QUANTUM GRAVITATIONAL EFFECTS ON STATISTICAL MECHANICS

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Dedication

To everyone who joined me on my journey from Mexico, Germany, the United States, and Canada. Thank you.

Abstract

Nowadays, the development of a consistent theory of quantum gravity (QG) continues to be one of the biggest challenges in physics, which has resulted in a number of potential candidates that have not been tested yet. However, phenomenology has joined this task in search of manifestations of the quantum effects of space-time. In particular, the generalized uncertainty principle (GUP) modifies the uncertainty relation between momentum and position, making room for a minimal length, as predicted by candidate theories of QG.

Inspired by the GUP, we derive Planck's distribution by considering a new quantization of the electromagnetic field. We elaborate on the thermodynamics of the blackbody resulting from Wien's law and the Stefan-Boltzmann law. We demonstrate that such thermodynamic laws are modified by the presence of a minimal length. Furthermore, we consider a momentum scale in classical statistical mechanics to see its potential effects in the construction of the microcanonical ensemble.

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Chapter 1

Quantum Gravity

> Last night I dreamed about you. What happened in detail I can hardly remember, all I know is that we merged into one another. I was you, you were me. Finally, you somehow caught fire.
>
> -Franz Kafka, *Quotes*

The last century witnessed the development of two revolutionary theories: General Relativity (GR) and Quantum Mechanics (QM). GR was formulated by Albert Einstein in 1915 [1]. Such a theory describes gravity by introducing a novel understanding of the nature of space-time. From its beginning, GR has been considered a challenging theory because of its mathematical formulation based on differential geometry. However, the vast number of predictions in agreement with observed phenomena have produced great confidence in GR as being the classical theory of gravity. Within this list of predictions, we can find the first tests of GR, namely, the precession of Mercury's perihelion [2]; the deflection of light by the sun [3]; and the gravitational redshift [4]. More recently, the number of tests has increased by the detection of gravitational waves [5], and the digital reconstruction of a black hole [6] including the one in the center of our galaxy [7]. Despite such a success, some phenomena point out discrepancies in the theory and suggest a more fundamental viewpoint about gravity. That fundamental theory may be QG. A primary motivation for QG is the unification of the four fundamental interactions. This idea is based on the philosophical thought of reductionism, in which various phenomena can be described in terms of much more fundamental ones [8]. This trend is recurrent in physics, as shown by the unification of the weak and electromagnetic forces [9] and electricity and magnetism unified into electromagnetism by special relativity. In addition to that, the lack of an early description of the universe [10], the final stage of black hole evaporation [11], and the non-renormalizability of gravity [12], drive the search for a more fundamental theory of gravity.

Historically, the first one who pointed out the need for a quantum theory of gravity was Albert Einstein [13]. Nonetheless, Léon Rosenfeld, a close collaborator of Niels Bohr, was the first to try to obtain a quantization of the gravitational field. In [14], by considering a system including electromagnetic and gravitational interactions and according to Heisenberg quantization, he found a severe problem related to the divergence of the gravitational energy. However, his attempt was only an approximation at the weak gravitational limit, as a flat background was fixed to allow the geometry of space-time to be ignored. Matvei Bronstein, a young Russian physicist in Lev Landau's group, was the first who understood deeply the problem of quantizing the gravitational field. In his Ph.D. dissertation, Bronstein showed the fundamental differences between QM and GR for all sorts of fields, including strong gravitational fields. He concluded that gravity does not allow high energy concentration in an arbitrary region of space-time. If a high concentration of energy in a region of space-time takes place, the system collapses into a black hole. He also showed that Riemannian geometry and the conventional quantization procedure are inadequate in the search for a theory of QG [15, 16].

Many attempts have arisen to achieve such a unification since then: String Theory (ST), Loop Quantum Gravity (LQG), Supergravity, Twistor Theory, Causal Dynamical Triangulation, and many more candidates. All of them aim to describe the quantum nature of gravity. Among those theories, we identify those that quantize a classical theory of gravity directly by applying covariant or canonical quantization procedures and those that opt to construct a unified theory of all fundamental interactions and see how gravity emerges at some energy scale. Some procedures involving direct quantization make use of canonical quantization. This procedure requires a Hamiltonian formulation of GR, a set of canonical variables, and commutation relations. If the configuration variables are named to be loop variables, LQG is derived. By comparison, covariant quantization implements 4-dimensional covariant methods such as perturbation theory, effective fields, and path integral. Despite the quantization procedures, most of the QG models agree on the prediction of a particle responsible for the gravitational interaction. The role played by this particle in mediating gravity is central in the quantization of the gravitational field. According to field theory, particles are described by irreducible representations of the Poincaré group. A linearized gravity theory is needed if one desires to study gravity in the context of Quantum Field Theory (QFT). That is because QFT, as well as QM, are theories with a fixed background. Once gravity is linearized and quantized, the gravitational field is described by a massless particle with spin-2. These characteristics are consistent with the interaction of gravity since it must be long-range and attractive. However, since this process implies a perturbative approach, it is only valid on some scales, and it faces substantial problems in the Planck regime, where a QG theory is expected to provide a complete description of gravitational phenomena.

As pointed out earlier, there is a real need to develop a quantum theory of gravity, which would allow a fundamental understanding of the phenomena involving gravitational interaction. Trying to reconcile GR and QM has been an arduous task for more than a hundred years. During this time, many researchers have explored different paths leading to QG. In the rest of this section, we will review some candidate theories of QG.

1.1 Supersymmetry

Symmetries play an essential role in modern physics. By Noether's theorem [17], a continuous symmetry that leaves the action invariant is associated with an invariant physical quantity. Such continuous symmetries can be space-time symmetry that acts over space-time coordinates, as described by the Poincaré group, or internal symmetries that act over fields, represented by the group any reducible Lie group, and transform fields as elements of a Hilbert space. The local excitation of these fields gives rise to elementary particles classified into two groups: bosons, particles with integer spin, symmetric wave functions under interchange of coordinates, and obeying the Bose-Einstein statistics; and fermions, particles with half-integer spin, antisymmetric wave function under interchange of coordinates, and obeying the Pauli exclusion principle. Supersymmetry is a special symmetry that links bosons and fermions. For each boson, there is a corresponding fermion and vice-versa, called a ”superpartner,” of equal mass and a difference in spin of 1/2. For example, the superpartner of the spin-2 field describing the graviton would be a spin-3/2 field describing the gravitino.

1.1.1 Superalgebra

Supersymmetry is generalized symmetry presented in many physical theories supported by both theoretical and phenomenological motivations. From a theoretical point of view, the Coleman-Mandula theorem [18] points out that supersymmetry is an extension of Poincaré invariance. Such an extension is introduced to mix space-time and internal symmetry generators in a Lie algebra. However, with the usual Lie algebra such a task is impossible, and a new extension of Lie algebra is required. Gelfand and Likhtman in [19] showed that by introducing the so-called graded Lie algebra is possible to mix in a non-trivial way the Poincaré algebra and an internal symmetry algebra. A graded Lie algebra of grade n can be defined as a sum of vector spaces $L = \bigoplus_{i=0}^{n} L_i$ where L_i is a subalgebra. Under this definition, supersymmetry forms a graded Lie algebra of grade $n = 1$, with $L = L_0 \bigoplus L_1$, where L_0 is the Poincaré algebra, and $L_1 = (Q_\alpha, \overline{Q}_{\dot{\alpha}})$ is the vector space expanded by the supersymmetry generators Q_α and $\overline{Q}_{\dot{\alpha}}$. The algebra of the generators is summarized as follows,

$$\begin{aligned} \{Q_\alpha, \overline{Q}_{\dot{\beta}}\} &= 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu, \quad \{Q_\alpha, Q^\beta\} = 0, \\ [Q_\alpha, M_{\mu\nu}] &= (\sigma_{\mu\nu})_\alpha^\beta Q_\beta, \quad [Q_\alpha, P_\mu] = 0, \end{aligned} \tag{1.1}$$

where $\sigma_{\mu\nu}$ are the Pauli matrices, and P_μ and $M_{\mu\nu}$ the Poincaré generators. Eq.(1.1) together with the Poincaré algebra form the superalgebra whit $N = 1$ since it only contains one generator of supersymmetry Q_α. From a phenomenological point of view, supersymmetry solves the hierarchy problem. Due to the cancellation of Planck-scale quantum corrections between partners and superpartners, the hierarchy between the electroweak scale and the Planck scale is established in this case in a natural way [20]. This is, it provides an explanation for the difference in the energy scale between the electroweak model (10^2 GeV) and the Planck scale (10^{19} GeV). However, supersymmetry is not an exact symmetry and must be broken at a certain energy scale. This is due to the lack of evidence of superpartners, which have not been detected in any experiment. Additionally, there is no evidence that supersymmetry is a symmetry of nature. Nonetheless, supersymmetry solves the cosmological constant problem and offers natural candidates for dark matter [21]. At this point, supersymmetry seems to solve many of the current problems in modern physics and provides a candidate for QG, supergravity.

1.1.2 Supergravity

Supergravity is an extension of GR that incorporates supersymmetry. One of the motivations for the study of supergravity is that it makes room for a well-behaved quantum theory at first and second loop orders due to the presence of the graviton's superpartner [22, 23]. By imposing local supersymmetry, gravity arises naturally. From the Yang-Mills formulation, the metric $g_{\mu\nu}$ plays the role of a field with spin-2. Once supersymmetry is applied, a spinorial field with spin-$3/2$ is introduced as the superpartner called the gravitino. Following the gauge formulation for gravity, it is necessary to introduce the tetrad field e_a^μ that allows coupling gravity to spinors and provides the local framework for describing local gravity, and the spin connection ω_μ^{ab} that plays the role of a gauge boson field. However, this last one is no dynamical. Those are auxiliary fields introduced for consistency. The supergravity action for $N = 1$ can be written as the sum of the actions of two different fields. Palatini action [24], which is the Einstein-Hilbert action written in terms of the tetrad and the spin connection, and the action of the gravitino given by the Rarita-Schwinger action in a curved space [25], that is the action of a free spin-$3/2$ field

$$S_{N=1} = S_{E.H}(e,\omega) + S_{R.S}(\psi). \tag{1.2}$$

Although supergravity is a promising theory due to good divergence behaviour and the incorporation of gravity into a Yang-Mill's theory, it requires the observation of superpartners. However, no particle experiments have found concrete evidence of their existence.

1.2 Loop Quantum Gravity

LQG is an attempt to formulate a QG theory and offers a possible conceptual framework in which GR and QFT merge together into a consistent theory of space-time. An important feature of this theory resides in the possibility of describing quantum space-time in a background-independent and non-perturbative fashion. In this section, we review the elements of this theory.

LQG is a canonical approach to QG; that is, it starts from a classic Hamiltonian formalism of gravity. A Hamiltonian formalism that plays a role in the canonical theory of QG is the ADM formalism, named after its authors: Richard Arnowitt, Stanley Deser, and Charles W. Misner [26]. This formalism introduces a foliation of space-time manifold $\mathcal{M}$ in hypersurfaces Σ_t of equal time. For each of these hypersurfaces, the canonical variables are the 3-dimensional metric h_{ab} and its conjugate momentum p^{ab}. Once the canonical variables are defined, a Hamiltonian is constructed, and the equation of motion for space-time can be derived. Additionally, constraints are introduced and dictate the dynamics between the different hypersurfaces

$$\mathbf{H}_{\perp} = 0, \qquad \mathbf{H}_a = 0, \tag{1.3}$$

where $\mathbf{H}_{\perp}$ is the Hamiltonian constraint that generates diffeomorphisms on the manifold $\mathcal{M}$ orthogonal to Σ_t, and $\mathbf{H}_a$ is the spatial constraint that generates diffeomorphisms that preserve Σ_t. Eq.(1.3) is obtained by applying the principle of path independence, which states that the change in all field variables is to be independent of the path that connects different hypersurfaces [26]. Once canonical variables are quantized, the variables are promoted to operators. As the Hamiltonian constraint depends on those operators, the analogue of the Schrödinger equation for space-time is derived. Such an equation is known as the Wheeler-DeWitt equation [27]. Unlike the Schrödinger equation in QM, which contains the time parameter explicitly, the constructions in the Hamiltonian formulation of gravity do not include any time parameter. That introduces the time problem in QG. Such a problem has been addressed in three different ways: choosing the definition of time before the quantization, after, or keeping the timeless as option [28].

The construction of the Hamiltonian is a significant problem because there is more than one version of the constraint Eq.(1.3) that depends on the configuration and momenta variables. Furthermore, some choices for the variables can lead to a non-linear Hamiltonian, which presents complications once the quantization procedure is applied. Therefore, the difficulty of this task depends on the choice of variables.

A new set of canonical variables was introduced by Abhay Ashtekar in 1986 [29]. Those variables are

$$\begin{aligned} E^a_i(x) &= \sqrt{e}\, e^a_i(x), \\ GA^i_a(x) &= \Gamma^i_a(x) + \gamma K^i_a(x), \end{aligned} \tag{1.4}$$

where $E^a_i(x)$ play the role of the canonical momenta and A^i_a of the configuration variables. The configuration variables are multiplied by G to have units corresponding to the inverse of a length. Furthermore, e^a_i are the local *dreibeine*, Γ^i_a a special type of $SU(2)$ Yang–Mills gauge theory connection related to the invariance of the metric, K^i_a the second fundamental form whose determinant is related to the Gaussian curvature, and γ is the Barbero–Immirzi parameter. Ashtekar's variables are formally promoted to operators via canonical quantization by imposing the following commutation relation

$$[\hat{A}^i_a(x), \hat{E}^b_j(y)] = 8\pi i\hbar\gamma\, \delta^i_j\, \delta^b_a\, \delta(x,y). \tag{1.5}$$

An alternative set of variables, the loop variables, were introduced by Carlo Rovelli and

Lee Smolin [30]. By considering a space of loops,

$$\mathcal{L}_p := \{\alpha : \ [0,1] \longrightarrow \Sigma \mid \alpha(0) = \alpha(1)\}, \tag{1.6}$$

the new variables are defined through the holonomy $U[A,\alpha]$, where A is the proper contraction of the Ashtekar configuration variables from Eq.(1.4) and α from Eq.(1.6). As the holonomy is defined as a linear transformation after a parallel transport along a closed-loop, $U[A,\alpha] \in SU(2)$ due to the connection on Σ_t. Formally, holonomies are defined as the path-ordered exponential product of the integral over the configuration variables around the α loop

$$U[A,\alpha] = \mathcal{P}\exp\left(G\int_\alpha A\right), \tag{1.7}$$

where E_i^a as the conjugated variable.

A significant result obtained using this formulation is geometry quantization. Because LQG is a quantum theory, the system's states are Hilbert space elements. These states can be decomposed as a combination of a basis. In the LQG formulation, the spin network $\mathcal{S}$ represents a complete orthonormal basis describing the quantum states of the gravitational field at the hypersurface Σ_t [31]. By introducing the area operator $\hat{A}$ [32] and applying it to the spin network state Ψ_s, a discrete spectrum is obtained

$$\hat{A}(\mathcal{S})\Psi_s = 8\pi l_{PL}^2 \gamma \sqrt{j_p(j_p+1)}\Psi_s, \tag{1.8}$$

where j_p is the spin associated with the link p of the spin network and l_{PL} is the Planck's length. In this manner, it is found that LQG has a minimum area of

$$A_{min} \propto l_{PL}^2. \tag{1.9}$$

This characteristic therefore caused the spin network to be interpreted as a graph which edges represented a quanta of area.

LQG is not a complete quantum theory of gravity and is still in the development phase. A weak point in the theory is its classical limit since it has not been possible to recover GR [33]. So far, there is no direct or indirect evidence that shows whether LQG is correct because the predictions are expected to describe how quantum phenomena manifest at the Planck scale, outside the range of our experiments. Regardless, LQG is a strong contender for a QG theory.

1.3 String theory

ST is an attempt to unify the four fundamental interactions into a single field theory framework. This feature distinguishes it from the other candidates already mentioned. As is well known, fields exhibit quantum properties at a specific energy scale. Thus, gravity is expected to emerge from ST.

Initially, ST was formulated as an attempt to explain the hadron spectrum, but the theory was set aside as a theory of the strong interaction with the rise of Quantum Chromodynamics. However, Scherk and Schwarz in [34] showed the possibility of implementing ST as a QG model since the spectrum of the string exhibits a massless field with spin-2, which contains the same expected properties as the graviton. ST at low energies leads to GR and gauge theories, which make it a strong contender to be a so-called theory of everything. Furthermore, ST incorporates new features of our universe, such as supersymmetry, extra dimensions, axions, and a new fundamental entity, the string. Such features have not been observed yet. However, these features may emerge beyond the standard model.

1.3.1 Relativistic String

The simplest system to be analysed and show the new tools that ST incorporates is a relativistic string that propagates in a fixed space-time. Within the space-time, the propagation of the string sweeps a region of the Minkowski space that can be visualized as a hypersurface $X(\tau,\sigma)$ parametrized by a timelike coordinate τ and a spacelike coordinate σ. This hypersurface is known as the *worldsheet* in the literature, with an induced metric $\gamma_{\alpha\beta}$.

