

Generalized Uncertainty Principle and Self-Adjoint Operators

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In this work we explore the self-adjointness of the GUP-modified momentum and Hamiltonian operators over different domains. In particular, we utilize the theorem by von-Neumann for symmetric operators in order to determine whether the momentum and Hamiltonian operators are self-adjoint or not, or they have self-adjoint extensions over the given domain. In addition, a simple example of the Hamiltonian operator describing a particle in a box is given. The solutions of the boundary conditions that describe the self-adjoint extensions of the specific Hamiltonian operator are obtained.

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I. INTRODUCTION

The Generalized Uncertainty Principle (henceforth abbreviated to GUP) is an outcome of modifications/corrections¹ to the conventional Heisenberg algebra satisfied by the two canonically conjugate observables: position \mathbf{x} and momentum \mathbf{p} . According to String Theory, the conventional Heisenberg algebra gains an extra term which is quadratic in momentum \mathbf{p} , in the Planck regime. In [2], the authors used the following modified Heisenberg algebra consistent with String Theory

$$[\hat{x}, \hat{p}] = i\hbar(1 + \beta p^2) . \quad (1)$$

Black Hole physics, and Doubly Special Relativity (DSR) propose a correction in the Planck regime with the extra term to be linear in momentum \mathbf{p} . In [3, 4], the authors considered modifications to the conventional Heisenberg algebra, which includes both linear and quadratic terms in momentum, namely

$$[\hat{x}, \hat{p}] = i\hbar[\delta_{ij} - \alpha(p\delta_{ij} + \frac{p_i p_j}{p}) + \alpha^2(p^2\delta_{ij} + 3p_i p_j)] . \quad (2)$$

Utilizing the following uncertainty relationship satisfied by any two operators \hat{A} and \hat{B}

$$\Delta\hat{A}\Delta\hat{B} \geq \frac{1}{2} \left\| \left\langle [\hat{A}, \hat{B}] \right\rangle \right\| \quad (3)$$

where $\Delta\hat{A}$ and $\Delta\hat{B}$ stand for the standard deviations of the corresponding operators, a GUP between position x and momentum p is obtained

$$\Delta x \Delta p \geq \frac{\hbar}{2} [1 - 2\alpha \langle p \rangle + 4\alpha^2 \langle p^2 \rangle] \quad (4)$$

where $l_{pl} \approx 10^{-35}$ m is the Planck length, $\alpha = \alpha_0 l_{pl}/\hbar$ is the GUP parameter, and it is normally assumed that $\alpha_0 = 1$. In [3, 4], the authors suggested an upper bound on α_0 by stating that its value cannot exceed 10^{17} which is precisely the electroweak length scale. This prediction comes about due to the fact that if α_0 were to be any larger, such an intermediate length scale would have been observed. Now, it is easy to verify the fact that the above two equations, i.e. Eqs.(2) and (4), predict a minimal measurable length Δx_{min} and a maximum measurable momentum Δp_{max}

$$\Delta x_{min} \propto \alpha_0 l_{pl} \quad (5)$$

$$\Delta p_{max} \propto \frac{M_{pl}c}{\alpha_0} \quad (6)$$

where M_{pl} is the Planck Mass. It can be shown that the following representations of the position and momentum operators satisfy the modified Heisenberg algebra given by Eqn.(2)

$$\hat{x}_i = \hat{x}_{0i} \quad (7)$$

$$\hat{p}_i = \hat{p}_{0i} (1 - \alpha \hat{p}_0 + 2\alpha^2 \hat{p}_0^2) \quad (8)$$

with \hat{x}_{0i} and \hat{p}_{0j} satisfying the ordinary canonical commutation relations $[\hat{x}_{0i}, \hat{p}_{0j}] = i\hbar\delta_{ij}$. Here, \hat{p}_0 can be interpreted as being the total momentum of a particle at low energies and having the standard representation, namely, in one dimension,

$$\hat{p}_0 = -i\hbar \frac{d}{dx} . \quad (9)$$

¹ Predicted by various quantum gravity theories (for a recent review see [1], and references there in).

Now, considering any non-relativistic Hamiltonian of the form

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\mathbf{r}) \quad (10)$$

we see that, due to Eq.(8), i.e., the GUP-modified momentum operator, every Hamiltonian of the form of Eq.(10) obtains higher order terms in α and \hat{p}_0 . A simple substitution of the new momentum operator into Eq.(10) yields

$$\hat{H} = \frac{\hat{p}_0^2}{2m} - \frac{\alpha}{m}\hat{p}_0^3 + \frac{5\alpha^2}{2m}\hat{p}_0^4 - \frac{2\alpha^3}{m}\hat{p}_0^5 + \frac{2\alpha^4}{m}\hat{p}_0^6 + V(\mathbf{r}) . \quad (11)$$

The aim of this work is to investigate the self-adjointness of the GUP-modified momentum and Hamiltonian operators, characterized by the powers of α up to $\mathcal{O}(\alpha^2)$. In section II, we briefly present some mathematical tools which will be used in next sections. In section III, we show in which domains the GUP-modified momentum operator is self-adjoint, in which domains it is not self-adjoint, and in which domains it has infinitely many self-adjoint extensions. We follow this analysis when the GUP-modified momentum operator has a linear term in the GUP parameter α , when it has a quadratic term in α , and when both terms, i.e., the linear and the quadratic in α , are present. In section IV, we perform the same analysis as presented in section III for the case of the Hamiltonian operator. Finally, in section V, we present our results.

II. MATHEMATICAL PRELIMINARIES

In this section, we briefly present all necessary definitions, theorems, and lemmas concerning linear operators [5–8].

Definition 1. Let \mathcal{V} a normed vector space. A linear mapping $\hat{A} : \mathcal{V} \rightarrow \mathcal{V}$ is called a **linear operator** in \mathcal{H} . The subspace of elements $x \in \mathcal{H}$ for which $\hat{A}x$ is defined is termed as the **domain** of \hat{A} and is denoted as $\mathcal{D}_{\hat{A}}$. The **range** of \hat{A} is the set of all elements $y \in \mathcal{V}$ such that $y = \hat{A}x$ holds, and is denoted as $\mathcal{R}_{\hat{A}}$.

At this point, it should be stressed that from now onwards we use the words **linear operator** and **operator** interchangeably.

Definition 2. Let \mathcal{V} be a normed vector space. Any linear mapping of \mathcal{V} into \mathcal{V} itself is called a **bounded operator** if the $\|\hat{A}\| < \infty$. The norm of an operator is defined as follows:

$$\|\hat{A}\| = \sup_{x \in \mathcal{D}_{\hat{A}} \|x\| \neq 0} \frac{\|\hat{A}x\|}{\|x\|} . \quad (12)$$

The above simply implies that we can always find a positive real constant, say \mathcal{M} , such that $\|\hat{A}x\| \leq \mathcal{M}\|x\|$. In the case when no such constant exists, we term the operator to be **unbounded**.

In this section, we focus more on the properties of unbounded operators since most of the operators encountered in Physics such as the momentum and the Hamiltonian are unbounded operators.

Example 1. Consider the differential operator $\frac{d}{dx}$ to be defined on the space of all differentiable functions on some interval $[a, b] \subset \mathbb{R}$, which is a subspace of $\mathcal{L}^2([a, b])$. Suppose we consider a sequence of functions $f_n(x) = \sin(nx)$, $n = 1, 2, 3, \dots$, defined on $[-\pi, \pi]$. Then

$$\|f_n\| = \sqrt{\int_{-\pi}^{\pi} (\sin nx)^2 dx} = \sqrt{\pi} < \infty \quad (13)$$

and

$$\left\| \frac{d}{dx} f_n \right\| = \sqrt{\int_{-\pi}^{\pi} (n \cos nx)^2 dx} = n\sqrt{\pi} . \quad (14)$$

From the above we see that, there is no real constant that can set an upper bound on $\left\| \frac{d}{dx} f_n \right\|$, hence we see that the differential operator is unbounded.

Since the GUP-modified momentum and Hamiltonian operators are differential operators, we will completely work with unbounded operators and for this reason the following definitions and theorems concern only unbounded operators.

Definition 3. Let \mathcal{H} be a normed vector space. Let \hat{A} be an operator such that $\mathcal{D}_{\hat{A}} \in \mathcal{H}$. Then \hat{A} is said to be *densely defined* if $\mathcal{D}_{\hat{A}}$ is dense in \mathcal{H} , i.e. $\forall \psi \in \mathcal{H}, \exists \in \mathcal{D}_{\hat{A}}$ a sequence ϕ_n which in norm converges to ψ

Definition 4. An operator $\hat{A} : \mathcal{H} \rightarrow \mathcal{H}$, with domain $\mathcal{D}_{\hat{A}} \in \mathcal{H}$, is said to be **closed** if its graph $\Gamma(\hat{A})$

$$\Gamma(\hat{A}) = \left\{ (x, y) \mid x \in \mathcal{D}_{\hat{A}}, y = \hat{A}x \right\} \quad (15)$$

is closed in the normed space $\mathcal{H} \times \mathcal{H}$.

