APPROXIMATIONS FOR SOME FUNCTIONS OF PRIMES

Fataneh Esteki
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ABSTRACT

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Fataneh Esteki
Department of Mathematics and Computer Science
University of Lethbridge
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Let

$$\pi(x) = \#\{p \leq x;\ p \text{ prime}\},$$

$$\theta(x) = \sum_{p \leq x} \log p,$$

and

$$\psi(x) = \sum_{p^k \leq x, k \geq 1} \log p.$$

This thesis studies different methods in establishing estimates for $\pi(x)$, $\theta(x)$, and $\psi(x)$. This is a summary of the main result of the thesis.

(i) A detailed exposition of a theorem of Rosser on the estimation of $\psi(x)$ is given. The theorem is written using parameters instead of the specific constants. So it conveniently will produce new estimates for $\psi(x)$ whenever new improvements in the values of the parameters occur.

As an example, our theorem, with current known values of parameters gives

$$0.98719x < \psi(x) < 1.012807x, \text{ for } x \geq e^{20}.$$

(ii) Different techniques for establishing explicit upper and lower bounds for $\theta(x)$ are studied. It is proved that

$$\theta(x) - x < \frac{1}{36269.2}x, \text{ for } x > 0.$$

(iii) The following sharp lower bounds for $\psi(x) - \theta(x)$ are established.

$$\psi(x) - \theta(x) > \sqrt{x}, \text{ for } 121 \leq x \leq e^{145.5},$$

and

$$\psi(x) - \theta(x) > 0.99997159\sqrt{x}, \text{ for } x \geq 121.$$

(*iv*) Several tables of upper and lower bounds for $\theta(x)$ based on different methods are generated.

(*v*) Different methods for establishing inequalities of the form

$$\psi(x) - \theta(x) < c_1 x^{1/2} + c_2 x^{1/3}$$

are studied and specific numerical examples are generated. As an example it is shown that

$$\psi(x) - \theta(x) < \left(1 + 1.2998600240 \times 10^{-9}\right) x^{1/2} + 1.0003 x^{1/3}, \text{ for } x \geq e^{100}.$$

(*vi*) Different methods for finding upper and lower bounds for $\pi(x)$ of the forms

$$\frac{x}{\log x - a}, \ a > 0,$$

and

$$\frac{x}{\log x}\left(1 + \frac{1!}{\log x} + \frac{2!}{\log^2 x} + \cdots + \frac{(\ell-1)!}{\log^{\ell-1} x} + \frac{c}{\log^{\ell} x}\right), \ \ell \in \mathbb{N}, \ c > 0,$$

are considered.

Also specific numerical examples are generated. As a sample it is established that for $x \geq 10^{11}$,

$$\frac{x}{\log x - 0.9999} < \pi(x) < \frac{x}{\log x - 1.0456},$$

$$\frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{1.9899}{\log^2 x}\right) < \pi(x) < \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2.296}{\log^2 x}\right).$$

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Dedicated to My Parents

Contents

Chapter 1

Introduction

1.1 Chebyshev functions

One of the important achievements of the ninetieth century mathematics is the proof of the Prime Number Theorem. This theorem gives the asymptotic density of prime numbers among integers. More precisely, let

$$\pi(x) = \#\left\{p \leq x,\ p \text{ prime}\right\}.$$

Then the Prime Number Theorem asserts that

$$\lim_{x\to\infty} \frac{(\log x)\pi(x)}{x} = 1.$$

The notation $f \sim g$ means

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1.$$

Using this notation the Prime Number Theorem can be written as

$$\pi(x) \sim \frac{x}{\log x},$$

as $x \to \infty$.

The Prime Number Theorem was originally conjectured in 1791 by Gauss and later on, in 1798, independently by Legendre. Gauss looked at the list of primes less than $3,000,000$ and observed that $\pi(x)$ can be approximated very closely by the function li(x), which is defined by

$$\mathrm{li}(x) = \int_0^x \frac{dt}{\log t} = \lim_{\eta \to 0^+} \left(\int_0^{1-\eta} + \int_{1+\eta}^x \right) \frac{dt}{\log t}.$$

By integration by parts on the integral defining li(x) one can show that the above integral is asymptotically equivalent to $x/\log x$. Legendre in his textbook in number theory asserted that $\pi(x)$ can be approximated by

$$\frac{x}{\log x - 1.08366}.$$

Hence Gauss and Legendre's observations announce the Prime Number Theorem. See [1] for more historical information.

The first attempt for proving the Prime Number Theorem was done by Chebyshev in 1848. Among other results, he established that if the ratio $(\log x)\pi(x)/x$ tends to a limit as x approaches to infinity then that limit must be one. Moreover he showed (see [11, p. 15]) that for all sufficiently large values of x,

$$(0.921\cdots)\frac{x}{\log x} \leq \pi(x) \leq (1.105\cdots)\frac{x}{\log x}. \tag{1.1}$$

For proving the above assertions Chebyshev introduced two new functions θ and ψ. These two functions played an important role in the development of the prime number theory. They can be defined as

$$\theta(x) = \sum_{p \leq x} \log p,$$

and

$$\psi(x) = \sum_{p^n \leq x, n \geq 1} \log p.$$

Chebyshev observed that proving any of

$$\theta(x) \sim x \text{ or } \psi(x) \sim x,$$

as $x \to \infty$, would imply the Prime Number Theorem. However he was not able to establish either of the above asymptotics.

A revolutionary idea for proving the Prime Number Theorem was introduced by Riemann in 1859. In his approach he used Euler's identity

$$\sum_{n=1}^{\infty} n^{-s} = \prod_{p} \left(1 + p^{-s} + p^{-2s} + \cdots\right) = \prod_{p} (1 - p^{-s})^{-1},$$

where the products run over all primes p . Riemann studied the above identity where s is a complex variable. He considered

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s},$$

for $\mathrm{Re}(s) > 1$ and verified that $\zeta(s)$ has an analytic continuation to the whole complex plane with the exception of a simple pole with residue 1 at $s = 1$. Moreover he proved that $\zeta(s)$ satisfies the functional equation

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s),$$

where $\Gamma(s) = \int_0^{\infty} x^{s-1}e^{-x}dx$ (see [11, Chapter III]). By using this equation one can show that $\zeta(s)$ vanishes at all the negative even integers. These zeros are called the trivial zeros of the zeta function. Riemann observed that all other zeros of $\zeta(s)$ are situated in the critical strip $0 \leq \mathrm{Re}(s) \leq 1$ (see [11, p. 58]). We denote these "so called" non-trivial zeros by $\rho = \beta + i\gamma$ where $0 \leq \beta \leq 1$ and $\gamma \in \mathbb{R}$.

Riemann observed that the distribution of the non-trivial zeros of $\zeta(s)$ had an important role in the proof of the Prime Number Theorem and designed a program for the proof of the Prime Number Theorem based on certain properties of the zeros of the Riemann zeta function. However he failed to deduce the Prime Number Theorem. More precisely, he suggested that to prove the Prime Number Theorem it was sufficient to prove that $\zeta(s)$ does not vanish on the line $\mathrm{Re}(s) = 1$. Moreover, he went one step further and conjectured that all non-trivial zeros of $\zeta(s)$ lie on the line $\mathrm{Re}(s) = 1/2$. This is the celebrated Riemann Hypothesis and it has been remained unsolved up to this day.

The Prime Number Theorem was finally proven in 1896 by Hadamard and independently by de La Vallée Poussin, following Riemann's suggested program.

Despite the fact that the Prime Number Theorem determines the asymptotic behavior of $\pi(x)$, it does not give explicit upper and lower bounds for $(\log x)\pi(x)/x$. In many applications we need to have such explicit bounds.

As an example Chebyshev established upper and lower bound for $\pi(x)$ given in (1.1). This enabled him to prove Bertrand's postulate which states that for x sufficiently large, there is a prime between x and $2x$.

To prove this, it suffices to show that for large x

$$\pi(2x) - \pi(x) \geq 1.$$

Using (1.1), for x sufficiently large, we obtain

$$\pi(2x) - \pi(x) \geq 0.921\frac{2x}{\log 2x} - 1.106\frac{x}{\log x}.$$

For x sufficiently large, the above can be written as

$$\pi(2x) - \pi(x) \geq \frac{1.842x}{\log x + 1/3\log x} - \frac{1.106x}{\log x}.$$

Since $(3/4)(1.842) - 1.106 > 0$ Bertrand's postulate follows.

This thesis studies different methods in establishing explicit estimates for $\pi(x)$, $\theta(x)$, and $\psi(x)$.

1.2 Relations between $\psi(x)$, $\theta(x)$, and $\pi(x)$

In order to study the counting function $\pi(x)$, we use Chebyshev functions $\psi(x)$ and $\theta(x)$ since they are easier to deal with in comparison to $\pi(x)$. We also introduce another function

$$T(x) = \sum_{n\leq x} \log n = \log([x]!),$$

where $[x]$ denotes the greatest integer function. The following lemma [16, p. 104] gives the basic relations between $\psi(x)$, $\theta(x)$, $T(x)$, and $\pi(x)$.

Lemma 1.1. *Let $x > 0$. We have*

$$\psi(x) = \sum_{p^n\leq x, n\geq 1} \log p = \sum_{p\leq x} \left[\frac{\log x}{\log p}\right] \log p, \tag{1.2}$$

$$T(x) = \log([x]!) = \sum_{n\geq 1} \psi\left(\frac{x}{n}\right) = \sum_{k,n\geq 1} \theta\left(\left(\frac{x}{n}\right)^{1/k}\right), \tag{1.3}$$

$$\psi(x) - \sqrt{x}\log x \leq \theta(x) \leq \psi(x), \tag{1.4}$$

and, for every $\varepsilon > 0$*,*

$$\frac{\theta(x)}{\log x} \leq \pi(x) \leq \frac{\theta(x)}{(1-\varepsilon)\log x} + x^{1-\varepsilon}. \tag{1.5}$$

Proof. We start by proving (1.2). Using definitions of $\psi(x)$ and $\theta(x)$ we see that

$$\psi(x) = \sum_{n\geq 1} \theta(x^{1/n}) = \sum_{n\geq 1} \sum_{p\leq x^{1/n}} \log p = \sum_{p^n\leq x, n\geq 1} \log p.$$

Since the summand is zero when $n > \log x/\log p$, we have

$$\psi(x) = \sum_{p\leq x} \left[\frac{\log x}{\log p}\right] \log p.$$

Next we prove (1.3). To do this we require to show that, for every $x \geq 1$,

$$[x]! = \prod_{p\leq x} p^{\alpha_p}, \text{ where } \alpha_p = \sum_{k=1}^{\infty} \left[\frac{x}{p^k}\right]. \tag{1.6}$$

To establish (1.6) it suffices to show that

$$\log([x]!) = \log\left(\prod_{p\leq x} p^{\alpha_p}\right) = \sum_{p\leq n} \left(\sum_{k=1}^{\infty} \left[\frac{n}{p^k}\right]\right) \log p,$$

where $n = [x]$. We have

$$\log([x]!) = \log(\prod_{d\leq n} d) = \sum_{d\leq n} \log d = \sum_{d\leq n} \sum_{p^k|d} \log p = \sum_{p^k\leq n} \log p \sum_{d\leq n, p^k|d} 1$$
$$= \sum_{p\leq n} \left(\sum_{k=1}^{\infty} \left[\frac{n}{p^k}\right]\right) \log p.$$

From which we deduce (1.6).

It follows from (1.6) that

$$\log([x]!) = \sum_{p\le x} \alpha_p \log p = \sum_{p\le x}\sum_{k=1}^{\infty}\left[\frac{x}{p^k}\right]\log p$$

$$= \sum_{p\le x}\sum_{k=1}^{\infty}\left(\sum_{n\le \frac{x}{p^k}} 1\right)\log p = \sum_{k,n\ge 1}\sum_{p\le \left(\frac{x}{n}\right)^{1/k}} \log p. \tag{1.7}$$

On the other hand

$$\sum_{n\ge 1}\psi\left(\frac{x}{n}\right) = \sum_{n\le x}\psi\left(\frac{x}{n}\right) = \sum_{n\le x}\sum_{p^k\le \frac{x}{n}, k\ge 1}\log p = \sum_{k,n\ge 1}\sum_{p\le \left(\frac{x}{n}\right)^{1/k}}\log p. \tag{1.8}$$

Moreover, following the definition of θ, we find that

$$\sum_{k,n\ge 1}\theta\left(\left(\frac{x}{n}\right)^{1/k}\right) = \sum_{k,n\ge 1}\sum_{p\le \left(\frac{x}{n}\right)^{1/k}}\log p. \tag{1.9}$$

The proof of (1.3) is immediate by putting together (1.7), (1.8), and (1.9).

We verify (1.4) by noting that

$$\psi(x) - \theta(x) = \sum_{n\ge 2}\theta(x^{1/n}) = \sum_{n\ge 2}\sum_{p^n\le x}\log p.$$

It follows that $\psi(x) - \theta(x) \ge 0$. Moreover we have

$$\sum_{n\ge 2}\sum_{p^n\le x}\log p \le \sum_{p\le\sqrt{x}}\sum_{2\le n\le \frac{\log x}{\log p}}\log p$$

$$\le \sum_{p\le\sqrt{x}}\left(\frac{\log x}{\log p}\right)\log p = \left(\sum_{p\le\sqrt{x}}1\right)\log x$$

$$\le \sqrt{x}\log x.$$

This implies

$$\psi(x) - \sqrt{x}\log x \le \theta(x).$$

We now prove the left-hand side of (1.5). We have

$$\theta(x) = \sum_{p\le x}\log p \le \log x\sum_{p\le x}1 = (\log x)\pi(x).$$

For the right-hand side of (1.5) we observe that

$$\theta(x) \geq \sum_{x^{1-\varepsilon} \leq p \leq x} \log p \geq \log(x^{1-\varepsilon}) \sum_{x^{1-\varepsilon} \leq p \leq x} 1 = (1-\varepsilon)(\log x)(\pi(x) - \pi(x^{1-\varepsilon})).$$

This together with the trivial bound $\pi(x^{1-\varepsilon}) \leq x^{1-\varepsilon}$ achieve the proof. □

The relations between $\psi(x)$, $\theta(x)$, and $\pi(x)$ in Lemma 1.1 suggest that by establishing an upper (or lower) bound for any of these functions we will be able to deduce an upper (or lower) bound for the other functions. This leads us to the following lemma.

Lemma 1.2. *Suppose that $f(x)$ and $g(x)$ are two distinct functions in the set*

$$\left\{ \frac{(\log x)\pi(x)}{x}, \frac{\theta(x)}{x}, \frac{\psi(x)}{x} \right\}.$$

If $b \leq f(x) \leq a$ when x is large enough then, for $\varepsilon > 0$, we have

$$b - \varepsilon \leq g(x) \leq a + \varepsilon,$$

for large values of x.

Proof. We consider the case $f(x) = \pi(x)$, and $g(x) = \theta(x)$, the proofs for other cases are similar.

Assume that $\pi(x) \leq ax/\log x$, when $x \geq x_0$. From this, by the left-hand side of (1.5) we see that $\theta(x) \leq ax$ when $x \geq x_0$.

We next assume that $bx/\log x \leq \pi(x)$ when $x \geq x_0$. We combine this, with the right-hand side of (1.5) to obtain

$$x(1-\varepsilon') \left(b - \frac{\log x}{x^{\varepsilon'}} \right) \leq \theta(x),$$

for $\varepsilon' > 0$. Since $\log x / x^{\varepsilon'}$ decreases to zero, for large x we have

$$x(1-\varepsilon')\,(b - \varepsilon'') \leq \theta(x).$$

This completes the proof. □

1.3 Explicit bounds for $\pi(x)$, $\theta(x)$, and $\psi(x)$

It is not difficult to find numerical upper and lower bounds for $\pi(x)$ and the Chebyshev functions. We next establish a numerical lower bound for $\pi(x)$. The main argument is due to Nair [15, p. 126].

Lemma 1.3. *For $x \geq 4$,*

$$0.173 \, \frac{x}{\log x} \leq \pi(x).$$

Proof. We let $d_n = \text{lcm}_{1 \leq m \leq n}\{m\}$, where lcm denotes the least common multiple, and

$$I = \int_0^1 x^n (1-x)^n dx.$$

Since

$$(1-x)^n = \sum_{r=0}^{n} (-1)^r \binom{n}{r} x^r,$$

we have

$$I = \int_0^1 \sum_{r=0}^{n} (-1)^r \binom{n}{r} x^{n+r} dx = \sum_{r=0}^{n} (-1)^r \binom{n}{r} \frac{1}{n+r+1}. \tag{1.10}$$

Since $x \in [0,1]$ we have $x(1-x) \leq 1/4$. Therefore

$$I \leq \frac{1}{4^n}. \tag{1.11}$$

Observe that every denominator in (1.10) is not greater than $2n+1$, therefore Id_{2n+1} is a positive integer. So together with (1.11) we find that

$$4^n \leq d_{2n+1}. \tag{1.12}$$

Let $\alpha \in \mathbb{N}$ such that p^α is the exact power of p which divides d_{2n+1}. Thus $p^\alpha \leq 2n+1$. This leads to

$$d_{2n+1} \leq \prod_{p \leq 2n+1} p^{\frac{\log(2n+1)}{\log p}}. \tag{1.13}$$

We combine (1.12) and (1.13) and take the logarithm to obtain

$$n \log 4 \leq \log d_{2n+1} \leq \sum_{p \leq 2n+1} \frac{\log(2n+1)}{\log p} \log p = \log(2n+1)\pi(2n+1).$$

It follows that

$$\frac{2n\log 2}{\log(2n+1)} \le \pi(2n+1). \tag{1.14}$$

Observe that

$$\frac{(2n-2)\log 2}{\log 2n} < \frac{(2n-2)\log 2}{\log(2n-1)} \le \pi(2n-1) \le \pi(2n).$$

Therefore

$$\frac{(2n-2)\log 2}{\log 2n} \le \pi(2n). \tag{1.15}$$

Putting together (1.14), and (1.15) gives that for every natural number $N \ge 2$,

$$\frac{(N-2)\log 2}{\log N} \le \pi(N). \tag{1.16}$$

In order to obtain a lower bound for $\pi(x)$ for $x \in \mathbb{R}$, we proceed as follows. We combine (1.16) with the facts that $[x] \le x$, and $x - 1 \le [x]$ to deduce

$$\frac{(x-3)\log 2}{\log x} \le \frac{([x]-2)\log 2}{\log[x]} \le \pi([x]) \le \pi(x).$$

Thus

$$\frac{(x-3)\log 2}{\log x} \le \pi(x).$$

We conclude by checking that, for $x \ge 4$, we have

$$\frac{x}{\log x}\left(\left(1-\frac{3}{4}\right)\log 2\right) \le \frac{(x-3)\log 2}{\log x}.$$

The left-hand side of the above inequality gives the desired constant 0.173. □

Next we present a lemma which gives an upper bound for $\psi(x)$ (see [16, p. 118]). The following argument is due to Mertens.

Lemma 1.4. *If* $x > 1$ *then*

$$\psi(x) < 2x.$$

Proof. By (1.3), we may write

$$\log([x]!) = \sum_{n\ge 1} \psi\left(\frac{x}{n}\right).$$

From this we find that

$$\log([x]!) - 2\log(\left[\frac{x}{2}\right]!) = \sum_{n\geq 1}\psi(x/n) - 2\sum_{n\geq 1}\psi(x/2n).$$

Hence

$$\log([x]!) - 2\log(\left[\frac{x}{2}\right]!) = \sum_{n\geq 1}(-1)^{n+1}\psi\left(\frac{x}{n}\right).$$

Since $\psi(x)$ is an increasing function, from the last identity we have

$$\psi(x) - \psi(x/2) < \log([x]!) - 2\log(\left[\frac{x}{2}\right]!). \tag{1.17}$$

Let $x > 14$. We define the integer N such that $2(N-1) < x \leq 2N$. We observe that

$$\log([x]!) - 2\log(\left[\frac{x}{2}\right]!) < \log((2N)!) - 2\log((N-1)!) = \log\left(\frac{(2N)!}{((N-1)!)^2}\right). \tag{1.18}$$

By induction we find that

$$\frac{(2N)!}{((N-1)!)^2} < e^{2(N-1)}, \text{ for } N \geq 8.$$

Thus

$$\log\left(\frac{(2N)!}{((N-1)!)^2}\right) < 2(N-1). \tag{1.19}$$

Putting together (1.17), (1.18), and (1.19) gives

$$\psi(x) - \psi(x/2) < x. \tag{1.20}$$

Observe that

$$\psi(x) = \sum_{n\geq 0}\left[\psi\left(\frac{x}{2^n}\right) - \psi\left(\frac{x}{2^{n+1}}\right)\right].$$

This combined with (1.20) give

$$\psi(x) \leq x\sum_{n\geq 0}\frac{1}{2^n} = 2x, \text{ for } x > 14.$$

We check that the inequality also holds for $1 < x \leq 14$. This completes the proof. □

Having established relations between $\pi(x)$, $\psi(x)$, and $\theta(x)$ in Lemma 1.1, a lower bound for $\pi(x)$ in Lemma 1.3 and an upper bound for $\psi(x)$ in Lemma 1.4, we can deduce the following theorem.

Theorem 1.5. *(i) For $x > 1$ we have*

$$\theta(x) \le \psi(x) < 2x.$$

(ii) For $x \ge e^{40}$ we have

$$\pi(x) < 2.956\frac{x}{\log x}.$$

(iii) For $x \ge e^{10}$ we have

$$0.017x \le \theta(x) \le \psi(x).$$

Proof. We begin with the upper bound for $\theta(x)$. Lemma 1.4 combined with (1.4) yields $\theta(x) < 2x$, for $x > 1$. Using this upper bound for the right-hand side of (1.5) gives

$$\pi(x) \le \frac{x}{\log x}\left(\frac{2}{1-\varepsilon} + \frac{\log x}{x^{\varepsilon}}\right). \tag{1.21}$$

Let $\varepsilon = 1/10$. Since $(\log x)/x^{1/10}$ decreases for $x > e^{10}$, then (1.21) gives

$$\pi(x) < 2.956\frac{x}{\log x}, \text{ for } x \ge e^{40}.$$

Next we put together Lemma 1.3 and the right-hand side of (1.5) to derive

$$x(1-\varepsilon)\left(0.173 - \frac{\log x}{x^{\varepsilon}}\right) \le \theta(x).$$

With $\varepsilon = 9/10$ the last inequality is transformed into

$$0.017x \le \theta(x), \text{ for } x \ge e^{10}.$$

From this, by (1.4), we deduce

$$0.017x \le \psi(x), \text{ for } x \ge e^{10}.$$

□

The above theorem establishes explicit upper and lower bounds for the Chebyshev functions and $\pi(x)$, however these bounds are rather poor and with more work one can replace them with more precise bounds. Our main goal in this thesis is to develop techniques which enable us to derive sharp explicit bounds for $\psi(x)$, $\theta(x)$, and $\pi(x)$.

1.4 Statement of the results

Although the Prime Number Theorem states that $\psi(x)$ is asymptotic to x, it does not give any information about the size of the error term $\psi(x) - x$. We are interested in finding the size of this error term. This is done in Theorem 2.4 of Chapter 2 which gives an exposition of Rosser's method [17] for finding an explicit error term in the Prime Number Theorem.

To explain our theorem we introduce the following notations.

Let $N(T)$ be the number of non-trivial zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ such that $0 < \gamma \leq T$. Let A be the height that the Riemann Hypothesis has been verified. In other words if $\zeta(\beta + i\gamma) = 0$ for $0 \leq \gamma \leq A$ then $\beta = 1/2$.

We assume there exists $r > 0$ such that for $\gamma > A$ we have

$$\beta < 1 - \frac{1}{r \log \gamma}.$$

We define

$$F(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8},$$

$$R(T) = d_1 \log T + d_2 \log \log T + d_3,$$

and assume that

$$|N(T) - F(T)| < R(T),$$

for $T \geq 2$ and reals $d_1,\ d_2$, and d_3. Moreover suppose that $F(A) \leq N(A)$. Also we define $c_1, \cdots, c_6$ as

$$c_1 = r(\log A)^2,$$

$$c_2 = r \log A,$$

$$c_3 = R(A),$$

$$c_4 = \frac{1}{2\pi} + \frac{d_1}{A \log \frac{A}{2\pi}} + \frac{d_2}{A(\log A)(\log \frac{A}{2\pi})},$$

$$c_5 = \frac{1}{\log(A/2\pi)},$$

and

$$c_6 = \log(A/2\pi).$$

In Chapter 2 we give a detailed proof of the following theorem.

Theorem 2.4. *Let $m \in \mathbb{N}$ and $a > 0$ such that*

$$\log a < \frac{c_1 m^2}{m + c_5}.$$

Suppose that

$$\sum_{\rho} \frac{1}{|\gamma|^{m+1}} \leq k,$$

$$\delta = 2 \left(a^{-1/2} k + \frac{c_3}{A^{m+1} a^{1/c_2}} + \frac{c_4(1 + mc_6)}{\left(1 - \frac{(m+c_5)\log a}{c_1 m^2}\right) m^2 A^m a^{1/c_2}} \right)^{1/(m+1)},$$

and

$$\varepsilon = \frac{\delta}{2} \left(\left(\frac{(1+\delta)^{m+1} + 1}{2} \right)^m + m \right).$$

If $1 + m\delta a < a$ we have

$$x(1 - \varepsilon) - \log 2\pi < \psi(x) < x(1 + \varepsilon) - \frac{1}{2} \log\left(1 - 1/x^2\right),$$

for $x \geq a$.

This theorem is essentially Theorem 21 of [17], however it is stated using parameters instead of the specific constants. So it conveniently will produce new estimates for $\psi(x)$ whenever new improvements in the values of the parameters occur. The above theorem plays a fundamental role in effective estimations of the functions of primes such as Chebyshev functions and $\pi(x)$. By employing this theorem, one can generate tables to provide bounds for $\psi(x)$ in various ranges.

We say some words on the proof of Theorem 2.4. Riemann's explicit formula represents the error term in the Prime Number Theorem as a sum over the non-trivial zeros of the Riemann zeta function. More precisely, we have

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \log 2\pi - \frac{1}{2} \log(1 - x^{-2}),$$

(see [5, p. 60]). So

$$\phi(x) = \psi(x) - x + \log 2\pi + \frac{1}{2}\log\left(1 - x^{-2}\right) = -\sum_{\rho} \frac{x^{\rho}}{\rho}$$

measures the error term in the Prime Number Theorem.

In order to bound $\phi(x)$ we require to estimate $\sum_{\rho} x^{\rho}/\rho$. This is not most convenient since the sum is not absolutely convergent. To resolve this difficulty we introduce

$$K_m(x,h) = \int_0^h dy_1 \int_0^h dy_2 \cdots \int_0^h \phi\left(x + y_1 + y_2 + \cdots + y_m\right) dy_m,$$

where $m \in \mathbb{N}$ and $h > 0$. Next we will bound $\psi(x)$ in terms of $K_m(x,h)$. We have

$$x(1-\varepsilon_1) - \log 2\pi - \frac{1}{2}\log\left(1 - x^{-2}\right) \le \psi(x) \le x(1+\varepsilon_2) - \log 2\pi - \frac{1}{2}\log\left(1 - x^{-2}\right),$$

where

$$\varepsilon_1 = \frac{K_m(x, -x\delta)}{(-x)^{m+1}\delta^m} + \frac{m\delta}{2},\ \ \varepsilon_2 = \frac{K_m(x, x\delta)}{x^{m+1}\delta^m} + \frac{m\delta}{2},$$

and $0 < \delta < (x-1)/xm$.

We note $\psi(x)$ can be related to the non-trivial zeros of $\zeta(s)$. More precisely we have

$$\int_1^x \psi(u)du = \frac{x^2}{2} - \sum_{\rho} \frac{x^{\rho+1}}{\rho(\rho+1)} - x\frac{\zeta'(0)}{\zeta(0)} + \frac{\zeta'(-1)}{\zeta(-1)} - \sum_{r=1}^{\infty} \frac{x^{1-2r}}{2r(2r-1)},$$

(see [11, p. 73]). This allows us to derive an expression for $K_m(x,h)$ in terms of the zeros of $\zeta(s)$. We have

$$K_m(x,h) = \sum_{\rho} \frac{1}{\rho(\rho+1)\cdots(\rho+m)} \left(\sum_{j=0}^{m} (-1)^{j+m+1} \binom{m}{j} (x+jh)^{\rho+m} \right).$$

Next by employing the properties of the zeros of $\zeta(s)$, we shall bound $K_m(x,h)$ by

$$K = \sum_{\rho} x^{\beta-1}/|\gamma|^{m+1}.$$

Finally we will prove that

$$\varepsilon_1, \varepsilon_2 < \frac{\delta}{2}\left(\left(\frac{(1+\delta)^{m+1}+1}{2} \right)^m + m \right),$$

provided that we have

$$K \leq (\delta/2)^{m+1}.$$

This will complete the proof of the theorem.

In Chapter 3 we derive upper and lower bounds for $\theta(x)$.

Upper bounds for $\theta(x)$ can be obtained by means of the lower bound for $\psi(x)-\theta(x)$ together with an estimation table for $\psi(x)$. More precisely, we proceed by considering intervals $(0, e^{b_1}]$, $[e^{b_1}, e^{b_2}]$, and $[e^{b_2}, \infty)$, where b_1 and b_2 are fixed positive constants, and do the following.

