

ASPECTS OF BLACK HOLE PHYSICS

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Dedication

To ELHAM.

To my parents.

Abstract

In this thesis, aspects of the physics of black holes are reviewed and new results in black hole thermodynamics are presented. First, general black hole solutions of Einstein's equations of general relativity are mentioned and a proof of conservation law of energy and momentum in general relativity is presented. Aspects of the laws of black hole mechanics and Hawking radiation are then studied. Two proposals which attempt to explain the origin of black hole entropy (the brick wall model and entanglement entropy) are then discussed. Finally, some recent work related to the possible production and detection of black holes in colliders is presented.

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Chapter 1

Introduction

As early as the late 18th century scientists speculated about the existence of stellar objects whose mass was confined to such a small space that the escape velocity of any other object from it would exceed the speed of light. For this to happen the mass density of the stellar object must be very large. This can occur if a star collapses under the pressure of its gravitational self-interaction. Our sun, for instance, would have to shrink to a ball with radius of approximately 3 km. The gravitational forces of such a collapsed star are so strong that nothing, not even light-rays, can emerge from it. These objects are now called black holes, and they have, ever since their early theoretical discovery in general relativity, fascinated physicists and non-physicists alike.

For many theoretical physicists black holes are a kind of laboratory, in which they put their ideas and theories to a test. In fact, many of the questions concerning black holes are related to a fundamental open problem of contemporary theoretical physics, namely that of reconciling quantum theory with the theory of gravity (general relativity). When

describing black holes, elements of both theories become relevant. Black holes therefore serve as probes for the yet unknown theory of quantum gravity: for instance, there are quantum mechanical radiations from black holes.

Einstein's theory of special relativity results from two statements - the two basic postulates of special relativity:

1. The speed of light is the same for all inertial observers, no matter what their relative velocities.
2. The laws of physics are the same in any inertial frame of reference. This means that the laws of physics observed by an observer traveling with a free relativistic particle must still be the same as those observed by an observer who is stationary in the laboratory.

Given these two statements, Einstein showed how definitions of momentum and energy must be refined and how quantities such as length and time must change from one observer to another in order to get consistent results for physical quantities such as a particle half-life. He derived the Lorentz transformations from the principle of relativity and the observed constancy of light speed, without assuming the presence of an ether.

In 1905, Einstein published a few articles, including one on light quanta, one on the foundations of the theory of special relativity, and another on Brownian motion. After his work on special relativity, Einstein started thinking about gravity and how to give it a relativistically invariant formulation. This work, after many trials and errors, culminated in his masterpiece, the general theory of relativity, presented in 1915/1916. It is clearly one of the greatest scientific achievements of all time, a beautiful theory derived from pure

thought and physical intuition, capable of explaining, even today, virtually every aspect of gravitational physics ever observed. Einstein's key insight was that gravity is not an external force like the other forces of nature but rather a manifestation of the curvature of spacetime itself. This realization, in its simplicity and beauty, has had a profound impact on theoretical physics as a whole, and Einstein's vision of a geometrization of all of physics is still with us today.

Despite the proposal of many novel ideas, the unification of quantum mechanics - which reigns in the domain of the very small - and general relativity - a superb description of the massive - remains a tantalizing future possibility. Quantum mechanics incorporates four classes of phenomena that classical physics cannot account for: (i) the quantization (discretization) of certain physical quantities, (ii) wave-particle duality, (iii) the uncertainty principle, and (iv) quantum entanglement. Since the early days of quantum theory, physicists have made many attempts to combine it with general theory of relativity. While quantum mechanics is entirely consistent with special relativity, serious problems (like non-renormalizability) emerge when one tries to join the quantum laws with general relativity. Resolving these inconsistencies has been a major goal of twentieth- and twenty-first-century physics.

1.1 Summary of thesis

General relativity plays its role when we are incorporating gravity or better to say when we are studying noninertial reference frames. One of the defining features of general relativity is the idea that the gravitational force is replaced by geometry. According

to general relativity the laws of physics must take the same form in all coordinate systems including those that are accelerating. We recover the laws of special relativity when we apply general relativity to inertial observers. Einstein's Field Equations describe how curvature is created. In chapter 2, we shall review the important results of Einstein's general theory of relativity, which predicts the objects of our primary interest: black holes.

In 1916, Karl Schwarzschild found his famous solution to Einstein's equations for a mass distribution consisting of a mass M located at the origin of spatial coordinates but an otherwise perfectly empty space. This solution gives rise to spherical objects of radius $2GM/c^2$ that have zero angular momentum and zero electric charge.

Such objects have been referred to as black holes since no physical entity - subject to the limit of the speed of light, c , as the maximum speed possible - can escape from their interior due to the mass's gravitational attraction. It is believed that this holds for massive particles as well as for massless photons and, in fact, any kind of information.

The theory of black holes is a well-developed subject in general relativity. Two results form the cornerstone of this theory: the uniqueness theorems and the laws of black hole mechanics. The uniqueness theorems state that, while a black hole can form from an asymmetric gravitational collapse, the asymptotic equilibrium configurations of Einstein-Maxwell gravity are axisymmetric and characterized by just three parameters, the total mass M , the total charge Q , and the angular momentum J . All other details of the matter and radiation that form the black hole are dissipated as gravitational and electromagnetic radiation in the process of collapse. The corresponding three-parameter class of equilibrium solutions is formed by the Kerr-Newman solutions. Different black hole solutions of Ein-

stein's equations are the subject of chapter 3, where we will also take a look at black hole solutions in higher dimensional spacetimes.

Actually, from far away, a black hole looks very much like a particle with a certain mass and charge with a horizon around it. This means that, once the black hole has settled down in its final state, all the details of the in-falling matter and radiation, which formed the black hole during the period of collapse, have been lost and is therefore permanently inaccessible to external observers. This is sometimes expressed by the one-liner: a black hole has no hair. It has been proposed that hairy black holes may be considered to be bound states of hairless black holes and solitons. In fact mass, charge, and angular momentum are the only properties a black hole can possess, or there are no features that distinguish one black hole from another, other than the three quantities above (black hole no hair theorem).

All the more surprising was Hawking's announcement that from the quantum field theoretical point of view, "black holes are not black". This statement has to be understood in the sense that if the spacetime surrounding a black hole is filled with a classical vacuum, nevertheless a thermal flux of energy or thermal radiation of particles, can be measured to come from it according to quantum field theory. The temperature of that radiation, called the Hawking temperature T_H , has been calculated to be

$$T_H = \frac{\hbar c^3}{8\pi GM}. \quad (1.1)$$

Under some very general assumptions, Hawking showed that black holes do emit particles due to a quantum effect. The effect responsible is called spontaneous pair production, in which the vacuum spontaneously emits a particle and an anti-particle. In fact, this is a non-rigorous picture. If gravity is weak, the particle and the anti-particle enjoy only

a very short lifetime as they almost immediately annihilate each other and the resulting energy is reabsorbed by the vacuum. On an average, energy therefore stays conserved. But when this particle/anti-particle-pair is subject to the strong gravitational forces just outside the black hole horizon, the anti-particle can fall into the black hole, giving the particle a chance to escape before facing annihilation. The predominantly positive energy modes carried by the escaping particles are measured by a distant observer as radiation. This radiation was found to be that characteristic of a black body at the so-called Hawking temperature. This thermal radiation does not reveal anything of the inner structure of the black hole: it captures only the random fluctuations of the vacuum near the horizon, polarized by the strong gravitational forces. This is a rather disturbing conclusion, for it implies that a black hole is a sink for information: if particles in well defined quantum states fall into the black hole, all of the information concerning their states is lost, because the black hole radiates thermally. Such an information loss seems to be in conflict with the quantum mechanical principle of unitary time-evolution. In chapter 4, the main discussion will be around the laws of black hole thermodynamics and especially the generalized second law. We will also consider the Hawking spectrum.

Recall that the thermodynamic properties, such as pressure, temperature, or entropy of an ideal gas, for instance, are explained in the context of statistical physics as averages of certain observables of an underlying quantum theory of microscopic degrees of freedom. In terms of statistical mechanics, the entropy, symbolized by S , describes the number of the possible microscopic configurations that are capable of yielding the observed

macroscopic description of the thermodynamic system. It is written as

$$S = k \ln \Omega, \quad (1.2)$$

where Ω represents the number of degrees of freedom or the number of the possible microscopic configurations of the system. k is the Boltzmann's constant. This constant has the values below:

$$\begin{aligned} k &= 1.3806505 \times 10^{-23} \text{ J/K} \\ &= 8.617343 \times 10^{-5} \text{ eV/K.} \end{aligned} \quad (1.3)$$

Although the concept of entropy was originally a thermodynamic construct, it has been adapted in other fields of study, including information theory and black hole physics. One should be able to do similar for black holes to make the analogy closer: to define some microscopic degrees of freedom and the associated entropy for black holes.

Providing a description of the microscopic degrees of freedom of a black hole (to see what the Ω s and $\ln \Omega$ s and then entropy are) is a great challenge for any candidate theory of quantum gravity. String theory, loop quantum gravity and the idea of entanglement have provided some exciting insights into the microscopic nature of black holes, but from different points of view. Entangled states were investigated in the famous paper of Einstein, Podolsky and Rosen (EPR) [1]. Quantum mechanics is “nonlocal”, in the sense that distant and non-interacting systems may be “entangled”, namely they can exhibit perfect and instantaneous correlations. Chapter 5 reviews one of the most significant aspects of the candidates above, which is entanglement. We study the basic ideas of entanglement and entanglement entropy in a quantum mechanics background. The brick wall model is another concept of interest

in this chapter. In this model the entropy arises from a thermal bath of quantum fields propagating outside the horizon. It should be noted that every calculation of statistical entropy encounters the problem of dealing with the behavior of the physical quantities near the horizon where they typically diverge. To remove these divergences a brick wall is introduced: a fixed boundary near the horizon within which the quantum field does not propagate.

After brick wall model, we raise a discussion about the possibility of checking the entropy-area concept using entanglement ideas. It is known that the entanglement entropy of a scalar field is proportional to the area. If the density matrix for a massless free field is traced over the degrees of freedom residing inside an imaginary sphere, then the resulting entropy is proportional to the area (and not the volume) of the sphere. However, that is when the scalar field is assumed to be in its ground state. Here we try to check the area law for the first excited state of N coupled oscillators.

We mention black hole solutions in higher dimensions at the end of chapter 3. We also consider aspects of black hole thermodynamics in chapters 5 and 4. In a brane world model, our visible, four-dimensional universe is entirely restricted to a brane inside a higher-dimensional space, called the bulk. The additional dimensions may be taken to be compact, in which case the observed universe contains the extra dimensions. One way to test the idea of the existence of higher dimensions experimentally is to try to make black holes in a lab and study their thermodynamical features. Future colliders and the possibility of producing and observing black holes in them is the topic of chapter 6. Electroweak and strong forces are confined to the usual 3 spatial dimensions. However, gravity propagates in

all dimensions. There are a couple of theories, one based on flat extra dimensions and the other on the warped extra dimensions hypothesis, which show that it is possible to lower the effective Planck Mass at short distances [2, 3, 4, 5]. We must consider the fact that in any $d \geq 5$, the Planck mass M_{Pl} has a value about the TeV scale, however, the size of the extra dimension is less than or equal to mm if $d \geq 6$. In comparison, the order of the energy expected in Large Hadron Collider (LHC) at CERN is about 10 TeV. So physicists believe that it might be possible to produce black holes at the LHC, if we have more than two extra dimensions: gravity plays its role in quantum mechanics scale.

The black holes that may be produced in a collider would be so small that they would rapidly decay as a result of Hawking radiation. The radiation spectrum contains all the standard model particles, which are emitted on our brane, as well as gravitons, which are also emitted into the extra dimensions. It is in fact expected that most of the initial energy is emitted during this phase into standard model particles [6] although this conclusion is still being debated (see, e.g. Ref. [7]). The collider black hole may decay completely, or leave a Planck mass remnant. In the remaining sections of chapter 6 we will also talk about two additional ideas which have some effects on the radiation of the collider black holes: the general uncertainty principle and thermal fluctuation effects.

The last chapter is dedicated to conclusions and an overview of the whole thesis. We will mention the important ideas of the thesis and will summarize the results.

Mathematically, spacetime is represented by a 4-dimensional differentiable manifold M and the metric is given as a second-rank tensor on M , conventionally denoted by g . In this thesis we will use a metric of signature $(+,-,-,-)$ in most parts. We also use the

Einstein summation convention. In Einstein notation, an index that is repeated twice in an equation implies a summation, and the summation symbol need not be included.

This allows a concise algebraic presentation of vector and tensor equations. For example, if u_i and v_j are the components of the vectors \vec{u} and \vec{v} , using the orthogonal unit vectors \hat{e}_i we have

$$\begin{aligned}\vec{u} \cdot \vec{v} &= \sum_i u_i \hat{e}_i \cdot \sum_j v_j \hat{e}_j \\ &\equiv u_i \hat{e}_i \cdot v_j \hat{e}_j \\ &= u_i v_j (\hat{e}_i \cdot \hat{e}_j).\end{aligned}\tag{1.4}$$

In most places the notation below has been chosen: Latin indices (i, j, \dots) run from 1 to 3 indicating spacelike components only; Greek indices (μ, ν, \dots) run from 0 to 3. We keep the units such that \hbar , G (gravitational constant) and c (speed of light in vacuum) will not be omitted in formulas.

Chapter 2

A brief introduction to general relativity

This chapter will be devoted to a brief discussion of general relativity, where we will try to cover at least the main ideas. Here, we mainly follow reference [8]. For certain topics we follow references [9] to [16].

While we shall attempt to give a comprehensive summary of the main theoretical results, we shall, in general, refrain from discussing applications in this chapter. The main reason being, applications invariably entail some lengthy background, the discussion of which would be a digression from the central theme of this chapter.

2.1 Principle of equivalence

From the special theory of relativity, we learn that the velocity of light is Lorentz-invariant. There are also many physical quantities which are not invariant but covariant

under Lorentz transformations. Energy E and momentum p transform as different components of a four-vector and electromagnetic fields transform as a tensor. In special relativity, a four-vector is a vector in a four-dimensional real vector space, called Minkowski space, whose components transform under spatial translations, rotations, and boosts (a change by a constant velocity to another inertial reference frame). For instance, the four-momentum P^μ can be written as

$$\begin{aligned} P^\mu &= \left(\frac{E}{c}, \mathbf{p} \right) \\ &= \left(\frac{E}{c}, p_x, p_y, p_z \right), \end{aligned} \quad (2.1)$$

where E is the total energy of the particle and \mathbf{p} represents its linear momentum vector. Lorentz transformations are linear transformations that preserve the spacetime interval between any two events in Minkowski space, while leaving the origin fixed. This set of transformations can be written as

$$(x^\mu)' = L^\mu_\nu x^\nu. \quad (2.2)$$

Such form of transformations is subject to the condition that the quantity

$$x^\mu x_\mu = c^2 t^2 - |\vec{x}|^2 \quad (2.3)$$

remains invariant. This condition is necessary to make sure that the coefficients L^μ_ν form an orthogonal tensor:

$$\begin{aligned} L_{\mu\nu} L^{\nu\sigma} &= \delta_\mu^\sigma \\ L^\mu_\sigma L^\nu_\sigma &= \delta^{\mu\nu} \\ L^\sigma_\mu L^\sigma_\nu &= \delta_{\mu\nu}. \end{aligned} \quad (2.4)$$

With v as the velocity of transforming from one coordinate frame to another one, we define ξ as

$$\frac{v}{c} = \tanh \xi. \quad (2.5)$$

Then the equations of transformation can be written as below:

$$\begin{aligned} x' &= x, & y' &= y \\ z' &= z \cosh \xi - ct \sinh \xi \\ t' &= t \cosh \xi - \frac{z}{c} \sinh \xi. \end{aligned} \quad (2.6)$$

The Lorentz transformation of an arbitrary tensor with arbitrary rank is given by

$$A'^{\mu\nu\dots}{}_{\rho\sigma\dots} = L_{\tau}^{\mu} L_{\eta}^{\nu} \dots L_{\rho}^{\kappa} L_{\sigma}^{\lambda} \dots A^{\tau\eta\dots}{}_{\kappa\lambda\dots}. \quad (2.7)$$

If one would be able to write both sides of a physical equation with Lorentz indices always transforming as vectors or tensors like above, then it is guaranteed that such equations keep their mathematical form unchanged in any reference frame.

To show an application of tensors in physics, we mention the field strength tensor of electromagnetism, which is written in terms of electric and magnetic fields:

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}. \quad (2.8)$$

The fields are transformed to a frame moving with constant relative velocity by

$$F'^{\mu\nu} = L_{\alpha}^{\mu} L_{\beta}^{\nu} F^{\alpha\beta}. \quad (2.9)$$

Again, L_{α}^{μ} is a Lorentz transformation.

We must have an inertial reference frame to be able to use special relativity. By contrast, one can not apply special relativity to non-inertial frames of reference. The magnitude or direction of the velocity of the object would not be constant any more and we might have some forces which cause the non-inertiality. Bodies are subject to so-called fictitious forces in non-inertial reference frames. These are forces that result from the acceleration of the reference frame itself and not from any physical force acting on the body. Examples of fictitious forces are the centrifugal force and the Coriolis force in rotating reference frames. Therefore, scientists living inside a box that is being rotated or otherwise accelerated can measure their acceleration by observing the fictitious forces on bodies inside the box.

Einstein's general theory modifies the distinction between inertial and non-inertial effects, by replacing special relativity's flat Lorentzian geometry with a curved non-Euclidean geometry. The latter can be described in a geometry so-called pseudo-Riemannian geometry. It deals with a broad range of geometries whose metric properties vary from point to point. In general relativity, the principle of inertia is replaced with the principle of geodesic motion, whereby objects move in a way dictated by the curvature of spacetime. As a consequence of this curvature, it is not a given in general relativity that inertial objects moving at a particular rate with respect to each other will continue to do so. This phenomenon means that inertial frames of reference do not exist globally as they do in Newtonian mechanics and special relativity. However, the general theory reduces to the special theory locally over sufficiently small regions of spacetime, where curvature effects become less important and the earlier inertial frame arguments can be applied. General relativity is the theory in

which we study the relationship among all these issues and it starts from the principle of equivalence.

According to the *weak principle of equivalence*, a non-inertial reference frame is equivalent to a certain gravitational field and there is an equivalence between inertial mass and gravitational mass. Inertial mass is a measure of the resistance of an entity to a change in its velocity relative to an inertial frame. However, the gravitational mass is a property of a physical object that quantifies the amount of matter and energy it is equivalent to. This equivalence has been proven experimentally by L. Eötvös with the accuracy of one part in 10^9 and by R. H. Dicke with the accuracy of one part in 10^{11} . Eötvös used a torsion balance to show what the combined effects of gravity and centrifugal force (due to Earth rotation) would be for two bodies made of different substances, having the same inertial mass, but slightly different gravitational masses (if any). His first instruments were similar to those of Mitchell, Cavendish, and Coulomb. The torsion balance was at that time a horizontal rod suspended at the center. Eötvös soon realized the potentialities of this simple device for measuring the difference between the two main curvatures of the very local equipotential surface, i.e. of the surface perpendicular in each point to the combined effects of gravity and the centrifugal force due to earth rotation. The Dicke's experiment follows the same ideas of the Eötvös's one. The only difference is that in Dicke's experiment we use the Sun's gravitational field. We also consider the fact that the Earth revolves around the Sun [17, 18]. There are also some plans to check this principle by a satellite (STEP or Satellite Test of the Equivalence Principle) with the accuracy of one part in 10^{17} .

Einstein expressed the above principle in a more precise way: in small enough

regions of spacetime, the laws of physics reduce to those of special relativity. This is called the *Einstein Equivalence Principle* or *Strong Principle of Equivalence*. In fact we define *unaccelerated* as *freely falling*. This point of view is the origin of the idea that gravity is not a force - a force is something which leads to acceleration, and our definition of zero acceleration is moving freely in the presence of whatever gravitational field happens to be around.

An event is described by the place where it occurred and the time when it occurred. Let's assume that one event happens at (t, x, y, z) and another infinitesimally separated event happens at $(t + dt, x + dx, y + dy, z + dz)$. In a flat spacetime, the interval ds between these two events is

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2. \quad (2.10)$$

Although this form remains the same under any transformation to an other inertial reference frame, however, we will end up with a different form upon transforming to a non-inertial system of reference.

For example

$$ds^2 = [c^2 - \Omega^2(x'^2 + y'^2)]dt^2 - dx'^2 - dy'^2 - dz'^2 + 2\Omega y' dx' dt - 2\Omega x' dy' dt \quad (2.11)$$

is the case of a transformation to uniformly rotating coordinates,

$$\begin{aligned} x &= x' \cos \Omega t - y' \sin \Omega t, \\ y &= x' \sin \Omega t + y' \cos \Omega t, \\ z &= z' \end{aligned} \quad (2.12)$$

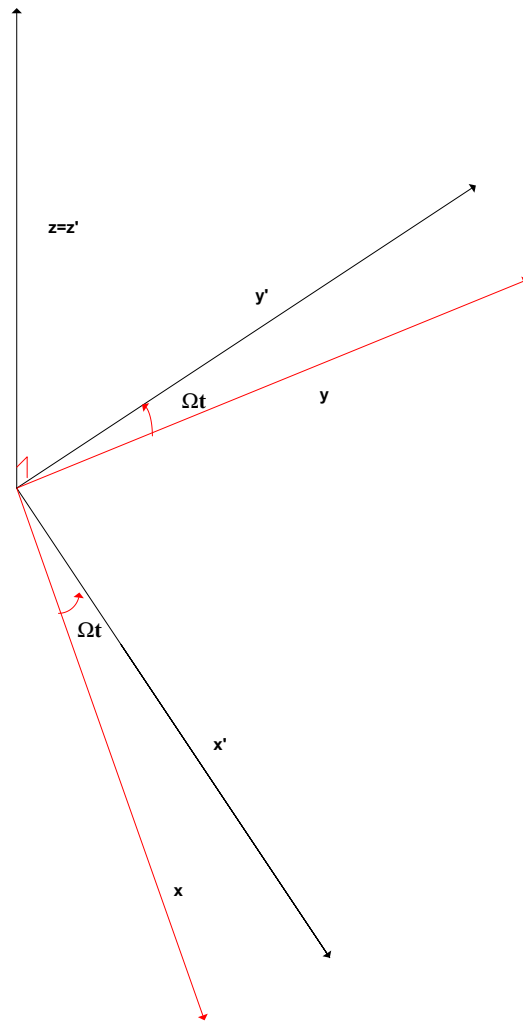


Figure 2.1: Plot of a transformation to uniformly rotating coordinates with angular velocity of Ωt .

where Ω is the constant angular velocity of the rotation, directed along the z axis. We align z and z' and the rotation is clockwise (Fig 2.1). In any case one can not write the equation (2.11) as a negative sum of squares of the four coordinate differentials. In the most general case we can write the interval in terms of a metric $g_{\mu\nu}$ and in tensor notation

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \quad (2.13)$$

A metric is a second-rank tensor on the spacetime manifold and all the geometrical information about our spacetime is contained in the metric $g_{\mu\nu}$. The contravariant metric tensor is the tensor $g^{\mu\nu}$ reciprocal to $g_{\mu\nu}$, given as

$$g_{\mu\sigma} g^{\sigma\nu} = \delta_\mu^\nu. \quad (2.14)$$

For the special case of cartesian coordinates in an inertial reference system we have

$$\begin{aligned} g_{00} &= 1 \\ g_{11} &= g_{22} = g_{33} = -1 \\ g_{\mu\nu} &= 0 \quad (\mu \neq \nu). \end{aligned} \quad (2.15)$$

Comparing with equation (2.10), we see that the coefficients of dx^2 , dy^2 and dz^2 are -1 , while the coefficient of $c^2 dt^2$ is 1 . There are also no terms like $dx dy$, $dx dz$, etc.

2.2 Particles in a gravitational field

According to special relativity the action of a free moving material point of mass m with velocity v is

$$S = -mc \int ds. \quad (2.16)$$

Its Lagrangian is therefore

$$L = -mc^2 \sqrt{1 - \frac{v^2}{c^2}}. \quad (2.17)$$

If we write the principle of least action for such particle, then we have

$$\delta S = -mc \delta \int ds = 0. \quad (2.18)$$

However, we try to find the equation of motion of a particle in a gravitational field by generalizing the differential equations for the free motion of a particle in special relativity.

For an inertial four-dimensional coordinate system we have $du^\mu/ds = 0$ or $du^\mu = 0$ where $u^\mu = dx^\mu/ds$ is the four velocity. The generalization in curvilinear coordinates would be

$$\frac{Du^\mu}{ds} = 0 \quad (2.19)$$

where for Du^μ (covariant differential of vector u^μ) we have

$$Du^\mu = \left(\frac{\partial u^\mu}{\partial x^\nu} + \Gamma_{\kappa\nu}^\mu u^\kappa \right) dx^\nu. \quad (2.20)$$

To define the connection coefficients $\Gamma_{\kappa\nu}^\mu$ we have to consider few facts. In curvilinear coordinates, the difference in the components of the two vectors after translating one of them to the point where the other is located will not coincide with their difference before the translation (du^μ). So there is a difference which we call δu^μ and Du^μ is then defined as the difference between du^μ and δu^μ :

$$Du^\mu = du^\mu - \delta u^\mu. \quad (2.21)$$

The value δu^μ , which is the change in the components of a vector under an infinitesimal parallel displacement depends on the values of the components themselves, with the coefficient

$\Gamma_{\kappa\nu}^{\mu}$:

$$\delta u^{\mu} = -\Gamma_{\kappa\nu}^{\mu} u^{\kappa} dx^{\nu}. \quad (2.22)$$

Substituting this equation and $du^{\mu} = (\partial u^{\mu}/\partial x^{\nu})dx^{\nu}$ in definition of $Du^{\mu}(= du^{\mu} - \delta u^{\mu})$, we get back to the equation (2.20).

$\Gamma_{\kappa\nu}^{\mu}$ are the so-called Christoffel symbols which are obviously not tensors,

$$\Gamma_{\mu'\lambda'}^{\nu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\lambda}}{\partial x^{\lambda'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \Gamma_{\mu\lambda}^{\nu} - \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\lambda}}{\partial x^{\lambda'}} \frac{\partial^2 x^{\nu'}}{\partial x^{\lambda} \partial x^{\mu}}. \quad (2.23)$$

This is not, of course, the tensor transformation law due to the second term on the right, though Christoffel symbols are functions of coordinates. For an inertial reference frame $\Gamma_{\kappa\nu}^{\mu} = 0$.

The so-called geodesic equations are

$$Du^{\mu} = \frac{du^{\mu}}{ds} + \Gamma_{\kappa\nu}^{\mu} u^{\kappa} \frac{dx^{\nu}}{ds} = 0 \quad (2.24)$$

or

$$\frac{d^2 x^{\mu}}{ds^2} + \Gamma_{\kappa\nu}^{\mu} \frac{dx^{\kappa}}{ds} \frac{dx^{\nu}}{ds} = 0. \quad (2.25)$$

The derivative $d^2 x^{\mu}/ds^2$ represents the four-acceleration of the particle. So one may call the quantity $-m\Gamma_{\kappa\nu}^{\mu} u^{\kappa} u^{\nu}$ the *four-force* acting on the particle in the gravitational field.

We also introduce the most general definition of the covariant derivative as

$$\begin{aligned} \nabla_{\sigma} T_{\nu_1 \nu_2 \dots \nu_l}^{\mu_1 \mu_2 \dots \mu_k} = & \quad \partial_{\sigma} T_{\nu_1 \nu_2 \dots \nu_l}^{\mu_1 \mu_2 \dots \mu_k} \\ & + \Gamma_{\sigma\lambda}^{\mu_1} T_{\nu_1 \nu_2 \dots \nu_l}^{\lambda \mu_2 \dots \mu_k} + \Gamma_{\sigma\lambda}^{\mu_2} T_{\nu_1 \nu_2 \dots \nu_l}^{\mu_1 \lambda \dots \mu_k} + \dots \\ & - \Gamma_{\sigma\nu_1}^{\lambda} T_{\lambda \nu_2 \dots \nu_l}^{\mu_1 \mu_2 \dots \mu_k} - \Gamma_{\sigma\nu_2}^{\lambda} T_{\nu_1 \lambda \dots \nu_l}^{\mu_1 \mu_2 \dots \mu_k}. \end{aligned} \quad (2.26)$$

The geodesic equation (2.24) is not valid for the ray of light as in this case the interval ds is zero. In special relativity the four-dimensional wave vector tangent to the

ray ($k^\mu = dx^\mu/d\lambda$ with λ as some parameter varying along the ray) shows the direction of propagation, and in vacuum the wave vector does not vary along the path, i.e. $dk^\mu = 0$. In the presence of a gravitational field this equation goes over into $Dk^\mu = 0$ or

$$\frac{dk^\mu}{d\lambda} + \Gamma^\mu_{\kappa\nu} k^\kappa k^\nu = 0. \quad (2.27)$$

One should be also able to calculate the Hamilton-Jacobi equation for a particle in a gravitational field

$$g^{\mu\nu} \frac{\partial S \partial S}{\partial x^\mu \partial x^\nu} - m^2 c^2 = 0. \quad (2.28)$$

In fact, we use the definition of the four-momentum of a particle (and its square) in the same field

$$\begin{aligned} p^\mu &= mcu^\mu \\ p_\mu p^\mu &= m^2 c^2 \end{aligned} \quad (2.29)$$

and we substitute $-\partial S/\partial x^\mu$ for p_μ .

