}^T) \otimes (e_p e_p^T)) + H_m(I_{p^m} \otimes (PP^T))H_m^T$$

$$= (p^m I_{p^m} - J_{p^m}) \otimes J_p + H_m H_m^T (I_{p^m} \otimes (pI_p - J_p))$$

Since $H_m H_m^T = p^{m-1}(J_{p^{m+1}} - I_{p^m} \otimes J_p) + p^m I_{p^{m+1}}$, then

$$C_m C_m^T = p^{m+1} I_{p^{m+1}} - J_{p^{m-1}}.$$

□

We are now ready to reconstruct the weighing matrices with classical parameters. The foundation of this construction will be the matrix C_m that has been obtained in Theorem 3.21.

Example 3.22 Let C be the constructed matrix in Example 3.19. Then, the following matrix will be a weighing matrix W$(13,9)$

$$W = \begin{bmatrix} 0_4 & \mathrm{W}(4,3) \otimes e_3^T \\ e_9 & C \end{bmatrix} = \left[\begin{array}{c|ccc|ccc|ccc|ccc}
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 \\
0 & 1 & 1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 \\
\hline
1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 \\
1 & 0 & 1 & -1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 \\
1 & 0 & 1 & -1 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 \\
\hline
1 & -1 & 0 & 1 & 0 & 1 & -1 & -1 & 0 & 1 & 1 & -1 & 0 \\
1 & -1 & 0 & 1 & -1 & 0 & 1 & 1 & -1 & 0 & 0 & 1 & -1 \\
1 & -1 & 0 & 1 & 1 & -1 & 0 & 0 & 1 & -1 & -1 & 0 & 1 \\
\hline
1 & 1 & -1 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & -1 & 0 & 1 \\
1 & 1 & -1 & 0 & -1 & 0 & 1 & 0 & 1 & -1 & 1 & -1 & 0 \\
1 & 1 & -1 & 0 & 1 & -1 & 0 & -1 & 0 & 1 & 0 & 1 & -1
\end{array}\right]$$

where

$$\mathrm{W}(4,3) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix}$$

In general, with a similar procedure as above, a weighing matrix with parameters (p^2+p+1, p^2) for any odd prime power p can be obtained.

Lemma 3.23 Let p be an odd prime power and C_1 be the matrix constructed in Lemma 3.20. Then, the following matrix is a $\mathrm{W}(p^2+p+1, p^2)$:

$$W = \left[\begin{array}{c|c} 0_{p+1} & \mathrm{W}(p+1,p)\otimes e_p^T \\ \hline e_{p^2} & C_1 \end{array}\right]. \tag{3.12}$$

Proof. It will be proven that $WW^T = p^2 I_{p^2+p+1}$.

$$WW^T = \left[\begin{array}{c|c} W_1 & W_2 \\ \hline W_3 & W_4 \end{array}\right]$$

where

$$W_1 = (\mathrm{W}(p+1,p)\otimes e_p^T)(\mathrm{W}(p+1,p)^T \otimes e_p) = (\mathrm{W}(p+1,p)\mathrm{W}(p+1,p)^T)\otimes(e_p^T e_p) = p^2 I_{p+1},$$

$$W_2 = (\mathrm{W}(p+1,p)\otimes e_p^T)C_1^T,$$

$$W_3 = C_1(\mathrm{W}(p+1,p)^T \otimes e_p),$$

and

$$W_4 = e_{p^2}e_{p^2}^T + C_1C_1^T = J_{p^2} + (p^2 I_{p^2} - J_{p^2}) = p^2 I_{p^2}.$$

Additionally, $\mathrm{W}(p+1,p)\otimes e_p^T = [K_{i,j}]$ is a block matrix containing blocks of dimension $1\times p$ such that each block, $K_{i,j}$, is equal to 0_p^T or e_p^T or $-e_p^T$. Also, the matrix $C_1 = [C_{i,j}]$

is a block matrix containing blocks of size $1 \times p$ where each block has one zero entry and $\frac{p-1}{2}$ entries equal to 1 and $\frac{p-1}{2}$ entries equal to -1. Thus, each entry of the matrix W_2 is as following

$$[(\mathrm{W}(p+1,p)\otimes e_p^T)C_1^T]_{i,j} = \sum_{z=1}^{p+1} K_{i,z}C_{z,j}^T = \sum_{z=1}^{p+1} K_{i,z}(C_{j,z})^T$$

Since the number of 1s and -1s in $C_{j,z}$ are the same and all the entries of $K_{i,z}$ are equal, $K_{i,z}(C_{j,z})^T = 0$. Then $[(\mathrm{W}(p+1,p)\otimes e_p^T)C_1^T]_{i,j} = 0$.

Therefore, $WW^T = p^2 I_{p^2+p+1}$ and W is a $\mathrm{W}(p^2+p+1,p^2)$. □

Theorem 3.24 Suppose p is an odd prime power. Let C_m be the constructed matrix in Theorem 3.21 for $m \geq 1$. Then the following matrix is a $\mathrm{W}(\sum_{i=0}^{m} p^i, p^m)$

$$W = \left[\begin{array}{c|c} 0_{(\sum_{i=0}^{m-1} p^i)} & W_1 \\ \hline e_{p^m} & C_m \end{array}\right] \tag{3.13}$$

where $W_1 = \mathrm{W}(\sum_{i=0}^{m-1} p^i, p^{m-1}) \otimes e_p^T$.