For a relativistic particle, the action is proportional to the worldline. Then, for a relativistic string, the action must be proportional to the area of the worldsheet given by

$$S = -T \int d\sigma d\tau \sqrt{-\det \gamma}, \tag{1.10}$$

where T is related to the tension of the string. Depending on the type of string we're contemplating, T, will vary. When the tension of the string is assumed to be much higher, often an order of magnitude or so below the Planck scale, strings will be relevant to quantum gravity $T \lessapprox M_{PL}^2$. Here, M_{PL} is the Planck mass.

Eq.(1.10) is known as the Nambu-Goto action. If the string is open, boundary conditions can be imposed at the string's end points. When Dirichlet conditions are set, a new object arises called a Dp-brane, which is an extended object with p spatial dimensions with $1 \leq p < 25$. A D25-brane is a space-filling brane in the bosonic string theory, where the number of spatial dimensions is 25. The string attaches both extremes to branes at the two ends, gaining its properties and conserving momentum [35].

Some problems that appear in the quantization process of the Nambu-Goto action are due to the presence of the square root in Eq.(1.10). Nevertheless, Alexander Polyakov studied a more general action already introduced by Stanley Deser and Bruno Zumino [36]. Although Polyakov was not the one who discovered the action, he did understand how to use it in the path integral, and as a result, the action bears his name. This action does not present similar drawbacks under path integral quantization. Indeed, Polyakov was able to quantize it. The Polyakov action is given by

$$S = \frac{T}{2} \int d\sigma \, d\tau \, \sqrt{h} \, h^{\alpha\beta} \, \partial_\alpha X^\mu \, \partial_\beta X^\nu \, g_{\mu\nu}(X), \tag{1.11}$$

where $h_{\alpha\beta}$ is a dynamical intrinsic metric on the worldsheet. As the intrinsic metric $h_{\alpha\beta}$ is not fixed, to recover the Nambu-Goto action from Polyakov is necessary to impose the following "on-shell" condition

$$\frac{\delta S}{\delta h_{\alpha\beta}} = 0, \tag{1.12}$$

that guarantees the string will be physical. Because of the freedom of the intrinsic world-sheet metric, as a very first option, a conformal flat Minkowski metric can be chosen. Such a choice reduces the equation of motions of the string to

$$\partial_\alpha \partial^\alpha X^\mu = 0, \tag{1.13}$$

where the solution is a superposition of flat waves of the form

$$X^\mu = \sum_{n=-\infty}^{\infty} \alpha_n^\mu e^{\frac{i2\pi n}{l}(\tau+\sigma)} + \tilde{\alpha}_n^\mu e^{\frac{-i2\pi n}{l}(\tau-\sigma)}. \tag{1.14}$$

Once the theory is quantized, the coefficients α_n^μ and $\tilde{\alpha}_n^\mu$ can be identified as the creation and annihilation operators [37]. However, some of the states that the theory contains have a negative norm, the so-called *ghosts*, that break the probabilistic interpretation of QM. Nevertheless, there is a procedure, the Faddeev-Popov procedure, which decouples such states from the theory. Initially, this procedure was introduced in the context of Yang-Mills theories [38]. To decouple the negative norm states that the bosonic string contains, the method requires a total of 26 dimensions to work. That was one of the first predictions of ST, the existence of extra dimensions.

1.3.2 Superstring Theory

As we mentioned, the bosonic ST describes only bosons fields in terms of the different oscillation modes of the string. In the attempt to introduce fermions into the spectrum of strings, supersymmetry was considered in order to generate fermionic fields $\psi(\sigma,\tau)$. Nevertheless, a supersymmetric ST inherits negative norm problems just like his bosonic ST counterpart. In order to decouple those negative norm states, the Faddeev-Popov procedure is used. However, because of supersymmetry, such a procedure only requires dimension

$D = 10$ to work. There are five alternative perturbative approaches for superstring theory, each with nine spatial and one temporal dimensions [39]:

- Type I string: the wordsheet is non-orientable for open and closed strings, and the theory contains one generator of supersymmetry.
- Type II string: It is further divided into types IIA and IIB. Both theories have an orientable worldsheet, and two generators of supersymmetry. The difference between the two theories lies in chirality. While type IIB is chiral, type IIA is not.
- The heterotic string: there are two heterotic theories that depend on the gauge group $SO(32)$ or $E_8 \times E_8$. In both type of theories, both left-moving and right-moving sectors can be described by world-sheet. However, it is heterotic because the left and right sectors bellow to different spaces with different dimension.

It is worth mentioning that there are no experiments that refute or support any prediction offered by ST. There is expectation that results may arise in the near future, once the technology improves. Nonetheless, ST is still a theory in progress. There are many aspects of it that are not quite well understood, besides the lack of a final version of the theory.

1.4 Black Hole Thermodynamics

One of the predictions of the Einstein field equations is that a sufficiently compact mass can deform a region of space-time, producing a strong gravitational field that even radiation cannot escape from. Such a region was named by Wheeler “a black hole”. A black hole is formed when a quantity of matter is compressed in a region of space-time smaller than its Schwarzschild radius, $r = 2GM/c^2$. For many years, black holes were not considered to be astronomical phenomena, but rather mathematical inconsistencies in Einstein’s theory due to the divergence of the field equations present in that region. However, the observation of stars and their evolutionary stages showed evidence supporting the existence of such objects as the last evolutionary stage of a massive body [40]. The existence of black holes represented a crisis since it showed that Einstein's theory predicts its failure by not being able to describe the region of space-time inside the black hole. The most amazing fact is that a black hole could be described as a thermodynamic system. The area property states that the area of a black hole horizon never decreases and implies that no physical allowed process can minimize the area, *i.e.*, $\delta A \geq 0$. This property allows one to establish a direct connection between the horizon area and the thermodynamic entropy. In such a connection the state variables are mass M, angular momentum J, and charge Q. Based on Hawking's area property in [41], Jacob Bekenstein argued in 1970 that the entropy of a black hole is related to its area A by analogy to the second law of thermodynamics [42].

Moreover, the laws of thermodynamics can be extended and formulated to black hole. The zeroth law states that the surface gravity κ of a black hole is constant over its event horizon, whereas the zeroth law of thermodynamics states that the temperature of a system in equilibrium is uniform. In addition, the first law for a rotating and charged black hole can be written as

$$dM = \frac{\kappa}{8\pi G} dA + \Omega dJ + \Phi dQ, \tag{1.15}$$

where Ω is the angular velocity of a rotating black hole, Φ is the electric potential, and G the Newton's constant. Those quantities remain constant over the event horizon for a stationary black hole [43]. Years later, the idea that black holes can be explored as thermodynamic systems gained support when Hawking showed that all black holes radiate as a blackbody with some temperature T_H [44]

$$T_H = \frac{\hbar\kappa}{2\pi k_B}. \tag{1.16}$$

Furthermore, Hawking radiation reduces the energy of a black hole, theorizing a scenario where black holes evaporate. Nevertheless, this radiation has not been observed by any experiment.

As shown, Eq.(1.16) mixes the universal gravitational constant G and Planck's constant $\hbar$, suggesting a strong connection between gravity and quantum mechanics to describe a black hole.

1.4.1 Quantum Effects in Black Holes

Black hole thermodynamics may offer a framework where gravity and quantum effects are considered together for describing processes in a semi-classical limit. That is, non-gravitational fields are considered quantum fields, but gravity is treated as a classical field. However, far from being a definitive framework, this approach faces a crucial problem related to the lack of a description during the final evaporation stage of a black hole [11]. According to Hawking's calculation, once quantum aspects of a black hole are taken into consideration, some classical results are no longer applicable. One of those results is Hawking's area theorem. By quantum effects, the black hole radiates particles, decreasing its area and, therefore, mass. Black hole evaporation is the name of this process. Another issue, the so-called information paradox, is connected to this evaporation process.

The information paradox establishes that the radiation of a black hole is an irreversible process, that is, the final state that describes the radiation is compatible with multiple initial states. That is because the radiation would be uniquely characterised by the Hawking temperature. Therefore, the initial states after evaporation will be indistinguishable. Many mathematical solutions to this problem have been proposed, including in [45–49]. However, as Hawking radiation has not been detected yet, There is no community-acceptable solution. In addition, by considering a black hole as a thermodynamic system, statistical mechanics may reconcile and merge QM and GR due to the existence of microscopic degrees of freedom that contribute to the thermodynamic observable. This possibility has been explored by looking for a statistical description of black holes that relates the number of microstates and entropy [50–54]. Due to the lack of a complete understanding of black hole statistical mechanics, another attempt, particularly striking for implementing a geometric formulation of statistical mechanics, has been investigated in the context of black holes. Such a formulation is called geometric information [55–58].

Geometric information is an attempt to understand the statistical properties of black holes through thermodynamics using differential geometry. In this formulation, thermodynamic information can be recovered using geometrical objects defined on a Riemannian manifold whose point-set are equilibrium thermodynamic states. An example of this is the curvature scalar R, which is a measure of the interaction between particles that conform to the system. Furthermore, R diverges as the system approaches critical points. According to [59–61], the same behaviour is obtained for black holes where R diverges at the limit of mechanical stability and suggests the existence of a microscopic structure whose constituents interact. This formulation may offer a straightforward connection and interpretation of the microscopic degrees of freedom in terms of the geometry the system generates.

Thus far, the ultimate goal of building a consistent quantum gravity theory has not been accomplished. The candidates we reviewed, rather than being definitive theories, are attempts that are still in progress. A future prospect is the development of a theory consistent both mathematically and experimentally with the quantum nature of gravitational phenomena.

Chapter 2

Phenomenology

> We degrade Providence too much by attributing our ideas to it out of annoyance at being unable to understand it.
>
> -Fyodor Dostoyevsky, *The Idiot*

At the outset of humanity, the study of nature has been one of the most remarkable tasks. Trying to understand phenomena that we once feared or dogmatized by attributing their manifestation to divine powers has led to an understanding of the world around us. Ancient civilizations developed their methods and theories to understand their existence as entities belonging to nature. Furthermore, we have realised that the world around us follows laws that we can reveal and understand. This knowledge has brought great benefit to humanity by allowing technological progress. Over time, these methods and systems became more and more refined, allowing the development of new concepts. Moreover, we have obtained information from objects of nature through assumptions and deductions to give an explanation based on fundamental principles. This process led to theories that could be falsified or corroborated through the scientific method. However, the certainty of the behaviour of nature depends on the assumptions we make to construct a theory. In his criticism of this method, Edmund Husserl, a German mathematician and philosopher, argued that theories may lead to faulty assumptions about nature. Edmund Husserl is considered one of the promoters of phenomenology, a philosophical current developed during the XX century [62]. In his phenomenological theory, he categorises objects into two types: the real-object that exists independently of us, that is, exists by itself, and the phenomenological-object. The latter is the one we must know, whose existence depends on the correlations between object and subject. However, it is possible to study a real-object from a phenomenological point of view. As mentioned above, the existence of real-objects is independent of humans. By accepting this fact, we are adopting the so-called natural attitude. However, by adopting a new attitude and doubting the existence of real-objects, we focus only on phenomenological-objects. Husserl calls this attitude "epojé." When practising the epojé, a phenomenological reduction is carried out and allows studying real-objects of nature as phenomenological-objects.

Phenomenological studies provide a comprehensive way of understanding science since it makes us adopt a new position of discovering instead of assuming. The discovery is mediated through the instrument, and the phenomenon will manifest itself in different ways depending on the type of instrumentation used. As a final comment, phenomenology in science should not be considered a threat to the formulation of theories but rather a new way of discovering phenomena in nature using the context in which they are described. That is, phenomenological studies can be adopted in different frameworks, trying to relate separate phenomena to each other in a consistent way with a theory but without deriving directly from it. Therefore, a phenomenological theory can be understood as a midpoint between theory and experiment.

2.1 Thermodynamics

Historically, thermodynamics started as an empirical formulation for understanding the processes that matter undergoes as induced by heat and work [63]. Understanding matter transformation replaced theological conceptions of matter as divine with a scientific vision based on experiments [64]. Under this new vision, it was revealed that matter follows simple laws expressed using mathematics, establishing a set of relationships between state variables such as volume, pressure, and temperature. However, this view was limited when choosing the variables that describe the system, with observation playing a fundamental role in such choice. Those variables are classified as intensive and extensive. Both classes of variables describe the system macroscopically.

The success archived by this new understanding of matter can be found in the gas laws, which are a series of relationships that describe the behaviour of a gas under different initial conditions [63]. The initial conditions give rise to a number of phenomena, each of which is consistently described by one of those gas laws. Therefore, a more general formulation that consistently describes all phenomena was required.

We can find the same argument in electromagnetism. In its early stages, electric and magnetic phenomena were described by different laws. However, Maxwell unified those separated phenomena under one theory, electromagnetism. In thermodynamics, a unification took place under two main concepts: energy and entropy. The industrial revolution promoted the study of thermodynamic systems. In particular, the steam engine is considered the driver of such a revolution. The attention this machine received can be seen most clearly in the work by Sadi Carnot, who identified the flow of heat as the cause of movement. Later, James Prescott Joule realised that heat is a source of energy and follows the same rules of mechanical conservation. Rudolf Clausius, following Carnot's writings on steam engines, introduced the concept of entropy as a fundamental quantity related to transformations [65]. All these novel concepts can be summarised in the laws of thermodynamics

- Zeroth law: If two systems are in thermal equilibrium with a third system, then they are in thermal equilibrium with each other.

- First law: When a system undergoes a state transformation, the algebraic sum of the different energy contributions is independent of the way in which the transformation occurs. This means that the difference in energy E depends only on the initial and final state

  $$dE = TdS - PdV + \sum_{i=1}^{N} \mu_i dN_i, \tag{2.1}$$

  where T is the temperature, S entropy, P pressure, V volume, μ_i and N_i are the chemical potential and number of particles for the $i-$th substance.

- Second law: The sum of the entropy changes of a closed system can never decrease

$$dS \geq 0. \tag{2.2}$$

Where it is assumed that T, V are positive, S and N_i are not negligible. In addition, it is considered a thermodynamic state well defined in equilibrium. Moreover, these variables that describe states are related through equations of state. These assumptions define the phenomenological nature of thermodynamics. Once the thermodynamic laws were established, thermodynamics achieved great success by having great applicability to different kinds of systems, including solids, liquids, gases, and radiation.

Despite the success of thermodynamics, its applicability to matter is limited. That is because thermodynamics does not describe the molecular level of matter. Therefore, a new theory that considers the behaviour of each of the molecules involved in a process was required. Thermodynamics gave rise to new fundamental theories able to understand the behaviour of matter from a microscopic point of view. This new approach began with the kinetic theory of gases, which describes the properties of a macroscopic system in terms of its microscopic constituents.

Rudolf Clausius, once again an important figure, introduced statistical tools to study microscopic properties. Moreover, Clausius's works represented a basis for Maxwell and Boltzmann to study the dynamics of gases based on different statistical distributions. The contact with thermodynamics arose through Boltzmann's H-theorem, which introduced a connection between entropy and molecular dynamics of the system [66]. The kinetic theory gave rise to a more sophisticated successor, the Gibbs ensemble theory, which introduced the fundamental concept of an ensemble.

2.2 Statistical Mechanics

Statistical mechanics is a branch of physics with the greatest applicability to all states of matter, such as solids, liquids, gases, compounds, and systems under extreme conditions.

Its objective is the study of macroscopic thermodynamic properties of different systems, considering classical or quantum microscopic properties and making use of statistical analyses. That is, thermodynamic properties can be obtained from the collective dynamics of the particles via statistical approaches.

The fundamental concept for the construction of statistical mechanics is the definition of an ensemble. An ensemble can be understood as a collection of mental copies of a thermodynamic system. Each of these copies is subject to the same thermodynamic conditions and receives the name of microstate. However, from a microscopic point of view, each of them has a different configuration. The members of an ensemble are represented by points in an N-dimensional phase space where $6N$ is the number of particles and evolves according to the Hamiltonian equations of motion. This geometric representation using the phase space of the ensemble allows us to study the system according to the density function $\rho(q, p; t)$. Such a function contains all the system's information and can be used to obtain thermodynamic properties. In order to obtain the value of the thermodynamic observable f, the ensemble average is taken. The ensemble average consists of calculating the value of f for all microstates compatible with the thermodynamic variables in equilibrium. This is because the thermodynamic properties will take different values in each microstate. Then, the average is taken, and it is postulated that such a value corresponds to the thermodynamic variable of the system.

In the following sections, we review the different types of ensembles. Furthermore, the fundamental connection between statistical properties and thermodynamic potentials is revised. In addition, the partition function is introduced, which turns out to be some of the most significant objects in statistical thermodynamics.

