Electromagnetic and gravitational scattering at Planckian energies

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The scattering of pointlike particles at very large center-of-mass energies and fixed low momentum transfers, occurring due to both their electromagnetic and gravitational interactions, is reexamined in the particular case when one of the particles carries a magnetic charge. At Planckian center-of-mass energies, when gravitational dominance is usually expected, the presence of magnetic charge is shown to produce dramatic modifications to the scattering cross section as well as to the holomorphic structure of the scattering amplitude.

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I. INTRODUCTION

In perturbative quantum field theory, all of the information about interactions is customarily relegated to the perturbing Hamiltonian, with the exactly integrable part corresponding to the free propagation of quanta (forward scattering). In situations where a well-defined perturbative domain is not available, such a decomposition of the original Hamiltonian into "free" and "interacting" parts is clearly not meaningful. It is more desirable, therefore, to formulate the theory in such a way that the part that is exactly tractable contains nontrivial information about the interaction, even though this may be semiclassical. In some cases, it turns out that there are kinematical regimes where in fact the semiclassical approximation is exact, permitting the calculation of scattering amplitudes without further approximations. We shall focus on two such cases in the sequel. The first deals with an electromagnetic system consisting of a point charge and a Dirac monopole, both of very small mass. The second is basically a generalization of the first, in which gravitational interactions between these particles is also taken into account.

It is well known that a local field theory of electromagnetism incorporating *both* electric and magnetic charges is not as easy to formulate as one with electric charges alone. Furthermore, if we assume that the electric charge is small, given essentially in terms of the fine structure constant, then the magnetic charge, by virtue of Dirac quantization, will certainly not be small. Thus, the sector of the theory with magnetic charge is not amenable to a perturbative treatment. However, there exists a kinematical region in which exact computation of the scattering amplitude of these particles is possible. The way this comes about is the following: if we imagine a situation in which the center-of-mass (c.m.) energy of the system is very high, while the momentum transfer between the scattering constituents is fixed at a relatively low value, then many of the degrees of freedom of the system decouple. The remaining degrees of freedom become strongly coupled and turn out to be accessible to exact analyses without further approximations. In the case of pure electric charge-charge scattering, the amplitude corresponds exactly to the one calculated in the so-called Eikonal approximation of quantum electrodynamics. In this case, of course, radiative corrections can be calculated perturbatively, unlike in the charge-monopole case.

When the c.m. energies approach Planckian values, quantum effects of general relativity can no longer be ignored. But, as of now, there is no fully satisfactory quantized theory of gravity. When one tries to quantize gravity from a local field theoretic viewpoint, one immediately runs into uncontrollable ultraviolet divergences. The string theory approach, though excellent from the standpoint of perturbation theory, is yet to be completely understood on a nonperturbative basis. Other approaches such as the Ashtekar formalism are not developed well enough for analyzing physical processes involving exchange of gravitational quanta. However, as we shall see, in the kinematical regime under consideration, amplitudes of several processes involving gravitational interactions become exactly calculable, despite the lack of a full quantum gravity theory. Furthermore, the interplay between gravitational and electromagnetic interactions become especially interesting in this kinematical regime when one of the particles is magnetically charged. In this case the fine structure constant of electromagnetism α does not evolve with the c.m. squared energy s, but increases with increasing the squared momentum transfer t. So if t is held fixed then α does not run at all. Thus, in the kinematical region of interest, one expects gravi-

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tational interactions to dominate over electromagnetism. With monopole-charge scattering, though, this is not the case as we show below.

We shall see that longitudinal and transverse degrees of freedom behave quite differently in the above situation, with the latter essentially dropping out of the problem. This will lead to the truncation of the full action of the theory under consideration (both for general relativity and quantum electrodynamics) to a two-dimensional action defined on the boundary of space time. In this sense, the theory has a distinctly topological nature and yet nontrivial dynamical results follow from it.

The paper is organized as follows. In the second section we review earlier literature on pure electric charge-charge scattering within the "shock-wave" picture. Scaling arguments leading to a truncation of the Maxwell action and eventually to the shock wave picture will be summarized for completeness. Then we introduce magnetic monopoles in the theory, and proceed to generalize the foregoing formalism to calculate the scattering amplitude. Particular attention will be paid to subtleties arising from problems such as the Dirac string singularity. In the third section gravity will be introduced and the interactions involving both electromagnetism and gravity will be studied. Once again we will motivate the discussion by considering the full Einstein action and how it gets simplified, in the absence of electromagnetism [1]. Next we discuss charge-charge and charge-monopole scattering at Planckian energies. The relative contributions of electromagnetic and gravitational scattering in the two cases will be contrasted in detail. We will also comment on the behavior of singularities, namely the poles in the scattering amplitude and how they differ from one process to another. We conclude with a number of observations on our results and future outlook.

II. ELECTROMAGNETIC SCATTERING AT HIGH ENERGIES

At sub-Planckian c.m. energies that are still large compared to the rest masses of the particles, the dominant physical processes originate from a truncated version of the original Maxwell action. The derivation of this truncation is first briefly sketched, and the resulting shock wave fields calculated in a frame where one of the particles moves almost luminally. The other scattering particle, assumed to be relatively slow, scatters off these fields with an exactly computable amplitude. The review of this material follows the treatment of Verlinde and Verlinde [2] and of Jackiw *et al.* [3], and is followed by generalization to the case of monopole-charge scattering.

A. Effective theory at high energies

Suppose there are two spinless charged particles moving at very high velocities, such that the center-of-mass energy $\sqrt{s}$ is very high. The action for the electromagnetic field is given by

$$S = -\frac{1}{4}\int d^4x\,(F_{\mu\nu}F^{\mu\nu}) \ , \qquad (1)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the second-rank electromagnetic field strength tensor and $A^\mu = \left(A^0, \vec{A}\right)$ is the electromagnetic four potential. At high center-of-mass energies and very low momentum transfer $\sqrt{t}$, the scattering is almost exclusively in the forward direction. Without loss of generality, if we assume the particles to move initially in the z direction, with four-momentum $p^\mu = (p^0, \vec{p})$, then we have, for lightlike particles, the energy $E \approx p_z \sim \sqrt{s}$ and $p_x = p_y \sim 0$. The square roots of s and t thus measure the typical momenta associated with the longitudinal and the transverse directions. Now, if we associate two length scales with the longitudinal and transverse directions, then the characteristic transverse length scale is much bigger than the longitudinal length scale. Thus, we scale the null coordinates $x^\pm$ such that $x^\alpha \to \lambda x^\alpha$ and $x^i \to x^i$, where α runs over the light cone indices $+,-$, while i signifies the transverse coordinates x, y. Under this scaling the A_μ s transform as $A_\alpha \to \lambda^{-1}A_\alpha$. The transverse A_i s remain unchanged. The transformed action now has the form

$$S = -\frac{1}{4}\int d^4x\left(\lambda^{-2}F_{\alpha\beta}F^{\alpha\beta} + 2F_{\alpha i}F^{\alpha i} + \lambda^2 F_{ij}F^{ij}\right) \ . \qquad (2)$$

The parameter λ may now be chosen to depend on s:

$$\lambda = \frac{k}{\sqrt{s}} \to 0 \ , \qquad (3)$$

where k is a finite constant having dimensions of energy. Then the limit $s \to \infty$ becomes equivalent to the limit $\lambda \to 0$. Thus, in this kinematical regime, the transverse part of the action with F_{ij} can be ignored and what we have left is an effective action of the form

$$S = -\frac{1}{4}\int d^4x\left(\lambda^{-2}F_{\alpha\beta}F^{\alpha\beta} + 2F_{\alpha i}F^{\alpha i}\right). \qquad (4)$$

Notice that in the partition function the fluctuations of the term $F_{\alpha\beta}F^{\alpha\beta}$ are suppressed in the imaginary exponent (due to the smallness of λ) and the configuration with the dominant contribution is $F_{\alpha\beta} = 0$, i.e., $F^{+-} = E_z = 0$ [4]. This shows that the electric field is localized in the transverse plane. Similarly, if we write the original action in the dual formalism, with the $F_{\mu\nu} \to \tilde{F}_{\mu\nu}$, then $\tilde{F}^{+-} = B_z = 0$. This brings us to the *shock wave picture*: fields due to processes characterized by longitudinal momenta that are overwhelmingly larger than transverse momenta are essentially confined to the plane (called the "shock front") perpendicular to the direction of motion of the source particles.