Proof. It will be shown that $WW^T = p^m I_{\sum_{i=0}^{m} p^i}$.

$$WW^T = \left[\begin{array}{c|c} W_2 & W_3 \\ \hline W_4 & W_5 \end{array}\right],$$

where

$$W_2 = (\mathrm{W}(\sum_{i=0}^{m-1} p^i, p^{m-1})\mathrm{W}(\sum_{i=0}^{m-1} p^i, p^{m-1})^T) \otimes e_p^T e_p,$$

$$W_3 = (\mathrm{W}(\sum_{i=0}^{m-1} p^i, p^{m-1}) \otimes e_p^T)C_m^T,$$

and

$$W_4 = C_m(\mathrm{W}(\sum_{i=0}^{m-1} p^i, p^{m-1})^T \otimes e_p),$$

$$W_5 = e_{p^m}e_{p^m}^T + C_m C_m^T.$$

Since $(\mathrm{W}(\sum_{i=0}^{m-1} p^i, p^{m-1})\mathrm{W}(\sum_{i=0}^{m-1} p^i, p^{m-1})^T) = p^{m-1}I_{\sum_{i=0}^{m-1} p^i}$ and $e_p^T e_p = p$. So,

$$(\mathrm{W}(\sum_{i=0}^{m-1} p^i, p^{m-1})\mathrm{W}(\sum_{i=0}^{m-1} p^i, p^{m-1})^T) \otimes e_p^T e_p = p^m I_{\sum_{i=0}^{m-1} p^i},$$

and

$$e_{p^m} e_{p^m}^T + CC^T = p^m I_{p^m}.$$

$\mathrm{W}(\sum_{i=0}^{m-1} p^i, p^{m-1}) \otimes e_p^T = [K_{i,j}]$ is a block matrix with blocks $K_{i,j}$ of size $1 \times p$ such that each block is either equal to $0_{1\times p}$, e_p^T, or $-e_p^T$. Similarly, $C_m = [C_{i,j}]$ is a block matrix with blocks $C_{i,j}$ of dimension $1 \times p$ such that each block has one zero entry and $\frac{p-1}{2}$ entries equal to -1 and $\frac{p-1}{2}$ entries equal to 1. Therefore,

$$[(\mathrm{W}(\sum_{i=0}^{m-1} p^i, p^{m-1}) \otimes e_p^T)C_m^T]_{i,j} = \sum_{z=1}^{\sum_{i=0}^{m-1} p^i} K_{i,z}C_{z,j}^T = \sum_{z=1}^{\sum_{i=0}^{m-1} p^i} K_{i,z}(C_{j,z})^T$$

Since the number of 1s and -1s in $C_{j,z}$ are the same and all the entries of $K_{i,z}$ are equal, thus $K_{i,z}(C_{j,z})^T = 0$ and consequently, $(\mathrm{W}(\sum_{i=0}^{m-1} p^i, p^{m-1}) \otimes e_p^T)C_m^T = 0$.

Therefore, $WW^T = p^m I_{\sum_{i=0}^{m} p^i}$ and that proves W is a $\mathrm{W}(\sum_{i=0}^{m} p^i, p^m)$. □

In summary in this chapter, using similar techniques and methods of constructions a class of embeddable quasi-residual designs was obtained and a reconstruction of a family of weighing matrices was introduced. In the following chapter, these quasi-residual designs will be discussed further.

Chapter 4

Constructing more Quasi-Residual BIBDs

In the first section, BIBD$(p^2, p(p+1), p+1, p, 1)$ will be used to construct quasi-residual designs with parameters

$$\left(p^2\frac{q^{m+1}-1}{q-1}, p(p+1)\frac{q^{m+1}-1}{q-1}, (p+1)q^m, pq^m, q^m\right)$$

where $q = p+1$ and p are prime powers.
In the second section, BIBD$\left(p^d, p\frac{p^d-1}{p-1}, \frac{p^d-1}{p-1}, p^{d-1}, \frac{p^{d-1}-1}{p-1}\right)$ will be extended to

$$\text{BIBD}\left(p^d\frac{q^{m+1}-1}{q-1}, p\frac{p^d-1}{p-1}\frac{q^{m+1}-1}{q-1}, q^m\frac{p^d-1}{p-1}, q^m p^{d-1}, q^m\frac{p^{d-1}-1}{p-1}\right)$$

where p is a prime power and $q = \frac{p^d-1}{p-1}$ is also a prime power for some positive integer d.

For some values of p, the integer $q = \frac{p^d-1}{p-1}$ will never be a prime power therefore, no quasi-residual designs with parameters

$$\text{BIBD}\left(p^d\frac{q^{m+1}-1}{q-1}, p\frac{p^d-1}{p-1}\frac{q^{m+1}-1}{q-1}, \frac{p^d-1}{p-1}q^m, p^{d-1}q^m, \frac{p^{d-1}-1}{p-1}q^m\right)$$

can be constructed with the proposed method for those values of p. In the last part of this chapter, an example for this case will be given.

Yury Ionin constructed designs with the above parameters in [11]. The method presented here is different from those in [11].

4.1 Extension of the Affine Geometries

For a clearer insights on the extension of parameters of a BIBD$(p^2, p^2+p, p+1, p, 1)$ to larger parameters, we begin with the following example.

Example 4.1 In this example, three disjoint incidence matrices of BIBD$(45,60,16,12,4)$ will be obtained using the incidence matrices $\{M_0, M_1, M_2\}$ of BIBD$(9,12,4,3,1)$ constructed in Example 3.6.

Let $q = 4$, since 4 is a prime power, then there exists a BGW$(5,4,3)$ over the cyclic group $G = \{1, \sigma, \sigma^2\}$, where σ is a bijection from $\{M_0, M_1, M_2\}$ to itself and $\sigma(M_i) = M_{i+1}$ for $i = 0, 1$ and $\sigma(M_2) = M_0$. Let W be the BGW$(5,4,3)$ over G such as

$$W = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & \sigma & \sigma^2 \\ 1 & 1 & 0 & \sigma^2 & \sigma \\ 1 & \sigma & \sigma^2 & 0 & 1 \\ 1 & \sigma^2 & \sigma & 1 & 0 \end{bmatrix}, \tag{4.1}$$

Then, the matrices $\{W \otimes M_0, W \otimes M_1, W \otimes M_2\}$ are the desired incidence matrices. To see this,

$$W \otimes M_0 = \left[\begin{array}{c|c|c|c|c} 0 & M_0 & M_0 & M_0 & M_0 \\ \hline M_0 & 0 & M_0 & M_1 & M_2 \\ \hline M_0 & M_0 & 0 & M_2 & M_1 \\ \hline M_0 & M_1 & M_2 & 0 & M_0 \\ \hline M_0 & M_2 & M_1 & M_0 & 0 \end{array}\right], \tag{4.2}$$