2.2.1 Microcanonical ensemble

The simplest system to start within ensemble theory is that of an isolated system, which is a system that does not exchange matter or energy with its surroundings, and whose volume is constant. Each phase space point defines a microstate accessible to the system with N molecules, fixed volume V, and energy E, and whose dynamics are determined by the Hamiltonian function $H(q,p)$. To evaluate the equilibrium properties that the system manifests, it is necessary to know the number of elements that exhibit these equilibrium properties. Furthermore, each microstate is assumed to be equally likely. This idea is summed up by the postulate of equal a priori probability. The phase space density of the microcanonical ensemble is as follows [67]

$$\rho_{mc} = \begin{cases} \frac{1}{\Omega} & \text{if } E \leq H(q,p) \leq E + \Delta E \\ 0 & \text{otherwise} \end{cases} \tag{2.3}$$

where Ω is the number of microstates with an energy range between E and $E + \Delta E$. By using the density function, the average value of thermodynamic observables can be obtained. This value corresponds to what is expected to be obtained after a measurement

$$\langle f \rangle = \int d^{3N}p\, d^{3N}q\, f(p,q)\, \rho_{mc}(p,q). \tag{2.4}$$

The fundamental quantity that connects the statistical view encoded into the probability density and the phenomenological view given by thermodynamics is entropy

$$S = k_B \ln\, \Omega, \tag{2.5}$$

where k_B is the Boltzmann constant. This definition justifies the fact that, in equilibrium thermodynamics, the most likely macrostate to manifest is the one that corresponds to the largest number of microstates. That is, for the state of equilibrium, the number of microstates is maximal Ω_{max}. In thermodynamics, entropy follows the same extremal principle. In equilibrium, the entropy is maximal. Therefore, Eq.(2.5) manifests the properties of a thermodynamic entropy defined by the second law of thermodynamics.

2.2.2 Canonical ensemble

The microcanonical ensemble represents an idealization because having an isolated system in nature with fixed energy is impossible. Real systems exhibit fluctuations due to the exchange of matter and energy with their surroundings. Thus, the next system to analyse is a closed system for which the conditions at the boundary allow the exchange of energy in the form of heat with its surroundings. Unlike an isolated system that is described by the state variable E, a closed system is described by a temperature T once the system reaches equilibrium. Thus, the system can access different microstates characterised by different energies E_i with a fixed T as a constraint. The probability $\mathcal{P}_i$ of each microstate manifesting depends on its energy

$$\mathcal{P}_i \propto \exp\{-\beta E_i\}, \tag{2.6}$$

that is, the probability decreases as E_i increases [67]. The postulate of equal a priori probability ensures that microstates with the same energy E_i are equally accessible. By extending such probability to phase-space, the probability density of the canonical ensemble is given by [67]

$$\rho_c = \frac{\exp\{-\beta H(p,q)\}}{\int d^{3N}q\, d^{3N}p\, \exp\{-\beta H(p,q)\}}, \tag{2.7}$$

where β is a Lagrange multiplier due to the system constraints. The entropy is defined now as the ensemble average of the probability density $S = -k_B \ln\, \rho_c$. By using Eq.(2.7), the average entropy is then

$$\langle S \rangle = \int d^{3N}q\, d^{3N}p\, \rho_c(p,q)\, [k_B \beta H(p,q) + k_B \ln Z], \tag{2.8}$$

where Z is the partition function defined as

$$Z = \int d^{3N}q\, d^{3N}p\, \exp\{-\beta H(p,q)\}. \tag{2.9}$$

The partition function represents the heart of statistical mechanics. It provides information on the number of microstates accessible to the system given the ensemble. Furthermore, the partition function is used to establish the connection with thermodynamics. For the canonical ensemble, the thermodynamic potential that establishes the connection between statistical mechanics and equilibrium thermodynamics is the Helmholtz free energy, defined as follows

$$F(T,V,N) = -k_B \ln Z. \tag{2.10}$$

Once this connection is established, all the thermodynamic information about the system can be derived.

The ensemble theory can be reformulated using statistical methods.This new ensemble formalism allows for a fundamental connection between the quantum mechanical energy levels available to a system and thermodynamic potentials. As previously mentioned, this connection is through the partition function. By considering a closed system in equilibrium, each microstate is described by the macroscopic thermodynamic variables (T,V,N). Because energy is not fixed, the microstates are defined by the energy available in the set $\{E_j\}$. Furthermore, degeneration is allowed. That is, different microstates can be in the same energy state. All this information is constained in the occupancy numbers a_j, which describe the number of microstates in the j-th energy state. Occupation numbers meet the following conditions

$$\sum_j a_j = \mathcal{A}, \qquad \sum_j a_j E_j = \mathcal{E}, \tag{2.11}$$

where $\mathcal{A}$ is the number of elements in the ensemble, and $\mathcal{E}$ is the total energy of the ensemble. The equation above means that a canonical ensemble can be understood as an isolated system, pointing out a connection with the microcanonical ensemble. Moreover, the postulate of equal a priori probability implies that each possible distribution of $\{a_j\}$ is equally probable. Each of the possible distributions is realisable in different ways. The number of ways W can be expressed through a multinomial coefficient, which assigns a weight in which the total number of members of the ensemble $\mathcal{A}$ can be arranged in a particular distribution $\{a_j\}$. The multinomial coefficient is expressed as follows

$$W = \frac{\mathcal{A}!}{\Pi_k a_k!}. \tag{2.12}$$

Eq.(2.12) can be maximised to obtain the most probable distribution. This maximisation process can be performed by applying Lagrange's method of undetermined multipliers under the conditions given by Eq.(2.11)

$$\delta \left\{ \ln W - \alpha \sum_k a_k - \beta \sum_k a_k E_k \right\} = 0 \tag{2.13}$$

where for a large number of microstates, Stirling's approximation can be adopted for W. This maximization gives the most probable distribution in terms of α and β, expressed as follows

$$\overline{a}_k = e^{-\alpha-1} e^{-\beta E_k}. \tag{2.14}$$

The occupation number a_j of the state j corresponds to the weight of the corresponding microstate in the ensemble. Therefore, $a_j/\mathcal{A}$ can be interpreted as the probability of the j-th state to take place. For the most probable distribution, this is shown as follows

$$\mathcal{P}_j = \frac{\overline{a}_j}{\mathcal{A}} = \frac{e^{-\beta E_j(N,V)}}{\sum_j e^{-\beta E_j(N,V)}}. \tag{2.15}$$

The sum in the denominator in Eq.(2.15) is identified with the canonical partition function

$$Z = \sum_j e^{-\beta E_j(N,V)}. \tag{2.16}$$

The sum extends over all microstates accessible to the system. As mentioned, each of these microstates is represented by a point in phase space. For a classical system, it becomes necessary to interchange the sum over all possible states with an integral over phase-space as well as the energy of each state with the Hamiltonian of the system as a function of the coordinates and momentum as shown

$$\sum_j \to \int d^{3N}q\, d^{3N}p, \qquad\qquad E_j(N,V) \to H(q_j,p_j). \tag{2.17}$$

In this way, both the partition function Eq.(2.9) and the probability density Eq.(2.7) are recovered.

2.2.3 Grand canonical ensemble

Unlike the microcanonical and canonical ensemble, the grand canonical ensemble represents a better way to describe a system where energy and the number of particles do not remain constant. In addition, this ensemble represents an advantage when studying mixtures, where each substance is characterised by a chemical potential μ_i. Elaborating on ensemble theory, the microstates accessible to the system are characterised by an energy E_i and a number of particles N. The probability $\mathcal{P}_{iN}$ of each microstate manifesting depends on its energy and its number of particles [67]

$$\mathcal{P}_{iN} \propto \exp\left\{-\beta E_i + \gamma N\right\}, \tag{2.18}$$

where β and γ are Lagrange multipliers. The phase space of a grand canonical ensemble is spanned by the momenta and positions of the N particles the system contains. Since this number is no longer fixed, the distribution density depends on the number of particles in that particular state. The grand canonical ensemble is described by the probability density

$$\rho_{gc} = \frac{\exp\{-\beta H(p,q) - \gamma N\}}{\sum_{N=0}^{\infty} \int d^{3N}q\, d^{3N}p\, \exp\{-\beta H(p,q) - \gamma N\}}. \tag{2.19}$$

Using Eq.(2.19), the average entropy can be obtained as follows

$$\langle S\rangle = \sum_{N=0}^{\infty}\int d^{3N}q\, d^{3N}p\, \rho_{gc}(p,q,N)\, [k_B\beta H(p,q) + k_B\gamma N + k_B \ln \mathcal{Q}], \tag{2.20}$$

where $\mathcal{Q}$ is the grand canonical partition function

$$\mathcal{Q} = \sum_{N=0}^{\infty}\int d^{3N}q\, d^{3N}p\, \exp\{-\beta H(p,q) - \gamma N\}. \tag{2.21}$$

For this description, the connection between statistical mechanics and thermodynamics in equilibrium is given by the grand potential, defined as follows

$$\Phi(T,V,\mu) = -k_B \ln\, \mathcal{Q}. \tag{2.22}$$

Once again, the importance of the partition function as a bridge between statistical properties and thermodynamic properties of the system is observed in Eq.(2.22).

Now, we can proceed to derive the partition function of the grand canonical ensemble using the statistical method in ensemble theory. The partition function was derived using the most probable distribution for a closed system in thermal equilibrium. The same procedure applies to an open system that is described by the state variables (T,V,μ) in equilibrium. The ensemble members are defined by a number N of particles they contain, and depending on that number there is a set of accessible energy $\{E_{Nj}\}$. Similarly, the occupation number a_{Nj} represents the number of ensemble members who have N particles in the j-th state. The distribution is the set of occupation numbers $\{a_{Nj}\}$, and each possible distribution meets the following conditions

$$\sum_N\sum_j a_{Nj} = \mathcal{A}, \qquad \sum_N\sum_j a_{Nj}E_{Nj} = \mathcal{E}, \qquad \sum_N\sum_j a_{Nj}N = \mathcal{N} \tag{2.23}$$

where $\mathcal{A}$ is the total number of members in the ensemble, $\mathcal{E}$ is the ensemble's energy, and $\mathcal{N}$

is the total number of particles the ensemble contains. The postulate of equal a priori probability implies that each distribution $\{a_{Nj}\}$ has the same probability. Nevertheless, each of those distributions is realisable differently. By introducing a multinomial coefficient, each distribution is assigned a weight

$$W = \frac{\mathcal{A}!}{\prod_N \prod_j a_{Nj}!}. \tag{2.24}$$

The multinomial coefficient is maximised to get the most probable distribution. By applying Lagrange's method of undetermined multipliers under the condition Eq.(2.23) and considering a large number of microstates, the maximisation procedure leads to the following expression

$$\delta \left\{ \ln W - \alpha \sum_N \sum_k a_{Nk} - \beta \sum_N \sum_k a_{Nk} E_{Nk} - \gamma \sum_N \sum_k a_{Nk} N \right\} = 0, \tag{2.25}$$

we get the most probable distribution in terms of the multipliers α, β, and γ

$$\overline{a}_{Nk} = e^{-\alpha-1} e^{-\beta E_k} e^{-\gamma N}. \tag{2.26}$$

In a similar way, the coefficient $a_{Nj}/\mathcal{A}$ is interpreted as the probability that a microstate in the state j with N particles appears between the $\mathcal{A}$ members of the ensemble. For the most probable distribution, that is

$$\mathcal{P}_{Nj} = \frac{\overline{a}_{Nj}}{\mathcal{A}} = \frac{e^{-\beta E_{Nj}(V)} e^{-\gamma N}}{\sum_N \sum_j e^{-\beta E_{Nj}(N)} e^{-\gamma N}}, \tag{2.27}$$

where the denominator of Eq.(2.27) is the grand canonical partition function

$$\mathcal{Q} = \sum_N \sum_j e^{-\beta E_{Nj}(V)} e^{-\gamma N}. \tag{2.28}$$

Eq.(2.22) can be recovered by performing the following changes

$$\sum_N \sum_j \to \sum_N \int d^{3N}q \, d^{3N}p, \qquad\qquad E_{Nj} \to H(q_j, p_j). \tag{2.29}$$

As has been mentioned previously, each ensemble describes a different thermodynamics system. This choice was based on the way the system interacts with its surroundings. However, it can be demonstrated that in the thermodynamic limit defined as

$$V \to \infty, \qquad\qquad N \to \infty, \qquad\qquad \frac{N}{V} = \text{constant}, \tag{2.30}$$

the ensembles are equivalent. The reason for this equivalence is that in the thermodynamic limit, the system experiences fluctuations on the order of $1/\sqrt{N}$. This can be exemplified in the case of the canonical ensemble, where each microstate is in a different energy state. However, the energy sharply peaked at the main value in the thermodynamic limit. Therefore, the total energy of the ensemble is the same for each microstate. Thus, each of them can be considered a microcanonical ensemble. Thus, the choice is reduced to mere convenience. However, technically speaking, the choice of an ensemble can significantly simplify the calculations. As seen throughout this section, the role played by the partition function as a bridge that connects the microscopic statistical properties of matter with thermodynamic variables leads us to consider this object as the heart of statistical mechanics. Under equilibrium conditions, three partition functions were constructed that depend on the microscopic degrees of freedom of the system. Furthermore, thermodynamics results as a direct consequence of statistical mechanics at equilibrium. Thus, thermodynamics can be understood as a manifestation of the equilibrium properties of matter and, therefore, a phenomenological realization of statistical mechanics.

2.3 Quantum Gravity Phenomenology

The current description of nature is due to two theories that seem to be incompatible with each other. On the one hand, GR is a classical theory that describes the gravitational field through curvature and neglects quantum properties of gravity. On the other hand, QM neglects the gravitational interaction between particles. However, we know there are possible scenarios in nature that require both theories. Among those scenarios, we have collisions between particles at high energies where the reciprocal gravitational interaction cannot be negligible. In one attempt to mix both contributions, QFT has been formulated in a curved space, where a classical and fixed background is considered, and the fields are described quantumly. However, this is considered an approach due to the fact that gravity is treated classically.

The first chapter was dedicated to different attempts to merge GR and QM in a theoretical framework allowing the description of possible quantum properties of gravity. Nevertheless, these theories predict that the quantum properties of gravity will be manifested through experiments at energy scales of the order of the Planck energy. Our current technology can not achieve such a scale. This limitation has led to a purely theoretical approach to the problem of quantum gravity, but it is worth remembering that physics, apart from being a theoretical science, is also experimental. The design of experiments that allow the measurement of quantum properties of gravity has led to several possible proposals even with the technology we have [68–72]. Such proposals opened the door to a phenomenological approach to gravity. Based on Husserl's philosophy, phenomenological studies of quantum gravity require a phenomenological reductionism through the epojé as we explained before. This leaves quantum gravity as a phenomenological-object that can manifest in different frameworks.

2.3.1 Minimal Length

The ancient sophists' fascination with understanding what the matter of the universe is made of led them to propose various substances as possible origins. This concept was the so-called $\alpha\rho\chi\eta$ that means origin. In turn, this concept was separated into two schools of thought. Pluralism establishes the existence of several elementary substances. An example of this is Empedocles' proposition, which said that the four fundamental elements, water, earth, air, and fire, gave rise to everything around us. In contrast to pluralism, monism establishes the existence of an elementary substance. However, the sophists struggled to explain how a single substance gave rise to the different elements. In this last line of thought, we find Democritus, who proposed the atom as an elementary substance. The atom could combine with others in different ways and give rise to all kinds of bodies. In this way, by introducing the atom as the fundamental block of reality, all kinds of bodies turn out to be manifestations of the different ways in which such fundamental blocks can be arranged.

The experimental proof came a long time later. John Dalton began a series of experiments that led him to conclude the atom's existence and propose the first atomic model. His experimental basis was studying the pressure exerted by a mixture of gases. The results led to Dalton's law of partial pressures, which states that the net pressure exerted by a mixture of non-interacting gases is the sum of the individual pressures. Moreover, part of his research on gases led him to propose his law of multiple proportions, which states that only certain proportions of gases can be combined. Both laws became the basis for proposing his atomic model. Despite pioneering atomic models, Dalton's model faces flaws. For example, it does not explain observations of cathode rays or radioactivity [73]. Once more sophisticated experiments took place, they revealed a structure different from the one proposed by Dalton. Furthermore, the discovery of subatomic particles gave rise to new and more precise atomic models that made the idea of the atom as a fundamental unit disappear. However, something can be resumed from Dalton's atomic model and the philosophical idea of Democritus, that is, the need for a fundamental block of reality.

Max Planck's solution to blackbody radiation introduced a new constant that defined quantum physics, $\hbar$, in the same way that Einstein's special relativity introduced the speed of light, c. Moreover, Max Planck proposed the construction of a units that are derived using those fundamental constants together with Newton's gravitational constant, G. Those units are known as the Planck units.

In an attempt to improve QM and explain the breakdown of Fermi's theory at high energies, Heisenberg argued that a minimal length must be introduced in the already revolutionary QM to be complete [74]. In parallel, the calculation of the transition amplitudes in different particle processes represented a major challenge due to the presence of divergences. As a consequence, the possible final state was no longer normalizable. The non-renormalizability function was used to support Heisenberg's argument for the need to include a minimal length as a limit to overcome such divergences. However, an unfavourable point for such an argument is that a minimal length as a cut-off brings invariance problems. That is, distinct inertial observers will assign a different value to the cut-off scale. Furthermore, the unitarity of operators breaks because there is no complete set of states that expand the operator space. The problem of infinite transition amplitudes has been highly studied by many people in the field. Independently, Feynman, Dyson, Schwinger, and Tomonaga developed procedures to cure the divergence called renormalization. One of these procedures is the regularization, in agreement with Heisenberg's idea of introducing a cut-off. However, in the renormalization context, introducing a cut-off does not present an invariance problem because the regularisation process is only momentary. This set of techniques presented an advantage over Heisenberg's idea of introducing a minimal length in the calculations. Nevertheless, the idea of the existence of a minimal length remained. To avoid the invariance problem related to a minimal length, Hartland Snyder proposed a change in the canonical commutation relations by introducing models of quantum mechanics in a non-commutative Hilbert space [75]. On the other hand, Alden Mead argued through a *gedankeexperiments* in favour of the modification of the commutation relation due to the effects of gravity, because gravity couples to everything [76]. Models with a minimal length have gained great relevance as regulators of divergences and possible manifestations of the quantum properties of space-time.