For non-relativistic mechanics, which is equal to the assumption of small velocities (and so weak gravitational field), we write the Lagrangian as

$$L = -mc^2 + \frac{mv^2}{2} - m\Phi \quad (2.30)$$

adding the constant $-mc^2$. The reason of adding this constant is to make sure that in the absence of the field, where the potential Φ is zero, we get back to the relativistic Lagrangian $L = -mc^2 \sqrt{1 - v^2/c^2}$ when $v/c \rightarrow 0$. Consequently, the nonrelativistic action function S for a particle in a gravitational field would be

$$S = \int L dt = -mc \int \left(c - \frac{v^2}{2c} + \frac{\Phi}{c} \right) dt. \quad (2.31)$$

Comparing this with the equation (2.18) one can guess the ds to be

$$ds = \left(c - \frac{v^2}{2c} + \frac{\Phi}{c} \right) dt \quad (2.32)$$

and for the limit $c \rightarrow \infty$ we get

$$ds^2 = (c^2 + 2\Phi)dt^2 - d\mathbf{r}^2 \quad (2.33)$$

with $\mathbf{v}dt=d\mathbf{r}$. Thus in the limiting case, the components of the metric tensor are as follows:

$$\begin{aligned} g_{00} &= 1 + \frac{2\Phi}{c^2} \\ g_{ij} &= -\delta_{ij} \\ g_{0j} &= 0. \end{aligned} \quad (2.34)$$

2.3 Gravitational field description

The Einstein equations of the gravitational field, which are the master equations in general relativity, can be obtained from the principle of least action. These equations can be written as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu} \quad (2.35)$$

or, in mixed components,

$$R^\nu_\mu - \frac{1}{2}\delta^\nu_\mu R = \frac{8\pi G}{c^4}T^\nu_\mu. \quad (2.36)$$

In the principle of least action

$$\delta(S_m + S_g) = 0 \quad (2.37)$$

the quantities S_m and S_g are the actions of matter and the gravitational field respectively.

The terms on the left hand side of the Einstein equations, which contains R^ν_μ and R originate from S_g . The term representing the matter contribution of the action gives the right

hand side (T_μ^ν). Basically these equations describe how stress-energy causes curvature of spacetime. Consider that we have to mention both terms of matter and gravitational field in the action: this is not the vacuum and we have both gravitational fields and matter which produces the gravity. $T_{\mu\nu}$ is the energy-momentum tensor, while $R_{\mu\nu}$ stands for Ricci tensor and R for scalar curvature. $R_{\mu\nu}$ and R are the direct result of contraction on curvature tensor

$$R_{\nu\rho\sigma}^\mu = \frac{\partial\Gamma_{\nu\sigma}^\mu}{\partial x^\rho} - \frac{\partial\Gamma_{\nu\rho}^\mu}{\partial x^\sigma} + \Gamma_{\kappa\rho}^\mu\Gamma_{\nu\sigma}^\kappa - \Gamma_{\kappa\sigma}^\mu\Gamma_{\nu\rho}^\kappa \quad (2.38)$$

$$R_{\mu\nu} = g^{\rho\sigma}R_{\rho\mu\sigma\nu} = R_{\mu\rho\nu}^\rho \quad (2.39)$$

$$R_{\mu\nu} = \frac{\partial\Gamma_{\mu\nu}^\rho}{\partial x^\rho} - \frac{\partial\Gamma_{\mu\rho}^\rho}{\partial x^\nu} + \Gamma_{\mu\nu}^\rho\Gamma_{\rho\sigma}^\sigma - \Gamma_{\mu\rho}^\sigma\Gamma_{\nu\sigma}^\rho. \quad (2.40)$$

So the Ricci tensor can be shown easily that is symmetric

$$R_{\mu\nu} = R_{\nu\mu}. \quad (2.41)$$

We get the scalar curvature by contracting $R_{\mu\nu}$

$$R = g^{\mu\nu}R_{\mu\nu} = g^{\mu\rho}g^{\nu\sigma}R_{\mu\nu\rho\sigma}. \quad (2.42)$$

The curvature tensor has the symmetry properties

$$R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma} = -R_{\mu\nu\sigma\rho}, \quad (2.43)$$

$$R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu} \quad (2.44)$$

which follow from the equation

$$R_{\mu\nu\rho\sigma} = \frac{1}{2} \left(\frac{\partial^2 g_{\mu\sigma}}{\partial x^\nu \partial x^\rho} + \frac{\partial^2 g_{\nu\rho}}{\partial x^\mu \partial x^\sigma} - \frac{\partial^2 g_{\mu\rho}}{\partial x^\nu \partial x^\sigma} - \frac{\partial^2 g_{\nu\sigma}}{\partial x^\mu \partial x^\rho} \right) + g_{\kappa\tau} (\Gamma_{\nu\rho}^\kappa \Gamma_{\mu\sigma}^\tau - \Gamma_{\nu\sigma}^\kappa \Gamma_{\mu\rho}^\tau). \quad (2.45)$$

One would also write the *permutation relation* and *Bianchi identity* for the curvature tensor,

$$R_{\mu\nu\rho\sigma} + R_{\mu\rho\nu\sigma} + R_{\mu\sigma\nu\rho} = 0, \quad (2.46)$$

$$R_{\mu\nu\rho;\sigma} + R_{\mu\sigma\nu;\rho} + R_{\mu\rho\sigma;\nu} = 0 \quad (2.47)$$

where the semicolons stand for covariant differentiation (see equation 2.20).

The components of the curvature tensor $R_{\mu\nu\rho\sigma}$ are not all independent. In fact, equations (2.43), (2.44) and (2.46) relate some components of each tensor. In a two dimensional space (a surface), the indices only accept numbers 1 and 2. Equation (2.43) indicates that in such case there is only one independent component. In three dimensional space indices run through values 1, 2, 3. We write the curvature tensor as R_{abcd} . The index pairs ab and cd run through three different sets of values: 12, 23, and 31. Permutation of indices in a pair ab or cd change the sign of the tensor component. R_{abcd} is also symmetric under interchange of these pairs (Eq. 2.44). There is no more restriction added by relation (2.46) here. Therefore, there are six independent components finally: three independent components with different pairs of indices and three components with identical pairs. In four dimensional spacetime, we have our normal $R_{\mu\nu\rho\sigma}$. The pairs of indices $\mu\nu$ and $\rho\sigma$ accept six different sets of values: 12, 13, 14, 23, 34, 42. Thus there are six components with identical and 15 with different pairs of indices. Among these 15 components, there are three of them with all the indices different, which are related by the identity (2.46). So if we have two from those three, we can find the latter one as well. Therefore, in here the curvature tensor has 20 ($= 6 + 14$) independent indices.

After contracting on the indices μ and ν of the equation (2.36), we get

$$R = -\frac{8\pi G}{c^4}T \quad (2.48)$$

where $R = R^\mu_\mu$ and $T = T^\mu_\mu$. Therefore, another form of the field equations would be

$$R_{\mu\nu} = \frac{8\pi G}{c^4} \left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T \right). \quad (2.49)$$

The Einstein equations reduce to the equation

$$R_{\mu\nu} = 0 \quad (2.50)$$

for vacuum ($T_{\mu\nu} = 0$). However, that does not imply that the spacetime is flat since $R^\mu_{\nu\rho\sigma}$ is not necessarily zero.

The principle of superposition is not valid for gravitational fields as the Einstein equations are nonlinear. We may be able to use this principle only approximately for classical Newtonian limit and weak fields which are described by linearized Einstein equations.

The numerical values of the constants in Einstein equations are as follows:

$$\begin{aligned} G &= 6.67 \times 10^{-8} \text{cm}^3 \cdot \text{g}^{-1} \cdot \text{s}^{-2} \\ \frac{8\pi G}{c^2} &= 1.86 \times 10^{-27} \text{cm} \cdot \text{g}^{-1}. \end{aligned} \quad (2.51)$$

2.4 Conservation of energy-momentum

One can prove that the divergence of the energy-momentum tensor is zero:

$$T^\nu_{\mu;\nu} = 0 \quad (2.52)$$

that implies the conservation of energy and momentum in general relativity. This equation can also be obtained if one applies the permutation relation (2.46) to the Einstein equation

(2.36). Hence the equations of the gravitational field (Einstein equations) also contain the law of conservation of energy and momentum, i.e. equations for the matter which produces the field. So the distribution and motion of the matter producing the gravitational field are completely related and must be determined at the same time as we find the field produced by the matter. However, this is quite different from the case of electromagnetism. The continuity equation expresses the conservation of the total electric charge without considering their distribution. Therefore the distribution and motion of the charges can be assigned arbitrarily, so long as the total charge is constant. Then by applying Maxwell's equations, one can find the electromagnetic field produced by the distribution.

The components of the energy-momentum tensor, $T_{\mu\nu}$, stand for a set of numbers: $T_{xx}, T_{xy}, T_{xz}, T_{xt}, T_{yy}, T_{yz}, T_{yt}, T_{zz}, T_{zt}$, and T_{tt} . T_{xx}, T_{yy} and T_{zz} measure the pressure in each of the three directions. T_{xt}, T_{yt} and T_{zt} measure how fast the matter is moving (its momentum). The component T_{tt} represents the amount of mass there is at a point (density). The remained three components T_{xy}, T_{xz} and T_{yz} measure the stresses in the matter [45]. Therefore, stress, pressure, and momentum all have some effect on the warping of spacetime. One of the most important properties of the energy-momentum tensor is the equation (2.52), $T_{\mu;\nu}^{\nu} = 0$. This implies the conservation of energy and momentum in general relativity. To prove that we start from the action integral and conservation of energy and momentum in special relativity.

Assume that Λ is Lagrangian density of an arbitrary system. That means if we integrate Λ over volume, the result is the Lagrangian. This Lagrangian density is some function of the quantities q (the generalized coordinates) and its first derivative $\frac{\partial q}{\partial x^{\mu}}$. Actually

Λ describes the state of the system. The derivative of q is taken with respect to coordinates and time. The derivative versus time gives the generalized velocity. For example, a bead constrained to move on a wire has only one degree of freedom. The generalized coordinate used to describe its motion is then $q_1 = l$, where l is the distance along the wire from some reference point on the wire. In special relativity, one can write the action integral for such system as

$$\begin{aligned} S &= \int \Lambda \left(q, \frac{\partial q}{\partial x^\mu} \right) dV dt \\ &= \frac{1}{c} \int \Lambda d\Omega. \end{aligned} \quad (2.53)$$

Here, dV is the spatial volume element. So the Lagrangian, as the spatial part of the action integral, is written in the form of:

$$L = \int \Lambda dV. \quad (2.54)$$

In action integral, we have also used the definition

$$d\Omega = c dt dV. \quad (2.55)$$

By varying S and using the principle of least action, the equations of motion in special relativity are obtained. δS can be written as

$$\delta S = \frac{1}{c} \int \left(\frac{\partial \Lambda}{\partial q} \delta q + \frac{\partial \Lambda}{\partial q_{,\mu}} \delta q_{,\mu} \right) d\Omega. \quad (2.56)$$

Here, we defined $q_{,\mu}$ as

$$q_{,\mu} \equiv \frac{\partial q}{\partial x^\mu}. \quad (2.57)$$

According to principle of least action, we have to put δS equal to zero:

$$\delta S = \frac{1}{c} \int \left[\frac{\partial \Lambda}{\partial q} \delta q + \frac{\partial}{\partial x^\mu} \left(\frac{\partial \Lambda}{\partial q_{,\mu}} \delta q \right) - \delta q \frac{\partial}{\partial x^\mu} \frac{\partial \Lambda}{\partial q_{,\mu}} \right] d\Omega = 0. \quad (2.58)$$

By means of the Gauss' theorem, one can simplify the integration over all the space for the second term of the integrand. The result is zero as we are varying the path only, not the endpoints ($\delta q = 0$ on the boundary of surface):

$$\int \frac{\partial}{\partial x^\mu} \left(\frac{\partial \Lambda}{\partial q_{,\mu}} \delta q \right) d\Omega = \oint \frac{\partial \Lambda}{\partial q_{,\mu}} \delta q dS_\mu = 0. \quad (2.59)$$

We then find the equations of motion from the remaining part of the equation (2.58):

$$\frac{\partial}{\partial x^\mu} \frac{\partial \Lambda}{\partial q_{,\mu}} - \frac{\partial \Lambda}{\partial q} = 0. \quad (2.60)$$

To derive the conservation of energy and momentum in special relativity, we have to expand $\frac{\partial \Lambda}{\partial x^\mu}$:

$$\frac{\partial \Lambda}{\partial x^\mu} = \frac{\partial \Lambda}{\partial q} \frac{\partial q}{\partial x^\mu} + \frac{\partial \Lambda}{\partial q_{,\nu}} \frac{\partial q_{,\nu}}{\partial x^\mu}. \quad (2.61)$$

We substitute the equivalent of $\frac{\partial \Lambda}{\partial q}$ from the equation of motion. We also note that

$$q_{,\nu,\mu} = q_{,\mu,\nu}. \quad (2.62)$$

Then we find

$$\begin{aligned} \frac{\partial \Lambda}{\partial x^\mu} &= \frac{\partial}{\partial x^\nu} \left(\frac{\partial \Lambda}{\partial q_{,\nu}} \right) q_{,\mu} \\ &= \frac{\partial}{\partial x^\nu} \left(q_{,\mu} \frac{\partial \Lambda}{\partial q_{,\nu}} \right). \end{aligned} \quad (2.63)$$

Left hand side can be written as

$$\frac{\partial \Lambda}{\partial x^\mu} = \delta_\mu^\nu \frac{\partial \Lambda}{\partial x^\nu}. \quad (2.64)$$

So the equation (2.63) can be simplified to

$$\frac{\partial}{\partial x^\nu} \left(q_{,\mu} \frac{\partial \Lambda}{\partial q_{,\nu}} - \delta_\mu^\nu \frac{\partial \Lambda}{\partial x^\nu} \right) = 0. \quad (2.65)$$

So the quantity in brackets is conserved. We define it as

$$T_{\mu}^{\nu} = q_{,\mu} \frac{\partial \Lambda}{\partial q_{,\nu}} - \delta_{\mu}^{\nu} \frac{\partial \Lambda}{\partial x^{\nu}} \quad (2.66)$$

and the conservation law is then

$$\frac{\partial}{\partial x^{\nu}} T_{\mu}^{\nu} = 0. \quad (2.67)$$

This is similar to the procedure in mechanics for deriving the conservation of energy. On the other hand, equation (2.67) asserts that the vector

$$P^{\mu} = \frac{1}{c} \int T^{\mu\nu} dS_{\nu} \quad (2.68)$$

is conserved. In fact, the vanishing of the four-divergence of a vector is equivalent to the conservation of the integral of the vector over a hypersurface which contains all of three-dimensional space. Here, the conserved vector P^{μ} must be identified with the four-vector of momentum of the system. The constant $\frac{1}{c}$ appears in front of the integral to make sure that $\int T^{00} dV$ is the total energy of the system.

We know from Riemannian geometry that on transforming to curvilinear coordinates x^i , the element of integration $d\Omega$ goes over to

$$d\Omega \rightarrow \sqrt{-g} d\Omega. \quad (2.69)$$

The quantity g is the inverse of the determinant of the reciprocal metric $g^{\mu\nu}$,

$$|g^{\mu\nu}| = \frac{1}{g} = \frac{1}{|g_{\mu\nu}|}. \quad (2.70)$$

Now we are able to write the action integral in curvilinear coordinates

$$S = \frac{1}{c} \int \Lambda \sqrt{-g} d\Omega. \quad (2.71)$$

In the limit of Minkowskian (flat) coordinates, $g = -1$ and S goes over into the equation (2.53). We transform to new coordinates which are different from coordinates x^μ by small quantities ξ^μ

$$x'^\mu = x^\mu + \xi^\mu. \quad (2.72)$$

The new metric in the transformed coordinates is

$$\begin{aligned} g'^{\mu\nu}(x'^\varsigma) &= g^{s\iota}(x^\varsigma) \frac{\partial x'^\mu}{\partial x^\varsigma} \frac{\partial x'^\nu}{\partial x^\iota} \\ &= g^{s\iota} \left(\delta_\varsigma^\mu + \frac{\partial \xi^\mu}{\partial x^\varsigma} \right) \left(\delta_\iota^\nu + \frac{\partial \xi^\nu}{\partial x^\iota} \right) \\ &= g^{s\iota} \delta_\varsigma^\mu \delta_\iota^\nu + g^{s\iota} \delta_\varsigma^\mu \frac{\partial \xi^\nu}{\partial x^\iota} + g^{s\iota} \delta_\iota^\nu \frac{\partial \xi^\mu}{\partial x^\varsigma} + g^{s\iota} \frac{\partial \xi^\mu}{\partial x^\varsigma} \frac{\partial \xi^\nu}{\partial x^\iota}. \end{aligned} \quad (2.73)$$

We approximate this as

$$g'^{\mu\nu}(x'^\varsigma) \approx g^{\mu\nu}(x^\varsigma) + g^{\mu\iota} \frac{\partial \xi^\nu}{\partial x^\iota} + g^{\nu\varsigma} \frac{\partial \xi^\mu}{\partial x^\varsigma}. \quad (2.74)$$

In fact we just ignored the term $g^{s\iota} \frac{\partial \xi^\mu}{\partial x^\varsigma} \frac{\partial \xi^\nu}{\partial x^\iota}$ in equation (2.73). The tensor $g'^{\mu\nu}$ is a function of x'^ς , while the tensor $g^{\mu\nu}$ is a function of the original coordinates x^ς . We expand $g'^{\mu\nu}(x'^\varsigma)$ in powers of ξ^ς , in order to represent all terms as functions of the same variables. We also neglect the terms of higher order in ξ^ς :

$$\begin{aligned} g'^{\mu\nu}(x'^\varsigma) &= g'^{\mu\nu}(x^\varsigma + \xi^\varsigma) \\ &= g'^{\mu\nu}(x^\varsigma) + \xi^\varsigma \frac{\partial g'^{\mu\nu}}{\partial x^\varsigma} + \dots \end{aligned} \quad (2.75)$$

Thus from the equations (2.74) and (2.75) we find

$$g'^{\mu\nu}(x^\varsigma) = g^{\mu\nu}(x^\varsigma) - \xi^\varsigma \frac{\partial g^{\mu\nu}}{\partial x^\varsigma} + g^{\mu\varsigma} \frac{\partial \xi^\nu}{\partial x^\varsigma} + g^{\nu\varsigma} \frac{\partial \xi^\mu}{\partial x^\varsigma}. \quad (2.76)$$

One can show that the sum of contravariant derivatives of the ξ^μ is as below:

$$\xi^{\mu;\nu} + \xi^{\nu;\mu} = -\xi^\varsigma \frac{\partial g^{\mu\nu}}{\partial x^\varsigma} + g^{\mu\varsigma} \frac{\partial \xi^\nu}{\partial x^\varsigma} + g^{\nu\varsigma} \frac{\partial \xi^\mu}{\partial x^\varsigma}. \quad (2.77)$$

Therefore, finally we obtain the simplified form for the transformation of the $g^{\mu\nu}$ as

$$g'^{\mu\nu} = g^{\mu\nu} + \delta g^{\mu\nu}. \quad (2.78)$$

Comparing with equation (2.76) we obtain

$$\delta g^{\mu\nu} = \xi^{\mu;\nu} + \xi^{\nu;\mu}. \quad (2.79)$$

The covariant components for the equations (2.78) and (2.79) to leading order are as below:

$$\begin{aligned} g'_{\mu\nu} &= g_{\mu\nu} + \delta g_{\mu\nu} \\ \delta g_{\mu\nu} &= -[\xi_{\mu;\nu} + \xi_{\nu;\mu}], \end{aligned} \quad (2.80)$$

which follows from:

$$g_{\mu\varsigma} g^{\nu\varsigma} = \delta_{\mu}^{\nu}. \quad (2.81)$$

Now we vary the action in the curvilinear coordinates (2.71) and write δS as

$$\delta S = \frac{1}{c} \int \left\{ \left[\frac{\partial \sqrt{-g} \Lambda}{\partial g^{\mu\nu}} \delta g^{\mu\nu} + \frac{\partial \sqrt{-g} \Lambda}{\partial \frac{\partial g^{\mu\nu}}{\partial x^{\varsigma}}} \delta \frac{\partial g^{\mu\nu}}{\partial x^{\varsigma}} \right] + \left[\frac{\partial \Lambda}{\partial q} \delta q + \frac{\partial \Lambda}{\partial q_{,\mu}} \delta q_{,\mu} \right] \right\} d\Omega. \quad (2.82)$$

According to the principle of least action (equations 2.56 and 2.58), integration over the second bracket gives zero. We also factorize $\delta g^{\mu\nu}$ in the first bracket and simplify δS :

$$\delta S = \frac{1}{c} \int \left\{ \frac{\partial \sqrt{-g} \Lambda}{\partial g^{\mu\nu}} - \frac{\partial}{\partial x^{\varsigma}} \frac{\partial \sqrt{-g} \Lambda}{\partial \frac{\partial g^{\mu\nu}}{\partial x^{\varsigma}}} \right\} \delta g^{\mu\nu} d\Omega. \quad (2.83)$$

$T_{\mu\nu}$ is defined as

$$\frac{1}{2} \sqrt{-g} T_{\mu\nu} = \frac{\partial \sqrt{-g} \Lambda}{\partial g^{\mu\nu}} - \frac{\partial}{\partial x^{\varsigma}} \frac{\partial \sqrt{-g} \Lambda}{\partial \frac{\partial g^{\mu\nu}}{\partial x^{\varsigma}}}. \quad (2.84)$$

Note that

$$g^{\mu\nu} \delta g_{\nu\mu} = -g_{\nu\mu} \delta g^{\mu\nu} \quad (2.85)$$

and therefore

$$T^{\mu\nu}\delta g_{\mu\nu} = -T_{\mu\nu}\delta g^{\mu\nu}. \quad (2.86)$$

So δS can be simplified to

$$\begin{aligned} \delta S &= \frac{1}{2c} \int T_{\mu\nu} \delta g^{\mu\nu} \sqrt{-g} d\Omega \\ &= -\frac{1}{2c} \int T^{\mu\nu} \delta g_{\mu\nu} \sqrt{-g} d\Omega. \end{aligned} \quad (2.87)$$

We then substitute $\xi^{\mu;\nu} + \xi^{\nu;\mu}$ for $\delta g^{\mu\nu}$ and make use of the symmetry of the tensor $T_{\mu\nu}$:

$$\begin{aligned} \delta S &= \frac{1}{2c} \int T_{\mu\nu} (\xi^{\mu;\nu} + \xi^{\nu;\mu}) \sqrt{-g} d\Omega \\ &= \frac{1}{c} \int T_{\mu\nu} \xi^{\mu;\nu} \sqrt{-g} d\Omega. \end{aligned} \quad (2.88)$$

To get to the equation (2.52), we have to write this equation in the mixed form of the tensor

$T_{\mu\nu}$:

$$\delta S = \frac{1}{c} \int (T_{\mu}^{\nu} \xi^{\mu})_{;\nu} \sqrt{-g} d\Omega - \frac{1}{c} \int T_{\mu;\nu}^{\nu} \xi^{\mu} \sqrt{-g} d\Omega. \quad (2.89)$$

One of the relations for the Christoffel symbols is

$$\Gamma_{\nu\mu}^{\mu} = \frac{1}{2g} \frac{\partial g}{\partial x^{\nu}} = \frac{\partial \ln \sqrt{-g}}{\partial x^{\nu}}. \quad (2.90)$$

This is coming from the fact that

$$\Gamma_{\mu\nu}^{\sigma} = \frac{1}{2} g^{\sigma\lambda} \left(\frac{\partial g_{\lambda\mu}}{\partial x^{\nu}} + \frac{\partial g_{\lambda\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} \right) \quad (2.91)$$

and so

$$\Gamma_{\mu\nu}^{\nu} = \frac{1}{2} g^{\nu\lambda} \frac{\partial g_{\nu\lambda}}{\partial x^{\mu}}. \quad (2.92)$$

On the other hand, the differential dg of the determinant g made up from the components of the metric tensor $g_{\mu\nu}$ is

$$\begin{aligned} dg &= gg^{\mu\nu} dg_{\mu\nu} \\ &= -gg_{\mu\nu} dg^{\mu\nu}. \end{aligned} \quad (2.93)$$

These two equations give us relation (2.90). From equation (2.20) and the explanation below it, we find the divergence of an arbitrary vector A^μ in curvilinear coordinates as

$$\begin{aligned} A^\mu_{;\mu} &= \frac{\partial A^\mu}{\partial x^\mu} + \Gamma^\mu_{\nu\mu} A^\nu \\ &= \frac{\partial A^\mu}{\partial x^\mu} + A^\nu \frac{\partial \ln \sqrt{-g}}{\partial x^\nu}. \end{aligned} \quad (2.94)$$

Simplifying this equation gives

$$A^\mu_{;\mu} = \frac{1}{\sqrt{-g}} \frac{\partial(\sqrt{-g} A^\mu)}{\partial x^\mu}. \quad (2.95)$$

Using this equation, the first integral of the equation (2.89) can be written as

$$\frac{1}{c} \int (T^\nu_\mu \xi^\mu)_{;\nu} \sqrt{-g} d\Omega = \frac{1}{c} \int \frac{\partial}{\partial x^\nu} (T^\nu_\mu \xi^\mu \sqrt{-g}) d\Omega. \quad (2.96)$$

This integral can be transformed into an integral over hypersurface, by means of the Gauss' theorem:

$$\frac{1}{c} \int \frac{\partial}{\partial x^\nu} (T^\nu_\mu \xi^\mu \sqrt{-g}) d\Omega = \frac{1}{c} \oint T^\nu_\mu \xi^\mu \sqrt{-g} dS_\nu = 0. \quad (2.97)$$

This integral is zero since the ξ^μ vanish at the limits of integration. On the other hand, according to the principle of least action δS is zero:

$$\delta S = -\frac{1}{c} \int T^\nu_{\mu;\nu} \xi^\mu \sqrt{-g} d\Omega = 0. \quad (2.98)$$

Since ξ^μ is arbitrary, it follows that

$$T_{\mu;\nu}^\nu = 0 \quad (2.99)$$

which is conservation of energy and momentum. In fact, this equation turns back to the conservation law (2.67), which is valid in flat metrics.