From the field theory standpoint, a charged scalar field theory coupled to electromagnetism also undergoes simplification in this kinematical regime: the action under the same scale transformation becomes

$$S = \int d^4x\left(D_\alpha\phi D^\alpha\phi^* + \lambda^2 D_i\phi D^i\phi^*\right). \qquad (5)$$

Once again, when neglecting terms of order λ^2, we see that only the longitudinal components of the gauge fields remain coupled. Thus, if we were to describe the gauge field interaction in terms of currents j^μ, then only the light cone components $j^\pm$ would be physically relevant. Furthermore, if these currents were to be associated with charges moving almost luminally, then

$$j_\pm = j_\pm\left(x^\pm, \vec{r}_\perp\right), \quad j^i(x) = 0 . \tag{6}$$

This allows us to define two functions k^+ and k^-, where

$$\begin{aligned} j_+ &= \partial_- k^-\left(x^-, \vec{r}_\perp\right), \\ j_- &= \partial_+ k^+\left(x^+, \vec{r}_\perp\right). \end{aligned} \tag{7}$$

In short, if we define a vector k such that

$$k(x) = k^+\left(x^+, \vec{r}_\perp\right) - k^-\left(x^-, \vec{r}_\perp\right) \tag{8}$$

then

$$j^\alpha = \epsilon^{\alpha\beta}\partial_\beta k, \tag{9}$$

where $\epsilon^{\alpha\beta}$ is antisymmetric and $\epsilon^{01} = 1$. The above form of j^α automatically ensures the current conservation $\partial_\alpha j^\alpha = 0$.

The flatness condition $F^{+-} = 0$ above admits a solution in terms of the light cone components of the gauge potential $A_\pm = \partial_\pm \Omega$. If, further, we impose the Landau gauge $\partial_\mu A^\mu = 0$, Ω obeys D'Alembert's equation

$$\partial_+\partial_-\Omega = 0 \tag{10}$$

which implies

$$\Omega = \Omega^+\left(x^+, \vec{r}_\perp\right) + \Omega^-\left(x^-, \vec{r}_\perp\right). \tag{11}$$

It is then easy to show that the electromagnetic Lagrange density

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j^\mu A_\mu$$

can be written as

$$\begin{aligned} \mathcal{L} = &-\frac{1}{2}\partial_-\Omega^-\vec{\nabla}^2\partial_+\Omega^+ - \frac{1}{2}\partial_+\Omega^+\vec{\nabla}^2\partial_-\Omega^- \\ &-\partial_+k^+\partial_-\Omega^- - \partial_-k^-\partial_+\Omega^+ \end{aligned} \tag{12}$$

which reduces to a total derivative in the light cone coordinates:

$$\begin{aligned} \mathcal{L} = &-\partial_-\left(\frac{1}{2}\Omega^-\ \vec{\nabla}^2\partial_+\Omega^+ + \partial_+k^+\ \Omega^-\right) \\ &-\partial_+\left(\frac{1}{2}\Omega^+\ \vec{\nabla}^2\partial_-\Omega^- + \partial_-k^-\Omega^+\right). \end{aligned} \tag{13}$$

This shows that the action $S = \int d^4x\mathcal{L}$ is a surface term defined on the boundary of null plane:

$$\begin{aligned} S = \oint d\tau \int d^2r_\perp \Bigg(&\frac{1}{2}\Omega^-\ \vec{\nabla}^2\dot{\Omega}^+ - \frac{1}{2}\Omega^+\ \vec{\nabla}^2\dot{\Omega}^- \\ &+\dot{k}^+\Omega^- - \dot{k}^-\Omega^+\Bigg). \end{aligned} \tag{14}$$

Here all the quantities are evaluated on the contour parametrized by the affine parameter τ. An overdot denotes $\partial/\partial\tau$. This shows that although the Lagrange density was reduced to a total derivative, the values of the gauge parameter at the boundary plays a significant role. In fact, they are the only dynamical degrees of freedom in the problem. This simplification of the action has its origin in the kinematics of the situation. On extremizing this action, the equations of motion obtained are

$$\nabla^2\dot{\Omega}^+ = -\dot{k}^+, \tag{15}$$

$$\nabla^2\dot{\Omega}^- = -\dot{k}^-, \tag{16}$$

which yields, on integration,

$$\begin{aligned} \Omega^+\left(x^+, \vec{r}_\perp\right) &= -\frac{1}{\nabla^2}\ k^+\left(x^+, \vec{r}_\perp\right), \\ \Omega^-\left(x^-, \vec{r}_\perp\right) &= -\frac{1}{\nabla^2}\ k^-\left(x^-, \vec{r}_\perp\right). \end{aligned} \tag{17}$$

It can be verified that these solutions are identical to those obained by solving the full set of Maxwell's equations with (7) as the source current. In other words, once again we arrive at the shock wave description of highly energetic charged particles. It can be shown [3] that exact scattering amplitude for charge-charge scattering, to be computed below semiclassically, can also be obtained from the above reduced action.

B. Charge-charge scattering

The foregoing analysis allows us to compute *exactly* the S matrix for the scattering of two highly energetic particles assumed to carry electric charge. Making use of Lorentz covariance of the theory, we will do the calculations in a special inertial frame in which one of the charges moves with velocity close to luminal, while the other is moving relatively slowly. The shock wave front due to the former extends over the entire transverse plane. Thus, the target particle, assumed to be moving in a direction opposite to that of the source, encounters this shock wave and its wave function acquires an Aharanov-Bohm-type phase factor. The overlap between the wave functions of the target particle before and after its encounter with the shock front leads to the scattering amplitude.

The potential of the lightlike particle can be found in various ways. First, we approach it from a very well known physical situation, namely, that of Cherenkov radiation. If a particle carrying an electric charge e' moves in the positive z direction in a dielectric medium with a dielectric constant ϵ, at a speed β greater than the speed of light in that medium, then it emits electromagnetic radiation. The charge carries with it a shock wave, in front of which all potentials and fields vanish. The vector potential due to this charge behind the shock wave is given by the formula (e.g., [5])

$$A_z(x,y,z) = \frac{\beta e'}{\sqrt{(z-\beta t)^2 + (1-\beta^2\epsilon)\, r_\perp{}^2}}, \tag{18}$$

A_x and A_y being zero. $r_\perp$ is the transverse distance from the charge given by $r_\perp{}^2 = x^2+y^2$. Thus $\vec{A}$ suffers a discontinuity across the shock front giving rise to singular fields. Now if we put $\epsilon = 1$, which means that the motion is in a vacuum, and take the limit $\beta \to 1$, then expression for $\vec{A}$ will be the quantity of our interest. Of course, now the charge will move exactly at the speed of light in a vacuum.