For an unbounded operator, one can define the corresponding *adjoint* operator in the following way:

Definition 5. The *adjoint*, \hat{A}^\dagger of an unbounded operator \hat{A} defined in a Hilbert space, is defined as

$$\forall x \in \mathcal{D}_{\hat{A}}, \forall y \in \mathcal{D}_{\hat{A}^\dagger} \quad \langle y \mid \hat{A}x \rangle = \langle \hat{A}^\dagger y \mid x \rangle . \quad (16)$$

Since, we only deal with dense domains, we will consider only densely defined unbounded operators.

Theorem 1. If \hat{A} is a densely defined operator, then its adjoint \hat{A}^\dagger is closed

Definition 6.

1. Let \hat{A} be an operator defined in the Hilbert space, \mathcal{H} . The \hat{A} is called **Hermitian** or **symmetric** if, $\forall x, y \in \mathcal{D}_{\hat{A}}$,

$$\langle y \mid \hat{A}x \rangle = \langle \hat{A}y \mid x \rangle . \quad (17)$$

2. An operator \hat{A} defined in a Hilbert space \mathcal{H} is said to be **self-adjoint** if it is densely defined over its domain and in form, $\hat{A} = \hat{A}^\dagger$.

Note. The equality $\hat{A} = \hat{A}^\dagger$, apart from the equality in form of the two operators, also implies that the respective domains of the operators should also coincide, i.e., $\mathcal{D}_{\hat{A}} = \mathcal{D}_{\hat{A}^\dagger}$.

In the case of bounded operators, one need not concern with the equality of domains because, the domain of a densely defined bounded operator can always be extended to the entire vector space. Therefore, a bounded Hermitian operator is also self-adjoint. However in the unbounded case, the situation is a little bit more subtle, since the operator being symmetric doesn't imply self-adjointness. We now describe the *von Neumann's Theorem* which is an indispensable tool in the analysis of self-adjointness of operators.

Note. From now onwards we consider only unbounded operators.

Definition 7. Let \hat{A} be a symmetric operator. Let

$$\mathcal{K}_+ = \ker(i - \hat{A}^\dagger) \quad (18)$$

$$\mathcal{K}_- = \ker(i + \hat{A}^\dagger) \quad (19)$$

where \mathcal{K}_+ and \mathcal{K}_- are called **deficiency subspaces** of \hat{A} and their dimensions, i.e., $n_+ = \dim[\mathcal{K}_+]$ and $n_- = \dim[\mathcal{K}_-]$ are referred to as the **deficiency indices** of \hat{A} .

Note. The deficiency indices of \hat{A} can be any positive integer and even infinite.

Definition 8. Let \hat{A} be an operator in a Hilbert space, \mathcal{H} . We say \hat{B} is an **extension** of \hat{A} if the following conditions hold

- $\mathcal{D}(\hat{A}) \subset \mathcal{D}(\hat{B})$
- $\hat{A}\phi = \hat{B}\phi, \forall \phi \in \mathcal{D}(\hat{A})$

i.e. $\hat{A} \subset \hat{B}$.

Given an operator \hat{A} in a Hilbert space and say \hat{B} is a closed symmetric extension of the same, then the following are true

- For $\phi \in \mathcal{D}(\hat{B}^\dagger)$

$$(\psi, \hat{B}^\dagger \phi) = (\hat{B}\psi, \phi) = (\hat{A}\psi, \phi) \quad (20)$$

for all $\psi \in \mathcal{D}(\hat{A})$. Thus, from the above we see that $\phi \in \mathcal{D}(\hat{A})$ and $\hat{B}^\dagger \phi = \hat{A}^\dagger \phi$ so

$$\hat{A} \subset \hat{B} \subset \hat{B}^\dagger \subset \hat{A}^\dagger. \quad (21)$$

von Neumann's Theorem

Theorem 2. Let \hat{A} be a closed symmetric operator with deficiency indices n_+ and n_- . Then,

- \hat{A} is self-adjoint if and only if $(n_+, n_-) = (0, 0)$.
- \hat{A} has self-adjoint extensions if and only if $n_+ = n_-$. These extensions are parametrized by an $n \times n$ unitary matrix.
- If $n_+ \neq n_-$, the \hat{A} has no self-adjoint extensions.

III. GUP-MODIFIED MOMENTUM

In this section we will apply von Neumann's theorem to the GUP-modified momentum operator [9]. For this reason, we first have to determine the functions $\psi_\pm(x)$ which satisfy the equation

$$\hat{p}\psi_\pm(x) = \pm i \frac{\hbar}{d} \psi_\pm(x) \quad (22)$$

where d is a positive constant introduced for dimensional reasons and which is homogeneous to some length. For the case of the total momentum of a particle at low energies and employing the standard representation, Eq.(22) becomes

$$-i\hbar \frac{d\psi_\pm(x)}{dx} = \pm i \frac{\hbar}{d} \psi_\pm(x). \quad (23)$$

It is easily seen that a solution to the above equation reads

$$\psi_\pm(x) = C_\pm \exp \left[\mp \frac{x}{d} \right]. \quad (24)$$

Over different domains, the deficiency indices and the self-adjointness of the momentum operator are described as follows:

- $\mathcal{D}(\hat{p}_0) = \mathcal{L}^2(\mathbb{R}) : (n_+, n_-) = (0, 0)$ and thus the operator is self-adjoint.
- $\mathcal{D}(\hat{p}_0) = \mathcal{L}^2[0, \infty) : (n_+, n_-) = (1, 0)$ and thus the operator is not self-adjoint.
- $\mathcal{D}(\hat{p}_0) = \mathcal{L}^2([0, L]) : (n_+, n_-) = (1, 1)$ and thus the operator has infinitely many self-adjoint extensions parametrized by a $U(1)$ group.

For future convenience, we will make all variables dimensionless. For this reason, using the quantity $\alpha\hbar$ which is the physical length scale introduced in GUP, we define a new dimensionless parameter ρ as follows

$$\rho = \frac{x}{\alpha\hbar}. \quad (25)$$

Therefore, the momentum operator \hat{p}_0 expressed in terms of the new variable ρ now reads

$$\hat{p}_0 = -i\hbar \frac{d}{dx} = -\frac{i}{\alpha} \frac{d}{d\rho}. \quad (26)$$

A. Momentum operator with linear term in α

Let us now consider as momentum operator in Eq.(22), the GUP-modified momentum operator which is linear in α , namely

$$\hat{p} = \hat{p}_0 - \alpha \hat{p}_0^2. \quad (27)$$

First we write Eq.(22) for the function ψ_+ and we get

$$\left(\hat{p}^\dagger - \frac{i}{\alpha} \right) \psi_+(x) = 0$$

which in terms of the dimensionless parameter ρ takes the form

$$\left(\frac{d^2}{d\rho^2} - i \frac{d}{d\rho} - i \right) \psi_+(\rho) = 0.$$

The characteristic equation of the above differential equation is

$$\lambda^2 - i\lambda - i = 0 \quad (28)$$

and so its roots are of the form

$$\lambda_+^1 = \frac{1}{4} \sqrt{-2 + 2\sqrt{17}} + i \left(\frac{1}{2} + \frac{1}{4} \sqrt{2 + 2\sqrt{17}} \right) = 0.6248105340 + 1.300242590 i \quad (29)$$

$$\lambda_+^2 = -\frac{1}{4} \sqrt{-2 + 2\sqrt{17}} + i \left(\frac{1}{2} - \frac{1}{4} \sqrt{2 + 2\sqrt{17}} \right) = -0.6248105340 - 0.3002425902 i. \quad (30)$$

Therefore, we obtain two linearly independent solutions for ψ_+

$$\psi_+^1(\rho) \propto \exp[(0.6248105340 + 1.300242590 i)\rho] \quad (31)$$

$$\psi_+^2(\rho) \propto \exp[(-0.6248105340 - 0.3002425902 i)\rho]. \quad (32)$$

Furthermore, over the different domains the deficiency index n_+ of the GUP-modified momentum operator which is linear in α is given as follows:

- $\mathcal{D}(\hat{p}) = \mathcal{L}^2(-\infty, \infty)$

In this domain, none of the functions, i.e., $\psi_+^1(x)$ and $\psi_+^2(x)$, is square integrable. So these functions do not belong to this space and thus $n_+ = 0$.

- $\mathcal{D}(\hat{p}) = \mathcal{L}^2(0, \infty)$

In this domain, only $\psi_+^2(x)$ has finite norm. So this is the only solution from the set which is square integrable and thus $n_+ = 1$.

- $\mathcal{D}(\hat{p}) = \mathcal{L}^2([a, b])$

Over the finite interval both solutions, i.e., $\psi_+^1(x)$ and $\psi_+^2(x)$, are square integrable. So these functions belong to this space and thus $n_+ = 2$.

