Entanglement as a source of black hole entropy

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Abstract. We review aspects of black hole thermodynamics, and show how entanglement of a quantum field between the inside and outside of a horizon can account for the area-proportionality of black hole entropy, provided the field is in its ground state. We show that the result continues to hold for Coherent States and Squeezed States, while for Excited States, the entropy scales as a power of area less than unity. We also identify location of the degrees of freedom which give rise to the above entropy.

1. Introduction

The field of black hole thermodynamics started around the time Bekenstein and Hawking argued that black holes have entropy and temperature, given respectively by [1]:

$$S_{BH} = \frac{A_H}{4\ell_{Pl}^2} \quad \text{and} \quad T_H = \frac{\hbar c^3}{8\pi G M} \ , \tag{1}$$

where M and A_H are the mass and horizon area of the black hole, and $\ell_{Pl} = \sqrt{G\hbar/c^3}$ is the Planck length in four dimensions. Together, they satisfy the following laws, which are analogous to the laws of classical thermodynamics [1] :

$$\begin{aligned} d(Mc^2) &= T_H dS + \Phi dQ + \Omega dJ \quad \text{(First law)} &(2)\\ \Delta S_{BH} &\geq 0 \quad \text{(Second law)} \ . &(3) \end{aligned}$$

The question naturally arises as to whether there are miscroscopic, elementary states, consistent with the black hole temperature and other macroscopic parameters, whose logarithm gives its entropy:

$$S_{BH} = \ln \Omega \ , \tag{4}$$

[1] In subsequent formulae, we will assume $c = 1 = \hbar$.

and if so, then what those states are. Another related question in this area of research is the following: is the area proportionality of the entropy, also know as the *Area Law* (AL) in Eq.(1) (as opposed to volume proportionality, which is the case for normal thermodynamic systems) more generic, or in fact universal, beyond the realm of black holes? This has given rise to the so-called *Holographic Hypothesis*, which conjectures that the information content of our universe may be encoded in a two dimensional screen which surrounds three dimensional space.

Finally, the expectation that black holes, after absorbing matter (and information contained therein) will radiate itself away in the form of pure thermal radiation at the temperature (1), has given rise to the *Information Loss Problem*: thermal radiation does not contain any information, except that of the temperature of the emitting source, so if the black hole disappears completely, does the information do so as well? While there is nothing in principle which prevents this form happening, it requires the quantum mechanics which governs the evolution to be non-unitary, since unitary evolution takes pure states to pure states. This eventually raises the question: *does quantum mechanics break down in the presence of a black hole?* To be able to address one or more of the above questions, one has to first explore origins of Black Hole Entropy and temperature. Among the various approaches that have been undertaken, the two most popular are: String Theory and Loop Quantum Gravity. In the first, degeneracy of strings attached to D-branes are claimed to give rise to the required degeneracy, while in the second, the spin-network states at the horizon are invoked for the same purpose. There have been some other interesting approaches as well [2, 3].

In addition to the above, there has been another idea, which goes as follows: consider a free scalar field, propagating in a spacetime containing a black hole. Now, do a partial tracing over the degrees of freedom (DOF) of the field which are inside the horizon. The resultant density matrix is mixed, due to the information being 'hidden' by the horizon (although the entire system, before tracing, is pure). When one computes the *Entanglement Entropy* associated with this, the latter turns out to be proportional to the horizon area [4, 5]! [The analytical proof for the area law have recently been shown in Refs. [6].] It is this approach which we will follow in this article. Here we will critically examine an important assumption that the authors had made in their analyses: that the scalar field is in its *Ground State* (GS). While this simplifying assumption is justified and even natural to make in a first calculation, we would like to see how important it really is to validity of the AL. Indeed we will find that while replacing the GS by generic *Coherent State* (CS) or a class of *Squeezed States* (SS) does not affect the AL, doing so with a class of *Excited States* (ES) does change the AL, and in fact renders the entropy proportional to a power of area less than unity.

One can ask as to why we consider a scalar field. While we do not have a completely satisfactory answer to this question, we are motivated by the fact that a significant part of gravitational perturbations in black hole backgrounds are scalar in nature [7]. So the scalar field does not have to be introduced *by hand* from outside.

This article is organised as follows: in the next section, we briefly review entanglement in quantum mechanics and its role in entropy. In Section 3, we set up the formalism for computing entropy of entangled scalar fields in flat spacetime, when the field is in its ground state. In Section 4, we extend our results to generic coherent states and a class of squeezed states. In Section 5, we examine excited states and their effects on entanglement entropy. In Section 6, we try to pin-point the location of the DOF which are responsible for entanglement entropy. In Section 7, we extend our results to black hole spacetimes. In Section 8, we conclude with a summary and open questions.