In the following two sections, two different models that introduce a minimal length are addressed. The first one, in the framework of special relativity, introduces length as a new invariant. This approach revises Einstein's postulates and the role played by the Lorentz group. The second model modifies the Heisenberg uncertainty principle, giving rise to a minimum uncertainty in the position.

2.4 Double Special Relativity

Double special relativity (DSR) is a phenomenological theory that modifies special relativity (SR) by incorporating a new invariant. This modification claims to introduce $\hbar$ into the structure of space-time and see the new features that emerge in the kinematics of particles. The way $\hbar$ is introduced is through the existence of a minimal length that is obtained using the fundamental constants

$$l_{PL} \equiv \sqrt{\frac{\hbar G}{c^3}} \approx 1.6 \times 10^{-35} \text{ m}. \tag{2.31}$$

This length scale marks a threshold beyond which Einstein's description of space-time breaks down, and new phenomena are expected to appear. If this proposal is correct, a QG would be reduced to DSR by the appropriate limit. However, some inconsistencies must be resolved once a minimal length is incorporated as a relativistic invariant. The most obvious one is the introduction of a preferred frame of reference where the minimal length takes its actual value. Thus, an observer could measure new phenomena over others by performing a transformation while breaking the principle of relativity. Therefore, a structural modification to SR is required to introduce a minimal length. To introduce a new invariant scale that all observers can agree on and have a theory consistent with the principle of relativity, a modification to Einstein's SR postulates is essential. It is worth noticing that

DSR reduces to SR in the limit $l_{PL} \to 0$ for consistency. With this in mind, different models have been developed that meet the aforementioned characteristics. So far, there are several models of DSR [77–79]. Each of these has successfully incorporated a new invariant into the SR structure. Among all these models, we can highlight the pioneering work by Amelino Camelia [78]. In his work, Amelino Camelia argues about the modification of the dispersion relation by introducing a minimum length as an invariant. Thus, the following dispersion was proposed

$$E^2 - c^2p^2 - m^2c^4 + f(E, p, l_{PL}) = 0, \tag{2.32}$$

where the function $f(E, p, l_{PL})$ can be established experimentally by measuring the departure from SR. The deformed dispersion relation Eq.(2.32) has an important implication in the speed of massless particles since it introduces a dependence on energy in the speed of light $\tilde{c} = cg(E)$, where $\tilde{c}$ is the variable speed, and c is the invariant that coincides with the speed of light at low energies. That is, the actual value of the invariant speed c can be established for any inertial observer in the limit $\frac{\lambda}{l_{PL}} \to \infty$ for the speed of light with wavelength λ.

The introduction of an invariant length scale has immediate consequences for the structure of space-time. The most important one is the violation of the Lorentz symmetry. By modifying the dispersion relation, the new theory must give rise to a set of new transformations that leave Eq.(2.32) invariant. The new transformations can be found by assuming that a continuous and invertible map F exists between the standard Lorentz transformation Λ_{SR} and the modified transformations Λ_{DSR}, as shown below

$$\Lambda_{DSR} = F^{-1} \circ \Lambda_{SR} \circ F, \tag{2.33}$$

where F is a non-unitary matrix and depends on the new invariant scale [80]. Under this procedure, the modified Lorentz transformations keep a minimal length invariant without introducing a preferred frame of reference. The new transformation group is known as κ-Lorentz, where κ is a deformation parameter on the standard Lorentz group. The deformed Lorentz group is included in Hopf algebras, and quantum groups that generalise the concept of Lie's algebras by having no linear relation between the generators [81, 82]. Additionally, since Lorentz transformations are the only transformations that leave the Minkowski metric invariant, a new space-time geometry should emerge. Such a space-time is called κ-Minkoswki space, and it presents a non-commutative geometry due to the symmetries of the deformed Lorentz group [83, 84].

With the improvement in the sensitivity of different experiments, the chance of measuring effects on the Planck scale may become possible shortly. Such effects will be comparable with the ratio between the characteristic wavelength of the system and Planck length, $\frac{\lambda}{l_{PL}}$. Therefore, testable modifications are expected at high energies. Astrophysical data sheds light on this theory by analysing the threshold energy of energetic rays. In this direction, SR predictions appear to be violated by analysing ultra high energy cosmic rays (UHECRs) [85, 86]. This type of anomaly open the possibility of an explanation by incorporating quantum properties into space-time. However, before taking this theory as something final, some structural problems must be solved [87–89]. Among such problems, one is related to multi-particle states. Such a problem is known as the 'soccer-ball problem' [90]. So far, the one-particle sector in DSR is well described. Nevertheless, once many-particle processes are considered, the momentum conservation rule has to be modified. Because the momentum transformation Λ_{DSR} is not linear, the momentum addition invariance is not well defined

$$\Lambda_{DSR}(p_1 + p_2) \neq \Lambda_{DSR}(p_1) + \Lambda_{DSR}(p_2). \tag{2.34}$$

One possible solution is to define a new non-linear momentum addition law to keep it invariant under the deformed Lorentz transformation. The soccer-ball problem has been addressed by many in the area. However, none of those solutions have been widely accepted

[91–93]. Since the momentum composition rule is not linear, it opens the possibility of a new geometry for momentum space [94–97].

The fate of Lorentz transformations at the Planck scale may offer a tentative vision of quantum properties of space-time by introducing a minimal length as a new invariant. Departures from SR at high energies observed in cosmic radiation may be a hint of a QG theory that considers new phenomena at high energies. However, a consistently phenomenological modification of SR is still in process.

2.5 Generalized Uncertainty Principle

One of the fundamental concepts in QM is the Heisenberg uncertainty principle (HUP). The HUP determines the limitations introduced in the observation of the different physical quantities that describe a system in QM. This means that it is not possible to determine the exact value of two observables simultaneously in an experiment. For a pair of observables in QM $\hat{A}$ and $\hat{B}$, represented by Hermitian operators on a Hilbert space and with a standard deviation given by

$$(\Delta A) = \sqrt{\langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2}, \qquad (\Delta B) = \sqrt{\langle \hat{B}^2 \rangle - \langle \hat{B} \rangle^2}, \tag{2.35}$$

the following relation known as Roberson-Schrödinger uncertainty relation can be derived

$$(\Delta A)^2 (\Delta B)^2 \geq \langle \frac{i}{2} [\hat{A}, \hat{B}] \rangle^2 + \left(\langle \frac{1}{2} \{\hat{A}, \hat{B}\} \rangle - \langle \hat{A} \rangle \langle \hat{B} \rangle \right)^2, \tag{2.36}$$

where

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}, \qquad \{\hat{A}, \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}, \tag{2.37}$$

such a relation applies to every observer described by a hermitian operator. If these operators represent the position $\hat{x}$ and momentum $\hat{p}$ of a system, the uncertainty principle is written as follows

$$\Delta x\,\Delta p \geq \frac{\hbar}{2}. \tag{2.38}$$

This uncertainty relationship prevents the position and momentum of a particle from being measured simultaneously with absolute precision. If a measurement is performed with arbitrary precision at the positions, i.e. $\Delta x \to 0$, satisfying the HUP would imply uncertainty at the moment $\Delta p \to \infty$. For position and momentum, the HUP is related to the canonical commutation relations

$$[\hat{x}_i, \hat{x}_j] = 0, \qquad [\hat{p}_i, \hat{p}_j] = 0, \qquad [\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}. \tag{2.39}$$

Mead pointed out in his gedankenexperiment [76] that gravity could play a significant role in processes involving short distances. Following his gedankenexperiment, an interaction between a photon and a particle in a region of size R is accounted for. In this scenario, he argued that the gravitational attraction exerted by the photon on the particle whose position and momentum are to be known would add to the uncertainty. Such an interaction causes an acceleration in the particle approximately given by

$$a \approx \frac{G\hbar\omega}{c^2R^2}, \tag{2.40}$$

where ω is the photon frequency. Because the photon's direction cannot be known with absolute certainty, an open window of angle ε is considered. Therefore, the acceleration and movement of the particle due to the interaction would not be known either, and an additional uncertainty would be introduced

$$\Delta x \geq \frac{G\hbar\omega}{c^4}\sin\varepsilon. \tag{2.41}$$

In the same way, the momentum of the photon is increased by $Gm\omega/R$, where m is the mass of the particle, due to the gravitational interaction. This increases the uncertainty in the particle momentum estimate

$$\Delta p \geq \frac{\hbar\omega}{c}\left(1+\frac{Gm}{R}\right)\sin\varepsilon, \tag{2.42}$$

and being $\Delta x \approx R\Delta p/m$ the position uncertainty in the region R, we obtain

$$\Delta x \geq \omega\left(\frac{R}{m}+G\right)\sin\varepsilon. \tag{2.43}$$

The idea of a modification in uncertainty has been explored using various gedankexperiments. For example, in black holes [98–102]. As mentioned before, one of the fundamental concepts in QM is the HUP. This principle can be derived from the most general uncertainty principle Eq(2.36). Any type of modification to the uncertainty principle is called the Generalized Uncertainty Principle (GUP). Many authors have tried various approaches to the GUP. In 1995, Kempf, Mangano, and Mann in [103] proposed the following GUP model

$$\Delta x\, \Delta p \geq \frac{\hbar}{2}[1+\alpha(\Delta x)^2+\beta(\Delta p)^2+\gamma], \tag{2.44}$$

where α and β are independent deformation parameters and $\gamma = \alpha\langle x\rangle^2+\beta\langle p\rangle^2$. Such parameters are not fixed, but in many models they are assumed to be of the order of unity. In [104], an analytical calculation for the parameters of a particular case was obtained by considering the GUP correction in the gravitational potential. However, it has been possible to set experimental bounds on the parameters according to different systems [105]. Eq.(2.44) is a modification to the HUP avoiding arbitrary precision measurement of positions because the term $\beta(\Delta p)^2$ grows faster as Δx decreases. Therefore, Δx can no longer be arbitrarily small, reaching a minimal value. As a result, the GUP introduces the minimal uncertainty in position and momentum. Thus, states with a smaller uncertainty are not considered physical states, that includes eigenstates. For the position operator, the latter can be seen as follows

$$(\Delta x)^2 = \langle\psi|\hat{x} - \langle\psi|\hat{x}|\psi\rangle|\psi\rangle \geq \Delta x_0, \tag{2.45}$$

where $|\psi\rangle$ is a quantum state. The minimal uncertainty does not exclude non-physical states. However, those states have infinite uncertainty since they do not belong to the operator's domain. A model proposed by Kempf, Mangano, and Mann that simplifies the problem of the representation of the Hilbert space is by considering only a minimum uncertainty in positions. That is, by setting $\alpha = 0$ in Eq.(2.44), the following GUP model is obtained

$$\Delta x\, \Delta p \geq \frac{\hbar}{2}(1 + \beta(\Delta p)^2 + \beta\langle\hat{p}\rangle^2). \tag{2.46}$$

In order to obtain the GUP model given by Eq.(2.46), the following commutation relation for position and momentum operators is considered in [103]

$$[\hat{x}, \hat{p}] = i\hbar(1 + \beta\hat{p}^2). \tag{2.47}$$

Eq.(2.46) introduces a limit on the possible allowed values for Δx and Δp

$$\Delta p = \frac{\Delta x}{\beta\hbar} \pm \sqrt{\left(\frac{\Delta x}{\beta\hbar}\right)^2 - \langle\hat{p}\rangle^2 - \frac{1}{\beta}}. \tag{2.48}$$

By extremizing the previous expression, a minimal uncertainty is obtained in the positions. Such an uncertainty reads as follows

$$\Delta x_{min} = \hbar\sqrt{\beta}\sqrt{1 + \beta\langle p\rangle^2}. \tag{2.49}$$

The deformed Heisemberg algebra Eq(2.47) has a representation in the momentum space given by

$$\hat{p}\ \psi(p) = p\ \psi(p), \tag{2.50}$$

$$\hat{x}\ \psi(p) = i\hbar(1+\beta\ p^2)\partial_p\psi(p), \tag{2.51}$$

where $\psi(p)$ is the momentum wave function and $\hat{x}$ and $\hat{p}$ are self-adjoint operators. It is worth mentioning that this representation is not unique and it depends on the ordering between operators [106]. For this particular representation, the eigenvalue equation for the position operator can be solved

$$i\hbar(1+\beta\ p^2)\partial_p\psi(p) = \lambda\ \psi_\lambda(p). \tag{2.52}$$

Nevertheless, such states are not physical states because of the minimal uncertainty in position.

As shown in this section, a minimal length modifies the HUP and gives rise to the GUP, which is associated with a commutation relation between position and momentum as in Eq.(2.47). The commutation relation will be satisfied regardless of the representation of the operators. However, it can also be studied how a minimum length modifies the representation of the operators. The particular momenta representation for the GUP model given in Eq. (2.46) has been given in Eq.(2.51), where the usual representation is recovered once $\beta \to 0$.

A more general GUP model was proposed in [107] in one dimension. For a 3-dimensional system, the modification to the commutation relation is given by

$$[\hat{q}_i, \hat{p}_j] = i\hbar[\delta_{ij} f(\hat{p}) + g_{ij}(\hat{p})], \tag{2.53}$$

where $f(\hat{p})$ is the deforming function and $g_{ij}(\hat{p})$ is a deforming matrix. It is worth noting that one of the requirements for momentum much less than the Planck momentum is the usual commutation relation to be recovered, $f(\hat{p}) \to 1$. Furthermore, for the elements, $g_{ij}(\hat{p})$ is required to be zero under the same condition. In momentum space, the most general representation of the position operator for this GUP model is given by

$$\hat{q}_i = i\hbar \left[\delta_{ij} f(\hat{p})\hat{Q}_j + g_{ij}(\hat{p})\hat{Q}_j\right], \tag{2.54}$$

where $\hat{Q}_j = \frac{d}{dp_j}$. However, this representation is not unique and depends on the order of the operators. For 1-dimensional case, the different operator order has been explored in [106].

In this chapter, we reviewed two phenomenological models of QG in different frameworks. DSR introduces a new independent observable in the structure of space-time. Such an invariant leads to reconsidering Einstein's postulates as well as the role of the Lorentz group at high energies to describe the symmetries of space. On the other hand, GUP modifies the Heisenberg algebra, modifying the uncertainty principle and giving rise to a minimal length. Both phenomenological models are open to the possibility of measuring deviations in both SR and QM that will be assigned to QG effects.

Chapter 3

Blackbody Radiation

> As soon as definite knowledge concerning any subject becomes possible, this subject ceases to be called philosophy, and becomes a separate science.
>
> -Bertrand Russell, *The Value of Philosophy*

In his revolutionary paper, titled "Über das Gesetz der Energieverteilung im Normalspectrum" [108], Max Planck described the properties that had been observed for the thermal radiation of blackbodies and for which it had not been possible to give a physical explanation within the classical framework. The solution to the blackbody problem came with the hypothesis of "quanta" introduced by Planck. Such a hypothesis laid the foundations for the development of QM by Erwin Schrödinger and Werner Heisenberg a quarter of a century later.

3.1 Thermal Radiation

The properties of the thermal radiation of bodies have been extensively studied [109]. Such a phenomenon can be defined as a cluster of particles that propagate in a medium or vacuum. The radiation that propagates in the form of electromagnetic waves receives the name electromagnetic radiation, whose properties at different frequencies have been studied over time. For example, Newton and his contemporaries elaborated on the study of visible light, showing that white light is dispersed using a prism. Later in time, in 1895, the German physicist Wilhelm Conrad Rötgen discovered x-rays. In the middle of the XIX century, the German physicist Gustav Robert Kirchhoff observed that molecules and atoms of different materials absorb and emit electromagnetic radiation at very characteristic frequencies. Furthermore, he assumed that electromagnetic radiation, when interacting with atoms and molecules, reaches a state of thermodynamic equilibrium at a distinctive temperature. This radiation is called thermal radiation. Such radiation is associated with two physical quantities: the energy per unit of volume $\rho(\nu)$; and the incident energy per unit area or intensity $I(\nu)$. Both quantities depend on the frequency and are related as follows

$$\rho(\nu) = \frac{4\pi I(\nu)}{c}. \tag{3.1}$$

In several observations, Kirchhoff noted that a body emits more radiation as its temperature increases. Moreover, the equilibrium state of radiation is not affected if matter is introduced or removed, [110]. That is, thermal radiation depends on temperature and not on the material, and therefore the energy density and intensity are also functions of temperature. This equilibrium state is reached when the emitted radiation and the absorbed radiation, are equal. Kirchhoff derived the following relationship by taking into account the relationship between emissivity $e(\nu, T)$, which is defined as the quantity of radiation it emits at a specific frequency, and absorptivity $a(\nu, T)$, which is defined as the ratio between the absorbed and received flux

$$\frac{e(\nu, T)}{a(\nu, T)} = I(\nu, T). \tag{3.2}$$

Here, we have introduced $I(\nu, T)$ to show the explicit temperature dependence on the intensity. Eq.(3.2) is known as Kirchhoff's law, where emissivity and absorptivity universally characterize a body. In particular, there is a type of body that absorbs all the radiation that hits it. For this particular body if the absorptivity function takes the value $a(\nu, T) = 1$, this body is called a blackbody.