- We numerically check that

$$\theta(x) < x, \text{ for } x \in (0, e^{b_1}].$$

- When $x \in [e^{b_1}, e^{b_2}]$, we estimate $\theta(x)$ by means of a lower bound for $\psi(x) - \theta(x)$. We give here some of the known inequalities for $\psi(x)-\theta(x)$ that can be employed to obtain such lower bounds.

$$\psi(x) - \theta(x) \geq \psi(x^{1/2}) + \theta(x^{1/3}),$$

$$\psi(x) - \theta(x) \geq \psi(x^{1/2}) + \psi(x^{1/3}) + \psi(x^{1/5}) - \psi(x^{6}),$$

  and

$$\psi(x) - \theta(x) \geq \psi(x^{1/2}) + \psi(x^{1/3}) + \psi(x^{1/5}) - \theta(x^{1/6}) - \psi(x^{1/30}).$$

- Finally when $x \in [e^{b_2}, \infty)$, we bound $\theta(x)$ by using the trivial inequality $\theta(x) \leq \psi(x)$.

We then consider the maximum of the upper bounds for $\theta(x)$ derived over the intervals $(0, e^{b_1}]$, $[e^{b_1}, e^{b_2}]$, $[e^{b_2}, \infty)$ as the upper bound of $\theta(x)$ for $x > 0$.

To explain our results more specifically, let

$$A^{-}(b)x < \psi(x) < A^{+}(b)x, \text{ for } x \geq e^{b}.$$

In other words $A^{+}(b)$ and $A^{-}(b)$ are upper and lower bounds for $\psi(x)/x$ on the interval $[e^{b}, \infty)$.

The following theorem shows that how one can improve a given upper bound for $\theta(x)$.

Theorem 3.12. *Let* b_1 *and* b_2 *be positive constants such that* $0 < b_1 \le 27.4$ *and* $b_1 < b_2$. *Let*

$$c_1 = \max_{x \in \left[e^{b_1}, e^{b_2}\right]} \left\{ A^{+}(b_1) - \frac{A^{-}(b_1/2)}{x^{1/2}} - \frac{A^{-}(b_1/3)}{x^{2/3}} - \frac{A^{-}(b_1/5)}{x^{4/5}} + \frac{\tilde{c}_0}{x^{5/6}} + \frac{A^{+}(b_1/30)}{x^{29/30}} \right\},$$

where $\tilde{c}_0$ *is an upper bound for* $\theta(x)/x$ *when* $x > 0$. *Then*

$$\theta(x) < c_0 x, \text{ for } x > 0,$$

where $c_0 = \max\{c_1, A^{+}(b_2)\}$.

The above will allow us to establish an upper bound for $\theta(x)$ valid for all $x > 0$ which surpasses [7, Proposition 5.1].

Example 3.13. $\theta(x) < (1 + 2.7571593586 \times 10^{-5})x, \text{ for } x > 0.$

We point out that in the process of establishing upper bounds for $\theta(x)$, we need lower bounds for $\psi(x) - \theta(x)$. It is known that

$$\psi(x) - \theta(x) > \sqrt{x}, \ 121 \le x \le 10^{16} \simeq e^{36.8},$$

(see [18, p. 73]). In Chapter 3, we will extend the above range significantly to obtain

Theorem 3.25. $\psi(x) - \theta(x) > \sqrt{x}, \ 121 \le x \le e^{145.5}.$

We next employ upper bounds for $\psi(x) - \theta(x)$ to generate several lower bounds tables for $\theta(x)$ over different ranges.

Finally, in Chapter 3, we present some techniques which allow us to derive inequalities in the form

$$\psi(x) - \theta(x) < c_2 x^{1/2} + c_3 x^{1/3},$$

for $c_2, c_3 > 0$. We will prove the following theorem.

Theorem 3.32. *Suppose that for* $x \ge e^b$ *there is a positive constant* ε *such that* $\varepsilon > A^{+}(b/2) - 1 > 0$ *and*

$$e^b \ge \left(\frac{4A^{+}(b/5)}{5\left(A^{+}(b/2) - 1 - \varepsilon\right)} \right)^{\frac{10}{3}}.$$

We let

$$h(x) = \left(A^+(b/2) - 1 - \varepsilon\right) x^{1/6} + A^+(b/3) + A^+(b/5)x^{-2/15}.$$

Then

$$\psi(x) - \theta(x) < (1+\varepsilon)x^{1/2} + h(e^b)x^{1/3}, \text{ for } x \geq e^b.$$

In Chapter 4 we demonstrate some techniques which give sharp estimates for $\pi(x)$ over different ranges. Note that by the Prime Number Theorem with the remainder we have

$$\pi(x) = \mathrm{li}(x) + O(xe^{-c\sqrt{\log x}}),$$

for some constant c. It follows from [13, p. 55] that if $2 \leq x < 10^{14}$ then $\pi(x) < \mathrm{li}(x)$. Therefore we will be able to establish upper bounds for $\pi(x)$ for $x \leq 10^{14}$ by establishing upper bounds for $\mathrm{li}(x)$. For example we prove the following inequality.

Corollary 4.5. $\pi(x) < \frac{x}{\log x}\left(1 + \frac{1.2762}{\log x}\right)$, for $51022 \leq x \leq 10^{14}$.

Next we assume that we have positive constants β and η_k and a natural number k such that

$$|\theta(x) - x| < \eta_k \frac{x}{\log^k x}, \text{ for } x \geq \beta.$$

We let $x_0 \geq \beta$, and introduce

$$J(x, \eta_k) = \pi(x_0) - \frac{\theta(x_0)}{\log x_0} + \frac{x}{\log x} + \eta_k \frac{x}{\log^{k+1} x} + \int_{x_0}^{x} \left(\frac{1}{\log^2 y} + \frac{\eta_k}{\log^{k+2} y}\right) dy,$$

We shall verify

$$J(x, -\eta_k) < \pi(x) < J(x, \eta_k),$$

for $k \geq 1$ and $x \geq x_0$. This enables us to estimate $\pi(x)$ by employing $J(x, \pm\eta_k)$.

Here we give a sample of our results on estimation of $\pi(x)$.

In Chapter 4, we prove that for $x \geq x_0$, there exist positive numbers d_1, d_2, and d_3 such that

$$\pi(x) < \frac{x}{\log x}\left(1 + \frac{d_1}{\log x}\right),$$

$$\pi(x) < \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{d_2}{\log^2 x}\right),$$

$$\pi(x) < \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2}{\log^2 x} + \frac{d_3}{\log^3 x}\right).$$

Admissible values for d_1, d_2, and d_3 are given in Table 1.1.

Table 1.1:

x_0	d_1	d_2	d_3
10^{11}	1.0902	2.296	7.9724
10^{12}	1.0830	2.267	7.8510
10^{15}	1.0640	2.208	7.5976

Also we will prove that for $x \geq 10^{11}$,

$$\pi(x) > \frac{x}{\log x}\left(1 + \frac{0.9999}{\log x}\right),$$

$$\pi(x) > \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{1.9899}{\log^2 x}\right),$$

and for $x \geq 10^{10}$,

$$\pi(x) > \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2}{\log^2 x} + \frac{5.2199}{\log^3 x}\right).$$

We also establish another form of bounds for $\pi(x)$. As a sample of our results we will show that

$$\frac{x}{\log x - 0.9989} < \pi(x) < \frac{x}{\log x - 1.0520}, \text{ for } x \geq 10^{10}.$$

Chapter 2

Bounds for $\psi(x)$

2.1 Elementary estimates of $\psi(x)$

Around 1850 Chebyshev introduced the function $\psi(x)$ and obtained numerical bounds for it. More precisely he proved that for every $\epsilon > 0$ and sufficiently large x,

$$(\nu - \epsilon)x \leq \psi(x) \leq (\frac{6}{5} + \epsilon)\nu x,$$

where

$$\nu = \log\left(\frac{2^{1/2}3^{1/3}5^{1/5}}{30^{1/30}}\right) = 0.92129\cdots . \tag{2.1}$$

Note that $(6/5)\nu = 1.105\cdots$. Over the years many authors attempted to improve the Chebyshev estimate and also make the above estimation effective. For example Erdős [8] proved that

$$\frac{\psi(x)}{x} < \log 4 = 1.38629\cdots, \text{ for } x > 0.$$

This bound has been improved later by Grimson and Hanson [9] who obtained

$$\frac{\psi(x)}{x} < \log 3 = 1.09861\cdots, \text{ for } x > 0.$$

Another improvement was given by Deshouillers [6]. He verified that

$$\frac{\psi(x)}{x} < 1.07715, \text{ for } x > 0,$$

and

$$0.92129 < \frac{\psi(x)}{x}, \text{ for } x \geq 59.$$

All the above estimates are obtained by elementary means and without employing the analytic properties of the Riemann zeta function. Later in this chapter we study the analytic estimates of $\psi(x)$, however before doing this we give an explicit version of Chebyshev's theorem as given by Landau in [14, Vol. 1, p.88].

Recall that by Lemma (1.3) we have

$$T(x) = \sum_{n \leq x} \log n = \sum_{n=1}^{\infty} \psi\left(\frac{x}{n}\right). \tag{2.2}$$

Using this we verify the following.

Lemma 2.1. *Let $x > 0$ and*

$$\alpha(x) = T(x) - T\left(\frac{x}{2}\right) - T\left(\frac{x}{3}\right) - T\left(\frac{x}{5}\right) + T\left(\frac{x}{30}\right).$$

Then

$$\nu x - 5(\log x + 1) \leq \alpha(x) \leq \nu x + 5(\log x + 1). \tag{2.3}$$

Proof. By partial summation formula [11, Theorem A, p.10] we have

$$T(x) = \sum_{n \leq x} \log n = [x] \log x - \int_1^x \frac{[t]}{t} dt,$$

where $[x]$ is the greatest integer less than or equal to x. Writing $[x] = x - \{x\}$, we have

$$T(x) = (x - \{x\}) \log x - \int_1^x \frac{t - \{t\}}{t} dt.$$

By splitting the integral at $x = 2$ we find that

$$T(x) = x \log x - x + U(x), \tag{2.4}$$

where

$$U(x) = \int_2^x \frac{\{t\}}{t} dt - \{x\} \log x - \log 2 + 2.$$

By using the fact that $0 \le \{x\} < 1$ we observe that

$$|U(x)| \le \log x + 1. \tag{2.5}$$

We combine (2.4) and (2.5) to obtain

$$|T(x) - x\log x + x| \le \log x + 1.$$

From the definition of $\alpha(x)$ and the above inequality we have

$$\begin{aligned}\left|\alpha(x) - x\log x + \frac{x}{2}\log\frac{x}{2} + \frac{x}{3}\log\frac{x}{3} + \frac{x}{5}\log\frac{x}{5} - \frac{x}{30}\log\frac{x}{30} + x - \frac{x}{2} - \frac{x}{3} - \frac{x}{5} + \frac{x}{30}\right| \\ \le \log x + 1 + \log\frac{x}{2} + 1 + \log\frac{x}{3} + 1 + \log\frac{x}{5} + 1 + \log\frac{x}{30} + 1 = 5(\log x + 1).\end{aligned}$$

Recognizing

$$\left|\alpha(x) - x\log\left(\frac{2^{1/2}3^{1/3}5^{1/5}}{30^{1/30}}\right)\right|,$$

in the left-hand side of the last inequality implies the proof.

□

In the next lemma we establish upper and lower bound for $\psi(x)$ in terms of $\alpha(x)$.

Lemma 2.2. *For $x \ge 2$ we have*

$$\alpha(x) < \psi(x) < \psi\left(\frac{x}{6}\right) + \alpha(x). \tag{2.6}$$

Proof. Using (2.2) we find that

$$\begin{aligned}\alpha(x) &= \sum_{n=1}^{\infty}\psi\left(\frac{x}{n}\right) - \sum_{n=1}^{\infty}\psi\left(\frac{x}{2n}\right) - \sum_{n=1}^{\infty}\psi\left(\frac{x}{3n}\right) - \sum_{n=1}^{\infty}\psi\left(\frac{x}{5n}\right) + \sum_{n=1}^{\infty}\psi\left(\frac{x}{30n}\right) \\ &= \psi(x) - \psi\left(\frac{x}{6}\right) + \psi\left(\frac{x}{7}\right) - \psi\left(\frac{x}{10}\right) + \psi\left(\frac{x}{11}\right) - \psi\left(\frac{x}{12}\right) + \psi\left(\frac{x}{13}\right) \\ &\quad -\psi\left(\frac{x}{15}\right) + \psi\left(\frac{x}{17}\right) - \psi\left(\frac{x}{18}\right) + \psi\left(\frac{x}{19}\right) - \psi\left(\frac{x}{20}\right) + \psi\left(\frac{x}{23}\right) \\ &\quad -\psi\left(\frac{x}{24}\right) + \psi\left(\frac{x}{29}\right) - \psi\left(\frac{x}{30}\right) + \cdots .\end{aligned}$$

We claim that

$$\alpha(x) = \sum_{n=1}^{\infty} A_n \psi\left(\frac{x}{n}\right), \tag{2.7}$$

where

$$A_n = \begin{cases} 1 & \text{if } gcd(n,30) = 1, \\ -1 & \text{if } gcd(n,30) \text{ has at least two distinct prime divisors,} \\ 0 & \text{otherwise.} \end{cases}$$

In order to verify this, we split integers in types:

(1) $\gcd(n, 30) = 1$ which gives $A_n = 1 - 0 - 0 - 0 + 0 = 1$.

(2) Only one of 2, 3 and 5 divides n. This gives $A_n = 1 - 1 - 0 - 0 + 0 = 0$.

(3) Only two of 2, 3 and 5 divides n. Hence we have $A_n = 1 - 1 - 1 - 0 + 0 = -1$.

(4) $30 \mid n$ which gives $A_n = 1 - 1 - 1 - 1 + 1 = -1$.

We now let $c_0 = 1 < c_1 < c_2 < \cdots$ be the sequence of all integers n such that $A_n \neq 0$. We observe that $A_{c_n} = (-1)^n$. Therefore (2.7) can be written as follows.

$$\alpha(x) = \sum_{n=1}^{\infty} (-1)^n \psi\left(\frac{x}{c_n}\right).$$

It follows that

$$\psi(x) - \psi\left(\frac{x}{6}\right) = \psi\left(\frac{x}{c_0}\right) - \psi\left(\frac{x}{c_1}\right) < \alpha(x) < \psi\left(\frac{x}{c_0}\right) = \psi(x),$$

from which the proof follows. □

Theorem 2.3. *For $x \geq 2$ we have*

$$\nu x - 5(\log x + 1) < \psi(x) < \frac{6}{5}\nu x + (3\log x + 5)(\log x + 1), \tag{2.8}$$

where ν being given by (2.1).

Proof. We begin with the right-hand side of (2.8). We combine (2.3) and (2.6) to deduce

$$\psi(x) - \psi\left(\frac{x}{6}\right) < \nu x + 5(\log x + 1).$$

Hence for all $n \in \mathbb{N}$

$$\begin{aligned} \psi\left(\frac{x}{6^n}\right) - \psi\left(\frac{x}{6^{n+1}}\right) &\leq \nu \frac{x}{6^n} + 5\left(\log \frac{x}{6^n} + 1\right) \\ &\leq \nu \frac{x}{6^n} + 5(\log x + 1). \end{aligned}$$

Summing on $n = 0, 1, \cdots, [\log x/\log 6]$, the sum on the left-hand side gives a telescoping series and we obtain

$$\begin{aligned}
\psi(x) &= \sum_{n=0}^{\left[\frac{\log x}{\log 6}\right]} \left(\psi\left(\frac{x}{6^n}\right) - \psi\left(\frac{x}{6^{n+1}}\right)\right) \\
&\le \nu x \sum_{n=0}^{\left[\frac{\log x}{\log 6}\right]} \frac{1}{6^n} + 5\left(\left[\frac{\log x}{\log 6}\right] + 1\right)(\log x + 1) \\
&< \frac{6}{5}\nu x + \left(\frac{5}{\log 6}\log x + 5\right)(\log x + 1).
\end{aligned}$$

We obtain the announced upper bound for $\psi(x)$ by noting that $5/\log 6 < 3$.

Now (2.3) combined with (2.6) gives the left-hand side of (2.8). This completes the proof. □

Note that a careful reading of the above proof allows us to tighten the upper bound to

$$1.2\nu x + 2.791 \log^2 x + 7.791 \log x + 5.$$

2.2 Analytic estimates of $\psi(x)$

As we saw in the previous section, the elementary method of Chebyshev will establish the explicit inequality

$$(\nu - \epsilon)x \le \psi(x) \le (\frac{6}{5}\nu + \epsilon)x,$$

for $\epsilon > 0$ and large values of x. From the Prime Number Theorem we know that

$$(1 - \epsilon)x < \psi(x) < (1 + \epsilon)x,$$

for large values of x. To establish the latter type inequality one needs to find explicit estimates for $\psi(x) - x$, the so called the error term of the Prime Number Theorem. Rosser [17] was the first who considered this problem. Rosser used numerical verification of the Riemann hypothesis together with an explicit zero-free region in study of the error term and developed an analytical method which allowed him to derive explicit estimates for functions involving primes. Rosser's estimates were more precise than the ones which have been established by elementry methods. His results were later improved by Rosser and Schoenfeld [18], [19] and Schoenfeld [20].

Among many other results, Rosser and Schoenfeld [18] proved that $\psi(x)/x$ takes its maximum at $x = 113$. More precisely

$$\psi(x) < 1.03883x, \text{ for } x > 0.$$

Explicit upper and lower bounds for $\psi(x)$ can be obtained by means of Theorem 21 of [17]. In this chapter we will give a detailed proof of this theorem. We start by setting up our notations.

We define

$$\phi(x) = \psi(x) - x + \log 2\pi + \frac{1}{2}\log\left(1 - 1/x^2\right).$$

$\phi(x)$ can be considered as the error term in the Prime Number Theorem. In Theorem 2.4, we give an upper and lower bounds for this error term.

Let m be a positive integer and x and h be positive reals. For $x > 1$ and $x+mh > 1$, we define the multiple integral

$$K_m(x,h) = \int_0^h dy_1 \int_0^h dy_2 \cdots \int_0^h \phi\left(x + y_1 + y_2 + \cdots + y_m\right) dy_m,$$

and

$$f_{m,n,a}(x,h,z) = \frac{K_m(x,h)}{h^n} + \frac{1}{2}nh^a - zh^{a-1}. \tag{2.9}$$

Since for $h > 0$, $y_i \in [0,h]$, and so we have $x+y_1+\cdots+y_m > x$. Thus $\phi(x+y_1+\cdots+y_m)$ exists provided that $x > 1$. Also for $h < 0$, $y_i \in [h,0]$ we have $x + y_1 + \cdots + y_m > x + mh$. Hence $\phi(x + mh)$ exists provided that $x + mh > 1$. So the above definitions are well defined.

Next we review some facts about the zeros of the Riemann zeta function. It is known that $\zeta(s)$ has zeros at all negative even numbers (see [11, p. 49]). This zeros are called the "trivial zeros" of $\zeta(s)$.

It is known that all the other zeros of $\zeta(s)$ are in the strip $0 < \mathrm{Re}(s) < 1$ (see [11, p. 58]). We denote the non-trivial zeros of $\zeta(s)$ by $\rho = \beta + i\gamma$, where $0 < \beta < 1$ and $\gamma \in \mathbb{R}$.

Let $N(T)$ be the number of non-trivial zeros of $\zeta(s)$ such that $0 < \gamma \leq T$.

Suppose that $A > 0$ is such that

$$\text{for } 0 < \gamma \leq A \text{ we have } \beta = \frac{1}{2}. \tag{2.10}$$

Also assume that $r > 0$ is such that for $\gamma > A$ we have

$$\beta < 1 - \frac{1}{r \log \gamma}. \tag{2.11}$$

We define

$$F(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8},$$

and

$$R(T) = d_1 \log T + d_2 \log \log T + d_3,$$

and assume that

$$|N(T) - F(T)| < R(T), \tag{2.12}$$

for $T \geq 2$ and real numbers $d_1,\ d_2$, and d_3. Moreover suppose that $F(A) \leq N(A)$. We also define $c_1, \cdots, c_6$ as follows.

$$c_1 = r(\log A)^2, \tag{2.13}$$

$$c_2 = r \log A, \tag{2.14}$$

$$c_3 = R(A), \tag{2.15}$$

$$c_4 = \frac{1}{2\pi} + \frac{d_1}{A \log \frac{A}{2\pi}} + \frac{d_2}{A(\log A)(\log \frac{A}{2\pi})}, \tag{2.16}$$

$$c_5 = \frac{1}{\log(A/2\pi)}, \tag{2.17}$$

and

$$c_6 = \log(A/2\pi). \tag{2.18}$$

We will prove the following theorem.

Theorem 2.4. *Let $m \in \mathbb{N}$ and $a > 0$ such that*

$$\log a < \frac{c_1 m^2}{m + c_5}.$$

Suppose that

$$\sum_{\rho} \frac{1}{|\gamma|^{m+1}} \leq k,$$

$$\delta = 2\left(a^{-1/2}k + \frac{c_3}{A^{m+1}a^{1/c_2}} + \frac{c_4(1+mc_6)}{\left(1 - \frac{(m+c_5)\log a}{c_1 m^2}\right) m^2 A^m a^{1/c_2}}\right)^{1/(m+1)},$$

and

$$\varepsilon = \frac{\delta}{2}\left(\left(\frac{(1+\delta)^{m+1}+1}{2}\right)^m + m\right).$$

If $1 + m\delta a < a$ *we have*

$$x(1-\varepsilon) - \log 2\pi < \psi(x) < x(1+\varepsilon) - \frac{1}{2}\log\left(1 - 1/x^2\right), \tag{2.19}$$

for $x \geq a$.

Before presenting the proof of this theorem, we give an overview of the three main steps of the argument.

Step 1: We first find a lower and upper bound for the error term of the Prime Number Theorem $\psi(x) - x$, in terms of $K_m(x,h)$. This is done in Proposition 2.11 by the following inequalities.

$$-\left(\frac{K_m(x,-x\delta)}{(-x)^{m+1}\delta^m} + \frac{m\delta}{2}\right)x - \log 2\pi - \frac{1}{2}\log\left(1 - \frac{1}{x^2}\right) \leq \psi(x) - x,$$

and

$$\psi(x) - x \leq \left(\frac{K_m(x,x\delta)}{x^{m+1}\delta^m} + \frac{m\delta}{2}\right)x - \log 2\pi - \frac{1}{2}\log\left(1 - \frac{1}{x^2}\right),$$

where $0 < \delta < (x-1)/xm$.

Step 2: In the next step we will establish the following explicit bound for $K_m(x,h)$,

$$|K_m(x,\pm x\delta)| < x^{m+1}\left((1+\delta)^{m+1}+1\right)^m K,$$

where

$$K = \sum_{\rho} \frac{x^{\beta-1}}{|\gamma|^{m+1}}.$$

Combining this with Proposition 2.11, in Proposition 2.15 we will derive lower and upper bounds for $\psi(x)$ in terms of the sum K.

Step 3: Next we bound K by splitting the sum between the zeros on the left of the 1/2-line and those on the left of the zero-free region:

$$K \leq x^{\frac{-1}{2}} \sum_{\rho} \frac{1}{|\gamma|^{m+1}} + \sum_{A<\gamma} f(\gamma),$$

where

$$f(\gamma) = \frac{x^{\frac{-1}{r \log \gamma}}}{\gamma^{m+1}}.$$

This is done in Lemma 2.16 by means of (2.11) and properties of the zeros of the zeta function. To find an effective upper bound for K it suffices to explicitly bound $\sum_{A<\gamma} f(\gamma)$. This has been done in Lemma 2.17 by applying partial summation formula on $\sum_{A<\gamma} f(\gamma)$. This lemma gives an upper bound for $\sum_{A<\gamma} f(\gamma)$. It is shown that

$$\sum_{A<\gamma} f(\gamma) < \frac{c_3}{A^{m+1} x^{1/c_2}} + c_4 \int_A^{\infty} f(y) \log \frac{y}{2\pi} dy.$$

Finally Lemma 2.18 furnishes an explicit upper bound for $\int_A^{\infty} f(y) \log(y/2\pi) dy$. Combining all these will establish the desired upper bound for K.

In the next three sections we will describe in details each step.

2.2.1 Comparing $\psi(x) - x$ with its average

In this section we compare $\psi(x) - x$ to its average $K_m(x, h)$. Proposition 2.11 is the main result of this section and is obtained by using Lemma 2.7 and 2.9 . These two lemmas give upper and lower bounds for $\phi(x)$ depending on "m-average" $K_m(x, h)$.

For proving them we need Lemma 2.5 and 2.6. Lemma 2.5 establishes a recurrence relation for $f_{m,n,a}$.

Lemma 2.5. *We have*

$$\int_0^h f_{m,n,a}(x, h, y_1 + y_2 + \cdots + y_n)\, dy_n = f_{m,n-1,a+1}(x, h, y_1 + y_2 + \cdots + y_{n-1}).$$

Proof. We use (2.9) to compute the integral

$$\int_0^h f_{m,n,a}\left(x,h,y_1+y_2+\cdots+y_n\right)dy_n$$
$$= \int_0^h \left(\frac{K_m(x,h)}{h^n} + \frac{1}{2}nh^a - (y_1+y_2+\cdots+y_n)h^{a-1}\right)dy_n$$
$$= \frac{K_m(x,h)}{h^{n-1}} + (\frac{n-1}{2})h^{a+1} - \left(y_1+\cdots+y_{n-1}\right)h^a$$
$$= f_{m,n-1,a+1}\left(x,h,y_1+y_2+\cdots+y_{n-1}\right).$$

□

Lemma 2.6. *We have*

$$K_m(x,h) = \int_0^h dy_1 \int_0^h dy_2 \cdots \int_0^h f_{m,n,a}\left(x,h,y_1+y_2+\cdots+y_n\right)dy_n. \tag{2.20}$$

Proof. We prove this lemma by induction on n.

For $n=1$ we use (2.9) and compute the right-hand side of (2.20) to get

$$\int_0^h f_{m,1,a}(x,h,y)dy = \frac{K_m(x,h)h}{h} + \frac{1}{2}h^a.h - h^{a-1}\frac{h^2}{2} = K_m(x,h).$$

We assume

$$K_m(x,h) = \int_0^h dy_1 \int_0^h dy_2 \cdots \int_0^h f_{m,n,a}\left(x,h,y_1+y_2+\cdots+y_n\right)dy_n,$$

and together with Lemma 2.5 we obtain

$$\int_0^h \cdots \int_0^h \left(\int_0^h f_{m,n+1,a}\left(x,h,y_1+\cdots+y_{n+1}\right)dy_{n+1}\right)dy_1\cdots dy_n$$
$$= \int_0^h \cdots \int_0^h f_{m,n,a+1}\left(x,h,y_1+\cdots+y_n\right)dy_1\cdots dy_n = K_m(x,h).$$

□

We now define

$$f_m(x,h,z) = f_{m,m,1}(x,h,z) = \frac{K_m(x,h)}{h^m} + \frac{mh}{2} - z.$$

Lemma 2.7. *If $h > 0$ then there exists $z \in [0, mh]$ such that*

$$\phi(x+z) \leq f_m(x,h,z). \tag{2.21}$$

Proof. Assume to the contrary that, for all $z \in [0, mh]$, we have

$$\phi(x+z) > f_m(x,h,z).$$

Then

$$\int_0^h \cdots \int_0^h \phi\left(x + y_1 + y_2 + \cdots + y_m\right) dy_1 dy_2 \cdots dy_m$$
$$> \int_0^h \cdots \int_0^h f_m\left(x, h, y_1 + \cdots + y_m\right) dy_1 dy_2 \cdots dy_m.$$

We recognize $K_m(x,h)$ in the left-hand side of the above inequality. However by Lemma 2.6 the right-hand side of it equals $K_m(x,h)$. This is a contradiction. □

Remark 2.8. *Formula* (2.21) *states that there exists $z_1 \in [0, mh]$ such that*

$$\phi(x+z_1) + z_1 \leq \xi_1,$$

where

$$\xi_1 = \frac{1}{h^m} \int_0^h \cdots \int_0^h \left(\phi\left(x + y_1 + y_2 + \cdots + y_m\right) + \left(y_1 + y_2 + \cdots + y_m\right)\right) dy_1 dy_2 \cdots dy_m.$$

Observe that

$$\begin{aligned} \frac{1}{h^m} \int_0^h \cdots \int_0^h \left(y_1 + y_2 + \cdots + y_m\right) dy_1 dy_2 \cdots dy_m &= \frac{m}{h^m} \int_0^h \cdots \int_0^h y_1 dy_1 \cdots dy_m \\ &= \frac{mh}{2}. \end{aligned}$$

Lemma 2.9. *If $h < 0$ then there exists $z \in [mh, 0]$ such that*

$$f_m(x,h,z) \leq \phi(x+z). \tag{2.22}$$

Proof. Assume to the contrary that for all $z \in [mh, 0]$ we have

$$\phi(x+z) < f_m(x,h,z).$$

Then

$$\int_h^0 \cdots \int_h^0 \phi\left(x + y_1 + y_2 + \cdots + y_m\right) dy_1 dy_2 \cdots dy_m$$
$$< \int_h^0 \cdots \int_h^0 f_m\left(x, h, y_1 + \cdots + y_m\right) dy_1 dy_2 \cdots dy_m.$$

By definition the left-hand side of the above inequality is $(-1)^m K_m(x,h)$. However by Lemma 2.6 the right-hand side of that equals $(-1)^m K_m(x,h)$. Thus we get a contradiction. □

Remark 2.10. *Formula* (2.22) *states that there exists* $z_2 \in [0, mh]$ *such that*

$$\phi(x - z_2) - z_2 \geq \xi_2,$$

where

$$\xi_2 = \frac{1}{h^m} \int_0^h \cdots \int_0^h \left(\phi\left(x + y_1 + y_2 + \cdots + y_m\right) - \left(y_1 + y_2 + \cdots + y_m\right)\right) dy_1 dy_2 \cdots dy_m.$$

We are now ready to give a bound for $\psi(x) - x$ in terms of the K_m's.