2.5 Derivation of Einstein's equations

To derive the Einstein's equations of general relativity, we can use the principle of least action (equation 2.37). First we need to write the action of the gravitational field S_g :

$$S_g = -\frac{c^3}{16\pi G} \int R\sqrt{-g}d\Omega. \quad (2.100)$$

Here R is the scalar curvature and the constant $-\frac{c^3}{16\pi G}$ has been chosen such that the non-relativistic limit yields the usual form of Newton's gravity law. Calculating the variation δS_g we have

$$\begin{aligned} \delta S_g &= -\frac{c^3}{16\pi G} \delta \int R\sqrt{-g}d\Omega \\ &= -\frac{c^3}{16\pi G} \delta \int g^{\mu\nu} R_{\mu\nu} \sqrt{-g}d\Omega \\ &= -\frac{c^3}{16\pi G} \int \{R_{\mu\nu} \sqrt{-g} \delta g^{\mu\nu} + R_{\mu\nu} g^{\mu\nu} \delta \sqrt{-g} + g^{\mu\nu} \sqrt{-g} \delta R_{\mu\nu}\} d\Omega. \end{aligned} \quad (2.101)$$

For the differential dg of the determinant g we have

$$\begin{aligned} dg &= gg^{\mu\nu} dg_{\mu\nu} \\ &= -gg_{\mu\nu} dg^{\mu\nu} \end{aligned} \quad (2.102)$$

and so

$$\begin{aligned}\delta\sqrt{-g} &= -\frac{1}{2\sqrt{-g}}\delta g \\ &= -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}.\end{aligned}\quad (2.103)$$

We substitute this in equation (2.101) and then we can simplify δS_g to

$$\delta S_g = -\frac{c^3}{16\pi G} \int (R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)\delta g^{\mu\nu} \sqrt{-g}d\Omega - \frac{c^3}{16\pi G} \int g^{\mu\nu} \delta R_{\mu\nu} \sqrt{-g}d\Omega. \quad (2.104)$$

To calculate $\delta R_{\mu\nu}$ we use a locally inertial system of coordinates. In such frame of reference

$\Gamma_{\nu\sigma}^{\mu} = 0$ as the covariant derivative is equal to the usual derivative. We make use of the

equation (2.40) and write $\delta R_{\mu\nu}$ as

$$\delta R_{\mu\nu} = \frac{\partial}{\partial x^{\rho}}\delta\Gamma_{\mu\nu}^{\rho} - \frac{\partial}{\partial x^{\nu}}\delta\Gamma_{\mu\rho}^{\rho} + \Gamma_{\mu\nu}^{\rho}\delta\Gamma_{\rho\sigma}^{\sigma} + \Gamma_{\rho\sigma}^{\sigma}\delta\Gamma_{\mu\nu}^{\rho} - \Gamma_{\mu\rho}^{\sigma}\delta\Gamma_{\nu\sigma}^{\rho} - \Gamma_{\nu\sigma}^{\rho}\delta\Gamma_{\mu\rho}^{\sigma}. \quad (2.105)$$

As $\Gamma_{\nu\sigma}^{\mu} = 0$ for any arbitrary μ, ν and σ , then in equation above all the terms except the

first two are zero. Then we have

$$\begin{aligned}g^{\mu\nu}\delta R_{\mu\nu} &= g^{\mu\nu} \left\{ \frac{\partial}{\partial x^{\rho}}\delta\Gamma_{\mu\nu}^{\rho} - \frac{\partial}{\partial x^{\nu}}\delta\Gamma_{\mu\rho}^{\rho} \right\} \\ &= g^{\mu\nu} \frac{\partial}{\partial x^{\rho}}\delta\Gamma_{\mu\nu}^{\rho} - g^{\mu\rho} \frac{\partial}{\partial x^{\rho}}\delta\Gamma_{\mu\nu}^{\nu}.\end{aligned}\quad (2.106)$$

We define w^{ρ} as

$$w^{\rho} = g^{\mu\nu}\delta\Gamma_{\mu\nu}^{\rho} - g^{\mu\rho}\delta\Gamma_{\mu\nu}^{\nu}. \quad (2.107)$$

So equation (2.106) can be simplified to this form:

$$g^{\mu\nu}\delta R_{\mu\nu} = \frac{\partial w^{\rho}}{\partial x^{\rho}}. \quad (2.108)$$

We actually used our assumption ($\Gamma_{\nu\sigma}^{\mu} = 0$) and considered the fact that the first derivatives

of $g^{\mu\nu}$ are zero due to their relations:

$$\frac{\partial g_{\mu\nu}}{\partial x^{\rho}} = \Gamma_{\nu,\mu\rho} + \Gamma_{\mu,\nu\rho}. \quad (2.109)$$

We then replace $\partial w^\rho / \partial x^\rho$ by $w^\rho_{;\rho}$ and use the generalized divergence in curvilinear coordinates (equation 2.95). The result is

$$g^{\mu\nu} \delta R_{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\rho} (\sqrt{-g} w^\rho) \quad (2.110)$$

or better to write

$$\sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} = \frac{\partial}{\partial x^\rho} (\sqrt{-g} w^\rho). \quad (2.111)$$

Now we can get back to the equation (2.104) and calculate the second integral containing $g^{\mu\nu} \delta R_{\mu\nu}$ in it. To do that we make use of equation (2.111) and write the integral as:

$$\int g^{\mu\nu} \delta R_{\mu\nu} \sqrt{-g} d\Omega = \int \frac{\partial (\sqrt{-g} w^\rho)}{\partial x^\rho} d\Omega. \quad (2.112)$$

By means of Gauss' theorem, the integral on the right hand side can be transformed into an integral of w^ρ over the hypersurface surrounding the whole four-volume

$$\int \frac{\partial (\sqrt{-g} w^\rho)}{\partial x^\rho} d\Omega = \oint \sqrt{-g} w^\rho dS_\rho. \quad (2.113)$$

This integral is equal to zero as the variations of the field are zero at the integration limits.

So the total variation of S_g is equal to

$$\delta S_g = -\frac{c^3}{16\pi G} \int (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) \delta g^{\mu\nu} \sqrt{-g} d\Omega. \quad (2.114)$$

From the principle of least action $\delta S_m + \delta S_g = 0$, where δS_m is given by equation (2.87):

$$-\frac{c^3}{16\pi G} \int \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{8\pi G}{c^4} T_{\mu\nu} \right) \delta g^{\mu\nu} \sqrt{-g} d\Omega = 0. \quad (2.115)$$

Because of the arbitrariness of $\delta g^{\mu\nu}$, that is the integrant which is zero,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}. \quad (2.116)$$

These are the Einstein's equations of general relativity. The Einstein's equations are non-linear and so the principle of superposition is not valid for gravitational field except in some approximations.

Chapter 3

Black hole solutions

Here we will give a brief review of different black hole solutions in general relativity. By solutions of Einstein's equations we mean metrics that produce $R_{\mu\nu}$ in a way that they satisfy equations (2.35). The solutions are provided in the usual 4-dimensional spacetime. However, there are some theories which predict the possibility of producing and observing black holes in colliders in case of the existence of higher dimensions. So, at the end of this chapter we will discuss higher dimensions, mostly following references [2, 3, 19].

3.1 A spherically symmetric solution

The most general spherically symmetric expression for ds^2 in spherical polar coordinates is

$$ds^2 = h(r, t)dr^2 + k(r, t)(\sin^2 \theta d\phi^2 + d\theta^2) + l(r, t)dt^2 + a(r, t)drdt \quad (3.1)$$

where a , h , k , l are functions of radius r and time t . The choice of the reference frame in general relativity is arbitrary. Thus, one can subject the coordinates to any transformation

which of course does not destroy the spherical symmetry of ds^2 . We use a frame in which

$$\begin{aligned} a(r, t) &= 0 \\ k(r, t) &= -r^2. \end{aligned} \tag{3.2}$$

The latter condition defines r in such a way that $2\pi r$ gives the circumference of a circle in the plane $\theta = \pi/2$ with the origin at the center of coordinates (here dl is the spatial length element)

$$\int_{circle} dl = r \int_{circle} d\phi = 2\pi r. \tag{3.3}$$

For the resulting metric one is able to calculate $R_{\mu\nu}$. Setting the energy-momentum tensor to zero, we end up with the unique *Schwarzschild solution* of Einstein equations as follows:

$$ds^2 = \left(1 - \frac{r_g}{r}\right) c^2 dt^2 - r^2(\sin^2 \theta d\phi^2 + d\theta^2) - \frac{dr^2}{1 - \frac{r_g}{r}}. \tag{3.4}$$

The quantity r_g is called *gravitational radius* or *Schwarzschild radius* and is defined by

$$r_g = \frac{2Gm}{c^2}. \tag{3.5}$$

There is a singularity at $r = r_g$. However, this is a coordinate singularity, which can be transformed away by switching to Kruskal coordinates or free-fall coordinates. A coordinate singularity is a singularity that only appears in certain choices of coordinate. A true singularity can not be removed by coordinate transformations. According to the cosmic censorship hypothesis, every singularity is hidden behind a horizon and cannot be probed. This horizon is a surface surrounding the mass of the black hole. The final position of the event horizon occurs at the black hole gravitational radius once all the mass of the collapsing star is located inside r_g . The topology of the horizon of a non-spinning black hole (like

Schwarzschild solution) is a sphere. The escape velocity at the horizon is equal to the speed of light.

Assume some idealized stars collapsing into black holes. One such star is a spherically-symmetric ball of dust (i.e. zero pressure fluid). There is a theorem which states that the metric outside the star is the Schwarzschild metric. This theorem, (known as Birkhoff's theorem [20]), says that any spherically symmetric vacuum solution is static, which effectively implies that it must be Schwarzschild. This solution is unique and it represents the changes of the gravitational field produced by any spherically-symmetric distribution of masses. The Schwarzschild solution is not valid any more if this spherical symmetry is broken.

We should point out the geometrical meaning of the coordinate r here. The circumference of a circle with its center at the center of the field is $2\pi r$. However, if one integrates over the square root of the spatial part of the Schwarzschild metric and between two points r_1 and r_2 along the same radius, the result would be greater than the value $\Delta r = r_2 - r_1$. So the physical length is more than Δr .

For the limit of large distances from the origin ($r \gg r_g$), we can also approximate ds^2 as

$$ds^2 \simeq ds_0^2 - \frac{2Gm}{c^2 r} (dr^2 + c^2 dt^2). \quad (3.6)$$

The first term(ds_0^2) is the minkowskian metric and the remaining comprises the small cor-

rections. As mentioned before, the limit is the limit of $g_{\mu\nu}$ when $r \rightarrow \infty$, which is

$$\begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (3.7)$$

In fact, equation (3.6) turns out to be equal to ds_0 , when $r \rightarrow \infty$. That is the asymptotic flatness of the Schwarzschild metric. Defining

$$r = \left(1 + \frac{r_g}{4\rho}\right)^2 \rho \quad (3.8)$$

we can change the form of the solution (3.4) into *isotropic spherical coordinates* of ρ , θ , and ϕ :

$$ds^2 = \left(\frac{1 - \frac{r_g}{4\rho}}{1 + \frac{r_g}{4\rho}}\right)^2 c^2 dt^2 - \left(1 + \frac{r_g}{4\rho}\right)^4 (d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2 \theta d\phi^2). \quad (3.9)$$

One can also introduce the isotropic Cartesian coordinates x , y , z as an approximation for large distances, i.e. $\rho \gg r_g$. We expand the terms in the equation (3.9) as

$$\begin{aligned} \left(\frac{1 - \frac{r_g}{4\rho}}{1 + \frac{r_g}{4\rho}}\right)^2 &= \left[\left(1 - \frac{r_g}{4\rho}\right) \left(1 + \frac{r_g}{4\rho}\right)^{-1}\right]^2 \\ &\simeq \left(1 - \frac{r_g}{2\rho} + \dots\right)^2 \\ &\simeq 1 - \frac{r_g}{\rho} + \dots \end{aligned} \quad (3.10)$$

and

$$\left(1 + \frac{r_g}{4\rho}\right)^4 \simeq 1 + \frac{r_g}{\rho}. \quad (3.11)$$

So these isotropic coordinates turns out to be

$$ds^2 = \left(1 - \frac{r_g}{\rho}\right) c^2 dt^2 - \left(1 + \frac{r_g}{\rho}\right) (dx^2 + dy^2 + dz^2). \quad (3.12)$$

Here, the volume element of the Cartesian coordinates is

$$\begin{aligned} dx dy dz &= (d\rho) \times (\rho d\theta) \times (\rho \sin(\theta) d\phi) \\ &= \rho^2 \sin(\theta) d\rho d\theta d\phi. \end{aligned} \quad (3.13)$$

At the end of this section, we introduce Penrose diagrams. In order to visualize the causal properties of a space-time, Penrose diagrams can be employed. For a Schwarzschild black hole, the Penrose diagram is illustrated in figure (3.1). $r = 0$ corresponds to the singularity. In addition to the normal horizon, through which light rays can fall, there appears to be a second horizon, a “past” horizon or antihorizon. In the Schwarzschild coordinate systems, this antihorizon existed only in the infinite past.

3.2 Rotating black holes

There is a solution of the Einstein’s equations in which the black hole rotates and it is charged as well:

$$ds^2 = \left(\frac{\Delta - a^2 \sin^2 \theta}{\rho^2} \right) c^2 dt^2 + 2\omega \varpi^2 dt d\phi - \varpi^2 d\phi^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2. \quad (3.14)$$

This is called the *Newman* or *Kerr-Newman solution* discovered by Newman et al in 1965. Here, we used the Boyer-Lindquist coordinates and we define the quantities applied in our metric as below:

$$\begin{aligned} \Delta &= a^2 + r^2 \alpha \\ \rho &= (r^2 + a^2 \cos^2 \theta)^{\frac{1}{2}} \end{aligned} \quad (3.15)$$

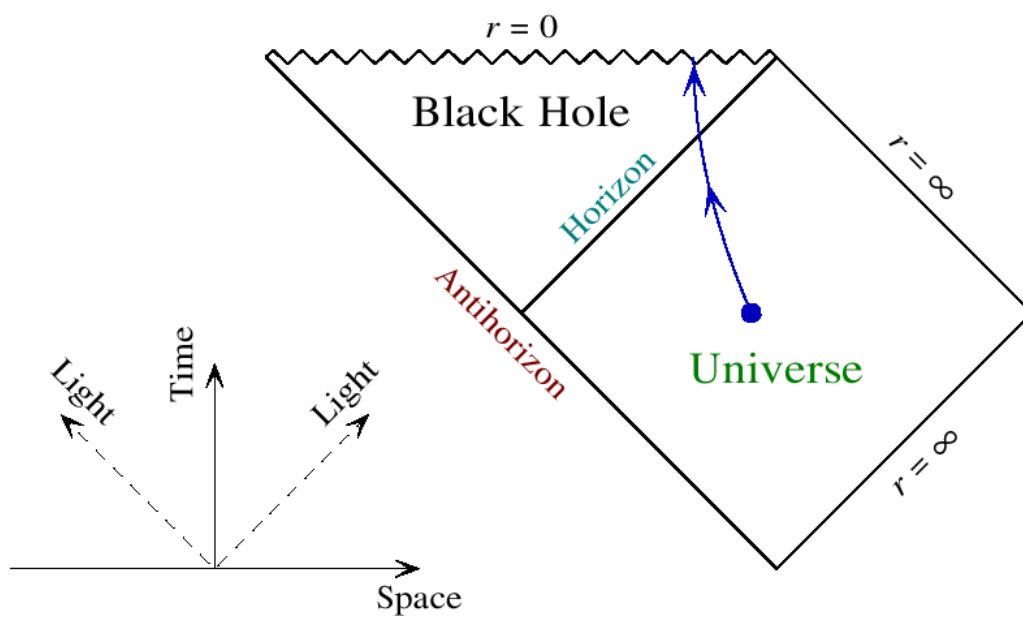


Figure 3.1: Penrose diagram of the Schwarzschild Black Hole.

along with the following definitions

$$\begin{aligned}\Sigma &= \sqrt{(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta} \\ \varpi &= \frac{\Sigma}{\rho} \sin \theta \\ \omega &= \frac{a(r^2 + a^2 - \Delta)}{\Sigma^2} c.\end{aligned}\tag{3.16}$$

We also require the definition of α :

$$\alpha = 1 - \frac{r_g}{r} + \frac{e^2}{r^2}\tag{3.17}$$

with the specific charge e as

$$e = \frac{\sqrt{k_e G}}{c^2} q.\tag{3.18}$$

The specific charge e is actually defined to have the same scale as r , which is length. The quantities a and r_g are defined in equations (3.25) and (3.5) respectively. k_e is the constant in Coulomb's law of electrostatics

$$k_e = \frac{1}{4\pi\epsilon_0}\tag{3.19}$$

where ϵ_0 is the permittivity constant for vacuum equal to

$$\epsilon_0 = 8.854 \times 10^{-12} \text{ F/m}.\tag{3.20}$$

Newman's result represents the most general stationary, axisymmetric solution of the Einstein's equations in the presence of an electromagnetic field in four dimensions.

3.2.1 Kerr solution

An axially (cylindrically) symmetric stationary solution of Einstein equations, known as the Kerr metric, represents the gravitational field of the uncharged rotating black

hole. This solution can be written in various coordinates [25]. Here, we use the Boyer-Linquist coordinates, which is the notation of the references [8] and [21]. The metric for such black hole is:

$$\begin{aligned}
 ds^2 &= \left(1 - \frac{r_g r}{\rho^2}\right) c^2 dt^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 \\
 &- \left(r^2 + a^2 + \frac{r_g r a^2}{\rho^2} \sin^2 \theta\right) \sin^2 \theta d\phi^2 + \frac{2cr_g r a}{\rho^2} \sin^2 \theta d\phi dt
 \end{aligned} \tag{3.21}$$

with

$$\begin{aligned}
 \Delta &= r^2 - r_g r + a^2 \\
 \rho^2 &= r^2 + a^2 \cos^2 \theta
 \end{aligned} \tag{3.22}$$

and r_g given by the equation (3.5). This is a special case of equation (3.14).

The Schwarzschild solution is a spherical symmetric metric. We want to have a slow rotation and so we add a weak disturbance to this spherical symmetry. To get this perturbative term, we consider the total angular momentum of the body J as the basic term required to show rotation. We also pay attention to the fact that this perturbation is not supposed to increase without limit as $r \rightarrow r_g$. This perturbation can be added to the Schwarzschild metric tensor as a small off-diagonal component. This component is [8]:

$$g_{03} = \frac{2GJ}{rc^2} \sin^2 \theta \tag{3.23}$$

with J as angular momentum. Now if we consider the r^{-1} order of the Kerr metric tensor components,

$$\begin{aligned}
 g_{00} &\approx 1 - \frac{r_g}{r} \\
 g_{03} &\approx \frac{r_g a c}{r} \sin^2 \theta
 \end{aligned} \tag{3.24}$$

we see that the first one is the same as g_{00} of the Schwarzschild solution. The second one is exactly presenting the equation (3.23), if the constants m and a are related to the angular momentum J by

$$J = mac. \quad (3.25)$$

While in Schwarzschild metric the conditions

$$g_{00} \rightarrow 0 \quad (3.26)$$

and

$$g_{11} \rightarrow -\infty \quad (3.27)$$

occur simultaneously and define the horizon surface

$$r = r_g, \quad (3.28)$$

however, this is not the case for the Kerr metric. For equation (3.21), the former condition happens when

$$\rho^2 = rr_g \quad (3.29)$$

and the latter holds when

$$\Delta = 0. \quad (3.30)$$

The larger of the two roots of each quadratic equation would be

$$r_0 = \frac{r_g}{2} + \sqrt{\left(\frac{r_g}{2}\right)^2 - a^2 \cos^2 \theta} \quad (3.31)$$

and

$$r_h = \frac{r_g}{2} + \sqrt{\left(\frac{r_g}{2}\right)^2 - a^2} \quad (3.32)$$

respectively.

The Kerr metric is not meaningful for $a > r_g/2$ which defines the upper limits for a and J (extremity bound):

$$\begin{aligned} a_{max} &= \frac{r_g}{2} \\ J_{max} &= \frac{mcr_g}{2}. \end{aligned} \quad (3.33)$$

If the black hole spins so fast that this constraint ($a \leq r_g/2$) is violated, the centrifugal forces, will exceed the gravitational forces and stop the collapse. So we can find the corresponding limiting values of the radii of the surfaces $r = r_0$ and $r = r_h$ as

$$\begin{aligned} r_0 &= \frac{r_g}{2}(1 + \sin \theta) \\ r_h &= \frac{r_g}{2}. \end{aligned} \quad (3.34)$$

The surface $r = r_h$ is the event horizon and it is a null hypersurface. So the world lines of particles or light rays can cross the hypersurface in only one direction and this direction is towards the interior of the horizon.

A null hypersurface is a hypersurface whose normal at every point is a null vector. If $f(x^\mu) = \text{const}$ defines a hypersurface, then its normal n_μ , which is directed along the four-gradient, can be written as

$$n_\mu = \frac{\partial f}{\partial x^\mu}. \quad (3.35)$$

So for a null hypersurface we have

$$n^\mu n_\mu = 0. \quad (3.36)$$

In other words, the direction of the normal lies in the surface itself. So the normal vector

is along the hypersurface

$$df = n_\mu dx^\mu = 0. \quad (3.37)$$

The element of length on the hypersurface in the same direction is then zero ($ds = 0$). So along this direction and at the given point, the hypersurface is tangent to the light cone constructed on the point. This happens to all the light cones constructed at each point of a null hypersurface and all of them lie entirely on one side of it. This is equivalent to the statement that this null hypersurface is a unidirectional passage for the world lines of particles or light rays. The condition of a null hypersurface (Eq 3.35) for the hypersurface of the form $f(r, \theta) = \text{const}$ in the Kerr field turns out to be

$$g^{11} \left(\frac{\partial f}{\partial r} \right)^2 + g^{22} \left(\frac{\partial f}{\partial \theta} \right)^2 = \frac{1}{\rho^2} \left[\Delta \left(\frac{\partial f}{\partial r} \right)^2 + \left(\frac{\partial f}{\partial \theta} \right)^2 \right] = 0. \quad (3.38)$$

This equation is only satisfied on the surface of horizon, where $\frac{\partial f}{\partial \theta} = 0$ and $\Delta = 0$. The surface $r = r_h$ is a sphere contained inside the other surface $r = r_0$ and touching that at the poles ($\theta = 0$ and $\theta = \pi$). The space between the horizon and the surface $r = r_0$ is called *ergosphere* and particles in this region must necessarily rotate around the axis of symmetry of the gravitational field ($\phi \neq \text{const} \quad \forall \phi$). That is different from the inside the Schwarzschild horizon in which all the particles should move radially towards the center: $r = \text{const}$ is not possible for particle motion. In fact in the ergosphere of the Kerr black hole, no particle can remain at rest relative to the reference frame of a distant observer. In the case that r , θ or ϕ are constants, the spacetime interval is not timelike, i.e. $ds^2 < 0$, while it should be timelike for the world line of a particle. However, $r = \text{const}$ is allowed for the ergosphere of the Kerr black hole and particles can move back and forth in radial direction.

In Kerr metric, there is a ring-shaped curvature singularity in the $z = 0$ plane with $\theta = \frac{\pi}{2}$. If we calculate the element of the spatial distance for outside of the ergosphere using the formula

$$dl^2 = \left(-g_{ij} + \frac{g_{0i}g_{0j}}{g_{00}} \right) dx^i dx^j, \quad (3.39)$$

we see that there exists a singularity at $r = r_0$:

$$dl^2 = \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\Delta \sin^2 \theta}{1 - \frac{rr_g}{\rho^2}} d\phi^2, \quad (3.40)$$

while the surface $r = r_0$ itself is not singular. That means while the surface of the ergosphere is not singular in the spacetime coordinates, it does have a singularity there in the purely spatial metric. Near this surface we have

$$g_{00} \rightarrow 0 \quad (3.41)$$

and so the difference in clocks

$$\Delta x^0 = - \oint \frac{g_{0i} dx^i}{g_{00}} \quad (3.42)$$

goes to infinity when they are synchronized along this closed contour. Equation (3.42) is valid from the fact that

$$\Delta x^0 = - \frac{g_{0i} dx^i}{g_{00}} \quad (3.43)$$

gives the difference in the values of the time x^0 for two simultaneous events occurring at infinitely near points.

When there is no gravitational mass, i.e. $m = 0$ (or $r_g = 0$), we should recover the Minkowski metric. In such case the metric turns out to be

$$ds^2 = c^2 dt^2 - \frac{\rho^2}{r^2 + a^2} dr^2 - \rho^2 d\theta^2 - (r^2 + a^2) \sin^2 \theta d\phi^2. \quad (3.44)$$

In fact if we consider it in the *oblate spheroidal coordinates* and use the transformations

$$\begin{aligned}x &= \sqrt{r^2 + a^2} \sin \theta \cos \phi \\y &= \sqrt{r^2 + a^2} \sin \theta \sin \phi \\z &= r \cos \theta\end{aligned}\tag{3.45}$$

we easily get back to the cartesian coordinates:

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2.\tag{3.46}$$

Equation (3.25) shows that when $m = 0$, then the quantity a is not necessarily zero. Our discussion here is consistent with the extremity bound (3.33) as the Kerr metric is no longer meaningful here. The Lorentz metric we got is representing a flat spacetime. It means that in addition to having an energy-momentum equal to zero,

$$T_{\mu\nu} = 0\tag{3.47}$$

or

$$R_{\mu\nu} = 0.\tag{3.48}$$

3.2.2 Reissner-Nordstrom solution

A Reissner-Nordstrom black hole [22, 23] is a special case of Kerr-Newman, which is a charged but non-rotating hole. That means if we put $J = 0$ in the Kerr-Newman metric, then we must get the Reissner-Nordstrom black hole which is expressed by:

$$ds^2 = \alpha c^2 dt^2 - \alpha^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2\tag{3.49}$$

with α to be the same as the equation (3.17). Metric is singular when $\alpha = 0$. So the roots of the equation $\alpha = 0$ give us (two) horizons for this charged black hole:

$$\begin{aligned}
 r_h &= r_{\pm} \\
 &= \frac{r_g \pm \sqrt{r_g^2 - 4e^2}}{2} \\
 &= \frac{1}{c^2} \left[Gm \pm (G^2 m^2 - k_e Gq^2)^{\frac{1}{2}} \right]
 \end{aligned} \tag{3.50}$$

with the condition

$$Gm^2 > k_e q^2. \tag{3.51}$$

The condition above gives a real solution for the horizons. However, if we assume the other case, that is

$$Gm^2 < k_e q^2, \tag{3.52}$$

then r_h has no real roots. So there is no horizon and the singularity at $r = 0$ is naked. This case is similar to $M < 0$ Schwarzschild. According to the cosmic censorship hypothesis this case could not occur in gravitational collapse [27, 28].

What we already have done in this chapter was a brief review of rotating and charged black hole solutions of Einstein's equations. However, all these solutions were in a 4-dimensional spacetime. In next section, we try to answer to this question that why study of higher dimensions is important for black hole physics. In fact higher dimensions open new windows to black hole physics and experimental high energy physics.

3.3 Higher dimensions?

Above the electroweak scale of the order of 10^2 GeV, electromagnetism and the weak nuclear force merge into a single electroweak force. The electroweak interactions have been probed at distances (the wavelength related to the energy above)

$$\lambda_{EW} \sim 10^{-18} \text{ m}, \quad (3.53)$$

while inverse square law of gravity has only been accurately measured in the $\sim 10^{-4}$ m range (or to be more accurate, reported $218 \mu\text{m}$) [44]. However, what are of significance when one would like to study the effects of quantum mechanics and gravity together are the Planck length

$$\ell_{Pl} \sim 10^{-35} \text{ m} \quad (3.54)$$

and the Planck mass

$$\begin{aligned} M_{Pl} &= 1.2209 \times 10^{19} \frac{\text{GeV}}{c^2} \\ &= 2.176 \times 10^{-8} \text{ kg}. \end{aligned} \quad (3.55)$$

The Planck mass is the mass of a black hole whose Schwarzschild radius multiplied by π equals its Compton wavelength. The radius of such a black hole is, roughly (because of a factor of 2), the Planck length. The Compton wavelength of an object of mass m is

$$\lambda = 2\pi \frac{\hbar}{mc}. \quad (3.56)$$

The Schwarzschild radius r_g is also given by equation (3.5). Then the Planck mass M_{Pl} can be found by solving the relation $\pi r_g = \lambda$ for mass, which turns out to be

$$M_{Pl} = \sqrt{\frac{\hbar c}{G}}. \quad (3.57)$$

If we put this Planck mass in equation (3.5) and neglect a factor of 2, then we get the Planck length, which is

$$\ell_{Pl} = \sqrt{\frac{\hbar G}{c^3}}. \quad (3.58)$$

The gravitational forces have not been probed at the Planck length and so such scale is of interest considering the *hierarchy problem*, i.e. the huge difference between the Planck scale M_{Pl} and the electroweak scale m_{EW} ,

$$m_{EW} = 246 \frac{GeV}{c^2}. \quad (3.59)$$

The first idea about extra dimensions dates back to at least the 1920s. At that time, Kaluza and Klein assumed that a curled-up fifth dimension is attached to the usual four-dimensional spacetime. Actually the aim was to unite the forces of electromagnetism and gravity. In their theory, they extended general relativity to a five-dimensional spacetime. The resulting equations can be separated out into three sets of equations, one of which is equivalent to Einstein field equations, another set equivalent to Maxwell's equations for the electromagnetic field and the final part an extra scalar field. We think that the four forces, (electromagnetic, weak, strong, and gravitational) were joined as a single superforce at the time of the Big Bang. In theory, the forces could be unified only if they were about the same strength under conditions of high energy. However, gravity is much weaker than the others.

Higher dimensions open new windows of solving the hierarchy problem. In higher dimensional theories, the approach is to look for a way to make the gravitational force comparable in strength to the other forces at an energy of about 1 TeV. So if the higher dimensions exist, then we can have an explanation for the hierarchy of energy scales. Forces

also may be unified at a certain accessible energy for the future colliders.