2. Entanglement entropy

Consider a quantum mechanical system, which can be decomposed into two subsystems u and v, as shown below [2]:

$$\boxed{\text{System}} = \boxed{\text{Subsystem } u} + \boxed{\text{Subsystem } v}$$

Correspondingly, the Hilbert space of the system is a Kronecker product of the subsystem Hilbert spaces:

$$\mathcal{H} = \mathcal{H}_u \otimes \mathcal{H}_v \ . \tag{5}$$

If $|u_i\rangle$ and $|v_j\rangle$ are eigenbases in $\mathcal{H}_u$ and $\mathcal{H}_v$ respectively, then $|u_i\rangle \otimes |v_j\rangle$ form an eigenbasis in $\mathcal{H}$, in terms of which a generic wavefunction $|\Psi\rangle$ in $\mathcal{H}$ can be expanded:

$$|\Psi\rangle = \sum_{ij} d_{ij} |u_i\rangle \otimes |v_j\rangle \in \mathcal{H} \ . \tag{6}$$

Note however, that in general, $|\Psi\rangle$ cannot be factorised into two wavefunctions, one in u and the other in v:

$$|\Psi\rangle \neq |\Psi_u\rangle \otimes |\Psi_v\rangle \ . \tag{7}$$

Such states are called *Entangled States* or *EPR states.* The ones that can be factorised, on the other hand, are called *Unentangled States.* For example, for a system with two spins, the following is an unentangled state:

$$|\uparrow\downarrow\rangle + |\uparrow\uparrow\rangle = |\uparrow\rangle \otimes (|\downarrow\rangle + |\uparrow\rangle) \ , \tag{8}$$

while the following is an entangled state:

$$|\downarrow\downarrow\rangle + |\uparrow\uparrow\rangle \neq |\ldots\rangle \otimes |\ldots\rangle \ . \tag{9}$$

Entangled states have a variety of uses, including in *Quantum Teleportation.*

Next, let us define density matrices. If the quantum-mechanical wavefunction of a system is known (however complex the system might be), its density matrix is defined as:

$$\rho \equiv |\Psi\rangle\langle\Psi| \ . \tag{10}$$

It can easily be verified that ρ satisfies the following properties:

$$|\rho| \geq 0 \ , \quad \rho^\dagger = \rho, \ \ \rho^2 = \rho \ . \tag{11}$$

The last property is known as *idempotency*, from which it follows that the eigenvalue p_n of the density matrix can only be 0 or 1. Thus the *Entanglement Entropy* or *Von Neumann Entropy*, defined as follows, vanishes :

$$S \equiv -Tr\left(\rho \ln \rho\right) = -\sum_n p_n \ln p_n = 0 \ . \tag{12}$$

Now, one can take the trace of ρ, only in the subsystem v, to find the *Reduced Density Matrix*, which is still an operator in subsystem u:

$$\rho_u = Tr_v(\rho) = \sum_l \langle v_l|\rho|v_l\rangle = \sum_{i,k,j} d_{ij} d^\star_{kj} |u_i\rangle\langle u_k| \ . \tag{13}$$

[2] for details, we refer the reader to the review [8].

For ρ_u, it can be shown that the following properties hold:

$$|\rho_u| \geq 0 \ , \quad \rho_u^\dagger = \rho_u \ , \quad \rho_u^2 \neq \rho_u \ . \tag{14}$$

That is, idempotency no longer holds. As a result, its eigenvalues now satisfy: $0 < p_{n(u)} < 1$, and the entanglement entropy is non-zero:

$$S_u \equiv -Tr_u \left(\rho_u \ln \rho_u\right) = -\sum_n p_{n(u)} \ln p_{n(u)} > 0 \ . \tag{15}$$

The ignorance resulting from tracing over one part of the system manifests itself as entropy. One important property of reduced density matrices is that if we traced over u instead and found the reduced density matrix ρ_v, then the latter would have the same set of non-zero eigenvalues. Consequently, the entanglement entropies are equal, being a common property of the entangled system: $S_v = S_u$.

3. Entanglement entropy of scalar fields

Now consider a free scalar field in (3+1)-dimensional flat space, with the following Hamiltonian:

$$H = \frac{1}{2} \int d^3x \left[\pi^2(x) + |\vec{\nabla}\varphi(\vec{x})|^2\right] \ . \tag{16}$$

Decomposing the field and its conjugate momentum into partial waves

$$\varphi(\vec{r}) = \sum_{lm} \frac{\varphi_{lm}(r)}{r} \ Y_{lm}(\theta,\phi) \ , \quad \pi(\vec{r}) = \sum_{lm} \frac{\pi_{lm}(r)}{r} \ Y_{lm}(\theta,\phi) \tag{17}$$

yields:

$$H = \sum_{lm} H_{lm} = \sum_{lm} \frac{1}{2} \int_0^\infty dr \left\{ \pi_{lm}^2(r) + x^2 \left[\frac{\partial}{\partial r}\left(\frac{\varphi_{lm}(r)}{r}\right)\right]^2 + \frac{l(l+1)}{r^2} \ \varphi_{lm}^2(r) \right\} \ . \tag{18}$$