Proposition 2.11. *If* $0 < \delta < (x-1)/xm$,

$$\varepsilon_1 = \frac{K_m(x, -x\delta)}{(-x)^{m+1}\delta^m} + \frac{m\delta}{2}, \text{ and } \varepsilon_2 = \frac{K_m(x, x\delta)}{x^{m+1}\delta^m} + \frac{m\delta}{2}, \tag{2.23}$$

then

$$x(1 - \varepsilon_1) - \log 2\pi - \frac{1}{2}\log\left(1 - \frac{1}{x^2}\right) \leq \psi(x) \leq x(1 + \varepsilon_2) - \log 2\pi - \frac{1}{2}\log\left(1 - \frac{1}{x^2}\right). \tag{2.24}$$

Proof. To prove the right-hand side of (2.24) we use the definitions of $\phi(x+z)$ and $f_m(x,h,z)$. Lemma 2.7 yields that there exists $z \in [0, mh]$ such that

$$\psi(x+z) - (x+z) + \log 2\pi + \frac{1}{2}\log\left(1 - \frac{1}{(x+z)^2}\right) \leq \frac{K_m(x,h)}{h^m} + \frac{mh}{2} - z. \tag{2.25}$$

For $z \geq 0$, we have the inequalities

$$\frac{-1}{2}\log\left(1 - \frac{1}{(x+z)^2}\right) \leq \frac{-1}{2}\log\left(1 - \frac{1}{x^2}\right), \tag{2.26}$$

and $\psi(x) \leq \psi(x+z)$. An application of this fact and (2.26) in (2.25) and replacing h by $x\delta$ in (2.25) yield

$$\psi(x) - (x+z) + \log 2\pi + \frac{1}{2}\log\left(1 - \frac{1}{x^2}\right) \leq \frac{K_m(x, x\delta)}{(x\delta)^m} + \frac{mx\delta}{2} - z.$$

By simplifying the above inequality we get

$$\psi(x) \leq x\left(1 + \frac{K_m(x, x\delta)}{x^{m+1}\delta^m} + \frac{m\delta}{2}\right) - \log 2\pi - \frac{1}{2}\log\left(1 - \frac{1}{x^2}\right).$$

This proves the inequality in the right-hand side of (2.24). For proving the inequality in the left-hand side of (2.24), note that by Lemma 2.9 there exists $z \in [mh, 0]$ such that

$$\psi(x+z) - (x+z) + \log 2\pi + \frac{1}{2}\log\left(1 - \frac{1}{(x+z)^2}\right) \geq \frac{K_m(x, h)}{h^m} + \frac{mh}{2} - z. \quad (2.27)$$

Note that for $z \leq 0$ we have

$$\frac{-1}{2}\log\left(1 - \frac{1}{(x+z)^2}\right) \geq \frac{-1}{2}\log\left(1 - \frac{1}{x^2}\right),$$

and $\psi(x) \geq \psi(x+z)$. We replace h by $-x\delta$ in (2.27) to obtain

$$\psi(x) - (x+z) + \log 2\pi + \frac{1}{2}\log\left(1 - \frac{1}{x^2}\right) \geq \frac{K_m(x, -x\delta)}{(-x\delta)^m} + \frac{-mx\delta}{2} - z.$$

Simplifying the above inequality yields

$$x\left(1 - \left(\frac{K_m(x, -x\delta)}{(-x)^{m+1}\delta^m} + \frac{m\delta}{2}\right)\right) - \log 2\pi - \frac{1}{2}\log\left(1 - \frac{1}{x^2}\right) \leq \psi(x),$$

which is the desired result. □

2.2.2 Relating $\psi(x) - x$ and the zeros of the zeta function

In this section we find an explicit bound for $K_m(x, h)$ in terms of the zeros of the zeta function. This is done by using Proposition 2.13 which provides an explicit formula. Then Proposition 2.14 gives a bound for $|K_m(x, \pm x\delta)|$ in terms of $K = \sum_\rho x^{\beta-1}/|\gamma|^{m+1}$. The last result of this section is Proposition 2.15 which gives bounds for $\psi(x)$ in terms of δ.

Proposition 2.12. *We have*

$$\int_0^h \phi(x+z)dz = \sum_\rho \frac{1}{\rho(\rho+1)} \left(x^{\rho+1} - (x+h)^{\rho+1}\right). \tag{2.28}$$

Proof. Using the definition of $\phi(x)$ we have

$$\phi(x+z) = \psi(x+z) - (x+z) + \log 2\pi + \frac{1}{2}\log\left(1 - \frac{1}{(x+z)^2}\right).$$

We split $\phi(x+z)$ into two parts and calculate the integral from 0 to h of them separately. By change of the variable we have

$$\int_0^h \psi(x+z)dz = \int_x^{x+h} \psi(u)du = \int_1^{x+h} \psi(u)du - \int_1^x \psi(u)du. \tag{2.29}$$

We now employ the classical explicit formula [11, page 73],

$$\int_1^x \psi(u)du = \frac{x^2}{2} - \sum_\rho \frac{x^{\rho+1}}{\rho(\rho+1)} - x\frac{\zeta'(0)}{\zeta(0)} + \frac{\zeta'(-1)}{\zeta(-1)} - \sum_{r=1}^\infty \frac{x^{1-2r}}{2r(2r-1)},$$

and apply it on the right-hand side of (2.29) to get

$$\int_0^h \psi(x+z)dz = \frac{h^2}{2} + hx - \sum_\rho \frac{(x+h)^{\rho+1} - x^{\rho+1}}{\rho(\rho+1)} - h\frac{\zeta'(0)}{\zeta(0)} - \sum_{r=1}^\infty \frac{(x+h)^{1-2r} - x^{1-2r}}{2r(2r-1)}. \tag{2.30}$$

Next we consider

$$\int_x^{x+h} \left(x + z - \log(2\pi) - \frac{1}{2}\log\left(1 - \frac{1}{(x+z)^2}\right)\right) dz.$$

By using the Taylor expansion

$$-\log\left(1 - \frac{1}{(x+z)^2}\right) = \sum_{r=1}^\infty \frac{1}{r(x+z)^{2r}},$$

it follows that

$$\begin{aligned}&\int_0^h \left(x + z - \log(2\pi) + \frac{1}{2}\sum_{r=1}^\infty \frac{1}{r(x+z)^{2r}}\right) dz \\ &= hx + \frac{h^2}{2} - h\log 2\pi - \frac{1}{2}\sum_{r=1}^\infty \frac{(x+h)^{1-2r} - x^{1-2r}}{r(2r-1)}.\end{aligned} \tag{2.31}$$

From [14, page 317] we have

$$\frac{\zeta'(0)}{\zeta(0)} = \log 2\pi.$$

Having this we now subtract (2.30) from (2.31) to obtain (2.28). □

Proposition 2.13. *We have*

$$K_m(x,h) = \sum_\rho \frac{1}{\rho(\rho+1)\cdots(\rho+m)} \left(\sum_{j=0}^{m} (-1)^{j+m+1} \binom{m}{j} (x+jh)^{\rho+m} \right). \tag{2.32}$$

Proof. The proof follows by induction on m.

By using the definition of $K_m(x,h)$ and Proposition 2.12 we find that

$$K_1(x,h) = \int_0^h \phi(x+z)dz = \sum_\rho \frac{1}{\rho(\rho+1)} \left(x^{\rho+1} - (x+h)^{\rho+1}\right)$$
$$= \sum_\rho \frac{1}{\rho(\rho+1)} \left(\sum_{j=0}^{1} (-1)^{j+2} \binom{1}{j} (x+jh)^{\rho+1} \right).$$

We now assume

$$K_m(x,h) = \sum_\rho \frac{1}{\rho(\rho+1)\cdots(\rho+m)} \left(\sum_{j=0}^{m} (-1)^{j+m+1} \binom{m}{j} (x+jh)^{\rho+m} \right). \tag{2.33}$$

We need to prove that

$$K_{m+1}(x,h) = \sum_\rho \frac{1}{\rho(\rho+1)\cdots(\rho+m+1)} \left(\sum_{j=0}^{m+1} (-1)^{j+m+2} \binom{m+1}{j} (x+jh)^{\rho+m+1} \right).$$

By using the definition of $K_{m+1}(x,h)$ we have

$$K_{m+1}(x,h) = \int_0^h \left(\int_0^h \cdots \int_0^h \phi\left(x+y_1+\cdots+y_{m+1}\right) dy_1 \cdots dy_m \right) dy_{m+1}$$
$$= \int_0^h K_m(x+y_{m+1},h) dy_{m+1}.$$

Next we apply (2.33) on the above equality to obtain

$$K_{m+1}(x,h) = \int_0^h \sum_\rho \frac{1}{\rho\cdots(\rho+m)} \sum_{j=0}^{m} \left((-1)^{j+m+1} \binom{m}{j} \left(x+y_{m+1}+jh\right)^{\rho+m} \right) dy_{m+1}.$$

Interchanging the sum and the integral is legitimate since

$$\left|\sum_\rho \frac{1}{\rho\cdots(\rho+m)} \sum_{j=0}^{m} \left((-1)^{j+m+1}\binom{m}{j}(x+y_{m+1}+jh)^{\rho+m}\right) dy_{m+1}\right|$$
$$= \left|\sum_\rho \frac{(x+y_{m+1})^{\rho+m}}{\rho\cdots(\rho+m)} \left(\sum_{j=0}^{m}\binom{m}{j}\left(1\pm\frac{jx\delta}{x+y_{m+1}}\right)^{\rho+m}\right)\right|$$
$$\leq \sum_\rho \frac{(x+h)^{m+1}}{|\gamma|^{m+1}}\left(\sum_{j=0}^{m}\binom{m}{j}(1+j\delta)^{m+1}\right),$$

noting that $\sum_\rho 1/|\gamma|^{m+1}$ is convergent for $m \geq 1$. Thus we obtain

$$K_{m+1}(x,h) = \sum_\rho \frac{1}{\rho\cdots(\rho+m)} \sum_{j=0}^{m}(-1)^{j+m+1}\binom{m}{j}\int_0^h (x+y_{m+1}+jh)^{\rho+m}\, dy_{m+1}.$$

It is now easy to compute the integral:

$$\int_0^h (x+y_{m+1}+jh)^{\rho+m}\, dy_{m+1} = \frac{(x+jh+h)^{\rho+m+1}-(x+jh)^{\rho+m+1}}{\rho+m+1}.$$

Therefore the sum over j equals

$$\sum_{j=0}^{m}(-1)^{j+m+1}\binom{m}{j}\frac{(x+(j+1)h)^{\rho+m+1}-(x+jh)^{\rho+m+1}}{\rho+m+1}.$$

The above can be rewritten as:

$$\frac{1}{\rho+m+1}\sum_{j=0}^{m}(-1)^{j+m+1}\binom{m}{j}(x+(j+1)h)^{\rho+m+1}$$
$$-\frac{1}{\rho+m+1}\sum_{j=0}^{m}(-1)^{j+m+1}\binom{m}{j}(x+jh)^{\rho+m+1}.$$

Changing the variable j to $j-1$ in the first sum of the above formula gives

$$\frac{1}{\rho+m+1}\sum_{j=1}^{m+1}(-1)^{j+m}\binom{m}{j-1}(x+jh)^{\rho+m+1}$$
$$-\frac{1}{\rho+m+1}\sum_{j=0}^{m}(-1)^{j+m+1}\binom{m}{j}(x+jh)^{\rho+m+1}.$$

The last expression can be transformed into

$$\frac{1}{\rho+m+1}\sum_{j=1}^{m}(-1)^{j+m}(x+jh)^{\rho+m+1}\binom{m+1}{j}$$

$$+\frac{1}{\rho+m+1}\left((-1)^{2m+1}(x+(m+1)h)^{\rho+m+1}-(-1)^{m+1}x^{\rho+m+1}\right), \tag{2.34}$$

since

$$\binom{m}{j-1}+\binom{m}{j}=\binom{m+1}{j}, \text{ for } 1\le j\le m.$$

To complete the proof we observe that (2.34) can be rewritten as

$$\frac{1}{(\rho+m+1)}\left(\sum_{j=0}^{m+1}(-1)^{j+m+2}\binom{m+1}{j}(x+jh)^{\rho+m+1}\right).$$

This leads to

$$K_{m+1}(x,h)=\sum_{\rho}\frac{1}{\rho(\rho+1)\cdots(\rho+m+1)}\left(\sum_{j=0}^{m+1}(-1)^{j+m+2}\binom{m+1}{j}(x+jh)^{\rho+m+1}\right).$$

□

Proposition 2.14. *If $\delta\ge 0$ then*

$$|K_m(x,\pm x\delta)|<x^{m+1}\left((1+\delta)^{m+1}+1\right)^m K, \tag{2.35}$$

where

$$K=\sum_{\rho}\frac{x^{\beta-1}}{|\gamma|^{m+1}}.$$

Proof. Replacing h by $\pm x\delta$ in Proposition 2.13 yields

$$K_m(x,\pm x\delta)=\sum_{\rho}\frac{x^{\rho+m}}{\rho(\rho+1)\cdots(\rho+m)}\left(\sum_{j=0}^{m}(-1)^{j+m+1}\binom{m}{j}(1\pm j\delta)^{\rho+m}\right).$$

From $|x^{\rho+m}|=x^{\beta+m}$ and $|\rho|=\sqrt{\beta^2+\gamma^2}\ge|\gamma|$, we find that

$$\left|\frac{x^{\rho+m}}{\rho(\rho+1)\cdots(\rho+m)}\right|<\frac{x^{\beta+m}}{|\gamma|^{m+1}}<\frac{x^{m+1}}{|\gamma|^{m+1}}. \tag{2.36}$$

Therefore by using the fact that $(1 + j\delta) < (1 + \delta)^j$ for $\delta > 0$ one can deduce

$$\left|\sum_{j=0}^{m}(-1)^{j+m+1}\binom{m}{j}(1 \pm j\delta)^{\rho+m}\right| < \sum_{j=0}^{m}\binom{m}{j}\left((1+\delta)^j\right)^{m+1}$$
$$= \sum_{j=0}^{m}\binom{m}{j}\left((1+\delta)^{m+1}\right)^j = \left((1+\delta)^{m+1}+1\right)^m.$$

Thus

$$\left|\sum_{j=0}^{m}(-1)^{j+m+1}\binom{m}{j}(1 + j\delta)^{\rho+m}\right| \leq \left((1+\delta)^{m+1}+1\right)^m. \tag{2.37}$$

The proof follows by putting together (2.32), (2.36), and (2.37). □

Proposition 2.15. *If $\delta \geq 2K^{1/(m+1)}$ and*

$$\epsilon = \frac{\delta}{2}\left(\left(\frac{(1+\delta)^{m+1}+1}{2}\right)^m + m\right),$$

then

$$\varepsilon_1 < \epsilon, \;\; \varepsilon_2 < \epsilon, \tag{2.38}$$

where ε_1 and ε_2 are defined in (2.23).

Proof. It follows from Proposition 2.14 that

$$|\varepsilon_1|, |\varepsilon_2| \leq \frac{\left((1+\delta)^{m+1}+1\right)^m K}{\delta^m} + \frac{m\delta}{2}. \tag{2.39}$$

Since $\delta \geq 2K^{1/(m+1)}$ we have $K < \frac{\delta^{m+1}}{2^{m+1}}$. Applying this upper bound for K, (2.39) is transformed into

$$\varepsilon_1, \varepsilon_2 < \frac{\left((1+\delta)^{m+1}+1\right)^m \delta^{m+1}}{\delta^m 2^{m+1}} + \frac{m\delta}{2} = \frac{\delta}{2}\left(\left(\frac{(1+\delta)^{m+1}+1}{2}\right)^m + m\right),$$

from which the proof follows. □

2.2.3 Explicit bounds for the sums over the zeros of the zeta function

In this section we will derive an upper bound for K by finding an upper bound for $\sum_{A<\gamma} f(\gamma)$ where

$$f(\gamma) = x^{-1/r\log\gamma}/\gamma^{m+1}.$$

Lemma 2.16. *We have*

$$K \le x^{\frac{-1}{2}} \sum_{\rho} \frac{1}{|\gamma|^{m+1}} + \sum_{A<\gamma} f(\gamma). \tag{2.40}$$

Proof. For $\beta \le 1/2$ we have $x^{\beta-1} \le x^{-1/2}$. Hence

$$\sum_{\beta\le\frac{1}{2}} \frac{x^{\beta-1}}{|\gamma|^{m+1}} \le \sum_{\beta\le\frac{1}{2}} \frac{x^{-1/2}}{|\gamma|^{m+1}} \le x^{-1/2} \sum_{\rho} \frac{1}{|\gamma|^{m+1}}. \tag{2.41}$$

If $\beta > 1/2$ it follows from (2.10), that $|\gamma| > A$. We now use (2.11) to deduce

$$x^{\beta-1} < x^{\frac{-1}{r\log\gamma}}.$$

From this, by the fact that both $\beta + i\gamma$ and $\beta - i\gamma$ are zeros of $\zeta(s)$, we find that

$$\sum_{\beta>\frac{1}{2}} \frac{x^{\beta-1}}{|\gamma|^{m+1}} = 2 \sum_{\beta>\frac{1}{2},\gamma>0} \frac{x^{\beta-1}}{|\gamma|^{m+1}} \le 2 \sum_{\beta>\frac{1}{2},\gamma>A} \frac{x^{\frac{-1}{r\log\gamma}}}{\gamma^{m+1}}. \tag{2.42}$$

Since $\beta + i\gamma$ and $1 - \beta + i\gamma = \beta' + i\gamma$ are zeros of $\zeta(s)$, we observe that

$$\begin{aligned} 2 \sum_{\beta>\frac{1}{2},\gamma>A} \frac{x^{\frac{-1}{r\log\gamma}}}{\gamma^{m+1}} &= \sum_{\beta>\frac{1}{2},\gamma>A} \frac{x^{\frac{-1}{r\log\gamma}}}{\gamma^{m+1}} + \sum_{\beta>\frac{1}{2},\gamma>A} \frac{x^{\frac{-1}{r\log\gamma}}}{\gamma^{m+1}} \\ &= \sum_{\beta>\frac{1}{2},\gamma>A} \frac{x^{\frac{-1}{r\log\gamma}}}{\gamma^{m+1}} + \sum_{\beta'<\frac{1}{2},\gamma>A} \frac{x^{\frac{-1}{r\log\gamma}}}{\gamma^{m+1}} \le \sum_{\gamma>A} \frac{x^{\frac{-1}{r\log\gamma}}}{\gamma^{m+1}}. \end{aligned}$$

Hence (2.42) can be transformed into

$$\sum_{\beta>\frac{1}{2}} \frac{x^{\beta-1}}{|\gamma|^{m+1}} \le \sum_{\gamma>A} \frac{x^{\frac{-1}{r\log\gamma}}}{\gamma^{m+1}}. \tag{2.43}$$

The proof is now complete by adding (2.41) to (2.43). □

Next we find an upper bound for $\sum_{A<\gamma} f(\gamma)$. In order to do this we use the partial summation formula for $\sum_{A<\gamma} f(\gamma)$. We also make use of the upper bound for $N(y)$ given in (2.12).

Lemma 2.17. *If* $\log x \le c_1(m+1)$ *we have*

$$\sum_{A<\gamma} f(\gamma) < \frac{c_3}{A^{m+1}x^{1/c_2}} + c_4 \int_A^\infty f(y) \log \frac{y}{2\pi} dy,$$

where c_1, c_2, c_3, *and* c_4 *are defined in* (2.13), (2.14), (2.15), *and* (2.16).

Proof. By employing the partial summation formula, we see that

$$\sum_{A<\gamma} f(\gamma) = -\int_A^\infty N(y) f'(y) dy - N(A) f(A). \tag{2.44}$$

We have

$$f'(y) = \frac{\exp\left(\frac{-\log x}{r \log y}\right)}{y^{m+2}} \left(\frac{\log x}{r(\log y)^2} - (m+1)\right). \tag{2.45}$$

Since $y > A$ and $\log x \le c_1(m+1) = r(\log A)^2(m+1)$, we find that

$$\frac{\log x}{r(\log y)^2} - (m+1) \le \frac{\log x}{r(\log A)^2} - (m+1) < 0.$$

We combine this with (2.45) to deduce that $-f'(y) > 0$. This enables us to use $N(y) \le F(y) + R(y)$ in (2.44). We obtain

$$\sum_{A<\gamma} f(\gamma) < -\int_A^\infty \left(F(y) + R(y)\right) f'(y) dy - N(A) f(A). \tag{2.46}$$

We use integration by parts on the above integral to derive

$$\int_A^\infty \left(F(y) + R(y)\right) f'(y) dy = -\left(F'(A) + R'(A)\right) f(A) - \int_A^\infty \left(F'(y) + R'(y)\right) f(y) dy.$$

From this and (2.46) we find that

$$\sum_{A<\gamma} f(\gamma) < \int_A^\infty \left(F'(y) + R'(y)\right) f(y) dy + R(A) f(A), \tag{2.47}$$

noting that $F(A) \le N(A)$. Observe that

$$R(A)f(A) = \frac{R(A)}{A^{m+1}x^{\frac{1}{r\log A}}} = \frac{c_3}{A^{m+1}x^{\frac{1}{c_2}}}, \tag{2.48}$$

and $F'(y) = \frac{1}{2\pi}\log(y/2\pi)$. This implies

$$\int_A^\infty F'(y)f(y)dy = \frac{1}{2\pi}\int_A^\infty f(y)\log\frac{y}{2\pi}dy. \tag{2.49}$$

Since $A < y$ we have

$$\frac{1}{y} < \frac{\log\frac{y}{2\pi}}{A\log\frac{A}{2\pi}}.$$

Therefore

$$R'(y) = \frac{d_1}{y} + \frac{d_2}{y\log y} \quad < \quad \left(\frac{d_1\log\frac{y}{2\pi}}{A\log\frac{A}{2\pi}} + \frac{d_2\log\frac{y}{2\pi}}{A(\log A)(\log\frac{A}{2\pi})}\right) = \alpha\log\frac{y}{2\pi},$$

where

$$\alpha = \frac{d_1}{A\log\frac{A}{2\pi}} + \frac{d_2}{A(\log A)(\log\frac{A}{2\pi})}.$$

By using the latter result it follows that

$$\int_A^\infty R'(y)f(y)dy < \alpha\int_A^\infty \log\frac{y}{2\pi}f(y)dy. \tag{2.50}$$

Putting together (2.47), (2.48), (2.49), and (2.50) completes the proof. □

The next lemma provides an upper bound for $\int_A^\infty f(y)\log(y/2\pi)dy$.

Lemma 2.18. *If* $\log x < c_1m^2/(m+c_5)$, *then*

$$\int_A^\infty f(y)\log\frac{y}{2\pi}dy < \frac{1+c_6m}{\left(1-\frac{(m+c_5)\log x}{c_1m^2}\right)m^2A^mx^{1/c_2}},$$

where $c_1,\ c_2,\ c_5,\ c_6$ are defined in (2.13), (2.14), (2.17) and (2.18).

Proof. Let

$$I = \int_A^\infty f(y)\log\frac{y}{2\pi}dy \quad = \quad \int_A^\infty \frac{1}{y^{m+1}}(\log\frac{y}{2\pi})\exp\left(\frac{-\log x}{r\log y}\right)dy.$$

Choosing $u' = \left(\log \frac{y}{2\pi}\right) \frac{1}{y^{m+1}}$ and $v = \exp\left(\frac{-\log x}{r \log y}\right)$ yields

$$u = \frac{-1}{y^m m^2}\left(1 + m \log \frac{y}{2\pi}\right),\ v' = \exp\left(\frac{-\log x}{r\log y}\right)\frac{\log x}{ry(\log y)^2}.$$

Using integrating by parts we find that

$$\begin{aligned} I &= \frac{\left(1+m\log\frac{A}{2\pi}\right)}{A^m m^2}\exp\left(\frac{-\log x}{r\log A}\right) + \int_A^\infty \exp\left(\frac{-\log x}{r\log y}\right)\frac{\left(1+m\log\frac{y}{2\pi}\right)\log x}{r(\log y)^2 m^2 y^{m+1}}dy \\ &= \frac{\left(1+m\log\frac{A}{2\pi}\right)}{A^m m^2}\exp\left(\frac{-\log x}{r\log A}\right) + \int_A^\infty \frac{\exp\left(\frac{-\log x}{r\log y}\right)\left(\log\frac{y}{2\pi}\right)}{y^{m+1}}\frac{\left(\frac{1}{\log\frac{y}{2\pi}}+m\right)\log x}{m^2 r(\log y)^2}dy \\ &= \frac{\left(1+m\log\frac{A}{2\pi}\right)}{A^m m^2}\exp\left(\frac{-\log x}{r\log A}\right) + \frac{\left(\frac{1}{\log\frac{y}{2\pi}}+m\right)\log x}{m^2 r(\log y)^2}\int_A^\infty f(y)\log\frac{y}{2\pi}dy. \end{aligned}$$

Since $A < y$ we have

$$\frac{\frac{1}{\log\frac{y}{2\pi}}+m}{(\log y)^2} \le \frac{\frac{1}{\log\frac{A}{2\pi}}+m}{(\log A)^2},$$

which gives

$$I < \frac{\left(1+m\log\frac{A}{2\pi}\right)}{A^m m^2}\exp\left(\frac{-\log x}{r\log A}\right) + \frac{\left(\frac{1}{\log\frac{A}{2\pi}}+m\right)\log x}{m^2 r(\log A)^2}\int_A^\infty f(y)\log\frac{y}{2\pi}dy. \qquad (2.51)$$

Following the definition of I, from (2.51) we find that

$$I\left(1 - \frac{\left(\frac{1}{\log\frac{A}{2\pi}}+m\right)\log x}{m^2 r(\log A)^2}\right) < \frac{\left(1+m\log\frac{A}{2\pi}\right)}{A^m m^2}\exp\left(\frac{-\log x}{r\log A}\right).$$

It follows that for

$$\log x < \frac{r(\log A)^2 m^2}{m + \frac{1}{\log\frac{A}{2\pi}}} = \frac{c_1 m^2}{m + c_5},$$

that

$$I < \frac{\left(1+m\log\frac{A}{2\pi}\right)}{A^m m^2}.\frac{\exp\left(\frac{-\log x}{r\log A}\right)}{\left(1 - \frac{\left(\frac{1}{\log\frac{A}{2\pi}}+m\right)\log x}{m^2 r(\log A)^2}\right)}.$$

□

2.2.4 Proof of Theorem 2.4

In this section we give the proof of Theorem 2.4. By use of Proposition 2.11, Proposition 2.15, Lemma 2.16, Lemma 2.17 and Lemma 2.18 we deduce Theorem 2.4 which gives explicit bounds for $\psi(x)$.

Proof. We begin with proving the right-hand side of (2.19). Observe that by Proposition 2.11 we have

$$x(1-\varepsilon_1) - \log 2\pi - \frac{1}{2}\log\left(1-\frac{1}{x^2}\right) \le \psi(x) \le x(1+\varepsilon_2) - \log 2\pi - \frac{1}{2}\log\left(1-\frac{1}{x^2}\right).$$

Noting that $-\log 2\pi < 0$ it remains to verify that

$$\varepsilon_2 < \varepsilon. \tag{2.52}$$

It follows from (2.38) that

$$\varepsilon_2 < \varepsilon = \frac{\delta}{2}\left(\left(\frac{(1+\delta)^{m+1}+1}{2}\right)^m + m\right),$$

provided that $K \le (\delta/2)^{m+1}$.