To formulate the idea of higher dimensions, one way is to use a brane world. Branes or p -branes are spatially extended objects and the variable p refers to the dimension of the brane. That is, a 0-brane is a zero-dimensional particle, a 1-brane is a string, a 2-brane is a “membrane”, etc. Every p -brane sweeps out a $(p + 1)$ -dimensional world-volume as it propagates through spacetime. In a brane-world, our four-dimensional universe is entirely restricted to a brane inside a higher-dimensional space, called the bulk. The additional dimensions may be taken to be compact. In the bulk model, other branes may be moving through this bulk. Interactions with the bulk, and possibly with other branes, can influence our brane. In the brane picture, electromagnetism and the weak and strong nuclear forces are localized on the brane, but gravity has no such constraint and can leak into the bulk.

The assumption of existence of n compact extra dimensions of length $\sim R$ can solve the hierarchy problem if gravity is modified at scales smaller than 1 mm [2, 3]. In this proposed scenario we assume a brane-world in which the Standard Model matter and gauge degrees of freedom reside on a 3-brane within a flat compact space of volume V_{D-4} . Gravity propagates in both the compact and non-compact dimensions. We refer to this as the “flat” or “ADD scenario” on behalf of the authors of the paper [3]: Arkani-Hamed, Dimopoulos, and Dvali.

One can easily apply the Gauss’s law in $D = 4 + n$ dimensions and find the gravitational potential energy that affects two masses m_1 and m_2 at a distance $r \ll R$ as

$$V(r) = -G_D \frac{m_1 m_2}{r^{n+1}} \quad (3.60)$$

with

$$r \ll R. \quad (3.61)$$

G_D is the D -dimensional Newton constant.

At far distances one can get back to the four-dimensional Newton potential. In such case we have

$$r^n \sim R^n. \quad (3.62)$$

The Newton's constants are defined by the Newton's force laws:

$$\begin{aligned} F_D(r) &= G_D \frac{m_1 m_2}{r^{n+2}}, \\ F_4(r) &= G_4 \frac{m_1 m_2}{r^2}. \end{aligned} \quad (3.63)$$

Again, by means of applying the $(4 + n)$ -dimensional Gauss's law we find the relation between the four-dimensional Newton's constant G_4 and D -dimensional Newton's constant G_D :

$$G_4 = \frac{S_{n+3}}{4\pi} \frac{G_D}{R^n} \quad (3.64)$$

with

$$V_{D-4} = R^n. \quad (3.65)$$

V_{D-4} is the volume of the extra dimensions. S_D is the surface area of the unit sphere in D spatial dimensions :

$$S_D = \frac{2\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2})}, \quad (3.66)$$

where $\Gamma(z)$ is Euler's Gamma function. These relations result from the Einstein action (equation 2.100) as well. We can also find the D -dimensional Plank mass, $M_{Pl,D}$, by equating the Schwarzschild radius of an object of mass m to its Compton wavelength ($\lambda = 2\pi \frac{\hbar}{mc}$),

and then we have

$$(M_{Pl,D})^{D-2} \sim \left(\frac{2\pi\hbar}{c} \right)^{n+1} \frac{c^2}{G_D}. \quad (3.67)$$

So we can relate $M_{Pl,D}$ to the effective four-dimensional Planck mass M_{Pl} by

$$M_{Pl}^2 \sim (M_{Pl,D})^{D-2} \left(\frac{cR}{2\pi\hbar} \right)^n. \quad (3.68)$$

If we impose the equal scales in our higher dimensional theory, which is

$$m_{EW} = M_{Pl,D}, \quad (3.69)$$

the hierarchy problem is solved. This condition relates the size R of the extra dimensions to the number of extra dimensions n as

$$R \sim \frac{2\pi\hbar}{c m_{EW}} \left(\frac{M_{Pl}}{m_{EW}} \right)^{\frac{2}{n}}. \quad (3.70)$$

For $n = 1$, i.e. when we have only one extra dimension, the deviations from Newtonian gravity would be over solar system distances. Pluto is roughly 38 AU from the Sun (1 AU = 149598000 km). So $n = 1$ is obviously excluded. The size of the extra dimension for $n = 2$ is $R \sim 0.3$ mm. Related experiments with torsion pendulums can be found in [44]. The Newtonian gravity would be modified at distances smaller than those currently probed by experiment, if n is greater than or equal to 3.

Another scenario, called “warped” or “RS scenario” (on behalf of Randall and Sundrum), arises from properties of warped extra-dimensional geometries [4, 5]. The weak scale is generated from a large scale of order the Planck scale through an exponential hierarchy. However, this exponential arises not from gauge interactions but from the background metric (which is a slice of AdS_5 spacetime). This mechanism relies on the existence of only a single additional dimension.

A metric showing warped geometry has the form below:

$$ds^2 = e^{2A(y)} dx_4^2 + g_{mn}(y) dy^m dy^n \quad (3.71)$$

with

$$dx_4^2 = \eta_{\mu\nu} dx^\mu dx^\nu \quad (3.72)$$

as the standard four-dimensional Minkowski line element. The coordinates y parameterize the extra dimensions of spacetime, with metric g_{mn} . Here, the higher dimensional discussion requires some changes to the summation convention introduced in the introduction. The sum on n and m is over the higher spatial indices only. The function e^A is the warp factor, and leads to scales for four-dimensional physics that depend on the location within the extra dimensions. Gravity propagates in both the compact and non-compact directions and the four-dimensional Newton's constant is related to the D -dimensional once again using the Einstein action

$$\frac{1}{G_4} = \frac{1}{G_D} \int d^{D-4}y \sqrt{g} e^{2A}. \quad (3.73)$$

In warped scenario, there is a single 3-brane with positive tension, embedded in a five-dimensional bulk spacetime. We also have another brane at a distance πr_c from the brane of interest (the quantity r_c will appear in the metric later). We take the branes to be the boundaries of a finite fifth dimension. Finally, this second brane is taken to infinity, to be removed from the physical set-up. x^μ represent coordinates for the familiar four dimensions, while $0 \leq \phi \leq \pi$ is the coordinate for an extra dimension, which is a finite

interval whose size is set by r_c . For the boundary conditions we have:

$$\begin{aligned} g_{\mu\nu}^{vis}(x^\mu) &= G_{\mu\nu}(x^\mu, \phi = \pi) \\ g_{\mu\nu}^{hid}(x^\mu) &= G_{\mu\nu}(x^\mu, \phi = 0). \end{aligned} \quad (3.74)$$

$g_{\mu\nu}^{vis}(x^\mu)$ and $g_{\mu\nu}^{hid}(x^\mu)$ are purely four-dimensional components of the bulk metric and $G_{\mu\nu}(x^\mu)$ is the five dimensional metric. The action describing the system above contains three terms :

$$S = S_{gravity} + S_{vis} + S_{hid}. \quad (3.75)$$

The terms are defined as below:

$$\begin{aligned} S_{gravity} &= \int d^4x \int_{-\pi}^{\pi} d\phi \sqrt{-G} \{-\Lambda + 2M^2 R\} \\ S_{vis} &= \int d^4x \sqrt{-g_{vis}} \{\mathcal{L}_{vis} - V_{vis}\} \\ S_{hid} &= \int d^4x \sqrt{-g_{hid}} \{\mathcal{L}_{hid} - V_{hid}\}. \end{aligned} \quad (3.76)$$

Then the Einstein's equation for such action is

$$\begin{aligned} \sqrt{-G} \left(R_{\mu\nu} - \frac{1}{2} G_{\mu\nu} R \right) &= -\frac{1}{4M^3} [\Lambda \sqrt{-G} G_{\mu\nu} + V_{vis} \sqrt{-g_{vis}} g_{\sigma\lambda}^{vis} \delta_\mu^\sigma \delta_\nu^\lambda \delta(\phi - \pi) \\ &+ V_{hid} \sqrt{-g_{hid}} g_{\sigma\lambda}^{hid} \delta_\mu^\sigma \delta_\nu^\lambda \delta(\phi)]. \end{aligned} \quad (3.77)$$

A five-dimensional metric $G_{\mu\nu}$ satisfying this relation has this form :

$$ds^2 = e^{-2kr_c\phi} \eta_{\mu\nu} dx^\mu dx^\nu - r_c^2 d\phi^2, \quad (3.78)$$

where k is a scale of order the Planck scale. This metric is a solution to Einstein's equations in a simple set-up with two 3-branes and appropriate cosmological terms. In the RS scenario, four-dimensional mass scales are related to five-dimensional input mass parameters and the

warp factor, $e^{-2kr_c\phi}$. As the source of the large hierarchy between the observed Planck and weak scales is an exponential function of the compactification radius, we do not require extremely large r_c to generate such large hierarchy.

This warped solution is quite distinct from the flat one:

1. The hierarchy between the compactification scale and fundamental five-dimensional Planck scale,

$$\mu_c \equiv \frac{1}{r_c} \tag{3.79}$$

is different from $(M_{Pl}/TeV)^{2/n}$.

2. There is one additional dimension, as opposed to $n \geq 2$.

From the significant results originating from the physical interpretations of this scenario [4, 5], one is the relation between the effective four-dimensional (reduced) Planck scale, $M_{Pl} = 2 \times 10^{18} \frac{GeV}{c^2}$, and the fundamental $(4+n)$ -dimensional Planck scale, $M_{Pl,n}$. Because the effective field is four-dimensional, we can explicitly perform the ϕ integral to obtain a purely four-dimensional action. From this we have

$$\begin{aligned} M_{Pl}^2 &= M_{Pl,n}^3 r_c \int_{-\pi}^{\pi} d\phi e^{-2kr_c|\phi|} \\ &= \frac{M_{Pl,n}^3}{k} \left[1 - e^{-2kr_c\pi} \right]. \end{aligned} \tag{3.80}$$

This is a very important result. It implies that M_{Pl} depends only weakly on r_c in the large kr_c limit. However, this very small effect of the exponential in determining the Planck scale, would not be so little in the determination of the visible sector masses. In fact it plays a crucial role as one can determine the physical masses by properly normalizing the fields:

$$m \equiv m_0 e^{-kr_c\pi}. \tag{3.81}$$

For a fundamental Higgs field $H(x)$, we have

$$S_{vis} \supset \int d^4x \sqrt{-g_{vis}} \{g_{vis}^{\mu\nu} D_\mu H^\dagger D_\nu H - \lambda(|H|^2 - v_0^2)^2\} \quad (3.82)$$

which has one mass parameter v_0 . Here the index *vis* stand for visible indicating that the action calculated is for the visible area. Substituting the metric we get

$$S_{vis} \supset \int d^4x \sqrt{-\bar{g}} e^{-4kr_c\pi} \{\bar{g}^{\mu\nu} e^{2kr_c\pi} D_\mu H^\dagger D_\nu H - \lambda(|H|^2 - v_0^2)^2\}. \quad (3.83)$$

Then we renormalize the wave function as

$$H \rightarrow e^{kr_c\pi} \quad (3.84)$$

and we get (*eff* stands for effective)

$$S_{eff} \supset \int d^4x \sqrt{-\bar{g}} \{\bar{g}^{\mu\nu} D_\mu H^\dagger D_\nu H - \lambda(|H|^2 - e^{-2kr_c\pi} v_0^2)^2\}. \quad (3.85)$$

Then for the physical mass scales we obtain

$$v \equiv v_0 e^{-kr_c\pi}. \quad (3.86)$$

The equation (3.81) relates any mass parameter m_0 on the visible 3-brane in the fundamental higher-dimensional theory to a physical mass m . Then with the assumption of

$$e^{kr_c\pi} \sim \mathcal{O}(10^{15}), \quad (3.87)$$

this mechanism produces TeV physical mass scales from fundamental mass parameters not far from the Planck scale, 10^{19} GeV. This geometric factor is exponential and so no very large hierarchies among the fundamental parameters, v_0 , k , M , and $\mu_c \equiv 1/r_c$ is required; in fact, we just need $kr_c \approx 11$.

In either scenario (flat or warped) the Planck scale can be in the TeV range. This point is of significance as the Large Hadron Collider (LHC) at CERN would reach this energy (14 TeV center of mass energy). Then one would expect to observe the gravitational effects in quantum mechanical scale [26]. If this is the case, then the description of physics in this regime requires a quantum theory of gravity. Black holes exhibit many features described well by semi-classical physics. However, a quantitative understanding of black holes with masses of the order of the Planck scale is still required.

Chapter 4

Black hole thermodynamics

Classical thermodynamics states as its second principle that entropy is an always increasing function of time in a closed system - and the universe is a closed system, as nothing can escape from it. In terms of statistical mechanics, the entropy describes the number of the possible microscopic configurations of the system. Black holes confront us with a fundamental problem: what happens to the information when a particle falls inside a black hole? Is the information lost there in the black hole? According to what we know from Hawking radiation, pairs of virtual particles/antiparticles are continuously created in the vicinity of the horizon of the black hole. Among these pairs, some of them will not be able to annihilate, because one of the particles has fallen back into the black hole. The outgoing particle carries energy with it, which means that the black hole radiates. It has a thermal spectrum with the same temperature as black body radiation

$$T = \frac{\kappa\hbar}{2\pi kc} \tag{4.1}$$

with k as Boltzman's constant.

Since the black hole radiates, it evaporates. The lifetime of the black hole, like its mass is finite. With the Hawking radiation, the area of the black hole decreases, since its mass decreases with the evaporation. This area is comparable to the entropy. In fact, the black hole entropy is proportional to its area A . The horizon area is a non-decreasing function of time ($dA \geq 0$). However, this statement violates the second law of thermodynamics by matter losing its entropy as it falls in, giving a decrease in entropy. Generalized second law introduced as total entropy is equal to the sum of black hole entropy and outside entropy.

We expect that black holes radiate in the form of pure thermal radiation at Hawking temperature. However, thermal radiation does not contain any information, except that of the temperature of the emitting source. So what will happen if the black hole disappears completely? Does the information disappear as well? This is the so-called Information Loss Problem. While there is nothing in principle which prevents this from happening, it requires the quantum mechanics which governs the evolution to be non-unitary, since unitary evolution takes pure states to pure states.

4.1 Laws of black hole mechanics

In thermodynamics, there are four laws of very general validity, and as such they do not depend on the details of the interactions or the systems being studied. These laws are as follows:

- If two thermodynamic systems are in thermal equilibrium with a third, they are also in thermal equilibrium with each other.

- The increase in the energy of a closed thermodynamic system is equal to the amount of energy added to the system by heating, minus the amount lost in the form of work done by the system on its surroundings.
- The total entropy of any isolated thermodynamic system tends to increase over time, approaching a maximum value.
- As a system asymptotically approaches absolute zero of temperature all processes virtually cease and the entropy of the system asymptotically approaches a minimum value.

The four laws of black hole mechanics state that:

- The horizon has constant surface gravity for a stationary black hole.
- The changes in energy/mass of a black hole is due to rotation (change in angular momentum of the black hole), electromagnetism (change in electric charge of the black hole), and change in the surface area of the black hole.
- The sum of the black hole entropy and the outside entropy is a non-decreasing function of time.
- It is not possible to form a black hole with vanishing surface gravity.

One can use the expression “black hole thermodynamics” to study the laws of black hole mechanics. One reason is that these laws are quite similar to the laws of thermodynamics.

In thermodynamics, the conservation law of energy is stated by the first law as

$$dE = TdS - PdV \tag{4.2}$$

with

$$\begin{aligned} E &= \text{energy} & T &= \text{temperature} & S &= \text{entropy} \\ P &= \text{pressure} & V &= \text{volume}. \end{aligned}$$

Bekenstein made a detailed discussion [30] to show how to get the first law of black hole mechanics through analogy with thermodynamics. Basically the first law of black hole thermodynamics would also be essentially a conservation law:

$$\delta M = \frac{1}{8\pi} \kappa \delta A + \Omega \delta J + \Phi \delta Q. \quad (4.3)$$

The physical quantities here are as follows:

$$\begin{aligned} M &= \text{black hole mass} & \kappa &= \text{black hole surface gravity} & A &= \text{black hole area} \\ \Omega &= \text{angular velocity} & J &= \text{angular momentum} & \Phi &= \text{electrostatic potential} \\ Q &= \text{electric charge} \end{aligned}$$

and the δ shows the changes in quantities.

The four laws of black hole mechanics are analogous to the four laws of thermodynamics if one compares temperature T and some multiple of the black hole surface gravity κ , as well as entropy S and the black hole area A multiplied by a coefficient [29]. The surface gravity κ of a Killing horizon (null hypersurface on which there is a null Killing vector field) is the acceleration, needed to keep an object at the horizon. In fact, the surface gravity of an astronomical object (planet, star, etc.) is the gravitational acceleration experienced at its surface. κ depends on the mass of the object and its radius. Then, one can say that

$$T = \epsilon \kappa \quad (4.4)$$

and

$$S = \eta A, \quad (4.5)$$

with

$$\epsilon\eta = \frac{1}{8\pi}. \quad (4.6)$$

So ϵ and η are the slopes of the plots of temperature versus surface gravity and entropy versus area respectively. Doing this, the $\kappa\delta A/(8\pi)$ term in the first law of black hole mechanics would have the role of the heat transfer term $T\delta S$ in the first law of thermodynamics and that is where the number $1/8\pi$ is coming from. In fact, it was after the discovery of the famous Bekenstein-Hawking entropy-area proportionality $S \equiv \frac{1}{4}A$ that the constants were set to be $\eta = \frac{1}{4}$ and $\epsilon = \frac{1}{2\pi}$ [30, 31].

In black hole thermodynamics one would say that the surface gravity, κ of a stationary black hole is constant over the horizon [33]. It is comparable with the zeroth law of thermodynamics which says that for a system in thermal equilibrium the temperature T is constant. An extended form of the zeroth law would express that over the horizon of any stationary black hole the quantities κ , Ω and also Φ are constant.

According to the second law, the area A of the event horizon of each black hole does not decrease with time [33]:

$$\delta A \geq 0. \quad (4.7)$$

So it is analogous to the second law of thermodynamics which states that for a closed system the entropy S never decreases

$$\delta S \geq 0. \quad (4.8)$$

For the third law one would say that it is impossible by any procedure (even an idealized one), to reduce κ to zero by a finite sequence of operations [33, 37]. The similar (classical weak) law in thermodynamics implies that when talking about temperature T , absolute zero is not reachable in a finite number of operations. The stronger (Planck) form of the third law of thermodynamics [38, 39, 40], which says that the entropy of a system goes to zero when the temperature goes to zero

$$\lim_{T \rightarrow 0^+} S = 0, \quad (4.9)$$

can not be compared with the (classical) third law of black hole [29]. Otherwise, in comparison with thermodynamics one could write

$$\lim_{\kappa \rightarrow 0^+} A = 0 \quad (4.10)$$

which is not a true statement.

4.2 The generalized second law

Conservation of energy requires that an isolated black hole must lose mass in order to compensate for the energy radiated to infinity by the Hawking process. Indeed, if we equate the rate of mass loss of the black hole to the energy flux at infinity due to particle creation, we arrive at the conclusion that an isolated black hole radiates away all of its mass within a finite time. During this process of black hole evaporation, the black hole area, A , will decrease. This is against the second law of black hole mechanics. So it looks that the standard second law is not complete. According to the generalized second law of black hole mechanics, the total entropy of the black hole plus the entropy of the matter outside the

black hole cannot decrease:

$$\Delta S_{bh} + \Delta S_m = \Delta(S_{bh} + S_m) > 0 \quad (4.11)$$

where ΔS_{bh} stands for the change in black hole entropy and ΔS_m implies the change in the common entropy in the black hole exterior. So the generalized entropy

$$S_{bh} + S_m \quad (4.12)$$

never decreases. The proposal of this law by Bekenstein was after publishing the proportionality of entropy and area ($S_{bh} \propto A$) [30, 31].

If the black hole did not emit any radiation, the second law would have been violated by immersing a black hole in a heat bath of sufficiently low temperature [33], since the black hole might lose entropy. However, Hawking showed that there exists a radiation for black holes [32, 35] and also for a black hole immersed in a heat bath of arbitrary temperature the generalized second law is held if one assumes the radiation thermalized to the temperature of the heat bath. A mathematical proof which verifies this law in case of any process involving a quasi-stationary semi-classical black hole can be found in the reference [43].

4.3 Hawking spectrum

When quantum mechanics is taken into account, the picture turns out to be different from the classical one and a black hole is not that black. Quantum tunneling will occur at the event horizon and the radiation coming out of the horizon escapes to infinity at a steady rate (constant number of particles per time) [32, 35]. More exciting is the fact

that this radiation has an exactly thermal spectrum.

In free field calculation for black hole emissions, Hawking showed that one can find the expectation value $\langle N \rangle$ for the number of particles of the j th species with charge q_j emitted in a wave mode stated by frequency or energy ω , axial quantum number (angular momentum) m , polarization or helicity p and spherical harmonic l as

$$\langle N \rangle = \Gamma_{s_j} \left\{ e^{\left[\frac{2\pi}{\kappa} (\omega - m\Omega - q_j\Phi) \right] \mp 1} \right\}^{-1}. \quad (4.13)$$

The plus sign is for fermions, while the minus sign holds for bosons. Ω is the angular frequency of rotation of the black hole and Φ is the potential of the event horizon.

In equation (4.13), Γ_{s_j} is presenting the greybody factor. The Hawking radiation is determined for each mode by the greybody factor, i.e. the absorption probability (by the black hole) of an incoming wave of the corresponding mode. In other words, the greybody factor is the fraction of the mode that would be absorbed if incident on the black hole. The greybody factors modify the spectrum of emitted particles from that of a perfect thermal black body even in $4D$. They quantify the probability of transmission of the particles through the curved space-time outside the horizon, and can be determined from the absorption cross section for the emitted particle species. At high energies the shape of the spectrum is like that of a black body. However, the low energy behavior of the grey-body factors is spin-dependent and also depends on the number of dimensions.

For a black hole with mass M , charge Q , and angular momentum J , these quantities are as follows:

$$\kappa = \frac{4\pi(r_+c^2 - GM)}{A} \quad (4.14)$$

which is the surface gravity,

$$\Omega = \frac{4\pi J}{MA}, \quad (4.15)$$

$$\Phi = \frac{4\pi Q r_+}{A} \quad (4.16)$$

with

$$r_+ = \frac{1}{c^2} \left[GM + \sqrt{G^2 M^2 - \frac{J^2 c^2}{M^2} - GQ^2} \right] \quad (4.17)$$

and

$$A = \frac{4\pi G}{c^4} \left[2GM^2 - Q^2 + 2\sqrt{G^2 M^2 - J^2 c^2 - GM^2 Q^2} \right] \quad (4.18)$$

which is the area of the event horizon.

Parker [47] has calculated the density matrix of the emitted particles and found that it, as well as the expected number in each mode (modes represent different energies), is precisely thermal. The temperature is the Hawking temperature and actually that is where we encounter the Planck constant \hbar for the first time in this chapter. We can also set the Boltzman's constant k in the Hawking temperature to be 1.

When a black hole radiates, the particles escape, and it loses a small amount of its energy and therefore its mass. The emission rate for a particle of spin s and mass $m \ll M$ from a black hole of mass M into a n -dimensional slice of the D -dimensional spacetime is described by the blackbody distribution

$$\frac{dN}{dt} = \frac{A(n)c(n)\Gamma_{s_i}(n)}{(2\pi\hbar c)^{n-1}} \frac{D^{n-1}k}{e^{\frac{E}{T} - (-1)^{2s}}}. \quad (4.19)$$

$c(n)$ is the number of degrees of freedom. Γ_{s_i} are the n -dimensional greybody factors [49, 50, 51]. $A(n)$ is the area of the induced black hole on the brane

$$A(n) = \Omega_{n-2} r_c^{n-2} \quad (4.20)$$

which is taken as the optical area in $n - 2$ dimensions. r_c is the optical radius [6]

$$r_c = \left(\frac{D-1}{2} \right)^{\frac{1}{D-3}} \sqrt{\frac{D-1}{D-3}} r_s \quad (4.21)$$

where r_s shows the horizon radius of the D -dimensional Schwarzschild black hole given by

$$\begin{aligned} r_s &= \omega_D \ell_{Pl} m^{\frac{1}{D-3}} \\ m &= \frac{M}{M_{Pl}}. \end{aligned} \quad (4.22)$$

Here M stands for mass and the dimensionless area factor is

$$\omega_D = \left[\frac{16\pi}{(D-2)\Omega_{D-2}} \right]^{\frac{1}{D-3}} \quad (4.23)$$

with Ω_{D-2} as the area of S^{D-2} . The emitted energy density distribution in n -dimensions, $\frac{d\mathcal{E}}{dt}$ is related to the black body energy density distribution ($= E \frac{dN}{dt}$) by

$$\frac{d\mathcal{E}}{dt} = \frac{\Omega_{n-3}}{(n-2)\Omega_{n-2}} E \frac{dN}{dt}. \quad (4.24)$$

By integrating this equation over the phase space and summing over all the particle species we obtain the total emitted energy per unit time. Then for a black hole emitting on an n -dimensional brane, the mass loss would be as follows [48]:

$$\frac{dm}{dt} = -\frac{1}{cM_{pl}} \bar{\sigma}_n A(n) T^n \quad (4.25)$$

where $\bar{\sigma}_n$ is the effective n -dimensional Stefan-Boltzmann constant,

$$\bar{\sigma}_n = \frac{\Omega_{n-3} \Gamma(n) \zeta(n)}{(2\pi\hbar c)^{n-1} (n-2)} \sum_i c_i(n) \Gamma_{s_i}(n) f_i(n). \quad (4.26)$$

The sum is over all particle flavors and c_i are the n -dimensional degrees of freedom of the individual species. ζ is the Riemann zeta function and $f_i(n)$ is defined as

$$\begin{aligned} f_i(n) &= 1 \text{ for bosons} \\ &= 1 - 2^{1-n} \text{ for fermions.} \end{aligned} \quad (4.27)$$

A D -dimensional spherically symmetric black hole of mass M is presented by the metric below:

$$ds^2 = +Xc^2dt^2 - Ydr^2 - r^2d\Omega_{D-2}^2 \quad (4.28)$$

with

$$X = 1 - \frac{16\pi G_D M}{(D-2)\Omega_{D-2}c^2r^{D-3}} \quad (4.29)$$

and

$$Y = \frac{1}{X}. \quad (4.30)$$

One can write the equation (4.25) in terms of m as

$$\frac{dm}{dt} = -\frac{\mu}{t_{Pl}}m^{\frac{-2}{D-3}}. \quad (4.31)$$

The Planck time is

$$t_{Pl} = \left(\frac{\hbar G_D}{c^{D+1}}\right)^{\frac{1}{D-2}} \quad (4.32)$$

and μ is defined by

$$\mu = \left(\frac{r_c}{r_s}\right)^{n-2} \left(\frac{D-3}{4\pi}\right)^n \frac{\bar{\sigma}_n \Omega_{n-2}}{\omega_D^2} \quad (4.33)$$

Now we can integrate over time to obtain the decay time

$$\tau = \mu^{-1} \left(\frac{D-3}{D-1}\right) m_i^{\frac{D-1}{D-3}} t_{Pl}. \quad (4.34)$$

In this formula masses are defined in units of Planck mass, which means

$$m_i = \frac{M_i}{M_{Pl}} \quad (4.35)$$

where M_i is the initial black hole mass. The decay time τ is not infinite and after a finite time the radiation stops. The D -dimensional Hawking temperature is [54]:

$$T = \left(\frac{D-3}{4\pi\omega_D}\right) M_{Pl}c^2 m^{\frac{1}{3-D}}. \quad (4.36)$$

This equation is a function of mass, and if we integrate $ds = T^{-1}c^2dM$ we can calculate the entropy

$$S = \frac{4\pi\omega_D}{D-2}m^{\frac{D-2}{D-3}} \quad (4.37)$$

or

$$S = \frac{D-3}{D-2} \frac{Mc^2}{T_H} \quad (4.38)$$

The black hole specific heat is also given as

$$C = -4\pi\omega_D m^{\frac{D-2}{D-3}}. \quad (4.39)$$

For a particle species i produced in black hole decay, the multiplicity (the number of quanta emitted during the evaporation) is given by [48]

$$N_i = N \frac{c_i \Gamma_{S_i} f_i(3)}{\sum_j c_j \Gamma_{S_j} f_j(3)} \quad (4.40)$$

where N is the total multiplicity

$$N = \frac{30\zeta(3)}{\pi^4} S \frac{\sum_i c_i \Gamma_{S_i} f_i(3)}{\sum_j c_j \Gamma_{S_j} f_j(4)} \quad (4.41)$$

and S would be the black hole entropy. It means that N , the total number of quanta emitted during the evaporation, is proportional to the entropy of the black hole.