The same result can be derived somewhat more formally following [3]. We consider the electromagnetic potential A_μ of a static charged particle with an electric charge e'. Then we give it a Lorentz boost β along the positive z axis. The gauge potential transforms accordingly following the laws of special relativity. On taking the limit $\beta \to 1$, the potential of the lightlike particle is found to be a pure gauge almost everywhere except on the shock plane where it has a discontinuity:

$$\tilde{A}^0 = \tilde{A}^z = -2e' \ \ln(\mu r_\perp)\ \delta(x^-),$$
$$\tilde{A}^i_\perp = 0, \qquad i = 1,2. \tag{19}$$

Here, μ is a dimensional parameter inserted to make the logarithm in Eq. (19) dimensionless. The potential $\tilde{A}^\mu$ is singular on the shock plane ($x^- = 0$) as was seen from the Cherenkov formula. Now this potential is gauge equivalent to the potential A'^μ where $A'^\mu = \tilde{A}^\mu + \partial^\mu\Lambda$, Λ being a Lorentz scalar. Choosing Λ to be $-2e'\theta(x^-)\ln\mu r_\perp$, we get

$$A'^0 = A'^3 = 0, \qquad \vec{A'}_\perp = -2e'\theta(x^-)\vec{\nabla}\ln\mu r_\perp. \tag{20}$$

We see that the gauged transformed vector potential is a pure gauge everywhere except on the hyperplane $x^- = 0$ which is also the shock plane. Thus, as one expects, the fields are nonvanishing only on this plane and are given by

$$E^i = \frac{2e' r^i_\perp}{r^2_\perp}\delta(x^-), \qquad E^z = 0,$$
$$B^i = -\frac{2e'\epsilon_{ij} r^j_\perp}{r^2_\perp}\delta(x^-), \qquad B^z = 0\ . \tag{21}$$

These singular field configurations cause an instantaneous interaction with the (slower) target particle. First, consider the classical motion of the slow particle. This reduces to solving the Lorentz force equation for the test charge e of mass m with given boundary conditions. Since it has negligible velocity, we use the nonrelativistic form of the equation and also neglect the $\vec{B}$-dependent piece. Thus we have

$$m\frac{d^2\vec{r}}{dt^2} = \frac{2ee'}{r^2_\perp}\delta(t-z)\ \vec{r}_\perp. \tag{22}$$

The solution to the above equation can be easily guessed. Without loss of generality, let us assume that at the initial time $t = 0$, e is almost stationary on the x axis at a distance b from the origin. As the electric fields are all directed radially, the impulse imparted to e should be along the positive x axis after which it starts moving in that direction with a uniform velocity. The δ function shows that the shock wave arrives from the left and hits it at $t = 0$. Being nonzero only at that instant, it also allows us to replace x and y by b and 0 respectively on the right of the equation. Inserting the constants correctly we have the solution

$$y(t) = z(t) = 0, \qquad x(t) = \frac{2ee'}{bm}t\ \theta(t) + b, \tag{23}$$

which clearly satisfies Eq. (22). This is the classical trajectory of the charge e. The total momentum transfer (or the impulse) is just $2ee'/b$ which is finite although the fields are singular on the shock plane.

Having solved the classical part, we now consider the quantum problem for the charge e. As stated earlier, we look at how the wave function changes under the influence of the other charge which effectively provides just a classical background field. For early times $t < z$ the particle is free and its wave function is just a plane wave given by

$$\psi_<\ (x^\pm, \vec{r}_\perp) = \psi_0 = \exp[ipx] \ \text{ for } x^- < 0\ , \tag{24}$$

with momentum eigenvalue p^μ. Immediately after the shock front passes by, its interaction with the gauge potential enters via the 'minimal coupling prescription' by which we replace all the ∂_μ's with $\partial_\mu - ieA_\mu$. The corresponding wave function acquires a multiplicative phase factor $\exp\left(ie\int dx^\mu A_\mu\right)$. Thus from Eq. (20), for $x^- > 0$, the modified wave function is

$$\psi_>(x^\pm, \vec{r}_\perp) = \exp\left[-iee'\ln(\mu^2 r^2_\perp)\right]\psi'_0 \quad \text{for } x^- > 0\ , \tag{25}$$

where ψ_0 and ψ'_0 are related through the continuity requirement

$$\psi_< = \psi_> \quad \text{at } x^- = 0. \tag{26}$$

Here it may be noted that the additional phase factor due to electromagnetic interaction is a function of $r_\perp$ only, which is the length of the radius vector from the particle on the shock plane. It does not depend on the angular variable. This is due to the fact that the electric field of an electrically charged particle is central in nature. The wave function $\psi_>$ can now be expanded in terms of the complete set of momentum eigenfunctions (plane waves) with suitable coefficients in the form [6]

$$\psi_> = \int dk_+ d^2k_\perp\ A(k_+, \vec{k}_\perp)$$
$$\times \exp[i\vec{k}_\perp\cdot\vec{r}_\perp - ik_+x^- - ik_-x^+] \tag{27}$$

with the on shell condition $k_+ = (k^2_\perp + m^2)/k_-$. Obviously the coefficients $A(k_+, k_\perp)$ are the probability amplitudes for finding the particle with momentum k^μ when an experiment is performed on it after it has undergone the shock wave interaction. So we proceed to calculate them by multiplying both sides of Eq. (27) by a plane wave and integrating over x^-. Using the orthonormality of the eigenfunctions, we get

$$A(k_+, k_\perp) = \frac{\delta(k_+ - p_+)}{(2\pi)^2} \times \int d^2 r_\perp \exp\{i[-2ee' \ln(\mu r_\perp) + \vec{q}\cdot\vec{r}_\perp]\} , \tag{28}$$

where $\vec{q} \equiv \vec{p}_\perp - \vec{k}_\perp$ is the transverse momentum transfer, k and p being the final and initial momenta respectively. The integration over the transverse x-y plane can be performed exactly [3] yielding the amplitude

$$f(s,t) = \frac{k_+}{4\pi k_0}\,\delta(k_+ - p_+)\frac{\Gamma(1-iee')}{\Gamma(iee')}\left(\frac{4}{-t}\right)^{1-iee'} , \tag{29}$$

where we have put in the canonical kinematical factors. $t \equiv -q^2$ is the transverse momentum transfer. With this amplitude, one can easily show that the scattering cross section is

$$\frac{d^2\sigma}{d\vec{k}_\perp^2} \sim \frac{(ee')^2}{t^2} , \tag{30}$$

where we have used a property of the Γ function, namely $|\Gamma(a+ib)| = |\Gamma(a-ib)|$, a and b being real.

It has been shown in [3] that this scattering amplitude is identical to the amplitude obtained in the eikonal approximation where virtual momenta of exchanged quanta are ignored in comparison to external momenta, leading to a resummation of a class of Feynman graphs [7]. Since, generically $ee' \sim \frac{1}{137}$, this approximation will receive usual perturbative radiative corrections. The second-order pole singularity in the cross section as $t \to 0$ is, of course, typical of processes where massless quanta are exchanged.

