VACUUM POLARIZATION AND HAWKING RADIATION

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Dedication

To my mother.

Abstract

Quantum gravity is one of the interesting fields in contemporary physics which is still in progress. The purpose of quantum gravity is to present a quantum description for spacetime at 10^{-33}cm or find the 'quanta' of gravitational interaction.. At present, the most viable theory to describe gravitational interaction is general relativity which is a classical theory. Semi-classical quantum gravity or quantum field theory in curved spacetime is an approximation to a full quantum theory of gravity. This approximation considers gravity as a classical field and matter fields are quantized. One interesting phenomena in semi-classical quantum gravity is Hawking radiation. Hawking radiation was derived by Stephen Hawking as a thermal emission of particles from the black hole horizon. In this thesis we obtain the spectrum of Hawking radiation using a new method.

Vacuum is defined as the possible lowest energy state which is filled with pairs of virtual particle-antiparticle. Vacuum polarization is a consequence of pair creation in the presence of an external field such as an electromagnetic or gravitational field. Vacuum polarization in the vicinity of a black hole horizon can be interpreted as the cause of the emission from black holes known as Hawking radiation. In this thesis we try to obtain the Hawking spectrum using this approach. We re-examine vacuum polarization of a scalar field in a quasi-local volume that includes the horizon. We study the interaction of a scalar field with the background gravitational field of the black hole in the desired quasi-local region. The quasi-local volume is a hollow cylinder enclosed by two membranes, one inside the horizon and one outside the horizon. The net rate of particle emission can be obtained as the difference of the vacuum polarization from the outer boundary and inner boundary of the cylinder. Thus we found a new method to derive Hawking emission which is unitary and well defined in quantum field theory.

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Chapter 1

Introduction

There are four fundamental forces in nature:

- **electromagnetic force:** is a long range force that acts on charged particles. For example, electromagnetism is responsible for binding electrons to the nucleus to form the atom. The electromagnetic force between two charged particles is a result of the exchange of photons. Therefore, photons are mediators of the electromagnetic force between charged particles.

- **weak nuclear force:** is a very short range interaction$\approx 10^{-18}m$ [1]. Weak interaction acts on all fermions which are spin-1/2 elementary particles. The mediators of the weak interaction are W^+, W^- and Z bosons which are known as weak bosons.

- **strong nuclear force:** acts on some fermions and bosons. Fermions are divided into two main categories, baryons and leptons. Also, main types of bosons are mesons and gauge bosons. Baryons and mesons collectively are named hadrons. Strong interaction affects hadrons. It is a short range interaction$\approx 10^{-15}m$, [1], and is the force that binds protons and neutrons to form a nucleus.

- **gravity:** is the force between massive bodies. It is long range and the graviton which has not been detected yet is responsible to mediate gravity.

Based on Grand Unified Theory(GUT), strong nuclear force, weak nuclear force and electromagnetic force are distinguishable at low temperature or equivalently at low energy. However, these three forces would be indistinguishable at high temperature. Among these forces gravity is the strange one. Gravity is not unified with the other three forces of nature and there is no successful theory to quantize it until now. In this thesis we are going to study gravitational interaction and will review theories that have been suggested to describe gravity.

Newton's description of gravity was the most acceptable theory of gravity until the 20th century. He justified gravity as a mutual force between massive bodies which is inversely proportional to the square of the distance between them. Newtonian mechanics is valid at the limit of low speeds, also the theory is frame-dependent. Newtonian laws of mechanics are invariant for inertial observers which are non-accelerating observers. In order to define a preferred frame of reference among the infinite number of inertial frames Newton proposed the concept of absolute space. However, absolute space contradicts the third law of Newton and therefore was rejected by physicists.

In 1905 Einstein announced his special theory of relativity that could solve the problem of preferred inertial frame of Newtonian mechanics by postulating that all inertial frames are the same to do all physical experiments. Special relativity is applicable for inertial frames or equivalently for flat spacetime. In order to include accelerating frames or the curved spacetime a generalization of special relativity is required.

In 1916 Einstein announced his general theory of relativity as a generalization of special theory of relativity. Some principles such as Mach's principle, equivalence principle and so on, inspired Einstein to formulate his general theory of relativity.

This theory provided a new approach to the concept of space and time. In general relativity, time is observer-dependent and is not universal. Also, gravity is justified as due to the geometry of spacetime not as a force between massive objects like the Newtonian perspective.

In chapter two we will review the Newtonian mechanics and its shortcomings. We will mention the main statements of special relativity and the reasons that it was proposed. Since the concept of space and time is changed, the transformation that relates different coordinate systems should also change. We will show that the Galilean transformation in Newtonian mechanics has been replaced by Lorentz transformations in special relativity. Then we describe a basic concept in Newtonian mechanics which is mass of a body. We introduce different types of mass and will see that there is a unique mass associated to an object and all different types of mass refer to a single entity. We will also describe different definitions of mass in special relativity.