Recall that by Lemma 2.40 we have

$$K \le x^{\frac{-1}{2}} \sum_{\rho} \frac{1}{|\gamma|^{m+1}} + \sum_{A<\gamma} f(\gamma).$$

We combine this with Lemma 2.17 and Lemma 2.18 to deduce

$$K \le x^{\frac{-1}{2}} \sum_{\rho} \frac{1}{|\gamma|^{m+1}} + \frac{c_3}{A^{m+1}x^{\frac{1}{c_2}}} + \frac{c_4(1+mc_6)}{\left(1-\frac{(m+c_5\log x)}{c_1 m^2}\right) m^2 A^m x^{\frac{1}{c_2}}}.$$

From this, by $\sum_\rho 1/|\gamma^{m+1}| \le k$, we find that

$$K \le x^{\frac{-1}{2}} k + \frac{c_3}{A^{m+1}x^{\frac{1}{c_2}}} + \frac{c_4(1+mc_6)}{\left(1-\frac{(m+c_5\log x)}{c_1 m^2}\right) m^2 A^m x^{\frac{1}{c_2}}}.$$

Since $x \ge a$ we may infer

$$K \le a^{\frac{-1}{2}} k + \frac{c_3}{A^{m+1}a^{\frac{1}{c_2}}} + \frac{c_4(1+mc_6)}{\left(1-\frac{(m+c_5\log a)}{c_1 m^2}\right) m^2 A^m a^{\frac{1}{c_2}}}.$$

Thus

$$K \leq \left(\frac{2}{2} \left(a^{\frac{-1}{2}} k + \frac{c_3}{A^{m+1} a^{\frac{1}{c_2}}} + \frac{c_4(1+mc_6)}{\left(1 - \frac{(m+c_5 \log a)}{c_1 m^2}\right) m^2 A^m a^{\frac{1}{c_2}}} \right)^{1/m+1} \right)^{m+1} = \left(\frac{\delta}{2}\right)^{m+1},$$

which achieves the proof.

For the left-hand side of (2.19), having established Proposition 2.11, we require to prove that

$$\varepsilon_1 < \varepsilon, \tag{2.53}$$

since $-\log(1 - 1/x^2) > 0$. The proof of (2.53) is identical to the proof of (2.52). □

We end this chapter by providing two examples based on specific numerical values of parameters in Theorem 2.4. The following lemma which provides explicit upper bounds for $\sum_\rho 1/|\gamma|^{m+1}$ where $m \leq 3$ will be used to determine values for k in our examples.

Lemma 2.19. *We have*

$$\sum_\rho \frac{1}{\gamma^2} < 0.0463, \; \sum_\rho \frac{1}{|\gamma|^3} < 0.00167, \text{ and } \sum_\rho \frac{1}{\gamma^4} < 0.0000744.$$

Proof. See [17, Lemma 17, p. 225]. □

Example 2.20. *We let $A = 1467.47747$, $r = 17.72$ (see [17, pp. 223-224]) and $m = 2$. We see that*

$$c_1 = 942.04939, \; c_2 = 129.20183, \; c_3 = 3.46700,$$
$$c_4 = 0.15917, \; c_5 = 0.18337, \text{ and } c_6 = 5.45342.$$

Observe that $c_1 m^2/m + c_5 \simeq 1725.862$. Thus we choose $a = \exp(20)$ so that $a < \exp(1725.862)$. Following the definition of δ and ε we obtain $\delta = 0.01288$ and $\varepsilon = 0.0195832$. Hence by using Theorem 2.4 we deduce

$$0.98041x < \psi(x) < 1.019584x, \text{ for } x \geq e^{20}.$$

Example 2.21. *Let* $A = 29538618432.236$[1]. *Then Wedeniwski has shown that* $N(A) = 10^{11}$ *and the Riemann Hypothesis is true up to height* A. *We let* $r = 5.70176$ *from [12], and* $m = 2$. *We observe that*

$$c_1 = 3314.10330,\ c_2 = 137.46352,\ c_3 = 6.30081,$$
$$c_4 = 0.15915,\ c_5 = 0.04490,\ \text{and } c_6 = 22.27108.$$

We find that $c_1m^2/m + c_5 \simeq 6482.666$. *Hence we need to choose* $a < \exp(6482.666)$. *We let* $a = \exp(20)$. *Following the definition of* δ *and* ε *we obtain* $\delta = 0.00846$ *and* $\varepsilon = 0.0128063$. *Hence by using Theorem 2.4 we deduce*

$$0.98719x < \psi(x) < 1.012807x,\ \text{for } x \geq e^{20}.$$

By comparing the results of the above examples we conclude that improving the values of A and r gives sharper results.

We can use Theorem 2.4 to generate tables of upper and lower bounds for $\psi(x)$. In this thesis Tables 5.1 and 5.2 give estimations for $\psi(x)$. The values given in these tables will be used throughout the thesis.

[1] http://www.zetagrid.net/zeta/math/zeta.result.100billion.zeros.html

Chapter 3

Bounds for $\theta(x)$

3.1 Introduction

In this section we assume that we have a table of upper and lower bounds for $\psi(x)$. Therefore the positive values $A^+(b)$ and $A^-(b)$ are given such that

$$A^-(b)x < \psi(x) < A^+(b)x, \text{ for } x \geq e^b. \tag{3.1}$$

The relation between $\theta(x)$ and $\psi(x)$ given by

$$\psi(x) = \sum_{k=1}^{\infty} \theta(x^{1/k}). \tag{3.2}$$

plays a key role in our estimates. The above can be derived directly from the definition of $\psi(x)$ and $\theta(x)$.

From (3.2) we deduce the trivial bound

$$\psi(x) \geq \theta(x), \text{ for } x > 0.$$

Using the Prime Number Theorem one can show the asymptotic behavior of $\theta(x)$ by

$$\theta(x) \sim x, \tag{3.3}$$

as $x \to \infty$. However (3.3) does not give any information about numerical estimates for $\theta(x)/x$. We here mention a result, due to Rosser and Schoenfeld [18, (5.4), page 77], on the estimation of $\theta(x)/x$ by elementry methods.

Proposition 3.1. *$\theta(x) < 1.11x$ for $x > 0$.*

Proof. By direct computation one can show that

$$\theta(x) < x \text{ for } 0 < x \leq 10^8. \tag{3.4}$$

For larger values of x we proceed as follows. From Chapter 2, Theorem 2.3 we may infer

$$\theta(x) \leq \psi(x) < 1.2\nu x + (3\log x + 5)(\log x + 1) \text{ for } x \geq 2,$$

where $\nu \simeq 0.92129\cdots$ is given in (2.1). Hence

$$\theta(x) < \left(1.2\nu + \frac{s(x)}{x}\right) x,$$

where

$$s(x) = (3\log x + 5)(\log x + 1).$$

We note that $1.2\nu + s(x)/x$ is a decreasing function for $x > 10^8$. Moreover,

$$1.2\nu + s(x)/x < 1.2\nu + s(10^8)/10^8 < 1.11 \text{ for } x > 10^8.$$

From this, and (3.4) we deduce that

$$\theta(x) < 1.11x \text{ for } x > 0.$$

□

It is possible to improve the above bound by elementary methods. For example Hanson [10] has showed that

$$\theta(x) < (\log 3)x < 1.0987x, \text{ for } x > 0.$$

This bound was improved again by Grimson and Hanson [9], who obtained

$$\theta(x) < 1.0508x, \text{ for } x > 0.$$

A breakthrough came in 1989 where Costa Pereira [4] by employing an elementary method proved that

$$\theta(x) < \frac{532}{531}x = 1.001884x \text{ for } x > 0.$$

In another front, as described in the previous chapter, Rosser [17] developed an analytic method which allowed him to obtain sharp numerical estimates for $\psi(x)/x$ and therefore for functions involving primes such as $\theta(x)/x$. Numerical upper and lower bounds for $\theta(x)/x$ were improved later by Rosser and Schoenfeld [18], [19], and Schoenfeld [20] by some refinements in their analytical method.

Table 3.1 summarizes the history of numerical upper bounds for $\theta(x)/x$ which are deduced by analytical methods.

Table 3.1:

Authors	c_0
Rosser & Schoenfeld [18]	$1 + 1.6240 \times 10^{-2}$
Rosser & Schoenfeld [19]	$1 + 1.1020 \times 10^{-3}$
Schoenfeld [20]	$1 + 8.1000 \times 10^{-5}$
Dusart[7]	$1 + 2.7579 \times 10^{-5}$

$$\theta(x) < c_0 x, \text{ for } x > 0.$$

The goal of this chapter is to develop techniques which give sharper estimates for $\theta(x)/x$ compare to those that we have in the literature.

We start by giving upper bound for $\theta(x)/x$. By a recent result of Dusart [7, Proposition 5.1, p. 4] we have

$$\theta(x) < x, \text{ for } 0 < x \leq 8 \times 10^{11} \simeq e^{27.40}. \tag{3.5}$$

We combine (3.5) with the trivial bound $\theta(x) \leq \psi(x)$ to obtain

$$\theta(x) < A^{+}(27.4)x, \text{ for } x > 0.$$

This leads to the following proposition.

Proposition 3.2. $\theta(x) < \left(1 + 2.841 \times 10^{-5}\right) x, \text{ for } x > 0.$

Thus we can assume that for $c_0 = 1 + 2.841 \times 10^{-5}$ we have

$$\theta(x) < c_0 x \text{ for } x > 0.$$

In this chapter we present several methods to improve this upper bound for $\theta(x)$.

Next we consider the problem of finding a numerical lower bound for $\theta(x)/x$. To do this we consider the identity

$$\psi(x) - \theta(x) = \sum_{k \geq 2} \theta(x^{1/k}).$$

From the above equation we can derive several forms of upper bounds for $\psi(x) - \theta(x)$, and consequently we will be able to deduce tables for lower bounds of $\theta(x)$ in different ranges. Table 3.2 provides the history of numerical lower bounds for $\theta(x)/x$ given by analytical methods.

Table 3.2:

Authors	$B^{-}(b)$	b
Rosser & Schoenfeld[18]	0.840000	4.70
Rosser & Schoenfeld[18]	0.980000	8.93
Rosser & Schoenfeld [19]	0.998684	14.10
Schoenfeld [20]	0.998697	13.97
Dusart [7]	0.999900	25.00

$$\theta(x) > B^{-}(b)x, \text{ for } x \geq e^{b}.$$

3.2 Upper bounds for $\theta(x)$ for $x > 0$

The following inequalities play important roles in establishing bounds for $\theta(x)$.

- **Dusart [7, Proposition 5.1]**

$$\theta(x) < x, \text{ for } 0 < x \leq 8 \times 10^{11} \simeq e^{27.40}. \tag{3.6}$$

- **Rosser-Schoenfeld [18, Theorem 24]**

$$x^{1/2} < \psi(x) - \theta(x), \text{ for } 121 \leq x \leq 10^{16} \simeq e^{36.8}. \tag{3.7}$$

- **Rosser-Schoenfeld [18, Theorem 19]**

$$x - 2x^{1/2} < \theta(x), \text{ for } 0 < x \le 1420.9 \text{ and } 1423 \le x \le 10^8. \tag{3.8}$$

The main goal of this section is studying methods for providing sharp upper bounds for $\theta(x)/x$. We point out that a lower bound for $\psi(x) - \theta(x)$ will result in an upper bound for $\theta(x)$ provided that we have estimates for $\psi(x)$ in different ranges. Using this fact we present the following theorem.

Theorem 3.3. *Let b_1 and b_2 be positive constants such that $b_1 \le 27.4$ and $b_1 < b_2$. Let*

$$c_1 = A^+(b_1) - \frac{A^-(\frac{b_1}{2})}{e^{b_2/2}},$$

and

$$c_0 = \max\{c_1, A^+(b_2)\}.$$

Then

$$\theta(x) < c_0 x, \text{ for } x > 0.$$

Proof.

- Let $0 < x < e^{b_1}$. By (3.6), we have $\theta(x) < x$.

- We let $x \in [e^{b_1}, e^{b_2}]$. From (3.2) we deduce that

$$\psi(x^{1/2}) = \sum_{k \ge 1} \theta(x^{1/2k}). \tag{3.9}$$

  Therefore

$$\psi(x) - \theta(x) = \sum_{k=2}^{\infty} \theta(x^{1/k}) = \sum_{k \ge 1} \theta(x^{1/2k}) + \sum_{k \ge 1} \theta(x^{1/2k+1}). \tag{3.10}$$

  The previous identity implies that

$$\psi(x^{1/2}) \le \psi(x) - \theta(x). \tag{3.11}$$

  From (3.11) and (3.1) we obtain, for $x \in [e^{b_1}, e^{b_2}]$,

$$\theta(x) \le \psi(x) - \psi(x^{1/2}) < \left(A^+(b_1) - \frac{A^-(\frac{b_1}{2})}{e^{\frac{b_2}{2}}} \right) x. \tag{3.12}$$

- Let $x \geq e^{b_2}$. By using the trivial inequality $\theta(x) \leq \psi(x)$ we have

$$\theta(x) < A^{+}(b_2)x. \tag{3.13}$$

We combine (3.6), (3.12), and (3.13) to obtain

$$\theta(x) < \max\left\{1, A^{+}(b_1) - \frac{A^{-}(\frac{b_1}{2})}{e^{\frac{b_2}{2}}}, A^{+}(b_2)\right\} x, \text{ for } x > 0.$$

Since $A^{+}(b) > 1$ for $b > 0$ we have

$$\theta(x) < \max\left\{A^{+}(b_1) - \frac{A^{-}(\frac{b_1}{2})}{e^{\frac{b_2}{2}}}, A^{+}(b_2)\right\} x, \text{ for } x > 0.$$

From which the proof follows.

□

Example 3.4. *Let $b_1 = 27.4$, and $b_2 = 28$. Then by Tables 5.1 and 5.2 we have*

$$A^{+}(b_1) = 1 + 2.841 \times 10^{-5}, \; A^{-}(\frac{b_1}{2}) = 0.9988024, \text{ and } A^{+}(b_2) = 1 + 2.224 \times 10^{-5}.$$

Hence by Theorem 3.3 we have

$$c_0 = c_1 = 1 + 2.7579467120 \times 10^{-5}.$$

We observe that in proving Theorem 3.3 we can use relation (3.7) in place of (3.11). This leads us to the following theorem.

Theorem 3.5. *Let b_1 and b_2 be positive constants such that $b_1 \leq 27.4$ and $b_1 < b_2 \leq 36.8$. Let*

$$c_1 = A^{+}(b_1) - \frac{1}{e^{b_2/2}},$$

and

$$c_0 = \max\{c_1, A^{+}(b_2)\}.$$

Then

$$\theta(x) < c_0 x, \text{ for } x > 0.$$

Example 3.6. *Let* $b_1 = 27.4$*, and* $b_2 = 28$*. Then by employing Theorem 3.5 and Tables 5.1 and 5.2 we see that*

$$c_0 = c_1 = 1 + 2.7578471281 \times 10^{-5}.$$

Next we describe an improvement of Theorem 3.3. The main idea in the next theorem is that in the middle range $[e^{b_1}, e^{b_2}]$ we use more precise lower bound for $\psi(x) - \theta(x)$ than the one used in the previous theorem. This idea is inspired from [7, Proposition 3.1].

Theorem 3.7. *Let* b_1 *and* b_2 *be positive constants such that* $b_1 \leq 27.4$ *and* $b_1 < b_2 \leq 55.26$*. Let*

$$c_1 = \max_{x \in [e^{b_1}, e^{b_2}]} \left\{ A^+(b_1) - A^-(b_1/2)x^{-1/2} - x^{-2/3} + 2x^{-5/6} \right\}.$$

Then for $x > 0$ *we have*

$$\theta(x) < c_0 x,$$

where $c_0 = \max\{c_1, A^+(b_2)\}$*.*

Proof. We let $e^{b_1} \leq x \leq e^{b_2}$. By using (3.9) and (3.10) we have

$$\psi(x^{1/2}) + \theta(x^{1/3}) \leq \psi(x) - \theta(x).$$

The above gives

$$\theta(x) \leq \psi(x) - \psi(x^{1/2}) - \theta(x^{1/3}). \tag{3.14}$$

For the lower bound of $\theta(x^{1/3})$ we use (3.8) provided that $1423^3 \leq x \leq 10^{24}$. Note that $\log 10^{24} \simeq 55.26$. From this, by (3.1) we see that (3.14) can be transformed into

$$\theta(x) < A^+(b_1)x - A^-(b_1/2)x^{1/2} - x^{1/3} + 2x^{1/6}. \tag{3.15}$$

Putting together (3.6), (3.13), and (3.15) achieves the proof. □

Example 3.8. *Let* $b_1 = 27.4$ *and* $b_2 = 28$*. Then by Tables 5.1 and 5.2 we have*

$$A^+(b_1) = 1 + 2.841 \times 10^{-5}, \ A^-(b_1/2) = 0.9988024, \text{ and } A^+(b_2) = 1 + 2.224 \times 10^{-5}.$$

Hence by Theorem 3.7 we have

$$c_0 = c_1 = 1 + 2.7571794847 \times 10^{-5}.$$

A comparison of results of Examples 3.4, 3.6, and 3.8 shows that we obtain improved result using Theorem 3.7.

We next present a lemma which gives a lower bound for $\psi(x) - \theta(x)$ in terms of ψ function. This result is due to Costa Pereira [3, p. 211].

Lemma 3.9. *For $x > 0$ we have*

$$\theta(x) \leq \psi(x) - \psi(x^{1/2}) - \psi(x^{1/3}) - \psi(x^{1/5}) + \psi(x^{1/6}). \tag{3.16}$$

Proof. We substitute (3.9) in (3.2) to obtain

$$\psi(x) = \theta(x) + \psi(x^{1/2}) + \sum_{n\geq 1} \theta(x^{1/2n+1}), \tag{3.17}$$

or equivalently

$$\psi(x) - \theta(x) = \psi(x^{1/2}) + \sum_{n\geq 1} \theta(x^{1/6n-3}) + \sum_{n\geq 1} \theta(x^{1/6n-1}) + \sum_{n\geq 1} \theta(x^{1/6n+1}). \tag{3.18}$$

Now (3.2) implies that

$$\psi(x^{1/3}) = \sum_{n\geq 1} \theta(x^{1/3n}) = \sum_{n\geq 1} \theta(x^{1/6n-3}) + \sum_{n\geq 1} \theta(x^{1/6n}),$$

or equivalently

$$\psi(x^{1/3}) - \sum_{n\geq 1} \theta(x^{1/6n}) = \sum_{n\geq 1} \theta(x^{1/6n-3}).$$

We employ the above identity in (3.18) to obtain

$$\psi(x) - \theta(x) = \psi(x^{1/2}) + \psi(x^{1/3}) - \sum_{n\geq 1} \theta(x^{1/6n}) + \sum_{n\geq 1} \theta(x^{1/6n-1}) + \sum_{n\geq 1} \theta(x^{1/6n+1}). \tag{3.19}$$

Observe that

$$\sum_{n\geq 1} \theta(x^{1/6n-1}) + \sum_{n\geq 1} \theta(x^{1/6n+1}) \geq \sum_{n\geq 1} \theta(x^{1/10n-5}) + \sum_{n\geq 1} \theta(x^{1/10n}) = \psi(x^{1/5}), \tag{3.20}$$

and

$$\psi(x^{1/6}) = \sum_{n\geq 1} \theta(x^{1/6n}). \tag{3.21}$$

(3.19) combined with (3.20) and (3.21) implies the result. □

Replacing (3.12) with (3.16) in Theorem 3.3 enables us to establish the following theorem.

Theorem 3.10. *Let b_1 and b_2 be positive constants such that $0 < b_1 \leq 27.4$ and $b_1 < b_2$. Let*

$$c_1 = \max_{x \in [e^{b_1}, e^{b_2}]} \left\{ A^+(b_1) - \frac{A^-(b_1/2)}{x^{1/2}} - \frac{A^-(b_1/3)}{x^{2/3}} - \frac{A^-(b_1/5)}{x^{4/5}} + \frac{A^+(b_1/6)}{x^{5/6}} \right\}.$$

Then

$$\theta(x) < c_0 x, \text{ for } x > 0,$$

where $c_0 = \max\{c_1, A^+(b_2)\}$.

Example 3.11. *Let $b_1 = 27.4$ and $b_2 = 28$. Then from Tables 5.1 and 5.2 we have*

$$A^+(b_1) = 1 + 2.841 \times 10^{-5},\ A^-(b_1/2) = 0.9988024,\ A^-(b_1/3) = 0.99343,$$

$$A^-(b_1/5) = 0.96764,\ A^+(b_1/6) = 1.03883, \text{ and } A^+(b_2) = 1 + 2.224 \times 10^{-5}.$$

It follows from Theorem 3.10 that

$$c_0 = c_1 = 1 + 2.7571594613 \times 10^{-5}.$$

We see that there is a slight improvement in the result of Theorem 3.10 compared to the result of Theorem 3.7.

We now establish a theorem which theoretically will provide a better upper bound for $\theta(x)/x$ for $x > 0$, compared to the previous results. The reason is that in the middle range $e^{b_1} \leq x \leq e^{b_2}$, we are using a better upper bound for $\theta(x)$, namely

$$\theta(x) \leq \psi(x) - \psi(x^{1/2}) - \psi(x^{1/3}) - \psi(x^{1/5}) + \theta(x^{1/6}) + \psi(x^{1/30}), \tag{3.22}$$

(see [2, p. 110]). This upper bound is sharper than (3.16) since

$$\psi(x^{1/6}) \geq \theta(x^{1/6}) + \psi(x^{1/30}).$$

We point out that for an upper bound for $\theta(x^{1/6})$ in (3.22) we can use

$$\theta(x^{1/6}) < \tilde{c}_0 x^{1/6}, \text{ for } x > 0,$$

where $\tilde{c}_0$ is an upper bound for $\theta(x)/x$, when $x > 0$. We have the following theorem.

Theorem 3.12. *Let b_1 and b_2 be positive constants such that $0 < b_1 \leq 27.4$ and $b_1 < b_2$. Let*

$$c_1 = \max_{x \in \left[e^{b_1}, e^{b_2}\right]} \left\{ A^+(b_1) - \frac{A^-(b_1/2)}{x^{1/2}} - \frac{A^-(b_1/3)}{x^{2/3}} - \frac{A^-(b_1/5)}{x^{4/5}} + \frac{\tilde{c}_0}{x^{5/6}} + \frac{A^+(b_1/30)}{x^{29/30}} \right\},$$

where $\tilde{c}_0$ is an upper bound for $\theta(x)/x$ when $x > 0$. Then

$$\theta(x) < c_0 x, \text{ for } x > 0,$$

where $c_0 = \max\{c_1, A^+(b_2)\}$.

Example 3.13. *Let $b_1 = 27.4$ and $b_2 = 28$. Then by Tables 5.1 and 5.2 we obtain*

$$A^+(b_1) = 1 + 2.841 \times 10^{-5}, \; A^-(b_1/2) = 0.9988024, \; A^-(b_1/3) = 0.9934300,$$

$$A^-(b_1/5) = 0.9676400, \; A^+(b_1/30) = 1.03883, \text{ and } A^+(b_2) = 1 + 2.2244 \times 10^{-5}.$$

From Example 3.11 we take

$$\tilde{c}_0 = 1 + 2.7571594613 \times 10^{-5}.$$

Thus by employing Theorem 3.12 we obtain

$$c_0 = c_1 = 1 + 2.7571593586 \times 10^{-5}.$$

Comparison of the results of previous examples in this section shows, as expected, that Theorem 3.12 gives the best upper bound for $\theta(x)$ in the range $x > 0$.

By having one extra condition we can rewrite Theorem 3.12 as follows.

Theorem 3.14. *Let b_1 and b_2 be positive constants such that $0 < b_1 \leq 27.4$ and $b_1 < b_2$. If $b_2/6 \leq 27.4$ then let*

$$c_1 = \max_{x \in \left[e^{b_1}, e^{b_2}\right]} \left\{ A^+(b_1) - \frac{A^-(b_1/2)}{x^{1/2}} - \frac{A^-(b_1/3)}{x^{2/3}} - \frac{A^-(b_1/5)}{x^{4/5}} + \frac{1}{x^{5/6}} + \frac{A^+(b_1/30)}{x^{29/30}} \right\}.$$

Then

$$\theta(x) < c_0 x, \text{ for } x > 0,$$

where $c_0 = \max\{c_1, A^+(b_2)\}$.

Example 3.15. *Let* $b_1 = 27.4$ *and* $b_2 = 28$. *Then by Tables 5.1 and 5.2 we obtain*

$$A^+(b_1) = 1 + 2.841 \times 10^{-5},\ A^-(b_1/2) = 0.9988024,\ A^-(b_1/3) = 0.9934300,$$

$$A^-(b_1/5) = 0.9676400,\ A^+(b_1/30) = 1.03883,\ \text{and } A^+(b_2) = 1 + 2.2244 \times 10^{-5}.$$

Thus by using Theorem 3.14 we obtain

$$c_0 = c_1 = 1 + 2.7571593584 \times 10^{-5}.$$

This result has a slight improvement compare to the result of Example 3.13.

We now turn our attention to an unpublished result of Costa Pereire[1] which states

$$\theta(x) < x,\ \text{for } 0 < x < 10^{16} \simeq e^{36.8}.$$

This result together with Theorem 3.12 yield the following example.

Example 3.16. *Let* $b_1 = 36.8$ *and* $b_2 = 37$. *From Tables 5.1 and 5.2 we have*

$$A^+(b_1) = 1 + 1.301 \times 10^{-9},\ A^-(b_1/2) = 1 - 2.841 \times 10^{-5},\ A^-(b_1/3) = 0.9988024,$$

$$A^-(b_1/5) = 0.99770,\ A^+(b_1/30) = 1.03883,\ \text{and } A^+(b_2) = 1 + 4.348 \times 10^{-7}.$$

From Example 3.13 we have $\tilde{c}_0 = 1 + 2.7571593586 \times 10^{-5}$, *therefore by Theorem 3.12 we have* $c_1 = 0.99999999205$ *and*

$$c_0 = 1 + 4.348 \times 10^{-7}.$$

The last example suggests that extending the range of the inequality from $(0, e^{27.4}]$ to $(0, e^{36.8}]$ significantly improves the upper bound for the function $\theta(x)/x$.

3.3 Upper bounds for $\theta(x)$ for $x \geq e^b$

Recall that (3.6) states

$$\theta(x) < x,\ \text{for } 0 < x \leq 8 \times 10^{11} \simeq e^{27.4}.$$

[1]http://mat.fc.ul.pt/ind/ncpereira/

Therefore it would make sense to consider upper bounds for $\theta(x)$ on $[e^b, \infty)$ only for $b \geq 27.4$.

Since the sharpest upper bound for $\theta(x)$ in the previous section comes from Theorem 3.12 we use the same argument with means of Tables 5.1 and 5.2 to deduce Table 5.3 which establishes the inequalities

$$\theta(x) < B^+(b)x, \text{ for } x > e^b, \tag{3.23}$$

for different values of b.

Our Table 5.3 gives more precise estimates for upper bound of $\theta(x)/x$ compared to the recent results of Dusart [7].

3.4 The inequality $\psi(x) - \theta(x) > d_0\sqrt{x}$

In this section we are intersted in finding a lower bound for $\psi(x) - \theta(x)$ in the form of a constant multiple of $\sqrt{x}$. Recall that by (3.7) we have

$$\psi(x) - \theta(x) > \sqrt{x}, \text{ for } 121 \leq x \leq 10^{16} \simeq e^{36.8}.$$

For $x > e^{36.8}$, we use (3.11) which states

$$\psi(x) - \theta(x) \geq \psi(x^{1/2}).$$

From this inequality, (3.1), and Table 5.1 we can derive, for $x \geq e^{36.8}$, that

$$\psi(x) - \theta(x) > A^-(36.8/2)\sqrt{x} = 0.9988024\sqrt{x}.$$

The above with (3.7) give the following.

$$\psi(x) - \theta(x) > 0.9988024\sqrt{x}, \text{ for } x \geq 121.$$

We shall improve this lower bound for $\psi(x) - \theta(x)$ in the next theorem, by employing a better estimate in place of (3.11).

Theorem 3.17. *Let* $0 < b_1 \leq 36.8$ *and let* $b_1 < b_2 \leq 55.26$. *Let*

$$d_0 = \min\left\{ \min_{[e^{b_1}, e^{b_2}]} \left\{ A^-(b_1/2) + \frac{1}{x^{1/6}} - \frac{2}{x^{1/3}} \right\}, A^-(b_2/2) \right\}.$$

Then

$$\psi(x) - \theta(x) > d_0\sqrt{x}, \text{ for } x \geq 121.$$

Proof. • Let $121 \le x \le e^{b_1}$. By (3.7), we have $\psi(x) - \theta(x) > \sqrt{x}$.

- We let $e^{b_1} < x < e^{b_2}$. From (3.17) we can infer

$$\psi(x) - \theta(x) \ge \psi(\sqrt{x}) + \theta(x^{1/3}). \tag{3.24}$$

For the lower bound of $\theta(x^{1/3})$ we use (3.8) provided that $1423^3 \le x \le 10^{24}$, noting that $\log 10^{24} \simeq 55.26$. For the lower bound of $\psi(\sqrt{x})$ we use (3.1). Hence, for $e^{b_1} < x < e^{b_2}$, (3.24) is transformed into

$$\psi(x) - \theta(x) > A^{-}(b_1/2)x^{1/2} + x^{1/3} - 2x^{1/6}.$$

This implies

$$\psi(x) - \theta(x) > \min_{[e^{b_1}, e^{b_2}]} \left(A^{-}(b_1/2) + \frac{1}{x^{1/6}} - \frac{2}{x^{1/3}} \right) x^{1/2}. \tag{3.25}$$

- Let $x > e^{b_2}$. Using (3.1) and (3.24) we find that

$$\psi(x) - \theta(x) \ge \psi(\sqrt{x}) > A^{-}(b_2/2)x^{1/2}. \tag{3.26}$$

Putting together (3.7), (3.25), and (3.26) achieves the proof.