The above matrix is a 45×60 matrix. It will be shown $(W \otimes M_0)(W \otimes M_0)^T = 4J_{45} + 12I_{45}$. Since $M_0 + M_1 + M_2 = J_{9\times 12}$, $M_0M_0^T = J_9 + 3I_9$, and $M_iM_j^T + M_jM_i^T = 3J_9 - 3I_9$ for $i \neq j$, then

$$(W\otimes M_0)(W\otimes M_0)^T = \left[\begin{array}{c|c|c|c|c} 4(J_9+3I_9) & M_0J_{12\times 9} & M_0J_{12\times 9} & M_0J_{12\times 9} & M_0J_{12\times 9} \\ \hline M_0J_{12\times 9} & 4(J_9+3I_9) & 4J_9 & 4J_9 & 4J_9 \\ \hline M_0J_{12\times 9} & 4J_9 & 4(J_9+3I_9) & 4J_9 & 4J_9 \\ \hline M_0J_{12\times 9} & 4J_9 & 4J_9 & 4(J_9+3I_9) & 4J_9 \\ \hline M_0J_{12\times 9} & 4J_9 & 4J_9 & 4J_9 & 4(J_9+3I_9) \end{array}\right],$$

And since each row of M_0 has 4 nonzero entries, then $M_0J_{12\times 9} = 4J_9$ so

$$(W\otimes M_0)(W\otimes M_0)^T = \left[\begin{array}{c|c|c|c|c} 4(J_9+3I_9) & 4J_9 & 4J_9 & 4J_9 & 4J_9 \\ \hline 4J_9 & 4(J_9+3I_9) & 4J_9 & 4J_9 & 4J_9 \\ \hline 4J_9 & 4J_9 & 4(J_9+3I_9) & 4J_9 & 4J_9 \\ \hline 4J_9 & 4J_9 & 4J_9 & 4(J_9+3I_9) & 4J_9 \\ \hline 4J_9 & 4J_9 & 4J_9 & 4J_9 & 4(J_9+3I_9) \end{array}\right],$$

This shows that all the off-diagonal entries of $(W\otimes M_0)(W\otimes M_0)^T$ are equal to 4 and all the diagonal entries are equal to 16 thus, the matrix $W\otimes M_0$ is the incidence matrix of BIBD$(45,60,16,12,4)$. The matrix $W\otimes M_0$ can be found in Appendix A.

Similarly, we can construct two other incidence matrices of BIBD$(45,60,12,16,4)$,

$$W\otimes M_1 = \left[\begin{array}{c|c|c|c|c} 0 & M_1 & M_1 & M_1 & M_1 \\ \hline M_1 & 0 & M_1 & M_2 & M_0 \\ \hline M_1 & M_1 & 0 & M_0 & M_2 \\ \hline M_1 & M_2 & M_0 & 0 & M_1 \\ \hline M_1 & M_0 & M_2 & M_1 & 0 \end{array}\right], \tag{4.3}$$

$$W\otimes M_2 = \left[\begin{array}{c|c|c|c|c} 0 & M_2 & M_2 & M_2 & M_2 \\ \hline M_2 & 0 & M_2 & M_0 & M_1 \\ \hline M_2 & M_2 & 0 & M_1 & M_0 \\ \hline M_2 & M_0 & M_1 & 0 & M_2 \\ \hline M_2 & M_1 & M_0 & M_2 & 0 \end{array}\right], \tag{4.4}$$

The sum of the three incidence matrices will be

$$\left[\begin{array}{c|c|c|c|c}
0 & M_0+M_1+M_2 & M_0+M_1+M_2 & M_0+M_1+M_2 & M_0+M_1+M_2 \\ \hline
M_0+M_1+M_2 & 0 & M_0+M_1+M_2 & M_1+M_2+M_0 & M_2+M_0+M_1 \\ \hline
M_0+M_1+M_2 & M_0+M_1+M_2 & 0 & M_2+M_0+M_1 & M_1+M_2+M_0 \\ \hline
M_0+M_1+M_2 & M_1+M_2+M_0 & M_2+M_0+M_1 & 0 & M_0+M_1+M_2 \\ \hline
M_0+M_1+M_2 & M_2+M_0+M_1 & M_1+M_2+M_0 & M_0+M_1+M_2 & 0
\end{array}\right],$$

which shows the constructed three matrices are disjoint.

Subsequently, a general theorem for any prime power p such that $q = p+1$ is a prime power is presented. Before stating the general theorem, the following definition and results must be stated.

Definition 4.2 (Group of Symmetries, [12]) Suppose (X,A) is a BIBD with parameters (v,b,r,k,λ). Let M be a set of $v \times b$ incidence matrices of (X,A) and G be a finite group of bijections from M to itself satisfying the following conditions

- $(\sigma X)(\sigma Y)^T = XY^T$ for all $X,Y \in M$ and all $\sigma \in G$.
- $\sum_{\sigma \in G} \sigma X = \frac{k|G|}{v} J$ for all $X \in M$.

Then G is a group of symmetries of M.

Example 4.3 Suppose (X,A) is a BIBD$(9,12,4,3,1)$ and let $M=\{M_0,M_1,M_2\}$ such that

$$M_0=\begin{bmatrix}1&0&0&1&0&0&1&0&0&1&0&0\\1&0&0&0&1&0&0&1&0&0&1&0\\1&0&0&0&0&1&0&0&1&0&0&1\\0&1&0&1&0&0&0&1&0&0&0&1\\0&1&0&0&1&0&0&0&1&1&0&0\\0&1&0&0&0&1&1&0&0&0&1&0\\0&0&1&1&0&0&0&0&1&0&1&0\\0&0&1&0&1&0&1&0&0&0&0&1\\0&0&1&0&0&1&0&1&0&1&0&0\end{bmatrix},$$

$$M_1=\begin{bmatrix}0&1&0&0&1&0&0&1&0&0&1&0\\0&1&0&0&0&1&0&0&1&0&0&1\\0&1&0&1&0&0&1&0&0&1&0&0\\0&0&1&0&1&0&0&0&1&1&0&0\\0&0&1&0&0&1&1&0&0&0&1&0\\0&0&1&1&0&0&0&1&0&0&0&1\\1&0&0&0&1&0&1&0&0&0&0&1\\1&0&0&0&0&1&0&1&0&1&0&0\\1&0&0&1&0&0&0&0&1&0&1&0\end{bmatrix},$$

$$M_2=\begin{bmatrix}0&0&1&0&0&1&0&0&1&0&0&1\\0&0&1&1&0&0&1&0&0&1&0&0\\0&0&1&0&1&0&0&1&0&0&1&0\\1&0&0&0&0&1&1&0&0&0&1&0\\1&0&0&1&0&0&0&1&0&0&0&1\\1&0&0&0&1&0&0&0&1&1&0&0\\0&1&0&0&0&1&0&1&0&1&0&0\\0&1&0&1&0&0&0&0&1&0&1&0\\0&1&0&0&1&0&1&0&0&0&0&1\end{bmatrix}.$$