3.2 Blackbody radiation

As previously mentioned, the classical theory was not able to describe thermal radiation. In 1890, Lummer and Pringsheim made one of the first observations of the radiation emitted by a body at a given temperature using a spectrometer. Such an observation found a very characteristic shape of the radiation spectrum. The radiation spectrum shows the maximum point of radiation for a particular frequency. Moreover, the radiation decreases as the frequency increases beyond the peak frequency. In addition, the peak frequency increases as a function of the temperature. Another feature observed was the shift towards higher frequencies as the temperature increases, that is

$$\nu_{max} \propto T, \tag{3.3}$$

where ν_{max} is the maximal frequency. The first attempt to derive an expression for the energy density was made by Rayleigh and Jeans [73]. Their results showed larger discrepancies between the experiment and classical theory. Assuming the validity of the classical electromagnetic theory, they considered a cavity with radiation enclosed in the form of electromagnetic waves. These waves are stationary, with nodes at the walls of the cavity. Using purely geometrical arguments, they counted the number of standing waves in the frequency range between ν and $\nu + d\nu$, obtaining the following expression

$$N(\nu)d\nu = \frac{8V\pi\nu^2}{c^3}d\nu, \tag{3.4}$$

where V is the cavity's volume. The next step was to calculate the average energy of each of the standing waves with a frequency ν. They implemented the energy equipartition theorem for this. Such a theorem relates the temperature of a system in thermal equilibrium with the average energy of its components. As a result, each degree of freedom in the system contributes an equal amount of energy to $k_B T/2$, according to [111]. In the case of a stationary electromagnetic wave, the amplitude is a degree of freedom. Nevertheless, as the wave has two polarizations, this counts for a total of two degrees of freedom. By taking all this into consideration, the energy density for the system is

$$\rho(\nu)d\nu = \frac{8\pi\nu^2 k_B T}{c^3}d\nu. \tag{3.5}$$

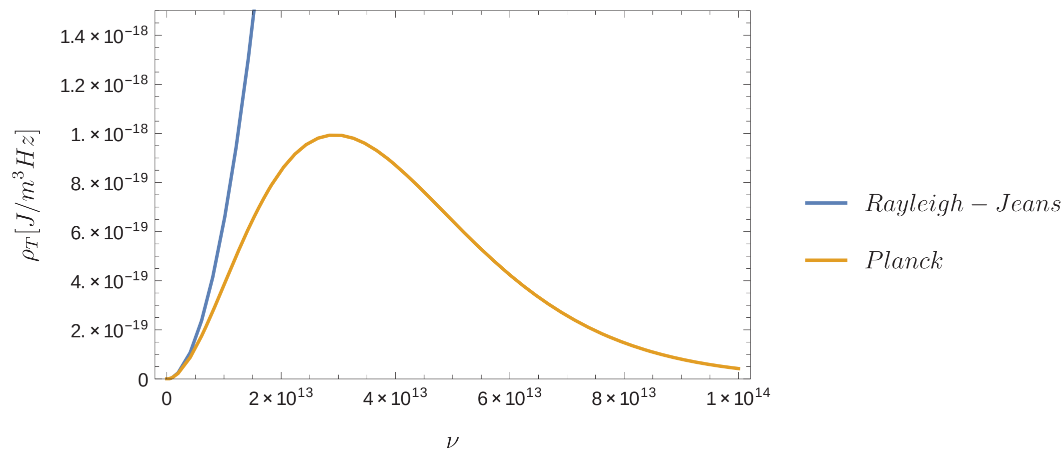

Figure 3.1: Rayleigh-Jeans prediction compared to Planck's prediction for the energy density of a blackbody in thermal equilibrium, $T = 500$ K.

The above expression predicts that the energy density for larger frequencies grows proportional to the square of the frequency. That is, the energy increases monotonically. This prediction conflicts with what Lummer and Pringsheim observed. This discrepancy between theory and experiment was named the ultraviolet catastrophe. This problem remained unresolved for several years until Max Planck tried to find a solution and reconcile theory with experiment. Planck assumed that the energy absorbed and emitted by the atoms in the walls of the cavity is proportional to the frequency of the mode, $\Delta E = h\nu$. By assuming the hypothesis of quanta and using the Boltzmann distribution, Planck obtained the correct expression describing the spectrum of the blackbody, given by

$$\rho_T(\nu)d\nu = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/k_BT}-1}d\nu. \tag{3.6}$$

It can be seen, when considering $k_BT \gg h\nu$ the exponential in the above expression can be expanded and obtain the Rayleigh and Jeans formula. Figure 3.1 shows the difference between Rayleigh-Jeans' prediction and Planck's prediction. At the low-frequency limit, both are close, but as the frequency increases, the difference is significant.

3.3 Radiation field

QM can be applied to electromagnetic theory to provide a more complete and accurate description of radiation phenomena. To achieve such a task, it is required to stop treating the electric and magnetic fields as classical variables and turn to a description of operators. The quantization procedure introduces quantum features into the radiation field. For example, the simultaneous definition of phase and amplitude is limited by the HUP [112]. To quantize the electromagnetic field, the classical Hamiltonian is needed to provide the transition to quantum theory with greater simplicity. This formalism, in turn, is based on a Lagrangian, which allows for the identification of configuration variables and their generalised momenta. Thus, the quantization procedure consists of promoting the generalised variables to operators and imposing a commutation relation. By starting from the Lagrangian of the electromagnetic field

$$\mathcal{L} = \frac{1}{8\pi}(|\mathbf{E}|^2 - |\mathbf{B}|^2). \tag{3.7}$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields, respectively. These fields can be written in terms of the potentials, where the vector potential A is defined along with the scalar potential ϕ by the equations

$$\mathbf{B} = \nabla \times \mathbf{A}, \qquad \mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}. \tag{3.8}$$

By choosing the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$, and the further constraint $\phi = 0$ that implies vacuum, the configuration variable q_i and the generalised momenta p_i can be identified as follows

$$q_i = A_i, \qquad p_i = \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = -\frac{1}{4\pi c}E_i. \tag{3.9}$$

In addition, the vector potential satisfies the wave equation

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = 0. \tag{3.10}$$

As a solution to the wave equation, the vector potential can be written as a superposition of plane waves given by the Fourier expansion over a collection of these modes [112]

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{2\sqrt{V}}\sum_k \mathbf{A}_\mathbf{k}(t)e^{i\mathbf{k}\cdot\mathbf{r}} + \mathbf{A}^*_\mathbf{k}(t)e^{-i\mathbf{k}\cdot\mathbf{r}}, \tag{3.11}$$

where the radiation field has been taken into account within a volume V. This volume is known as the quantization cavity of side L with periodic boundary conditions. Furthermore, the sum goes over all the wave numbers **k** by considering all directions. The classical Hamiltonian is obtained by performing a Legendre transformation on Eq.(3.7). Furthermore, electric and magnetic fields can be expanded into plane waves using the Eq.(3.8). Thus, the Hamiltonian adopts the following form

$$H = \frac{1}{8\pi}\int |\mathbf{E}|^2 + |\mathbf{B}|^2 \, dV = \frac{1}{8\pi}\sum_\mathbf{k} |\mathbf{k}|^2 |\mathbf{A}_\mathbf{k}|^2. \tag{3.12}$$

The Hamiltonian shows that the modes do not interfere with each other. Then, each mode is independent and facilitates the quantization procedure for the radiation field since each mode is quantized as a harmonic oscillator [113]. Thus, the mechanical variables are identified for each harmonic oscillator labelled by the wave vector **k**

$$p_\mathbf{k} = \frac{-i\omega_\mathbf{k}}{4\pi c^2}\mathbf{A}_\mathbf{k}, \qquad\qquad q_\mathbf{k} = \mathbf{A}_\mathbf{k}. \tag{3.13}$$

Therefore, the Hamiltonian of the system can be written as a sum over the wave vector for each harmonic oscillator, where the coordinates and momenta have already been identified from Eq.(3.13)

$$H = 2\pi c^2 \sum_{\mathbf{k}} p_{\mathbf{k}} \cdot p_{\mathbf{k}}^* + \frac{1}{8\pi c^2}\omega_{\mathbf{k}}^2 q_{\mathbf{k}} \cdot q_{\mathbf{k}}^*. \tag{3.14}$$

Having the generalised coordinates and knowing that each mode can be treated as a harmonic oscillator, each of them can be quantized. Therefore, coordinates and momenta are promoted to operators and a commutation relation is imposed [112]. The following are the commutation relation

$$\begin{aligned} [q_{\mathbf{k}}, p_{\mathbf{k}'}^\dagger] &= i\hbar\delta_{\mathbf{k},\mathbf{k}'}, \quad [q_{\mathbf{k}}^\dagger, p_{\mathbf{k}'}] = i\hbar\delta_{\mathbf{k},\mathbf{k}'}, \\ [q_{\mathbf{k}}, p_{\mathbf{k}'}] &= i\hbar\delta_{\mathbf{k},-\mathbf{k}'}, \quad [q_{\mathbf{k}}^\dagger, p_{\mathbf{k}'}^{*\dagger}] = i\hbar\delta_{\mathbf{k},-\mathbf{k}'}. \end{aligned} \tag{3.15}$$

For convenience, the following dimensionless operators are defined

$$P_{\mathbf{k}} = \sqrt{\frac{4\pi c^2}{\hbar\omega_{\mathbf{k}}}} p_{\mathbf{k}}, \qquad Q_{\mathbf{k}} = \sqrt{\frac{\omega_{\mathbf{k}}}{4\pi c^2 \hbar}} q_{\mathbf{k}}. \tag{3.16}$$

These are known as the quadrature operators of the electromagnetic field. Such operators allow us to rewrite the Hamiltonian as follows

$$H_{\mathbf{k}} = \frac{\hbar\omega_{\mathbf{k}}}{2}(P_{\mathbf{k}}^\dagger P_{\mathbf{k}} + Q_{\mathbf{k}}^\dagger Q_{\mathbf{k}}), \tag{3.17}$$

with the proper commutation relation

$$[P_{\mathbf{k}}, Q_{\mathbf{k}'}] = i \tag{3.18}$$

The quantum theory of radiation has many important applications, one of them being in the radiating cavity problem. Inspired by the GUP, the radiation field is studied in the presence of a minimal length. This length is expected to modify the commutation relation between the generalized coordinates and momenta of the electromagnetic field and have subsequent consequences for blackbody radiation. According to [114], the modification imposed on the commutation of the electromagnetic field's coordinates, $q_{\mathbf{k}}$ and momenta $p_{\mathbf{k}}$, is given by

$$[q_{\mathbf{k}}, p_{\mathbf{k}'}] = i\hbar\delta_{\mathbf{k},\mathbf{k}'}[1+\gamma_{EM}^2 p_{\mathbf{k}}^2]. \tag{3.19}$$

The GUP-inspired modification between the coordinates and generalized momenta introduces the minimal uncertainty possible into the vector field. This can be seen in the selection of the coordinates Eq.(3.13). It is worth noticing that the units of the generalized momentum of the electromagnetic field is (momentum)/(mass)$^{1/2}$. This can be shown making a dimensional analysis. The transverse electric field $\vec{E}_T$ is written in terms of $q_{\mathbf{k}}$ and $p_{\mathbf{k}}$ as [112]

$$\vec{E}_T = \sum_{\vec{k}} \sqrt{\frac{1}{\varepsilon_0 V}} \vec{\varepsilon}_k \left[\omega^k q_{\mathbf{k}} \sin\theta - p_{\mathbf{k}} \cos\theta\right], \tag{3.20}$$

where k is the magnitude of the wavenumber used to label the different modes and ω^k is the corresponding frequency. Furthermore, θ is the phase angle, $\vec{\varepsilon}_k$ the polarization vector, and V the cavity volume. Therefore, unlike the deformation parameter in a typical GUP model, for the electromagnetic field operators we muss to modify the deformation parameter as

$$\gamma_{EM} = \frac{\gamma_0}{\sqrt{M_{PL} c}}, \tag{3.21}$$

where $M_{PL} = \sqrt{\frac{\hbar c}{G}}$ is the Planck mass and γ_0 a dimensionless parameter related to the scale. As shown in the preceding section, the radiation field can be quantized by considering the field as a collection of independent harmonic oscillators. The energy spectrum for a harmonic oscillator with a GUP quadratic model has been determined in [103, 115]. The modified energy is

$$E_n^k = \hbar\omega^k \left\{ \left(n+\frac{1}{2}\right) + \frac{\hbar\omega^k}{4}\gamma_{EM}^2 (1+2n+2n^2) \right\}. \tag{3.22}$$

Here, we use the same notation where k denotes the wavenumber used to label the different modes, ω^k denotes the corresponding frequency, and n is the energy level. This notation will be helpful once the abundance of different modes of radiation enclosed in the cavity is considered. Here, it can be noticed that the energy difference between the ground state and a state n is

$$\Delta E_n^k = \hbar\omega^k n + \zeta^k n(n+1), \tag{3.23}$$

where we can identify the term $\zeta^k = \frac{1}{2}(\hbar\omega^k\gamma_{EM})^2$. This term refers to the energy modification caused by GUP. It is worth observing that in the limit $\gamma_{EM} \to 0$, we recover the usual expression for the energy difference.

3.4 Modified Planck's Law

Dissatisfied with the way Planck's law was derived, Satyendra Nath Bose, an Indian physicist, achieved one of the most surprising attainment during the so-called quantum revolution. He derived the Planck distribution using statistical arguments [116]. These arguments considered a gas of photons, a concept already introduced by Einstein in his photoelectric effect paper [117], in thermodynamic equilibrium. The Bose method led to a new statistical distribution that counts the number of photons per level of energy. Later, such statistic was generalised by Einstein by considering systems, giving rise to the so-called Bose-Einstein statistics, obeyed by particles with integer spin [111]. Here, we follow the argument introduced by Bose to derive Planck's distribution [116]. Considering a statistics analysis, we introduce a distribution Z_n^k for the number of modes in the state n with a particular frequency range $[\omega^k, \omega^k + d\omega^k]$. By introducing the energy Eq.(3.22), the total energy relative to the ground state can be written in terms of such a distribution as

$$\begin{aligned} E &= \sum_k (\Delta E_1^k Z_1^k + \Delta E_2^k Z_2^k + \Delta E_3^k Z_3^k + \ldots), \\ &= \sum_k \left[\hbar\omega^k N^k + \zeta^k (N^k + M^k)\right], \end{aligned} \tag{3.24}$$

where

$$N^k = \sum_n n Z_n^k, \qquad M^k = \sum_n n^2 Z_n^k. \tag{3.25}$$

For any distribution Z_n^k, we define A^k as the system's occupation number in the range $[\omega^k, \omega^k + d\omega^k]$, as stated by

$$A^k = \sum_n Z_n^k. \tag{3.26}$$

The probability of observing a particular distribution Z_n^k is related to the number of different ways that a particular distribution can be formed; that is, each possible distribution will be assigned a certain statistical weight. Thus, the following multinomial coefficient is introduced

$$W = \prod_k \frac{A^k!}{\prod_n Z_n^k!}. \tag{3.27}$$

For a large number of modes, the following expression is adopted by using Stirling's approximation

$$\log W = \sum_k A^k \log A^k - \sum_k \sum_n Z_n^k \log Z_n^k. \tag{3.28}$$

The quantity $\log W$ can be understood as a probability distribution of the microstates following Boltzmann formulation [118]. Thus, maximising such a quantity corresponds to looking for the equilibrium condition for the system, as its maximum value is related to the most probable distribution [111]. Let us refer to the distribution that maximises W as $\overline{Z}_n^k$. Furthermore, the system is subject to two constraints, i.e. Eq.(3.24) and Eq.(3.26). By applying Lagrange's method of undetermined multipliers, the maximal distribution can be obtained

$$\delta\left\{\log W+\sum_k\tau^k\sum_n Z_n^k+\alpha\sum_k\left[\hbar\omega^k N^k+\zeta^k(N^k+M^k)\right]\right\}=0, \tag{3.29}$$

where τ^k and α are Lagrange multipliers. The variation with respect to Z_n^k gives

$$\sum_k\sum_n\delta Z_n^k\left\{1+\log Z_n^k+\tau^k+\alpha\left[n\hbar\omega^k+\zeta^k n(1+n)\right]\right\}=0. \tag{3.30}$$