□

Example 3.18. *Let $b_1 = 36.8$. We divide the interval $[e^{36.8}, e^{55.26}]$ into the subintervals, $I_1 = [e^{36.8}, e^{40}]$, $I_2 = [e^{40}, e^{44}]$, $I_3 = [e^{44}, e^{49}]$, $I_4 = [e^{49}, e^{54}]$, and $I_5 = [e^{54}, e^{55.26}]$. We calculate $\left\{A^{-}(b_1/2) + \frac{1}{e^{b_2/6}} - \frac{2}{e^{b_1/3}}\right\}$ for each of $I_i = [e^{b_1}, e^{b_2}]$ with $i = 1, 2, 3, 4, 5$ separately. It follows that*

$$\min_{[e^{36.8}, e^{55.26}]} \left\{ A^{-}(b_1/2) + \frac{1}{x^{1/6}} - \frac{2}{x^{1/3}} \right\} = 1 + 4.949 \times 10^{-6}.$$

For $b_2 = 55.26$ we have $A^{-}(b_2/2) = 0.99997159$, so it follows from Theorem 3.17 that $d_0 = 0.99997159$. This result is comparable to Dusart's [7, Proposition 3.1] who obtained $d_0 = 0.9999$.

The results of the above example establishes the following inequality that improves a result of Rosser and Schoenfeld [18, Theorem 24, p.73].

Corollary 3.19. $\psi(x) - \theta(x) > \sqrt{x}$, for $121 \le x \le e^{55.26}$.

3.5 Lower bounds for $\theta(x)$ for $x \geq e^b$

In this section we establish some theorems which enable us to generate tables for lower bounds of $\theta(x)$. Our approach for obtaining lower bounds for $\theta(x)$ is different from the upper bounds, since for the upper bound of $\theta(x)$ we know that $\theta(x) < x$ over a large interval, however an analogous inequality for the lower bound does not exist.

Lower bounds for $\theta(x)$ can be attained provided that we have upper bounds for $\psi(x) - \theta(x)$. For an upper bound of $\psi(x) - \theta(x)$ we can employ the inequality

$$\psi(x) - \theta(x) < 2\psi(x^{1/2}). \tag{3.27}$$

This can be obtained directly from (3.9) and (3.10). A more precise estimates

$$\psi(x) - \theta(x) < \psi(x^{1/2}) + \psi(x^{1/3}) + \psi(x^{1/5}),$$

due to Costa Pereira [3, p. 211], is given in Lemma 3.26. Also from Theorem 2 of [2, p. 110] we have

$$\psi(x) - \theta(x) < \psi(x^{1/2}) + \psi(x^{1/3}) + \psi(x^{1/5}) - \theta(x^{1/6}) + \theta(x^{1/7}) - \psi(x^{1/30}).$$

This a better upper bound for $\psi(x) - \theta(x)$ compared to (3.27) and (3.5). In addition to these inequalities we need estimations of $\psi(x)$ over different ranges.

Next we establish a result which gives a lower bound for $\theta(x)$.

Theorem 3.20. *Let $b > 2.694$ and c_0 be a positive constant satisfying $\theta(x)/x < c_0$ for $x > 0$. Let*

$$B^-(b) = A^-(b) - \frac{c_0\left(\frac{b}{\log 2} - 1\right)}{e^{b/2}}.$$

Then we have

$$\theta(x) > B^-(b)x,$$

for $x \geq e^b$.

Proof. From (3.10) we have

$$\psi(x) - \theta(x) = \sum_{k \geq 2} \theta(x^{1/k}).$$

Therefore

$$\psi(x) - \theta(x) \le \sum_{k=2}^{\left[\frac{\log x}{\log 2}\right]} c_0 x^{1/k} \le c_0 \left(\frac{\log x}{\log 2} - 1\right) x^{1/2}.$$

We combine this with (3.1) to obtain

$$\theta(x) > \left(A^-(b) - c_0 \left(\frac{\log x}{\log 2} - 1\right) x^{-1/2}\right) x, \tag{3.28}$$

for $x \ge e^b$. Note that $(\log x/\log 2 - 1)\, x^{-1/2}$ decreases if $x \ge e^{2.694}$. □

Example 3.21. *By using Theorem 3.20 and Tables 5.1 and 5.2 we generate Tables 5.4, 5.5, 5.6, with* $c_0 = 1 + 2.7579 \times 10^{-5}$ *and Tables 5.7, 5.8 and 5.9 with* $c_0 = 1 + 2.7571593586 \times 10^{-5}$.

Refining the method which is used in Theorem 3.20 enables us to states the following.

Theorem 3.22. *Let* $b > 0$, $k_0 \ge 3$ *be an integer, and* c_0 *be a positive constant satisfying* $\theta(x)/x < c_0$ *for* $x > 0$. *We let*

$$B^-(b, k_0) = A^-(b) - \sum_{2 \le k \le k_0 - 1} \min\{A^+(b/k), c_0\} e^{b/k - 1} - c_0 e^{b/k_0 - 1} \left(\frac{b}{\log 2} - k_0 + 1\right).$$

If

$$b \ge \frac{1}{k_0 - 1}\left(k_0 + (k_0 - 1)^2 \log 2\right), \tag{3.29}$$

then

$$\theta(x) > B^-(b, k_0)x, \ \text{ for } x \ge e^b.$$

Proof. From (3.10) we have

$$\psi(x) - \theta(x) = \sum_{2 \le k \le k_0 - 1} \theta(x^{1/k}) + \sum_{k \ge k_0} \theta(x^{1/k}).$$

It follows that

$$\psi(x) - \theta(x) < \sum_{2 \le k \le k_0 - 1} \theta(x^{1/k}) + c_0 x^{1/k_0} \left(\frac{\log x}{\log 2} - k_0 + 1\right).$$

Therefore

$$\theta(x) > \psi(x) - \sum_{2\le k\le k_0-1} \theta(x^{1/k}) - c_0 x^{1/k_0}\left(\frac{\log x}{\log 2} - k_0 + 1\right). \tag{3.30}$$

Using the trivial bound $\theta(x^{1/k}) \le \psi(x^{1/k})$ with (3.1) allows us to rewrite relation (3.30) as follows.

$$\theta(x) > \left(A^-(b) - \sum_{2\le k\le k_0-1} \min\{A^+(b/k), c_0\} x^{1/k-1} - c_0 x^{1/k_0-1}\left(\frac{\log x}{\log 2} - k_0 + 1\right)\right) x.$$

By (3.29), $x^{1/k_0-1}\left(\frac{\log x}{\log 2} - k_0 + 1\right)$ decreases with $x \ge 1$ and so the proof follows. □

Example 3.23. *Let $k_0 = 3$. If $b \ge 2.887$ then Theorem 3.22 implies that*

$$\theta(x) > B^-(b,3)x, \text{ for } x \ge e^b, \tag{3.31}$$

where

$$B^-(b,3) = A^-(b) - \frac{\min\{A^+(b/2), c_0\}}{e^{b/2}} - \frac{c_0\left(\frac{b}{\log 2} - 2\right)}{e^{2b/3}}.$$

From (3.31)*, Tables 5.1 and 5.2, and $c_0 = 1 + 2.7571593586 \times 10^{-5}$ from Example 3.13, we generate Tables 5.10, 5.11 and 5.12.*

Note that according to the values of Table 5.2, for $x < e^{56}$ we have $c_0 < A^+(b/2)$, and for $x \ge e^{56}$ we have $c_0 < A^+(b/2)$.

Example 3.24. *Let $k_0 = 4$. If $b \ge 3.413$, by Theorem 3.22 we obtain*

$$\theta(x) > B^-(b,4)x, \text{ for } x \ge e^b,$$

where

$$B^-(b,4) = A^-(b) - \frac{\min\{A^+(b/2), c_0\}}{e^{b/2}} - \frac{\min\{A^+(b/3), c_0\}}{e^{2b/3}} - \frac{c_0\left(\frac{b}{\log 2} - 3\right)}{e^{3b/4}}.$$

From Tables 5.1, 5.2, and $c_0 = 1 + 2.7571593586 \times 10^{-5}$ from Example 3.13, we get Tables 5.13, 5.14, and 5.15.

Note that we have $c_0 < A^+(b/3)$ for $x < e^{84}$ and $c_0 > A^+(b/3)$ for $x \ge e^{84}$ (see Table 5.2).

We would like to point out that by considering $k_0 = 4$ we obtain better results comparing to the case $k_0 = 3$ in Example 3.23.

The next theorem gives an improvement of a result of Rosser and Schoenfeld [18, Theorem 24, p.73].

Theorem 3.25. $\psi(x) - \theta(x) > \sqrt{x}$, for $121 \le x < e^{145.5}$.

Proof. We have already proved, Corollary 3.19, that

$$\psi(x) - \theta(x) > \sqrt{x}, \text{ for } 121 \le x < e^{55.26}.$$

We start by breaking interval $x > e^{55.2}$ into the following subintervals.

$$[e^{55.26}, e^{60}],\ [e^{60}, e^{69.4}],\ [e^{69.4}, e^{70}],\ [e^{70}, e^{87.8}],\ [e^{87.8}, e^{103.6}],$$

$$[e^{103.6}, e^{122.6}],\ [e^{122.6}, e^{143.7}],\ [e^{143.7}, e^{145.4}],\ [e^{145.4}, \text{ and } e^{145.5}].$$

In each of the above intervals by means of relation (3.24) and Tables 5.13 and 5.14 which are our best results for the lower bounds of $\theta(x)$ so far, we verify that

$$\psi(x) - \theta(x) > \sqrt{x}$$

to deduce the result. □

We now establish the following lemma which plays a fundamental role in our next method in finding lower bounds for $\theta(x)$.

Lemma 3.26. *For $x > 0$ we have*

$$\psi(x) - \theta(x) < \psi(x^{1/2}) + \psi(x^{1/3}) + \psi(x^{1/5}). \tag{3.32}$$

Proof. By (3.19) we have

$$\psi(x) - \theta(x) = \psi(x^{1/2}) + \psi(x^{1/3}) - \sum_{n\ge1} \theta(x^{1/6n}) + \sum_{n\ge1} \theta(x^{1/6n-1}) + \sum_{n\ge1} \theta(x^{1/6n+1}).$$

Since $\theta(x)$ is an increasing function we have

$$-\sum_{n\ge1} \theta(x^{1/6n}) + \sum_{n\ge1} \theta(x^{1/6n+1}) < 0.$$

It follows that

$$\psi(x) - \theta(x) \leq \psi(x^{1/2}) + \psi(x^{1/3}) + \sum_{n\geq 1} \theta(x^{1/6n-1}). \tag{3.33}$$

By applying the fact that

$$\sum_{n\geq 1} \theta(x^{1/6n-1}) \leq \sum_{n\geq 1} \theta(x^{1/5n}),$$

on (3.33), we deduce

$$\psi(x) - \theta(x) \leq \psi(x^{1/2}) + \psi(x^{1/3}) + \sum_{n\geq 1} \theta(x^{1/5n}). \tag{3.34}$$

Since

$$\psi(x^{1/5}) = \sum_{n\geq 1} \theta(x^{1/5n}),$$

we find that (3.34) can be transformed into

$$\psi(x) - \theta(x) \leq \psi(x^{1/2}) + \psi(x^{1/3}) + \psi(x^{1/5}),$$

and the proof follows. □

Observe that (3.32) can be rewritten as

$$\theta(x) > \psi(x) - \psi(x^{1/2}) - \psi(x^{1/3}) - \psi(x^{1/5}). \tag{3.35}$$

We now replace (3.28) with (3.35) in Theorem 3.20 to derive the following.

Theorem 3.27. *Let $b > 0$. Then*

$$\theta(x) > B^{-}(b)x, \text{ for } x \geq e^{b},$$

where

$$B^{-}(b) = A^{-}(b) - \frac{A^{+}(b/2)}{e^{b/2}} - \frac{A^{+}(b/3)}{e^{2b/3}} - \frac{A^{+}(b/5)}{e^{4b/5}}.$$

Example 3.28. *By using Theorem 3.27 and Tables 5.1 and 5.2 we obtain Tables 5.16, 5.17 and 5.18. These tables provide our best results for the lower bounds. Moreover our results for $b \leq 100$ are sharper than Dusart [7, p. 15].*

We turn next to the Theorem 2 of Cook [2, p. 110] which states

$$\theta(x) \geq \psi(x) - \psi(x^{1/2}) - \psi(x^{1/3}) - \psi(x^{1/5}) + \theta(x^{1/6}) + \psi(x^{1/30}) - \theta(x^{1/7}), \qquad (3.36)$$

valid for $x > 0$.

We remark here to obtain more precise lower bound for $\theta(x)/x$ we can replace (3.28) with (3.36) in Theorem 3.20. For lower bound of $\theta(x^{1/6})$ we can use Tables 5.16, 5.17, and 5.18. For upper bound of $\theta(x^{1/7})$ we use the results of Table 5.3.

3.6 The inequality $\psi(x) - \theta(x) < c_2 x^{1/2} + c_3 x^{1/3}$

In this section we find upper bounds for $\psi(x) - \theta(x)$ in the form $c_2 x^{1/2} + c_3 x^{1/3}$ with $c_2, c_3 > 0$. This is done first by using upper bounds for $\theta(x)$ which is obtained in the previous sections. In particular we make a use of relation (3.23). Once this is done we verify Theorem 3.32 and Theorem 3.34 using the estimations for $\psi(x)$. Table 3.3 gives the history of numerical upper bounds for $\psi(x) - \theta(x)$ which are obtained by different authors.

Table 3.3:

Authors	c_2	c_3	α
Rosser & Schoenfeld [18]	1.02	3	0
Rosser & Schoenfeld [19]	1.001102	3	0
Schoenfeld [20]	1.001093	3	0
Costa Pereira [3]	1.001	1.1	$e^{36.84}$
Costa Pereira [3]	1.001	1	e^{36}
Dusart[7]	1.00007	1.78	0

$$\psi(x) - \theta(x) < c_2 x^{1/2} + c_3 x^{1/3}, \text{ for } x > \alpha.$$

Theorem 3.29. *Let* $b \geq 14.080$, *and let*

$$f(x) = 1 + \left(\frac{\log x}{\log 2} - 3\right) x^{-1/12}.$$

Assume that c_0 be an upper bound for $\theta(x)/x$ when $x > 0$. Then

$$\psi(x) - \theta(x) < B^+(b/2)x^{1/2} + c_0 f(e^b)x^{1/3}, \text{ for } x \geq e^b.$$

Proof. We write

$$\begin{aligned}\sum_{k=3}^{\left[\frac{\log x}{\log 2}\right]} \theta(x^{1/k}) < c_0 \sum_{k=3}^{\left[\frac{\log x}{\log 2}\right]} x^{1/k} &< c_0 \left(x^{1/3} + \left(\left[\frac{\log x}{\log 2}\right] - 3 \right) x^{1/4} \right) \\ &< c_0 \left(1 + \left(\frac{\log x}{\log 2} - 3 \right) x^{-1/12} \right) x^{1/3}.\end{aligned}$$

Since $f(x)$ is a decreasing function on $x \geq e^b$ where $b \geq 14.080$, we have

$$\sum_{k=3}^{[\frac{\log x}{\log 2}]} \theta(x^{1/k}) < c_0 f(e^b)x^{1/3}. \tag{3.37}$$

From (3.10) it follows that

$$\psi(x) - \theta(x) = \theta(x^{1/2}) + \sum_{k\geq 3} \theta(x^{1/k}). \tag{3.38}$$

In the above identity we bound $\theta(x^{1/2})$ by using (3.23). To bound $\sum_{k\geq 3} \theta(x^{1/k})$ we employ (3.37). The proof is now immediate. □

In order to improve the upper bound for $\psi(x) - \theta(x)$ given in the above theorem we divide $\sum_{k\geq 3} \theta(x^{1/k})$ into two sums and then estimates each of the sums separately. This enables us to establish the following theorem.

Theorem 3.30. *Let k_0 be a positive integer ≥ 3, $b > 0$, and c_0 satisfies $\theta(x)/x < c_0$ for $x > 0$. Let*

$$g(b, k_0) = \min\left\{A^+(b/k), c_0\right\} \sum_{k=3}^{k_0} e^{b(\frac{1}{k} - \frac{1}{3})} + c_0 \left(\frac{b}{\log 2} - k_0 \right) e^{b(\frac{1}{k_0+1} - \frac{1}{3})}.$$

Assume there exists positive constant b_0 such that $g(b, k_0)$ decreases for $b \geq b_0$. Then we have

$$\psi(x) - \theta(x) < B^+(b/2)x^{1/2} + g(b, k_0)x^{1/3}, \text{ for } x \geq e^b.$$

Proof. We have

$$\sum_{k\geq 3}\theta(x^{1/k})=\sum_{k=3}^{k_0}\theta(x^{1/k})+\sum_{k=k_0+1}^{\left[\frac{\log x}{\log 2}\right]}\theta(x^{1/k}).$$

Using $\theta(x)\leq\psi(x)$, we rewrite the last expression as follows.

$$\sum_{k\geq 3}\theta(x^{1/k})\leq\sum_{k=3}^{k_0}\min\left\{A^+(b/k),c_0\right\}x^{1/k}+\sum_{k=k_0+1}^{\left[\frac{\log x}{\log 2}\right]}c_0x^{1/k}=$$

$$\left(\min\left\{A^+(b/k),c_0\right\}\sum_{k=3}^{k_0}x^{\left(\frac{1}{k}-\frac{1}{3}\right)}+c_0\left(\frac{\log x}{\log 2}-k_0\right)x^{\left(\frac{1}{k_0+1}-\frac{1}{3}\right)}\right)x^{1/3}.$$

From this and (3.38) the proof follows. Note that the coefficient of $x^{1/3}$ in the above inequality is a decreasing function on $[e^b,\infty)$. □

Example 3.31. *Let $k_0=4$. If $b\geq 10.28$ then*

$$g(b,4)=\min\{A^+(b/3),c_0\}+\frac{\min\{A^+(b/4),c_0\}}{e^{b/2}}+\frac{c_0}{e^{2b/15}}\left(\frac{b}{\log 2}-4\right).$$

Therefore

$$\psi(x)-\theta(x)<B^+(b/2)x^{1/2}+g(b,4)x^{1/3},\ \text{for } x\geq e^b.$$

Observe that with $c_0=1+2.7571593586\times 10^{-5}$ from Example 3.13, for $b<84$ we have $c_0<A^+(b/3)$ and for $b<112$ we have $c_0<A^+(b/4)$. Using this information we generate Tables 3.4 and 3.5.

The idea of the following theorem is inspired by Costa Pereire [3].

Theorem 3.32. *Suppose that for $x\geq e^b$ there is a positive constant ε such that $\varepsilon>A^+(b/2)-1>0$ and*

$$e^b\geq\left(\frac{4A^+(b/5)}{5\left(A^+(b/2)-1-\varepsilon\right)}\right)^{\frac{10}{3}}.$$

We let

$$h(x)=\left(A^+(b/2)-1-\varepsilon\right)x^{1/6}+A^+(b/3)+A^+(b/5)x^{-2/15}.$$

Then

$$\psi(x)-\theta(x)<(1+\varepsilon)x^{1/2}+h(e^b)x^{1/3},\ \text{for } x\geq e^b.$$

Table 3.4:

b	$g(b,4)$	b	$g(b,4)$	b	$g(b,4)$
18.45	2.9326	27.4	1.9205	39	1.2884
18.50	2.9259	28	1.8704	40	1.2594
18.70	2.8990	29	1.7920	41	1.2331
19.00	2.8589	30	1.7196	42	1.2094
19.50	2.7926	31	1.6529	43	1.1879
20	2.7271	32	1.5916	44	1.1685
21	2.5992	33	1.5355	45	1.1511
22	2.4764	34	1.4842	46	1.1354
23	2.3593	34.53	1.4588	47	1.1212
24	2.2484	35	1.4373	48	1.1085
25	2.1441	36	1.3946	49	1.0971
26	2.0463	37	1.3557	50	1.0868
27	1.9551	38	1.3204	55	1.0493

$$\psi(x) - \theta(x) < x^{1/2} + g(b,4)x^{1/3}, \text{ for } x \geq e^b.$$

Proof. Since, for $x \geq e^b$,

$$\psi(x^{1/2}) < A^+(b/2)x^{1/2},\ \psi(x^{1/3}) < A^+(b/3)x^{1/3}, \text{ and } \psi(x^{1/5}) < A^+(b/5)x^{1/5},$$

by (3.32) we infer that

$$\psi(x) - \theta(x) < A^+(b/2)x^{1/2} + A^+(b/3)x^{1/3} + A^+(b/5)x^{1/5}.$$

The last inequality can be written as

$$\psi(x) - \theta(x) < (1+\varepsilon)x^{1/2} + \left(\left(A^+(b/2) - 1 - \varepsilon\right) x^{1/6} + A^+(b/3) + A^+(b/5)x^{-2/15}\right) x^{1/3}. \tag{3.39}$$

Since $A^+(b/2) - 1 - \varepsilon < 0$ and

$$x \geq \left(\frac{4A^+(b/5)}{5\left(A^+(b/2) - 1 - \varepsilon\right)}\right)^{\frac{10}{3}} \text{ for } x \geq e^b,$$

Table 3.5:

b	$B^+(b/2)$	$g(b,4)$
60	$1 + 9.2276182110 \times 10^{-6}$	1.0278
70	$1 + 1.0397503550 \times 10^{-6}$	1.0087
75	$1 + 4.2919385438 \times 10^{-7}$	1.0048
100	$1 + 1.2998600240 \times 10^{-9}$	1.0003
120	$1 + 3.9169369500 \times 10^{-11}$	1.0001

$$\psi(x) - \theta(x) < B^+(b/2)x^{1/2} + g(b,4)x^{1/3}, \text{ for } x \geq e^b.$$

$h(x)$ (the coefficient of $x^{1/3}$ in (3.39)) decreases on $x \geq e^b$. Thus

$$\psi(x) - \theta(x) < (1+\varepsilon)x^{1/2} + h(e^b)x^{1/3},$$

for $x \geq e^b$. □

Example 3.33. *Let $b = 100$. Using Table 5.2 we have*

$A^+(b/2) = 1+1.301\times10^{-9}$, $A^+(b/3) = 1+2.545\times10^{-6}$, and $A^+(b/5) = 1+6.123\times10^{-4}$.

Table 3.6 is generated by Theorem 3.32 and the above values.

Table 3.6:

ε	$h(e^{100})$
5.9070×10^{-8}	0.0002
5.9000×10^{-8}	0.0014
5.0000×10^{-8}	0.1572
1.0000×10^{-8}	0.8500
1.5000×10^{-9}	1.0223
1.4000×10^{-9}	1.0223
1.3020×10^{-9}	1.0223

$$\psi(x) - \theta(x) < (1+\varepsilon)x^{1/2} + h(e^{100})x^{1/3}, \text{ for } x \geq e^{100}.$$

By a similar argument we can establish an upper bound for $\psi(x)-\theta(x)$ over a finite range.

Theorem 3.34. *Assume that an interval I is the union of a finite collection of intervals $I_k = [m_k, n_k]$ and there are positive constants L^+ and $\varepsilon \geq 0$ such that $A_k^+(b/2) > 1+\varepsilon$. If*

$$m_k \geq \left(\frac{4A_k^+(b/5)}{5\left(A_k^+(b/2) - 1 - \varepsilon\right)} \right)^{10/3},$$

and

$$L^+ \geq \max_{[m_k,n_k]} \left\{ \left(A_k^+(b/2) - 1 - \varepsilon\right) n_k^{1/6} + A_k^+(b/3) + A_k^+(b/5) n_k^{-2/15} \right\},$$

then

$$\psi(x) - \theta(x) < (1+\varepsilon)x^{1/2} + L^+ x^{1/3} \text{ for } x \in I.$$

Proof. Since

$$\psi(x^{1/2}) < A_k^+(b/2)x^{1/2}, \ \psi(x^{1/3}) < A_k^+(b/3)x^{1/3}, \ \psi(x^{1/5}) < A_k^+(b/5)x^{1/5} \text{ for } x \in I_k,$$

from (3.32) it follows that

$$\psi(x) - \theta(x) < A_k^+(b/2)x^{1/2} + A_k^+(b/3)x^{1/3} + A_k^+(b/5)x^{1/5}, \text{ for } x \in I_k.$$

The last expression, for $x \in I_k$, can be rewritten as

$$\psi(x) - \theta(x) < (1+\varepsilon)x^{1/2} + \left((A_k^+(b/2) - 1 - \varepsilon)x^{1/6} + A_k^+(b/3) + A_k^+(b/5)x^{-2/15}\right) x^{1/3}.$$

Note that the coefficient of $x^{1/3}$ in the above expression increases if $A_k^+(b/2) - 1 - \varepsilon > 0$ and

$$x \geq \left(\frac{4A_k^+(b/5)}{5\left(A_k^+(b/2) - 1 - \varepsilon\right)} \right)^{10/3} \text{ for } x \in I_k.$$

Thus

$$\psi(x) - \theta(x) < (1+\varepsilon)x^{1/2} + L^+ x^{1/3} \text{ for } x \in I.$$

□

We finish this chapter with an example demonstrating the previous theorem.

Example 3.35. *We divide $[e^{100}, e^{140}]$ into four equal subintervals. By employing Table 5.2 and Theorem 3.34 with $\varepsilon = 0$ we obtain Table 3.7.*

Table 3.7:

Number	m_k	n_k	$A_k^+(b/2)$	$A_k^+(b/3)$	$A_k^+(b/5)$	L^+
1	e^{100}	e^{110}	$1 + 1.3010 \times 10^{-9}$	$1 + 2.5450 \times 10^{-6}$	$1 + 6.123 \times 10^{-4}$	1.1193
2	e^{110}	e^{120}	$1 + 1.4810 \times 10^{-10}$	$1 + 6.7750 \times 10^{-7}$	$1 + 2.706 \times 10^{-4}$	1.0718
3	e^{120}	e^{130}	$1 + 3.9170 \times 10^{-11}$	$1 + 1.1630 \times 10^{-7}$	$1 + 1.183 \times 10^{-4}$	1.1007
4	e^{130}	e^{140}	$1 + 3.9170 \times 10^{-11}$	$1 + 3.0110 \times 10^{-8}$	$1 + 5.121 \times 10^{-5}$	1.5328

$$\psi(x) - \theta(x) < x^{1/2} + L^+ x^{1/3}, \text{ for } x \in [m_k, n_k].$$

Chapter 4

Bounds for $\pi(x)$

4.1 Introduction

Let $\pi(x)$ be the number of primes not exceeding x. Chebyshev was the first who established the true order of $\pi(x)$. As we described in Chapters 1 and 2, by using elementary methods, he proved that for every $\epsilon > 0$ and sufficiently large x we have

$$(\nu - \epsilon)\frac{x}{\log x} \le \pi(x) \le (\frac{6}{5}\nu + \epsilon)\frac{x}{\log x},$$

where $\nu \simeq 0.921292022934$ is given in (2.1).

In order to approximate $\pi(x)$ we can use estimates on $\psi(x)$ and $\theta(x)$. Recall that by (1.2) we have

$$\psi(x) = \sum_{p^n \le x, n \ge 1} \log p = \sum_{p \le x} \left[\frac{\log x}{\log p}\right] \log p. \tag{4.1}$$

Using the trivial relation

$$[y] \le y < [y] + 1 \le 2[y], \text{ for } y > 1,$$

we find that

$$\left[\frac{\log x}{\log p}\right] \le \frac{\log x}{\log p} \le \left[\frac{\log x}{\log p}\right] + 1.$$

Therefore

$$\sum_{p \le x} \left[\frac{\log x}{\log p}\right] \log p \le \log x \sum_{p \le x} 1 \le \sum_{p \le x} \left[\frac{\log x}{\log p}\right] \log p + \sum_{p \le x} \log p.$$

Combining this with (4.1) and the trivial inequality $\theta(x) \leq \psi(x)$ yields

$$\psi(x) \leq \pi(x) \log x \leq \psi(x) + \theta(x) \leq 2\psi(x). \tag{4.2}$$

Assuming that

$$A^-(b)x < \psi(x) < A^+(b)x \text{ and } B^-(b)x < \theta(x) < B^+(b)x \text{ for } x \geq e^b,$$

(4.2) is transformed into

$$A^-(b)\frac{x}{\log x} < \pi(x) < \left(A^+(b) + B^+(b)\right)\frac{x}{\log x} \text{ for } x \geq e^b. \tag{4.3}$$

The last expression gives upper and lower bounds for $\pi(x)$ provided that we have bounds for $\theta(x)$ and $\psi(x)$. The bounds given by (4.3) are not sharp. Note that upper bound given by (4.3) are always bigger than 2.

In this chapter we will devise methods that give sharper bounds for $\pi(x)$. These estimates are established by using estimates for $\theta(x)$. The fundamental relation between $\pi(x)$ and $\theta(x)$ are given in the following lemma.