Next, discretise along the radial direction with lattice spacing a, such that $r \to r_i = ia$; $r_{i+1} - r_i = a$. The lattice is terminated at a large but finite N ('size of the universe'), and an intermediate point n is chosen, which we call the *horizon*, and which separates the lattice points between an 'inside' and an 'outside'. As of now, there is nothing intrinsically special about this point, although later we will associate it with a black hole horizon. The discretisation is depicted below:

$$\Big|\underbrace{\underbrace{\bullet}_{1}\ \underbrace{\bullet}_{2} - \bullet - \bullet - \quad \underbrace{\bullet}_{n(\text{Horizon})}}_{\text{Inside}} \quad \underbrace{- \bullet - \bullet - \underbrace{\bullet}_{j} - \bullet - \bullet - \underbrace{\bullet}_{N}}_{\text{Outside}}\Big|$$

The discretised version of Hamiltonian (18) takes the form:

$$H_{lm} = \frac{1}{2a} \sum_{j=1}^{N} \left[\pi_{lm,j}^2 + \left(j + \frac{1}{2}\right)^2 \left(\frac{\varphi_{lm,j}}{j} - \frac{\varphi_{lm,j+1}}{j+1}\right)^2 + \frac{l(l+1)}{j^2} \varphi_{lm,j}^2\right] \ . \tag{19}$$

Note that it resembles the Hamiltonian of a set of N coupled harmonic oscillators, with the interaction between them contained in the off-diagonal elements of the matrix K_{ij} (the coordinates x_i replace the field variables φ_{lm}):

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \frac{1}{2}\sum_{i,j=1}^{N} x_i K_{ij} x_j \tag{20}$$

The density Matrix, tracing over the first n of N oscillators ($x \equiv x_{n+1}, \ldots, x_N$), is given by:

$$\rho\left(x; x'\right) = \int \prod_{i=1}^{n} dx_i \; \psi(x_1, \ldots, x_n; x_{n+1}, \ldots, x_N) \; \psi^\star(x_1, \ldots, x_n; x_{n+1}, \ldots, x'_N) \tag{21}$$

where ($\underline{\mathrm{x}} = Ux$, $UKU^T =$ Diagonal). The wavefunction

$$\psi(x_1, \ldots, x_N) = \prod_{i=1}^{N} N_i \; H_{\nu_i}\left(k_{D\;i}^{\frac{1}{4}} \; \underline{\mathrm{x}}_i\right) \exp\left(-\frac{1}{2} k_{D\;i}^{\frac{1}{2}} \; \underline{\mathrm{x}}_i^2\right) \tag{22}$$

denotes the most general eigenstate of Hamiltonian (20), being a product of N harmonic oscillator wavefunctions. Note that $\rho_{out}^2 \neq \rho_{out}$, since ρ_{out} is mixed, although the full state is pure. Consequently, the entanglement entropy $= -Tr\left(\rho \ln \rho\right) > 0$. For the scalar field Hamiltonian (19), K_{ij} can be read-off:

$$\begin{aligned} K_{ij} \;\; &= \;\; \frac{1}{i^2}\left[l(l+1)\delta_{ij} + \frac{9}{4}\,\delta_{i1}\delta_{j1} + \left(N - \frac{1}{2}\right)^2 \delta_{iN}\delta_{jN} + \left(\left(i + \frac{1}{2}\right)^2 + \left(i - \frac{1}{2}\right)^2\right)\delta_{i,j(i\neq 1,N)}\right] \\ &\quad - \left[\frac{(j+\frac{1}{2})^2}{j(j+1)}\right]\delta_{i,j+1} - \left[\frac{(i+\frac{1}{2})^2}{i(i+1)}\right]\delta_{i,j-1} \; . \end{aligned} \tag{23}$$