At the end of this chapter, as an example of a check of the black hole laws, we consider the case of the Reissner-Nordström black hole of mass M and charge Q in D -spacetime dimensions with the metric:

$$ds^2 = +Xc^2dt^2 - Ydr^2 - r^2d\Omega_{D-2} \quad (4.42)$$

where

$$X = 1 - \frac{16\pi G_D M}{(D-2)c^2\Omega_{D-2}r^{D-3}} + \frac{16\pi G_D Q^2}{(D-2)(D-3)c^4r^{2(D-3)}} \quad (4.43)$$

and

$$X = \frac{1}{Y}. \quad (4.44)$$

Therefore, the horizon radius, electrostatic potential at the horizon, Hawking temperature and entropy are given by [46, 19]:

$$\begin{aligned} r_+^{D-3} &= \frac{8\pi G_D M}{(D-2)c^2 \Omega_{D-2}} + \sqrt{\left(\frac{8\pi G_D M}{(D-2)c^2 \Omega_{D-2}}\right)^2 - \frac{2G_D Q^2}{(D-2)(D-3)c^4}} \\ \Phi &= \sqrt{\frac{2(D-3)}{D-2}} \frac{Q}{r_+^{D-3}} \\ T_H &= \frac{(D-3)\hbar c}{2\pi r_+^{D-2}} \sqrt{\left(\frac{8\pi G_D M}{(D-2)c^2 \Omega_{D-2}}\right)^2 - \frac{2G_D Q^2}{(D-2)(D-3)c^4}} \\ S_{BH} &= \frac{c^3 A}{4G\hbar} = \frac{A}{4\ell_{Pl}^2} = \frac{\Omega_{D-2} r_+^{D-2}}{4\ell_{Pl}^{D-2}}. \end{aligned} \quad (4.45)$$

A is the black hole horizon area. G_D represents the D -dimensional Newton's constant as

$$G_D = \frac{\hbar^{D-3}}{c^{D-5} M_{Pl}^{D-2}}. \quad (4.46)$$

ℓ_{Pl} is the D -dimensional Planck length and Ω_{D-2} the area of S^{D-2} .

T_H is constant over the black hole horizon which shows the validity of the zeroth law. One can also prove that $d(Mc^2) = T_H dS_{BH} + \Phi dQ$ and $\Delta S_{BH} \geq 0$. These are nothing but the first and second law, respectively.

Chapter 5

Entanglement and area

The concept of entanglement has played an important role in quantum physics ever since its discovery last century. In black hole physics, entanglement entropy has been recognized as a candidate to explain black hole entropy. In this chapter we discuss the motivation behind this approach, and present the implications of entanglement entropy to study entanglement in black hole physics. The main goal will be the study of entropy-area relation by means of our powerful tool: entanglement.

Let us think of the well-known double-slit experiment. The reason that we want to discuss it here is that recently some experiments have been done to relate this fundamental experiment to entanglement. This experiment uses the phenomena of interference, produced by light incident on a double slit, to investigate the quantum mechanical principle of complementarity between the wave and particle characteristics of light. Using a special state of light, Walborn and his coworkers [55] created an interference pattern, made a “which-way” measurement which destroyed the interference, and then erased the “which-

way” marker, bringing the interference back. This experiment clearly displays the way in which nature is counterintuitive on the quantum scale and makes it clear that our ways of thinking based on our everyday experiences in the classical world are often completely inadequate to understand the quantum world.

It has long been known that which-path information and visibility of interference fringes are complementary quantities: any distinguishability between the paths of an interferometer destroys the quality (visibility) of the interference fringes. Originally, it was thought that the uncertainty principle was the mechanism responsible for the absence of interference fringes due to a which-path measurement. Bohr showed that the uncertainty in the knowledge of the double slits initial position was of the same order of magnitude as the space between the interference minima and maxima: interference fringes were washed out due to the uncertainty principle [56]. More recently, Scully and Druhl [57] and Scully, Englert, and Walther [58] have shown that, in certain cases, we can attribute this loss of interference not to the uncertainty principle but to quantum entanglement between the interfering particles and the measuring apparatus.

Disregarding internal degrees of freedom, we can represent the state of particles exiting an interferometer by

$$|\Psi\rangle = \frac{1}{\sqrt{2}} [|\Psi_1(\mathbf{r})\rangle + |\Psi_2(\mathbf{r})\rangle], \quad (5.1)$$

where $|\Psi_1(\mathbf{r})\rangle$ and $|\Psi_2(\mathbf{r})\rangle$ represent the amplitude for the particles to take path 1 or 2, respectively. The probability distribution for one-particle detection at a point \mathbf{r} is given by

$|\langle \mathbf{r} | \Psi \rangle|^2$. Ignoring the normalization coefficients we have :

$$\begin{aligned} |\langle \mathbf{r} | \Psi \rangle|^2 &= [|\langle \mathbf{r} | \Psi_1(\mathbf{r}) \rangle + \langle \mathbf{r} | \Psi_2(\mathbf{r}) \rangle|^2 \\ &= |\langle \mathbf{r} | \Psi_1(\mathbf{r}) \rangle|^2 + |\langle \mathbf{r} | \Psi_2(\mathbf{r}) \rangle|^2 + \langle \Psi_1(\mathbf{r}) | \mathbf{r} \rangle \langle \mathbf{r} | \Psi_2(\mathbf{r}) \rangle + \langle \Psi_2(\mathbf{r}) | \mathbf{r} \rangle \langle \mathbf{r} | \Psi_1(\mathbf{r}) \rangle. \end{aligned} \quad (5.2)$$

The cross terms $\langle \Psi_1(\mathbf{r}) | \mathbf{r} \rangle \langle \mathbf{r} | \Psi_2(\mathbf{r}) \rangle$ and $\langle \Psi_2(\mathbf{r}) | \mathbf{r} \rangle \langle \mathbf{r} | \Psi_1(\mathbf{r}) \rangle$ are responsible for interference.

The introduction of an apparatus M capable of marking the path taken by a particle without disturbing $|\Psi_1(\mathbf{r})\rangle$ or $|\Psi_2(\mathbf{r})\rangle$ can be represented by the expansion of the Hilbert space of the system in the following way:

$$|\Psi\rangle = \frac{1}{\sqrt{2}} [|\Psi_1(\mathbf{r})\rangle |M_1\rangle + |\Psi_2(\mathbf{r})\rangle |M_2\rangle], \quad (5.3)$$

where M_j is the state of the which-path marker corresponding to the possibility of passage through the path j . The which-path marker has become entangled with the two possible particle states. A 100% effective which-path marker is prepared such that $|M_1\rangle$ is orthogonal to $|M_2\rangle$. In this case, a measurement of M_i reduces $|\Psi\rangle$ to the appropriate state for the passage of the particle through path 1 or 2. However, the disappearance of the interference pattern is not dependent on such a measurement. The presence of the which-path marker alone is sufficient to make the two terms on the right-hand side of equation (5.3) orthogonal and thus there will be no cross terms in $|\langle \mathbf{r} | \Psi \rangle|^2$. Therefore, it is enough that the which-path information is available to destroy interference. Moreover, provided that $|\Psi_1(\mathbf{r})\rangle$ and $|\Psi_2(\mathbf{r})\rangle$ are not significantly perturbed by the observer, one can erase the which-path information and recover interference by correlating the particle detection with an appropriate measurement on the which-path markers. Such a measurement is known as *quantum erasure*. The quantum erasure experiment is a double-slit experiment in which

particle entanglement and selective polarization are used to determine which slit a particle goes through by measuring the particle's entangled partner. This entangled partner never enters the double slit experiment. The quantum erasure effectively erases the which-path information (and restores interference) without altering the double-slit experiment.

We can study the superposition principle by stressing the expansion postulate. Any wave function Ψ could be expanded in a complete set of eigenfunctions of any hermitian operator. If $u_a(x)$ form a complete, orthonormal set, which means

$$Au_a(x) = au_a(x) \quad (5.4)$$

with A and a as (Hermitian) operator and eigenvalue respectively, and

$$\int_{-\infty}^{+\infty} dx u_a^*(x)u_b(x) = \delta_{ab} \quad (5.5)$$

then for any square-integrable wave function we can write

$$\Psi(x) = \sum_a C_a u_a(x). \quad (5.6)$$

The coefficient can be found using the orthonormality of the eigenfunctions,

$$C_a = \int_{-\infty}^{+\infty} dx u_a^*(x)\Psi(x). \quad (5.7)$$

A measurement of the eigenvalues of A , on the system which $\Psi(x)$ describes, must lead to one of the eigenvalues. In fact, the probability of finding a particular measurement (for example the value a) is given by

$$P(a) = |C_a|^2. \quad (5.8)$$

Another point is that the measurement causes the system to be projected into the state described by $u_a(x)$. These assertions (random nature of the results of the measurement,

the probability interpretation of the expansion coefficients, and the projection caused by a measurement), do not directly follow from the Schrödinger equation

$$i\hbar \frac{d}{dt} |\Psi\rangle = H|\Psi\rangle. \quad (5.9)$$

We follow the discussion by considering the singlet state of the form

$$X_{singlet} = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle). \quad (5.10)$$

This state is said to be entangled since one can not write it as a tensor product of two states

$$|\dots\rangle \otimes |\dots\rangle. \quad (5.11)$$

As an example of a non-entangled state we assume an addition of the two states $|\uparrow\downarrow\rangle$ and $|\uparrow\uparrow\rangle$. In such case we are able to write this addition as a tensor product of two states

$$|\uparrow\downarrow\rangle + |\uparrow\uparrow\rangle = |\uparrow\rangle \otimes (|\downarrow\rangle + |\uparrow\rangle), \quad (5.12)$$

which is not entangled. In equation (5.10) if we carry out a measurement on particle (1), it does not result in a communication from that particle to particle (2). The two measurements of the x -component of the spins can be carried out simultaneously - that is, when the separation between the locations of the particles is spacelike. An important paper related to this matter is called EPR on behalf of a publication by A. Einstein, B. Podolsky and N. Rosen [1] which discusses significant postulates:

1. Every element of physical reality must have a counterpart in a complete physical theory.
2. If without disturbing the system we can predict, with certainty, the value of some observable, then we can associate an element of reality with this observable.

Suppose we measure the z -component of the spin of particle (1) and we get the eigenvalue $+\hbar/2$. Then without disturbing the system one can say that the z -component of the spin of particle (2) is $-\hbar/2$. According to EPR, we may thus associate with the z -component of spin of particle (2) an element of physical reality, meaning that particle (2) must have had $S_z = -\hbar/2$ all along, since the measurement on particle (1)- undertaken at an arbitrary distance from where particle (2) is being measured- could not possibly affect its state. The same argument would go through if we measure the x -component instead of the z -component. As another case suppose that Alice measures S_z and obtains $+\hbar/2$ (let's call it state I). Now, instead of measuring the S_z as well, Bob measures the S_x . According to quantum mechanics, when the system is in state I, Bob's S_x measurement will have a 50% probability of producing $+\hbar/2$ and a 50% probability of $-\hbar/2$. Furthermore, it is fundamentally impossible to predict which outcome will appear until Bob actually performs the measurement.

According to the EPR criteria, either quantum mechanics is not a complete theory or there must exist a nonlocal interaction between the two particles. The nonlocality does not contradict special relativity, since no message can be sent from one particle to the other during the performance of these incompatible experiments.

Bohr's response to EPR issue was that quantum mechanics was different from classical mechanics, in that it was meaningless to try to simultaneously assign values to physical quantities represented by operators that do not commute (Here S_x and S_z do not commute). This was an example of what he called *complementary*, a view that there are some aspects in quantum mechanics that cannot all be determined simultaneously in one experiment (like

wave/particle characteristic of matter that can not be observed simultaneously). Then, a number of people talked about the existence of *hidden variables*, which would in some sense describe a linkage between the two *separate* states. In quantum mechanics, a local hidden variable theory is one in which distant events are assumed to have no instantaneous effect on local ones. Of course, the modification should not affect the results of experiments.

5.1 Entanglement

We mentioned two examples in last section to introduce the entanglement: equation (5.10) which illustrates an entangled state and relation (5.12) which presents a non-entangled state. Here we try to answer a question: what is the meaning of entanglement [62]? Assume a quantum mechanical state which comprises two other states u and v :

$$\mathcal{H} = \mathcal{H}_u \otimes \mathcal{H}_v. \quad (5.13)$$

The tensor product \otimes indicates that the Hilbert space \mathcal{H} which comprises two other Hilbert spaces, \mathcal{H}_u and \mathcal{H}_v . \mathcal{H}_u and \mathcal{H}_v represent the subsystems and \mathcal{H} stands for the main system. Let's assume $|\Psi_u\rangle$ and $|\Psi_v\rangle$ as the wave functions in the subsystems \mathcal{H}_u and \mathcal{H}_v respectively

$$\begin{aligned} |\Psi_u\rangle &\in \mathcal{H}_u \\ |\Psi_v\rangle &\in \mathcal{H}_v. \end{aligned} \quad (5.14)$$

Then one can expand them by means of the eigenvectors of the subsystems as

$$\begin{aligned} |\Psi_u\rangle &= \sum a_i |u_i\rangle \\ |\Psi_v\rangle &= \sum b_i |v_i\rangle. \end{aligned} \quad (5.15)$$

The wave function of the entangled system can be written as

$$\begin{aligned} |\Psi\rangle &= \sum_{ij} d_{ij} |u_i\rangle \otimes |v_j\rangle, \\ |\Psi\rangle &\in \mathcal{H}. \end{aligned} \quad (5.16)$$

In general, however, this entangled state is not equal to the tensor product of the wave functions of the subsystems,

$$|\Psi\rangle \neq |\Psi_u\rangle \otimes |\Psi_v\rangle. \quad (5.17)$$

But an unentangled state can be written like equation (5.12), for instance.

The density matrix of a pure state with state vector $|\Psi\rangle$ is defined by

$$\rho \equiv |\Psi\rangle\langle\Psi|. \quad (5.18)$$

It has the properties below:

$$\begin{aligned} |\rho| &\geq 0 \\ \rho^\dagger &= \rho \\ \rho^2 &= \rho. \end{aligned} \quad (5.19)$$

The last property means that the eigenvalues of ρ are 0 and 1,

$$p_n = 0, 1. \quad (5.20)$$

Then one can define the entanglement entropy by

$$S \equiv -Tr(\rho \ln \rho) = -\sum_n p_n \ln p_n = 0. \quad (5.21)$$

However, one can trace over one of the subsystems to get the reduced density matrix as

$$\rho_u = Tr_v(\rho) = \sum_l \langle v_l | \rho | v_l \rangle = \sum_{j,i,k} d_{ij} d_{kj}^* |u_i\rangle \langle u_k|. \quad (5.22)$$

Now the properties of this reduced density matrix are

$$\begin{aligned} |\rho| &\geq 0 \\ \rho^\dagger &= \rho \\ \rho^2 &\neq \rho. \end{aligned} \tag{5.23}$$

The last property is the only one different for a reduced traced over density matrix and the density matrix of a pure state. As a result of this, the state of an entangled subsystem cannot be described in terms of a state vector in the corresponding Hilbert space, and one is forced to use the density matrix language. In this case one says that the subsystem is in a mixed state. Pure states, associated with definite state vectors, are on the other hand always associated with idempotent density operators.

Here, in the mixed states, the eigenvalues of ρ are not limited to 0 and 1,

$$0 < p_{n(u)} < 1 \tag{5.24}$$

and the entanglement entropy also is not necessarily zero,

$$S \equiv -Tr_u(\rho_u \ln \rho_u) = -\sum_n p_{n(u)} \ln p_{n(u)} > 0. \tag{5.25}$$

Choosing to trace over the subsystem u or v does not have any effect on the value of the entropy, since the set of the eigenvalues are equal for both u and v subsystems. So tracing over \mathcal{H}_u or \mathcal{H}_v gives identical entropy.

To prove that the nonzero eigenvalues of the subsystems u and v are equal, we diagonalize the matrix of ρ_u in the u_i basis by means of a unitary transformation U

$$\sum_k \rho_{u,ik} U_{kl} = \lambda_l U_{il}. \tag{5.26}$$

By a unitary transformation we mean that U has the following characteristics:

$$\begin{aligned}\sum_i U_{il}U_{ij}^* &= \delta_{lj} \\ \sum_i U_{li}U_{ji}^* &= \delta_{jl}.\end{aligned}\tag{5.27}$$

The element of the ρ_u matrix is

$$\rho_{u,ik} = \sum_j d_{ij}d_{kj}^*.\tag{5.28}$$

So the unitary transformation can be written as

$$\sum_{kj} d_{ij}d_{kj}^*U_{kl} = \lambda_l U_{il}.\tag{5.29}$$

If we multiply both sides of the equation above by $d_{i'l}^*$ and sum over the index i , then we obtain

$$\sum_{ijk} d_{i'l}^*d_{ij}d_{kj}^*U_{kl} = \lambda_l \sum_i d_{i'l}^*U_{il}.\tag{5.30}$$

We define the components of the vector V

$$V = (V_{1l}, V_{2l}, \dots, V_{jl}, \dots)\tag{5.31}$$

as

$$V_{jl} = \sum_k d_{kj}^*U_{kl}.\tag{5.32}$$

One can write the elements of ρ_v as

$$\rho_{v,mn} = \sum_i d_{im}d_{in}^*.\tag{5.33}$$

So V_{jl} is an eigenvector of ρ_v with the eigenvalue λ_l ,

$$\sum_j \rho_{v,lj}V_{jl} = \lambda_l V_{ll}.\tag{5.34}$$

Consequently, both ρ_u and ρ_v have the same eigenvalues λ . If the nonvanishing eigenvalues are non-degenerate, then all these vectors are automatically orthogonal. And if, moreover, the set of eigenvectors with nonvanishing eigenvalues is still not complete, one still may complete the bases with additional orthogonal vectors which are again eigenvectors of the respective reduced densities with eigenvalue zero.

5.2 The brick wall model

A significant part of black hole physics is concerned with their dynamical behavior which includes studying thermal properties and information content. Thermal properties of a black hole depend on its mass. The mass of black hole is a function of time. So as the black hole loses its mass through the Hawking radiation, the thermal properties including its entropy change. A basic ingredient in black hole thermodynamics is the notion of entropy. The entropy S of a standard non-extreme black hole obeys the well-known Bekenstein-Hawking area law, which means it is proportional to the area of the horizon. The contribution to the entropy of quantum fields in black hole backgrounds was studied using different methods such as the WKB approximation [63] and the path integral method [64]. However, the entropy due to quantum fields in the black hole background introduces divergences which are interpreted as renormalizations of the gravitational coupling constant G .

The brick wall model is another method which tries to represent an explanation for the entanglement-area proportionality law for black holes. The basic ideas of the brick wall model are reviewed in the references [65] and [66] and we review this method in this

section. Here, our system comprises two parts: the region inside the brick wall and the region outside the brick wall. So according to our discussion in the last chapter, we can write the quantum mechanical state as $\mathcal{H} = \mathcal{H}_u \otimes \mathcal{H}_v$, where \mathcal{H}_u and \mathcal{H}_v represent the inside and outside of the brick wall.

In the approach taken by 't Hooft, the entropy of a thermal gas of particles outside the event horizon of a Schwarzschild black hole is considered using the WKB approximation. The problem is solved in a black hole geometry which is a fixed classical background and the quantum fields propagate on it. The calculation involves divergences coming from the number of modes close to the event horizon, which were regulated by using a *brick wall*, namely a cut-off just outside the event horizon. He calculated the leading order divergences in the entropy, and found that they were proportional to the area of the event horizon multiplied by h^{-2} , where h is the proper distance of the brick wall from the event horizon. There are also some works done to extend the original brick wall model to more general black holes and dimensions other than four (for example see [67]).

Let us assume some point r_1 near the horizon of a black hole (r_+) as

$$r_1 = r_+ + h \tag{5.35}$$

with

$$\begin{aligned} h &> 0 \\ r_+ &= 2M. \end{aligned} \tag{5.36}$$

Here we are using a system in which $G = c = 1$. We give the multiplicity N to the scalar

field $\Phi_i(r, \theta, \phi, t)$ with $i = 1, \dots, N$. At $r = r_1$ we put the boundary condition

$$\Phi_i(r, \theta, \phi, t) = 0 \quad (5.37)$$

for $r \leq r_1$. Our plan is to find the thermodynamic properties of this system, particularly the entropy. The Lagrange density \mathcal{L} for a Schwarzschild metric is

$$\mathcal{L}(x, t) = \left(1 - \frac{2M}{r}\right)^{-1} \partial_t \Phi_i^2 - \left(1 - \frac{2M}{r}\right) \partial_r \Phi_i^2 - r^{-2} \partial_\Omega \Phi_i^2. \quad (5.38)$$

The total Lagrangian reads

$$\int_{r_1}^L dr \int d\Omega r^2 \mathcal{L}(r, \Omega, t) \quad (5.39)$$

with

$$d\Omega = \sin \theta d\theta d\varphi. \quad (5.40)$$

The field equation for the scalar field with mass m is obtained from the Lagrangian as

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \Phi) - m^2 \Phi = 0. \quad (5.41)$$

So the field equation for energy levels $E(n, l)$ of bosons Φ_i is

$$\left(1 - \frac{2M}{r}\right)^{-1} E^2 \Phi + \frac{1}{r^2} \partial_r r (r - 2M) \partial_r \Phi - \left(\frac{l(l+1)}{r^2} + m^2\right) \Phi = 0. \quad (5.42)$$

As the singularity in the second term at $r = 2M$ is too close to the cutoff point $r = r_1$, we need to smooth it by means of some mathematical trick

$$r - 2M = e^\sigma \quad (5.43)$$

and then

$$\left[r E^2 + \frac{1}{r^2} \partial_\sigma r \partial_\sigma - e^\sigma \left(\frac{l(l+1)}{r^2} + m^2 \right) \right] \Phi = 0 \quad (5.44)$$

with

$$r = 2M + e^\sigma. \quad (5.45)$$

The energy spectrum of this equation (or in other words the number of radial modes n) is given by

$$\pi n = \int_{r_1}^L dr k(r, l, E). \quad (5.46)$$

Here the quantum numbers $n > 0$, l and $l_3 = -l, \dots, l$ are integers. Then the total number ν of wave solutions with energy not exceeding E is then given by the function $g(E)$ as

$$\begin{aligned} g(E) &= \pi \nu \\ &= \int (2l + 1) dl \pi n \\ &= \int_{r_1}^L dr \left(1 - \frac{2M}{r}\right)^{-1} \int (2l + 1) dl \sqrt{E^2 - \left(1 - \frac{2M}{r}\right) \left(\frac{l(l+1)}{r^2} + m^2\right)} \end{aligned} \quad (5.47)$$

Every energy level may be occupied by any non-negative number of quanta. The free energy F at an inverse temperature β is given by

$$e^{-\beta F} = \sum e^{-\beta E} = \prod_{i=1}^N \prod_{n, l, l_3} \frac{1}{1 - e^{-\beta E(n, l, l_3)}} \quad (5.48)$$

Then we have

$$\beta F = N \sum_{\nu} \log(1 - e^{-\beta E}). \quad (5.49)$$

Solving this summation in this form is not basically possible. So we use the continuous form of summation, which is integral. The summation is over ν and so the integral also must be over ν . The good point about it is that ν is a function of energy ($\pi \nu = g(E)$). So we can

change the parameter of integration to energy and integrate from zero to infinity:

$$\begin{aligned}
\pi\beta F &= N \int dg(E) \log(1 - e^{-\beta F}) \\
&= -N \int_0^\infty dE \frac{\beta g(E)}{e^{\beta E} - 1} \\
&= -\beta N \int_0^\infty dE \int_{r_1}^L dr \left(1 - \frac{2M}{r}\right)^{-1} \int (2l+1) dl \\
&\times (e^{\beta E} - 1)^{-1} \sqrt{E^2 - \left(1 - \frac{2M}{r}\right) \left(m^2 + \frac{l(l+1)}{r^2}\right)}
\end{aligned} \tag{5.50}$$

where

$$r_1 = 2M + h. \tag{5.51}$$

The integral is taken over those values for which the square root exists. In the approximation

$$m^2 \ll \frac{2M}{\beta^2 h} \tag{5.52}$$

and

$$L \gg 2M \tag{5.53}$$

the main contributions to this integral are found to be

$$F \approx -\frac{2\pi^3 N}{45h} \left(\frac{2M}{\beta}\right)^4 - \frac{2}{9\pi} L^3 N \int_m^\infty \frac{dE (E^2 - m^2)^{\frac{3}{2}}}{e^{\beta E} - 1}. \tag{5.54}$$

There are two terms in the solution. The first term diverges linearly as $h \rightarrow 0$. The second term is the contribution from the vacuum surrounding the black hole at great distances (which should be discarded). So the total energy U and the entropy S are

$$U = \frac{\partial}{\partial \beta}(\beta F) = \frac{2\pi^3}{15h} \left(\frac{2M}{\beta}\right)^4 N \tag{5.55}$$

and

$$S = \beta(U - F) = \frac{8\pi^3}{45h} 2M \left(\frac{2M}{\beta}\right)^3 N \tag{5.56}$$

respectively. The main point here is that the result obtained for the entropy S is proportional to the area [68]

$$S \propto A. \quad (5.57)$$

To prove that, we assume the general metric below:

$$ds^2 = g_{tt}(r)dt^2 - g_{rr}(r)dr^2 - g_{\theta\theta}(r)d\Omega_2 \quad (5.58)$$

with

$$d\Omega_2 = d\theta^2 + \sin^2\theta d\varphi^2 \quad (5.59)$$

as the line element on a unit sphere. General arguments imply that the metric functions depend only on a radial coordinate r in $(3+1)$ spacetime dimensions and that is the reason we assumed such metric here. We make a Laurent expansion of the metric around the horizon $r \sim r_h = r_+$ as

$$\begin{aligned} g_{tt}(r) &= (r - r_h)^a \sum_{i=0}^{\infty} a^i(r_h)(r - r_h)^i, \\ g_{rr}(r) &= (r - r_h)^b \sum_{i=0}^{\infty} b^i(r_h)(r - r_h)^i, \\ g_{\theta\theta}(r) &= (r - r_h)^c \sum_{i=0}^{\infty} c^i(r_h)(r - r_h)^i, \end{aligned} \quad (5.60)$$

where the coefficients a^0 , b^0 and c^0 are assumed not to be zero and a , b , c denote the leading exponents of Laurent expansion for each metric function. By defining the area of the brick wall surface

$$A = \int d\Omega_2 g_{\theta\theta}|_{r=r_h+h} \quad (5.61)$$

and the invariant distance to the brick wall

$$h = \int_{r_h}^{r_h+\epsilon} dr \sqrt{g_{rr}} \quad (5.62)$$

one can show that the entropy is

$$S \propto \frac{1}{45} \left(\frac{a}{2}\right)^3 \left(\frac{2}{b+2}\right)^2 h^{-2} \frac{A}{4\pi}. \quad (5.63)$$

So the leading contribution to entropy is proportional to the area, but diverges in h . This term is considered as the renormalization effect to the gravitational constant G in the Bekenstein-Hawking entropy. Note that

$$\lim_{h \rightarrow 0} S = 0 \quad (5.64)$$

which means the leading contribution to entropy is zero when h approaches zero.

5.3 Black hole entropy and entanglement entropy

The brick wall model represents an explanation for the black hole entropy-area proportionality. Another candidate for such discussion is the entanglement entropy. The basic idea is to find the entanglement entropy between the region inside the black hole horizon and the region outside the black hole.

Let us suppose a system is in a pure state $|\Psi\rangle$. The full space of states is the Hilbert space \mathcal{H} . We span this using two subspaces \mathcal{H}_u and \mathcal{H}_v by means of the states within the horizon $|u_i\rangle$ and states outside $|v_j\rangle$ respectively (u and v show the dependence of each state to the subspaces). This makes sense as an observer who is confronted with a horizon cannot make measurements on the system beyond that. The Hilbert space is $\mathcal{H} = \mathcal{H}_u \otimes \mathcal{H}_v$ and the general state $|\Psi\rangle$ is then written as $|\Psi\rangle = \sum_{i,j} d_{ij} |u_i\rangle |v_j\rangle$. Since ρ corresponds to a pure state, $\rho^2 = \rho$. An observer constrained to subspace \mathcal{H}_u appears to be in a mixed state described by the density matrix ρ_u . Diagonalizing ρ_u , it can be expressed

in a new basis $|i\rangle_u$ for \mathcal{H}_u as

$$\rho_u = Tr_v \rho = \sum w_i |i\rangle_{uu} \langle i|. \quad (5.65)$$

The quantities w_i are eigenvalues of ρ_u in this basis with the condition that

$$0 \leq w_i \leq 1. \quad (5.66)$$

An observer in \mathcal{H}_v sees the density matrix ρ_v . The nonzero eigenvalues of ρ_u and ρ_v are the same (even if they have different dimensions). Consequently, the entanglement entropies of the two subspaces must be equal: $S_u = \sum w_i \ln w_i = S_v$. If there is equal likelihood to be in each state $|i\rangle_u$, then $w_i = 1/N_u$, where N_u is the dimension of \mathcal{H}_u . Then one sees that

$$S_u = \ln N_u. \quad (5.67)$$

In that case, the entanglement entropy coincides with the log of the number of microstates, which is the common definition of entropy for a thermal ensemble.