Lemma 4.1.

$$\pi(x) = \frac{\theta(x)}{\log x} + \int_2^x \frac{\theta(t)dt}{t\log^2 t}.$$

Proof. Let

$$a_n = \begin{cases} 1 & \text{if } n \text{ is prime,} \\ 0 & \text{otherwise .} \end{cases}$$

Following the definition of a_n we see that

$$\pi(x) = \sum_{p\leq x} 1 = \sum_{1<n\leq x} \frac{a_n \log n}{\log n}.$$

By applying the partial summation formula in the last sum and using definition of $\theta(x)$ we obtain

$$\begin{aligned}\pi(x) = \frac{\sum_{n\leq x} a_n \log n}{\log x} + \int_1^x \frac{\sum_{t\leq x} a_t \log t}{t\log^2 t}dt \\ = \frac{\theta(x)}{\log x} + \int_2^x \frac{\theta(t)dt}{t\log^2 t}.\end{aligned}$$

□

Using Lemma 4.1 we will be able to find upper and lower bounds for $\pi(x)$. This is done in Theorem 4.7 by means of estimates for $\theta(x)$. By this theorem in our hand we will be able to derive upper and lower bounds for $\pi(x)$ in a variety of forms.

4.2 Upper bounds over finite ranges

Let

$$\mathrm{li}(x) = \int_0^x \frac{dt}{\log t}.$$

Recall that Gauss conjectured, later proved by Hadamard and de la Vallée Poussin, that a good approximation for $\pi(x)$ can be given by $\mathrm{li}(x)$. The Prime Number Theorem with the remainder states that

$$\pi(x) = \mathrm{li}(x) + O(xe^{-c\sqrt{\log x}}), \tag{4.4}$$

for some positive constant c. Integrating $\mathrm{li}(x)$ by parts gives

$$\mathrm{li}(x) = \frac{x}{\log x} + \frac{1!x}{\log^2 x} + \cdots + \frac{(n-1)!x}{\log^n x} + O\left(\frac{x}{\log^{n+1} x}\right), \tag{4.5}$$

(see [11, page 65]). Hence we can expect to have

$$\mathrm{li}(x) < \frac{x}{\log x} + \frac{1!x}{\log^2 x} + \cdots + \frac{(\ell-1)!x}{\log^\ell x} + \frac{cx}{\log^{\ell+1} x}, \text{ for } x \geq x_0,$$

where $c > 0$ and ℓ is a natural number.

For the case $\ell = 1$ we have the following.

Proposition 4.2. *Suppose there exists positive constant c_1 such that $c_1 > 1$ and a positive constant x_0 such that $x_0 \geq \exp(2c_1/(c_1-1))$ and*

$$\mathrm{li}(x_0) < \frac{x_0}{\log x_0}\left(1 + \frac{c_1}{\log x_0}\right).$$

Then for $x \geq x_0$ we have

$$\mathrm{li}(x) < \frac{x}{\log x}\left(1 + \frac{c_1}{\log x}\right).$$

Proof. It suffices to show that

$$\mathrm{li}'(x) < \left(\frac{x}{\log x} + \frac{c_1 x}{\log^2 x}\right)', \text{ for } x \geq x_0.$$

Hence we need to show that

$$\frac{c_1 - 1}{\log^2 x} - \frac{2c_1}{\log^3 x} > 0,$$

which is true provided that $c_1 > 1$ and $x > \exp(2c_1/(c_1-1))$. □

Example 4.3. *Using Proposition 4.2 we choose* $c_1 = 1.01$ *so that* c_1 *be larger than* 1. *This gives* $x > \exp(202) \simeq 5.34 \times 10^{87}$. *To extend the range of* x *we increase the value of* c_1 *to obtain Table 4.1.*

Table 4.1:

x_0	c_1
51022	1.2762
10^{11}	1.0902
10^{12}	1.0817
10^{15}	1.0637

$$\mathrm{li}(x) < \frac{x}{\log x}\left(1 + \frac{c_1}{\log x}\right), \text{ for } x \geq x_0.$$

Next we mention a lemma which connect $\pi(x)$ and $\mathrm{li}(x)$ together.

Lemma 4.4. *For* $2 \leq x < 10^{14}$ *we have*

$$\pi(x) < \mathrm{li}(x).$$

Proof. See [13, p. 55]. □

By using Lemma 4.4, with the results of Example 4.3 we can establish upper bounds for $\pi(x)$ over finite ranges. For example we have the following.

Corollary 4.5. $\pi(x) < \frac{x}{\log x}\left(1 + \frac{1.2762}{\log x}\right)$, for $51022 \leq x \leq 10^{14}$.

We finish this section by describing a relation, given in [17, Lemma 4] between the inequality $\theta(x) < x$ and $\pi(x) < \mathrm{li}(x)$.

Lemma 4.6. *If* $\theta(x) < x$ *for* $e^{2.4} \leq x \leq K$, *then* $\pi(x) < \mathrm{li}(x)$ *for* $e^{2.4} \leq x \leq K$.

Proof. Since $\theta(x) < x$ for $e^{2.4} \leq x \leq K$, by Lemma 4.1 we have

$$\pi(x) < \frac{x}{\log x} + \int_{e^{2.4}}^{x} \frac{dt}{\log^2 t} + \int_{2}^{e^{2.4}} \frac{\theta(t)dt}{t\log^2 t}, \text{ for } e^{2.4} \leq x \leq K. \tag{4.6}$$

Now by applying integration by parts on the first integral in (4.6) we have

$$\pi(x) < \mathrm{li}(x) + \frac{e^{2.4}}{2.4} - \mathrm{li}(e^{2.4}) + \int_2^{e^{2.4}} \frac{\theta(t)dt}{t\log^2 t}.$$

Now the result follows since by numerical computation (see [17, Lemma 1]) we can show that

$$\frac{e^{2.4}}{2.4} - \mathrm{li}(e^{2.4}) + \int_2^{e^{2.4}} \frac{\theta(t)dt}{t\log^2 t} < 0.$$

□

4.3 Upper bounds for $\pi(x)$

The next result is our main tool in establishing bounds for $\pi(x)$.

Theorem 4.7. *Let $k \in \mathbb{N}$ and suppose that we have positive constants β and η_k such that*

$$|\theta(x) - x| < \eta_k \frac{x}{\log^k x} \quad \text{for } x \geq \beta. \tag{4.7}$$

Let

$$J(x, \eta_k) = \pi(x_0) - \frac{\theta(x_0)}{\log x_0} + \frac{x}{\log x} + \eta_k \frac{x}{\log^{k+1} x} + \int_{x_0}^{x} \left(\frac{1}{\log^2 y} + \frac{\eta_k}{\log^{k+2} y} \right) dy,$$

where $x_0 \geq \beta$. Then

$$J(x, -\eta_k) < \pi(x) < J(x, \eta_k), \tag{4.8}$$

for $x \geq x_0$.

Proof. By Lemma 4.1 we have

$$\pi(x) = \frac{\theta(x)}{\log x} + \int_2^x \frac{\theta(t)dt}{t\log^2 t}. \tag{4.9}$$

Since

$$\pi(x_0) - \frac{\theta(x_0)}{\log x_0} = \int_2^{x_0} \frac{\theta(t)dt}{t\log^2 t},$$

relation (4.9) can be transformed into

$$\pi(x) = \pi(x_0) - \frac{\theta(x_0)}{\log x_0} + \frac{\theta(x)}{\log x} + \int_{x_0}^{x} \frac{\theta(t)dt}{t\log^2 t}.$$

By means of the upper and lower bounds for $\theta(x)$ given in (4.7) we obtain

$$J(x, -\eta_k) < \pi(x) < J(x, \eta_k).$$

□

In order to bound $\pi(x)$ from the above, it would be enough to find an upper bound for $J(x, \eta_k)$.

Let

$$T_{\ell,c}(x) = \frac{x}{\log x}\left(1 + \frac{1!}{\log x} + \frac{2!}{\log^2 x} + \cdots + \frac{(\ell-1)!}{\log^{\ell-1} x} + \frac{c}{\log^{\ell} x}\right), \qquad (4.10)$$

$\ell \in \mathbb{N}$ and $c > 0$. Our strategy is to choose ℓ and c in such a way that

$$J(x, \eta_k) < T_{\ell,c}(x), \text{ for } x \geq x_0.$$

Let

$$G_{\ell,c,k}(x) = T_{\ell,c}(x) - J(x, \eta_k),$$

or equivalently

$$G_{\ell,c,k}(x) = \frac{x}{\log^2 x} + \frac{2!x}{\log^3 x} + \cdots + \frac{(\ell-1)!x}{\log^{\ell} x} + \frac{cx}{\log^{\ell+1} x} - R_0 - \frac{\eta_k x}{\log^{k+1} x} - \int_{x_0}^{x}\left(\frac{1}{\log^2 y} + \frac{\eta_k}{\log^{k+2} y}\right) dy,$$

where

$$R_0 = \pi(x_0) - \frac{\theta(x_0)}{\log x_0}.$$

We are interested in finding ℓ, c, k, x_0, and x_1 that satisfy $G_{\ell,c,k}(x) > 0$ for $x \geq x_1$.

In order to achieve this goal, it would be enough to choose $\ell, k \in \mathbb{N}$, and $c, x_0, x_1 > 0$ such that

$$G_{\ell,c,k}(x_1) > 0, \qquad (4.11)$$

and

$$G'_{\ell,c,k}(x) = \frac{c - \ell!}{\log^{\ell+1} x} - \frac{c(\ell+1)}{\log^{\ell+2} x} - \frac{\eta_k}{\log^{k+1} x} + \frac{k\eta_k}{\log^{k+2} x} > 0, \text{ for } x \geq x_1. \qquad (4.12)$$

To discuss the possibility of existence of $\ell, k \in \mathbb{N}$, and $c, x_0, x_1 > 0$ in such a way that (4.11) and (4.12) hold we consider the relations which might hold between ℓ and k and establish the following propositions.

Proposition 4.8. *Let c be a positive constant and $\ell, k \in \mathbb{N}$.*

(i) If $\ell > k$ then $G_{\ell,c,k}(x)$ is a decreasing function for large values of x.

(ii) If $\ell = k$, and $c > \ell! + \eta_\ell = k! + \eta_k$, then $G_{\ell,c,\ell}(x)$ is an increasing function for large values of x.

(iii) If $\ell < k$ and $c > \ell!$, then $G_{\ell,c,k}(x)$ is an increasing function for large values of x.

Proof. (*i*) We know by (4.12) that

$$G'_{\ell,c,k}(x) = \frac{c - \ell!}{\log^{\ell+1} x} - \frac{c(\ell+1)}{\log^{\ell+2} x} - \frac{\eta_k}{\log^{k+1} x} + \frac{k\eta_k}{\log^{k+2} x}. \tag{4.13}$$

If $\ell > k$ then in the above expression the dominant term will be $-\eta_k / \log^{k+1} x$. Since the coefficient of this term is negative the function $G_{\ell,c,k}(x)$ eventually will decrease for large values of x.

(*ii*) We note that when $\ell = k$ then

$$G'_{\ell,c,\ell}(x) = \frac{c - \ell! - \eta_\ell}{\log^{\ell+1} x} + \frac{\ell\eta_\ell - c\ell - c}{\log^{\ell+2} x}. \tag{4.14}$$

Since the coefficient of dominant term in the above equation is $c - \ell! - \eta_\ell$, the function $G_{\ell,c,\ell}(x)$ eventually will increase for large values of x provided that $c > \ell! + \eta_\ell$.

(*iii*) Since $\ell < k$, from (4.13) we see that if $c > \ell!$ then the dominant term $(c - \ell!)/\log^{\ell+1} x$ has a positive coefficient. Thus if $c > \ell!$ then $G_{\ell,c,k}(x)$ will increase when x is large enough. □

Proposition 4.9. *Let c be a positive number and $\ell, k \in \mathbb{N}$.*

(i) If $k = \ell$ and $c > \ell! + \eta_\ell = k! + \eta_k$ then $G_{\ell,c,\ell}(x)$ is an increasing function for

$$x > \exp\left(\frac{c\ell + c - \ell\eta_\ell}{c - \ell! - \eta_\ell}\right).$$

(ii) If $k = \ell + 1$ and $c > \ell!$ then $G_{\ell,c,\ell+1}(x)$ is an increasing function for

$$x > \exp\left(\frac{c(\ell+1) + \eta_{\ell+1} + \sqrt{(c(\ell+1) - \eta_{\ell+1})^2 + 4\eta_{\ell+1}\ell!}}{2(c - \ell!)}\right).$$

(iii) Let $k = \ell + 2$, $c > \ell!$, and

$$x_3 = \exp\left(\frac{c(\ell+1) + \sqrt{c^2(\ell+1)^2 + 3(c - \ell!)\eta_{\ell+2}}}{3(c - \ell!)}\right).$$

If there exists positive constant x_2 *such that*

$$(c-\ell!)\log^3 x_2 - c(\ell+1)\log^2 x_2 - \eta_{\ell+2}\log x_2 + (\ell+2)\eta_{\ell+2} > 0,$$

then $G_{\ell,c,\ell+2}(x)$ *is an increasing function for* $x > \max\{x_2, x_3\}$.

Proof. (i) First we consider the case that $k=\ell$. In this case by (4.14) we have

$$G'_{\ell,c,\ell}(x) = \frac{c-\ell!-\eta_\ell}{\log^{\ell+1} x} + \frac{\ell\eta_\ell - c\ell - c}{\log^{\ell+2} x}.$$

Now if $c > \ell! + \eta_\ell$ and $x > \exp\left((c\ell + c - \ell\eta_\ell)/(c-\ell!-\eta_\ell)\right)$ then $G'_{\ell,c,\ell}(x) > 0$, and (i) follows.

(ii) Next for the case that $k = \ell+1$ we have

$$G'_{\ell,c,\ell+1}(x) = \frac{c-\ell!}{\log^{\ell+1} x} - \frac{c(\ell+1)+\eta_{\ell+1}}{\log^{\ell+2} x} + \frac{(\ell+1)\eta_{\ell+1}}{\log^{\ell+3} x}. \tag{4.15}$$

Simplifying the above expression gives $G'_{\ell,c,\ell+1}(x) > 0$, provided that

$$(c-\ell!)\log^2 x - (c(\ell+1)+\eta_{\ell+1})\log x + (\ell+1)\eta_{\ell+1} > 0. \tag{4.16}$$

Since $c > \ell!$ and

$$x > \exp\left(\frac{c(\ell+1)+\eta_{\ell+1}+\sqrt{(c(\ell+1)-\eta_{\ell+1})^2 + 4\eta_{\ell+1}\ell!}}{2(c-\ell!)}\right),$$

then (4.16) is positive and (ii) follows.

(iii) If $k = \ell+2$ then

$$G'_{\ell,c,\ell+2}(x) = \frac{c-\ell!}{\log^{\ell+1} x} - \frac{c(\ell+1)}{\log^{\ell+2} x} - \frac{\eta_{\ell+2}}{\log^{\ell+3} x} + \frac{(\ell+2)\eta_{\ell+2}}{\log^{\ell+4} x}.$$

From the above we find that $G'_{\ell,c,\ell+2}(x) > 0$ provided that

$$(c-\ell!)\log^3 x - c(\ell+1)\log^2 x - \eta_{\ell+2}\log x + (\ell+2)\eta_{\ell+2} > 0. \tag{4.17}$$

Now let

$$g(x) = (c-\ell!)\log^3 x - c(\ell+1)\log^2 x - \eta_{\ell+2}\log x + (\ell+2)\eta_{\ell+2}.$$

It follows that

$$g'(x) = \frac{1}{x}\left(3(c-\ell!)\log^2 x - 2c(\ell+1)\log x - \eta_{\ell+2}\right),$$

which is positive if

$$x > \exp\left(\frac{c(\ell+1) + \sqrt{c^2(\ell+1)^2 + 3(c-\ell!)\eta_{\ell+2}}}{3(c-\ell!)}\right)$$

and $c > \ell!$. From this and the fact that $g(x_2) > 0$, we conclude that (4.17) holds for $x > \max\{x_2, x_3\}$ and the proof of (iii) follows. □

With these two propositions in hand we can consider numerical choices for ℓ and k.

- Let $\ell = k = 1$. In this case by (4.14) we have

$$G'_{1,c,1}(x) = \frac{c-1-\eta_1}{\log^2 x} + \frac{\eta_1 - 2c}{\log^3 x}.$$

Hence by using Proposition 4.9, part (i) if $c > \eta_1 + 1$, and $x > \exp((2c-\eta_1)/(c-\eta_1-1))$ then $G_{1,c,1}(x)$ is an increasing function.

Example 4.10. *With* $\eta_1 = 0.001$ *valid for* $x \geq 908994923$ *[7, Theorem 5.2, p. 4] we can choose* $x_0 = 10^{11}$ *and* $c = 1.002$*, so that* $c > 1 + \eta_1$*. This gives us* $x > \exp(2003) \simeq 7.8 \times 10^{869}$*. To extend the range of* x *we increase the value of* c*. Using the same* η_1 *we can establish Table 4.2.*

Table 4.2:

α	c_1
10^{11}	1.0920
10^{12}	1.0868
10^{15}	1.0648

$$\pi(x) < \frac{x}{\log x}\left(1 + \frac{c_1}{\log x}\right), \text{ for } x \geq \alpha.$$

- Let $\ell = 1$ and $k = 2$. Hence by (4.15) we find that

$$G'_{1,c,2}(x) = \frac{c-1}{\log^2 x} - \frac{2c+\eta_2}{\log^3 x} + \frac{2\eta_2}{\log^4 x}.$$

Now we employ Proposition 4.9, part (ii) to conclude that if $c > 1$ and

$$x > \exp\left(\frac{2c+\eta_2+\sqrt{(2c-\eta_2)^2+8\eta_2}}{2(c-1)}\right),$$

then $G_{1,c,2}(x)$ will be an increasing function.

Example 4.11. *By [7, Theorem 5.2, p. 4], we have for* $x \geq 7713133853$ *that* $\eta_2 = 0.01$. *We choose* $x_0 = 10^{11}$, *and* $c = 1.001$ *noting that c must be larger than one. This gives* $x > \exp(2012) \simeq 6.4 \times 10^{873}$. *In order to extend the range of x we increase the value of c to establish Table 4.3.*

Table 4.3:

α	c_1
10^{11}	1.0910
10^{12}	1.0830
10^{15}	1.0640

$$\pi(x) < \frac{x}{\log x}\left(1+\frac{c_1}{\log x}\right), \text{ for } x \geq \alpha$$

Comparing the results of Examples 4.10 and 4.11 shows that by using a bigger k in definition of $J(x, \eta_k)$ we will get an improved upper bound for $\pi(x)$.

Remark 4.12. *Using the fact that* $\pi(x) < \mathrm{li}(x)$ *for* $2 \leq x < 10^{14}$, *from Lemma 4.4, with the result of Example 4.3 for* $x \geq 10^{11}$ *yield*

$$\pi(x) < \frac{x}{\log x}\left(1+\frac{1.0902}{\log x}\right), \text{ for } 10^{11} \leq x \leq 10^{14}. \quad (4.18)$$

We now let $\ell = 1$, $k = 2$, *and* $\eta_2 = 0.01$ *from [7, Theorem 5.2, p. 2] to establish the following.*

$$\pi(x) < \frac{x}{\log x}\left(1+\frac{1.0690}{\log x}\right), \text{ for } x \geq 10^{14}. \quad (4.19)$$

With comparing the relations (4.18), *and* (4.19) *we find that*

$$\pi(x) < \frac{x}{\log x}\left(1 + \frac{1.0902}{\log x}\right), \text{ for } x \geq 10^{11}.$$

This shows that $\mathrm{li}(x)$ *can provide a better approximation for* $\pi(x)$ *over* $2 \leq x < 10^{14}$, *compared to* $J(x, \eta_k)$, *since the above gives an improved result compared to the result of Example 4.11 in the given range.*

- Let $\ell = 2$ and $k = 2$. It follows from (4.14) that

$$G'_{2,c,2}(x) = \frac{c - 2 - \eta_2}{\log^3 x} + \frac{2\eta_2 - 3c}{\log^4 x}.$$

Moreover part (i) of Proposition 4.9 shows that we can choose $c > 2 + \eta_2$ and $x > \exp\left((3c - 2\eta_2)/(c - 2 - \eta_2)\right)$ in order to have $G'_{2,c,2}(x) > 0$.

Example 4.13. *With* $\eta_2 = 0.01$ *valid for* $x \geq 7713133853$ *[7, Theorem 5.2, p. 4] we choose* $x_0 = 10^{11}$ *and* $c = 2.02$ *so that* $c > 2 + \eta_2$. *We obtain* $x > \exp(604) \simeq 2.1 \times 10^{262}$. *To obtain a larger range for* x *we increase the value of* c. *We get Table 4.4.*

Table 4.4:

α	c_1
10^{11}	2.296
10^{12}	2.267
10^{15}	2.208

$$\pi(x) < \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{c_1}{\log^2 x}\right), \text{ for } x \geq \alpha$$

- Next we consider the case that $\ell = 3$ and $k = 3$. From (4.14) we find that

$$G'_{3,c,3}(x) = \frac{c - 3! - \eta_3}{\log^4 x} + \frac{3\eta_3 - 4c}{\log^5 x}.$$

Moreover Proposition 4.9, part (i), implies that if $c > 3! + \eta_3$ then $G_{3,c,3}(x)$ is an increasing function for

$$x > \exp\left(\frac{4c - 3\eta_3}{c - 3! - \eta_3}\right).$$

Example 4.14. *For $x \geq 158822621$ we have $\eta_3 = 0.78$ [7, Theorem 5.2, p. 4]. Since $c > 3! + \eta_3$ we can choose $c = 6.79$. Hence $x > \exp(2482) \simeq 8.3 \times 10^{1077}$. To obtain the larger range for x we increase the value of c. This gives Table 4.5.*

Table 4.5:

$x_0 = \alpha$	c_1
10^{11}	7.9724
10^{12}	7.8510
10^{15}	7.5976

$$\pi(x) < \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2}{\log^2 x} + \frac{c_1}{\log^3 x}\right), \text{ for } x \geq \alpha$$

Here we note that for large x we can make c_1 as close to 6 as we want. To do this we need to find a smaller value for η_3.

Proposition 4.15. *For $x \geq \exp(6000)$, we can choose $\eta_3 \geq 0.016023$.*

Proof. Following the proof of Theorem 5.2 of [7] we should choose η_3 such that

$$\log^3 x\sqrt{\frac{8}{\pi}}X^{1/2}\exp(-X) \leq \eta_3, \tag{4.20}$$

where $X = \sqrt{\log x/R}$ and $R = 5.70176$. Let $Q(x) = \log^3 xX^{1/2}\exp(-X)$ or equivalently

$$Q(x) = \frac{\log^3 x(\log x)^{1/4}\exp(-\sqrt{\log x/R})}{R^{1/4}}.$$

It follows that

$$Q'(x) = \frac{\exp(\sqrt{-\log x/R})(\log x)^{9/4}}{R^{1/4}x}\left(\frac{13}{4} - \frac{1}{2}\sqrt{\frac{\log x}{R}}\right),$$

which is negative when $x \geq \exp(1373.6)$. Since the left-hand side of (4.20) is a decreasing function for $x \geq \exp(1373.6)$ we can establish $\eta_3 \geq 0.016023$ for $x \geq \exp(6000)$. $\square$

We now let $\ell = k = 3$. By part (i) of Proposition 4.9 we choose $c > 3! + 0.016023$ and $x > \exp(3440000)$ to obtain

$$\pi(x) < \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2}{\log^2 x} + \frac{6.01603}{\log^3 x}\right).$$

Remark 4.16. *Here we describe how one can obtain an upper bound for $\pi(x_0) - \theta(x_0)/\log x_0$, where $x_0 > 0$. We proceed as follows.*

By Lemma 4.1 we can infer

$$\pi(x_0) - \frac{\theta(x_0)}{\log x_0} = \int_2^{x_0} \frac{\theta(t)}{t\log^2 t}dt.$$

This implies

$$\pi(x_0) - \frac{\theta(x_0)}{\log x_0} = \int_2^{\xi} \frac{\theta(t)}{t\log^2 t}dt + \int_{\xi}^{x_0} \frac{\theta(t)}{t\log^2 t}dt,$$

where ξ is the largest number such that the inequality $\theta(x) < x$ holds. Assuming that $\xi \simeq e^b$, for some $b > 0$ and using the fact that

$$\theta(x) < B^+(b)x, \text{ for } x > e^b,$$

we obtain

$$\pi(x_0) - \frac{\theta(x_0)}{\log x_0} \leq \int_2^{\xi} \frac{dt}{\log^2 t} + B^+(b)\int_{\xi}^{x_0} \frac{dt}{\log^2 t}. \tag{4.21}$$

The right-hand side of the last expression is computable. This gives us the upper bound for $\pi(x_0) - \theta(x_0)/\log x_0$. This is useful especially when we are dealing with large values of x.

We assume $\xi \simeq e^{27.4}$. From (3.6) we know that $\theta(x) < x$ for $x \in (0, \xi]$. By Example 3.13 we have

$$\theta(x) < \left(1 + 2.7571593586 \times 10^{-5}\right)x, \text{ for } x \geq e^{27.4}.$$

We let $c_0 = 1 + 2.7571593586 \times 10^{-5}$ and

$$\kappa_1 = \int_2^{x_0} \frac{dt}{\log^2 t} + c_0\int_{e^{27.4}}^{x_0} \frac{dt}{\log^2 t}.$$

Then (4.21) *is transformed into*

$$\pi(x_0) - \frac{\theta(x_0)}{\log x_0} \le \kappa_1.$$

We also, by direct computation, have

$$\pi(x_0) - \frac{\theta(x_0)}{\log x_0} \simeq \kappa_2.$$

A sample of values of κ_1 *and* κ_2 *for different* x_0 *is given in Table 4.6.*

Table 4.6:

x_0	κ_1	κ_2
10^{12}	1.416750519×10^9	1.41674029×10^9
10^{14}	$1.028414206 \times 10^{11}$	$1.02838602 \times 10^{11}$
10^{15}	$8.916304196 \times 10^{11}$	8.9160595×10^{11}

$$\pi(x_0) - \frac{\theta(x_0)}{\log x_0} \le \kappa_1 \text{ and } \pi(x_0) - \frac{\theta(x_0)}{\log x_0} \simeq \kappa_2$$

The next proposition leads to a new form of upper and lower bounds for $\pi(x)$.

Proposition 4.17. (*i*) *There exists* $\omega_1 > 0$ *such that*

$$\frac{x}{\log x - 1 + (\log x)^{-1}} < \pi(x) < \frac{x}{\log x - 1 - \omega_1(\log x)^{-1}},$$

when x *is large enough.*

(*ii*) *We have*

$$\lim_{x\to\infty} \left(\frac{x}{\pi(x)} - \log x + 1 \right) = 0.$$

Proof. Combining (4.4) and (4.5) gives

$$\pi(x) = \frac{x}{\log x} + \frac{1!x}{\log^2 x} + \frac{2!x}{\log^3 x} + \cdots + \frac{(\ell-1)!x}{\log^\ell x} + O\left(\frac{x}{\log^{\ell+1} x}\right).$$

This implies that there exist $\omega_1 > 0$ and $\delta_1 > 0$ such that for $x > \delta_1$ we have

$$\pi(x) \leq \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{\omega_1 x}{\log^3 x} = \frac{x}{\log x}\left(1 + \frac{1 + \omega_1/\log x}{\log x}\right). \tag{4.22}$$

Observe that

$$\frac{x}{\log x}\left(1 + \frac{b}{\log x}\right) < \frac{x}{\log x - b}.$$

Using this, (4.22) can be transformed into

$$\pi(x) < \frac{x}{\log x - 1 - \omega_1(\log x)^{-1}}, \tag{4.23}$$

for $x > \delta_1$. From (4.4), and (4.5) we also can infer there exist $\omega_2 > 0$ and $\delta_2 > 0$ such that for $x > \delta_2$ we have

$$\frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2x}{\log^3 x} - \frac{\omega_2 x}{\log^4 x} \leq \pi(x). \tag{4.24}$$

Using the fact that

$$\frac{x}{\log x - 1 + (\log x)^{-1}} < \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2x}{\log^3 x} - \frac{\omega_2 x}{\log^4 x},$$

when x is large enough, allows us to deduce from (4.24).

$$\frac{x}{\log x - 1 + (\log x)^{-1}} < \pi(x), \tag{4.25}$$

for large x. Relation (4.23) combined with (4.25) gives

$$\frac{x}{\log x - 1 + (\log x)^{-1}} < \pi(x) < \frac{x}{\log x - 1 - \omega_1(\log x)^{-1}},$$

when x is large enough. This proves (i).

Taking the limit of both sides of the last expression and using squeeze theorem complete the proof of (ii). □

Here we point out one of the consequences of Proposition 4.17.

Proposition 4.18. *For all $a > 0$, there exists a positive constant δ_3 such that for $x > \delta_3$ we have*

$$\frac{x}{\log x - 1 + a} < \pi(x) < \frac{x}{\log x - 1 - a}.$$

Proof. By Proposition 4.17, we have for all $a > 0$ there is a $\delta_3 > 0$ such that for $x > \delta_3$

$$\left| \frac{x}{\pi(x)} - \log x + 1 \right| < a.$$

This implies

$$-a < \frac{x}{\pi(x)} - \log x + 1 < a.$$

Simplifying the above gives the assertion. □

Now we are ready to present the following theorem which gives an upper bound for $\pi(x)$ in the form $T(x,a) = x/(\log x - a)$, where a is a positive constant.