Note that the last two terms denote (nearest-neighbour) interaction and originates in the derivative term in (18). Schematically,

$$K = \begin{pmatrix} \times & \times & & & & & \\ \times & \times & \times & & & & \\ & \times & \times & \times & & & \\ & & \times & \times & \times & & \\ & & & \times & \times & \times & \\ & & & & \times & \times & \times \\ & & & & & \times & \times \end{pmatrix} \tag{24}$$

where the off-diagonal terms represent interactions. The GS of the above Hamiltonian is given by:

$$\psi(x_1, \ldots, x_N) = \prod_{i=1}^{N} N_i \exp(-\frac{1}{2} k_{Di}^{\frac{1}{2}} \underline{\mathrm{x}}_i^2) \tag{25}$$

which follows from (22), by setting all ν_i to zero. The corresponding density matrix (21) can be evaluated exactly (the suffix 0 signifies GS):

$$\rho_0(x, x') \sim \exp\left[-(x^T \gamma x + x'^T \gamma x')/2 + x^T \beta x\right] \tag{26}$$

where:

$$\Omega \sim K^{1/2} = \left(\begin{array}{cc} A & B \\ B^T & C \end{array} \right) \ , \ \beta = \frac{1}{2} B^T A^{-1} B \ , \ \gamma = C - \beta \ . \tag{27}$$

Note that B and β are non-zero if and only if there are interactions. The gaussian nature of the above density matrix lends itself to a series of diagonalisations [3] , such that it reduces to a product of $(N-n)$, 2-oscillator density matrices, in each of which one oscillator is traced over [5]:

$$\rho_0(x, x') \sim \prod_{i=1}^{N-n} \exp\left[-\frac{v_i^2 + v_i'^2}{2} + \bar{\beta}_i v_i v_i' \right] \ . \tag{28}$$

The corresponding entropy is given by:

$$S = \sum_{i=1}^{N-n} \left(-\ln[1-\xi_i] - \frac{\xi_i}{1-\xi_i} \ln \xi_i \right) \ \left[\xi_i = \frac{\bar{\beta}_i}{1 + \sqrt{1 - \bar{\beta}_i^2}} \right] \ . \tag{29}$$

Thus, for the full Hamiltonian $H = \sum_{lm} H_{lm}$, the entropy is:

$$S = \sum_{l=0}^{\infty} (2l+1) S_l \tag{30}$$

where the degeneracy factor $(2l+1)$ follows from spherical symmetry of the Hamiltonian. In practice, we will replace the upper bound of the sum in the above to a large value l_{max}. For the interaction matrix (23), the above entropy, computed numerically, turned out to be [4, 5]:

$$S = 0.3 (n + 1/2)^2 \equiv 0.3 \left(\frac{R}{a} \right)^2 \ . \tag{31}$$

The logarithm of the entropy versus the logarithm of the radius of the horizon is plotted in Figure 1, and the slope is seen to be 2.

4. Coherent and Squeezed States

4.1. Coherent state

In this section, we first examine the situation when GS in (25) is replaced by coherent states. A CS ('shifted' ground state) is defined as the following, for which $\Delta p \Delta x = \hbar/2$:

$$\begin{aligned} \psi(x_1, \ldots, x_N)_{CS} &= \prod_{i=1}^{N} N_i \exp(-\frac{1}{2} k_{Di}^{\frac{1}{2}} (\underline{x}_i - \alpha_i)^2) &(32) \\ &= \prod_{i=1}^{N} \exp(-i \underline{p}_i \alpha_i) N_i \exp(-\frac{1}{2} k_{Di}^{\frac{1}{2}} \underline{x}_i^2) &(33) \end{aligned}$$

where $\underline{p}_i = -i\partial/\partial \underline{x}_i$. Next, defining $\tilde{x} \equiv x - U^{-1}\alpha$, $d\tilde{x} = dx$, it is easy to show that the density Matrix for the CS can be written as:

$$\rho_{CS}\left(x; x'\right) = \int \prod_{i=1}^{n} dx_i \ \psi_{CS}(x_1, \ldots; x_{n+1}, \ldots) \ \psi^\star_{CS}(x_1, \ldots; x_{n+1}, \ldots) = \rho\left(\tilde{x}; \tilde{x}'\right) \ . \tag{34}$$

[3] $V\gamma V^T = \mathrm{diag}$, $\bar{\beta} \equiv \gamma_D^{-\frac{1}{2}} V \beta V^T \gamma_D^{-\frac{1}{2}}$, $W\bar{\beta}W^T = \mathrm{diag}$, $v_i \in v \equiv W^T (V\gamma V^T)^{\frac{1}{2}} V T$.

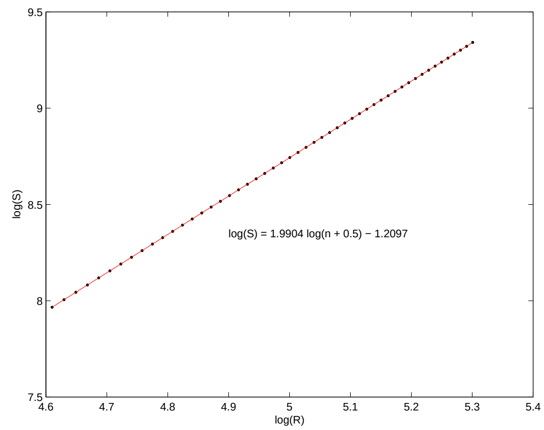

Figure 1. $\ln(S)$ vs $\ln(R)$ for GS. The slope is 2.