Let $G=\{1,\sigma,\sigma^2\}$, where $\sigma(M_0)=M_1,\sigma(M_1)=M_2$ and $\sigma(M_2)=M_0$. Consequently,

$\sigma^2(M_0) = M_2, \sigma^2(M_1) = M_0$ and $\sigma^2(M_2) = M_1$, $1(M_0) = M_0, 1(M_1) = M_1$ and $1(M_2) = M_2$. G will be a group of symmetries of M, since it satisfies the conditions in Definition 4.2:

- Suppose $M_i, M_j \in M$ and $\rho \in G$. According to the construction of G, $\rho = \sigma^t$ for some $t = 0, 1, 2$. Therefore, $(\sigma^t M_i)(\sigma^t M_j)^T = (M_x)(M_y)^T$ for some $x, y \in \{0, 1, 2\}$ and by Proposition 3.13 $M_x M_y^T = M_i M_j^T$.
- $\sum_{\rho \in G} \rho M_i = \sum_{t=0}^{2} \sigma^t M_i = M_1 + M_2 + M_0 = J$.

Theorem 2.4 from [11] will be stated below.

Theorem 4.4 Let M be a set of incidence matrices of (v, b, r, k, λ) BIBDs. Let G be a finite group of bijections $M \longrightarrow M$ satisfying conditions (i) $(\sigma X)(\sigma Z)^T = XZ^T$ for all $X, Z \in M$ and all $\sigma \in G$ and (ii) $\sum_{\sigma \in G} \sigma X = \frac{k|G|}{v} J$ for all $X \in M$. Let W be a BGW(w, l, μ) over G with $kr\mu = v\lambda l$. Then, for any $X \in M$, $W \otimes X$ is the incidence matrix of a BIBD with parameters $(vw, bw, rl, kl, \lambda l)$.

Thus, the following theorem can be concluded.

Theorem 4.5 Suppose p is a prime power such that $q = p + 1$ is also a prime power and $M = \{M_0, M_1, ..., M_{p-1}\}$ is the set of p incidence matrices of BIBD $\left(p^2, p^2 + p, p + 1, p, 1\right)$ constructed in Lemma 3.7. Then, there exists a quasi-residual design

$$\text{BIBD}\left(p^2 \frac{q^{m+1} - 1}{q - 1}, p(p+1)\frac{q^{m+1} - 1}{q - 1}, q^{m+1}, pq^m, q^m\right)$$

for any positive integer m.

Proof. Let $G =< \sigma >$ where σ is the bijection from M to itself and $\sigma(M_i) = M_{i+1}$ for $i = 0, .., p-2$ and $\sigma(M_{p-1}) = M_0$. It will be proven that G satisfies the conditions in Theorem 4.4. For any $\rho \in G$, $\rho = \sigma^r$ for some $r \in \{0, 1, ..., p-1\}$. Let M_i and M_j be two arbitrary elements of M thus

$$(\sigma^r M_i)(\sigma^r M_j)^T = M_\alpha M_\beta^T,$$

for some $M_\alpha, M_\beta \in G$. Since $\sigma^r M_i = M_\alpha$ and $\sigma^r M_j = M_\beta$ then by Proposition 3.11,

$$M_\alpha M_\beta^T = M_i M_j^T,$$

Additionally, by the construction of the quasi-residual designs in Lemma 3.7 and the group G,

$$\sum_{\rho \in G} \rho(M_i) = J_{p^2 \times (p^2+p)}.$$

Let W be a BGW with parameters $\left(\frac{q^{m+1}-1}{q-1}, q^m, q^m - q^{m-1}\right)$ over G. Then, by Theorem 4.4, $W \otimes M_i$ is an incidence matrix of the quasi-residual

$$\text{BIBD}\left(p^2 \frac{q^{m+1}-1}{q-1}, p(p+1)\frac{q^{m+1}-1}{q-1}, q^{m+1}, pq^m, q^m\right)$$

for any positive integer m. □

The above theorem gives rise to the question that whether the constructed quasi-residual designs are embeddable. Firstly, using the ideas in [14], an appropriate class of quasi-derived BIBDs with the parameters

$$\left(q^{m+1}, pq\frac{q^{m+1}-1}{q-1}, q^{m+1}-1, q^m, q^m-1\right),$$

where p and $q = p+1$ are prime powers, will be obtained and later the sufficient conditions for the embeddablity of the quasi-residual BIBDs will be represented.

In the following example the quasi-derived BIBD$(16, 60, 15, 4, 3)$ will be constructed.

Example 4.6 Let A be an orthogonal array in 4 symbols, $\{1,2,3,4\}$ of dimensions 16×5. Note that by Proposition 2.32 this orthogonal array exists.

$$A = \begin{bmatrix} 1&1&1&1&1 \\ 1&2&2&2&2 \\ 1&3&3&3&3 \\ 1&4&4&4&4 \\ 2&1&4&2&3 \\ 2&2&3&1&4 \\ 2&3&2&4&1 \\ 2&4&1&3&2 \\ 3&1&2&3&4 \\ 3&2&1&4&3 \\ 3&3&4&1&2 \\ 3&4&3&2&1 \\ 4&1&3&4&2 \\ 4&2&4&3&1 \\ 4&3&1&2&4 \\ 4&4&2&1&3 \end{bmatrix}$$

By replacing symbol 1 by the first row, symbol 2 by the second row, 3 by the third row and 4 by the fourth row of $D_0 = (I_4 \otimes e_3^T)$ we will obtain the incidence matrix of the quasi-derived design D of dimension 16×60.

$$D_0 = \begin{bmatrix} 1&1&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&1&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&1&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&1&1 \end{bmatrix}$$

and D is equal to

$$\left[\begin{array}{cccccccccccc|cccccccccccc|cccccccccccc|cccccccccccc|cccccccccccc}
1&1&1&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0\\
1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0\\
1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0\\
1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&1&1&1\\
0&0&0&1&1&1&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0\\
0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1\\
0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0\\
0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&1&1&1&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1\\
0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&1&1&1&0&0&0\\
0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&1&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&1&1&1&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&1&1&1&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1\\
0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&1&1&1&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0
\end{array}\right]$$

There are 15 and 4 nonzero entries in each row and column of D, respectively and the inner product of any two distinct rows of D is equal to 3. Therefore, D is the incidence matrix of the quasi-derived $\mathrm{BIBD}(16,60,15,4,3)$.