Proposition 4.19. *Let a be a positive constant and $k \in \mathbb{N}$. Let*

$$V_a(x,k) = (a-1)\log^{k+2} x - a^2 \log^{k+1} x - \eta_k \log^3 x +$$
$$(k\eta_k + 2a\eta_k)\log^2 x - (a^2\eta_k + 2ak\eta_k)\log x + k\eta_k a^2.$$

Suppose that there exists a positive constant $x_4 \geq x_0$ such that $J(x_4, \eta_k) < T(x_4, a)$ and $V_a(x,k) > 0$ for $x > x_4$. Then

$$\pi(x) < \frac{x}{\log x - a},$$

for $x \geq x_4$.

Proof. By (4.8) we know that

$$\pi(x) < J(x, \eta_k).$$

Hence it suffices to show that

$$J(x, \eta_k) < T(x, a).$$

In order to do this we consider

$$G_a(x,k) = T(x,a) - J(x, \eta_k).$$

We see that $G_a'(x,k) = V_a(x,k)/\log^{k+2} x(\log x - a)^2$. Since $G_a'(x,k) > 0$ for $x \geq x_4$ and $J(x_4, \eta_k) < T(x_4, a)$, we infer that

$$J(x, \eta_k) < T(x, a)$$

for $x \geq x_4$. □

Here we consider some special cases for k.

- Let $k = 1$. Hence

$$V_a(x,1) = (a-1-\eta_1)\log^3 x + (-a^2+\eta_1+2a\eta_1)\log^2 x - (a^2\eta_1+2a\eta_1)\log x + \eta_1 a^2.$$

If $a > 1+\eta_1$ and

$$x > \exp\left(\frac{a^2-\eta_1-2a\eta_1+\sqrt{(a^2-\eta_1-2a\eta_1)^2+3(a-1-\eta_1)(a^2\eta_1+2a\eta_1)}}{3(a-1-\eta_1)}\right),$$

then $V_a(x,1)$ is an increasing function.

Example 4.20. *With $\eta_1 = 0.001$, valid for $x \geq 908994923$, [7, Theorem 5.2, p. 4], we choose $x_0 = 10^{10}$, and $a = 1.002$ so that $a > 1+\eta_1$. This gives $x > 66 \times 10^{290}$. We increase the values of a to obtain*

$$\pi(x) < \frac{x}{\log x - 1.0520}, \text{ for } x \geq 10^{10}.$$

- Let $k = 2$. This implies

$$V_a(x,2) = (a-1)\log^4 x - (a^2+\eta_2)\log^3 x + (2\eta_k+2a\eta_2)\log^2 x - (a^2\eta_2+4a\eta_2)\log x + 2\eta_2 a^2.$$

We are now looking for the possibility of existence of $a, d_1 > 0$ such that $V_a(x,2)$ increases for $x > d_1$. We require to choose $a > 1$ and $x > 0$ such that

$$x > \exp\left(\frac{3(a^2+\eta_2)+\sqrt{9(a^2+\eta_2)+24(2\eta_2+2a\eta_2)(a-1)}}{12(a-1)}\right),$$

and $V_a'(x,2) > 0$.

Example 4.21. *For $x \geq 7713133853$ we have $\eta_2 = 0.01$ [7, Theorem 5.2, p. 4]. We choose $x_0 = 10^{11}$, and $a = 1.001$ so that $a > 1$. We obtain $x > 56 \times 10^{220}$. In order to extend the range we increase the values of a. We obtain*

$$\pi(x) < \frac{x}{\log x - 1.0456}, \text{ for } x \geq 10^{11}.$$

4.4 Lower bounds over finite ranges

First we state a lemma which plays an important role in finding lower bounds for $\pi(x)$ over finite ranges.

Lemma 4.22. *We have*

$$\int_{\sqrt{x}}^{x} \frac{dt}{\log t} = \mathrm{li}(x) - \mathrm{li}(x^{1/2}) < \pi(x), \text{ for } 11 \le x \le 10^8. \tag{4.26}$$

Proof. See [18, Theorem 16, p. 72]. □

By using Lemma 4.22, we will be able to verify the following theorem.

Theorem 4.23. *We have*

$$\frac{x}{\log x}(1 + \frac{1}{\log x}) < \pi(x), \text{ for } 188 \le x \le 10^8.$$

Proof. Using (4.26), it suffices to prove that

$$\frac{x}{\log x}(1 + \frac{c}{\log x}) < \int_{\sqrt{x}}^{x} \frac{dt}{\log t},$$

where $c = 1$. We consider the difference between the two functions in the last expression to get

$$I_c(x) = \int_{\sqrt{x}}^{x} \frac{dt}{\log t} - \frac{x}{\log x}(1 + \frac{c}{\log x}).$$

We see that

$$I'_c(x) = \frac{1-c}{\log^2 x} + \frac{2c}{\log^3 x} - \frac{1}{\sqrt{x}\log x}.$$

Let $c = 1$. In this case $I_1(x)$ and $I'_1(x)$ are positive for $x \ge 188$. This leads to

$$\frac{x}{\log x}(1 + \frac{1}{\log x}) < \pi(x), \text{ for } 188 \le x \le 10^8,$$

as desired. □

4.5 Lower bounds for $\pi(x)$

Since by (4.8) we have

$$\pi(x) > J(x, -\eta_k),$$

in order to bound $\pi(x)$ from below we require to find a lower bound for $J(x, -\eta_k)$. Let $T_{\ell,c}(x)$ be given by (4.10). We will look for $\ell \in \mathbb{N}$ and $c, x_0 > 0$, such that for $x \geq x_0$, $T_{\ell,c}(x) < J(x, -\eta_k)$.

Let us denote

$$S_{\ell,c,k}(x) = T_{\ell,c}(x) - J(x, -\eta_k).$$

Simplifying the above expression gives

$$S_{\ell,c,k}(x) = \frac{x}{\log^2 x} + \frac{2!x}{\log^3 x} + \cdots + \frac{(\ell-1)!x}{\log^\ell x} + \frac{cx}{\log^{\ell+1} x} - R_0 + \frac{\eta_k x}{\log^{k+1} x} - \int_{x_0}^{x} \left(\frac{1}{\log^2 y} - \frac{\eta_k}{\log^{k+2} y} \right) dy,$$

where $R_0 = \pi(x_0) - \theta(x_0)/\log x_0$. In order to make $T_{\ell,c}(x)$ a lower bound for $J(x, -\eta_k)$ we expect to find positive constant x_0, x_1, c and $\ell, k \in \mathbb{N}$ such that $S_{\ell,c,k}(x_1) < 0$ and

$$S'_{\ell,c,k}(x) = \frac{c - \ell!}{\log^{\ell+1} x} - \frac{c(\ell+1)}{\log^{\ell+2} x} + \frac{\eta_k}{\log^{k+1} x} - \frac{k\eta_k}{\log^{k+2} x} < 0, \text{ for } x \geq x_1. \tag{4.27}$$

We now consider the following proposition.

Proposition 4.24. *Let c be a positive constant and $\ell, k \in \mathbb{N}$.*

(i) If $\ell > k$ then $S_{\ell,c,k}(x)$ is an increasing function for large values of x.

(ii) If $\ell = k$ and $c < \ell! - \eta_\ell = k! - \eta_k$ then $S_{\ell,c,\ell}(x)$ is a decreasing function for large values of x.

(iii) If $\ell < k$ and $c < \ell!$ then $S_{\ell,c,k}(x)$ is a decreasing function for large values of x.

Proof. (*i*) We turn our attention to (4.27), which states

$$S'_{\ell,c,k}(x) = \frac{c - \ell!}{\log^{\ell+1} x} - \frac{c(\ell+1)}{\log^{\ell+2} x} + \frac{\eta_k}{\log^{k+1} x} - \frac{k\eta_k}{\log^{k+2} x}.$$

If $\ell > k$ then the dominant term in the last expression would be $\eta_k/\log^{k+1} x$. The coefficient of this term is positive. Hence the function $S_{\ell,c,k}(x)$ will increase when x is large enough.

(*ii*) By means of Relation (4.27) we find that

$$S'_{\ell,c,\ell}(x) = \frac{c - \ell! + \eta_\ell}{\log^{\ell+1} x} - \frac{\ell\eta_\ell + c\ell + c}{\log^{\ell+2} x}, \tag{4.28}$$

provided that $\ell = k$. In this case if we choose $c < \ell! - \eta_l$ then the function $S_{\ell,c,\ell}(x)$ will eventually decrease when x is large enough. This is true since the dominant terms will be negative.

(*iii*) By (4.27), we find that the dominant term, which is $(c - \ell!)/\log^{\ell+1} x$, has a negative coefficient if $c < \ell!$. Therefore $S_{\ell,c,k}(x)$ will decrease for large x provided that $c < \ell!$. □

Proposition 4.25. *Let c be a positive number and $\ell, k \in \mathbb{N}$.*

(*i*) *If $k = \ell$ and $c < \ell! - \eta_\ell = k! - \eta_k$ then $S_{\ell,c,\ell}(x)$ is a decreasing function for*

$$x > \exp\left(\frac{c\ell + c + \ell\eta_l}{c - \ell! + \eta_l}\right).$$

(*ii*) *If $k = \ell + 1$ and $c < \ell!$ then $S_{\ell,c,\ell+1}(x)$ is a decreasing function for*

$$x > \exp\left(\frac{c(\ell+1) - \eta_{\ell+1} + \sqrt{(c(\ell+1) + \eta_{\ell+1})^2 + 4\eta_{\ell+1}(\ell+1)(c - \ell!)}}{2(c - \ell!)}\right).$$

(*iii*) *Let $k = \ell + 2$, $c < \ell!$, and*

$$x_7 = \exp\left(\frac{c(\ell+1) + \sqrt{c^2(\ell+1)^2 - 3(c - \ell!)\eta_{\ell+2}}}{3(c - \ell!)}\right).$$

If there exists a positive constant x_6 such that $x_7 \geq x_6$, and

$$(c - \ell!)\log^3 x_6 - c(\ell+1)\log^2 x_6 + \eta_{\ell+2}\log x_6 - (\ell+2)\eta_{\ell+2} < 0,$$

then $S_{\ell,c,\ell+2}(x)$ is a decreasing function for $x > x_7$.

Proof. (*i*) For the case $k = \ell$ we have by (4.28)

$$S'_{\ell,c,\ell}(x) = \frac{c - \ell! + \eta_\ell}{\log^{\ell+1} x} - \frac{\ell\eta_\ell + c\ell + c}{\log^{\ell+2} x}. \tag{4.29}$$

In this case $S'_{\ell,c,\ell}(x) < 0$ if $c < \ell! - \eta_\ell$ and $x > \exp\left((c\ell + c + \ell\eta_\ell)/(c - \ell! + \eta_\ell)\right)$.

(ii) We now consider the case that $k = \ell + 1$. By (4.27) we find that

$$S'_{\ell,c,\ell+1}(x) = \frac{c - \ell!}{\log^{\ell+1} x} + \frac{\eta_{\ell+1} - c(\ell+1)}{\log^{\ell+2} x} - \frac{(\ell+1)\eta_{\ell+1}}{\log^{\ell+3} x}. \tag{4.30}$$

This gives $S'_{\ell,c,\ell+1}(x) < 0$ when

$$(c - \ell!) \log^2 x + (\eta_{\ell+1} - c(\ell+1)) \log x - (\ell+1)\eta_{\ell+1} < 0.$$

Now we choose $c < \ell!$ and

$$x > \exp\left(\frac{c(\ell+1) - \eta_{\ell+1} + \sqrt{(c(\ell+1) + \eta_{\ell+1})^2 + 4\eta_{\ell+1}(\ell+1)(c - \ell!)}}{2(c - \ell!)} \right).$$

Hence $S'_{\ell,c,\ell+1}(x) < 0$ and (ii) follows.

(iii) We now consider the case $k = \ell + 2$. We have by (4.27)

$$S'_{\ell,c,\ell+2}(x) = \frac{c - \ell!}{\log^{\ell+1} x} - \frac{c(\ell+1)}{\log^{\ell+2} x} + \frac{\eta_{\ell+2}}{\log^{\ell+3} x} - \frac{(\ell+2)\eta_{\ell+2}}{\log^{\ell+4} x}.$$

Therefore $S'_{\ell,c,\ell+2}(x) < 0$ if

$$w(x) = (c - \ell!) \log^3 x - c(\ell+1) \log^2 x + \eta_{\ell+2} \log x - (\ell+2)\eta_{\ell+2} < 0. \tag{4.31}$$

We see that

$$w'(x) = \frac{1}{x} \left(3(c - \ell!) \log^2 x - 2c(\ell+1) \log x + \eta_{\ell+2} \right).$$

The above expression is negative provided that

$$x > \exp\left(\frac{c(\ell+1) + \sqrt{c^2(\ell+1)^2 - 3(c - \ell!)\eta_{\ell+2}}}{3(c - \ell!)} \right),$$

and $c < \ell!$. Thus by using the fact that $w(x_6) < 0$, we find that (4.31) holds. □

We proceed by considering different numerical choices for ℓ and k.

- Let $\ell = k = 1$. We have by (4.29)

$$S'_{1,c,1}(x) = \frac{c - 1 + \eta_1}{\log^2 x} - \frac{\eta_1 + 2c}{\log^3 x}.$$

  We also have by Proposition 4.25 that if $c < 1 - \eta_1$ and $x > \exp(2c + \eta_1/c - 1 + \eta_1)$ then $S_{1,c,1}(x)$ is a decreasing function.

Example 4.26. *With* $\eta_1 = 0.001$ *valid for* $x \geq 908994923$ *[7, Theorem 5.2, p. 4], we choose* $x_0 = 10^{10}$ *and* $c = 0.9989$ *so that* $c < 1 - \eta_1$. *This gives the following result.*

$$\pi(x) > \frac{x}{\log x}\left(1 + \frac{0.9989}{\log x}\right), \text{ for } x \geq 10^{10}.$$

- Let $k = 2$ and $\ell = 1$. In this case by (4.30), we see that

$$S'_{1,c,2}(x) = \frac{c-1}{\log^2 x} + \frac{\eta_2 - 2c}{\log^3 x} - \frac{2\eta_2}{\log^4 x}.$$

Using part (ii) of Proposition 4.25 follows that if $1 > c$ and

$$x > \exp\left((2c - \eta_2 + \sqrt{(2c+\eta_2)^2 + 8\eta_2(c-1)})/2(c-1)\right),$$

then $S_{1,c,2}(x)$ is a decreasing function.

Example 4.27. *With* $\eta_2 = 0.01$ *valid for* $x \geq 7713133853$ *[7, Theorem 5.2, p. 4], we choose* $x_0 = 10^{11}$, *and* $c = 0.9999$ *so that* $c < 1$. *Hence*

$$\pi(x) > \frac{x}{\log x}\left(1 + \frac{0.9999}{\log x}\right), \text{ for } x \geq 10^{11}.$$

- Let $k = \ell = 2$. By (4.29) we can infer

$$S'_{2,c,2}(x) = \frac{c - 2 + \eta_2}{\log^3 x} - \frac{2\eta_2 + 3c}{\log^4 x}.$$

If $c < 2 - \eta_2$ and $x > \exp\left((3c + 2\eta_2)/(c - 2 + \eta_2)\right)$ then $S'_{2,c,2}(x) < 0$.

Example 4.28. *For* $x \geq 7713133853$ *we have* $\eta_2 = 0.01$, *[7, Theorem 5.2, p. 4]. We choose* $x_0 = 10^{11}$, *and* $c = 1.9899$, *so that* $c < 2 - 0.01$, *to get*

$$\pi(x) > \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{1.9899}{\log^2 x}\right), \text{ for } x \geq 10^{11}.$$

- We now consider the case that $k = \ell = 3$. Using (4.29) we find that

$$S'_{3,c,3}(x) = \frac{c - 6 + \eta_3}{\log^4 x} - \frac{3\eta_3 + 4c}{\log^5 x}.$$

We have $S'_{3,c,3}(x) < 0$ if $c < 6 - \eta_3$ and $x > \exp\left((4c + 3\eta_3)/(c - 6 + \eta_3)\right)$.

Example 4.29. *With* $\eta_3 = 0.78$ *valid for* $x \geq 158822621$ *[7, Theorem 5.2, p. 4], we choose* $x_0 = 10^{10}$*, and* $c = 5.2199$ *so that* $c < 5.22$*. Hence*

$$\pi(x) > \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2}{\log^2 x} + \frac{5.2199}{\log^3 x}\right), \text{ for } x \geq 10^{10}.$$

We next consider lower bounds for $\pi(x)$ in the form $T(x,a) = x/(\log x - a)$, where a is a positive number.

Proposition 4.30. *Let* a *be positive and* $k \in \mathbb{N}$*. We denote by*

$$h_a(x,k) = (a-1)\log^{k+2} x - a^2 \log^{k+1} x + \eta_k \log^3 x - (k\eta_k + 2a\eta_k)\log^2 x + (a^2\eta_k + 2ak\eta_k)\log x - k\eta_k a^2.$$

Assume that there exist a positive constants $x_5 \geq x_0$ *such that* $T(x_5, a) < J(x_5, -\eta_k)$ *and* $h_a(x,k) < 0$ *for* $x > x_5$*. Then*

$$\frac{x}{\log x - a} < \pi(x),$$

for $x \geq x_5$*.*

Proof. By (4.8), we know that

$$J(x, -\eta_k) < \pi(x).$$

Therefore we require to prove that

$$T(x,a) < J(x, -\eta_k).$$

In order to do this we consider

$$S_a(x,k) = T(x,a) - J(x, -\eta_k).$$

We see that $S_a'(x,k) = h_a(x,k)/\log^{k+2} x(\log x - a)^2$. Since $S_a'(x,k) < 0$ for $x \geq x_5$ and $T(x_5, a) < J(x_5, -\eta_k)$, we infer that

$$T(x,a) < J(x, -\eta_k),$$

for $x \geq x_5$. □

We now consider some specific choices for k.

- Let $k = 1$. This gives

  $$h_a(x,1) = (a-1+\eta_1)\log^3 x - (a^2+\eta_1+2a\eta_1)\log^2 x + (a^2\eta_1+2a\eta_1)\log x - \eta_1 a^2.$$

  In this case if $a < 1-\eta_1$ and

  $$x > \exp\left(\frac{a^2+\eta_1+2a\eta_1+\sqrt{(a^2+\eta_1+2a\eta_1)^2-3(a-1+\eta_1)(a^2\eta_1+2a\eta_1)}}{3(a-1+\eta_1)}\right),$$

  then $h_a(x,1)$ is a decreasing function.

  Example 4.31. *With $\eta_1 = 0.001$ valid for $x \geq 908994923$ [7, Theorem 5.2, p. 4], we choose $x_0 = 10^{10}$, and $a = 0.9989$ so that $a < 1-\eta_1$. We obtain in this case*

  $$\frac{x}{\log x - 0.9989} < \pi(x), \text{ for } x \geq 10^{10}.$$

- Let $k = 2$. We see that

  $$h_a(x,2) = (a-1)\log^4 x + (-a^2+\eta_2)\log^3 x - (2k\eta_2+2a\eta_2)\log^2 x + (a^2\eta_2+4a\eta_2)\log x - 2\eta_2 a^2.$$

  Here we consider those $a, d_2 > 0$ such that $h_a(x,2)$ decreases for $x > d_2$. We choose $a < 1$ and $x > 0$ such that

  $$x > \exp\left(\frac{3(a^2-\eta_2)+\sqrt{9(a^2-\eta_2)^2+24(2\eta_2+2a\eta_2)(a-1)}}{12(a-1)}\right),$$

  and $h'_a(x,2) < 0$.

  The following example shows that $k = 2$ provides a better estimates for $\pi(x)$ in the smaller range compare to the result of Example 4.31.

  Example 4.32. *By [7, Theorem 5.2, p. 4], for $x \geq 7713133853$, we have $\eta_2 = 0.01$. We choose $x_0 = 10^{11}$, and $a = 0.9999$ so that $a < 1$. We get*

  $$\frac{x}{\log x - 0.9999} < \pi(x), \text{ for } x \geq 10^{11}.$$

Chapter 5

Tables

In this chapter we present tables which provide lower and upper bounds for $\psi(x)/x$ and $\theta(x)/x$ over different ranges.

Table 5.1 gives upper and lower bounds for $\psi(x)/x$ when $x \leq \exp(20)$. We deduced this table by means of Table II of [3, p. 220] and Table 6.3 of [7, p. 15].

Table 5.2 is taken from [7, p. 15].

Table 5.3 provides upper bounds for $\theta(x)/x$ when $x \geq e^b$, for different values of $b \geq 27.4$. To generate this table we used Theorem 3.12 which states

Theorem 3.12. *Let b_1 and b_2 be positive constants such that $0 < b_1 \leq 27.4$ and $b_1 < b_2$. Let*

$$c_1 = \max_{x\in\left[e^{b_1},e^{b_2}\right]} \left\{ A^+(b_1) - \frac{A^-(b_1/2)}{x^{1/2}} - \frac{A^-(b_1/3)}{x^{2/3}} - \frac{A^-(b_1/5)}{x^{4/5}} + \frac{\tilde{c}_0}{x^{5/6}} + \frac{A^+(b_1/30)}{x^{29/30}} \right\},$$

where $\tilde{c}_0$ is an upper bound for $\theta(x)/x$ when $x > 0$. Then

$$\theta(x) < c_0 x, \text{ for } x > 0,$$

where $c_0 = \max\{c_1, A^+(b_2)\}$.

The above will allow us to establish an upper bound for $\theta(x)$ valid for all $x > 0$.

Example 3.13. $\theta(x) < (1 + 2.7571593586 \times 10^{-5})x, \text{ for } x > 0.$

We employed Theorem 3.20 to generate lower bounds for $\theta(x)/x$ for $x \geq e^b$ and $b > 2.694$.

Theorem 3.20. *Let $b > 2.694$, $x \geq e^b$, and c_0 be a positive constant satisfying $\theta(x)/x < c_0$ for $x > 0$. Let*

$$B^-(b) = A^-(b) - \frac{c_0\left(\frac{b}{\log 2} - 1\right)}{e^{b/2}}.$$

Then we have

$$\theta(x) > B^-(b)x.$$

We considered $c_0 = 1 + 2.7579 \times 10^{-5}$ to generate Tables 5.4, 5.5 and 5.6.

We used a more precise $c_0 = 1 + 2.7571593586 \times 10^{-5}$ to generate Tables 5.7, 5.8 and 5.9.

By comparing the results of these tables we observed that by employing an improved c_0 we can obtain improved results.

By refining the method used in Theorem 3.20 we proved the folowing theorem.

Theorem 3.22. *Let $b > 0$, k_0 be an integer ≥ 3, and c_0 be a positive constant satisfying $\theta(x)/x < c_0$ for $x > 0$. We let*

$$B^-(b, k_0) = A^-(b) - \sum_{2 \leq k \leq k_0 - 1} \min\{A^+(b/k), c_0\}e^{b/k - 1} - c_0 e^{b/k_0 - 1}\left(\frac{b}{\log 2} - k_0 + 1\right).$$

If

$$b \geq \frac{1}{k_0 - 1}\left(k_0 + (k_0 - 1)^2 \log 2\right),$$

then

$$\theta(x) > B^-(b, k_0)x, \text{ for } x \geq e^b.$$

We generated Tables 5.10, 5.11 and 5.12 using Theorem 3.22 with $k_0 = 3$ and observed that the results are improved compare to the previous tables.

We next replaced $k_0 = 3$ with $k_0 = 4$ in Theorem 3.22 to obtain Tables 5.13, 5.14 and 5.15 and observed that by using a bigger k_0 the results improved.

Our best results on the lower bound for $\theta(x)/x$ when $x \geq e^b$ and $b > 0$ were obtained in Tables 5.16, 5.17 and 5.18. We generated these tables by using the following theorem.

Theorem 3.27. *Let $b > 0$ and $x \geq e^b$. Then*

$$\theta(x) > B^-(b)x,$$

where

$$B^-(b) = A^-(b) - \frac{A^+(b/2)}{e^{b/2}} - \frac{A^+(b/3)}{e^{2b/3}} - \frac{A^+(b/5)}{e^{4b/5}}.$$

We mention here that for $b > 100$ all the values of our tables for lower bounds of $\theta(x)$ are equal to the values of Table 6.3. of [7, p. 15].

Table 5.1: $A^{-}(b)x < \psi(x) < A^{+}(b)x$, for $x \geq \exp(b)$

b	$A^{-}(b)$	$A^{+}(b)$	b	$A^{-}(b)$	$A^{+}(b)$
3.14	0.86583	1.03883	9.25	0.99343	1.00458
3.72	0.90602	1.03883	9.38	0.99486	1.00458
4.08	0.92237	1.03883	9.88	0.99643	1.00458
4.62	0.94842	1.03883	10.10	0.99643	1.00297
4.74	0.94842	1.03591	10.38	0.99703	1.00297
5.31	0.94842	1.02728	10.39	0.99770	1.00297
5.43	0.96764	1.02728	10.68	0.99770	1.00291
5.69	0.96764	1.02117	11.00	0.99770	1.00237
5.83	0.97494	1.02117	11.01	0.99770	1.00182
6.16	0.97494	1.01802	11.02	0.99770	1.00157
6.35	0.97870	1.01802	11.16	0.99787	1.00157
6.50	0.97870	1.01386	11.17	0.99816	1.00157
6.53	0.97870	1.01364	11.40	0.99851	1.00157
7.27	0.98708	1.01364	11.48	0.99851	1.00153
7.40	0.98708	1.01196	11.54	0.99851	1.00144
7.43	0.98708	1.00990	11.96	0.99851	1.00121
7.89	0.98828	1.00990	12.08	0.99870	1.00121
7.97	0.98828	1.00744	12.35	0.99870	1.0011976
8.00	0.98828	1.00662	12.63	0.9988024	1.0011976
8.11	0.99002	1.00662	18.43	0.998807	1.001193
8.15	0.99227	1.00662	18.44	0.9988115	1.0011885
8.17	0.99237	1.00662	18.45	0.9988161	1.0011839
8.29	0.99237	1.00649	18.50	0.9988385	1.0011615
8.60	0.99330	1.00649	18.70	0.9989235	1.0010765
8.77	0.99330	1.00543	19.00	0.99903839	1.00096161
8.87	0.99330	1.00517	19.50	0.99919989	1.00080011
8.92	0.99343	1.00517	20	0.99938770	1.0006123

Table 5.2: $|\psi(x) - x| < \varepsilon x$ for $x \geq e^b$

b	ε	b	ε
20	6.123×10^{-4}	100	2.903×10^{-11}
21	4.072×10^{-4}	200	2.838×10^{-11}
22	2.706×10^{-4}	300	2.772×10^{-11}
23	1.792×10^{-4} -	400	2.706×10^{-11}
24	1.183×10^{-4}-	500	2.641×10^{-11}
25	7.789×10^{-5}	600	2.575×10^{-11}
26	5.121×10^{-5}	1000	2.315×10^{-11}
27	3.368×10^{-5}	1250	2.153×10^{-11}
27.4	2.841×10^{-5}	1500	1.991×10^{-11}
28	2.224×10^{-5}	2000	1.671×10^{-11}
29	1.451×10^{-5}	2200	1.544×10^{-11}
30	9.414×10^{-6}	2500	1.355×10^{-11}
31	6.099×10^{-6}	2800	1.169×10^{-11}
32	3.944×10^{-6}	3000	1.047×10^{-11}
33	2.545×10^{-6}	3200	9.267×10^{-12}
34	1.640×10^{-6}	3300	8.658×10^{-12}
34.53	1.293×10^{-6}	3400	8.083×10^{-12}
35	1.055×10^{-6}	3455	7.750×10^{-12}
36	6.775×10^{-7}	3500	7.488×10^{-12}
37	4.348×10^{-7}	3600	6.930×10^{-12}
38	2.793×10^{-7}	3700	6.351×10^{-12}
39	1.805×10^{-7}	3750	6.080×10^{-12}
40	1.163×10^{-7}	3800	5.821×10^{-12}
41	7.414×10^{-8}	3850	5.533×10^{-12}
42	4.723×10^{-8}	3900	5.259×10^{-12}
43	3.011×10^{-8}	3950	4.999×10^{-12}
44	1.932×10^{-8}	4000	4.751×10^{-12}
45	1.234×10^{-8}	4050	4.496×10^{-12}
46	7.839×10^{-9}	4100	4.231×10^{-12}
47	5.026×10^{-9}	4150	3.981×10^{-12}
48	3.190×10^{-9}	4200	3.746×10^{-12}
49	2.038×10^{-9}	4300	3.308×10^{-12}
50	1.301×10^{-9}	4400	2.844×10^{-12}
55	1.481×10^{-10}	4500	2.445×10^{-12}
60	3.917×10^{-11}	4700	1.774×10^{-12}
70	2.929×10^{-11}	5000	9.562×10^{-13}
75	2.920×10^{-11}	10000	6.341×10^{-18}