Remarkably, the density matrix has the same function form as the GS density matrix (26), albeit in terms of the tilded variables. However, this means that the eigenfunctions will be identical to those for the GS, as a function of the tilded variables. Consequently, the eigenvalues, and hence the entropy will be the same as those of the GS [10]!

4.2. Squeezed state

SS are also minimum uncertainty packets, for which for which $\Delta p \Delta x = \hbar/2$. However, they are characterised by $\Delta p \gg 1$ or $\Delta x \ll 1$ (or vice-versa). Here, we consider a class of SS, characterised by a single parameter r, and given by the following wavefunction:

$$\psi_{SS}(x_1, \ldots, x_N) = r^{N/2} \prod_{i=1}^{N} N_i \exp\left[- \sum_i r \kappa_{Di}^{1/2} \underline{x}_i^2 \right] . \tag{35}$$

In this case, defining a new variable as: $\tilde{x} \equiv \sqrt{r}\ \underline{x}$, $d\tilde{x} = \sqrt{r}\ d\underline{x}$, the density matrix becomes:

$$\rho_{SS}\left(x; x'\right) = \int \prod_{i=1}^{n} dx_i\ \psi_{SS}(x_1, \ldots; x_{n+1}, \ldots)\ \psi^\star_{SS}(x_1, \ldots; x_{n+1}, \ldots) = \rho\left(\tilde{x}; \tilde{x}'\right) . \tag{36}$$

Once again, it has the same functional form, and the entropy is same as that for the ground state [10].

5. First excited state

Although, ideally one would like to compute the entanglement entropy for the most general eigenstate (or superpositions of such eigenstates), given by Eq.(22), this turned out to be a formidable task. Instead, to examine the role of excitations which do not form minimum uncertainty packets, we consider the following wavefunction, which is a superposition of N wavefunctions, each of which signifies *exactly* one oscillator in the first excited state [4].

$$\psi_1(x_1 \ldots x_N) \quad = \quad \sum_{i=1}^{N} a_i N_i H_1\left(k_{Di}^{\frac{1}{4}} \underline{x}_i \right) \exp\left[-\frac{1}{2} \sum_j k_{Dj}^{\frac{1}{2}}\ \underline{x}_j^2 \right] \tag{37}$$

[4] The analysis for two harmonic oscillators was done in [9].

$$= \sqrt{2} \quad \left(a^T K_D^{\frac{1}{2}} \underline{\mathrm{x}}\right) \ \psi_0\left(x_1, \ldots, x_N, \right) \tag{38}$$

Where $a^T = (a_1, \ldots, a_N)$, and normalisation requires $a^T a = 1$. The density matrix can be written as:

$$\rho(x; x') = 2 \int \prod_{i=1}^{n} dx_i \left[x'^T \Lambda x\right] \psi_0\left(x_i; x\right) \psi_0^\star\left(x_i; x'\right) \ . \tag{39}$$

This can be evaluated *exactly*, yielding:

$$\rho(x, x') = \left[1 - \frac{1}{2}\left(x^T \Lambda_\gamma t + x'^T \Lambda_\gamma x'\right) + x^T \Lambda_\beta x'\right] \rho_0(x, x') \tag{40}$$

where

$$\Lambda = U^T \ K_D^{\frac{1}{4}} \ a \ a^T \ K_D^{\frac{1}{4}} \ U \equiv \begin{pmatrix} \Lambda_A & \Lambda_B \\ \Lambda_B^T & \Lambda_C \end{pmatrix} \tag{41}$$

Without further approximations, (40) cannot be factorised into 2-particle density matrices. However, noting that ρ_0 in the above is a gaussian (given by (26)), decaying virtually to zero beyond its few sigma limits, we check whether quantities $\epsilon_{1,2}$, defined below, are small for any given Λ matrix. That is, whether:

$$\epsilon_1 \equiv x_{max}^T \Lambda_\beta x_{max} \ll 1 \ , \ \epsilon_2 \equiv x_{max}^T \Lambda_\gamma x_{max} \ \ll 1 \tag{42}$$

where

$$x_{max}^T = \left(\frac{3(N-n)}{\sqrt{2Tr(\gamma - \beta)}}\right)(1, 1, \ldots) \tag{43}$$

corresponding to 3σ limits in the argument of ρ_0. If both $\epsilon_{1,2} \ll 1$, then we can make the following approximation:

$$1 - \frac{1}{2}\left(x^T \Lambda_\gamma x + x'^T \Lambda_\gamma x'\right) + x^T \Lambda_\beta x' \approx \exp\left[-\frac{1}{2}\left(x^T \Lambda_\gamma x + x'^T \Lambda_\gamma x'\right) + x^T \Lambda_\beta x'\right] \ , \tag{44}$$

with which (40) can be written as:

$$\rho(x, x') \approx \exp\left(-\frac{1}{2}\left(x^T \gamma' x + x'^T \gamma' x'\right) + x^T \beta' x'\right) \qquad \left[\beta' \equiv \beta + \Lambda_\beta, \gamma' \equiv \gamma + \Lambda_\gamma\right] \ . \tag{45}$$