C. Charge-monopole scattering

Now that we have calculated the amplitude of the scattering of two charges, one can inquire as to what changes, if any, will take place if we replace one of the charges by a Dirac magnetic monopole. This question is worth pursuing for various reasons. First of all, the (albeit imagined) existence of monopoles will imply that the Maxwell equations assume a more symmetric form, due to the property of duality of field strengths and electric and magnetic charges. Within quantum mechanics, as Dirac has shown, monopoles offer a unique explanation of the quantized nature of electric charge. But, as is well known, introduction of monopoles in the theory brings in other problems such as singularities in the vector potential. It will be interesting to see how one can deal with them in the present formalism and investigate the range of validity of the shock wave picture in this context. One should also keep in mind the fact that a satisfactory local quantum field theory for monopoles is still lacking. Further, given Dirac's quantization condition, monopole elelctrodynamics cannot be understood in perturbative terms around some noninteracting situation. Thus, as advertised earlier, the shock wave picture may be one of the few important probes available for such processes.

Recall, however, following [8] that it is not possible to choose a single non-singular potential to describe the field of the monopole everywhere. We need at least two such potentials, each being well behaved in some region and being related by a local gauge transformation in the overlapping region. In spherical polar coordinates, these potentials can be chosen as [8]

$$\begin{aligned} \vec{A}^{\,\mathrm{I}} &= \frac{g}{r\sin\theta}(1-\cos\theta)\hat{\phi} , \qquad 0 \le \theta < \pi, \\ \vec{A}^{\,\mathrm{II}} &= \frac{-g}{r\sin\theta}(1+\cos\theta)\hat{\phi} , \qquad 0 < \theta \le \pi . \end{aligned} \tag{31}$$

The Dirac strings associated with the two potentials are along the semi-infinite lines $\theta = \pi$ and 0 respectively, i.e., along the negative and positive halves of the z axis. $\vec{A}^{\mathrm{I}}$ and $\vec{A}^{\mathrm{II}}$ become singular along these two lines respectively. It may be noted that here we have made the gauge choice $A^0 = 0$, and have chosen an orientation of our coordinates such that only the x and y components survive. In the region $-\pi < \phi < \pi$, where either of $\vec{A}^{\,\mathrm{I}}$ or $\vec{A}^{\,\mathrm{II}}$ may be used, they are related by a gauge transformation with the gauge parameter $2g\phi$. It can be readily verified that

$$\vec{\nabla}\times\vec{A}^{\,\mathrm{I}} = \vec{\nabla}\times\vec{A}^{\,\mathrm{II}} = \frac{g}{r^2}\hat{r}. \tag{32}$$

Here the curls are taken in the respective regions of validity of the potentials. In the following calculations, for convenience we shall work with $\vec{A}^{\,\mathrm{I}}$ only, but all subsequent results will be independent of this particular choice.

As in the last section, we give the monopole a Lorentz boost of magnitude β along the positive z axis. It can be shown that if Eqs. (31) are rewritten in Cartesian coordinates, then $\vec{A}^{\,\mathrm{I}}$ transforms to [9]

$${}^{\beta}\vec{A}^{\mathrm{I}}_i = \frac{-g\epsilon_{ij}r^j_\perp}{r^2_\perp}\left[1 - \frac{z-\beta t}{R_\beta}\right] . \tag{33}$$

Before proceeding further, let us examine the behavior of the Dirac strings under Lorentz boosts. For this purpose it is convenient to write Eq. (33) in the form

$${}^{\beta}\vec{A}^{\mathrm{I}} = \frac{g}{r_\perp}\left[1 - \frac{z-\beta t}{R_\beta}\right]\hat{\phi} . \tag{34}$$

On the z axis ($\theta = 0$ or π), we have $r_\perp \to 0$ implying that $R_\beta \to |z-\beta t|$. Thus the above equation reduces to

$${}^{\beta}\vec{A}^{\mathrm{I}} = \frac{g}{r_\perp}\left[1 - \mathrm{sgn}(z-\beta t)\right] . \tag{35}$$

Thus for $z > \beta t$, i.e., in front of the boosted monopole the vector potential vanishes, while it becomes singular behind it ($z < \beta t$). It is as if the monopole drags the

Dirac string along with it and as in the static case, the semi-infinite line of singularity originates from it. Similarly, by looking at the boosted potential $\vec{A}^{\rm II}$, it can be easily verified that for this, the string is always in front of the monopole and "pushed" by it as it moves. These results also hold in the limit $\beta \to 1$, i.e., for the potential,

$$\vec{A}^{\rm I}_0 \equiv \lim_{\beta\to 1} {}^{\beta}\vec{A}^{\rm I}_i = \frac{2g}{r_\perp}\,\theta(x^-)\hat{\phi}. \tag{36}$$

The corresponding electromagnetic fields are

$$B^i = \frac{2gr^i_\perp}{r^2_\perp}\delta(x^-),\quad B^z = 0,$$
$$E^i = \frac{2g\epsilon_{ij}r^j_\perp}{r^2_\perp}\delta(x^-),\quad E^z = 0\,. \tag{37}$$

Unlike the fields of a charge in motion, here the magnetic field is radial, whereas the electric field is circular on the shock plane. Here also $\vec{A}^{\rm I}_0$ is a pure gauge everywhere except on the null plane $x^- = 0$. It may be noted that the above $\vec{E}$ and $\vec{B}$ fields can be obtained by making the following transformations in (21): $e' \to g$, $\vec{E} \to \vec{B}$ and $\vec{B} \to -\vec{E}$. This is a consequence of the duality symmetry in Maxwell's equations incorporating monopoles.

As before let us now calculate the classical trajectory of the charge under the influence of the monopole shock wave. Here the nonrelativistic Lorentz force equation for e becomes

$$m\frac{d^2\vec{r}}{dt^2} = \frac{2eg}{r^2_\perp}\delta(t-z)\,[\hat{x}y - \hat{y}x], \tag{38}$$

where we have ignored the velocity of the slow test charge. Imposing identical boundary conditions for the charged particle e as before and taking into account the fact that in this case the momentum transfer will be along the y axis (now that the $\vec{E}$ field lines are circles on the shock plane) the solution $\vec{r}(t)$ is

$$x(t) = b\,,\quad z(t) = 0\,,\quad y(t) = \frac{2eg}{bm}t\theta(t). \tag{39}$$

In this case the impulse is $2eg/bm$.

In the quantum case, the charge e interacts instantaneously with the monopole shock wave, the net effect being a gauge rotation in the wave function of the former. To compute this explicitly, we proceed as follows [9]. We first rewrite $\vec{A}^{\rm I}_0$ in (36) as a total derivative in the form

$$\vec{A}^{\rm I}_0 = 2g\theta(x^-)\vec{\nabla}\phi\,. \tag{40}$$

We note in passing that the gauge potentials for a luminally boosted electric charge (20) and monopole (40), both given as total derivatives on the transverse plane, form the real and imaginary parts respectively of the gradient of the holomorphic function ln$z$ with $z \equiv r_\perp e^{i\phi}$, with ϕ now the azimuthal angle on the transverse plane.

For $t < z$, i.e., before the arrival of the monopole with its shock front, the wave function of the charge e is once again the plane wave

$$\psi_<(x^\pm,\vec{r}_\perp) = \psi_0 \quad \text{for } x^- < 0\,. \tag{41}$$

After encountering the shock wave, it is modified by the gauge-potential-dependent phase factor. The final form of the wave function is

$$\psi_>(x^\pm,\vec{r}_\perp) = \exp[i2eg\phi]\,\psi_0' \quad \text{for } x^- > 0 \tag{42}$$

by virtue of the potential (40) with the usual requirement of continuity. At this point we make the additional assumption of Dirac quantization, namely, for an interacting monopole-charge system, the magnitudes of their electric and magnetic charge must be constrained by the relation

$$e\,g = \frac{n}{2},\qquad n = 0, \pm 1, \pm 2, \ldots\,. \tag{43}$$

Thus we get

$$\psi_> = e^{in\phi}\psi_0'\,. \tag{44}$$

This sort of phase factor in the small angle scattering of a monopole and a charge was first found by Goldhaber [10]. It depends on the angular variable ϕ only. This may be a reflection of the noncentral nature of the classical charge-monopole interaction.