Second we write Eq.(22) for the function ψ_- and we get

$$\left(\hat{p}^\dagger + \frac{i}{\alpha} \right) \psi_-(x) = 0$$

which in terms of the dimensionless parameter ρ takes the form

$$\left(\frac{d^2}{d\rho^2} - i \frac{d}{d\rho} + i \right) \psi_-(\rho) = 0 .$$

The characteristic equation of the above differential equation is

$$\lambda^2 - i\lambda + i = 0 \tag{33}$$

and so its roots are of the form

$$\begin{aligned} \lambda_-^1 &= \frac{1}{4}, \sqrt{-2 + 2\sqrt{17}} + i \left(\frac{1}{2} - \frac{1}{4} \sqrt{2 + 2\sqrt{17}} \right) = 0.6248105340 - 0.3002425902 i \\ \lambda_-^2 &= -\frac{1}{4} \sqrt{-2 + 2\sqrt{17}} + i \left(\frac{1}{2} + \frac{1}{4} \sqrt{2 + 2\sqrt{17}} \right) = -0.6248105340 + 1.300242590 i . \end{aligned}$$

Therefore, we obtain two linearly independent solutions for ψ_-

$$\psi_-^1(\rho) \propto \exp[(0.6248105340 - 0.3002425902 i)\rho] \tag{34}$$

$$\psi_-^2(\rho) \propto \exp[(-0.6248105340 + 1.300242590 i)\rho] . \tag{35}$$

Furthermore, over the different domains the deficiency index n_- of the GUP-modified momentum operator which is linear in α is given as follows:

- $\mathcal{D}(\hat{p}) = \mathcal{L}^2(-\infty, \infty)$

In this domain, none of the functions, i.e., $\psi_-^1(x)$ and $\psi_-^2(x)$, is square integrable. So these functions do not belong to this space and thus $n_- = 0$.

- $\mathcal{D}(\hat{p}) = \mathcal{L}^2(0, \infty)$

In this domain, only $\psi_-^2(x)$ has finite norm. So this is the only solution from the set which is square integrable and thus $n_- = 1$.

- $\mathcal{D}(\hat{p}) = \mathcal{L}^2([a, b])$

Over the finite interval both solutions, i.e., $\psi_-^1(x)$ and $\psi_-^2(x)$, are square integrable. So these functions belong to this space and thus $n_- = 2$.

Finally, employing von Neumann's theorem, the self-adjointness of the GUP-modified momentum operator which is linear in α is described as follows :

- $\mathcal{D}(\hat{p}) = \mathcal{L}^2(-\infty, \infty) : (n_+, n_-) = (0, 0)$ and thus the momentum operator is self-adjoint.
- $\mathcal{D}(\hat{p}) = \mathcal{L}^2(0, \infty) : (n_+, n_-) = (1, 1)$ and thus the momentum operator has infinitely many self-adjoint extensions.
- $\mathcal{D}(\hat{p}) = \mathcal{L}^2([a, b]) : (n_+, n_-) = (2, 2)$ and thus the momentum operator has infinitely many self-adjoint extensions.

B. Momentum Operator with quadratic term in α

We now consider as momentum operator in Eq.(22), the GUP-modified momentum operator which is quadratic in α , namely

$$\hat{p} = \hat{p}_0 + 2\alpha^2 \hat{p}^3$$

and we adopt the analysis of the previous subsection in order to study the self-adjointness of this GUP-modified momentum operator.

First we write Eq.(22) for the function ψ_+ and we get

$$\left(\hat{p}^\dagger - \frac{i}{\alpha}\right) \psi_+(x) = 0$$

which in terms of the dimensionless parameter ρ takes the form

$$\left(2 \frac{d^3}{d\rho^3} - \frac{d}{d\rho} - 1\right) \psi_+(\rho) = 0.$$

The characteristic equation of the above differential equation is

$$2\mu^3 - \mu - 1 = 0 \tag{36}$$

and so its roots are of the form

$$\mu_+^1 = 1 \tag{37}$$

$$\mu_+^2 = -\frac{1}{2} - \frac{1}{2}i \tag{38}$$

$$\mu_+^3 = -\frac{1}{2} + \frac{1}{2}i. \tag{39}$$

Therefore, we obtain three linearly independent solutions for ψ_+

$$\psi_+^1(\rho) \propto \exp(\rho) \tag{40}$$

$$\psi_+^2(\rho) \propto \exp\left[\left(-\frac{1}{2} - \frac{1}{2}i\right)\rho\right] \tag{41}$$

$$\psi_+^3(\rho) \propto \exp\left[\left(-\frac{1}{2} + \frac{1}{2}i\right)\rho\right]. \tag{42}$$

Furthermore, over the different domains the deficiency index n_+ of the GUP-modified momentum operator which is in α is given as follows:

- $\mathcal{D}(\hat{p}) = \mathcal{L}^2(-\infty, \infty)$

In this domain, none of the functions, i.e., $\psi_+^1(x)$, $\psi_+^2(x)$, and $\psi_+^3(x)$, is square integrable. So these functions do not belong to this space and thus $n_+ = 0$.

- $\mathcal{D}(\hat{p}) = \mathcal{L}^2(0, \infty)$

In this domain, only $\psi_+^2(x)$ and $\psi_+^3(x)$ have finite norm. So there are only two solutions from the set which are square integrable and thus $n_+ = 2$.

- $\mathcal{D}(\hat{p}) = \mathcal{L}^2([a, b])$

Over the finite interval both solutions, i.e., $\psi_+^1(x)$, $\psi_+^2(x)$, and $\psi_+^3(x)$, are square integrable. So these functions belong to this space and thus $n_+ = 3$.

Second we write Eq.(22) for the function ψ_- and we get

$$\left(\hat{p}^\dagger + \frac{i}{\alpha}\right) \psi_-(x) = 0$$

which in terms of the dimensionless parameter ρ takes the form

$$\left(2 \frac{d^3}{d\rho^3} - \frac{d}{d\rho} + 1\right) \psi_-(\rho) = 0 .$$

The characteristic equation of the above differential equation is

$$2\mu^3 - \mu + 1 = 0 \tag{43}$$

and so its roots are of the form

$$\mu_-^1 = -1 \tag{44}$$

$$\mu_-^2 = \frac{1}{2} - \frac{1}{2}i \tag{45}$$

$$\mu_-^3 = \frac{1}{2} + \frac{1}{2}i . \tag{46}$$

Therefore, we obtain three linearly independent solutions for ψ_-

$$\psi_-^1(x) \propto \exp(-\rho) \tag{47}$$

$$\psi_-^2(x) \propto \exp\left[\left(\frac{1}{2} - \frac{1}{2}i\right)\rho\right] \tag{48}$$

$$\psi_-^3(x) \propto \exp\left[\left(\frac{1}{2} + \frac{1}{2}i\right)\rho\right] . \tag{49}$$

Furthermore, over the different domains the deficiency index n_- of the GUP-modified momentum operator which is quadratic in α is given as follows:

- $\mathcal{D}(\hat{p}) = \mathcal{L}^2(-\infty, \infty)$

In this domain, none of the functions, i.e., $\psi_-^1(x)$, $\psi_-^2(x)$, and $\psi_-^3(x)$, is square integrable. So these functions do not belong to this space and thus $n_- = 0$.

- $\mathcal{D}(\hat{p}) = \mathcal{L}^2(0, \infty)$

In this domain, only $\psi_-^1(x)$ has finite norm. So this is the only solution from the set which is square integrable and thus $n_- = 1$.

- $\mathcal{D}(\hat{p}) = \mathcal{L}^2([a, b])$

Over the finite interval all three solutions, i.e., $\psi_-^1(x)$, $\psi_-^2(x)$, and $\psi_-^3(x)$, are square integrable. So these functions belong to this space and thus $n_+ = 3$.

Finally, employing von Neumann's theorem, the self-adjointness of the GUP-modified momentum operator which is quadratic in α is described as follows :

- $\mathcal{D}(\hat{p}) = \mathcal{L}^2(-\infty, \infty) : (n_+, n_-) = (0, 0)$ and thus the momentum operator is self-adjoint.
- $\mathcal{D}(\hat{p}) = \mathcal{L}^2(0, \infty) : (n_+, n_-) = (2, 1)$ and thus the momentum operator is not self-adjoint.
- $\mathcal{D}(\hat{p}) = \mathcal{L}^2([a, b]) : (n_+, n_-) = (3, 3)$ and thus the momentum operator has infinitely many self-adjoint extensions.