Lemma 4.7 Suppose p and $q = p+1$ are prime powers. Then, there exists a quasi-derived

$$\mathrm{BIBD}\left(q^{m+1}, pq\frac{q^{m+1}-1}{q-1}, q^{m+1}-1, q^m, q^m-1\right).$$

Proof. Suppose D is one of the constructed incidence matrices of $\mathrm{BIBD}(p+1, p^2+p, p, 1, 0)$ in Lemma 3.15. Let A be an orthogonal array of dimension $q^{m+1} \times \frac{q^{m+1}-1}{q-1}$ on q symbols for which any two distinct rows agree in $\frac{q^m-1}{q-1}$ columns. Let $\overline{D}$ be the resulting matrix from replacing the q symbols of A by the rows of D, respectively. Then, $\overline{D}$ is the desired incidence matrix. It will be shown that

$$\overline{D}\overline{D}^T = (q^m-1)J + \left(q^{m+1}-q^m\right)I.$$

$\overline{D} = [\overline{D}_{x,y}]$ is a block matrix containing blocks $\overline{D}_{x,y}$ of dimension $1 \times (p^2+p)$ such that each block is equal to a row of D. Thus, the entry (x,y) of the matrix $\overline{D}\overline{D}^T$ is

$$(\overline{D}\overline{D}^T)_{x,y} = \sum_{z=1}^{\frac{q^{m+1}-1}{q-1}} \overline{D}_{x,z}(\overline{D}_{y,z})^T.$$

If $x = y$ since each row of D has p nonzero entries, then $\overline{D}_{x,z}(\overline{D}_{x,z})^T = p = q-1$ therefore,

$$(\overline{D}\overline{D}^T)_{x,x} = q^{m+1} - 1,$$

and if $x \neq y$ since distinct rows of D are disjoint and two distinct rows of A agree in $\frac{q^m-1}{q-1}$ columns, then

$$(\overline{D}\overline{D}^T)_{x,y} = \sum_{z=1}^{\frac{q^{m+1}-1}{q-1}} \overline{D}_{x,z}(\overline{D}_{y,z})^T = p\frac{q^m-1}{q-1} = q^m - 1.$$

Therefore, $\overline{D}\overline{D}^T = (q^m-1)J + (q^{m+1}-q^m)I$ which proves that $\overline{D}$ is the incidence matrix of BIBD $\left(q^{m+1}, pq\frac{q^{m+1}-1}{q-1}, q^{m+1}-1, q^m, q^m-1\right)$. □

The following example shows that the constructed BIBD$(45,60,16,12,4)$ of Example 4.6 is embeddable.

Example 4.8 Suppose $R_0 = W \otimes M_0$ is the constructed incidence matrix of the quasi-residual BIBD$(45,60,16,12,4)$ in Example 4.1. It will be shown that R_0 is embeddable. Let D be the constructed quasi-derived BIBD$(16,60,15,4,3)$ in Example 4.6. Then, the following matrix is an incidence matrix of SBIBD$(61,16,4)$.

$$S = \begin{bmatrix} 0_{45} & R_0 \\ e_{16} & D \end{bmatrix} \tag{4.5}$$

It will be proven that $SS^T = 4J_{61} + 12I_{61}$. The matrix S can be found in Appendix A.

$$SS^T = \begin{bmatrix} R_0R_0^T & R_0D^T \\ DR_0^T & J_{16} + DD^T \end{bmatrix}$$

By Examples 4.1 and 4.6, $R_0R_0^T = 4J_{45} + 12I_{45}$, $DD^T + J_{16} = 4J_{16} + 12I_{16}$, also each two distinct rows of D and R_0 intersect in 4 columns, thus $DR_0^T = 4J_{16\times 45}$ (for a better understanding look at the matrices D and R_0 in Appendix A).

Lemma 4.9 Let p be a prime power such that $q = p+1$ is also a prime power. Let R_i be one of the incidence matrices of the quasi-residual BIBD with parameters

$$\left(p^2\frac{q^{m+1}-1}{q-1}, pq\frac{q^{m+1}-1}{q-1}, q^{m+1}, pq^m, q^m\right)$$

that has been constructed in Theorem 4.5. Then, R_i is embeddable for each i.

Proof. Using the notation in Theorem 4.5, the following matrix is the incidence matrix of SBIBD $\left(pq\frac{q^{m+1}-1}{q-1}+1, q^{m+1}, q^m\right)$

$$B_i = \left[\begin{array}{c|c} 0_{p^2\frac{q^{m+1}-1}{q-1}} & R_i \\ \hline e_{q^{m+1}} & \overline{D} \end{array}\right], \tag{4.6}$$

where $\overline{D}$ is the incidence matrix of

$$\text{BIBD}\left(q^{m+1}, pq\frac{q^{m+1}-1}{q-1}, q^{m+1}-1, q^m, q^m-1\right)$$

in Lemma 4.7. It is sufficient to show $B_i{B_i}^T = (q^m)J + (q^{m+1}-q^m)I$.

$$B_i{B_i}^T = \left[\begin{array}{c|c} R_iR_i^T & R_i\overline{D}^T \\ \hline \overline{D}R_i^T & e_{q^{m+1}}e_{q^{m+1}}^T + \overline{D}\overline{D}^T \end{array}\right],$$

R_i is the incidence matrix of $\left(p^2\frac{q^{m+1}-1}{q-1}, pq\frac{q^{m+1}-1}{q-1}, q^{m+1}, pq^m, q^m\right)$, thus

$$R_iR_i^T = (q^m)J + (q^{m+1}-q^m)I.$$

Additionally, by Lemma 4.7, $\overline{DD}^T + J_{q^{m+1}} = q^m J + (q^{m+1} - q^m)I$.