Previously, there have been several attempts to relate the entropy and thermodynamics of the black hole to entanglement [63, 69, 70, 71]. The area (as opposed to volume) proportionality of black hole entropy has been an interesting issue and the entanglement entropy is one of the best candidates for explaining the problem. In fact recently some papers have proposed that the black hole entropy can always be identified with entanglement entropy [72, 73]. A straightforward approach is one based on finding the entanglement entropy for an imaginary box containing coupled harmonic oscillators and expressing the possible connection with the physics of black holes [74, 75]. To do that the ground state density matrix for a massless free field is traced over the degrees of freedom residing inside an imaginary sphere and the resulting entropy is shown to be proportional to the area (and

not the volume) of the sphere. The simplified form of the free scalar field approach is the harmonic oscillators discussion in a non-continuous lattice.

Suppose we have a free scalar field written as

$$H = \frac{1}{2} \int d^3x \left[\pi^2(x) + |\vec{\nabla}\varphi(\vec{x})|^2 \right] \quad (5.68)$$

where π is the momentum conjugate to φ . The partial wave decomposition can be written in terms of spherical harmonics:

$$\begin{aligned} \varphi(\vec{r}) &= \sum_{lm} \frac{\varphi_{lm}(r)}{r} Y_{lm}(\theta, \phi) \\ \pi(\vec{r}) &= \sum_{lm} \frac{\pi_{lm}(r)}{r} Y_{lm}(\theta, \phi). \end{aligned} \quad (5.69)$$

The real spherical harmonics Z_{lm} , which are orthonormal and complete,

$$\begin{aligned} Z_{l0} &= Y_{l0} \\ Z_{lm} &= \sqrt{2} \operatorname{Re} Y_{lm} \quad \text{for } m > 0 \\ Z_{lm} &= \sqrt{2} \operatorname{Im} Y_{lm} \quad \text{for } m < 0 \end{aligned} \quad (5.70)$$

are useful here. Orthogonality of spherical harmonics applied to (5.69) leads to

$$\begin{aligned} \varphi_{lm}(r) &= r \int d\Omega Z_{lm}(\theta, \phi) \varphi(\vec{r}) \\ \pi_{lm}(r) &= r \int d\Omega Z_{lm}(\theta, \phi) \pi(\vec{r}). \end{aligned} \quad (5.71)$$

These operators are hermitian and satisfy the canonical commutation relations

$$[\varphi_{lm}(r), \pi_{l'm'}(r')] = i\delta_{ll'}\delta_{mm'}\delta(r - r'). \quad (5.72)$$

Then for the Hamiltonian we have

$$\begin{aligned} H &= \sum_{lm} H_{lm} \\ &= \sum_{lm} \frac{1}{2} \int_0^\infty dr \left\{ \pi_{lm}^2(r) + r^2 \left[\frac{\partial}{\partial r} \left(\frac{\varphi_{lm}(r)}{r} \right) \right]^2 + \frac{l(l+1)}{r^2} \varphi_{lm}^2(r) \right\} \end{aligned} \quad (5.73)$$

with

$$r = |\vec{r}|. \quad (5.74)$$

Now we want to replace the continuous space by a lattice of discrete points with spacing a .

That means for the coordinate we require to assume

$$r \rightarrow r_i \quad (5.75)$$

and

$$r_{i+1} - r_i = a. \quad (5.76)$$

Then we put the system in a spherical box of radius L

$$L = (N + 1)a. \quad (5.77)$$

Now the discrete H_{lm} can be written as

$$H_{lm} = \frac{1}{2a} \sum_{j=1}^N \left[\pi_{lm,j}^2 + \left(j + \frac{1}{2} \right)^2 \left(\frac{\varphi_{lm,j}}{j} - \frac{\varphi_{lm,j+1}}{j+1} \right)^2 + \frac{l(l+1)}{j^2} \varphi_{lm,j}^2 \right] \quad (5.78)$$

where

$$\varphi_{lm,N+1} = 0. \quad (5.79)$$

Here, we have defined

$$\begin{aligned} \varphi_{lm,j} &\equiv \varphi_{lm,j}(r_j) \\ \pi_{lm,j} &\equiv \pi_{lm,j}(r_j). \end{aligned} \quad (5.80)$$

We also consider that $\varphi_{lm,j}$ and $\pi_{lm,j}$ do not have any physical dimension. They are hermitian, with the canonical commutation relations

$$[\varphi_{lm,j}, \pi_{l'm',j'}] = i\delta_{ll'}\delta_{mm'}\delta_{jj'}. \quad (5.81)$$

We can think of our imaginary box as a black hole with horizon radius na . N and n are positive integers and

$$n < N. \quad (5.82)$$

One can trace over the first n sites and find the entanglement entropy for the black hole.

The same form of equation (5.78) can be applied again as

$$H = \frac{1}{2} \sum_{i=1}^N p_i^2 + \frac{1}{2} \sum_{i,j=1}^N x_i K_{ij} x_j, \quad (5.83)$$

which is the hamiltonian of N -coupled oscillators. We trace over the first n of N oscillators and find a density matrix which we call ρ_{out} as it is traced over:

$$\begin{aligned} \rho_{out}(x_{n+1}, \dots, x_N; x'_{n+1}, \dots, x'_N) = \\ \int \prod_{i=1}^n dx_i \psi_0(x_1, \dots, x_n, x_{n+1}, \dots, x_N) \psi_0^*(x_1, \dots, x_n, x'_{n+1}, \dots, x'_N). \end{aligned} \quad (5.84)$$

ψ_0 is the ground state wave function. In such case

$$\rho_{out}^2 \neq \rho_{out} \quad (5.85)$$

and so the ρ_{out} represents a mixed state, though the full state is pure. Then we can find the entanglement entropy for this mixed state.

5.4 Two coupled oscillators

In this section we start with a simpler system of harmonic oscillators. We illustrate the general results discussed above for the special case of two coupled oscillators, i.e. $N = 2$ [74]. The Hamiltonian for such a system is

$$H = \frac{1}{2} [p_1^2 + p_2^2 + k_0(x_1^2 + x_2^2) + k_1(x_1 - x_2)^2]. \quad (5.86)$$

The last term in the Hamiltonian $[k_1(x_1 - x_2)^2]$ is responsible for coupling the oscillators. It shows the relative position of two oscillators. We write the pure state $\psi_0(x_1, x_2)$ of the two coupled harmonic oscillators as

$$\begin{aligned} \psi_0(x_1, x_2) &= \psi_0(x_+) \psi_0(x_-) \\ &= \frac{(\omega_+ \omega_-)^{\frac{1}{4}}}{\pi^{\frac{1}{2}}} \exp \left[\frac{-(\omega_+ x_+^2 + \omega_- x_-^2)}{2} \right]. \end{aligned} \quad (5.87)$$

The two states $\psi_0(x_+)$ and $\psi_0(x_-)$ are the ground states of the two harmonic oscillator systems. This equation resembles to the expansion of the general state by means of two other states. Other quantities are defined as

$$\begin{aligned} x_{\pm} &= \frac{x_1 \pm x_2}{\sqrt{2}}, \\ \omega_+ &= \sqrt{k_0}, \\ \omega_- &= \sqrt{k_0 + 2k_1}. \end{aligned} \quad (5.88)$$

We calculate the density matrix by tracing over the first oscillator of the two coupled oscillators

$$\begin{aligned}
\rho_{out}(x_2, x'_2) &= \int_{-\infty}^{\infty} dx_1 \psi_0(x_1, x_2) \psi_0^*(x_1, x'_2) \\
&= \sqrt{\frac{\gamma - \beta}{\pi}} \exp\left[-\frac{\omega_+ x_+^2 + \omega_- x_-^2}{2}\right] \\
&= \sqrt{\frac{\gamma - \beta}{\pi}} \exp\left[-\gamma \frac{(x_2 + x'_2)}{2} + \beta x_2 x'_2\right]
\end{aligned} \tag{5.89}$$

with

$$\beta = \frac{\omega_-(1 - R^2)^2}{4(1 + R^2)} \tag{5.90}$$

and

$$\gamma = \frac{1 + 6R^2 + R^4}{4(1 + R^2)}. \tag{5.91}$$

We define R as

$$R^2 = \frac{\omega_+}{\omega_-} \tag{5.92}$$

and then ξ and α are written as

$$\xi = \left(\frac{1 - R}{1 + R}\right)^2 \tag{5.93}$$

and

$$\alpha = \omega_- R. \tag{5.94}$$

The term $\psi_0(x_1, x_2) \psi_0^*(x_1, x'_2)$ presents a mixed state and that's why we expect a nonzero entanglement entropy. For the eigenvalue equation we have

$$\int_{-\infty}^{\infty} dx' \rho(x, x') f_n(x') = p_n f_n(x). \tag{5.95}$$

One can easily show that

$$f_n(x) = H_n(\sqrt{\alpha}x) \exp\left(\frac{-\alpha x^2}{2}\right), \tag{5.96}$$

$$p_n = (1 - \xi)\xi^n \quad (5.97)$$

satisfy the integral equation above. Here ξ is defined by equation (5.93). Applying equation (5.21) we calculate the entanglement entropy,

$$S(R) = - \sum_n^{\infty} p_n \ln p_n = - \ln(1 - \xi) - \frac{\xi}{1 - \xi} \ln \xi \neq 0. \quad (5.98)$$

Now consider equation (5.83) for N coupled oscillators and matrix K . The normalized ground state wave function is

$$\psi_0(x_1, \dots, x_N) = \pi^{-N/4} (\det \Omega)^{1/4} \exp\left(-\frac{x \cdot \Omega \cdot x}{2}\right) \quad (5.99)$$

with Ω as the square root of K . Assume

$$K = U^T K_D U \quad (5.100)$$

where K_D is diagonal and U is orthogonal, then

$$\Omega = U^T K_D^{1/2} U. \quad (5.101)$$

In this case the traced over density matrix would be

$$\rho_{out}(x_{n+1}, \dots, x_N; x'_{n+1}, \dots, x'_N) = \prod_{i=n+1}^N \rho_i(z_i, z'_i) \quad (5.102)$$

and so the entropy is

$$S(K_{ij}) = \sum_i S(\xi_i). \quad (5.103)$$

The term $\rho_i(z_i, z'_i)$ is calculated for two oscillators with one being traced over, and $S(\xi_i)$ is the entropy related to this term. $\rho_i(z_i, z'_i)$ is an exponential function of the form

$$\rho_i(z_i, z'_i) = e^{-\frac{z^2 + z'^2}{2} + \beta'_i z_i z'_i}. \quad (5.104)$$

Each term in this product is identical to the ρ_{out} of equation (5.89), with

$$\begin{aligned}\gamma &\rightarrow 1 \\ \beta &\rightarrow \beta'_i.\end{aligned}\tag{5.105}$$

Comparing the equations (5.78) and (5.83) we see that the matrix K_{ij} is

$$\begin{aligned}K_{ij} &= \frac{1}{i^2} \left[l(l+1) \delta_{ij} + \frac{9}{4} \delta_{i1} \delta_{j1} + \left(N - \frac{1}{2}\right)^2 \delta_{iN} \delta_{jN} + \left(\left(i + \frac{1}{2}\right)^2 + \left(i - \frac{1}{2}\right)^2 \right) \delta_{i,j(i \neq 1, N)} \right] \\ &- \left[\frac{(j + \frac{1}{2})^2}{j(j+1)} \right] \delta_{i,j+1} - \left[\frac{(i + \frac{1}{2})^2}{i(i+1)} \right] \delta_{i,j-1}.\end{aligned}\tag{5.106}$$

In this case Srednicki [74] plotted the graph of entropy versus \mathcal{R}^2 and it turned out to be a straight line. The factor \mathcal{R}^2 is proportional to area of the spherical box (\mathcal{R} is the radius) and so the entropy for this case is proportional to area of the horizon

$$S = 0.3 \left(\frac{\mathcal{R}}{a} \right)^2,\tag{5.107}$$

where a is the lattice spacing. This is a significant result because one would normally expect that the calculated entanglement entropy can be a candidate for the entropy of a black hole.

5.5 Excited states

In the last section we reviewed works showing that the entanglement entropy could explain the source of the entropy of a black hole. However, all the literature above is generated with the assumption that the coupled oscillators are in their ground states. So it makes sense to ask about the situation when the oscillators are not necessarily in their ground states [76].

Let us now consider the excited states of the N harmonic oscillators discussed in the previous section. The corresponding wave-function is:

$$\psi(x_1, \dots, x_N) = \prod_{i=1}^N N_i H_{\nu_i} \left(k_D^{\frac{1}{4}} \underline{x}_i \right) \exp \left(-\frac{1}{2} k_D^{\frac{1}{2}} \underline{x}_i^2 \right), \quad (5.108)$$

where

$$N_i = \frac{k_D^{\frac{1}{4}}}{\pi^{1/4} \sqrt{2^{\nu_i} \nu_i!}}. \quad (5.109)$$

$K_D \equiv UKU^T$ is a diagonal matrix ($U^T U = I_N$) with elements k_{Di} , $\underline{x} = Ux$, $\Omega = U^T K_D^{\frac{1}{2}} U$, such that $|\Omega| = |K_D|^{\frac{1}{2}}$, $x^T = (x_1, \dots, x_N)$, $\underline{x}^T = (\underline{x}_1, \dots, \underline{x}_N)$ and ν_i ($i = 1 \dots N$) are indices of the Hermite polynomials (H_{ν}). Note that the frequencies are ordered such that $k_{Di} > k_{Dj}$ for $i > j$. The density matrix, tracing over first n of the N field points, is given by:

$$\begin{aligned} \rho(t; t') &= \int \prod_{i=1}^n dx_i \psi(x_1, \dots, x_n; t_1, \dots, t_{N-n}) \psi^*(x_1, \dots, x_n; t'_1, \dots, t'_{N-n}) \\ &= \int \prod_{i=1}^n dx_i \exp \left[-\frac{x^T \cdot \Omega \cdot x}{2} \right] \times \prod_{i=1}^N N_i H_{\nu_i} \left(k_D^{\frac{1}{4}} \underline{x}_i \right) \\ &\quad \times \exp \left[-\frac{x'^T \cdot \Omega \cdot x'}{2} \right] \times \prod_{j=1}^N N_j H_{\nu_j} \left(k_D^{\frac{1}{4}} \underline{x}'_j \right), \end{aligned} \quad (5.110)$$

The definitions used in the density matrix above are

$$\begin{aligned} t_j &\equiv x_{n+j} \\ j &= 1 \dots (N-n). \end{aligned} \quad (5.111)$$

This implies:

$$\begin{aligned} x^T &= (x_1, \dots, x_n, t_1, \dots, t_{N-1}) \\ &= (x_1, \dots, x_n; t) \end{aligned} \quad (5.112)$$

with

$$t \equiv t_1, \dots, t_{N-n}. \quad (5.113)$$

Also we have

$$\Omega = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}. \quad (5.114)$$

A, B, C are $n \times n$, $n \times (N-n)$ and $(N-n) \times (N-n)$ matrices respectively. The evaluation for the integral of the product of $2N$ Hermite polynomials is, in general, non-trivial. In order to keep the calculations tractable, we consider two specific physical cases: (i) coherent states and (ii) superposition of ground and first excited states.

The coherent state wave function for a single harmonic oscillator has the following form:

$$\psi_{CS}(x, a) \equiv \psi_0(x - a) = e^{-i\hat{p}a} \psi_0(x) \quad (5.115)$$

The expectation of the position operator, w.r.t the coherent state wave function, oscillates in time with an amplitude a and the state has the minimum allowable uncertainty ($\hbar = 1$)

$$\Delta p \Delta x = \frac{1}{2}, \quad (5.116)$$

same as that of the ground state. For two coupled oscillators, the corresponding coherent state is:

$$\psi_{CS}(x_1, x_2) \equiv \psi_{CS}(x_+, a) \psi_{CS}(x_-, b) = \psi_0(x_+ - a) \psi_0(x_- - b). \quad (5.117)$$

Defining $\tilde{x}_2 = x_2 - (a - b)/\sqrt{2}$, it is easy to show that the corresponding density matrix retains the same form as (5.89), albeit in terms of these new variables:

$$\rho_{CS}(x_2, x'_2) = \rho_{out}(\tilde{x}_2, \tilde{x}'_2). \quad (5.118)$$

Thus, from Eqs. (5.96,5.97), it follows that the eigenfunctions are $f_n(\tilde{x})$ and eigenvalues remain unchanged (p_n), and we get the rather surprising result that the entropy is the same as that for the ground state, though its not the ground state itself. So coherent state and ground state are equientropy states.

Next, we consider the superposition of the ground and first excited state of the 2-harmonic oscillators system:

$$\psi(x_1, x_2) = \alpha_1 \psi_1(x_+) \psi_0(x_-) + \beta_1 \psi_0(x_+) \psi_1(x_-) + \gamma_1 \psi_0(x_+) \psi_0(x_-) \quad (5.119)$$

where

$$\alpha_1^2 + \beta_1^2 + \gamma_1^2 = 1. \quad (5.120)$$

In this equation

$$\psi_n(x) = N_n(\omega) e^{-\omega^2 x^2/2} H_n(\sqrt{\omega} x), \quad N_n(\omega) = \left(\frac{\omega}{\pi}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} \quad (5.121)$$

is the n^{th} excited state of an oscillator. Although from the identity of particles one would expect $\alpha_1 = \beta_1$, we do not impose such a condition at this point. From (5.110), the density matrix follows:

$$\rho(x_2, x'_2) = \rho_0(x_2, x'_2) [A (x_2^2 + x'^2_2) + B x_2 x'_2 + C (x_2 + x'_2) + D], \quad (5.122)$$

where $\rho_0(x_2, x'_2)$ is the ground state density matrix given by Eq. (5.89), and the constants

are given as:

$$\begin{aligned}
A &= \alpha_1^2 a + \beta_1^2 a_3 + \alpha_1 \beta_1 a_4, & B &= \alpha_1^2 b + \beta_1^2 b_3 + \alpha_1 \beta_1 b_4, & C &= \gamma_1 (\alpha_1 a_6 + \beta_1 a_7), \\
D &= \alpha_1^2 c + \beta_1^2 c_3 + \alpha_1 \beta_1 c_4 + \gamma^2, & a_6 &= \frac{2\sqrt{\omega_-} R}{1+R^2}, & a_7 &= -\frac{2\sqrt{\omega_-} R^2}{1+R^2}, \\
a &= \frac{R^2(1-R^2)(3+R^2)\omega_-}{4(1+R^2)^2}, & b &= \frac{R^2(5+2R^2+R^4)\omega_-}{2(1+R^2)^2}, & c &= \frac{R^2}{1+R^2}, \\
a_3 &= -\frac{(1-R^2)(1+3R^2)\omega_-}{4(1+R^2)^2}, & b_3 &= \frac{(1+2R^2+5R^4)\omega_-}{2(1+R^2)^2}, & c_3 &= \frac{1}{1+R^2}, \\
a_4 &= \left(\frac{1-R^2}{1+R^2}\right)^2 \frac{\omega_- R}{2}, & b_4 &= -\frac{R(1+6R^2+R^4)\omega_-}{(1+R^2)^2}, & c_4 &= \frac{2R}{1+R^2}.
\end{aligned} \tag{5.123}$$

It can be verified that: $Tr(\rho) = \int_{-\infty}^{\infty} dx_2 \rho(x_2, x_2) = \alpha_1^2 + \beta_1^2 + \gamma_1^2 = 1$. To find the eigenvalues of the density matrix (5.122), we follow the general procedure outlined in ref. [77]. First, we expand $\rho(x_2, x'_2)$ in terms of general harmonic oscillator's eigenstates (although, in principle any complete set of functions should suffice):

$$\rho(x_2, x'_2) = \sum_{n=0}^{\infty} h_n(x_2) g_n(x'_2), \quad h_m(x_2) = N_m(\alpha) \exp\left(-\frac{\alpha x_2^2}{2}\right) H_m(\sqrt{\alpha} x_2). \tag{5.124}$$

Inverting, we get:

$$\begin{aligned}
g_m(x'_2) &= \int_{-\infty}^{\infty} dx_2 \rho(x_2, x'_2) h_m(x_2) \\
&= p_m N_m e^{-\frac{\gamma x_2'^2}{2} - \frac{(\beta x_2')^2}{2(\gamma+\alpha)}} [(B_1 x'_2 + E_1) H_{m+1}(\sqrt{\alpha} x'_2) + (C_1 x_2'^2 + D_1 + F_1 x'_2) H_m(\sqrt{\alpha} x'_2)],
\end{aligned} \tag{5.125}$$

where

$$\begin{aligned}
B_1 &= -\sqrt{\alpha} \left[\frac{2\bar{a}\gamma}{\gamma^2 - \alpha^2} + \frac{\bar{b}}{\sqrt{\gamma^2 - \alpha^2}} \right], & C_1 &= \frac{2\bar{a}\gamma}{\gamma - \alpha} + \bar{b} \sqrt{\frac{\gamma + \alpha}{\gamma - \alpha}}, & D_1 &\equiv D_{11} + D_{12}, \\
D_{11} &= \frac{\bar{a}}{\gamma + \alpha} + \bar{c}, & D_{12} &= -\frac{2\bar{a}\alpha}{\gamma^2 - \alpha^2}, & E &= -\frac{\bar{d}\sqrt{\alpha}}{\sqrt{\gamma^2 - \alpha^2}}, & F_1 &= \bar{d} \left[1 + \sqrt{\frac{\gamma + \alpha}{\gamma - \alpha}} \right]
\end{aligned} \tag{5.126}$$

$$\bar{a} = \alpha_1^2 a + \beta_1^2 a_3 + \alpha_1 \beta_1 a_4, \quad \bar{b} = \alpha_1^2 b + \beta_1^2 b_3 + \alpha_1 \beta_1 b_4, \quad \bar{c} = \alpha_1^2 c + \beta_1^2 c_3 + \alpha_1 \beta_1 c_4 + \gamma^2, \quad \bar{d} = \alpha_1 a_6 + \beta_1 a_7,$$

and a_i, b_i have been defined in Eq. (5.123).

The next step is to define the matrix equivalent of ρ , i. e.,

$$\begin{aligned} \alpha_{pm} &\equiv \int_{-\infty}^{\infty} dx g_m(x) h_p(x) \\ &= p_m \left[\left(D_{11} + pD_{12} + \frac{B_1(p+1)}{\sqrt{\alpha}} + \frac{C_1(2p+1)}{2\alpha} \right) \delta_{pm} + \frac{C_1}{2\alpha} \sqrt{(p+1)(p+2)} \delta_{p,m-2} \right. \\ &\quad \left. + \sqrt{p(p-1)} \left(\frac{B_1}{\sqrt{\alpha}} + \frac{C_1}{2\alpha} \right) \delta_{p,m+2} + F_1 \sqrt{\frac{p+1}{2\alpha}} \delta_{p+1,m} + \left(E_1 \sqrt{2p} + F_1 \sqrt{\frac{p}{2\alpha}} \delta_{p-1,m} \right) \delta_{p-1,m} \right]. \end{aligned} \quad (5.127)$$

Although formally diagonalisable, the eigenvalues l_p of the above pentadiagonal matrix are most easily found numerically. With MAPLE, using up to 40×40 matrices, we verified that it has unit trace.

$$Tr(\alpha_{pm}) = \sum_{m=0}^{\infty} \alpha_{mm} = 1, \quad \alpha_{mm} = p_m \left[\left(D_{11} + \frac{B_1}{\sqrt{\alpha}} + \frac{C_1}{2\alpha} \right) + m \left(D_{12} + \frac{B_1}{\sqrt{\alpha}} + \frac{C_1}{\alpha} \right) \right], \quad (5.128)$$

The corresponding entropy as function of α_1, β_1, R defined as:

$$S(\alpha_1, \beta_1, R) = - \sum_{p=0}^{\infty} l_p \ln l_p \quad (5.129)$$

was also computed numerically, and for all $\alpha_1, \beta_1 \neq 0$ it was found that $S(\alpha_1, \beta_1, R) \geq S(0, 0, R)$, where $S(0, 0, R)$ is the ground state entropy. The equality holds *only* in the uncoupled limit $R = 1$ and $\alpha_1 = \beta_1$. These features are visible in figure (5.1), where we have plotted entropies for the excited state [$\alpha_1 = \beta_1 = 1/\sqrt{2}, \gamma_1 = 0$] as well as the ground state. In brief, any amount of excited state in the superposition increases the entropy.

5.6 N coupled oscillators

Reference [78] shows some recent results. As one more step forward, N harmonic oscillators are traced with one of them in the first excited state and the others are in

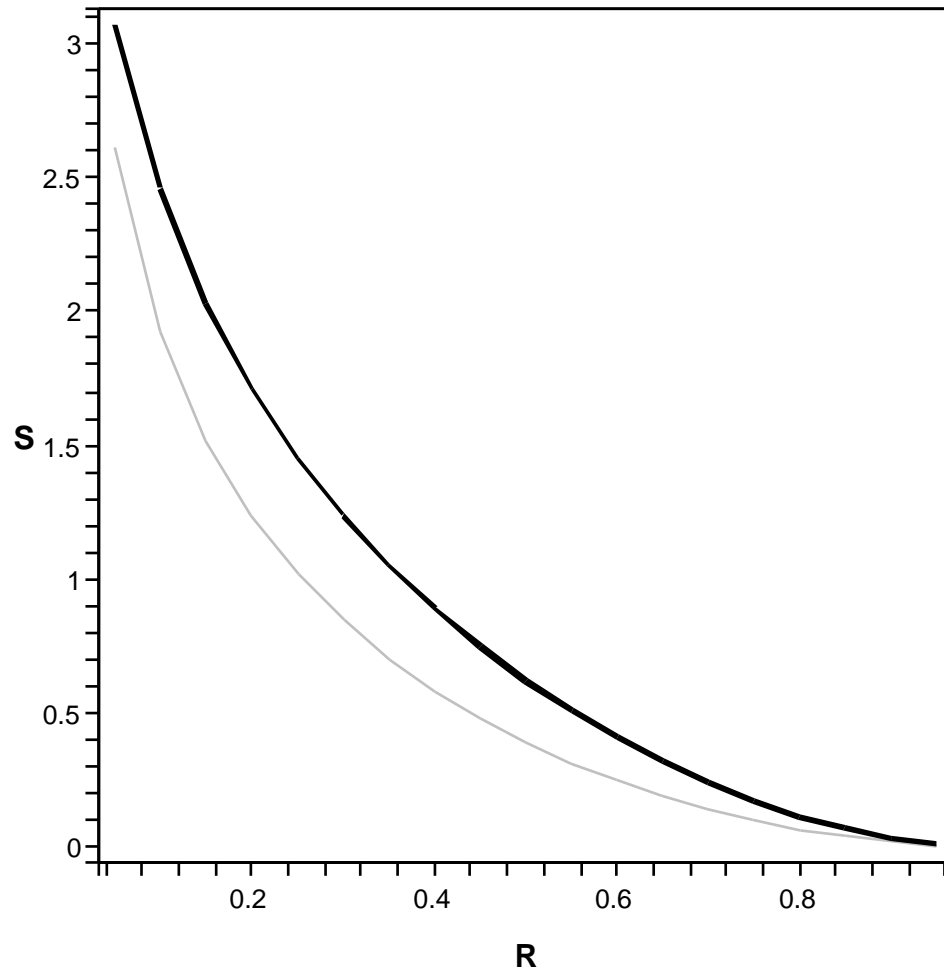


Figure 5.1: Plots of the entanglement entropy of the excited state $S(1/\sqrt{2}, 1/\sqrt{2}, R)$ (black curve) and entanglement entropy of the ground state $S(0, 0, R)$ (grey curve) vs. R . Note that excited state entropy is greater than the ground state entropy for all $R < 1$.

the ground state. The exact density matrix for the discretized scalar field with any one harmonic oscillator in the first excited state while the rest are in the ground state is found. We assume that there is a linear superposition of N wave functions. Each wave function has exactly one harmonic oscillator in the first excited state and the rest ($N - 1$) are in their ground state. The wave function for this set up is

$$\begin{aligned}\psi_1(x_1 \dots x_N) &= \left| \frac{\Omega}{4\pi^N} \right|^{\frac{1}{4}} \sum_{i=1}^N a_i H_1 \left(k_{Di}^{\frac{1}{4}} \underline{x}_i \right) \exp \left[-\frac{1}{2} k_{Di}^{\frac{1}{2}} \underline{x}_i^2 \right] \\ &= \sqrt{2} \left(a^T K_D^{\frac{1}{2}} \underline{x} \right) \psi_0(x_1, \dots, x_N,) \end{aligned} \quad (5.130)$$

with

$$a^T = (a_1, \dots, a_N) \quad (5.131)$$

as expansion coefficients. Normalization of ψ_1 gives

$$a^T a = 1. \quad (5.132)$$

To find the entanglement entropy we need the density matrix presented in equation (5.110).