C. Momentum operator with linear and quadratic terms in α

We now consider as momentum operator in Eq.(22), the GUP-modified momentum operator which has linear and quadratic terms in α , namely

$$\hat{p} = \hat{p}_0 - \alpha \hat{p}_0^2 + 2\alpha^2 \hat{p}_0^3. \quad (50)$$

First we write Eq.(22) for the function ψ_+ and we get

$$\left(\hat{p}^\dagger - \frac{i}{\alpha} \right) \psi_+(x) = 0$$

which in terms of the dimensionless parameter ρ takes the form

$$\left(2i \frac{d^3}{d\rho^3} + \frac{d^2}{d\rho^2} - i \frac{d}{d\rho} - i \right) \psi_+(\rho) = 0.$$

The characteristic equation of the above differential equation is

$$2i\nu^3 + \nu^2 - i\nu - i = 0 \quad (51)$$

and so its roots are of the form

$$\nu_+^1 = \frac{1}{6} \sqrt[3]{54 + 8i + 3\sqrt{303 + 96i}} + \frac{5}{6} \frac{1}{\sqrt[3]{54 + 8i + 3\sqrt{303 + 96i}}} + \frac{i}{6} \quad (52)$$

$$= 0.9676154706 + 0.1976442834i$$

$$\nu_+^2 = -\frac{1}{12} \sqrt[3]{54 + 8i + 3\sqrt{303 + 96i}} - \frac{5}{12} \frac{1}{\sqrt[3]{54 + 8i + 3\sqrt{303 + 96i}}} + \frac{1}{6}i \quad (53)$$

$$+ \frac{1}{2}i\sqrt{3} \left(\frac{1}{6} \sqrt[3]{54 + 8i + 3\sqrt{303 + 96i}} - \frac{5}{6} \frac{1}{\sqrt[3]{54 + 8i + 3\sqrt{303 + 96i}}}i \right) \quad (54)$$

$$= -0.5258014144 + 0.6865159449i$$

$$\nu_+^3 = -\frac{1}{12} \sqrt[3]{54 + 8i + 3\sqrt{303 + 96i}} - \frac{5}{12} \frac{1}{\sqrt[3]{54 + 8i + 3\sqrt{303 + 96i}}} + \frac{1}{6}i \quad (55)$$

$$- \frac{1}{2}i\sqrt{3} \left(\frac{1}{6} \sqrt[3]{54 + 8i + 3\sqrt{303 + 96i}} - \frac{5}{6} \frac{1}{\sqrt[3]{54 + 8i + 3\sqrt{303 + 96i}}}i \right) \quad (56)$$

$$= -0.4418140560 - 0.3841602281i.$$

Therefore, we obtain three linearly independent solutions for ψ_+

$$\psi_+^1(\rho) \propto \exp[(0.9676154706 + 0.1976442834i)\rho] \quad (57)$$

$$\psi_+^2(\rho) \propto \exp[(-0.5258014144 + 0.6865159449i)\rho] \quad (58)$$

$$\psi_+^3(\rho) \propto \exp[(-0.4418140560 - 0.3841602281i)\rho] \quad (59)$$

Furthermore, over the different domains the deficiency index n_+ of the GUP-modified momentum operator with linear and quadratic terms in α is given as follows:

- $\mathcal{D}(\hat{p}) = \mathcal{L}^2(-\infty, \infty)$

In this domain, none of the functions, i.e., $\psi_+^1(x)$, $\psi_+^2(x)$, and $\psi_+^3(x)$, is square integrable. So these functions do not belong to this space and thus $n_+ = 0$.

- $\mathcal{D}(\hat{p}) = \mathcal{L}^2(0, \infty)$

In this domain, only $\psi_+^2(x)$ and ψ_+^3 have finite norm. So there are only two solutions from the set which are square integrable and thus $n_+ = 2$.

- $\mathcal{D}(\hat{p}) = \mathcal{L}^2([a, b])$

Over the finite interval all three solutions, i.e., $\psi_+^1(x)$, $\psi_+^2(x)$, and ψ_+^3 , are square integrable. So these functions belong to this space and thus $n_+ = 3$.

Second we write Eq.(22) for the function ψ_- and we get

$$\left(\hat{p}^\dagger + \frac{i}{\alpha}\right) \psi_-(\rho) = 0$$

which in terms of the dimensionless parameter ρ takes the form

$$\left(2i \frac{d^3}{d\rho^3} + \frac{d^2}{d\rho^2} - i \frac{d}{d\rho} + i\right) \psi_+(\rho) = 0.$$

The characteristic equation of the above differential equation is

$$2i\nu^3 + \nu^2 - i\nu + i = 0 \quad (60)$$

and so its roots are of the form

$$\nu_-^1 = \frac{1}{6} \sqrt[3]{-54 + 8i + 3\sqrt{303 - 96i}} + \frac{5}{6} \frac{1}{\sqrt[3]{-54 + 8i + 3\sqrt{303 - 96i}}} + \frac{i}{6} \quad (61)$$

$$= 0.5258014150 + 0.6865159455 i$$

$$\nu_-^2 = -\frac{1}{12} \sqrt[3]{-54 + 8i + 3\sqrt{303 - 96i}} - \frac{5}{12} \frac{1}{\sqrt[3]{-54 + 8i + 3\sqrt{303 - 96i}}} + \frac{1}{6} i \quad (62)$$

$$+ \frac{1}{2} i \sqrt{3} \left(\frac{1}{6} \sqrt[3]{-54 + 8i + 3\sqrt{303 - 96i}} - \frac{5}{6} \frac{1}{\sqrt[3]{-54 + 8i + 3\sqrt{303 - 96i}}} \right) \quad (63)$$

$$= 0.4418140560 - 0.3841602291 i$$

$$\nu_-^3 = -\frac{1}{12} \sqrt[3]{-54 + 8i + 3\sqrt{303 - 96i}} - \frac{5}{12} \frac{1}{\sqrt[3]{-54 + 8i + 3\sqrt{303 - 96i}}} + \frac{1}{6} i \quad (64)$$

$$- \frac{1}{2} i \sqrt{3} \left(\frac{1}{6} \sqrt[3]{-54 + 8i + 3\sqrt{303 - 96i}} - \frac{5}{6} \frac{1}{\sqrt[3]{-54 + 8i + 3\sqrt{303 - 96i}}} \right) \quad (65)$$

$$= -0.9676154710 + 0.1976442837 i.$$

Therefore, we obtain three linearly independent solutions for ψ_+

$$\psi_-^4(\rho) \propto \exp[(0.5258014150 + 0.6865159455 i)\rho] \quad (66)$$

$$\psi_-^5(\rho) \propto \exp[(0.4418140560 - 0.3841602291 i)\rho] \quad (67)$$

$$\psi_-^6(\rho) \propto \exp[(-0.9676154710 + 0.1976442837 i)\rho] \quad (68)$$

Furthermore, over the different domains the deficiency index n_- of the GUP-modified momentum operator with linear and quadratic terms in α is given as follows:

- $\mathcal{D}(\hat{p}) = \mathcal{L}^2(-\infty, \infty)$

In this domain, none of the functions, i.e., $\psi_-^4(x)$, $\psi_-^5(x)$, and ψ_-^6 , is square integrable. So these functions do not belong to this space and thus $n_- = 0$.

- $\mathcal{D}(\hat{p}) = \mathcal{L}^2(0, \infty)$

In this domain, only $\psi_-^6(x)$ has finite norm. So this is the only solution from the set which is square integrable and thus $n_- = 1$.

- $\mathcal{D}(\hat{p}) = \mathcal{L}^2([a, b])$,

Over the finite interval all three solutions, i.e., $\psi_-^4(x)$, $\psi_-^5(x)$, and ψ_-^3 , are square integrable. So these functions belong to this space and thus $n_- = 3$.

Finally, employing von Neumann's theorem, the self-adjointness of the GUP-modified momentum operator which is quadratic in α is described as follows:

- $\mathcal{D}(\hat{p}) = \mathcal{L}^2(-\infty, \infty) : (n_+, n_-) = (0, 0)$ and thus the momentum operator is self-adjoint.
- $\mathcal{D}(\hat{p}) = \mathcal{L}^2(0, \infty) : (n_+, n_-) = (2, 1)$ and thus the momentum operator is not self-adjoint.
- $\mathcal{D}(\hat{p}) = \mathcal{L}^2([a, b]) : (n_+, n_-) = (3, 3)$ and thus the momentum operator has infinitely many self-adjoint extensions.

All results produced in this section are briefly presented in Table I (all A's, B's, C's, and D's are constants).

TABLE I: Results for GUP-modified Momentum operator

Operator	$\psi_+(\rho)$	$\psi_-(\rho)$		(n_+, n_-)	
			$(-\infty, +\infty)$	$[0, +\infty)$	$[a, b]$
\hat{p}_0	$A_1 \exp[-\rho]$	$A_2 \exp[\rho]$	(0, 0)	(1, 0)	(1, 1)
$\hat{p}_0(1 - \alpha\hat{p}_0)$	$B_1 \exp[\lambda_+^1 \rho]$	$B_3 \exp[\lambda_-^1 \rho]$	(0, 0)	(1, 1)	(2, 2)
	$B_2 \exp[\lambda_+^2 \rho]$	$B_4 \exp[\lambda_-^2 \rho]$			
$\hat{p}_0(1 + 2\alpha^2\hat{p}_0^2)$	$C_1 \exp[\mu_+^1 \rho]$	$C_4 \exp[\mu_-^1 \rho]$	(0, 0)	(2, 1)	(3, 3)
	$C_2 \exp[\mu_+^2 \rho]$	$C_5 \exp[\mu_-^2 \rho]$			
	$C_3 \exp[\mu_+^3 \rho]$	$C_6 \exp[\mu_-^3 \rho]$			
$\hat{p}_0(1 - \alpha\hat{p}_0^2 + 2\alpha^2\hat{p}_0^3)$	$D_1 \exp[\nu_+^1 \rho]$	$D_4 \exp[\nu_-^1 \rho]$	(0, 0)	(2, 1)	(3, 3)
	$D_2 \exp[\nu_+^2 \rho]$	$D_5 \exp[\nu_-^2 \rho]$			
	$D_3 \exp[\nu_+^3 \rho]$	$D_6 \exp[\nu_-^6 \rho]$			

IV. GUP-MODIFIED HAMILTONIAN

In this section we will apply von Neumann's theorem to the GUP-modified Hamiltonian operator. For this reason, we first have to determine the functions $\psi_{\pm}(x)$ [9] which satisfy the equation

$$\hat{H}\psi_{\pm}(x) = \pm ik_0^2\psi_{\pm}(x) \quad (69)$$

where k_0 is a positive constant.