$\overline{D} = [\overline{D}_{x,y}]$ is a block matrix containing blocks $\overline{D}_{x,y}$ of dimension $1 \times pq$ such that each block $\overline{D}_{x,y}$ is equal to one of the rows of the matrix D in Lemma 4.7. Let M_i be the incidence matrix of BIBD$(p^2, p^2+p, p+1, p, 1)$ and

$$W = \mathrm{BGW}\left(\frac{q^{m+1}-1}{q-1}, q^m, q^{m+1} - q^m\right)$$

such that $R_i = W \otimes M_i$.

$R_i = [(R_i)_{x,y}]$ is a block matrix containing blocks of size $p^2 \times (p^2+p)$ such that each block $(R_i)_{x,y}$ is equal to M_x for some $x \in \{0, ..., p-1\}$ or $0_{p^2 \times (p^2+p)}$. Thus, the matrix $\overline{D}R_i^T$ is also a block matrix including blocks $(\overline{D}R_i^T)_{x,y}$ of size $1 \times p^2$.

$$(\overline{D}R_i^T)_{x,y} = \sum_{w=1}^{\frac{q^{m+1}-1}{q-1}} \overline{D}_{x,w}((R_i)_{y,w})^T$$

Since W has q^m nonzero entries and by Lemma 3.17, M_i and D are the residual and derived parts of the symmetric design SBIBD$(p^2+p+1, p+1, 1)$. Then

$$\sum_{w=1}^{\frac{q^{m+1}-1}{q-1}} (\overline{D})_{x,w}((R_i)_{y,w})^T = q^m J_{p^2 \times (pq)}.$$

Then $\overline{D}R_i^T = q^m J$. Therefore, $B_i {B_i}^T = (q^m)J + (q^{m+1} - q^m)I$ and B_i is an incidence matrix of

$$\mathrm{SBIBD}\left(pq\frac{q^{m+1}-1}{q-1} + 1, q^{m+1}, q^m\right).$$

□

4.2 An Extension of the Constructed Quasi-Residual Designs

In the previous section, the parameters of affine geometries were extended to larger parameters and larger quasi-residual designs were constructed. In section 4.1 the assumption was that for the prime power p, $q = p+1$ is a prime power. In this section the assumption will be that for the prime power p, $q = \frac{p^d-1}{p-1}$ is a prime power for $d > 1$.

In the following it will be shown how the parameters of the constructed quasi-residual designs in Theorem 3.10 can be extended using Ionin's method in [11].

Theorem 4.10 Suppose p is a prime power and $q = \frac{p^d-1}{p-1}$ is also a prime power for some positive integer d. Let $M = \{M_0, M_1, ..., M_{p-1}\}$ be the set of the p incidence matrices of BIBD$(p^d, p\frac{p^d-1}{p-1}, \frac{p^d-1}{p-1}, p^{d-1}, \frac{p^{d-1}-1}{p-1})$ in Theorem 3.12. Then there exists a quasi-residual design with parameters

$$\left(p^d\frac{q^{m+1}-1}{q-1}, p\frac{p^d-1}{p-1}\frac{q^{m+1}-1}{q-1}, q^m\frac{p^d-1}{p-1}, q^m p^{d-1}, q^m\frac{p^{d-1}-1}{p-1}\right) \tag{4.7}$$

for any $m \geq 1$.

Proof. Suppose G is the cyclic group $G = <\sigma>$ where σ is a bijection from M to M such that $\sigma(M_{p-1}) = M_0$, $\sigma(M_i) = M_{i+1}$ for $i = 0, .., p-2$. Suppose W is a

$$\text{BGW}(\frac{q^{m+1}-1}{q-1}, q^m, q^m - q^{m-1})$$

over the group G for any $m \geq 1$.

The cyclic group G satisfies the conditions of Theorem 4.4 since for any $\rho \in G$ and $M_i, M_j \in M$

$$(\rho M_i)(\rho M_j)^T = (\sigma^x M_i)(\sigma^x M_j)^T = M_\gamma M_\theta^T$$

for some $x, \gamma, \theta \in \{0, 1, .., p-1\}$. On the other hand, according to the construction of the incidence matrices in M, since $\sigma^x(M_i) = M_\gamma, \sigma^x(M_j) = M_\theta$ then by Proposition 3.13, $M_\gamma M_\theta^T = M_i M_j^T$. Lastly, $\sum_{x=0}^{p-1} \sigma^x M_i = J$ for any $M_i \in M$ by Corollary 3.13. Therefore, by Theorem 4.4 the quasi-residual design with parameters (4.7) exists and the set of their incidence matrices is as following

$$\{W \otimes M_i : i \in \{0,1,...,p-1\}\}.$$

□

Similar to the previous section, it can be proven that the constructed quasi-residual BIBDs in Theorem 4.10 are embeddable.

For each prime power p the quasi-residual designs

$$\text{BIBD}\left(p^d, p\frac{p^d-1}{p-1}, \frac{p^d-1}{p-1}, p^{d-1}, \frac{p^{d-1}-1}{p-1}\right) \tag{4.8}$$

has been obtain in Theorem 3.12 and the quasi-derived designs

$$\text{BIBD}\left(\frac{p^d-1}{p-1}, p\frac{p^d-1}{p-1}, \frac{p^d-1}{p-1}-1, \frac{p^{d-1}-1}{p-1}, \frac{p^{d-1}-1}{p-1}-1\right) \tag{4.9}$$

has been constructed in Lemma 3.16 which by Theorem 3.18, they are the residual and derived parts of

$$\text{SBIBD}\left(p\frac{p^d-1}{p-1}+1, \frac{p^d-1}{p-1}, \frac{p^{d-1}-1}{p-1}\right). \tag{4.10}$$

These quasi-residual and quasi-derived designs can be used for the construction of

$$\text{SBIBD}\left(pq\frac{q^{m+1}-1}{q-1}+1, \frac{p^d-1}{p-1}q^m, \frac{p^{d-1}-1}{p-1}q^m\right) \tag{4.11}$$

where $q = \frac{p^d-1}{p-1}$ is a prime power for some positive integer d.