The above being of the same form as the gaussian density matrix, but with shifted parameters, can be factorised once again into two particle density matrices, and the associated entanglement entropy can be evaluated. The rest of the computation is numerical (done using MATLAB). We choose the following values for parameters: $N = 300,\ n = 100 - 200,\ o = 10 - 50$, where o signifies the number of non-zero entries in the vector: $a^T = (1/\sqrt{o})\,(0, \cdots, 0, 1, \cdots, 1\ )$. For the above choice, it was seen that $\epsilon_{1,2} \leq 10^{-3}$. We give the results of our computation in the form of relevant graphs. Figure 2 shows the logarithm of the entropy for the GS and ES for $o = 10, 30, 40, 50$. We see that for the ES, the entropy scales as A^α, with lesser α for higher o. It is less than unity for any $o > 0$. The AL does not seem to hold! Next, in Figure 3, we plot $(2l+1)S_l$ vs l for the GS as well as for various ES and several n. It is seen that for the GS, there is a peak at $l = 0$ (s-wave), followed by another one around $l \approx 50 - 100$, the degeneracy factor of $(2l+1)$ being responsible for the latter, For ES, the first peak shifts to $l > 0$, with the shift being greater for larger o. The graphs do not show a second peak, although there seems to be an increase towards higher values of l. Thus, higher partial waves are seen to get excited with greater excitations.

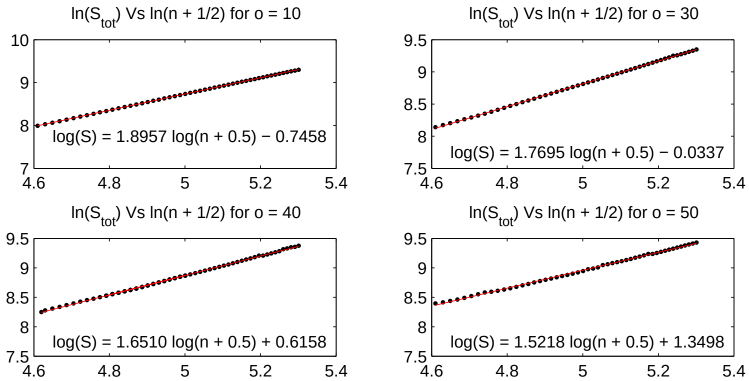

Figure 2. Plot of $\ln(S)$ vs $\ln(R)$ for ES. The slope is less than 2.

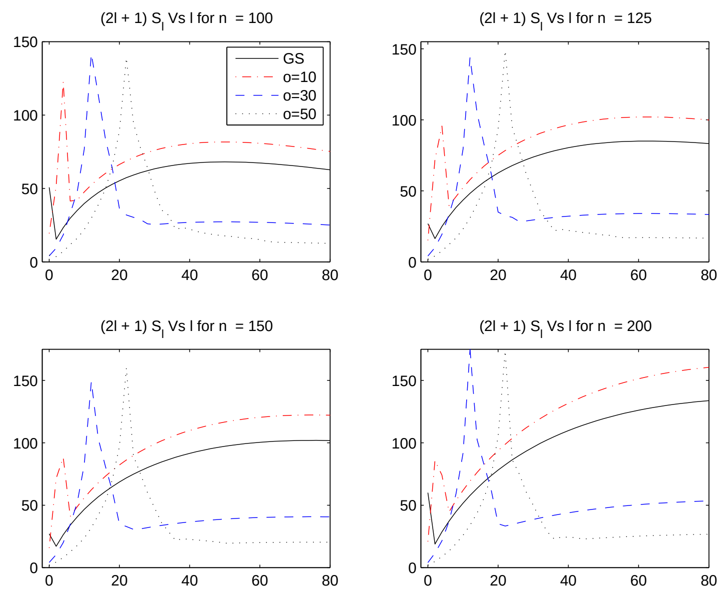

Figure 3. Plot of $(2l+1)S_l$ vs l for GS and different ES for various n ($N = 300$).