Table 5.3:

b	$A^+(b)-1$	$B^+(b)-1$
27.4	2.841×10^{-5}	$2.7571593586 \times 10^{-5}$
28	2.224×10^{-5}	$2.1732219494 \times 10^{-5}$
29	1.451×10^{-5}	$1.4202391092 \times 10^{-5}$
30	9.414×10^{-6}	$9.2276182110 \times 10^{-6}$
31	6.099×10^{-6}	$5.9860534540 \times 10^{-6}$
32	3.944×10^{-6}	$3.8755452080 \times 10^{-6}$
33	2.545×10^{-6}	$2.5035063110 \times 10^{-6}$
34	1.64×10^{-6}	$1.6081751250 \times 10^{-6}$
34.53	1.293×10^{-6}	$1.2678462010 \times 10^{-6}$
35	1.055×10^{-6}	$1.0397503550 \times 10^{-6}$
36	6.775×10^{-7}	$6.6825416415 \times 10^{-7}$
37	4.348×10^{-7}	$4.2919385438 \times 10^{-7}$
38	2.793×10^{-7}	$2.7590026736 \times 10^{-7}$
39	1.805×10^{-7}	$1.7843819931 \times 10^{-7}$
40	1.163×10^{-7}	$1.1504949810 \times 10^{-7}$
41	7.414×10^{-8}	$7.3381515722 \times 10^{-8}$
42	4.723×10^{-8}	$4.6770020626 \times 10^{-8}$
43	3.011×10^{-8}	$2.9830984342 \times 10^{-8}$
44	1.932×10^{-8}	$1.9150785460 \times 10^{-8}$
45	1.234×10^{-8}	$1.2237360906 \times 10^{-8}$
46	7.839×10^{-9}	$7.7767507227 \times 10^{-9}$
47	5.026×10^{-9}	$4.9882427539 \times 10^{-9}$
48	3.190×10^{-9}	$3.1671002529 \times 10^{-9}$
49	2.038×10^{-9}	$2.0241103614 \times 10^{-9}$
50	1.301×10^{-9}	$1.2998600240 \times 10^{-9}$
55	1.481×10^{-10}	$1.4800642220 \times 10^{-10}$
60	3.917×10^{-11}	$3.9169369500 \times 10^{-11}$
70	2.929×10^{-11}	$2.9289948200 \times 10^{-11}$

$$\theta(x) < B^+(b)x,\ b \geq 27.4,\ \text{and } c_0 = 1 + 2.7571593586 \times 10^{-5}$$

Table 5.4:

b	$1 - A^{-}(b)$	$1 - B^{-}(b)$
3.14	0.13417	0.6605518160
3.72	0.09398	0.6181170445
4.08	0.07763	0.5829609620
4.62	0.05158	0.5146714400
4.74	0.05158	0.5038872356
5.31	0.05158	0.4495323608
5.43	0.03236	0.4185993624
5.69	0.03236	0.3933218256
5.83	0.02506	0.3725668615
6.16	0.02506	0.3415901967
6.35	0.02130	0.3206005776
6.50	0.02130	0.3073652964
6.53	0.02130	0.3047595804
7.27	0.01292	0.2368833791
7.40	0.01292	0.2274258296
7.43	0.01292	0.2252864072
7.89	0.01172	0.1932948365
7.97	0.01172	0.1883211027
8.00	0.01172	0.1864845931
8.11	0.00998	0.1781432469
8.15	0.00773	0.1735440019
8.17	0.00763	0.1722795516
8.29	0.00763	0.1654340534
8.60	0.00670	0.1479142945
8.77	0.00670	0.1394637735
8.87	0.00670	0.1346991800
8.92	0.00657	0.1322429398
9.25	0.00657	0.1177948390
9.38	0.00514	0.1110881915
9.88	0.00357	0.0912436287

$\theta(x) > B^{-}(b)x, \text{ and } c_0 = 1 + 2.7579 \times 10^{-5}$

Table 5.5:

b	$1 - A^{-}(b)$	$1 - B^{-}(b)$
10.10	0.00357	0.084145362400
10.38	0.00297	0.075269754780
10.39	0.00230	0.074319146590
10.68	0.00230	0.066604726550
11.00	0.00230	0.058983634130
11.01	0.00230	0.058759590530
11.02	0.00230	0.058536371740
11.16	0.00213	0.055326439100
11.17	0.00184	0.054825277410
11.40	0.00149	0.049829570820
11.48	0.00149	0.048305193790
11.54	0.00149	0.047191654680
11.96	0.00149	0.040067380120
12.08	0.00130	0.038043123360
12.35	0.00130	0.034213605340
12.63	0.0011976	0.030542074070
18.43	0.001193	0.003640523620
18.44	0.0011885	0.003625245410
18.45	0.0011839	0.003609913830
18.50	0.0011615	0.003534548520
18.70	0.0010765	0.003248816700
19.00	0.00096161	0.002863737710
19.50	0.00080011	0.002323540430
20	0.00061230	0.001831498840
21	0.0004072	0.001186409267
22	0.0002706	0.000767310437
23	0.0001792	0.000495085148
24	0.0001183	0.000318758497
25	0.00007789	0.000204850797
26	0.00005121	0.000131476672

$$\theta(x) > B^{-}(b)x, \text{ and } c_0 = 1 + 2.7579 \times 10^{-5}$$

Table 5.6:

b	$1 - A^{-}(b)$	$1 - B^{-}(b)$
27	0.00003368	0.000084342128000
27.4	0.00002841	0.000070536398900
28	0.00002224	0.000054167809140
29	0.00001451	0.000034602835080
30	0.000009414	0.000022042256450
31	0.000006099	0.000014026108490
32	0.000003944	0.000008914392758
33	0.000002545	0.000005658170956
34	0.00000164	0.000003587961957
34.53	0.000001293	0.000002811774684
35	0.000001055	0.000002272725711
36	6.775×10^{-7}	0.000001438060801
37	4.348×10^{-7}	$9.094306345000 \times 10^{-7}$
38	2.793×10^{-7}	$5.752613814000 \times 10^{-7}$
39	1.805×10^{-7}	$3.649124512400 \times 10^{-7}$
40	1.163×10^{-7}	$2.311255038300 \times 10^{-7}$
41	7.414×10^{-8}	$1.455888276690 \times 10^{-7}$
42	4.723×10^{-8}	$9.165986698400 \times 10^{-8}$
43	3.011×10^{-8}	$5.772159828000 \times 10^{-8}$
44	1.932×10^{-8}	$3.646972720000 \times 10^{-8}$
45	1.234×10^{-8}	$2.298593136000 \times 10^{-8}$
46	7.839×10^{-9}	$1.444413548000 \times 10^{-8}$
47	5.026×10^{-9}	$9.122015082000 \times 10^{-9}$
48	3.19×10^{-9}	$5.728823911000 \times 10^{-9}$
49	2.038×10^{-9}	$3.610909344000 \times 10^{-9}$
50	1.301×10^{-9}	$2.275054363000 \times 10^{-9}$
55	1.481×10^{-10}	$2.362787808000 \times 10^{-10}$
60	3.917×10^{-11}	$4.708318353000 \times 10^{-11}$
70	2.929×10^{-11}	$2.935241522273 \times 10^{-11}$
75	2.92×10^{-11}	$2.920549670115 \times 10^{-11}$
100	2.903×10^{-11}	$2.903000002745 \times 10^{-11}$

$$\theta(x) > B^{-}(b)x, \text{ and } c_0 = 1 + 2.7579 \times 10^{-5}$$

Table 5.7:

b	$1 - A^{-}(b)$	$1 - B^{-}(b)$
3.14	0.13417	0.6605518121
3.72	0.09398	0.6181170406
4.08	0.07763	0.5829609582
4.62	0.05158	0.5146714300
4.74	0.05158	0.5038872322
5.31	0.05158	0.4495323578
5.43	0.03236	0.4185993596
5.69	0.03236	0.3933218229
5.83	0.02506	0.3725668589
6.16	0.02506	0.3415901943
6.35	0.02130	0.3206005754
6.50	0.02130	0.3073652942
6.53	0.02130	0.3047595783
7.27	0.01292	0.2368833775
7.40	0.01292	0.2274258281
7.43	0.01292	0.2252864056
7.89	0.01172	0.1932948352
7.97	0.01172	0.1883211014
8.00	0.01172	0.1864845918
8.11	0.00998	0.1781432457
8.15	0.00773	0.1735440007
8.17	0.00763	0.1722795504
8.29	0.00763	0.1654340522
8.60	0.00670	0.1479142935
8.77	0.00670	0.1394637726
8.87	0.00670	0.1346991786
8.92	0.00657	0.1322429388
9.25	0.00657	0.1177948382
9.38	0.00514	0.1110881907
9.88	0.00357	0.0912436280

$$\theta(x) > B^{-}(b)x, \text{ and } c_0 = 1 + 2.7571593586 \times 10^{-5}$$

Table 5.8:

b	$1 - A^{-}(b)$	$1 - B^{-}(b)$
10.10	0.00357	0.084145361800
10.38	0.00297	0.075269754240
10.39	0.00230	0.074319146050
10.68	0.00230	0.066604726070
11.00	0.00230	0.058983633710
11.01	0.00230	0.058759590110
11.02	0.00230	0.058536371330
11.16	0.00213	0.055326438710
11.17	0.00184	0.054825277100
11.40	0.00149	0.049829570460
11.48	0.00149	0.048305193500
11.54	0.00149	0.047191654340
11.96	0.00149	0.040067379840
12.08	0.00130	0.038043123090
12.35	0.00130	0.034213605100
12.63	0.0011976	0.030542073860
18.43	0.001193	0.003640523600
18.44	0.0011885	0.003625245390
18.45	0.0011839	0.003609913810
18.50	0.0011615	0.003534548500
18.70	0.0010765	0.003248816680
19.00	0.00096161	0.002863737700
19.50	0.00080011	0.002323540410
20	0.00061230	0.001831498826
21	0.0004072	0.001186409263
22	0.0002706	0.000767310434
23	0.0001792	0.000495085146
24	0.0001183	0.000318758495
25	0.00007789	0.000204850796
26	0.00005121	0.000131476671

$$\theta(x) > B^{-}(b)x, \text{ and } c_0 = 1 + 2.7571593586 \times 10^{-5}$$

Table 5.9:

b	$1 - A^{-}(b)$	$1 - B^{-}(b)$
27	0.00003368	0.0000843421280000
27.4	0.00002841	0.0000705363985807
28	0.00002224	0.0000541678089016
29	0.00001451	0.0000346028349296
30	0.000009414	0.0000220422563541
31	0.000006099	0.0000140261084296
32	0.000003944	0.0000089143927203
33	0.000002545	0.0000056581709321
34	0.00000164	0.0000035879619422
34.53	0.000001293	0.0000028117746725
35	0.000001055	0.0000022727257010
36	6.775×10^{-7}	0.0000014380607949
37	4.348×10^{-7}	$9.094306308963 \times 10^{-7}$
38	2.793×10^{-7}	$5.752613791776 \times 10^{-7}$
39	1.805×10^{-7}	$3.649124498671 \times 10^{-7}$
40	1.163×10^{-7}	$2.311255029741 \times 10^{-7}$
41	7.414×10^{-8}	$1.455888271392 \times 10^{-7}$
42	4.723×10^{-8}	$9.165986665455 \times 10^{-8}$
43	3.011×10^{-8}	$5.772159806553 \times 10^{-8}$
44	1.932×10^{-8}	$3.646972706462 \times 10^{-8}$
45	1.234×10^{-8}	$2.298593127459 \times 10^{-8}$
46	7.839×10^{-9}	$1.444413542969 \times 10^{-8}$
47	5.026×10^{-9}	$9.122015051276 \times 10^{-9}$
48	3.19×10^{-9}	$5.728823891751 \times 10^{-9}$
49	2.038×10^{-9}	$3.610909331823 \times 10^{-9}$
50	1.301×10^{-9}	$2.275054354867 \times 10^{-9}$
55	1.481×10^{-10}	$2.362787800737 \times 10^{-10}$
60	3.917×10^{-11}	$4.708318346468 \times 10^{-11}$
70	2.929×10^{-11}	$2.935241522227 \times 10^{-11}$
75	2.92×10^{-11}	$2.920549670111 \times 10^{-11}$
100	2.903×10^{-11}	$2.903000002745 \times 10^{-11}$

$\theta(x) > B^{-}(b)x,$ and $c_0 = 1 + 2.7571593586 \times 10^{-5}$

Table 5.10:

b	$1 - A^{-}(b)$	$A^{+}(b/2) - 1$	$1 - B^{-}(b, 3)$
6.16	0.02506	0.03883	0.16793831
6.35	0.02130	0.03883	0.15245829
6.50	0.02130	0.03883	0.14377440
6.53	0.02130	0.03883	0.14209653
7.27	0.01292	0.03883	0.09812400
7.40	0.01292	0.03883	0.09293137
7.43	0.01292	0.03883	0.09177408
7.89	0.01172	0.03883	0.07462449
7.97	0.01172	0.03883	0.07217223
8.00	0.01172	0.03883	0.07127551
8.11	0.00998	0.03883	0.06635136
8.15	0.00773	0.03883	0.06298300
8.17	0.00763	0.03883	0.06233155
8.29	0.00763	0.03883	0.05912854
8.60	0.00670	0.03883	0.05071396
8.77	0.00670	0.03883	0.04705486
8.87	0.00670	0.03883	0.04503820
8.92	0.00657	0.03883	0.04393589
9.25	0.00657	0.03883	0.03808045
9.38	0.00514	0.03883	0.03459231
9.88	0.00357	0.03591	0.02624007
10.10	0.00357	0.03591	0.02375610
10.38	0.00297	0.03591	0.02037193
10.39	0.00230	0.03591	0.01960969
10.68	0.00230	0.02728	0.01713145
11.00	0.00230	0.02728	0.01479604
11.01	0.00230	0.02728	0.01472915

$\theta(x) > B^{-}(b, 3)x$ for $x \geq e^{b}$ with $c_0 = 1 + 2.7571593586 \times 10^{-5}$

Table 5.11:

b	$1 - A^{-}(b)$	$A^{+}(b/2) - 1$	$1 - B^{-}(b, 3)$
11.02	0.00230	0.02728	0.01466260
11.16	0.00213	0.02728	0.01359660
11.17	0.00184	0.02728	0.01324508
11.40	0.00149	0.02117	0.01156568
11.48	0.00149	0.02117	0.01113973
11.54	0.00149	0.02117	0.01083186
11.96	0.00149	0.02117	0.00893017
12.08	0.00130	0.02117	0.00827036
12.35	0.00130	0.01802	0.00731718
12.63	0.0011976	0.01802	0.00636170
18.43	0.001193	0.00517	0.00140136
18.44	0.0011885	0.00517	0.00139570
18.45	0.0011839	0.00517	0.00138996
18.50	0.0011615	0.00517	0.00136192
18.70	0.0010765	0.00458	0.00125586
19.00	0.00096161	0.00458	0.00111348
19.50	0.00080011	0.00458	0.00091522
20	0.00061230	0.00458	0.00069958
21	0.0004072	0.00297	0.00045744
22	0.0002706	0.00291	0.00029958
23	0.0001792	0.00153	0.00019595
24	0.0001183	0.00121	0.00012801
25	0.00007789	0.0011976	0.00008353
26	0.00005121	0.0011976	0.00005450
27	0.00003368	0.0011976	0.00003560
27.4	0.00002841	0.0011976	0.00002996
28	0.00002224	0.0011976	0.00002337

$\theta(x) > B^{-}(b, 3)x$ for $x \geq e^{b}$ with $c_0 = 1 + 2.7571593586 \times 10^{-5}$

Table 5.12:

b	$1 - A^{-}(b)$	$A^{+}(b/2) - 1$	$1 - B^{-}(b, 3)$
29	0.00001451	0.0011976	0.00001518
30	0.000009414	0.0011976	0.00000981
31	0.000006099	0.0011976	0.00000633
32	3.944×10^{-6}	0.0011976	4.079992×10^{-6}
33	2.545×10^{-6}	0.0011976	2.625702×10^{-6}
34	1.64×10^{-6}	0.0011976	1.687997×10^{-6}
34.53	1.293×10^{-6}	0.0011976	1.329472×10^{-6}
35	1.055×10^{-6}	0.0011976	1.083604×10^{-6}
36	6.775×10^{-7}	0.0011976	6.945780×10^{-7}
37	4.348×10^{-7}	0.0011839	4.450150×10^{-7}
38	2.793×10^{-7}	0.0010765	2.854187×10^{-7}
39	1.805×10^{-7}	0.00096161	1.841706×10^{-7}
40	1.163×10^{-7}	0.00080011	1.185048×10^{-7}
41	7.414×10^{-8}	0.0006123	7.546581×10^{-8}
42	4.723×10^{-8}	0.0006123	4.802811×10^{-8}
43	3.011×10^{-8}	0.0004072	3.059088×10^{-8}
44	1.932×10^{-8}	0.0004072	1.960998×10^{-8}
45	1.234×10^{-8}	0.0002706	1.251499×10^{-8}
46	7.839×10^{-9}	0.0002706	7.944666×10^{-9}
47	5.026×10^{-9}	0.0001792	5.089842×10^{-9}
48	3.19×10^{-9}	0.0001792	3.228592×10^{-9}
49	2.038×10^{-9}	0.0001183	2.061339×10^{-9}
50	1.301×10^{-9}	0.0001183	1.315120×10^{-9}
55	1.481×10^{-10}	0.00002841	1.492492×10^{-10}
60	3.917×10^{-11}	0.00001451	3.926394×10^{-11}
70	2.929×10^{-11}	0.000001293	$2.9290631100000000 \times 10^{-11}$
75	2.92×10^{-11}	$4.348.10^{-7}$	$2.9200051775870000 \times 10^{-11}$
100	2.903×10^{-11}	$2.038.10^{-9}$	$2.9030000000192877 \times 10^{-11}$

$\theta(x) > B^{-}(b, 3)x$ for $x \geq e^{b}$ with $c_0 = 1 + 2.7571593586 \times 10^{-5}$

Table 5.13:

b	$1 - A^-(b)$	$A^+(b/2) - 1$	$A^+(b/3) - 1$	$1 - B^-(b, 4)$
6.16	0.02506	0.03883	0.03883	0.13563550
6.35	0.02130	0.03883	0.03883	0.12169854
6.50	0.02130	0.03883	0.03883	0.11425836
6.53	0.02130	0.03883	0.03883	0.11283079
7.27	0.01292	0.03883	0.03883	0.07496668
7.40	0.01292	0.03883	0.03883	0.07080003
7.43	0.01292	0.03883	0.03883	0.06987640
7.89	0.01172	0.03883	0.03883	0.05614184
7.97	0.01172	0.03883	0.03883	0.05424841
8.00	0.01172	0.03883	0.03883	0.05355841
8.11	0.00998	0.03883	0.03883	0.04937870
8.15	0.00773	0.03883	0.03883	0.04627573
8.17	0.00763	0.03883	0.03883	0.04575591
8.29	0.00763	0.03883	0.03883	0.04332757
8.60	0.00670	0.03883	0.03883	0.03679340
8.77	0.00670	0.03883	0.03883	0.03409150
8.87	0.00670	0.03883	0.03883	0.03261366
8.92	0.00657	0.03883	0.03883	0.03177408
9.25	0.00657	0.03883	0.03883	0.02754352
9.38	0.00514	0.03883	0.03883	0.02464478
9.88	0.00357	0.03591	0.03883	0.01830895
10.10	0.00357	0.03591	0.03883	0.01659456
10.38	0.00297	0.03591	0.03883	0.01409502
10.39	0.00230	0.03591	0.03883	0.01336251
10.68	0.00230	0.02728	0.03883	0.01169377
11.00	0.00230	0.02728	0.03883	0.01014143
11.01	0.00230	0.02728	0.03883	0.01009728

$\theta(x) > B^-(b, 4)x$ for $x \geq e^b$ with $c_0 = 1 + 2.7571593586 \times 10^{-5}$

Table 5.14:

b	$1 - A^{-}(b)$	$A^{+}(b/2) - 1$	$A^{+}(b/3) - 1$	$1 - B^{-}(b, 4)$
11.02	0.00230	0.02728	0.03883	0.01005337
11.16	0.00213	0.02728	0.03883	0.00929392
11.17	0.00184	0.02728	0.03883	0.00896357
11.40	0.00149	0.02117	0.03883	0.00774560
11.48	0.00149	0.02117	0.03883	0.00746915
11.54	0.00149	0.02117	0.03883	0.00726986
11.96	0.00149	0.02117	0.03883	0.00604905
12.08	0.00130	0.02117	0.03883	0.00556034
12.35	0.00130	0.01802	0.03883	0.00495807
12.63	0.0011976	0.01802	0.03883	0.00432124
18.43	0.001193	0.00517	0.02117	0.00131959
18.44	0.0011885	0.00517	0.02117	0.00131441
18.45	0.0011839	0.00517	0.02117	0.00130913
18.50	0.0011615	0.00517	0.01802	0.00128340
18.70	0.0010765	0.00458	0.01802	0.00118596
19.00	0.00096161	0.00458	0.01802	0.00105478
19.50	0.00080011	0.00458	0.01802	0.00087141
20	0.00061230	0.00458	0.01364	0.00066693
21	0.0004072	0.00297	0.01364	0.00043937
22	0.0002706	0.00291	0.01364	0.00028963
23	0.0001792	0.00153	0.00990	0.00019050
24	0.0001183	0.00121	0.00744	0.00012503
25	0.00007789	0.0011976	0.00649	0.00008191
26	0.00005121	0.0011976	0.00649	0.00005362
27	0.00003368	0.0011976	0.00517	0.00003513
27.4	0.00002841	0.0011976	0.00517	0.00002959
28	0.00002224	0.0011976	0.00458	0.00002311
29	0.00001451	0.0011976	0.00458	0.00001504
30	0.000009414	0.0011976	0.00458	0.00000973
31	0.000006099	0.0011976	0.00297	0.00000629

$\theta(x) > B^{-}(b,4)x$ for $x \geq e^{b}$ with $c_0 = 1 + 2.7571593586 \times 10^{-5}$

Table 5.15:

b	$1 - A^{-}(b)$	$A^{+}(b/2) - 1$	$A^{+}(b/3) - 1$	$1 - B^{-}(b, 4)$
32	3.944×10^{-6}	0.0011976	0.00297	4.058674×10^{-6}
33	2.545×10^{-6}	0.0011976	0.00291	2.614315×10^{-6}
34	1.64×10^{-6}	0.0011976	0.00157	1.681924×10^{-6}
34.53	1.293×10^{-6}	0.0011976	0.00153	1.325123×10^{-6}
35	1.055×10^{-6}	0.0011976	0.00144	1.080370×10^{-6}
36	6.775×10^{-7}	0.0011976	0.00121	6.928583×10^{-7}
37	4.348×10^{-7}	0.0011839	0.00121	4.441010×10^{-7}
38	2.793×10^{-7}	0.0010765	0.0011976	2.849343×10^{-7}
39	1.805×10^{-7}	0.00096161	0.0011976	1.839139×10^{-7}
40	1.163×10^{-7}	0.00080011	0.0011976	1.183689×10^{-7}
41	7.414×10^{-8}	0.0006123	0.0011976	7.539398×10^{-8}
42	4.723×10^{-8}	0.0006123	0.0011976	4.799016×10^{-8}
43	3.011×10^{-8}	0.0004072	0.0011976	3.057085×10^{-8}
44	1.932×10^{-8}	0.0004072	0.0011976	1.959942×10^{-8}
45	1.234×10^{-8}	0.0002706	0.0011976	1.250943×10^{-8}
46	7.839×10^{-9}	0.0002706	0.0011976	7.941735×10^{-9}
47	5.026×10^{-9}	0.0001792	0.0011976	5.0882992×10^{-9}
48	3.19×10^{-9}	0.0001792	0.0011976	3.2277801×10^{-9}
49	2.038×10^{-9}	0.0001183	0.0011976	2.0609120×10^{-9}
50	1.301×10^{-9}	0.0001183	0.0011976	1.3148960×10^{-9}
55	1.481×10^{-10}	0.00002841	0.0011976	$1.4924023 \times 10^{-10}$
60	3.917×10^{-11}	0.00001451	0.00080011	$3.9263590 \times 10^{-11}$
70	2.929×10^{-11}	0.000001293	0.0001792	$2.9290630600000000 \times 10^{-11}$
75	2.92×10^{-11}	4.348×10^{-7}	0.0001183	$2.9200051755810000 \times 10^{-11}$
100	2.903×10^{-11}	2.038×10^{-9}	0.000002545	$2.9030000000192875 \times 10^{-11}$

$\theta(x) > B^{-}(b, 4)x$ for $x \geq e^{b}$ with $c_0 = 1 + 2.7571593586 \times 10^{-5}$

Table 5.16:

b	$1 - A^{-}(b)$	$1 - B^{-}(b)$
3.14	0.86583	0.5626112
3.72	0.09398	0.3956690
4.08	0.07763	0.3208603
4.62	0.05158	0.2282260
4.74	0.05158	0.2161899
5.31	0.05158	0.1695963
5.43	0.03236	0.1424471
5.69	0.03236	0.1271021
5.83	0.02506	0.1124738
6.16	0.02506	0.0974278
6.35	0.02130	0.0862456
6.50	0.02130	0.0809440
6.53	0.02130	0.0799384
7.27	0.01292	0.0515832
7.40	0.01292	0.0488752
7.43	0.01292	0.0482785
7.89	0.01172	0.0391046
7.97	0.01172	0.0379192
8.00	0.01172	0.0374884
8.11	0.00998	0.0342301
8.15	0.00773	0.0314511
8.17	0.00763	0.0310911
8.29	0.00763	0.0295911
8.60	0.00670	0.0252255
8.77	0.00670	0.0235808
8.87	0.00670	0.0226842
8.92	0.00657	0.0221243
9.25	0.00657	0.0195691
9.38	0.00514	0.0172545
9.88	0.00357	0.0127973

$$\theta(x) > B^{-}(b)x.$$

Table 5.17:

b	$1 - A^{-}(b)$	$1 - B^{-}(b)$
10.10	0.00357	0.0117680
10.38	0.00297	0.0100255
10.39	0.00230	0.0093178
10.68	0.00230	0.0082692
11.00	0.00230	0.0073337
11.01	0.00230	0.0073067
11.02	0.00230	0.0072804
11.16	0.00213	0.0067534
11.17	0.00184	0.0064389
11.40	0.00149	0.0055404
11.48	0.00149	0.0053724
11.54	0.00149	0.0052511
11.96	0.00149	0.0045030
12.08	0.00130	0.0041284
12.35	0.00130	0.0037475
12.63	0.0011976	0.0033107
18.43	0.001193	0.0012982
18.44	0.0011885	0.0012932
18.45	0.0011839	0.0012881
18.50	0.0011615	0.0012623
18.70	0.0010765	0.0011682
19.00	0.00096161	0.0010403
19.50	0.00080011	$8.61147120 \times 10^{-4}$
20	0.00061230	$6.59666455 \times 10^{-4}$
21	0.0004072	$4.35713633 \times 10^{-4}$
22	0.0002706	$2.87806650 \times 10^{-4}$
23	0.0001792	$1.89577557 \times 10^{-4}$
24	0.0001183	$1.24569772 \times 10^{-4}$
25	0.00007789	$8.16814039 \times 10^{-5}$
26	0.00005121	$5.35038530 \times 10^{-5}$

$$\theta(x) > B^{-}(b)x$$

Table 5.18:

b	$1-A^{-}(b)$	$1-B^{-}(b)$
27	0.00003368	3.50683372×10^{-5}
27.4	0.00002841	2.95458300×10^{-5}
28	0.00002224	2.30805718×10^{-5}
29	0.00001451	1.50190705×10^{-5}
30	0.000009414	9.72237782×10^{-6}
31	0.000006099	6.28583999×10^{-6}
32	0.000003944	4.05722264×10^{-6}
33	0.000002545	2.61362101×10^{-6}
34	0.00000164	1.68159396×10^{-6}
34.53	0.000001293	1.32491000×10^{-6}
35	0.000001055	1.08021440×10^{-6}
36	6.775×10^{-7}	6.92786332×10^{-7}
37	4.348×10^{-7}	4.44067934×10^{-7}
38	2.793×10^{-7}	2.84918855×10^{-7}
39	1.805×10^{-7}	1.83906681×10^{-7}
40	1.163×10^{-7}	1.18365442×10^{-7}
41	7.414×10^{-8}	7.53922725×10^{-8}
42	4.723×10^{-8}	4.79894152×10^{-8}
43	3.011×10^{-8}	3.05704494×10^{-8}
44	1.932×10^{-8}	1.95992434×10^{-8}
45	1.234×10^{-8}	1.25093295×10^{-8}
46	7.839×10^{-9}	7.94169478×10^{-9}
47	5.026×10^{-9}	5.08827735×10^{-9}
48	3.19×10^{-9}	3.22777082×10^{-9}
49	2.038×10^{-9}	2.06090658×10^{-9}
50	1.301×10^{-9}	1.31489294×10^{-9}
55	1.481×10^{-10}	1.49240144×10^{-10}
60	3.917×10^{-11}	3.92635820×10^{-11}
70	2.929×10^{-11}	$2.929063052000000000\times10^{-11}$
75	2.92×10^{-11}	$2.920005175577000000\times10^{-11}$
100	2.903×10^{-11}	$2.903000000019287499\times10^{-11}$

$$\theta(x) > B^{-}(b)x$$

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