As each δZ_n^k is arbitrary, we can impose the curly brackets to vanish obtaining

$$\overline{Z}_n^k=B^k e^{-\alpha\hbar\omega^k n}e^{-\alpha\zeta^k n(n+1)}, \tag{3.31}$$

where

$$B^k=e^{-1-\tau^k}. \tag{3.32}$$

Unlike the usual case, the distribution Eq.(3.31) contains an additional term which is related to the modification due to the GUP. However, corrections are expected to be very small at low temperatures and frequencies. Such an observation allows us to introduce an approximation where the argument of the second exponential is considered to be much less than unity in Eq.(3.31). This approximation will impose certain conditions on both the frequency and the Lagrange multiplier, α, which will be later identified as the inverse of the temperature. Therefore, the most probable distribution can be approximated as follows

$$\overline{Z}_n^k=B^k e^{-\alpha\hbar\omega^k n}\left[1-\alpha\zeta^k n(n+1)\right]. \tag{3.33}$$

Due to the approximation, there is a maximum value for n^k. Exceeding such a limit

$$n_{max}^k=\left\lfloor\left(\frac{1}{\alpha\zeta^k}+\frac{1}{4}\right)^{1/2}-\frac{1}{2}\right\rfloor, \tag{3.34}$$

the approximation is no longer valid. As a result, a maximum value for n^k corresponds to a maximum value for energy

$$E^k_{max} = \hbar\omega^k + \zeta^k \left\{ \left(\frac{1}{\alpha\zeta^k} + \frac{1}{4} \right)^{1/2} + \frac{1}{2} \right\}. \tag{3.35}$$

Having the most probable distribution Eq.(3.33), we can proceed by following Bose's approach [116] in order to obtain the energy density for photons enclosed in a cavity. The approximation Eq.(3.33) allows us to obtain an analytical expression for Eq.(3.25) and Eq.(3.26) using the convergence of the geometric series and its derivatives, see Appendix A. For the mean occupation number in the range $[\omega^k, \omega^k + d\omega^k]$, we obtain the following expression

$$\begin{aligned} A^k &= \sum_n \overline{Z}^k_n \\ &= \frac{B^k}{(1-e^{-\alpha\hbar\omega^k})^3} \left[(1-e^{-\alpha\hbar\omega^k})^2 - 2\alpha\zeta^k e^{-\alpha\hbar\omega^k} \right], \end{aligned} \tag{3.36}$$

which can be solved for B^k. Then, substituting in Eq.(3.25), we find

$$\begin{aligned} N^k &= \sum_n n\overline{Z}^k_n \\ &= \frac{A^k}{e^{\alpha\hbar\omega^k} - 1} \left\{ \frac{(1-e^{-\alpha\hbar\omega^k})^2 - 2\alpha\zeta^k(1+2e^{-\alpha\hbar\omega^k})}{(1-e^{-\alpha\hbar\omega^k})^2 - 2\alpha\zeta^k e^{-\alpha\hbar\omega^k}} \right\}, \end{aligned} \tag{3.37}$$

and for M^k

$$\begin{aligned} M^k &= \sum_n n^2\overline{Z}^k_n \\ &= \frac{A^k e^{-\alpha\hbar\omega^k}}{(1-e^{-\alpha\hbar\omega^k})^2} \left\{ \frac{(1-e^{-\alpha\hbar\omega^k})^2(1+e^{-\alpha\hbar\omega^k}) - 2\alpha\zeta^k(1+7e^{-\alpha\hbar\omega^k}+4e^{-2\alpha\hbar\omega^k})}{(1-e^{-\alpha\hbar\omega^k})^2 - 2\alpha\zeta^k e^{-\alpha\hbar\omega^k}} \right\}. \end{aligned} \tag{3.38}$$

The sum was calculated from zero to infinity in the previous expressions, exceeding the value n_{max} derived in Eq. (3.34). This was to compute the sum using the convergence of the geometric series. In fact, the contribution to the sum due to levels above n_{max} are negligible. Rewriting the sum over the distributions A^k as

$$\sum_{n=0}^{\infty} \overline{Z}_n^k = \sum_{n=0}^{n_{max}} \overline{Z}_n^k + \sum_{n=n_{max+1}}^{\infty} \overline{Z}_n^k, \tag{3.39}$$

where the second term is identified as the rest $\mathcal{R}$ of the sum. The sum is calculated from zero to infinity by introducing a new counter, $m = n + n_{max} + 1$, yielding the following expression

$$\mathcal{R} = -\frac{e^{-\frac{\sqrt{\alpha}(2-3\sqrt{\alpha\zeta^k})}{\sqrt{2}\gamma_{EM}}}\left[2\sqrt{\alpha\zeta^k}\left(e^{\frac{\sqrt{2\alpha}\sqrt{\alpha\zeta^k}}{\gamma_{EM}}}-1\right)+\alpha\zeta^k\right]}{\left(e^{\frac{\sqrt{2\alpha}\sqrt{\alpha\zeta^k}}{\gamma_{EM}}}-1\right)^3} \tag{3.40}$$

Expanding in power series and for $\alpha\zeta^k \ll 1$, the dominant term is of the form

$$\mathcal{R} \sim -\frac{\gamma_{EM}^2 e^{-\frac{\sqrt{2\alpha}}{\gamma_{EM}}}\left(4\sqrt{\alpha}+\sqrt{2}\gamma_{EM}\right)}{4\alpha^{3/2}\sqrt{\alpha\zeta^k}}. \tag{3.41}$$

It is then easy to see that $\mathcal{R} \ll 1$ when $\gamma_{EM}/\sqrt{\alpha} \ll 1$. At this point, since no information is present yet regarding the meaning of α, the two relations, namely $\alpha \ll E_{Pl}/(\hbar\omega^k)^2$ and $\alpha^{-1/2} \ll E_{Pl}$, have to be treated as constraints on α for the approximations leading to Eqs.(3.36-3.38) to be valid.

The number of modes per unit volume in the blackbody cavity for a particular wavenumber range $[k, k+dk]$ is given by the Rayleigh-Jeans expression

$$\mathfrak{R}(k)dk = \frac{k^2}{\pi^2}\, dk. \tag{3.42}$$

Geometrically, this expression corresponds to the number of points in an octant of a spherical shell with thickness dk, and radius [111]. Once the GUP is considered, in terms of the wavenumber, the expression remains the same due to its geometrical meaning. However, Planck-scale variation of the dispersion relation is present in quantum gravity. Therefore, the group velocity $d\omega/dk$ is deformed, and as a consequence c varies as a function of energy. Many observations of gamma rays have been made, motivated by the energy dependence of the speed of light and as a possible test of the effects of quantum gravity [119,120]. In particular, those observations focus on measuring the time delay of several photons with different energies emitted by the same source at the same time. Additionally, it was possible to establish a limit on the energy where violations of the Lorentz invariance are expected. One of these effects would be the variable speed of light. Based on these findings, we motivate the idea that the deformation in the group velocity may be smaller than predicted and allow us to consider the usual relationship between wave number and frequency, $d\omega/dk = c$. Therefore, the number of photons in a frequency range $[\omega^k, \omega^k + d\omega^k]$ can be written as follows

$$g(\omega)d\omega^k = \frac{V(\omega^k)^2}{\pi^2 c^3} d\omega^k, \tag{3.43}$$

where V is the cavity volume. Following the approach in [116], the total number of cells is identified with the number of possible ways of placing a photon in the relevant volume. Thus, it is possible to find that the number of cells allowed for a frequency $[\omega^k, \omega^k + d\omega^k]$ is

$$A^k = \frac{V(\omega^k)^2}{\pi^2 c^3} d\omega^k. \tag{3.44}$$

In relation to statistical thermodynamics, according to the definition of Boltzmann entropy, the entropy S can be defined using the probability function W from Eq.(3.28)

$$S = k_B \left\{ \alpha E - \sum_k 3A^k \log(1 - e^{-\alpha\hbar\omega^k}) - A^k \log\left[(1 - e^{-\alpha\hbar\omega^k})^2 - 2\alpha\zeta^k e^{-\alpha\hbar\omega^k}\right] \right\}, \tag{3.45}$$

where k_B is the Boltzmann constant. The thermodynamics relation $\frac{\partial S}{\partial E} = 1/T$ is used to obtain an expression for the Lagrange multiplier α, $\alpha = 1/k_B T$. With the Lagrange multiplier identified, we can write the condition in terms of temperature $E_P \gg (\hbar\omega^k)^2/k_B T$. Such conditions are consistent with the approximation introduced above, since the Planck scale will not be accessible. By considering the radiation enclosed in a volume V, with a large abundance of modes, each mode is identified with its wave vector k. The sum over k in Eq.(3.24) can be approximated to an integral introducing the energy density $\rho_T(\omega)$ as

$$E = \sum_k \left[\hbar\omega^k N^k + \zeta^k (N^k + M^k)\right] = V \int_0^\infty \rho_T(\omega)\, d\omega. \tag{3.46}$$

It is now convenient to write the energy density as a function of wavelength. To do so, we consider that the two expressions for the energy density are related by $\rho_T(\lambda) = -\rho_T(\omega)\frac{d\omega}{d\lambda}$. In this way, the energy density becomes

$$\begin{aligned}\rho_T(\lambda) = \frac{8\pi hc}{\lambda^5} &\left\{ \frac{1 + \frac{hc}{2\lambda}\gamma_{EM}^2}{e^{\frac{hc}{\lambda k_B T}} - 1} \left[\frac{(e^{\frac{hc}{\lambda k_B T}} - 1)^2 - \frac{(hc\gamma_{EM})^2}{\lambda^2 k_B T} e^{\frac{hc}{\lambda k_B T}} (2 + e^{\frac{hc}{\lambda k_B T}})}{(e^{\frac{hc}{\lambda k_B T}} - 1)^2 - \frac{(hc\gamma_{EM})^2}{\lambda^2 k_B T} e^{\frac{hc}{\lambda k_B T}}} \right] \right. \\ &\left. + \frac{\frac{hc}{2\lambda}\gamma_{EM}^2 e^{\frac{hc}{\lambda k_B T}}}{(e^{\frac{hc}{\lambda k_B T}} - 1)^2} \left[\frac{(e^{\frac{hc}{\lambda k_B T}} - 1)^2 (e^{\frac{hc}{\lambda k_B T}} + 1) - \frac{(hc\gamma_{EM})^2}{\lambda^2 k_B T} e^{\frac{hc}{\lambda k_B T}} (4 + 7e^{\frac{hc}{\lambda k_B T}} + e^{\frac{2hc}{\lambda k_B T}})}{(e^{\frac{hc}{\lambda k_B T}} - 1)^2 - \frac{(hc\gamma_{EM})^2}{\lambda^2 k_B T} e^{\frac{hc}{\lambda k_B T}}} \right] \right\}. \end{aligned} \tag{3.47}$$

It is worth noting that in the limit $\gamma_{EM} \to 0$, the extra terms that carry the modification on the energy density vanish. In such a limit, $\rho_T(\lambda)$ consistently reduces to the usual expression for Planck's distribution

$$\rho_{0T}(\lambda) = \frac{8\pi\hbar c}{\lambda^5} \frac{1}{e^{\frac{hc}{\lambda k_B T}} - 1}. \tag{3.48}$$

In Figure 3.2, we show the relative modification between the energy density modified by GUP and the standard expression, defined as

$$\frac{\Delta\rho_T}{\rho_{0T}} = \frac{\rho_{0T} - \rho_T}{\rho_{0T}}, \tag{3.49}$$

for several values of the temperature. As it can be observed, the larger difference in energy

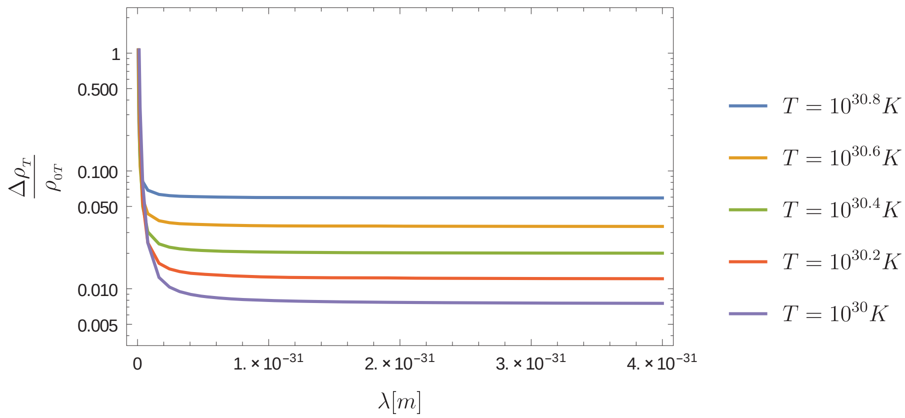

Figure 3.2: Relative modification of the energy density $\frac{\Delta\rho_T}{\rho_{0T}}$ for different temperatures. For higher temperatures, the difference between the standard and GUP-modified energy densities increases.

takes place at higher temperatures. This implies that modifications in the energy density are temperature-dependent. This is consistent with thermodynamics because temperature is the conjugate variable of the energy which has been modified by the GUP Eq.(3.46).

3.5 Wien's Law

In the blackbody distribution, Wien's law establishes a relationship between a given temperature and the wavelength of the maximum of the distribution. In particular, the higher the temperature of the blackbody, the lower the wavelength of the maximum. Wien's Law can be deduced by finding the maximum of the distribution in Eq.(3.47). According to the standard theory, in the limit $\gamma_{EM} \to 0$, this procedure leads to constant quantity as a solution for λ_0 given by $x = \frac{hc}{\lambda_0 k_B T} = 5 + W(0, -5e^{-5})$, where $W(z)$ is the Lambert W function. In the present case, we will consider the approximated expression

$$x = \frac{hc}{\lambda_0 k_B T}\left(1 - \frac{\delta\lambda}{\lambda_0}\right), \tag{3.50}$$

where λ_0 is the wavelength that satisfies Wien's law in the standard cases and $\delta\lambda$ is the shift on the wavelength of the maximum due to GUP. In order to simplify the expression, an expansion up to the first order in $k_B T \gamma_{EM}^2$ is considered. Such an approximation is justified

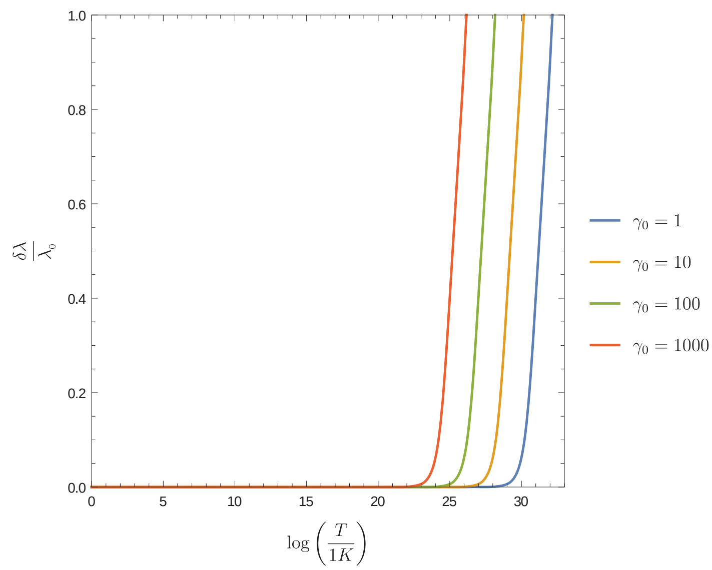

Figure 3.3: Relative shift $\frac{\delta\lambda}{\lambda_0}$ as a function of the temperature for several values of the parameter γ_0.

for temperatures much smaller than the Planck temperature. By differentiating Eq.(3.47) with respect to λ and imposing the maximum condition, we obtain

$$k_B T \gamma_{EM}^2 x e^x \left[x^2(e^{2x}+4e^x+1) - 8x(e^{2x}-1) + 6(e^x-1)^2\right] - xe^x(e^x-1)^2 + 5(e^x-1)^3 = 0. \tag{3.51}$$

Eq.(3.51) can be solved numerically for $\frac{\delta\lambda}{\lambda_0}$ with the condition $\frac{\delta\lambda}{\lambda_0} < 1$. Figure 3.3 shows the temperature dependence of relative shift for the wavelength of the maximum for several values of γ_0. We notice that the modification grows with the temperature, reaching the value $\frac{\delta\lambda}{\lambda_0} = 1$ near Planck's temperature for $\gamma_{EM} = 1$. For such a value and beyond, the approximation in Eq.(3.50) cannot be considered valid. We notice that for much lower temperatures, the modification $\delta\lambda$ becomes 0.