For the simple case of the Hamiltonian of a free particle, i.e., $\hat{H} = -D^2$, where D is the differential d/dx in the standard representation, Eq.(69) becomes

$$-D^2\psi_{\pm}(x) = \pm ik_0^2\psi_{\pm}(x) . \quad (70)$$

It is easily seen that the linearly independent solutions to the above equation are of the form

$$\psi_{\pm}(x) = a_{\pm} \exp[k_{\pm}x] + b_{\pm} \exp[-k_{\pm}x] \quad (71)$$

where $k_{\pm} = \frac{(1 \mp i)}{\sqrt{2}} k_0$.

Over different domains, the deficiency indices and the self-adjointness of the Hamiltonian operator are described as follows:

- $\mathcal{D}(\hat{H}_0) = \mathcal{L}^2(\mathbb{R}) : (n_+, n_-) = (0, 0)$ and thus the operator is self-adjoint.
- $\mathcal{D}(\hat{H}_0) = \mathcal{L}^2[0, \infty) : (n_+, n_-) = (1, 1)$ and thus the operator has infinitely many self-adjoint extensions parametrized by a $U(1)$ group.
- $\mathcal{D}(\hat{H}_0) = \mathcal{L}^2([0, L]) : (n_+, n_-) = (2, 2)$ and thus the operator has infinitely many self-adjoint extensions parametrized by a $U(2)$ group.

At this point it is noteworthy that since the quantity $(m\alpha^2)^{-1}$ (with m to be the mass of the particle) is a characteristic energy scale of GUP, we will use it from now on instead of k_0 .

A. Hamiltonian with linear term in α

We now consider as momentum operator in Eq.(69), the GUP-modified Hamiltonian operator which is linear in α , namely

$$\hat{H} = \frac{\hat{p}_0^2}{2m} - \frac{\alpha}{m} \hat{p}_0^3 \quad (72)$$

and we adopt the analysis of the previous section in order to study the self-adjointness of this GUP-modified Hamiltonian operator.

First we write Eq.(69) for the function ψ_+ and we get

$$\left(\hat{H}^\dagger - \frac{i}{\alpha^2 m} \right) \psi_+(x) = 0$$

which in terms of the dimensionless parameter ρ takes the form

$$\left(2i \frac{d^3}{d\rho^3} + \frac{d^2}{d\rho^2} + 2i \right) \psi_+(\rho) = 0 .$$

The characteristic equation of the above differential equation is

$$2i \lambda^3 + \lambda^2 + 2i = 0 \quad (73)$$

and so its roots are of the form

$$\begin{aligned}\lambda_+^1 &= \frac{1}{6} \sqrt[3]{-108 - i + 6\sqrt{324 + 6i}} - \frac{1}{6} \frac{1}{\sqrt[3]{-108 - i + 6\sqrt{324 + 6i}}} + \frac{i}{6} \\ &= -0.9722421029 + 0.1634946287 i\end{aligned}\quad (74)$$

$$\lambda_+^2 = -\frac{1}{12} \sqrt[3]{-108 - i + 6\sqrt{324 + 6i}} + \frac{1}{12} \frac{1}{\sqrt[3]{-108 - i + 6\sqrt{324 + 6i}}} + \frac{1}{6} i \quad (75)$$

$$+ \frac{1}{2} i \sqrt{3} \left(\frac{1}{6} \sqrt[3]{-108 - i + 6\sqrt{324 + 6i}} + \frac{1}{6} \frac{1}{\sqrt[3]{-108 - i + 6\sqrt{324 + 6i}}} \right) \quad (76)$$

$$= 0.4835224689 + 1.058350192 i$$

$$\lambda_+^3 = -\frac{1}{12} \sqrt[3]{-108 - i + 6\sqrt{324 + 6i}} + \frac{1}{12} \frac{1}{\sqrt[3]{-108 - i + 6\sqrt{324 + 6i}}} + \frac{1}{6} i \quad (77)$$

$$- \frac{1}{2} i \sqrt{3} \left(\frac{1}{6} \sqrt[3]{-108 - i + 6\sqrt{324 + 6i}} + \frac{1}{6} \frac{1}{\sqrt[3]{-108 - i + 6\sqrt{324 + 6i}}} \right) \quad (78)$$

$$= 0.4887196335 - 0.7218448203 i .$$

Therefore, we obtain three linearly independent solutions for ψ_+

$$\psi_+^1(\rho) \propto \exp [(-0.9722421029 + 0.1634946287 i)\rho] \quad (79)$$

$$\psi_+^2(\rho) \propto \exp [(0.4835224689 + 1.058350192 i)\rho] \quad (80)$$

$$\psi_+^3(\rho) \propto \exp [(0.4887196335 - 0.7218448203 i)\rho] . \quad (81)$$

Furthermore, over the different domains, the deficiency index n_+ of the GUP-modified Hamiltonian operator which is linear in α is given as follows:

- $\mathcal{D}(\hat{H}) = \mathcal{L}^2(-\infty, \infty)$

In this domain, none of the functions, i.e., $\psi_+^1(x)$, $\psi_+^2(x)$, and ψ_+^3 , is square integrable. So these functions do not belong to this space and thus $n_+ = 0$.

- $\mathcal{D}(\hat{H}) = \mathcal{L}^2[0, \infty)$

In this domain, only $\psi_+^1(x)$ has finite norm. So this is the only solution from the set which is square integrable and thus $n_+ = 1$.

- $\mathcal{D}(\hat{H}) = \mathcal{L}^2([a, b])$

Over the finite interval all three solutions, i.e., $\psi_+^1(x)$, $\psi_+^2(x)$, and ψ_+^3 , are square integrable. So these functions belong to this space and thus $n_+ = 3$.

Second we write Eq.(22) for the function ψ_- and we get

$$\left(\hat{H}^\dagger + \frac{i}{\alpha^2 m} \right) \psi_-(x) = 0$$

which in terms of the dimensionless parameter ρ takes the form

$$\left(2i \frac{d^3}{d\rho^3} + \frac{d^2}{d\rho^2} - 2i \right) \psi_-(\rho) = 0 .$$

The characteristic equation of the above differential equation is

$$2i \lambda^3 + \lambda^2 - 2i = 0 \quad (82)$$

and so its roots are of the form

$$\lambda_-^1 = \frac{1}{6} \sqrt[3]{108 - i + 6\sqrt{324 - 6i}} - \frac{1}{6} \frac{1}{\sqrt[3]{108 - i + 6\sqrt{324 - 6i}}} + \frac{i}{6} \quad (83)$$

$$= 0.9722396188 + 0.1634946804i$$

$$\lambda_-^2 = -\frac{1}{12} \sqrt[3]{108 - i + 6\sqrt{324 - 6i}} + \frac{1}{12} \frac{1}{\sqrt[3]{108 - i + 6\sqrt{324 - 6i}}} + \frac{1}{6}i \quad (84)$$

$$+ \frac{1}{2}i\sqrt{3} \left(\frac{1}{6} \sqrt[3]{108 - i + 6\sqrt{324 - 6i}} + \frac{1}{6} \frac{1}{\sqrt[3]{108 - i + 6\sqrt{324 - 6i}}} \right) \quad (85)$$

$$= -0.4835212697 + 1.058348130i$$

$$\lambda_-^3 = -\frac{1}{12} \sqrt[3]{108 - i + 6\sqrt{324 - 6i}} + \frac{1}{12} \frac{1}{\sqrt[3]{108 - i + 6\sqrt{324 - 6i}}} + \frac{1}{6}i \quad (86)$$

$$- \frac{1}{2}i\sqrt{3} \left(\frac{1}{6} \sqrt[3]{108 - i + 6\sqrt{324 - 6i}} + \frac{1}{6} \frac{1}{\sqrt[3]{108 - i + 6\sqrt{324 - 6i}}} \right) \quad (87)$$

$$= -0.4887183483 - 0.7218428106i. \quad (88)$$

Therefore, we obtain three linearly independent solutions for ψ_-

$$\psi_-^1(\rho) \propto \exp[(0.9722396188 + 0.1634946804i)\rho] \quad (89)$$

$$\psi_-^2(\rho) \propto \exp[(-0.4835212697 + 1.058348130i)\rho] \quad (90)$$

$$\psi_-^3(\rho) \propto \exp[(-0.4887183483 - 0.7218428106i)\rho]. \quad (91)$$

Furthermore, over the different domains the deficiency index n_- of the GUP-modified Hamiltonian operator which is linear in α is given as follows:

- $\mathcal{D}(\hat{H}) = \mathcal{L}^2(-\infty, \infty)$

In this domain, none of the functions, i.e., $\psi_-^1(x)$, $\psi_-^2(x)$, and $\psi_-^3(x)$, is square integrable. So these functions do not belong to this space and thus $n_- = 0$.

- $\mathcal{D}(\hat{H}) = \mathcal{L}^2[0, \infty)$

In this domain, only $\psi_-^2(x)$ and $\psi_-^3(x)$ have finite norm. So there are only two solutions from the set which are square integrable and thus $n_- = 2$.