Expanding $\psi_>$ in plane waves as before we get an integral expression for the scattering amplitude as follows:

$$A(k_+,k_\perp) = \frac{\delta(k_+ - p_+)}{(2\pi)^2}\int d^2r_\perp \exp[i\,(n\phi + \vec{q}\cdot\vec{r}_\perp)]\,. \tag{45}$$

Once again $\vec{q} \equiv p_\perp - k_\perp$ is the momentum transfer and as before we have the dispersion relation $k_+ = (k^2_\perp + m^2)/k_-$. By conveniently choosing the orientation of the transverse axes as in the previous section, the angular integration gives $(1/q^2)\int_0^\infty d\rho\;\rho J_n(\rho)$, where $J_n(\rho)$ is the Bessel function of order n. This integral is also standard [11] and the result is

$$\left(\frac{1}{-t}\right)\frac{2\Gamma(1+\frac{n}{2})}{\Gamma(\frac{n}{2})}\,. \tag{46}$$

Here we note an important difference from the previously calculated charge-charge amplitude. There the arguments of the Γ functions were complex, whereas in this case they are real. In fact, the amplitude in this case is simply

$$f(s,t) = \frac{k_+}{2\pi k_0}\,\delta(k_+ - p_+)\left(\frac{n}{-t}\right)\,, \tag{47}$$

where we have incorporated the canonical kinematical factors. Such factorization makes the expression for the amplitude simple. We observe that it is proportional to the monopole strength n. It follows that the scattering cross section becomes

$$\frac{d^2\sigma}{d\vec{k}^2_\perp} \sim \frac{n^2}{t^2}. \tag{48}$$

It may be mentioned that we would have obtained the same result if we had used the second of the gauge potentials in (31) and performed the Lorentz boost etc. One way to see this is by noting that the potentials, boosted to $\beta \approx 1$, are both gauge equivalent to a gauge potential A'_μ given by

$$\vec{A}'_\perp = 0 = A'_+ , \quad A'_- = 2g\phi\delta(x^-) \text{ everywhere} . \tag{49}$$

The apparent disappearance of the Dirac string singularity in this gauge is a red herring; the gauge transformation has flipped the Dirac string onto the shock plane, thus preventing it from being manifest. More importantly, the gauge potential, though globally defined functionally, is not single-valued, being a monotonic function of a periodic angular variable. Thus, the singularity has been traded in for non-single-valuedness. Of course, the theory of fields that are not single-valued functions is in no way easier to formulate than that for singular fields. It is interesting to note further that for subluminal boost velocities one cannot obtain a globally defined potential A'^μ in any gauge.

We would like to make a few more remarks at this point. First of all, if we choose another Lorentz frame in which the electric charge is lightlike while the monopole is moving slowly, we would get identical results for the scattering amplitude. The easiest way to see this is to use the *dual* formalism wherein one introduces a gauge potential A^M_μ such that the dual field strength $\tilde{F}_{\mu\nu} \equiv \partial_\mu A^M_\nu - \partial_\nu A^M_\mu$. If this gauge potential is used to define electric and magnetic fields, then the standard field tensor $F^{\mu\nu}$ must satisfy a Bianchi identity of the form $\partial_\mu F^{\mu\nu} = 0$ which would then imply that the gauge potential due to a point charge must have a Dirac string singularity. Further, the monopole will behave identically to the point charge of the usual formalism, so that our method above is readily adapted to produce identical consequences. Second, one can also treat the scattering of two Dirac monopoles in the same kinematical limit exactly as in Sec. II B, using this dual formalism. This would yield a result identical to the one for the electric charge case, with e and e' being replaced by g and g', the monopole charges. Finally, having dealt with particles carrying either electric or magnetic charge, it is straightforward to extend our calculations when one of them is a dyon, that is, it has both electric and magnetic charge. The electromagnetic fields on the shock front of the boosted dyon will be the superposition of the fields produced by a fast charge and a monopole. Also, depending upon the nature of the charge on the other particle (electric or magnetic), one must employ the usual or the dual formalism.

With the above observations we are in a position to address the problem of dyon-dyon scattering in this formalism. Consider two dyons (e_1, g_1) and (e_2, g_2), where the ordered pair denotes its electric and magnetic charge contents respectively. Let us assume that the first one is ultrarelativistic. By means of an electromagnetic duality transformation we can "rotate" a dyon by an angle θ, so that the new values of electric and magnetic charges become e' and g'. In terms of the old quantities they can be expressed in matrix notation as

$$\begin{pmatrix} e' \\ g' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} e_1 \\ g_1 \end{pmatrix} \tag{50}$$

and

$$\begin{pmatrix} e \\ g \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} e_2 \\ g_2 \end{pmatrix} . \tag{51}$$

Now, physical observables do not depend on the parameter θ. We can make use of this symmetry and choose it to be such that

$$\tan\theta = \frac{g_2}{e_2}. \tag{52}$$

This implies that the first dyon transforms to

$$e' = \frac{e_1 e_2 + g_1 g_2}{\sqrt{e_2^2 + g_2^2}}, \quad g' = \frac{-e_1 g_2 + g_1 e_2}{\sqrt{e_2^2 + g_2^2}} \tag{53}$$

while for the second dyon

$$e = \sqrt{e_2^2 + g_2^2}, \quad g = 0. \tag{54}$$

This shows that the slow test dyon has been rotated to a pure electric charge. Then from the results derived previously, the total phase shift in its wave function after being hit by the shock wave of the dyon (e', g') is $\left[ee' \ln \mu^2 {r_\perp}^2 + 2eg'\phi\right]$. Having found this, we can express this in terms of the parameters of the two dyons we started with. The result is $\left[(e_1 e_2 + g_1 g_2) \ln \mu^2 {r_\perp}^2 - 2\,(e_1 g_2 - g_1 e_2)\,\phi\right]$. The calculation of the scattering amplitude now becomes straightforward. It may be noted that the quantities $(e_1 e_2 + g_1 g_2)$ and $(e_1 g_2 - g_1 e_2)$ are the only combinations of the electric charges e_1, e_2 and the magnetic charges g_1, g_2 that are invariant under duality rotations [12]. Thus it is remarkable that the total phase shift and hence the scattering amplitude depends only on these combinations. Alternatively, we could also have made the choice $\tan\theta = -e_2/g_2$, in which case e would become zero and the second dyon transforms into a monopole. Obviously these different choices are merely for convenience and the scattering amplitude does not depend on it. Thus dyon-dyon scattering can always be reduced to dyon-charge or dyon-monopole scattering. Also note that by a duality rotation the usual Dirac quantization condition gets transformed into the generalized expression

$$e_1 g_2 - e_2 g_1 = \frac{n}{2}.$$

This implies that the second term in the phase shift becomes $n\phi$ as in the charge-monopole scattering case.