3.6 Stefan-Boltzmann Law

The Stefan-Boltzmann law describes the total power radiated by a cavity with a volume V at an absolute temperature T. The law establishes that the radiance of a blackbody, that is, the amount of energy radiated per unit of surface, is proportional to the fourth power of the absolute temperature

$$R_T = \sigma T^4, \tag{3.52}$$

where the proportionality constant, called the Stefan-Boltzmann constant, is given by the following quantity $\sigma = \frac{2\pi^5 k_B^4}{15h^3c^2}$. This law can be derived by integrating Eq.(3.47) and using the relation between spectral radiance and the energy density, $R_T(\lambda)d\lambda = \frac{c}{4}\rho_T(\lambda)d\lambda$. As done before, Eq.(3.47) is expanded up to the first order in $k_B T\gamma_{EM}^2$. The total energy per unit volume is then

$$\begin{aligned} R_T &= \frac{2\pi^5 k_B^4}{15h^3c^2}T^4 - \frac{2\pi k_B^4}{h^3c^2}(k_B T\gamma_{EM}^2)T^4 \int_0^\infty \frac{x^4 e^x(1+x-e^x+xe^x)}{(e^x-1)^3}dx \\ &= \frac{2\pi^5 k_B^4}{15h^3c^2}T^4(1-16k_B T\gamma_{EM}^2). \end{aligned} \tag{3.53}$$

By substituting the value of the Stefan-Boltzmann constant, the following equation is derived

$$R_T = \sigma T^4(1-16k_B T\gamma_{EM}^2). \tag{3.54}$$

In Figure 3.4, we show the radiance for both the GUP modification and the ordinary case. This is the first order modification of the Stefan-Boltzmann law by including a minimal measurable length. The total energy radiated is less than in the usual case. This result is in agreement with previous results in DSR applied to thermodynamics, where due to the presence of an invariant energy scale, the number of available microstates is reduced. This is observed in a decrease in entropy [121]. In our case, states characterised by short wavelengths carry higher energy values than in the standard case. Therefore, those states are less likely to take place. Thus, their contribution to energy is smaller. It worth noticing that in the limit $\gamma_{EM} \to 0$, the usual Stefan-Boltzmann law for the blackbody Eq.(3.52) is recovered.

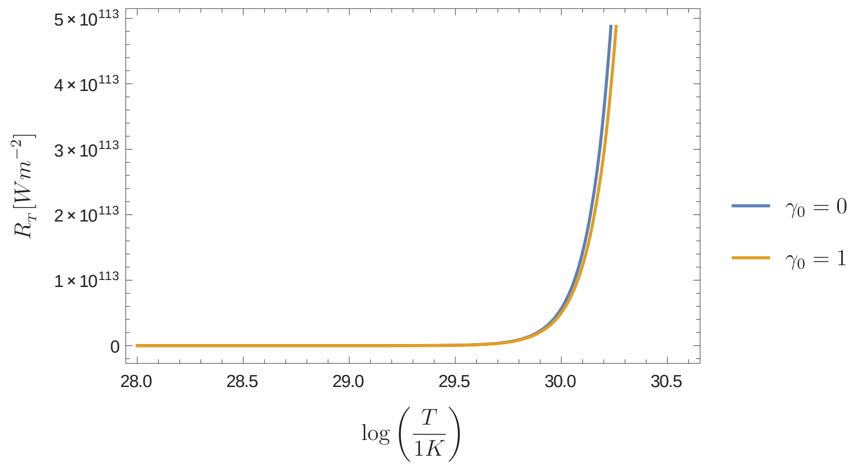

Figure 3.4: Plot of the radiance of a blackbody R_T. The solid blue line represents the Stefan-Boltzmann law in the ordinary theory. The solid orange line corresponds to the modified law in Eq.(3.54)

Statistical mechanics, as well as thermodynamics, may offer indirect evidence of quantum gravity effects related to a minimal measurable length. The effects of such a length modifies the potentials as well as the laws established in both disciplines. As shown in this chapter, the GUP introduced in the field of radiation introduced corrections in the quanta energy. We showed that such corrections modify Planck's law and, consequently, Wien's law and Botlzmann's law. These corrections are dependent on temperature, and we expect they will manifest at high temperatures.

Chapter 4

Statistical Mechanics

> Ich komme immer mehr zu der Überzeugung, dass die Nothwendigkeit unserer Geometrie nicht bewiesen werden kann, wenigstens nicht vom menschlichen Verstande noch für den menschlichen Verstand. Vielleicht kommen wir in einem andern Leben zu andern Einsichten in das Wesen des Raumes, die uns jetzt unerreichbar sind. Bis dahin müsste man die Geometrie nicht mit der Arithmetik, die rein a prioristeht, sondern etwa mit der Mechanik in gleichen Rang setzen.
>
> -Carl Friedrich Gauß, *Quotes*

The effort of many to understand the movement of bodies culminated in the publication of Newton's *Principia*, where he introduced the three laws that govern dynamics. The dynamic behaviour of a system is described by equations of motion that are deterministic in nature. That means, if the initial conditions are specified in advance, position and velocity can always be obtained. Coordinates and momenta span a $6N$-dimensional space in the Hamiltonian formulation of classical mechanics (CM), where $6N$ is the number of degrees of freedom [122]. This space is called phase space. Thus, the dynamics of a system results in a trajectory in phase space determined by the Hamiltonian function through the following equations of motion

$$\dot{Q}_i = \frac{\partial H(Q,P)}{\partial P_i}, \qquad \dot{P}_i = -\frac{\partial H(Q,P)}{\partial Q_i}. \tag{4.1}$$

If the variables Q_i and P_i satisfy the following Poisson algebra

$$\{Q_i, Q_j\} = 0 \qquad \{P_i, P_j\} = 0 \qquad \{Q_i, P_j\} = \delta_{ij}, \tag{4.2}$$

then, any transformation to a new set of variables that preserves the Poisson algebra, and therefore Hamilton's equations, is said to be canonical. Consider a system with canonical coordinate Q_i and conjugate momenta P_i, where $i = 1, ..., 3N$. To determine the trajectory in phase space, a total of $6N$ linear equations must be specified and solved. Solving the $6N$ equations does not turn out to be convenient or attractive. However, there is a method for studying the dynamics of such systems by introducing the phase space density. Such an idea gives rise to Liouville's theorem. Liouville's theorem states that the phase space density ρ of a system defined in a neighbourhood of a point remains constant in time [122]. As the density depends on the point where it is defined, its total time derivative is given by

$$\frac{d\rho}{dt} = \{\rho, H\} + \frac{\partial \rho}{\partial t}. \tag{4.3}$$

The time evolution of the system can be thought of as a canonical transformation generated by the Hamiltonian. Therefore, if we consider the time evolution of an infinitesimal volume of the system along its trajectory, by the Poincaré theorem it remains invariant and it can not vary with time [122]. That is, the Jacobian matrix $\mathcal{M}$, which relates the volume before and after the transformation, satisfies the so-called symplectic condition

$$|\mathcal{M}|^2 |J| = |J|, \tag{4.4}$$

where J is a symplectic matrix. Thus, if the phase space density is defined as the number of points $d\mathcal{N}$ in a region dV and both are constant, the density must be constant in time

$$\frac{d\rho}{dt} = 0, \tag{4.5}$$

and proves Liouville's theorem. Eq.(4.4) introduces the symplectic matrix that plays an significant role in CM. That is, CM can be formulated in terms of such a matrix, which introduces a symplectic structure to the Hamiltonian formalism. The abstract formulation of CM gives rise to such a type of geometry, which is an area of study within differential geometry.

A symplectic manifold is defined in differential geometry as the pair $(\mathcal{M},\omega)$, where $\mathcal{M}$ is a differential manifold, and ω is a 2-form [123]

$$\omega = \sum_{i=1}^{n} dP_i \wedge dQ_i, \tag{4.6}$$

where $\{Q^i\}$ are the local coordinates defined over $\mathcal{M}$ and $\{P_i\}$ are the components of vectors in the cotangent bundle $T^*\mathcal{M}$ spanned by $\{dQ^i\}$ [123]. The dynamics of a system can be related to differential geometry concepts, for example, the trajectories that a system takes are related to the vector field over $\mathcal{M}$ generated by the Hamiltonian function. As a result, we can introduce a Hamiltonian 1-form dH and a vector field X_H that are related by

$$dH = \omega(X_H.\cdot). \tag{4.7}$$

Since the action of the 2-form ω on two vector fields X_f and X_g is defined by the Poisson brackets, $\{f,g\} = \omega(X_f, X_g)$, we can write the Hamiltonian 1-form as

$$dH = \frac{\partial H}{\partial P_i}\frac{\partial}{\partial Q_i} - \frac{\partial H}{\partial Q_i}\frac{\partial}{\partial P_i}, \tag{4.8}$$

from which the equations of motion defined by the Hamiltonian flow X_H are

$$Q_i = \frac{\partial H}{\partial P_i}, \qquad P_i = -\frac{\partial H}{\partial Q_i}. \tag{4.9}$$

As it is showed, this formalism leads us to the Hamilton equations in a natural way.

Liouville's theorem also finds its generalization. In symplectic geometry, the phase space is a manifold with a natural volume element, called the volume form, constructed from the wedge product power of the symplectic form in Eq.(4.6)

$$dV = \omega^n. \tag{4.10}$$

Liouville's theorem states the invariance of phase space volume under transformation generated by the Hamiltonian. That is, the volume form remains invariant under the Hamiltonian flow $\mathcal{X}_H$. A property that allows us to prove Liouville's theorem is the preservation of the symplectic structure under Hamiltonian flow. This invariance can be expressed in terms of the Lie derivative in the following way $\mathcal{L}_{\mathbf{X}_H}\omega = 0$ [123]. Therefore, the volume form is also invariant.

As reviewed, symplectic geometry turns out to be the abstraction and generalisation of the Hamiltonian formalism of CM. The scope of this formulation is to use the concepts and tools of differential geometry that allow us to study the type of structure with which the manifold has been equipped. In this case, the structure is given by a 2-form. On the other hand, another type of manifold widely studied is the Riemannian manifolds, whose structure is provided by the metric [124]. This structure introduces the concepts of distance and curvature. Therefore, if a manifold is equipped with different types of structures, the geometry they describe is less general. A symplectic manifold can be generalised by introducing an additional metric structure, giving rise to so-called generalised Hamiltonian spaces. [125].

A Hamiltonian space is defined as a pair $(\mathcal{M}, H)$, where $\mathcal{M}$ is a differentiable manifold equipped with a Hamiltonian function $H : T^*\mathcal{M} \to \mathbb{R}$, and $T^*\mathcal{M}$ is the cotangent bundle. As it was mentioned, equipping the manifold with a metric structure introduces the notion of curvature. Such a metric depends on position and momentum, $g(Q,P)$. The components of the metric are connected to the Hamiltonian function and can be obtained in the following way

$$g^{ij} = \frac{\partial^2 H}{\partial P_i \partial P_j}. \tag{4.11}$$

Since the notion of curvature has been introduced, it is studied by introducing connections. A connection is an element in differential geometry that allows comparing and studying the geometry at different points on a manifold. The definition of connection is not unique and depends on the type of structure defined on the manifold. Furthermore, a connection can be introduced by different approaches, such as parallel transport, covariant derivative, and the horizontal distribution of bundles [126]. The last approach will be adopted in the case of Hamiltonian spaces since it allows the separation between coordinate and momenta bundles. Thus, in this case, the connection is given by [125]

$$N_{ij} = \frac{1}{4}\left(\{g_{ij}, H\} - g_{ik}\dot{\partial}^k\dot{\partial}_j H - g_{jk}\dot{\partial}^k\dot{\partial}_i H\right), \tag{4.12}$$

where the notation $\dot{\partial}_i \equiv \frac{\partial}{\partial P_i}$ is used. The connection introduces a new symplectic structure given by

$$\theta = \sum_{i=1}^{n} \delta P_i \wedge dQ_i, \tag{4.13}$$

where $\delta P_i = dP_i - N_{ij}dQ_j$. The Hamilton equation also find their general form by defining a Hamiltonian flow as in Eq.(4.7), but considering the new symplectic structure

$$\dot{Q}_i = \frac{\partial H}{\partial P_i}, \qquad \frac{\delta P_i}{dt} = -\frac{\delta H}{\delta Q_i}. \tag{4.14}$$

In the same way, the volume form can be obtained as the power product of the 2-form in Eq.(4.13), which by Liouville's theorem is a invariant under Hamiltonian flow

$$dV = \theta^n. \tag{4.15}$$

Symplectic geometry results in great applicability in different areas of physics, such as CM, QM, Optics, GR, and Statistical Mechanics.

4.1 Classical GUP

In QM, the uncertainty principle is related to the commutation rules between position and momentum Eq.(2.39). The commutation rules bear a resemblance to the Poisson brackets, Eq.(4.2). The first to notice this resemblance was Dirac. In his analysis, he found that the mathematical structure of Hamiltonian mechanics could be adapted to quantum mechanics by introducing the following prescription

$$\{A,B\} \leftrightarrow \frac{1}{i\hbar}[\hat{A},\hat{B}]. \tag{4.16}$$

This connection between QM and CM has inspired the study of a minimal length in the CM framework where effects appear at a dynamical level [107, 127–129]. At this point, it is worth arguing the validity of the relation Eq.(4.16) in the presence of a minimal length. In the classical limit, that is $\hbar \to 0$, the Planck length $l_{PL} \to 0$. That must necessarily imply that a minimal length of the order of the Planck length is not suitable in CM. However, starting from the fact that the Dirac prescription maps the Heisenberg algebra to the Poisson algebra, we can take as valid the mapping of the deformed Heisenberg algebra to a deformed Poisson algebra due to the presence of a classical scale. As a consequence, such a scale deforms the Poisson brackets and therefore the canonical relationship between position and momentum. The deformed Poisson brackets are given below

$$\{q_i,q_j\} = R_{ij}(q,p), \qquad \{p_i,p_j\} = 0, \qquad \{q_i,p_j\} = h_{ij}(q,p), \tag{4.17}$$

where $R_{ij}(q,p)$ and $h_{ij}(q,p)$ are the deformation functions in a tensor-fashion. The form of $h_{ij} = \delta_{ij}f(p) + g_{ij}(p)$ was introduced previously via the equation Eq.(2.53) and will be taken up for the rest of this section in its classical form. In recent years, the study of new geometries, particularly non-commutative geometry to reconcile QM and GR, has brought many into this area. This has already been introduced in the section dedicated to DSR by introducing the κ-Minkowski space. However, this idea can be generalized. In the birth of differential geometry, Gauss noted that the presence of a scale leads to non-trivial geometries. As it has been shown, the Poisson algebra related to the GUP through the classical limit indicates that the dynamics of a system differs from the usual dynamics described by Hamiltonian mechanics. This is because the variables (q_i, p_i) are no longer canonical. In order to recover the canonical brackets Eq.(4.2), a non-canonical map is proposed between the variables (q_i, p_i) *i.e.* the physical variables associated with a deformed algebra, and a set of variables (Q_i, P_i), which we will call auxiliary variables. In [130, 131] it has been shown that this type of map introduces a duality between theories with GUP and QM with a curved momentum space. Curved space theories have also been studied in the context of DSR [78, 95, 97, 132].

In symplectic geometry, Darboux's theorem, introduced by the French mathematician Jean Gaston Darboux, ensures the existence of a set of canonical local variables in a symplectic manifold [133]. By choosing that set of canonical variables as (Q_i, p_i), where Q_i is the auxiliary position and p_i the physical momentum, the following algebra is satisfied

$$\{Q_i, Q_j\} = 0, \qquad \{p_i, p_j\} = 0, \qquad \{Q_i, p_j\} = \delta_{ij}. \tag{4.18}$$

Since Q_i and p_i are canonical, once they are promoted to operators, they satisfy the Heisenberg algebra. Therefore, the representation of the auxiliary position in momentum space takes the usual form. Let us consider the following transformation between the physical and canonical variables

$$p_i = P_i \qquad q_i(Q, P) = h_{ij}(p) Q_j, \tag{4.19}$$

By assuming a non-vanishing $\det h$, we can also consider $Q_j = (h^{-1})_{ji} q_i$ as the inverse transformation. Introducing a new set of canonical variables simplifies the non-commutative treatment of phase-space. However, it has been mentioned that this treatment leads to a geometry of the phase space that is not trivial. We can see this by obtaining the geodesic equation for the physical coordinates

$$\frac{d^2 q^i}{dt^2} = 0, \tag{4.20}$$

where $i = 1, ..., 3N$. Such equations for the physical positions imply a flat geometry. That is because we are not making any relativistic considerations and we are considering the particles in the system do not interact. Therefore, momentum and position space have to be flat. In terms of the canonical variables, the geodesic equation takes the following form

$$\frac{d^2Q^\alpha}{dt^2}+\frac{\partial Q^\alpha}{\partial q^i}\left[\frac{\partial^2 q^i}{\partial Q^\beta \partial Q^\gamma}\dot{Q}^\beta\dot{Q}^\gamma+2\frac{\partial^2 q^i}{\partial Q^\beta \partial p^\gamma}\dot{Q}^\beta\dot{p}^\gamma+\frac{\partial^2 q^i}{\partial p^\beta \partial p^\gamma}\dot{p}^\beta\dot{p}^\gamma\right]=0. \tag{4.21}$$