- $\mathcal{D}(\hat{H}) = \mathcal{L}^2([a, b])$

Over the finite interval both solutions, i.e., $\psi_-^1(x)$, $\psi_-^2(x)$, and $\psi_-^3(x)$, are square integrable. So these functions belong to this space and thus $n_- = 3$.

Finally, employing von Neumann's theorem, the self-adjointness of the GUP-modified Hamiltonian operator which is linear in α is described as follows :

- $\mathcal{D}(\hat{H}) = \mathcal{L}^2(-\infty, \infty) : (n_+, n_-) = (0, 0)$. and thus the Hamiltonian operator is self-adjoint.
- $\mathcal{D}(\hat{H}) = \mathcal{L}^2(0, \infty) : (n_+, n_-) = (1, 2)$ and thus the Hamiltonian operator is not self-adjoint.
- $\mathcal{D}(\hat{H}) = \mathcal{L}^2([a, b]) : (n_+, n_-) = (3, 3)$ and thus the Hamiltonian operator has infinitely many self-adjoint extensions parametrized by a $U(3)$ group.

B. Hamiltonian with quadratic term in α

Let us now consider as momentum operator in Eq.(69), the GUP-modified Hamiltonian operator which is quadratic in α , namely

$$\hat{H} = \frac{\hat{p}_0^2}{2m} + \frac{5\alpha^2}{2m}\hat{p}_0^4 \quad (92)$$

First we write Eq.(69) for the function ψ_+ and we get

$$\left(\hat{H}^\dagger - \frac{i}{\alpha^2 m} \right) \psi_+(x) = 0$$

which in terms of the dimensionless parameter ρ takes the form

$$\left(5 \frac{d^4}{d\rho^4} - \frac{d^2}{d\rho^2} - 2i \right) \psi_+(\rho) = 0 .$$

The characteristic equation of the above differential equation is

$$5\mu^4 - \mu^2 - 2i = 0 \quad (93)$$

and so its roots are of the form

$$\begin{aligned} \mu_+^1 &= \frac{1}{20} \sqrt{40 + 40\sqrt{1+40i}} = 0.7938622795 + 0.2781709154i \\ \mu_+^2 &= -\frac{1}{20} \sqrt{40 + 40\sqrt{1+40i}} = -0.7938622795 - 0.2781709154i \\ \mu_+^3 &= \frac{1}{20} \sqrt{40 - 40\sqrt{1+40i}} = 0.3259261872 - 0.6775441980i \\ \mu_+^4 &= -\frac{1}{20} \sqrt{40 - 40\sqrt{1+40i}} = -0.3259261872 + 0.6775441980i . \end{aligned}$$

Therefore, we obtain four linearly independent solutions for ψ_+

$$\begin{aligned} \psi_+^1(\rho) &\propto \exp[(0.7938622795 + 0.2781709154i)\rho] \\ \psi_+^2(\rho) &\propto \exp[(-0.7938622795 - 0.2781709154i)\rho] \\ \psi_+^3(\rho) &\propto \exp[(0.3259261872 - 0.6775441980i)\rho] \\ \psi_+^4(\rho) &\propto \exp[(-0.3259261872 + 0.6775441980i)\rho] . \end{aligned}$$

Furthermore, over the different domains, the deficiency index n_+ of the GUP-modified Hamiltonian operator which is quadratic in α is given as follows:

- $\mathcal{D}(\hat{H}) = \mathcal{L}^2(\mathbb{R})$

In this domain, none of the functions, i.e., $\psi_+^1(x)$, $\psi_+^2(x)$, ψ_+^3 , and ψ_+^4 , is square integrable. So these functions do not belong to this space and thus $n_+ = 0$.

- $\mathcal{D}(\hat{H}) = \mathcal{L}^2[0, \infty)$

In this domain, only $\psi_+^2(x)$ and ψ_+^4 have finite norm. So there are only two solutions from the set which is square integrable and thus $n_+ = 2$.

- $\mathcal{D}(\hat{H}) = \mathcal{L}^2([a, b])$

Over the finite interval all four solutions, i.e., $\psi_+^1(x)$, $\psi_+^2(x)$, ψ_+^3 , and ψ_+^4 , are square integrable. So these functions belong to this space and thus $n_+ = 4$.

Second we write Eq.(22) for the function ψ_- and we get

$$\left(\hat{H}^\dagger + \frac{i}{\alpha^2 m} \right) \psi_-(\rho) = 0$$

which in terms of the dimensionless parameter ρ takes the form

$$\left(5 \frac{d^4}{d\rho^4} - \frac{d^2}{d\rho^2} + 2i \right) \psi_-(\rho) = 0 .$$

The characteristic equation of the above differential equation is

$$5 \mu^4 - \mu^2 + 2i = 0 \tag{94}$$

and so its roots are of the form

$$\begin{aligned} \mu_-^1 &= \frac{1}{20} \sqrt{40 + 40 \sqrt{1 - 40} i} = 0.7938622795 - 0.2781709154 i \\ \mu_-^2 &= -\frac{1}{20} \sqrt{40 + 40 \sqrt{1 - 40} i} = -0.7938622795 + 0.2781709154 i \\ \mu_-^3 &= \frac{1}{20} \sqrt{40 - 40 \sqrt{1 - 40} i} = 0.3259261872 + 0.6775441980 i \\ \mu_-^4 &= -\frac{1}{20} \sqrt{40 - 40 \sqrt{1 - 40} i} = -0.3259261872 - 0.6775441980 i . \end{aligned}$$

Therefore, we obtain four linearly independent solutions for ψ_-

$$\begin{aligned} \psi_-^1(\rho) &\propto \exp [(0.7938622795 - 0.2781709154 i)\rho] \\ \psi_-^2(\rho) &\propto \exp [(-0.7938622795 + 0.2781709154 i)\rho] \\ \psi_-^3(\rho) &\propto \exp [(0.3259261872 + 0.6775441980 i)\rho] \\ \psi_-^4(\rho) &\propto \exp [(-0.3259261872 - 0.6775441980 i)\rho] . \end{aligned}$$

Furthermore, over the different domains the deficiency index n_- of the GUP-modified Hamiltonian operator which is quadratic in α is given as follows:

- $\mathcal{D}(\hat{H}) = \mathcal{L}^2(\mathbb{R})$

In this domain, none of the functions, i.e., $\psi_-^1(x)$, $\psi_-^2(x)$, $\psi_-^3(x)$, and ψ_-^4 , is square integrable. So these functions do not belong to this space and thus $n_- = 0$.

- $\mathcal{D}(\hat{H}) = \mathcal{L}^2[0, \infty)$

In this domain, only $\psi_-^2(x)$ and ψ_-^4 have finite norm. So there are only two solutions from the set which is square integrable and thus $n_- = 2$.

- $\mathcal{D}(\hat{H}) = \mathcal{L}^2([a, b])$

Over the finite interval all four solutions, i.e., $\psi_-^1(x)$, $\psi_-^2(x)$, ψ_-^3 , and ψ_-^4 , are square integrable. So these functions belong to this space and thus $n_- = 4$.

Finally, employing von Neumann's theorem, the self-adjointness of the GUP-modified Hamiltonian operator which is quadratic in α is described as follows :

- $\mathcal{D}(\hat{H}) = \mathcal{L}^2(-\infty, \infty) : (n_+, n_-) = (0, 0)$ and thus the Hamiltonian operator is self-adjoint.
- $\mathcal{D}(\hat{H}) = \mathcal{L}^2(0, \infty) : (n_+, n_-) = (2, 2)$ and thus the Hamiltonian operator has infinitely many self-adjoint extensions parametrized by a $U(2)$ group.
- $\mathcal{D}(\hat{H}) = \mathcal{L}^2([a, b]) : (n_+, n_-) = (4, 4)$ and thus the momentum operator has infinitely many self-adjoint extensions parametrized by a $U(4)$ group.

C. Hamiltonian operator with linear and quadratic terms in α

We now consider as momentum operator in Eq.(69), the GUP-modified Hamiltonian operator which has linear and quadratic terms in α , namely

$$\hat{H} = \frac{\hat{p}_0^2}{2m} - \frac{\alpha}{m} \hat{p}_0^3 + \frac{5\alpha^2}{2m} \hat{p}_0^4. \quad (95)$$

First we write Eq.(69) for the function ψ_+ and we get

$$\left(\hat{H}^\dagger - \frac{i}{\alpha^2 m} \right) \psi_+(x) = 0$$

which in terms of the dimensionless parameter ρ takes the form

$$\left(5 \frac{d^4}{d\rho^4} - 2i \frac{d^3}{d\rho^3} - \frac{d^2}{d\rho^2} - 2i \right) \psi_+(\rho) = 0.$$

The characteristic equation of the above differential equation is

$$5\nu^4 - 2i\nu^3 - \nu^2 - 2i = 0. \quad (96)$$

Numerically solving the above characteristic equation, its roots are of the form

$$\nu_+^1 = -0.766585832834522 - 0.178090020398075 i \quad (97)$$

$$\nu_+^2 = -0.329812078138930 + 0.785252227529734 i \quad (98)$$

$$\nu_+^3 = 0.311994530947961 - 0.603212912845495 i \quad (99)$$

$$\nu_+^4 = 0.784403380025491 + 0.396050705713836 i. \quad (100)$$

Therefore, we obtain four linearly independent solutions for ψ_+

$$\psi_+^1(\rho) \propto \exp [(-0.766585832834522 - 0.178090020398075 i)\rho]$$

$$\psi_+^2(\rho) \propto \exp [(-0.329812078138930 + 0.785252227529734 i)\rho]$$

$$\psi_+^3(\rho) \propto \exp [(0.311994530947961 - 0.603212912845495 i)\rho]$$

$$\psi_+^4(\rho) \propto \exp [(0.784403380025491 + 0.396050705713836 i)\rho].$$

Furthermore, over the different domains, the deficiency index n_+ of the GUP-modified Hamiltonian operator which has linear and quadratic terms in α is given as follows:

- $\mathcal{D}(\hat{H}) = \mathcal{L}^2(-\infty, \infty)$

In this domain, none of the functions, i.e., $\psi_+^1(x)$, $\psi_+^2(x)$, ψ_+^3 , and ψ_+^4 , is square integrable. So these functions do not belong to this space and thus $n_+ = 0$.