Finally, we can ask the question as to what happens if we consider a massive vector field, e.g., that described by the Proca Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\mu^2}{2} A_\mu A^\mu. \tag{55}$$

The solution in the static limit for A^μ in the Lorentz gauge is given by

$$A^0 = \frac{e' \exp(-\mu r)}{r}, \qquad A^i = 0, \tag{56}$$

where e' is a point charge at rest. Formally we can apply a Lorentz boost to this potential and try to take the limit $\beta \to 1$. The result is

$${}^\beta A^\mu = \eta^\mu \frac{e' \exp\left(-\mu R_\beta/\sqrt{1-\beta^2}\right)}{R_\beta} \tag{57}$$

which vanishes identicaly when we take the limit $\beta \to 1$. Thus no shock wave emerges in this case and there are no δ-function electromagnetic fields on the null plane $x^- = 0$. This observation can also be understood as follows. In the formulation of the boundary field theory in Sec. II A it was shown that the gauge parameter $\Omega(\Omega^+, \Omega^-)$ was the only dynamical degree of freedom in the theory and the corresponding equations of motion yielded the shock wave picture. On the other hand, the Lagrangian of the massive vector field does not have the required gauge invariant structure to admit such a parameter. This accounts for the absence of the shock wave.

III. ELECTROMAGNETIC VERSUS GRAVITATIONAL SCATTERING AT PLANCKIAN ENERGIES

A. Gravity at Planckian energies

At Planckian c.m. energies the Einstein action also undergoes a truncation akin to the electromagnetic situation [1]. We briefly sketch how this comes about before summarizing results on the shock wave geometry and gravitational scattering. We define the Planck length $l_{\rm Pl}$ to be the inverse of the Planck energy. If $s \gg t$, then the longitudinal momenta determined by the center-of-mass energy $\sqrt{s}$ is much higher than the typical transverse momenta which depends on t. Now, if $\sqrt{s} \approx M_{\rm Pl}$, then correspondingly, the characteristic length scales associated with the longitudinal direction $l_\| \approx l_{\rm Pl}$, while the transverse length scales $l_\perp \gg l_{\rm Pl}$. We would also take the coordinates to be dimensionless, in which case the metric tensor $g_{\mu\nu}$ assumes the dimensions of $(\text{length})^2$. With an appropriate coordinate choice, the metric tensor may be cast into the form

$$g_{\mu\nu} = \begin{pmatrix} g_{\alpha\beta} & 0 \\ 0 & h_{ij} \end{pmatrix} . \tag{58}$$

Now we make the ansatz that only those components of the metric become physically relevant which are of the same order of magnitude as the typical length scales of the system. In other words, $g_{\alpha\beta} \sim {l_\|}^2$ and $h_{ij} \sim {l_\perp}^2$. If we define two dimensionless metrics $\hat{g}_{\alpha\beta}$ and $\hat{h}_{ij}$ such that

$$\begin{aligned} g_{\alpha\beta} &= {l_\|}^2\, \hat{g}_{\alpha\beta}, \\ h_{ij} &= {l_\perp}^2\, \hat{h}_{ij} \ , \end{aligned} \tag{59}$$

then it follows that $\hat{g}$ and $\hat{h}$ are of the order of unity. With these assumptions the usual Einstein action

$$S_E[g] = -\frac{1}{G}\int d^4x\, \sqrt{g}\, R \tag{60}$$

splits up into two parts in the form

$$S_E[g] = S_\|[g,h] + S_\perp[h,g], \tag{61}$$

where

$$S_\|\left[g,h\right] = -\frac{1}{G}\int \sqrt{g}\left(\sqrt{h}\, R_h + \frac{1}{4}\sqrt{h}h^{ij}\partial_i g_{\alpha\beta}\partial_j g_{\gamma\delta}\epsilon^{\alpha\gamma}\epsilon^{\beta\delta}\right) \tag{62}$$

and

$$S_\perp[h,g] = -\frac{1}{G}\int \sqrt{h}\left(\sqrt{g}\, R_g + \frac{1}{4}\sqrt{g}g^{\alpha\beta}\partial_\alpha h_{ij}\partial_\beta h_{kl}\epsilon^{ik}\epsilon^{jl}\right). \tag{63}$$

It can be shown that substitution of Eq. (59) in the above gives the relation

$$S_\|[g,h] = (l_\|/l_{\rm Pl})^2\, S_\|\left[\hat{g},\hat{h}\right], \tag{64}$$

$$S_\perp[h,g] = (l_\perp/l_{\rm Pl})^2\, S_\perp\left[\hat{h},\hat{g}\right], \tag{65}$$

where we have used $G \sim l_{\rm Pl}^2$. From the length estimates made earlier, we see that the $S_\|$ part of the action is strongly coupled with coupling constant $g_\| = \left(l_{\rm Pl}/l_\|\right)^2$ whereas the $S_\perp$ part has a weak coupling $g_\perp = (l_{\rm Pl}/l_\perp)^2$. This shows us that as far as the transverse directions are concerned (governed by $g_\perp$), the physics is essentially classical, due to the weak coupling. In fact, the partition function is dominated by configurations for which $S_\perp = 0$. It can be shown that here too one gets a zero curvature constraint:

$$R_{+-} = 0. \tag{66}$$

Once again we are able to justify using a semiclassical method to deal with such a situation, with the strongly coupled part of the action $S_\|$ being treated exactly.

B. Spacetime around a lightlike particle

The spacetime geometry that emerges for a particle boosted to velocities close to luminal, is expected to emerge from the coupling of the above truncated action to a suitably constrained matter energy-momentum tensor. This has been done in Ref. [1]. Identical answers can however be obtained by a process of *boosting* the static (Schwarzschild) metric due to a point particle, adopted in Ref. [6]; we sketch this approach below.

Essentially this boosting means the mapping of a solution of Einstein equation with a lightlike particle, namely Minkowski space, to another Minkowski space but with one of the null coordinates shifted nontrivially, now without any lightlike particle present [13]. It is argued below how this can be interpreted as a gravitational shock wave.

Once again we choose to carry out the analysis in a Lorentz frame in which the velocity of one particle is very much greater than that of the other. We know that the space-time around a point particle is spherically symmetric and is described by what is known as the Schwarzchild metric. If we assume the mass m of the particle to be small, then it is given in the Minkowski coordinates (T, x, y, Z) by

$$\begin{aligned} ds^2 &= -\left(1-\frac{2Gm}{R}\right)\, dT^2 \\ &\quad + \left(1+\frac{2Gm}{R}\right)\, \left(dx^2+dy^2+dZ^2\right), \end{aligned} \tag{67}$$

where $R = \sqrt{x^2+y^2+Z^2}$ and $m \ll R/G$ [13]. If the above coordinate system is moving with a relative velocity β with respect to coordinates (t, x, y, z) then the two are related by a Lorentz transformation of the form

$$\begin{aligned} T &= t\cosh\theta - z\sinh\theta, \\ Z &= -t\sinh\theta + z\cosh\theta. \end{aligned} \tag{68}$$