6. Where are the degrees of freedom?

In this section, we address the question: where are the DOF, which give rise to entanglement entropy for black holes?[5]

To this end, we re-consider the interaction matrix K in (24), and set to zero, *by hand*, the interaction (off-diagonal) elements except for a small window (of say size 3×3), and move the window from the origin through the horizon, to the outer edge of the spherical box, as shown below:

[5] In Ref. [11], the authors have studied the scaling properties of energy fluctuations close to the boundary.

$$K = \begin{pmatrix} \times &&&&&&\\ &\times&&&&&\\ &&\times&&&&\\ &&&\vert\overline{\times}&\overline{\times}&\quad\vert&\\ &&&\vert\times&\times&\times\vert&\\ &&&\vert&\underline{\times}&\underline{\times}\vert&\\ &&&&&&\times\end{pmatrix} \tag{46}$$

With the above modified matrix, now we compute the entanglement entropy, and plot it as a function of the position of the centre of the window (the green dotted boxes). The result is shown in Figure 4 (the solid red curve). Note that when the window lies entirely outside or inside the hrizon, the entropy is zero. When the window just enters the horizon, the entropy starts to rise, and when it is symmetrically placed between the inside and outside, the entropy rises to 100% of its value, and decreases again as it is further moved. The above results confirm the intuition that entanglement between inside and outside the horizon is the source of this entropy, and shows how it rises from zero to its maximum value as the entangled DOF are gradually incorporated. In Figure 5, we probe this further, by keeping one extremity of the

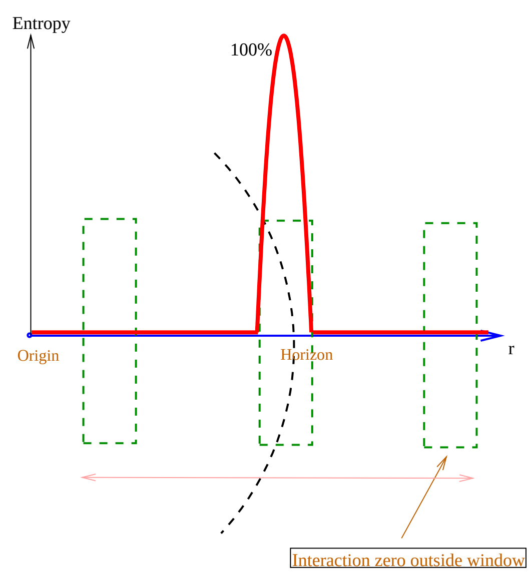

Figure 4. Entanglement entropy comes from the near-horizon DOF.

window just outside the horizon, while increasing its width from zero to the horizon radius. We see that with a width of one lattice spacing, about 85% of the total entropy is obtained, and within a width of about $8 - 10$ lattice spacings, it reaches 100%. While this shows that most of the entropy comes from DOF very close to the horizon, a small part (about 15% in this case), has its origin deeper inside. The latter is understood to indirectly affect the DOF near the boundary via the nearest-neighbour terms in (24). We expect a similar analysis for ES to give us a better understanding of the scaling of the entropy with area in that case.

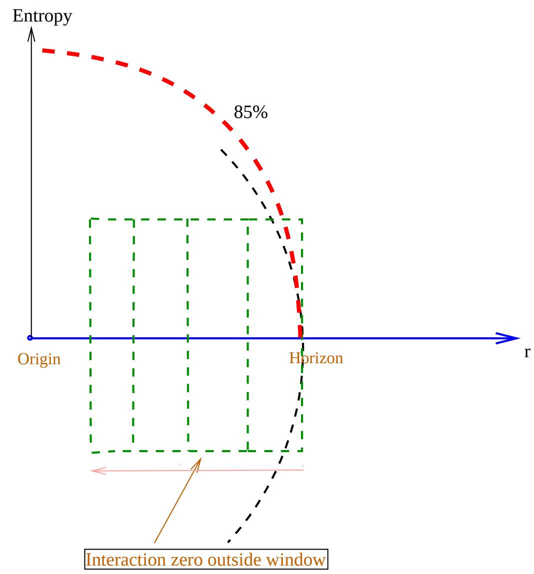

Figure 5. Rise of entropy as interaction region is increased.

7. Generalisation to black hole spacetimes

In this section, we generalise the framework of our calculations, so that they are applicable to black hole spacetimes. We start with the Schwarzschild metric with a horizon at $r = r_0$:

$$ds^2 = -f(r)dt^2 + \frac{dr}{f(r)} + r^2 d\Omega^2 \ , \quad f(r) = 1 - \frac{r_0}{r} \tag{47}$$

and define the following coordinate transformation from (t, r) to (τ, R) coordinates:

$$\tau = t + r_0 \left[\ln \left(\frac{1 - \sqrt{r/r_0}}{1 + \sqrt{r/r_0}} \right) + 2\sqrt{\frac{r}{r_0}} \right] \ , \ R = \tau + \frac{2 \ r^{\frac{3}{2}}}{3\sqrt{r_0}} \ , \ r = \left[\frac{3}{2} (R - \tau) \right]^{\frac{2}{3}} r_0^{\frac{1}{3}} \ , \tag{48}$$

such that (47) transforms to the following metric, in *Lemaitre coordinates*:

$$ds^2 = -d\tau^2 + \frac{dR^2}{\left[\frac{3}{2r_0} (R - \tau) \right]^{\frac{2}{3}}} + \left[\frac{3}{2r_0} (R - \tau) \right]^{\frac{4}{3}} r_0^{\frac{2}{3}} \ d\Omega^2 \ . \tag{49}$$