Lemma 4.11 Suppose p is a prime power and $q = \frac{p^d-1}{p-1}$ is also a prime power for some positive integer d. Then there exists a quasi-derived BIBD with the following parameters

$$\text{BIBD}\left(q^{m+1}, pq\frac{q^{m+1}-1}{q-1}, q^{m+1}-1, q^m\frac{p^{d-1}-1}{p-1}, q^m\frac{p^{d-1}-1}{p-1}-1\right).$$

Proof. Let D be one of the incidence matrices of the derived part of

$$\text{SBIBD}\left(p\frac{p^d-1}{p-1}+1, \frac{p^d-1}{p-1}, \frac{p^{d-1}-1}{p-1}\right),$$

that has been constructed in Lemma 3.16.

By Proposition 2.32, suppose A is an orthogonal array of dimension $q^{m+1} \times \frac{q^{m+1}-1}{q-1}$ on q symbols for which any two distinct rows agree in $\frac{q^m-1}{q-1}$ columns. Let $\overline{D}$ be the resulting matrix from replacing q symbols of A by the rows of D, respectively. Then, $\overline{D}$ is the desired incidence matrix.

It will be shown that $\overline{DD}^T = \left(q^m\frac{p^{d-1}-1}{p-1}-1\right)J + \left(q^{m+1} - q^m\frac{p^{d-1}-1}{p-1}\right)I$.

$\overline{D} = [(\overline{D})_{x,y}]$ is a block matrix containing blocks $(\overline{D})_{x,y}$ of size $1 \times pq$ such that each block is equal to a row of the incidence matrix D. Then each entry of the matrix $\overline{DD}^T$ is as following

$$(\overline{DD}^T)_{x,y} = (\overline{D})_x \cdot (\overline{D})_y,$$

where $\overline{D}_x$, $\overline{D}_y$ are the xth and yth rows of the matrix $\overline{D}$ and "·" refers to the inner product.

For diagonal entries of $\overline{DD}^T$ $x = y$ so

$$(\overline{DD}^T)_{x,x} = (\overline{D})_x \cdot (\overline{D})_x = \sum_{z=1}^{\frac{q^{m+1}-1}{q-1}} (\overline{D})_{x,z} \cdot (\overline{D})_{x,z} = \sum_{z=1}^{\frac{q^{m+1}-1}{q-1}} q-1 = q^{m+1}-1,$$

since each block $\overline{D}_{x,z}$ is equal to a row of the incidence matrix D and each row of D has $q-1$ nonzero entries.

For the off diagonal entries of $\overline{D}\overline{D}^T$ $x \neq y$ so

$$(\overline{D}\overline{D}^T)_{x,y} = (\overline{D})_x \cdot (\overline{D})_y = \sum_{z=1}^{\frac{q^{m+1}-1}{q-1}} (\overline{D})_{x,z} \cdot (\overline{D})_{x,z} = (q-1)\frac{q^m-1}{q-1} + \left(\frac{q^{m+1}-1}{q-1} - \frac{q^m-1}{q-1}\right) p \frac{p^{d-1}-1}{p-1}$$

$$= q^m \frac{p^{d-1}-1}{p-1} - 1,$$

since each two distinct rows of A agree in $\frac{q^m-1}{q-1}$ and they differ in $\left(\frac{q^{m+1}-1}{q-1} - \frac{q^m-1}{q-1}\right)$ columns and for each pair of distinct rows of D, they intersect in $p\frac{p^{d-2}-1}{p-1}$ entries.

Thus, the off-diagonal entries of $\overline{D}\overline{D}^T$ are equal to $q^m \frac{p^{d-1}-1}{p-1} - 1$ and the diagonal entries are $q^{m+1}-1$, therefore,

$$\overline{D}\overline{D}^T = \left(q^m \frac{p^{d-1}-1}{p-1} - 1\right) J + \left(q^{m+1} - q^m \frac{p^{d-1}-1}{p-1}\right) I.$$

□

Theorem 4.12 The quasi-residual designs in the Theorem 4.10 are embeddable.

Proof. Using the notation in Theorem 4.10 and Lemma 4.11, let $R_i = W \otimes M_i$. Then the following matrix is the incidence matrix of SBIBD $\left(pq\frac{q^{m+1}-1}{q-1} + 1, q^{m+1}, q^m \frac{p^{d-1}-1}{p-1}\right)$

$$B_i = \left[\begin{array}{c|c} 0_{p^d \frac{q^{m+1}-1}{q-1}} & R_i \\ \hline e_{q^{m+1}} & \overline{D} \end{array}\right] \tag{4.12}$$

for each $i \in \{0, 1, .., p-1\}$ where $\overline{D}$ is the constructed matrix in Lemma 4.11.

It is sufficient to show

$$B_i {B_i}^T = \left(q^m \frac{p^{d-1}-1}{p-1}\right) J_{pq\frac{q^{m+1}-1}{q-1}+1} + \left(q^{m+1} - q^m \frac{p^{d-1}-1}{p-1}\right) I_{pq\frac{q^{m+1}-1}{q-1}+1}.$$

Now

$$B_i{B_i}^T = \left[\begin{array}{c|c} R_iR_i^T & R_i\overline{D}^T \\ \hline \overline{D}R_i^T & e_{q^{m+1}}e^T_{q^{m+1}} + \overline{DD}^T \end{array}\right].$$

$R_iR_i^T = \left(q^m\frac{p^{d-1}-1}{p-1}\right)J + \left(q^{m+1} - q^m\frac{p^{d-1}-1}{p-1}\right)I$ since R_i is the constructed incidence matrix in Theorem 4.10.