Now one can put ψ_1 in this density matrix and get

$$\begin{aligned}\rho(t, t') &= 2 \int \prod_{i=1}^n dx_i (x'^T \Lambda x^T) \psi_0(x_i; t) \psi_0^*(x_i; t') \\ &= \rho_0(t, t') \text{Tr}(\Lambda_A A^{-1}) \\ &\times \left[1 - \frac{1}{2} (t^T \Lambda_\gamma t + t'^T \Lambda_\gamma t') + t^T \Lambda_\beta t' \right]. \end{aligned} \quad (5.133)$$

Λ is a $N \times N$ matrix defined by

$$\Lambda = U^T K_D^{\frac{1}{4}} a a^T K_D^{\frac{1}{4}} U \equiv \begin{pmatrix} \Lambda_A & \Lambda_B \\ \Lambda_B^T & \Lambda_C \end{pmatrix}, \quad (5.134)$$

and $\Lambda_A, \Lambda_B, \Lambda_C$ are $n \times n, (N - n) \times n, (N - n) \times (N - n)$ matrices respectively. We also have the following definitions:

$$\begin{aligned}\Lambda_\gamma &= \frac{2\Lambda_B^T (A^{-1}B) - B^T (A^{-1})^T \Lambda_A A^{-1}B}{\text{Tr}(\Lambda_A A^{-1})} \\ \Lambda_\beta &= \frac{2\Lambda_C + B^T [A^{-1}]^T \Lambda_A A^{-1}B - \Lambda_B^T A^{-1}B - B^T [A^{-1}]^T \Lambda_B}{\text{Tr}(\Lambda_A A^{-1})}.\end{aligned}\quad (5.135)$$

A and B are defined in equation (5.114). The authors have shown that the approximation

$$\begin{aligned}1 - \frac{1}{2} (t^T \Lambda_\gamma t + t'^T \Lambda_\gamma t') + t^T \Lambda_\beta t' \\ \simeq \exp \left[-\frac{1}{2} (t^T \Lambda_\gamma t + t'^T \Lambda_\gamma t') + t^T \Lambda_\beta t' \right]\end{aligned}\quad (5.136)$$

is valid for a large value of N ($N > 60$) within 1% error for $a^T = \frac{1}{\sqrt{o}}(0, \dots, 0, 1, \dots, 1)$ with the last o columns being non-zero. One reason to use such approximation is that unlike the ground state (5.89), the excited state density matrix (5.133) contains non-exponential terms. Indeed, without such approximation, this makes the calculation of the entanglement entropy impossible. The density matrix can therefore be simplified to

$$\begin{aligned}\rho(t, t') &= \left[\frac{|\Omega|}{|A| \pi^{N-n}} \right]^{\frac{1}{2}} \text{Tr}(\Lambda_A A^{-1}) \\ &\times \exp \left(-\frac{1}{2} (t^T \gamma' t + t'^T \gamma' t') + t^T \beta' t' \right).\end{aligned}\quad (5.137)$$

This mathematical form is exactly the same as for the ground state, with the replacements

$$\begin{aligned}\beta &\rightarrow \beta' \\ \gamma &\rightarrow \gamma'\end{aligned}\quad (5.138)$$

where

$$\begin{aligned}\beta' &\equiv \beta + \Lambda_\beta, \\ \gamma' &\equiv \gamma + \Lambda_\gamma.\end{aligned}\quad (5.139)$$

So the entropy is calculated in the same way as for the ground state. The reported result shows that the energy of these excited states are about 30 – 60% higher than the ground state energy. For a constant number of oscillators N , with increasing the number of traced over oscillators n the coefficient of the area increases. Also the power of area become less than one and deceases with the increase of n .

In figure (5.2), $\log(S)$ versus $\log(\mathcal{R}/a)$ has been plotted. From the best-fit curves, we see that for $o = 10, 20, 30, 40, 50$, $S = 0.4744(\mathcal{R}/a)^{0.9479}$, $0.6331(\mathcal{R}/a)^{0.9223}$, $0.9669(\mathcal{R}/a)^{0.8848}$, $1.8511(\mathcal{R}/a)^{0.8255}$, $4.002(\mathcal{R}/a)^{0.7571}$ respectively. Thus, although the coefficient in front increases, the power decreases with the number of excited states. Also, for large enough areas, the ground state (or closely related generalized coherent states or squeezed states) entropy is greater than the excited state entropy. So it might be possible that if the entanglement entropy of a superposition of the ground state and excited state is computed, it would (at least approximately) be a sum of the ground state entropy (the area law) and the excited state entropy, in which case, the latter can be interpreted as (power-law) corrections to the area law.

In figure (5.3), the entropy for each partial wave, $(2l+1)S_l$, has been plotted versus l , for $N = 300$ and various values of n . For each n , $o = 10, 30, 50$ has been applied. It can be seen that for the ground state, there is a maxima at $l = 0$, after which $(2l + 1)S_l$ decreases. Once it reaches a minimum, it starts to rise again, due to the large degeneracy factor $(2l + 1)$. For the excited state, however, a sharp maximum occurs between $l = 5$ and $l = 30$, depending on the parameter o .

The Bekenstein-Hawking area law determines the proportionality constant as $1/\ell_{Pl}^2$.

However, the analysis above, indicates that the constant of proportionality and the power of the area can depend on the choice of the state of the scalar field. An immediate question related to this discussion is then: which states determine the Bekenstein-Hawking entropy?

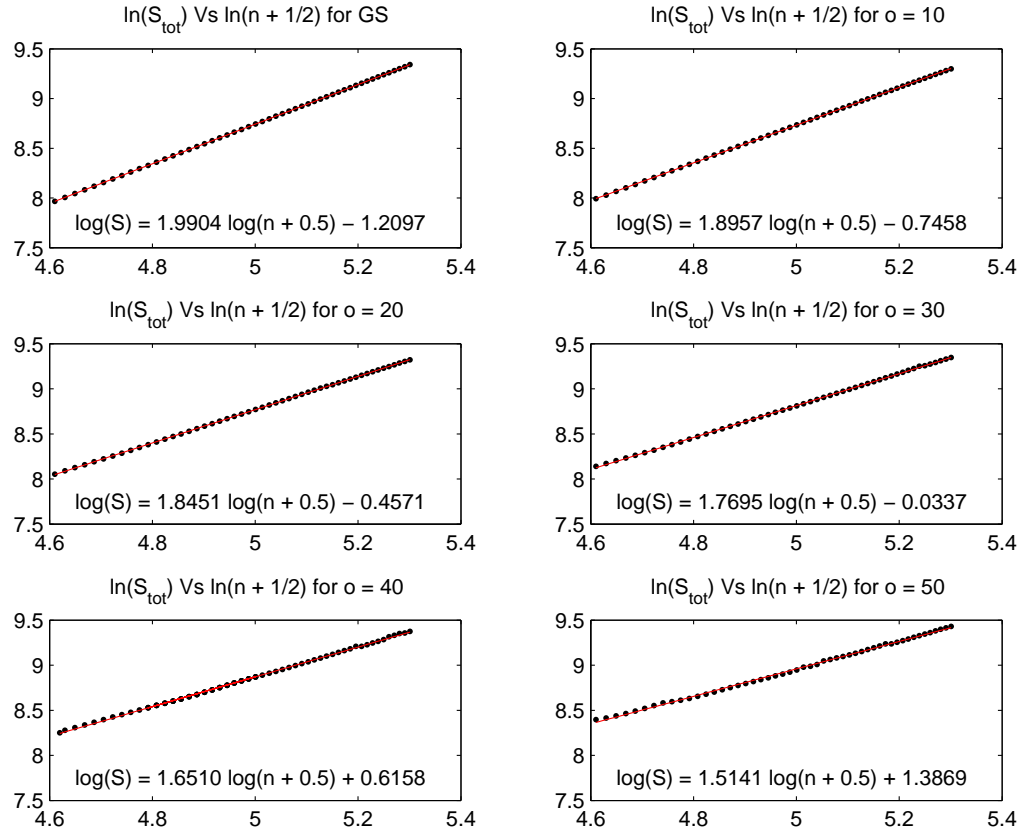


Figure 5.2: Logarithm of ground state and excited state entropies versus the radius of the sphere (\mathcal{R}/a) i. e. $\mathcal{R} = a(n + 1/2)$ for $N = 300$ and $100 \leq n \leq 200$. The maximum value of l is selected such that $[S(l_{\text{max}}) - S(l_{\text{max}} - 5)]/S(l_{\text{max}} - 5) < 10^{-3}$. The numerical error in the total entropy is less than 0.1%.

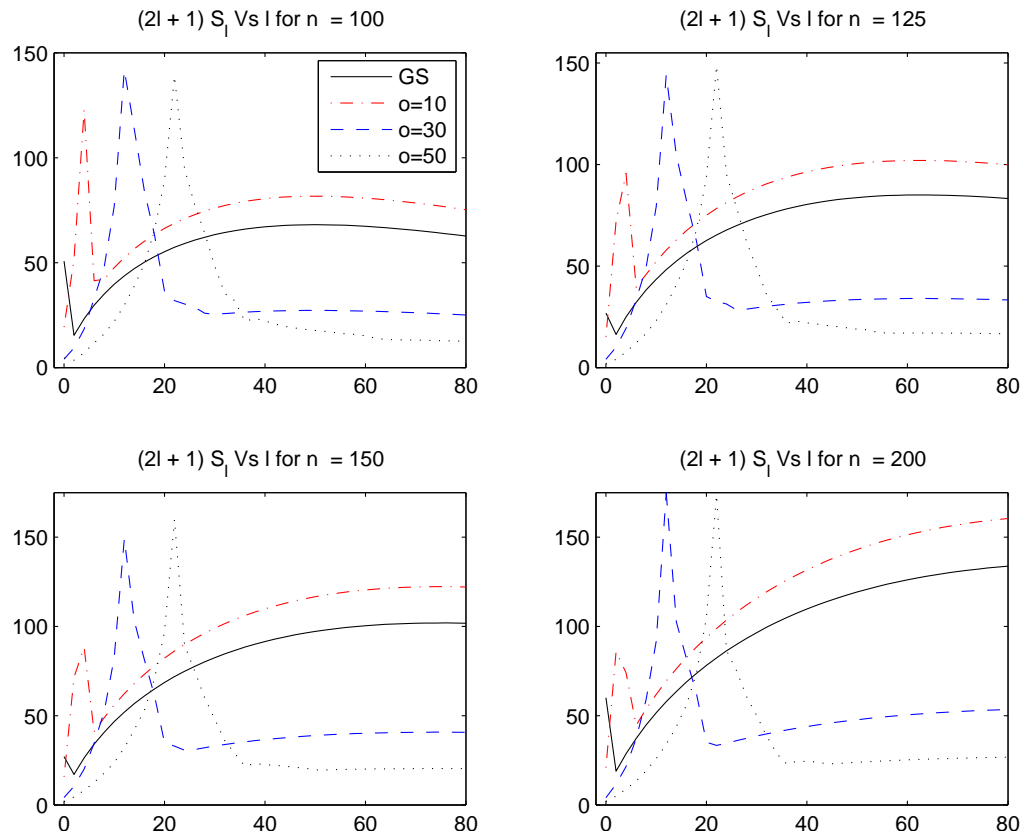


Figure 5.3: Plot of the distribution of entropy per partial wave $[(2l+1)S_l]$ for ground state (solid-curves) and excited state (dotted-curves). To illustrate the difference between the ground state and excited state (and that all curves can be fitted in the same graph), the ground state entropy per partial wave has been multiplied by a factor of 5, while the $o = 10$ and $o = 30$ curves have been multiplied by factors of 6 and 2 respectively in each plots.

Chapter 6

Evidence of black hole

thermodynamics in future colliders

While there is little doubt that black holes exist, we do not have any unambiguous, direct evidence for their existence so far. Many astronomers believe that quasars are powered by black holes (from slightly above the Chandrasekhar limit of $1.5 M_{\odot}$ to millions of M_{\odot} with M_{\odot} as the solar mass), and that there are supermassive ($\sim 10^6 M_{\odot}$) black holes at the centers of many galaxies, including our own. Three main black hole regimes are [79, 80] :

- Stellar-mass black holes formed after the death of some normal stars ($\sim 4 - 15 M_{\odot}$).
- Super-massive black holes formed in the centres of galaxies as a result of the processes of galactic dynamics. Collapse of super-massive stars or relativistic star clusters might also produce such black holes. The mass of super-massive black holes is almost $10^6 - 10^{10} M_{\odot}$.

- Black holes formed as a result of fluctuations or phase transitions in the early universe when conditions were so extreme that black holes of all masses might have been produced.

The best indirect evidence of the existence of black holes is the spectrum and periodicity in binary systems. Astronomers are also looking for flares of large objects falling into supermassive black holes. The X-ray emission from the accretion disks of black holes not only provides a powerful diagnostic of accretion disk physics, it also provides the most efficient means of detecting black holes, especially in external galaxies. With the launch of Chandra [81] and XMM-Newton [82], it is now possible to detect significant numbers of black hole X-ray binary systems in nearby galaxies, as well as study the X-ray spectra of supermassive black holes in greater detail. People also hope to observe gravitational waves from black hole collisions in the LIGO and VIRGO projects [83, 84, 85]. Here we discuss some aspects of the possibilities of production and observation of black holes at the Large Hadron Collider [86].

It has been estimated that the super-TeV particle colliders will be black hole factories at a production rate of a few per second. After production, a black hole starts to decay and radiates all its excess electric and magnetic multipoles. The initial shape of the black hole would be highly asymmetric, which settles down to a stationary state then. This is called the *balding phase*. The final state of the balding phase in four dimensions would be a Kerr-Newman black hole. After this phase we will have a spinning phase and then it starts to lose angular momentum which is called *spin-down phase*. The radiation in this phase, which carries away the angular momentum, is mainly along the equatorial plane [87, 88].

The remaining degrees of freedom is radiated after the spin-down phase through Hawking radiation. This phase is called *Schwarzschild phase* [89, 90, 51, 91] which we believe will end in either an explosion or leave something else behind, which we refer to as a remnant [92, 93].

6.1 Black holes at accelerators

According to the equation (3.68), $M_{Pl}^2 \sim (M_{Pl,D})^{D-2} \left(\frac{cR}{2\pi\hbar}\right)^n$, length of compact extra dimensions can be written as

$$R \sim \frac{2\pi\hbar}{c M_{Pl,D}} \left(\frac{M_{Pl}}{M_{Pl,D}}\right)^{\frac{2}{n}}. \quad (6.1)$$

n is the number of extra dimensions. We put $M_{Pl,D}$ as

$$M_{Pl,D} \sim 1 \text{ TeV} \quad (6.2)$$

and we get

$$R \propto \begin{cases} 8 \times 10^{12} \text{m} & \text{if } n = 1 \\ 0.2 \text{ mm} & \text{if } n = 2 \\ 3 \text{ nm} & \text{if } n = 3 \\ 6 \times 10^{-12} \text{m} & \text{if } n = 4 \end{cases} \quad (6.3)$$

The significance of the energy 1 TeV is that the next generation of colliders are expected to have this energy and above. Newton's law has not been tested for distances less than ~ 1 mm (as of 1998) and the fundamental Planck scale could be as low as 1 TeV for $n > 1$.

The main idea about the production of the black hole in a collider is as follows: when the energy of the center of mass reaches the fundamental Planck scale, a black hole

is formed and we approximate the cross section by the area of a “black disk”,

$$\sigma \sim \pi R_s^2 \sim 1\text{TeV}^{-2} \sim 10^{-38} m^2 \sim 100\text{pb}, \quad (6.4)$$

where σ is the cross section and R_s is the black disk radius. To be more precise, we can use the Schwarzschild radius R_s of an $(4+n)$ -dimensional black hole [19] :

$$R_s = \frac{1}{\sqrt{\pi} M_{Pl}} \left[\frac{M_{BH}}{M_{Pl}} \left(\frac{8\Gamma(\frac{n+3}{2})}{n+2} \right) \right]^{\frac{1}{n+1}} \quad (6.5)$$

with the assumption that the extra dimensions are large ($\gg R_s$). Now consider two partons moving in opposite directions and having the center of mass energy

$$\sqrt{s} = M_{BH}. \quad (6.6)$$

In the semiclassical view, one can say that if the impact parameter is less than the radius above (R_s), then a black hole with mass M_{BH} is produced. The total cross section is approximated by the area of a disk of radius R_s ,

$$\sigma(M_{BH}) \sim \pi R_s^2 = \frac{1}{M_{Pl}^2} \left[\frac{M_{BH}}{M_{Pl}} \left(\frac{8\Gamma(\frac{n+3}{2})}{n+2} \right) \right]^{\frac{2}{n+1}}. \quad (6.7)$$

The dependence of the cross section on n is weak and would be hardly noticeable on the logarithmic scale [98]. In equation (6.7), if the parton center of mass energy, which is equal to M_{BH} , reaches the fundamental Planck scale $M_{Pl} \sim TeV$ then the cross section is of order of $TeV^{-2} = 400 pb$. 1 barn (b) is equal to 10^{-28} square meters (m^2), so

$$1 pb = 10^{-40} m^2. \quad (6.8)$$

At the LHC, with the total center of mass energy $\sqrt{s} = 14 TeV$, production of black holes is then expected.

After formation, the decay happens governed by the Hawking temperature T_H of the black hole. Hawking radiation is proportional to the inverse of the radius of the black hole:

$$T_H = M_{Pl} \left(\frac{M_{Pl}}{M_{BH}} \frac{n+2}{8\Gamma\frac{n+3}{2}} \right)^{\frac{1}{n+1}} \frac{n+1}{4\sqrt{\pi}}. \quad (6.9)$$

If we increase the parton collision energy, the formed black hole would be heavier and its decay products get colder. Another point is that the wavelength corresponding to the Hawking radiation is

$$\lambda = \frac{2\pi}{T_H} \quad (6.10)$$

which is larger than the size of black hole. Thus, to first approximation the black hole behaves as a point-radiator [98].

An important limitation is that the semi-classical approach is strictly valid only when

$$M_{BH} \gg M_{Pl}, \quad (6.11)$$

however we are using

$$M_{BH} > M_{Pl} \quad (6.12)$$

instead, but there is no way around this limitation without a knowledge of quantum gravity.

6.2 General uncertainty principle and thermal fluctuations effects

The general uncertainty principle, which is an extension of the quantum mechanical uncertainty principle, originates from the fact that a minimum length is expected in

quantum gravity [94]. The basic ideas behind this minimum length are quantum mechanics, special relativity and general relativity. Thus, classical notions such as causality or distance between events cannot be expected to be applicable at this scale. Such length, as a lower bound to any output of a position measurement, seems to be a model-independent feature of quantum gravity. The general uncertainty principle is represented as

$$\Delta x_i \geq \frac{\hbar}{\Delta p_i} \left[1 + \left(\alpha' l_{Pl} \frac{\Delta p_i}{\hbar} \right)^2 \right] \quad (6.13)$$

with l_{Pl} as the Planck length and α' as a dimensionless constant of order one. This principle can be derived in few contexts (for example string theory [95]). The relation above gives a minimum length as

$$\Delta x_{min} \equiv 2\alpha' l_{Pl}. \quad (6.14)$$

In fact the string regime is recovered by setting $\Delta x_{min} = \Delta x_i$ in equation (6.13). In section (4.3), we mentioned some thermodynamical properties of black holes in D-dimensions. Now we make our black hole by assuming a $(D-1)$ -dimensional cube of size $2r_s$ (r_s is the horizon radius of the D-dimensional Schwarzschild black hole given by equation (4.22)) radiating a massless Hawking particle. One locates such a particle with the uncertainty of

$$\Delta x \sim 2r_s \quad (6.15)$$

or

$$\Delta x = 2Kr_s \quad (6.16)$$

with K as a correction factor of order one that can be calculated for the spherical geometry of the horizon. We approximate the Hawking temperature with the energy uncertainty of

the emitted particle and its associate linear momentum (Boltzman constant = 1),

$$T \sim \Delta E \sim c\Delta p. \quad (6.17)$$

The general uncertainty principle (Equ. 6.13) with the equality sign is a quadratic equation of Δp_i . So if we solve it for Δp_i , the uncertainty in linear momentum can be written as

$$\Delta p = \frac{2\hbar}{\Delta x} \left[\frac{1}{1 + \sqrt{1 - \frac{4l_{Pl}^2 \alpha'^2}{\Delta x^2}}} \right]. \quad (6.18)$$

To find Hawking temperature [93], we use equation (6.17). We substitute Δx from equation (6.16) into relation (6.18) and simplify it to

$$T = \frac{2T_0}{1 + \sqrt{1 - \frac{\alpha^2}{\omega_D^2 \left(\frac{M}{M_{Pl}}\right)^{2/(D-3)}}}} \quad (6.19)$$

with

$$T_0 = \left(\frac{D-3}{4\pi\omega_D} \right) M_{Pl} c^2 \left(\frac{M}{M_{Pl}} \right)^{\frac{1}{3-D}}. \quad (6.20)$$

ω_D is the dimensionless area factor which is defined in equation (4.23) as $\omega_D = \left[\frac{16\pi}{(D-2)\Omega_{D-2}} \right]^{\frac{1}{D-3}}$.

The other constant is α determined as

$$\alpha = \frac{\alpha'}{K}. \quad (6.21)$$

There is a significant point in equation (6.19). This equation illustrates that the general uncertainty principle increases the characteristic temperature of the black hole. Thus, when we consider the general uncertainty principle effect on the Hawking radiation, we get a hotter, shorter-lived black hole with a smaller entropy. That means the black hole evaporates more rapidly. The black hole radiates till reaching a remnant with the minimum mass equal to

$$M_{min} = \frac{D-2}{8\Gamma\left(\frac{D-1}{2}\right)} (\alpha\sqrt{\pi})^{D-3} M_{Pl}. \quad (6.22)$$

Black holes with mass less than M_{min} do not exist, since their horizon radius would fall below the minimum allowed length. Therefore, the black hole temperature is undefined for $M < M_{min}$. To see more details we mention some other physical quantities, e.g. the emission rate on a four-dimensional brane. To find the emission rate, we substitute the temperature T from relation (6.19) into equation (4.25), which is $\frac{dm}{dt} = -\frac{1}{cM_{pl}}\bar{\sigma}_n A(n)T^n$ (for definitions of $\bar{\sigma}$ and $A(n)$ please see the details under this formula in chapter 4). Then the mass rate is represented by

$$\frac{dm}{dt} = 16 \left(\frac{dm}{dt} \right)_0 \left(1 + \sqrt{1 - \frac{\alpha^2}{\omega_D^2 m^{\frac{2}{D-3}}}} \right)^{-4}. \quad (6.23)$$

The quantity m is defined as

$$m = \frac{M}{M_{Pl}} \quad (6.24)$$

and $(dm/dt)_0$ is the mass loss when considering the usual uncertainty principle ($\alpha = 0$) implied by the equation (4.25). The mass rate for an arbitrary n -dimensional brane is then

$$\frac{dm}{dt} = 2^n \left(\frac{dm}{dt} \right)_0 \left(1 + \sqrt{1 - \frac{\alpha^2}{\omega_D^2 m^{\frac{2}{D-3}}}} \right)^{-n}. \quad (6.25)$$

It is also possible to calculate the difference in decay time. The normal decay time, τ_0 , was defined in equation (4.34) as $\tau_0 = \mu^{-1} \left(\frac{D-3}{D-1} \right) m_i^{\frac{D-1}{D-3}} t_{Pl}$. To find τ_{GUP} (decay time after considering general uncertainty principle), we integrate the first order term of Taylor expansion of equation (6.25) over time. This Taylor expansion is

$$\frac{dm}{dt} = -\frac{\mu}{t_{Pl} m^{\frac{2}{D-3}}} \left[1 + \frac{\alpha^2 n}{4\omega_D^2 m^{\frac{2}{D-3}}} + \dots \right]. \quad (6.26)$$

The quantity μ is given by relation (4.33). t_{Pl} is also the Planck time, $t_{Pl} = \left(\frac{\hbar G_D}{c^{D+1}}\right)^{\frac{1}{D-2}}$.

Integrating equation (6.26) over time yields

$$\tau_{GUP} = \frac{1}{\mu} \left(\frac{D-3}{D-1}\right) \left\{ \left[m_i^{\frac{D-1}{D-3}} - \frac{n(D-1)\alpha^2}{4(D-3)\omega_D^2} m_i \right] - \left[1 - \frac{n(D-1)}{4(D-3)} \right] \left(\frac{\alpha}{\omega_D}\right)^{D-1} \right\} t_{Pl}. \quad (6.27)$$

When the initial mass M_i is much bigger than the Planck mass M_{Pl} ($m_i = \frac{M_i}{M_{Pl}} \gg 1$), then the term $\left[1 - \frac{n(D-1)}{4(D-3)} \right] \left(\frac{\alpha}{\omega_D}\right)^{D-1}$ can be ignored. So the difference in the decay time can be written as

$$\frac{\Delta\tau_{GUP}}{\tau_0} \equiv \frac{\tau_{GUP} - \tau_0}{\tau_0} = -\frac{n(D-1)\alpha^2}{4(D-3)\omega_D^2} m_i^{\frac{-2}{D-3}}. \quad (6.28)$$

To find the entropy we use the formula $dS = T^{-1}c^2dM$. We have the temperature T as a function of mass M from equations (6.19) and (6.20). The entropy turns out to be

$$S = 2\pi\omega_D \left(\frac{\alpha}{\omega_D}\right)^{D-2} I\left(1, D-4, \frac{\omega_D m^{\frac{1}{D-3}}}{\alpha}\right) \quad (6.29)$$

and I is an integral defined by

$$I(m, n, x) = \int_1^x dz z^n (z + \sqrt{z^2 - 1})^m. \quad (6.30)$$

The heat capacity is calculated through

$$C = T \frac{\partial S}{\partial T} \quad (6.31)$$

which gives

$$C = -2\pi\omega_D m^{\frac{D-2}{D-3}} \left(1 + \sqrt{1 - \frac{\alpha^2}{\omega_D^2 m^{\frac{2}{D-3}}}} \right) \sqrt{1 - \frac{\alpha^2}{\omega_D^2 m^{\frac{2}{D-3}}}} \quad (6.32)$$

It vanishes at the endpoint (no heat exchange with the environment). The temperature of the endpoint also would be the maximum possible temperature of the equation (6.19).

In such situation the black hole is characterized by a Planck-size remnant. So mass M would be equal to M_{min} in this case. To maximize T , we need to maximize the numerator and minimize the denominator for $M = M_{min}$ in relation (6.19). The minimum value of the denominator is 1. The maximum value of the numerator is $2T_0$. So the maximum temperature would be

$$T_{max} = 2T_0|_{M=M_{min}}. \quad (6.33)$$

The general uncertainty principle prevents black holes from evaporating completely similarly to how the standard uncertainty principle prevents the hydrogen atom from collapsing.

Equations (4.40) and (4.41) illustrate the relation between the multiplicity of particles produced in black hole decay with entropy S . The entropy corrected with general uncertainty principle is smaller than the standard entropy. Then black hole evaporation involves a smaller number of particles with an increase in the average energy of the particles and the Planck-scale remnant would have a zero heat capacity. In fact the expected approach in LHC is a Planck scale in size of TeV. Numerical results [92] show that general uncertainty principle could increase the minimum black hole formation energy beyond a TeV.