θ is called the rapidity which is related to the boost velocity by the relation

$$\tanh\theta = \beta. \tag{69}$$

Now to take the limit $\beta \to 1$ or alternatively $\theta \to \infty$, we also set

$$m = 2p_0\, e^{-\theta}, \tag{70}$$

where the rest energy of the particle is $2p_0 > 0$. This parametrization is consistent with the fact that the mass of the particle must exponentially vanish as its velocity approaches that of light. When we substitute Eq. (68) in Eq. (67), we have the metric due to a particle moving at the speed of light in the x^+ direction (i.e., along $x^- = 0$). In terms of the light cone and the transverse coordinates this metric becomes

$$\begin{aligned} ds^2 &= \left(1+\frac{2Gm}{R}\right)\left[-dx^-\, dx^+ + dx^2 + dy^2\right] \\ &\quad + \frac{4Gm}{R}\left[\frac{p_0}{m}dx- + \frac{m}{4p_0}dx^+\right]^2, \end{aligned} \tag{71}$$

with

$$R^2 = x^2+y^2+\left(\frac{p_0}{m}x^- - \frac{m}{4p_0}x^+\right)^2. \tag{72}$$

Using this and neglecting terms of order m or above, we get the limiting form of the metric:

$$\lim_{m\to 0}\, ds^2 = -dx^-\left(dx^+ - 4Gp_0\frac{dx^-}{|x^-|}\right) + dx^2 + dy^2, \tag{73}$$

where the limit is evaluated at $x^- \neq 0$ and (x^+, x, y) fixed. Defining a new set of coordinates through the relation

$$\begin{aligned} dx'^+ &= dx^+ - \frac{4Gp_0\, dx^-}{|x^-|}, \\ dx'^- &= dx^-\,, \\ dx'^i &= dx^i\,, \end{aligned} \tag{74}$$

we observe that the above metric is just a flat Minkowski metric:

$$ds^2 = -dx'^-\, dx'^+ + dx'^2 + dy'^2. \tag{75}$$

The crucial point to note here is that the metric suffers a discontinuity at $x^- = 0$ through the term $|x^-|^{-1}$. Now, taking the leading-order terms in Eq. (72), it can be shown that $dx^-/x^- = dR/R$, which gives

$$dx'^+ = dx^+ - \theta(x^-)\,\frac{4Gp_0\, dR}{R}. \tag{76}$$

A solution of the above equation near the null plane ($|x^-| \to 0$) is

$$x'^+ = x^+ + 2Gp_0\,\theta(x^-)\ln\left(\mu^2 {r_\perp}^2\right). \tag{77}$$

Note that the coordinates x^- and x^i remain unchanged. This step function at the null plane $x^- = 0$ is the gravitational equivalent of the electromagnetic shock wave. There we had a similar discontinuity in the gauge potential A^μ. Here we have two flat regions of space-time corresponding to $t < z$ and $t > z$ which are glued together at the null plane $t = z$ (or $x^- = 0$). However there is a shift of coordinates at this plane given by Eq. (77). It is as if a two-dimensional flat space-time on the t-z plane is cut along the line $t = z$ and pasted back again after being shifted along this line by the amount given above.

Now that we have found the metric around a lightlike particle, in principle we should be able to predict the behavior of another (slower) test particle encountering it. Since the sole effect of the gravitational shock wave is the cutting and pasting of the Minkowski space along the null direction $x^- = 0$ after a shift of the x^+ coordinate, it is easy to see that the test particle wave function will acquire a phase factor upon passing through this shock front. One more remark is in order at this point. The logarithmic singularity in the expression for the shift in the coordinate x^+ in Eq. (77) causes an infinite time delay of all interactions via virtual particle exchanges. This shows that it is the shock wave interactions which dominate over all standard field theoretic effects such as particle creation via brehmstrahlung etc. However, as we shall show later, the gravitational shock wave may not dominate in all situations where other interactions mediated also by shock wave fronts exist.

C. Gravitational scattering

To begin with we will assume the particles to be neutral and as before, also spinless. We look at the behavior of the wave function of a slow test particle in the background metric of the lightlike particle carrying with it a "gravitational" shock wave. Before the arrival of the shock wave ($x^- < 0$), the test particle is in a flat space time as derived in the last section. Thus, as before, its quantum mechanical wave function is a plane wave of the form

$$\psi_< (x^\pm, \vec{r}_\perp) = e^{ipx} \tag{78}$$

with definite momentum p^μ. This can be written in terms of the lightcone and transverse coordinates as

$$\psi_<(x^\pm, r_\perp) = \exp(i\,[p_\perp x_\perp - p_+ x^- - p_- x^+]). \tag{79}$$

On encountering the shock wave, it is transported to another flat space-time defined by $x^- > 0$ which is related to the previous one by a shift in the x^+ coordinates. From the explicit expression for this shift in Eq. (77) we see that the wave function immediately gets modified into

$$\psi_>(x^\pm, \vec{r}_\perp) = \exp\{i\,[p_\perp x_\perp - p_-\,(x^+ + 2Gp_0 \ln r_\perp{}^2)]\}, \tag{80}$$

which is also a plane wave but in the new coordinates. We have put $\mu = 1$ in Eq. (77) and evaluated the above at $x^- = 0^+$. Noting that the factor $2Gp_- p_0$ can be written as Gs, the phase shift in the final wave function is $-Gs \ln r_\perp^2$. But this is just the electromagnetic phase shift that we got in the last section in the case of charge-charge scattering with Gs replacing the earlier coupling ee'. This implies that the scattering amplitude will also be the same as the previous case with this replacement. Consequently we have, for the gravitational scattering of the two particles,

$$f(s,t) = \frac{k_+}{4\pi k_0}\,\delta(k_+ - p_+)\frac{\Gamma(1-iGs)}{\Gamma(iGs)}\left(\frac{4}{-t}\right)^{1-iGs} . \tag{81}$$

The corresponding cross section is

$$\frac{d^2\sigma}{d\vec{k}_\perp^2} \sim \frac{G^2 s^2}{t^2}. \tag{82}$$

Despite the striking similarity with electromagnetism, there is an important difference here. The coupling is now proportional to s, the square of the center-of-mass energy. The above cross section seems to increase without limit with increase of s, thus violating unitarity. To understand this, we must note that at super-Planckian energies one expects gravitational collapse and inelastic processes to take place. Hence the above expression fails to be a faithful representation of the actual scattering and one has to invoke a full theory of quantum gravity at such extreme energies [6]. Similar arguments hold good for all the other cross sections found in this paper.

Another important point to note is the structure of poles in the scattering amplitude (81). It seems that there is a "bound state" spectrum at

$$Gs = -iN \quad , \qquad N = 1, 2, 3, \ldots \quad .$$

It has been remarked in [14] that the t dependence of the residues of the poles can be expressed as polynomials in t with degree $N-1$. Thus, the largest spins of the bound states are $N-1$. This is similar to the Regge behavior of hadronic resonances, albeit with an imaginary slope. It remains to be seen whether these poles are "physical" in the sense they correspond to resonant states or as argued in [1] are just artifacts of our kinematical approximations. Nevertheless, we will show in the subsequent sections that the introduction of electromagnetism does have an effect on their location in the complex s plane.