$\overline{DD}^T = \left(q^m\frac{p^{d-1}-1}{p-1} - 1\right)J + \left(q^{m+1} - q^m\frac{p^{d-1}-1}{p-1}\right)I$ by Lemma 4.11 so

$$\overline{DD}^T + J = \left(q^m\frac{p^{d-1}-1}{p-1}\right)J + \left(q^{m+1} - q^m\frac{p^{d-1}-1}{p-1}\right)I,$$

The last step is to show $\overline{D}R_i^T = q^m\frac{p^{d-1}-1}{p-1}J$. $\overline{D} = [(\overline{D})_{x,y}]$ is a block matrix containing blocks $(\overline{D})_{x,y}$ of size $1 \times pq$ such that each block is equal to a row of D, the incidence matrix of

$$\text{BIBD}\left(\frac{p^d-1}{p-1}, p\frac{p^d-1}{p-1}, \frac{p^d-1}{p-1} - 1, \frac{p^{d-1}-1}{p-1}, \frac{p^{d-1}-1}{p-1} - 1\right).$$

Additionally, R_i is a block matrix including blocks $(R_i)_{x,y}$ of dimension $p^d \times pq$ such that each block is equal to M_r for some $r \in \{0,1,2,...,p-1\}$. Therefore, $\overline{D}R_i^T$ is also a block matrix containing blocks $(\overline{D}R_i^T)_{x,y}$ of dimension $1 \times p^d$

$$(\overline{D}R_i^T)_{x,y} = \sum_{w=1}^{\frac{q^{m+1}-1}{q-1}} (\overline{D})_{x,w}(R_i)^T_{y,w}$$

Since W has only q^m nonzero entries in each row then, for a fixed integer y, there are only q^m nonzero blocks of $(R_i)_{y,w}$ for any $w \in \{1,2,..,\frac{q^{m+1}-1}{q-1}\}$.Also, each block $\overline{D}_{x,w}$ is equal to a row of the incidence matrix D. Then for any nonzero block $(\overline{D})_{x,w}$, $(\overline{D})_{x,w}(R_i)^T_{y,w} = (D_\alpha)(M_r)^T$ for some $\alpha \in \{1,2,...,\frac{p^d-1}{p-1}\}, r \in \{0,1,..,p-1\}$ where D_α denotes αth row of D. Then

$$(\overline{D}R_i^T)_{x,y} = \sum_{w=1}^{\frac{q^{m+1}-1}{q-1}} (\overline{D})_{x,w}(R_i)^T_{y,w} = q^m(\frac{p^{d-1}-1}{p-1})J_{1\times p^d}.$$

Therefore, $\overline{D}R_i^T = q^m \frac{p^{d-1}-1}{p-1} J_{q^{m+1} \times p^d \frac{q^{m+1}-1}{q-1}}$, which completes the proof.

$$B_i {B_i}^T = \left(q^m \frac{p^{d-1}-1}{p-1}\right) J_{pq\frac{q^{m+1}-1}{q-1}+1} + \left(q^{m+1} - q^m \frac{p^{d-1}-1}{p-1}\right) I_{pq\frac{q^{m+1}-1}{q-1}+1}.$$

Thus B_i is the desired incidence matrix. □

Remark 4.13 The embeddablity of the incidence matrices R_i in Theorem 4.10 can only be proven using the introduced quasi-derived incidence matrices in Lemma 4.11 and not *any* incidence matrix of

$$\text{BIBD}\left(q^{m+1}, pq\frac{q^{m+1}-1}{q-1}, q^{m+1}-1, q^m\frac{q^{d-1}-1}{q-1}, q^m\frac{q^{d-1}-1}{q-1}-1\right).$$

Remark 4.14 For $p = 3$, $q = p+1 = 4$ is a prime power thus, Lemma 4.9 applies. Although, for $p = 5$, $q = p+1 = 6$ is not a prime power, $q = \frac{p^3-1}{p-1} = 31$ is a prime power and Theorem 4.12 applies. For $p = 9$, p is a prime power but $q = \frac{p^m-1}{p-1}$ is not a prime power for any m and Theorem 4.12 does not apply.

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Appendix A

Examples

Example 4.8. BIBD$(45,60,16,12,4)$

$$W \otimes M_0 =$$

$$\left[\begin{array}{cccccccccccc|cccccccccccc|cccccccccccc|cccccccccccc|cccccccccccc}
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&0&0&1&0&0&1&0&1&0&0&0&1&0&0&1&0&0&1&0&1&0&0&0&1&0&0&1&0&0&1&0&1&0&0&0&1&0&0&1&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&1&0&0&1&1&0&0&0&0&1&0&0&1&0&0&1&1&0&0&0&0&1&0&0&1&0&0&1&1&0&0&0&0&1&0&0&1&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&1&0&0&0&1&0&1&0&1&0&0&0&1&0&0&0&1&0&1&0&1&0&0&0&1&0&0&0&1&0&1&0&1&0&0&0&1&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&0&1&1&0&0&0&1&0&0&1&0&0&0&1&1&0&0&0&1&0&0&1&0&0&0&1&1&0&0&0&1&0&0&1&0&0&0&1&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&1&0&0&0&1&0&0&1&0&0&0&1&1&0&0&0&1&0&0&1&0&0&0&1&1&0&0&0&1&0&0&1&0&0&0&1&1&0&0&0&1&0\\
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0&0&1&0&1&0&1&0&0&0&0&1&1&0&0&0&0&1&0&1&0&1&0&0&0&1&0&1&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&0&0&0&1\\
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1&0&0&0&0&1&0&0&1&0&0&1&0&0&1&0&1&0&0&1&0&0&1&0&0&1&0&1&0&0&1&0&0&1&0&0&1&0&0&0&0&1&0&0&1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&1&0&0&0&1&0&0&0&1&1&0&0&0&0&1&1&0&0&0&1&0&0&0&1&0&1&0&0&0&1&1&0&0&0&1&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&1&0&0&0&1&1&0&0&1&0&0&1&0&0&0&1&0&0&0&1&0&0&1&0&0&1&1&0&0&0&1&0&0&1&0&0&1&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
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0&0&1&0&0&1&0&1&0&1&0&0&0&1&0&0&1&0&1&0&0&0&0&1&1&0&0&1&0&0&0&0&1&0&1&0&0&0&1&0&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{array}\right]$$

Example 4.8. SBIBD$(61,16,4)$

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\left[\begin{array}{c|cccccccccccc|cccccccccccc|cccccccccccc|cccccccccccc|cccccccccccc}
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&0&0&1&0&0&1&0&1&0&0&0&1&0&0&1&0&0&1&0&1&0&0&0&1&0&0&1&0&0&1&0&1&0&0&0&1&0&0&1&0&0&1&0\\
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0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&1&0&0&0&1&0&0&1&0&0&0&1&1&0&0&0&1&0&0&1&0&0&0&1&1&0&0&0&1&0&0&1&0&0&0&1&1&0&0&0&1&0\\
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0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&1&0&1&0&0&0&0&1&0&0&1&0&1&0&1&0&0&0&0&1&0&0&1&0&1&0&1&0&0&0&0&1&0&0&1&0&1&0&1&0&0\\
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\end{array}\right]
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