- $\mathcal{D}(\hat{H}) = \mathcal{L}^2[0, \infty)$

In this domain, only $\psi_+^1(x)$ and ψ_+^2 have finite norm. So there are only two solutions from the set which are square integrable and thus $n_+ = 2$.

- $\mathcal{D}(\hat{H}) = \mathcal{L}^2([a, b])$

Over the finite interval all four solutions, i.e., $\psi_+^1(x)$, $\psi_+^2(x)$, ψ_+^3 , and ψ_+^4 , are square integrable. So these functions belong to this space and thus $n_+ = 4$.

Second we write Eq.(69) for the function ψ_- and we get

$$\left(\hat{H}^\dagger + \frac{i}{\alpha^2 m} \right) \psi_+(x) = 0$$

which in terms of the dimensionless parameter ρ takes the form

$$\left(5 \frac{d^4}{d\rho^4} - 2i \frac{d^3}{d\rho^3} - \frac{d^2}{d\rho^2} + 2i \right) \psi_+(\rho) = 0 .$$

The characteristic equation of the above differential equation is

$$5 \lambda^4 - 2i \lambda^3 - \lambda^2 + 2i = 0 . \quad (101)$$

Numerically solving the above characteristic equation, its roots are of the form

$$\nu_-^1 = -0.784403380025491 + 0.396050705713836 i \quad (102)$$

$$\nu_-^2 = -0.311994530947961 - 0.603212912845495 i \quad (103)$$

$$\nu_-^3 = 0.329812078138930 + 0.785252227529734 i \quad (104)$$

$$\nu_-^4 = 0.766585832834522 - 0.178090020398075 i . \quad (105)$$

Therefore, we obtain four linearly independent solutions for ψ_-

$$\psi_-^1(\rho) \propto \exp[(-0.784403380025491 + 0.396050705713836 i)\rho]$$

$$\psi_-^2(\rho) \propto \exp[(-0.311994530947961 - 0.603212912845495 i)\rho]$$

$$\psi_-^3(\rho) \propto \exp[(0.329812078138930 + 0.785252227529734 i)\rho]$$

$$\psi_-^4(\rho) \propto \exp[(0.766585832834522 - 0.178090020398075 i)\rho] .$$

Furthermore, over the different domains, the deficiency index n_+ of the GUP-modified Hamiltonian operator which has linear and quadratic terms in α is given as follows:

- $\mathcal{D}(\hat{H}) = \mathcal{L}^2(-\infty, \infty)$

In this domain, none of the functions, i.e., $\psi_-^1(x)$, $\psi_-^2(x)$, ψ_-^3 , and ψ_-^4 , is square integrable. So these functions do not belong to this space and thus $n_- = 0$.

- $\mathcal{D}(\hat{H}) = \mathcal{L}^2[0, \infty)$

In this domain, only $\psi_-^1(x)$ and ψ_-^2 have finite norm. So there are only two solutions from the set which are square integrable and thus $n_- = 2$.

- $\mathcal{D}(\hat{H}) = \mathcal{L}^2([a, b])$

Over the finite interval all four solutions, i.e., $\psi_-^1(x)$, $\psi_-^2(x)$, ψ_-^3 , and ψ_-^4 , are square integrable. So these functions belong to this space and thus $n_- = 4$.

Finally, employing von Neumann's theorem, the self-adjointness of the GUP-modified Hamiltonian operator which is linear in α is described as follows :

- $\mathcal{D}(\hat{H}) = \mathcal{L}^2(-\infty, \infty) : (n_+, n_-) = (0, 0)$. and thus the Hamiltonian operator is self-adjoint.
- $\mathcal{D}(\hat{H}) = \mathcal{L}^2(0, \infty) : (n_+, n_-) = (2, 2)$ and thus the Hamiltonian operator has infinitely many self-adjoint extensions parametrized by a $U(2)$ group.
- $\mathcal{D}(\hat{H}) = \mathcal{L}^2([a, b]) : (n_+, n_-) = (4, 4)$ and thus the Hamiltonian operator has infinitely many self-adjoint extensions parametrized by a $U(4)$ group.

All results produced in this section are briefly presented in Table II (all \tilde{A} 's, \tilde{B} 's, \tilde{C} 's, and \tilde{D} 's are constants).

TABLE II: Results for GUP-modified Hamiltonian operator

Operator	$\psi_+(\rho)$	$\psi_-(\rho)$	(n_+, n_-)		
			$(-\infty, \infty)$	$[0, \infty)$	$[a, b]$
$\frac{1}{2m}\hat{p}_0^2$	$\tilde{A}_1 \exp [+k_+x]$	$\tilde{A}_3 \exp [+k_-x]$	(0, 0)	(1, 1)	(2, 2)
	$\tilde{A}_2 \exp [-k_+x]$	$\tilde{A}_4 \exp [-k_-x]$			
$\frac{1}{2m}\hat{p}_0^2 - \frac{\alpha}{m}\hat{p}_0^3$	$\tilde{B}_1 \exp [\lambda_+^1\rho]$	$\tilde{B}_4 \exp [\lambda_-^1\rho]$	(0, 0)	(1, 2)	(3, 3)
	$\tilde{B}_2 \exp [\lambda_+^2\rho]$	$\tilde{B}_5 \exp [\lambda_-^2\rho]$			
	$\tilde{B}_3 \exp [\lambda_+^3\rho]$	$\tilde{B}_6 \exp [\lambda_-^3\rho]$			
$\frac{1}{2m}\hat{p}_0^2 + \frac{5\alpha^2}{2m}\hat{p}_0^4$	$\tilde{C}_1 \exp [\mu_+^1\rho]$	$\tilde{C}_5 \exp [\mu_-^1\rho]$	(0, 0)	(2, 2)	(4, 4)
	$\tilde{C}_2 \exp [\mu_+^2\rho]$	$\tilde{C}_6 \exp [\mu_-^2\rho]$			
	$\tilde{C}_3 \exp [\mu_+^3\rho]$	$\tilde{C}_7 \exp [\mu_-^3\rho]$			
	$\tilde{C}_4 \exp [\mu_+^4\rho]$	$\tilde{C}_8 \exp [\mu_-^4\rho]$			
$\frac{\hat{p}_0^2}{2m} - \frac{\alpha}{m}\hat{p}_0^3 + \frac{5\alpha^2}{2m}\hat{p}_0^4$	$\tilde{D}_1 \exp [\nu_+^1\rho]$	$\tilde{D}_5 \exp [\nu_-^1\rho]$	(0, 0)	(2, 2)	(4, 4)
	$\tilde{D}_2 \exp [\nu_+^2\rho]$	$\tilde{D}_6 \exp [\nu_-^2\rho]$			
	$\tilde{D}_3 \exp [\nu_+^3\rho]$	$\tilde{D}_8 \exp [\nu_-^3\rho]$			
	$\tilde{D}_4 \exp [\nu_+^4\rho]$	$\tilde{D}_8 \exp [\nu_-^4\rho]$			