The above geodesic equation can be given a more compact form by introducing the vector $\Phi^\mu=(Q^\alpha,p^\beta)$ allowing us to rewrite the equation in the following way

$$\frac{d^2\Phi^\mu}{dt^2}+\Gamma^\mu_{\beta\gamma}\dot{\Phi}^\beta\dot{\Phi}^\gamma=0, \tag{4.22}$$

where $\Gamma^\mu_{\beta\gamma}$ is the connection. One can note that the vector Φ is an element of canonical phase space and the vector ϕ to the physical space. By elaborating on the transformation law between both vectors, the $6N$-bein e^k_μ is identified

$$e^k_\mu=\frac{\partial\phi^k}{\partial\Phi^\mu}. \tag{4.23}$$

Greek indices are used for canonical variables and Latin indices for the physical ones. Using the $6N$-bein, the metric is given by

$$g_{\mu\nu}=\delta_{ij}e^i_\mu e^j_\nu. \tag{4.24}$$

The $6N$-bein can be calculated using Eq.(4.19), and takes the following explicit form,

$$e=\begin{pmatrix} h & H \\ 0 & \mathbb{I} \end{pmatrix}, \tag{4.25}$$

where

$$H^i_\mu=\frac{\partial h^i_\alpha}{\partial p^\mu}Q^\alpha. \tag{4.26}$$

Each of the metric elements is a block $3N \times 3N$. Thus, the metric can be written using Eq.(4.24) as

$$g = \begin{pmatrix} h^T h & h^T H \\ H^T h & H^T H + \mathbb{I} \end{pmatrix}. \tag{4.27}$$

Once the metric is obtained, we can calculate its determinant, which will help us to build an integral measure. As it has been assumed that the existence of the inverse transformation for h_{ij}, the blocks of the matrix are invertible. Therefore, the determinant can be calculated in the following way

$$\begin{aligned} \det g &= \det(h^T h)\det(H^T H + \mathbb{I} - H^T h(h^T h)^{-1} h^T H) \\ &= \det h^T h \\ &= [\det h]^2. \end{aligned} \tag{4.28}$$

As mentioned earlier, inspired by the GUP, it is possible to introduce a deformation in CM by introducing a classical length. Furthermore, according to Gauss's argument, such a scale leads to a non-trivial geometry. We then introduce a momentum scale π which is related to the scale at which the deformation functions have relevance. In other words, if $\frac{p}{\pi} \to 0$, where p is the characteristic momentum of the system, the deformation is negligible and the Poisson algebra must be recovered. Therefore, we write the transformation matrix as follows

$$h^i_j = \delta^i_j f\left(\frac{|p|^2}{\pi^2}\right) + \frac{p^i p_j}{|p|^2} g\left(\frac{|p|^2}{\pi^2}\right), \tag{4.29}$$

where the determinant is given by

$$\det h = f^3\left(\frac{|p|^2}{\pi^2}\right). \tag{4.30}$$

By considering a system made of N free particles, the dynamics of each of them is not affected by the rest. That is, the transformation of coordinates for each of them depends only on their momentum. With this in mind, the metric for the position space can be written as follows

$$g = \begin{pmatrix} f(|p_{(1)}|^2)\mathbb{I}_{3\times 3} & 0 & \cdots & 0 \\ 0 & f(|p_{(2)}|^2)\mathbb{I}_{3\times 3} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f(|p_{(N)}|^2)\mathbb{I}_{3\times 3} \end{pmatrix}, \tag{4.31}$$

where we have introduce the notation $|p_{(i)}|$ to designate the magnitude of the momentum of the i-th particle. Thus, the phase space measure is expressed by

$$\begin{aligned} d\mu &= \det(\mathrm{h}) d^{3N}Q\, d^{3N}p, \\ &= \prod_{i=1}^{N} f^3\left(\frac{|p_{(i)}|^2}{\pi^2}\right) d^{3N}Q\, d^{3N}p. \end{aligned} \tag{4.32}$$

In order to demonstrate the results obtained, we use the following deformation function $f(p) = \left(1 + \frac{|p_{(i)}|^2}{\pi^2}\right)$. Such a modification is related to the quadratic model for the GUP. Then, the measure in phase space is given by

$$d\mu = \prod_{i=1}^{N} \left(1 + \frac{|p_{(i)}|^2}{\pi^2}\right)^3 d^{3N}Q\, d^{3N}p. \tag{4.33}$$

Higher terms of the modification factor can be ignored if the momentum scale π^2 is assumed to be larger than the momentum magnitude of every single particle $|p_{(i)}|$ in the system. Thus, we can write the following approximated expression for the measure

$$d\mu = \prod_{i=1}^{N} \left(1 + 3\frac{|p_{(i)}|^2}{\pi^2}\right) dQ^{3N}\, d^{3N}p. \tag{4.34}$$

We require that the usual measure of phase space be recovered in the limit $\pi \to \infty$. Likewise, in this limit, the physical variables recover their canonical form according to the transformation Eq.(4.19).

4.2 Microcanonical ensemble

The relationship between mathematics and physics has a long history. The book of nature, according to Galileo, is written in the language of mathematics. While physics studies the world around us, geometry describes shapes and sizes. In its abstract form, geometry allows us to visualise many physical concepts. Furthermore, geometry is the basis of many modern theories. Among these theories, we find electrodynamics, GR, QFT, CM, and statistical mechanics. Statistical mechanics has its foundations in symplectic geometry. The geometrical tools in classical statistical mechanics have allowed us to represent complex systems as points in phase-space whose dynamics are governed by a Hamiltonian. In addition, the application of Liouville's theorem has relevance in equilibrium statistical mechanics.

In the previous section, we reviewed Hamiltonian mechanics in the presence of a momentum scale. This scale modifies the geometry of the phase-space, from which we derive the new integral measure. The study of blackbody radiation in Chapter 3 with modifications due to GUP inspires us to study the same effects in classical statistical mechanics. For that, we start with the microcanonical ensemble whose members represent mental copies of an isolated system in thermodynamic equilibrium, that is, each member of the ensemble is described by the variables (E,V,N). Once a modification inspired by the GUP is considered in CM, it deforms the canonical relationship between momentum and position to Eq.(4.17). Such variables span a deformed phase space. By introducing a set of canonical variables, we recovered the symplectic structure in phase space and, therefore, the geometric structure of statistical mechanics. Elaborating on the microcanonical ensemble, we can use the measure we derived in terms of canonical variables, Eq.(4.34). We know that all members of such an ensemble are represented by points on the same hypersurface at constant energy E. Therefore, the microcanonical phase space density depends on the canonical variables $\rho_{MC}(Q,p)$. For points belonging to the hypersurface with equal energy, the phase space density is constant according to the postulate of equal a priori probability. For points that do not belong to such a hypersurface, the density vanishes.

Using the phase space approximated measure derived in Eq.(4.34), we have the following normalization condition for the phase space density, where we can omit the correction of order $O(|p_{(i)}|^4)$ and higher

$$\int \prod_{i=1}^{N} \left(1 + 3\frac{|p_{(i)}|^2}{\pi^2}\right) \rho_{MC}(Q,p)\; dQ^{3N}\; dp^{3N} = 1. \tag{4.35}$$

The fact that the set of variables (Q_i, p_i) are canonical guarantees invariance under canonical transformations. In ensemble theory, thermodynamic properties are obtained by taking the ensemble average. The value of an observable $f(Q,p)$ is obtained in each possible microstate, and the average value is assumed to correspond to the value of the thermodynamic variable

$$\langle f \rangle = \int \prod_{i=1}^{N} \left(1 + 3\frac{|p_{(i)}|^2}{\pi^2}\right) \rho_{MC}(Q,p) f(Q,p)\; d^{3N}Q\; d^{3N}p. \tag{4.36}$$

For a microcanonical ensemble, the contact between equilibrium statistical mechanics and thermodynamics is through entropy, defined in Eq.(2.5). The average of entropy is given by

$$\langle S \rangle = \int \prod_{i=1}^{N} \left(1 + 3\frac{|p_{(i)}|^2}{\pi^2}\right) \rho_{MC}(Q,p) k \ln \Omega\; d^{3N}Q\; d^{3N}p, \tag{4.37}$$

where Ω is the number of microstates that belong to the microcanonical ensemble of an isolated system in thermodynamic equilibrium. Using the normalization condition Eq.(4.35), this reduces to the usual average value for entropy $\langle S \rangle = k_B \ln \Omega$. This result is in agreement with what was expected, due to the use of canonical variables and the preserved nature of the symplectic structure. However, this result is expected to change in terms of the physical variables (q_i, p_i).

Canonical phase space points represent microstates in classical statistical mechanics. For an isolated system, the phase space density is constant. The properties of phase space are based on symplectic geometry, which can be generalised by introducing different geometric structures. In particular, the study of a minimal length in Classical Statistical Mechanics inspired by the GUP deforms Poisson algebra by breaking with the canonical relation of coordinates and momenta. In order to recover this structure, we introduce a set of canonical variables, which induce curvature in the phase space.

Chapter 5

Conclusion

> ¿Cómo nos rehabilitaremos? Hay quienes recaen al llegar a la cima de una montaña, al terminar su obra maestra, al afeitarse sin un solo tajito; no toda recaída va de arriba a abajo, porque arriba y abajo no quieren decir gran cosa cuando ya no se sabe dónde se está. Probablemente Ícaro creía tocar el cielo cuando se hundió en el mar epónimo, y Dios te libre de una zambullida tan mal preparada.
>
> -Julio Cortázar, *Me caigo me levanto*

Einstein's theory predicts that gravity is due to the curvature of space-time. Such a gravitational nature does not exhibit quantum behavior. Even though particles have mass and generate a gravitational field and, therefore, a curvature, the quantum nature of particles conflicts with the determinism of gravity. There are many reasons to believe that gravity does have quantum properties, and a QG theory is required. For example, classical electrodynamics presents singularities that are swept away in the quantum version via regularization, and it is expected that singularities in GR can also be swept away by a QG theory. Many candidates have been developed in order to predict the quantum properties of gravity. However, the energy scale at which gravity exhibits such properties seems out of reach. Regarding the experimental part, tests on QG have received attention in the past decades. Such tests are expected to characterize QG effects that cannot be explained by GR and QM separately. Phenomenology and well designed experiments may make testing QG possible even with our limited technology. It is worth using electromagnetism and QM to argue about this point. Historically, the first to realize the quantization of electric charge was Faraday in his experiments with electrolysis. In later years, Millikan established the value of the elementary charge. In QM, emission and absorption lines, the double split, and even the blackbody radiation suggest new (quantum) properties beyond direct observations of quanta. This introduces the possibility of measuring the properties of QG without having a fundamental theory. However, we need to be careful when applying phenomenological models of gravity to different systems because many of the results that can be obtained will not reflect the quantum nature of gravity.

In this thesis, we focus on the phenomenological aspects of QG that emerge in the context of thermodynamics and statistical mechanics. Such theories may offer indirect evidence of quantum gravitational effects related to a minimal measurable length. We showed that such effects emerge in Wien's law and Boltzmann's law for the blackbody and ensemble theory in statistical mechanics. Below, we include a brief summary of the content of this thesis and future perspectives.

5.1 Blackbody Radiation

In chapter 3, we analyzed the effects of a minimal measurable length on the blackbody spectrum. To do so, we have considered a quantization procedure for the electromagnetic field inspired by GUP models. These corrections were introduced in the commutator of the quadrature operators of the electromagnetic field. One of the effects of such a procedure is that of modifying the dependence of the quanta of energy on the frequency. Following the statistical method introduced by Bose [116], the most probable distribution $\overline{Z}_n^k$ was derived. The modified distribution matches the standard expression in the limit $\gamma_{EM} \to 0$. By considering $\alpha\zeta^k \ll 1$, where ζ^k is the term that carry the GUP modification and $\alpha = 1/k_BT$, we introduced the approximation

$$\overline{Z}_n^k = B^k e^{-\alpha\hbar\omega^k n}\left[1 - \alpha\zeta^k n(n+1)\right]. \tag{5.1}$$

A maximum value of n_{max} is introduced by such an approximation. When exceeding such a value, the approximation is not valid. This is because the n_{max} was calculated using the destruction function's zeros. If that number is surpassed, the distribution includes n values that result in a negative probability. Because these microstates are not physical and the distribution function is always positive, $\overline{Z}_n^k$ brakes for such n values. However, we discover that the contribution for n values greater than the maximum value can be ignored when computing the occupation number A^k.This was demonstrated by computing the rest Eq.(3.41), from which we obtained the condition $E_{PL} \gg (\hbar\omega^k)^2/k_B T$ for T and ω values where the approximation is valid. We obtained the modified Planck's distribution using Bose's method, which consistently reduces to the usual distribution for $\gamma_{EM} \to 0$.

Elaborating on the thermodynamics of the blackbody, we found that the modified energy density at any given temperature results in being smaller than the energy density in the standard theory. In particular, the model in Eq.(3.19) implies that larger wave numbers, and therefore shorter wavelengths, are associated with a value of momentum, and therefore of energy, larger than in the ordinary case. Hence, since short wavelengths are characterized by higher values of energy than in the standard case, they also result in being harder to excite, contributing less to the energy distribution.

We discover that GUP effects shift the maximum of the distribution based on Wien's law using the modified energy density. We observed that such a modification depends on the temperature. For much smaller temperatures than the Planck temperature, the modification goes to zero. The modified Stefan-Boltzmann law was obtained by integrating the spectral radiance related to the modified energy density. The results suggest that the total energy radiated is lower than in the ordinary case at high temperatures. This effect is consistent with DSR results for a photon gas [121]. We notice that the modifications to Wien's and Stefan-Botlzmann's law due to the effects of a minimal measurable length are temperature-dependent. Both modifications have the potential to be observed at high energies.

Currently, the calibration of instruments that measure thermal radiation offers advances in the construction of cavities. In particular, in metrology centers, cavities are designed to withstand high temperatures, above 3500 K, which could offer an opportunity to test the modifications on blackbody radiation [134]. These cavities are made of graphite and some variants of it. The cavity is heated through an electric current, reaching high temperatures. The emissivity of these cavities is around 0.995, which is very close to the emissivity of a blackbody. However, the only drawback mentioned is the useful life time, which is between 200 and 400 hours on average due to the high temperatures.

5.2 Statistical Mechanics

In Chapter 4, we elaborated on classical statistical mechanics by considering a minimal length inspired by the GUP. Moreover, we argued on the validity of a minimal length in the classical limit, which allowed us to introduce a function deforming the Poisson brackets

$$\{q_i, q_j\} = R_{ij}(q,p), \qquad \{p_i, p_j\} = 0, \qquad \{q_i, p_j\} = h_{ij}(q,p). \tag{5.2}$$

We revised the symplectic geometry of phase space, and its generalization to spaces with curvature. The reason was due to the dualism between theories with minimal length and theories with curved momentum space [131]. We introduced a the following map between the physical and auxiliary variables. To preserve the symplectic structure and recover the Poisson brackets,

$$p_i = P_i \qquad\qquad q_i(Q,P) = h_{ij}(p) Q_j. \tag{5.3}$$

The set of variables Q_i, p_i fulfill the Poisson algebra, implying that the symplectic structure is restored and that any transformation to a new set of variables that preserves the Poisson algebra, and therefore, the Hamilton's equation is canonical. This new set spans a curved phase space. That is shown by calculating the geodesic equation Eq.(4.21), where physical space was assumed to be flat. We found that the metric for a system made up of N particles only depends on momenta, in agreement with theories of curved momentum space [78, 95, 97, 132]. Using the determinant of the metric, we were able to construct the measure of phase space.

We introduced the phase space density to construct the microcanonical ensemble. As we introduced a set of canonical variables, Liouville's theorem is satisfied. Therefore, the phase space density is constant for a system at equal energy. Nonetheless, we found that the entropy assumes its usual form once the average is taken over each microstate.

In the future, using the microcanonical ensemble, we can construct the canonical and grand canonical ensemble through perturbation theory [135] This will allow us to study systems exchanging energy and matter with the environment. In addition, a modification to the energy equipartition law will be considered.

5.3 Final Remarks

The search for a quantum theory of gravity is one of the most arduous tasks in modern physics. Its construction represents a theoretical as well as an experimental challenge, with little conclusive achievement obtained at the moment. However, rather than discouraging us, this motivates us to continue working to achieve the ultimate goal of a definite theory of quantum gravity. Such an objective still seems to be a puzzle, with many of the possible candidates offering predictions at energy scales that are not currently available. Without a doubt, in order to continue with the scientific method, it is necessary to test these candidates. However, phenomenology seems to fill the gap between experiments and theory by offering an opportunity to obtain empirical support.

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Appendix A

Appendix

In this appendix we show different mathematical series, specifically the geometry series and its derivatives. If the argument $x < 1$ is considered, the series will be finite and agree to a finite value. These series are used in Chapter 3

$$
\begin{aligned}
\sum_{n}^{\infty} e^{-xn} &= \frac{1}{1-e^{-x}}, \\
\sum_{n=0}^{\infty} n e^{-xn} &= \frac{e^{-x}}{(1-e^{-x})^2}, \\
\sum_{n=0}^{\infty} n^2 e^{-xn} &= e^{-x}\frac{1+e^{-x}}{(1-e^{-x})^3}, \\
\sum_{n=0}^{\infty} n^3 e^{-xn} &= e^{-x}\frac{1+4e^{-x}+e^{-2x}}{(1-e^{-x})^4}, \\
\sum_{n=0}^{\infty} n^4 e^{-xn} &= e^{-x}\frac{1+11e^{-x}+11e^{-2x}+e^{-3x}}{(1-e^{-x})^5}.
\end{aligned}
\tag{A.1}
$$