D. Charge-charge versus gravitational scattering

After having considered the pure gravitational scattering, we introduce electromagnetic interactions in the following way. In addition to their mass, we now assume the particles to carry electric charges e and e', e being the charge of the slow test particle. Then the charge e' will also have an electromagnetic shock wave associated with it. The electric and magnetic fields on the shock front are those found in the previous section, given in Eq. (21). We assume that the resultant effect of the combined shock wave (gravitational and electromagnetic) on the test particle is to produce a phase shift in its wave function which is the sum of the individual phase shifts. This tacitly presumes the independence of the gravitational and electromagnetic shock waves. We shall not attempt to prove this supposition at this point except to note that this assumption has also been made without explicit mention in previous works [3,6,9]. However, it can be justified rigorously for a variety of situations [15]. Both phase shifts being proportional to $\ln \mu^2 r_\perp^2$, the net effect is succinctly captured by the shift $Gs \to Gs + ee'$, with the final form of the wave function after is crosses the null plane $x^- = 0$ being

$$\psi_>(x^\pm, x_\perp) = \exp\ \left[-i\,(ee' + Gs)\ln \mu^2 r_\perp^2 + ipx\right] . \tag{83}$$

Consequently, the scattering amplitude becomes

$$f(s,t) = \frac{k_+}{4\pi k_0}\,\delta(k_+ - p_+) \times\frac{\Gamma(1-iee'-iGs)}{\Gamma(iee'+iGs)}\left(\frac{4}{-t}\right)^{1-iee'-iGs} . \tag{84}$$

This gives the cross section

$$\frac{d^2\sigma}{d\vec{k}_\perp^2} \sim \frac{1}{t^2}\,(ee' + Gs)^2 . \tag{85}$$

To compare the relative magnitudes of the two terms, we recall that the electromagnetic coupling constant ee' evolves only with t through radiative corrections and not with s. Thus in the kinematical regime that we are considering, it remains fixed at its low energy value. For example, if the particles carry one electronic charge each, then $ee' \sim 1/137$. On the other hand, at Planck scales, the second term in the cross section is of order unity. This shows that gravity is the principal contributor in the scattering process and electromagnetic effects can be treated as small perturbations. Likewise, the poles of the scattering amplitude (84) are shifted by $O(\alpha)$ corrections to the pure gravity poles. Observe that these poles appear only when gravitational interactions are taken into account, because it is only in this case that the interaction is a (monotonically increasing) function of energy.

E. Charge-monopole versus gravitational scattering

Motivated by the conclusions of the last section, we now proceed to investigate whether they undergo any modifications when we assume one of the particles to carry a magnetic charge. In other words, will gravity still dominate over electromagnetic interactions at Planckian energies? With the replacement of the electric charge e' of the fast moving particle by a magnetic charge g, the fields on the electromagnetic shock front are given by Eq. (37). As before, when it crosses the charge e, we add the gravitational and electromagnetic phase shifts in its wave function. While the former is still $-Gs \ln r_\perp{}^2$, the latter, as seen from Eq. (2.39), is now $in\phi$. Thus, charge-monopole electromagnetic effects cannot be incorporated by a shift of Gs, in contrast to the charge-charge case. Thus the wave function assumes the form

$$\psi_>(x^\pm, x_\perp) \ = \ \exp\left[i\left(n\phi - Gs\ln\mu^2 r_\perp^2 + ipx\right)\right] . \quad (86)$$

Because of the azimuthal dependence, the calculation of the overlap with momentum eigenstates has to be done *ab initio*. Clearly, the relevant integral for the evaluation of $f(s,t)$ is

$$\int d^2 r_\perp \ \exp[i(n\phi - Gs \ \ln\mu^2 r_\perp^2 + \vec{q}\cdot\vec{r}_\perp)].$$

Once again, the integration over ϕ is readily done, and the above reduces to

$$\frac{1}{q^2}\int_0^\infty d\rho \ \rho^{1-2iGs} J_n(\rho) \ . \quad (87)$$

Here $J_n(\rho)$ is the Bessel function of order n. The above integral is again a standard one [11] and finally we get the amplitude

$$f(s,t) = \frac{k_+}{4\pi k_0} \ \delta(k_+ - p_+) \ \left(\frac{n}{2} - iGs\right) \times \frac{\Gamma(\frac{n}{2} - iGs)}{\Gamma(\frac{n}{2} + iGs)} \left(\frac{4}{-t}\right)^{1-iGs} \quad (88)$$

and hence the cross section

$$\frac{d^2\sigma}{d\vec{k}_\perp^2} \ \sim \ \frac{1}{t^2}\left(\frac{n^2}{4} + G^2 s^2\right) \ . \quad (89)$$

Since n is at least of order unity, it is clear from the above expression, that for $\sqrt{s} \approx M_{\rm Pl}$, both terms are of the same order of magnitude. This means that unlike charge-charge scattering, even at Planck scale gravity is no longer the dominant shock wave interaction. Electromagnetism with monopoles becomes equally important. This dramatic difference from the charge-charge case is a consequence of the Dirac quantization condition, which restricts the values of e and g from being arbitrarily small. In fact, the above may be considered a rephrasal of the strong coupling aspects of the monopole sector in electromagnetism and of the gravitational interactions at Planck scale. As already mentioned earlier, gravitational effects would indeed tend to dominate for $Gs \gg 1$ if the Dirac quantum number n is held fixed. But it is far from clear if, in this circumstance, the simple-minded semiclassical analysis performed above will go through without modification. Indeed, as explained in Ref. [6], super-Planckian energies will most probably entail real black hole collisions with the ensuing technical complications.

Returning once more to the analytic structure of $f(s,t)$, we see that now they occur at

$$Gs = -i\left(N + \frac{n}{2}\right) \ ,$$

that is, a shift in s by half-odd integral values. Once again, the spectrum of these "bound states" is no longer a perturbation on the spectrum in the pure gravity situation. More interestingly, notwithstanding claims in the literature (cf. [1]) that the 't Hooft poles are artifacts of the large impact parameter approximation, the shift observed above due primarily to the monopoles strongly suggest another possibility: the *Saha phenomenon* [16]. Recall that this implies that any charge-monopole pair composed of spinless particles will, as a consequence of Dirac quantization, possess a half-odd integral quantized (field) angular momentum. If we blithely regard the integer N, which also occurs in the spectrum of bound states in pure gravitational scattering, as the *spin* of the states, then it is enticing to consider the shift by one-half the Dirac quantum number n in the charge-monopole case to be the extra spin that the system would pick up in accord with Saha's predictions. Further, if one speculatively associates the Regge-like behavior observed in purely gravitational scattering with the spectrum of some string theory (albeit with imaginary slope parameter), then the spectrum with charge-monopole electromagnetic scattering can as well be speculated to correspond to some *supersymmetric* string theory. In any event the role of electric-magnetic duality, were we to actually discern any such string structures, can hardly be overemphasized.

IV. CONCLUSION

While reinforcing the general result that at c.m. energies of the order of the Planck scale and low momentum transfer, two-particle scattering is primarily a shock wave phenomenon with standard exchange processes relegated to relative unimportance, our work emphasizes the role of electromagnetic shock waves associated with the magnetic monopole sector. Since this sector is generically a strong coupling one akin to gravity at Planckian energies, it is not surprising that the contributions of the two interactions to the cross section are comparable. While similar cross sections have been computed for gravity within string theories [17] that are ostensibly correct theories of quantum gravity with tractable ultraviolet behavior, it will be interesting to see if the recently proposed "dual" strings [18] (or some modification thereof) exhibit the behaviour observed above. The major advantage of the shock wave picture is its universality in dealing with gauge particle exchanges within this, albeit somewhat restricted, kinematical region. Even when a well-defined local field theory is not available, nontrivial physical information can indeed be obtained within this picture. The task that remains then is to formulate the theory in such a way that a systematic procedure is available to compute corrections to the predictions given by this picture.

The assumption of decoupling of electromagnetic and gravitational shock waves that we have made above, of course, warrants justification, even though similar assumptions have been tacitly made in earlier work. This decoupling will be crucial if one wishes to apply the shock wave picture to analyze gravitational collapse and Hawking radiation from black holes [19], where the relevant particles carry electric and magnetic charge, or we have charged particles scattering off charged black holes. It appears that such decoupling naturally happens for particles interacting via electromagnetism and gravity within the framework of general relativity. However, no such statement can be made for particles whose fields are derived from dilaton gravity [15].

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