V. A SIMPLE EXAMPLE

In this section we present a simple example of the GUP-modified Hamiltonian with a linear term in α [3]

$$H = \frac{p_0^2}{2m} - \frac{\alpha}{m}p_0^3. \quad (106)$$

We follow the analysis of [9] in order to describe the self-adjoint extensions of the specific Hamiltonian. We choose the domain of the operator to be the positive semi-axis. Therefore, we obtain (we use $*$ to denote complex conjugates)

$$\begin{aligned}
(H^\dagger\phi, \psi) - (\phi, H^\dagger\psi) &= \int_0^L \left[\left(-\frac{\hbar^2}{2m} \frac{d^2\phi^*}{dx^2} + \frac{i\alpha\hbar^3}{m} \frac{d^3\phi^*}{dx^3} \right) \psi - \phi^* \left(-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} - \frac{i\alpha\hbar^3}{m} \frac{d^3\psi}{dx^3} \right) \right] \\
&= \int_0^L \left[\frac{\hbar^2}{2m} \left(\phi^* \frac{d^2\psi}{dx^2} - \psi \frac{d^2\phi^*}{dx^2} \right) \left(\frac{d^3\phi^*}{dx^3} \psi + \phi^* \frac{d^3\psi}{dx^3} \right) \right] \\
&= \int_0^L \left[\frac{\hbar^2}{2m} \frac{d}{dx} \left(\phi^* \frac{d\psi}{dx} - \psi \frac{d\phi^*}{dx} \right) + \frac{i\alpha\hbar^3}{m} \left(\frac{d^2}{dx^2} (\phi^*\psi) - 3 \frac{d\phi^*}{dx} \frac{d\psi}{dx} \right) \right] \\
&= \frac{\hbar^2}{2m} [(\phi^*(L)\psi'(L) - \psi(L)\phi'^*(L)) - (\phi^*(0)\psi'(0) - \psi(0)\phi'^*(0))] \\
&\quad + \frac{i\alpha\hbar^3}{m} [(\phi''^*(L)\psi(L) + \phi^*(L)\psi''(L) - \phi'^*(L)\psi'(L)) \\
&\quad - (\phi''^*(0)\psi(0) + \phi^*(0)\psi''(0) - \phi'^*(0)\psi'(0))]
\end{aligned} \tag{107}$$

which for $\phi = \psi$ reduces to

$$\begin{aligned}
(H^\dagger\phi, \phi) - (\phi, H^\dagger\phi) &= \frac{2i\hbar^2}{2m} \frac{1}{2i} [L(\phi^*(L)\phi'(L) - \phi(L)\phi'^*(L)) - L(\phi^*(0)\phi'(0) - \phi(0)\phi'^*(0))] \\
&\quad + \frac{i\alpha\hbar^3}{mL^2} [L^2(\phi''^*\phi(L) + \phi^*(L)\phi''(L) - \phi'^*(L)\phi'(L)) - L^2(\phi''^*(0)\phi(0) + \phi^*(0)\phi''(0) - \phi'^*(0)\phi'(0))] \\
&= \kappa [(|L\phi'(0) - i\phi(0)|^2 + |L\phi'(L) + i\phi(L)|^2) - |L\phi'(0) + i\phi(0)|^2 - |L\phi'(L) - i\phi(L)|^2] \\
&\quad + \frac{\alpha\hbar}{2L} (|L^2\phi''(0) + \phi(0)|^2 + |L^2\phi''(L) + \phi(L)|^2 - |L^2\phi''(0) - \phi(0)|^2 - |L^2\phi''(L) - \phi(L)|^2 \\
&\quad + L^2|\phi'(L)|^2 - L^2|\phi'(0)|^2)
\end{aligned} \tag{108}$$

where $\kappa = i\hbar^2/mL$, as in [9]. In addition, we have introduced factors of L for dimensional reasons and the following identities were employed

$$\begin{aligned}
\frac{1}{2i} (xy^* - yx^*) &= \frac{1}{4} (|x + iy|^2 - |x - iy|^2) \\
2(xy^* + yx^*) &= |x + y|^2 - |x - y|^2.
\end{aligned}$$

It is evident that if $\alpha = 0$, then Eq.(108) reduces to Eq.(30) of [9], as expected.

To bring out the $U(3)$ invariance of the self-adjoint extensions of the Hamiltonian, defined by $H^\dagger\phi, \psi) - (\phi, H^\dagger\psi)$, one might, for example, use a 5-dimensional representation of $U(3)$ [we scale both sides of Eq.(108) by κ , and in the following, $A = \alpha\hbar/2L$]

$$\begin{pmatrix} L\phi'(0) - i\phi(0) \\ L\phi'(L) + i\phi(L) \\ A(L^2\phi''(0) + \phi(0)) \\ A(L^2\phi''(L) + \phi(L)) \\ A\phi'(L) \end{pmatrix} = U \begin{pmatrix} L\phi'(0) + i\phi(0) \\ L\phi'(L) - i\phi(L) \\ A(L^2\phi''(0) - \phi(0)) \\ A(L^2\phi''(L) + \phi(L)) \\ A\phi'(0) \end{pmatrix}. \tag{109}$$

The construction of U could be involved. However, note that corresponding to $U = I$, the 5×5 identity matrix, $\phi(0) = \phi(L) = 0$ (like we assumed in [3]), $\phi''(0), \phi''(L)$ arbitrary, and $|\phi'(0)| = |\phi'(L)|$ is a valid solution set of $(H^\dagger\phi, \phi) - (\phi, H^\dagger\phi) = 0$. From [3], we get ($\ell \equiv 2\alpha\hbar$)

$$\begin{aligned}
\phi' &= iA \left[k' e^{ik'x} - \frac{1}{\ell} e^{ix/\ell} \right] - iB \left[k'' e^{-ik''x} + \frac{1}{\ell} e^{ix/\ell} \right] \\
&= 2iAk \left[\cos kx + \frac{ik\ell}{2} \sin kx \right] - Ak^3 \ell x \cos kx \\
&\quad + iC \left[k e^{ikx} + \frac{1}{\ell} e^{ix/\ell} \right].
\end{aligned} \tag{110}$$

using $A + B + C = 0$, $k' = k(1 + k\alpha\hbar)$ and $k'' = k(1 - k\alpha\hbar)$, and simplifying. Then the condition $|\phi'(0)| = |\phi'(L)|$ translates to,

$$(-1)^n \left[2iAk \cdot e^{ik^2\ell L/2} + iC \left(k + \frac{1}{\ell} e^{i2t\pi} \right) \right] = e^{i\chi} \left[2iAk + iC \left(k + \frac{1}{\ell} \right) \right] \quad (111)$$

where $2t\pi \equiv \pi(p - n) + 2\epsilon_i\theta_C$, where $\epsilon_1 = 1$ and $\epsilon_2 = 0$, corresponding to solutions (20) and (21) of [3] respectively, and χ is an arbitrary phase. This can be simplified to

$$(-1)^n e^{i\phi} \left[2iAk + \frac{iC}{\ell} e^{i(2t\pi - \Phi)} \right] = e^{i\chi} \left[2iAk + \frac{iC}{\ell} \right] \quad (112)$$

where $\Phi = k^2\ell L/2$ and we have used $k + 1/\ell \approx 1/\ell$. The above admits of the solution

$$\chi = n\pi + \Phi, \quad 2t\pi - \phi = 2p_1\pi, \quad p_1 \in \mathbb{N}. \quad (113)$$

At this point, it is noteworthy that the above solution show that the quantization conditions (20) and (21) of [3] form a subset of the general solution (likely with more parameters).

VI. CONCLUSIONS

In this work we explore the self-adjointness of the GUP-modified momentum and Hamiltonian operators over different domains. The domains under study are: (a) the whole real axis, (b) the positive semi-axis, and (c) a finite interval. In order to utilize the von Neumann's theorem, we first obtain the functions ψ_{\pm} , second we compute the dimensions n_{\pm} of the deficiency subspaces of the GUP-modified momentum and Hamiltonian operators, and finally we infer whether the operators are self-adjoint or not, or they have infinitely many self-adjoint extensions. This analysis is adopted for all three cases of GUP-modified momentum and Hamiltonian operators, namely with a linear term in the GUP parameter α , with a quadratic term in α , and with both terms, i.e., the linear and the quadratic in α , to be included. It is noteworthy that the GUP-modified momentum operator with both terms in α to be included is self-adjoint operator when its domain is the whole real axis, it is not self-adjoint operator when its domain is the positive semi-axis, and it has infinitely many self-adjoint extensions when its domain is a finite interval. Furthermore, the GUP-modified Hamiltonian operator with both terms in α to be included is self-adjoint operator when its domain is the whole real axis, and it has infinitely many self-adjoint extensions when its domain is the positive semi-axis or a finite interval. At this point, it should be stressed that the self-adjoint extensions of different domains are parametrized by different unitary groups. Finally, a simple example of the Hamiltonian for a particle in a box is given and the solutions for the boundary conditions which describe the self-adjoint extensions of the specific operator are obtained.

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- [1] G. Amelino-Camelia, Living Rev. Rel. **16**, 5 (2013) [arXiv:0806.0339 [gr-qc]].
 - [2] A. Kempf, G. Mangano and R. B. Mann, Phys. Rev. D **52**, 1108 (1995) [hep-th/9412167].
 - [3] A. F. Ali, S. Das and E. C. Vagenas, Phys. Lett. B **678**, 497 (2009) [arXiv:0906.5396 [hep-th]].
 - [4] A. F. Ali, S. Das and E. C. Vagenas, arXiv:1001.2642 [hep-th].
 - [5] L. Debnath. and P. Mikusinski, *Introduction to Hilbert Spaces with Applications*, Academic Press. INC., 1990.
 - [6] E. Kreyszig, *Introductory Functional Analysis with Applications*, John Wiley and Sons. Inc., 1978.
 - [7] N. Boccara, *Functional Analysis: An Introduction for Physicists*, Academic Press, INC., 1990.
 - [8] M. Reed and B. Simon *Methods of Modern Mathematical Physics II: Fourier Analysis, Self-Adjointness*, Academic Press, 1975.
 - [9] G. Bonneau, J. Faraut and G. Valent, Am. J. Phys. **69**, 322 (2001) [quant-ph/0103153].