The Hamiltonian for a free scalar field in the above spacetime can be written as [6]:

$$H(\tau) = \frac{1}{2} \int_\tau^\infty dR \left[\frac{2\pi(\tau, R)^2}{3(R - \tau)} + \frac{3}{2} \ r \ (R - \tau) \left(\partial_R \phi(\tau, R) \right)^2 \right] \ . \tag{50}$$

Next, choose a *fixed* Lemaitre time, say $\tau = 0$ and perform the following field redefinitions:

$$\pi(r) = \sqrt{r}\pi_1(r) \ , \quad \phi(r) = \frac{\phi_1(r)}{r} \ . \tag{51}$$

[6] the current analysis is done for $l = 0$. Generalisation to any $l > 0$ is straightforward.

Then, at that *fixed time*, the Hamiltonian (50) transforms to:

$$H(0) = \frac{1}{2} \int_0^\infty dr \left[\pi_1(r)^2 + r^2 \left(\partial_r \frac{\phi_1}{r} \right)^2 \right] . \tag{52}$$

Note that this is identical to the Hamiltonian in flat spacetime, Eq.(18). Also, it follows from Eq.(48) that the horizon now corresponds to $R = \frac{2}{3} r_0$. Thus, one is free to trace over either the region $R = 0 \to \frac{2}{3} r_0$ or over the region $R = \frac{2}{3} r_0 \to \infty$, and all our results go through for a fixed τ. Once again, the AL holds for GS, CS and SS, while it does not for ES. We also show that although the degrees of freedom near the horizon contribute most to the entropy, the DOF deep inside must also be taken into account for the AL.

8. Summary and open questions

In this article, we have shown that entanglement between DOF inside and outside the horizon is a viable source of black hole entropy. However, while the AL holds for the GS, CS, and SS, this entropy scales as an area power less than unity, when the considered wavefunction is a superposition of first ES. Furthermore, while the DOF near the horizon contribute most to the entropy, those farther away give a small contribution as well. We would like to investigate this further for ES, as that might provide a physical understanding of the deviation from the AL. We would also like understand the phhenomena of Hawking radiation and information loss in this icture. We hope to report on this elsewhere [12].

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References

[1] J. D. Bekenstein, Phys. Rev. **D7** (1973) 2333; S. W. Hawking, Commun. Math. Phys. **43** (1975) 199; S. W. Hawking, Phys. Rev. **D13** (1976) 191.
[2] A. Strominger, C. Vafa, Phys. Lett. **B379**, 99 (1996); A. Ashtekar *et al*, Phys. Rev. Lett. **80**, 904 (1998); S. Carlip, ibid **88**, 241301 (2002); A. Dasgupta, Class. Quant. Grav. **23** (2006) 635 [arXiv:gr-qc/0505017].
[3] R. Brustein, M. B. Einhorn and A. Yarom, JHEP **0601** (2006) 098; D. Marolf and A. Yarom, *ibid* 141.
[4] L. Bombelli, R. K. Koul, J. Lee, R. Sorkin, Phys. Rev. **D34** (1986) 373.
[5] M. Srednicki, Phys. Rev. Lett. **71** (1993) 666 [arXiv:hep-th/9303048].
[6] M.B. Plenio *et al*, Phys. Rev. Lett. **94** (2005) 060503; M. Cramer *et al*, Phys. Rev. **A73** (2006) 012309.
[7] S. Chandrasekhar, *The Mathematical Theory of Black Holes*, Clarendon Press, Oxford (1992), *Chapter 4*.
[8] P. Tommasini, E. Timmermans, A. F. R. de Toledo Piza, arXiv: quant-ph/9709052.
[9] M. Ahmadi, S. Das, S. Shankaranarayanan, Can. J. Phys. **84(S2)**, 1 (2006) [arXiv:hep-th/0507228].
[10] S. Das, S. Shankaranarayanan, Phys. Rev. **D73** (2006) 121701 [axXiv:gr-qc/0511066].
[11] R. Brustein, D. H. Oaknin and A. Yarom, Phys. Rev. D **70** (2004) 044043
[12] S. Das and S. Shankaranarayanan, `Work in progress`.