Another significant effect originates from thermal fluctuations. The Bekenstein-Hawking entropy is determined with the canonical entropy of the system. A partition function for a canonical ensemble can be written as

$$Z(\beta) = \int_0^\infty \rho(E)e^{-\beta E} dE \quad (6.34)$$

with β as the inverse of temperature:

$$\beta = \frac{1}{T}. \quad (6.35)$$

The inverse Laplace transformation for the partition function at fixed E yields the density of states:

$$\begin{aligned}\rho(E) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} Z(\beta) e^{\beta E} d\beta \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{S(\beta)} d\beta.\end{aligned}\quad (6.36)$$

$S(\beta)$ is defined as

$$S(\beta) = \ln Z(\beta) + \beta E. \quad (6.37)$$

We expand $S(\beta)$ around the saddle point $\beta_0 (= 1/T_0)$, where T_0 is the equilibrium temperature:

$$S = S_0 + \frac{1}{2}(\beta - \beta_0)^2 S_0'' + \dots \quad (6.38)$$

Here the parameters are defined as

$$\begin{aligned}S_0 &= S(\beta_0) \\ S_0' &= \left. \frac{\partial S(\beta)}{\partial \beta} \right|_{\beta=\beta_0} \\ S_0'' &= \left. \frac{\partial^2 S(\beta)}{\partial \beta^2} \right|_{\beta=\beta_0}.\end{aligned}\quad (6.39)$$

So ρ can be written as

$$\begin{aligned}\rho(E) &= \frac{e^{S_0}}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\frac{(\beta-\beta_0)^2 S_0''}{2}} d\beta \\ &= \frac{e^{S_0}}{\sqrt{2\pi S_0''}}.\end{aligned}\quad (6.40)$$

The corrected entropy is given by the logarithm of the density of states $\rho(E)$:

$$S = \ln \rho(E) = S_0 - \frac{1}{2} \ln S_0'' + \dots \quad (6.41)$$

The definitions of energy and specific heat are as follows :

$$\begin{aligned} E &= -\frac{\partial \ln Z(\beta)}{\partial \beta} \Big|_{\beta_0} \\ C &= \frac{\partial E}{\partial T} \Big|_{\beta=\beta_0}. \end{aligned} \quad (6.42)$$

From these two, we can find S''_{β} :

$$\begin{aligned} S''(\beta) &= \frac{1}{Z} \left[\frac{\partial^2 Z(\beta)}{\partial \beta^2} \right] - \frac{1}{Z^2} \left[\frac{\partial Z(\beta)}{\partial \beta} \right]^2 \\ &= \langle E^2 \rangle - \langle E \rangle^2 \\ &= CT^2. \end{aligned} \quad (6.43)$$

Thus, we obtain

$$S = \ln \rho = S_0 - \frac{1}{2} \ln CT^2 + \dots \quad (6.44)$$

This formula applies to any thermodynamic system in equilibrium. Back reaction also will correct non-equilibrium situation. However, it is not applicable to Schwarzschild black holes because of its negative specific heat. But the entropy corrections can be shown to be logarithmic by either assuming a small cosmological constant, or by putting the black hole into a finite box. Such entropy is presented by

$$S_{Thermo} = S_0 - k \ln S_0 \quad (6.45)$$

where k is a positive constant of order unity [93, 96, 97]. Then the correction to the black hole entropy is

$$\Delta S_{Thermo} = -k \ln S_0. \quad (6.46)$$

The corrected Hawking temperature is obtained from the first law of black hole thermody-

namics:

$$T'' = \frac{(D-3)}{4\pi\omega_D} m^{\frac{-1}{D-3}} \left[1 + \frac{k(D-2)}{4\pi\omega_D} m^{-\frac{D-2}{D-3}} + \dots \right] M_{Pl} c^2. \quad (6.47)$$

The first order corrected specific heat is also

$$C = C_0 \left[1 - \frac{k(D-1)(D-2)}{4\pi\omega_D} m^{-\frac{D-2}{D-3}} \right] \quad (6.48)$$

with C_0 as

$$C_0 = -4\pi\omega_D m^{\frac{D-2}{D-3}}. \quad (6.49)$$

The first-order correction to Hawking temperature and specific heat then lead to

$$\Delta T_{Thermo} = \frac{k(D-2)(D-3)}{16\pi^2\omega_D^2} m^{-\frac{D-1}{D-3}} M_{Pl} c^2, \quad (6.50)$$

$$\Delta C = -C_0 \frac{k(D-1)(D-2)}{4\pi\omega_D} m^{-\frac{D-2}{D-3}}. \quad (6.51)$$

This C vanishes for a non-zero mass of

$$m_0 = \left[\frac{k(D-1)(D-2)}{4\pi\omega_D} \right]^{\frac{D-3}{D-2}} \quad (6.52)$$

which shows a mass which black hole reaches when becomes thermodynamically stable.

However, near this mass our first order approximation of equation (6.45) breaks down due

to a large increase in thermal fluctuations. The Stefan-Boltzman law is obtained from

equation (6.47):

$$\frac{dm}{dt} = -\frac{\mu}{t_{Pl} m^{\frac{2}{D-3}}} \left[1 + \frac{kn(D-2)}{4\pi\omega_D} m^{-\frac{D-2}{D-3}} + \dots \right]. \quad (6.53)$$

Then the decay rate first-order correction is implied by

$$\Delta \left(\frac{dm}{dt} \right) = \left(\frac{dm}{dt} \right)_0 \frac{kn(D-2)}{4\pi\omega_D} m^{-\frac{D-2}{D-3}} \quad (6.54)$$

where n again shows the number of dimensions of the spacetime of the brane. The expression for the time decay is obtained if we integrate equation (6.53) over time:

$$\tau_2 = \mu^{-1} \left(\frac{D-3}{D-1} \right) m_i^{\frac{D-1}{D-3}} \left[1 - \frac{kn(D-1)(D-2)}{4\pi\omega_D} m_i^{-\frac{D-2}{D-3}} + \dots \right] t_{Pl}. \quad (6.55)$$

The decay time correction would then be

$$\frac{\Delta\tau_2}{\tau_0} \equiv \frac{\tau_2 - \tau_0}{\tau_0} = - \frac{kn(D-1)(D-2)}{4\pi\omega_D} m_i^{-\frac{(D-2)}{D-3}}. \quad (6.56)$$

So eventually one can say that the thermal fluctuation effect is not negligible for Planck scale black holes and it increases the black hole temperature and decreases the decay time and entropy. That means this effect would reduce the probability of observing black holes at colliders, even if they are produced successfully.

Chapter 7

Summary

In this thesis, aspects of black hole physics were discussed. First of all we demonstrated the fundamental ideas of Einstein's theory of general relativity including the principle of equivalence, geodesic equations of motion, Einstein's equations and their derivation and conservation of energy-momentum. Then we concentrated on the black hole solutions of general relativity. We mentioned Kerr-Newman solution (as the most general solution) and its properties. Then we reviewed two special cases of rotating (Kerr) and charged (Reissner-Nordstrom) black holes. Black holes are dynamic and they radiate. We studied the laws of black hole mechanics and Hawking radiation in the following chapter. Recently black holes are considered as candidates to be produced and observed in high energy colliders. The idea behind that is the possibility of existence of higher dimensions. But, what parts of our point of view of black holes and their properties need to be changed if higher dimensions exist? We answer this question in the last section of chapter 4. Another important question is raised when we consider the black hole entropy: what is the nature of black

hole entropy? There are many candidate theories as the answer of this question and we review two significant ones: the brick wall model and entanglement entropy. Our attention was the solution including treatment of entanglement entropy and its probable connection with black hole entropy. Another issue discussed in this thesis is the possibility of detecting higher dimensions experimentally, if they exist. Thermal properties of black holes change if we consider higher dimensions. In fact, even if the higher dimensions exist, we still require to know whether we can detect their evidence in our colliders or not.

General relativity is based on a set of fundamental principles. The laws of physics must be the same for all observers (accelerated or not) and they must take the same form in all coordinate systems. The world lines of particles unaffected by physical forces besides gravity are timelike or null geodesics of spacetime. In other words, inertial motion is geodesic motion. The laws of special relativity apply locally for all inertial observers. Einstein's equations describe the curvature of spacetime and make the relation between the curvature and energy-momentum.

A noninertial reference frame is equivalent to a certain gravitational field. This is the well-known principle of equivalence in general relativity. This principle was the starting point for the development of general relativity. However, it ended up being a consequence of the general principle of relativity (similarity of the laws of physics for accelerated and non-accelerated observers) and the principle that inertial motion is geodesic motion. Experiments have been performed several times to check this principle and they are still being performed. So far it is compatible with almost all observations to date. One exception might be the so-called Pioneer anomaly [100]. The Pioneer anomaly or Pioneer effect is the

observed deviation from expectations of the trajectories of various unmanned spacecraft visiting the outer solar system, notably number 10 and 11 of the Pioneer program.

The black hole solutions of the Einstein equations of general relativity provide us with a classical view of black hole physics. We categorize the black hole solutions according to the significant physical quantities which appear in the metric: mass, electric charge and angular momentum. The Schwarzschild solution is the simplest one - it neither rotates nor contains any electric charge. On the other hand there is the most complicated one which rotates and has electric charge. This is called the Kerr-Newmann solution. In addition to the horizon surface this black hole has a region called ergosphere. Special cases of Kerr-Newmann solution are those which rotates but are not charged (Kerr black holes) and those which contain electric charge and do not rotate (Reissner-Nordstrom).

The force of gravity in black holes is so strong that even light can not escape from it. However, at such a scale (large black holes), gravity is still much weaker than other fundamental forces in nature. This leads to the hierarchy problem: why the weak force is 10^{32} times stronger than gravity? Recently solutions of hierarchy problem have been investigated in higher dimensions. That is one of the reasons that the solutions of black hole metrics in higher dimensions became significant. There are two scenarios which provide solutions for the hierarchy problem: flat and warped. One difference between these two scenarios is the assumed compact space related to the higher dimensions, which can be flat or warped respectively. In higher dimensional theories, the approach is to look for a way to make the gravitational force comparable in strength to the other forces. If this strength is about 1 TeV , which is about the energy of particle colliders, then experimental

physicists are also interested to see the gravitational effects in such scale. Then at this energy, both the quantum mechanical and gravitational effects would be illustrated. One of these effects can be the production and observation of black holes in colliders. Experiments are planned to test the existence of higher dimensions. In fact if higher dimensions exist, there are some hopes of production and observation of black holes at Large Hadron Collider (LHC) in CERN as the experimental impact parameter would be reduced in this case.

We reviewed the laws of black hole mechanics. They look just like the laws of thermodynamics in general physics, assuming a linear relation between temperature and entropy of a thermodynamical system and surface gravity and area of a black hole, respectively. Also, the second law of thermodynamics is replaced with the generalized second law for black hole mechanics. The generalized second law states that the sum of the entropy outside the black hole and the entropy of the black hole itself will not decrease.

Quantum mechanical effects on black hole mechanics are not studied until we consider Hawking radiation, where \hbar appears in the Hawking temperature. The particles tunnel out of the black hole's horizon and escape the gravitational field of the black hole. However, with some probability they may be absorbed again. So one can calculate the rate of radiation and decay time as well as other thermodynamical properties like entropy and heat capacity. All these calculations can be done for an arbitrary dimension D .

The entropy of the black hole is assumed to be proportional to its area. This is quite different from the entropy-volume proportionality in classical thermodynamics. There are some theories as candidates for explaining this proportionality. The brick wall model is one such theory. It assumes a boundary very close to the horizon and then it calculates the

free energy and total energy of the system. Entropy is computed by subtracting the free energy from the total energy and multiplying that by the inverse of temperature. In this case the entropy turns out to be proportional to area.

Another approach is well-known as entanglement entropy. The discussion is presented by studying scalar fields. This selection makes our calculations easier, though basically it is possible to do the calculations for vector or tensor fields as well. We assume an imaginary spherical box of radius R containing n coupled harmonic oscillators inside connected to $N - n$ coupled harmonic oscillators outside the box. One can define the ground state wave function for the oscillators and calculate the density matrix traced over the oscillators inside the box. Then one can show that the entanglement entropy defined by such density matrix is proportional to the area of the spherical box.

For a system of two coupled oscillators, if the ground state oscillator wave functions are replaced by generalized coherent states, the entropy remains unchanged. If one of the oscillators is in its first excited state, then the entropy would increase as much as 50%. We can then model a free scalar field by means of assuming N -coupled oscillators together. In such case the area law still holds if the oscillators are in generalized coherent states and a class of squeezed states. In fact the entanglement entropy exactly equals that of ground state. If we consider superpositions of a number of wave-functions, each of which has exactly one harmonic oscillator in the first excited state, the entanglement entropy is proportional to a power of the area, but the power is now less than unity. The more terms there are in the superposition, the less is this power. So the lower N is, the less entropy we have in a field theory. One important work which can be done is to do the same calculations for

entanglement entropy to check its proportionality with area in higher dimensions ($D > 4$).

The last chapter of the thesis has been dedicated to the production and observation of black holes in future colliders. The proposed theory of extra dimensions makes the fundamental basis for the experiments planned to be done at LHC, CERN. If the number of extra dimensions is more than 6, then it is predicted that the TeV scale colliders would be a factory of black holes at a rate of at least one black hole per second. However, there are other issues like general uncertainty principle and thermal fluctuations effects involved which are not negligible. The general uncertainty principle plays its role when one adds the correction terms to the usual uncertainty principle of quantum mechanics. As the Hawking radiation comprises particles with some uncertainty in their energy and momentum then the discussion of general uncertainty principle can be applied to such radiated particles. Thermal fluctuations are also there when we expand the entropy of a thermal system around the equilibrium state. In fact as the Hawking radiation spectrum is thermal the discussion of such expansion is valid for black holes. However, both matters above have a negative effect on the radiation of a black hole. They give us black holes which are hotter and radiates faster. So we will have a less decay time and less entropy. In fact, even if the black holes would be produced in colliders, other effects may reduce the probability of observing them.

Bibliography

- [1] Einstein, A., Podolsky, B., Rosen, N.(1935). Can quantum-mechanical description of physical reality be considered complete? *Phys. Rev.*, **47**, 777.
- [2] Antoniadis, I., Arkani-Hamed, N., Dimopoulos, S. (1998). New dimensions at a millimeter to a fermi and superstrings at a Tev. *Phys. Lett. B* **436**, 257. (*hep-ph/9804398*).
- [3] Arkani-Hamed, N., Dimopoulos, S., Dvali, G. R. (1999). Phenomenology, astrophysics and cosmology of theoris with submillimeter dimensions and Tev scale quantum gravity. *Phys. rev. D* **59**, 086004. (*hep-ph/9807344*).
- [4] Randall, L., Sundrum, R. (1999). A large mass hierarchy from a small extra dimension. *Phys. Rev. Lett.* **83**, 3370. (*hep-ph/9905221*).
- [5] Randall, L., Sundrum, R. (1999). An alternative to compactification. *Phys. Rev. Lett* **83**, 4690. (*hep-th/9906064*).
- [6] Emparan, R., Horowitz, G. T., Myers, R. C. (2000). Black holes radiate mainly on the brane. *Phys. Rev. Lett.* **85**, 499. (*hep-th/0003118*).

- [7] Cardoso, V., Cavaglia, M., Gualtieri, L. (2006). Hawking emission of gravitons in higher dimensions: non-rotating black holes. *JHEP* **0602**, 021. (*hep-th/0512116*).
- [8] Landau, L. D., Lifshitz, E. M. (1975). *The classical theory of fields*. Toronto: Pergamon Press.
- [9] Misner, C. W., Thorne, K. S., Wheeler, J. A. (1973). *Gravitation*. New York: W. H. Freeman and Company.
- [10] Carroll, S. M. (1997). Lecture notes on general relativity. *gr-qc/9712019*.
- [11] 't Hooft, G. (2002). Introduction to general relativity. <http://www.phys.uu.nl/~thoof/lectures/genrel.pdf>.
- [12] Waner, S. (2002). Introduction to differential geometry and general relativity. <http://people.hofstra.edu/faculty/Stefan-Waner/diff-geom/tc.html/>.
- [13] Waite, D. (2005). Modern relativity. <http://www.geocities.com/zcphysicsms/>.
- [14] Dunsby, P.(1996). Tensors and relativity. <http://vishnu.mth.uct.ac.za/omei/gr/>.
- [15] Townsend, P. K. (1997). Black holes. *gr-qc/9707012*.
- [16] Hadrava, P. (2002). General relativity. <http://sunkl.asu.cas.cz/had/gr.html>.
- [17] <http://www.kfki.hu/eotvos/onehund.html>.
- [18] <http://www.kfki.hu/eotvos/stepcikk.html>.
- [19] Myers, R. C., Perry, M. J. (1986). Black holes in higher dimensional space-times. *Annals. Phys.*, **172**, 304.

- [20] Johansen, N. V., Ravndal, F. (2006). On the discovery of Birkhoff's theorem. *Gen. Rel. Grav.* **38**, 537. (*physics/0508163*).
- [21] Camenzind, M., Müller, A. (2004). General relativity. <http://www.lsw.uni-heidelberg.de/projects/theory/resource.htm>.
- [22] Reissner, H. (1916). Über die Eigengravitation des elektrischen Felds nach den Einsteinschen Theorie. *Ann. Phys.* **50**, 106.
- [23] Nordström, G. (1918). On the energy of the gravitational field in Einsteins theory. *Proc. Kon. Ned. Akad. Wet.* **20**, 1238.
- [24] Mitra, A. (1999). Kruskal coordinates and mass of Schwarzschild black holes. *astro-ph/9904162*.
- [25] Pelavas, N., Neary, N., Lake, K. (2001). Properties of the instantaneous Ergo Surface of a Kerr Black Hole. *Class. Quant. Grav.* **18**, 1319. (*gr-qc/0012052*).
- [26] <http://public.web.cern.ch/Public/Welcome.html>.
- [27] Krolak, A. (1986). Towards the proof of the cosmic censorship hypothesis. *Class. Quant. Grav.* **3**, 267.
- [28] Wagh, S. M. (2002). Classical formulation of Cosmic Censorship Hypothesis. *gr-qc/0201041*.
- [29] Page, D. N. (2004). Hawking radiation and black hole thermodynamics. *hep-th/0409024*.
- [30] Bekenstein, J. D. (1973). Black holes and entropy. *Phys. Rev. D.* **7**, 2333.

-
- [31] Bekenstein, J. D. (1974). Generalized second law of thermodynamics in black-hole physics. *Phys. Rev. D* **9**, 3292.
- [32] Hawking, S. W. (1975). Particle creation by black holes. *Commun. math. Phys.* **43**, 199.
- [33] Bardeen, J. M., Carter, B., Hawking, S. W. (1973). The four laws of black hole mechanics. *Commun. math. Phys.* **31**, 161.
- [34] Hawking, S. W. (1976). Black holes and thermodynamics. *Phys. Rev. D* **13**, 191.
- [35] Hawking, S. W. (1974). Black Hole Explosions. *Nature* **248**, 30.
- [36] Hawking, S. W. (2005). Information Loss in Black Holes. *hep-th/0507171*.
- [37] Israel, W. (1971). Third law of black-hole dynamics: a formulation and proof. *Phys. Rev. Lett.* **57**, 397.
- [38] Planck, M. (1945). *Treatise on Thermodynamics*. (New York: Dover) (3rd edn translated from the seventh German edition).
- [39] Belgiorno, F. (2003). Notes on the third law of thermodynamics: I. *J. Phys. A: Math. Gen.* **36**, 8165.
- [40] Belgiorno, F. (2003). Notes on the third law of thermodynamics: II. *J. Phys. A: Math. Gen.* **36**, 8195.
- [41] Das, S. (2004). Black hole thermodynamics: entropy, information and beyond. *hep-th/0403202*.

- [42] Zurek, W. H., Thorne, K. S. (1985). Statistical mechanical origin of the entropy of a rotating, charged black hole. *Phys. Rev. Lett.* **54**, 2171.
- [43] Frolov, V. P., Page, D. N. (1993). Proof of the generalized second law for quasistationary semiclassical black holes. *Phys. Rev. Lett.* **71**, 3902.
- [44] Hoyle, C. D., Kapner, D. J., Heckel, B. R., Adelberger, E. G., Gundlach, J. H., Schmidt, U., Swanson, H. E. (2004). Submillimeter tests of the gravitational inverse-square law. *Phys. Rev. D* **70**, 042004.
- [45] <http://www.black-holes.org/numrel1.html>.
- [46] Tangherlini, F. R. (1963). Schwarzschild field in n dimensions and the dimensionality of space problem. *Nuovo Cimento*, **XXVII**, 636.
- [47] Parker, L. (1975). Probability distribution of particles created by a black hole. *Phys. Rev. D* **12**, 1519.
- [48] Cavaglià, M. (2003). Black hole multiplicity at particle colliders. *Phys. Lett. B* **569** 7. (*hep-ph/0305256*).
- [49] Kanti, P., March-Russell, J. (2002). Calculable corrections to brane black hole decay: The scalar case. *Phys. Rev. D* **66** 024023. (*hep-ph/0203223*).
- [50] Kanti, P., March-Russell, J. (2003). Calculable corrections to brane black hole decay II: greybody factors for Spin 1/2 and 1. *Phys. Rev. D* **67** 104019. (*hep-ph/0212199*).
- [51] Ida, D., Oda, K., Park, S. C. (2003). Rotating black holes at future colliders: Greybody factors for brane fields. *Phys. Rev. D* **67** 064025. (*hep-th/0212108*).

- [52] Cremmer, E., Julia, B.(1979). The $SO(8)$ supergravity. *Nucl. Phys. B*, **159**, 141.
- [53] De Wit, B., Nicolai, H.(1982). N=8 supergravity. *Nucl. Phys. B*, **208**, 323.
- [54] Argyres, P. C., Dimopoulos, S., March-Russell, J. (1998). Black holes and sub-millimeter dimensions. *Phys.Lett. B* **441**, 96. (*hep-th/9808138*).
- [55] Walborn, S. P. , Terra Cunha, M. O., Padua, S., Monken, C. H. (2002). Double-slit quantum eraser. *Phys. Rev. A*, **65**, 033818.
- [56] Jammer, M. (1974). *The Philosophy of Quantum Mechanics*. Wiley, New York.
- [57] Scully, M. O., Druhl, K. (1982). Quantum eraser: A proposed photon correlation experiment concerning observation and “delayed choice” in quantum mechanics. *Phys. Rev. A* **25**, 2208.
- [58] Scully, M. O., Englert, B. G. , Walther, H. (1991). Quantum optical tests of complementarity. *Nature (London)*, **351**, 111.
- [59] Horowitz, G. T.(2000). Comments on Black Holes in String Theory. *Class. Quant. Grav.*, **17**, 1107. (*hep-th/9910082*).
- [60] Horowitz, G., Strominger, A. (1991). Black strings and p-branes. *Nucl. Phys. B* **360**, 197.
- [61] Duff, M., Khuri, R., Lu, J. (1995). String Solitons. *Phys. Rep.* **259**, 213. (*hep-th/9412184*).
- [62] Tommasini, P., Timmermans, E., de Toledo Piza, A. F. R. (1997). The Hydrogen atom as an entangled electron-proton system. *quant-ph/9709052*.

- [63] Susskind, L., Uglum, J. (1994). Black hole entropy in canonical quantum gravity and superstring theory. *Phys. Rev. D* **50**, 2700. (*hep-th/9401070*).
- [64] de Alwis, S. P., Ohta, N. (1995). Thermodynamics of quantum fields in black hole backgrounds. *Phys. Rev. D* **52**, 3529.
- [65] 't Hooft, G.(1996). The scattering matrix approach for the quantum black hole. *gr-qc/9607022*.
- [66] 't Hooft, G.(1985). On the quantum structure of a black hole. *Nucl. Phys. B* **256**, 727.
- [67] Mann, R. B., Tarasov, L., Zelnikov, A. (1992). Brick walls for black holes. *Class. Quant. Grav.* **9**, 1487.
- [68] Kenmoku, M., Nandi, K. K., Shigemoto, K. (2005). Analysis of black hole entropy in brick wall model. *Class. Quant. Grav.* **22**, 3923. (*gr-qc/0505129*).
- [69] Callan, C. G., Wilczek, F. (1994). On Geometric Entropy. *Phys. Lett. B* **333**, 55. (*hep-th/9401072*).
- [70] Kabat, D. (1995). Black Hole Entropy and Entropy of Entanglement. *Nucl. Phys. B* **453**, 281. (*hep-th/9503016*).
- [71] Frolov, V. P., Fursaev, D. V. (1998). Thermal Fields, Entropy, and Black Holes. *Class. Quant. Grav.* **15**, 2041. (*hep-th/9802010*).
- [72] Einhorn, M. B. (2005). The black hole information paradox. *hep-th/0510148*.

- [73] Einhorn, M. B.(2005). Entanglement Interpretation of Black Hole Entropy in String Theory. *hep-th/0508217*.
- [74] Srednicki,M. (1993). Entropy and area. *Phys. Rev. Lett.* **71**, 666. (*hep-th/9303048*).
- [75] Bombelli, L., Koul, R. K., Lee, J., Sorkin, R. D. (1986). A quantum source of entropy for black holes. *Phys. Rev. D* **34**, 373.
- [76] Ahmadi, M., Das, S., Shankaranarayanan, S. (2006). Is entanglement entropy proportional to area? *Can. J. Phys.* **84(S2)**, 1. (*hep-th/0507228*).
- [77] Morse, P. M., Feshbach, H. (1953). *Methods of Theoretical Physics*, McGraw-Hill: New York, 950-951.
- [78] Das, S., Shankaranarayanan, S. (2006). How robust is the entanglement entropy-area relation? *Phys. Rev. D* **73**, 121701 (*gr-qc/0511066*).
- [79] Maoz, E. (1998). Dynamical constraints on alternatives to supermassive black holes in galactic nuclei. *ApJ* **494**, L181.
- [80] Celotti, A., Miller, J. C., Sciamia, D. W. (1999). Astrophysical evidence for the existence of black holes. *astro-ph/9912186*.
- [81] <http://chandra.harvard.edu/>.
- [82] <http://xmm.vilspa.esa.es/>.
- [83] <http://www.ligo.caltech.edu>.
- [84] <http://wwwcascina.virgo.infn.it/>.

- [85] Voss, R., Tauris, T. M. (2003). Galactic distribution of merging neutron stars and black holes. *Mon. Not. R. Astron. Soc.* **342**, 1169.
- [86] <http://lhc.web.cern.ch/lhc/>.
- [87] Page, D. N. (1976). Particle emission rates from a black hole: Massless particles from an uncharged, nonrotating hole. *Phys. Rev. D* **13**, 198.
- [88] Page, D. N. (1976). Particle emission rates from a black hole. II. Massless particles from a rotating hole. *Phys. Rev. D* **14**, 3260.
- [89] Harris, C. M., Kanti, P. (2003). Hawking radiation from a (4+n)-dimensional black hole: exact results for the Schwarzschild phase *JHEP* **0310**, 014.
- [90] Nomura, H., Yoshida, S., Tanabe, M., Maeda, K. (2005). The fate of a five-dimensional rotating black hole via hawking radiation. *Prog. Theor. Phys.* **114**, 707. (*hep-th/0502179*).
- [91] Ida, D., Oda, K., Park, S. C. (2004). Erratum: Rotating black holes at future colliders: Greybody factors for brane fields [Phys. Rev. D 67, 064025 (2003)]. *Phys. Rev. D* **69**, 049901.
- [92] Cavaglià, M., Das, S., Maartens, R. (2003). Will we observe black holes at the LHC? *Class. Quant. Grav.* **20**, L205.
- [93] Cavaglià, M., Das, S. (2004). How classical are TeV-scale black holes? *Class. Quant. Grav.* **21**, 4511.

- [94] Garay, L. J. (1995). Quantum gravity and minimum length. *Int. J. Mod. Phys. A* **10**, 145 (*gr-qc/9403008*).
- [95] Konishi, K., Paffuti, G., Provero, P. (1990) Minimum physical length and the generalized uncertainty principle in string theory. *Phys. Lett. B* **234**, 276.
- [96] Das, S., Majumdar, P., Bhaduri, R. K.(2002). General logarithmic corrections to black hole entropy. *Class. Quant. Grav.* **19**, 2355 (*hep-th/0111001*).
- [97] Bhaduri, R. K., Tran, M. N., Das, S. (2004). On the microcanonical entropy of a black hole. *Phys. Rev. D* **69**, 104018 (*gr-qc/0312023*).
- [98] Dimopoulos, S., Landsberg, G. (2001.) Black holes at the LHC. *Phys. Rev. Lett.* **87**, 161602. (*hep-ph/0106295*).
- [99] Cardoso, V., Berti, E., Cavaglia, M. (2005). What we (don't) know about black-hole formation in high-energy collisions. *Class. Quant. Grav.* **22**, L61.
- [100] The Pioneer Explorer Collaboration: Dittus, H., Turyshev, S. G., Lammerzahn, C., Theil, S., Foerstner, R., Johann, U., Ertmer, W., Rasel, E., Dachwald, B., Seboldt, W., Hehl, F. W., Kiefer, C., Blome, H. J., Kunz, J., Giulini, D., Bingham, R., Kent, B., Sumner, T. J., Bertolami, O., Paramos, J., Rosales, J. L., Christophe, B., Foulon, B., Touboul, P., Bouyer, P., Reynaud, S., Brillet, A., Bondu, F., Samain, E., de Matos, C. J., Erd, C., Grenouilleau, J.C., Izzo, D., Rathke, A., Anderson, J. D., Asmar, S. W., Lau, E. L., Nieto, M. M., Mashhoon, B. (2005). A Mission to Explore the Pioneer Anomaly. *gr-qc/0506139*.