A brief review of the principles that motivated Einstein to postulate his general relativity as a generalization of special relativity is presented in chapter two. We shall talk about the key ideas of Mach's principle, principle of equivalence and principle of general covariance and explain how they helped Einstein to build his theory.

We will see that the principle of general covariance implies that the physical laws must have tensorial form in order to be invariant in all frames of reference. Therefore, our next task would be to study tensors. Tensors live on manifolds which can be interpreted as the background spacetime. We will discuss rank of tensors, their transformation rules and then will introduce important tensors in general relativity. The most essential tensor in general relativity is Einstein tensor $G_{\mu\nu}$ and we will describe all the tensors required to understand Einstein tensor. Einstein field equations which

are a set of equations to describe gravity as a result of the curvature of spacetime are

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G T_{\mu\nu}}{c^2},$$

we will talk about all the terms in the above equation and will see that field equations manifestly express that the cause of gravity is geometry of spacetime. We shall finish chapter two by explaining geodesics which are the generalization of the straight lines of the flat spacetime and demonstrate possible trajectories that physical particles can take in spacetimes with non vanishing curvatures.

We already expressed that Einstein's field equations are alternatives to Newton's law of gravity. Thus, in order to understand the nature of gravity one has to solve the field equations. The first solution for these equations for the case $T_{\mu\nu} = 0$ that describes the field equations in the vacuum was found by Karl Schwarzschild. Vacuum field equation describes the case where there are no matter fields in the region of spacetime under consideration. Chapter three is devoted to the analysis of the vacuum solution of the field equations. The solution of the vacuum equation depicts a black hole which is a singularity in spacetime and has strong gravitational field. A black hole affects its surrounding spacetime as a result of its strong gravitational field. A classical black hole tends to absorb whatever is in its gravitational field and even light can not escape the gravitational field of a black hole. There is a surface associated with each black hole know as the event horizon. Nothing can go out of the event horizon and no observer outside the event horizon can get information from inside the horizon. We will see that for a Schwarzschild black hole the event horizon is located at $r = 2GM$ (c=1), where r shows the radial distance from the center of the hole, M is the mass of the black hole and G is the gravitational constant.

We will explain the concept of singularity that describes the infinities of the spacetime line element. In chapter three we will explain the singularities of the Schwarzschild spacetime. We shall observe that $r = 2GM$ is a coordinate singularity and is removable while $r = 0$ is an intrinsic singularity related to the geometry of spacetime. Coordinate singularity at $r = 2GM$ can become a regular point by applying a proper coordinate transformation. Eddington-Finkelstein coordinates were suggested by Arthur Eddington and David Finkelstein in order to remove the singularity at $r = 2GM$. This coordinate is well-behaved at the horizon, however, the singularity at $r = 0$ still exists in Eddington-Finkelstein coordinates. Therefore, the Schwarzschild solution can be extended to $r = 0$ by choosing Eddington-Finkelstein coordinates.

The next interesting topic that is covered in this chapter is the Kruskal solution and then Penrose diagrams. The Kruskal coordinates are coordinates which are regular everywhere outside the singularity at $r = 0$. We will introduce the Kruskal coordinates and will write the Schwarzschild line element in terms of them.

Now that the coordinate singularity is removed, we will describe a way to study the infinities of spacetime. One can bring the entire spacetime onto a compact region and study the structure of infinity by applying a proper conformal transformation. Penrose diagram was introduced by Roger Penrose in an effort to bring in the infinities of the spacetime to finite positions. The process is based on conformal transformation of the physical metric of the spacetime. First we shall review the procedure to construct Penrose diagram for Minkowski spacetime and then for the Kruskal solution.

Another essential area of study in black hole physics is 'thermodynamics of black holes'. One can assign corresponding laws of thermodynamics to the black holes. The rest of chapter three covers the laws of black hole thermodynamics and how one can compare them to the classical laws of thermodynamics. We will observe that

according to the laws of thermodynamics black holes must have entropy.

Up to now, we have treated black holes as classical objects that just absorb whatever is in their gravitational field. However, if one takes into account quantum considerations black holes will possess some interesting properties. Quantum effects imply non-zero temperature for the black holes and therefore black holes are not black anymore. They emit a radiation called Hawking radiation which is predicted to have a black body spectrum. Hawking radiation can not be justified classically, since nothing can escape the gravitational field of a classical black hole. In chapter four we talk about QFT in curved spacetime which will be used in chapter five to derive the Hawking radiation spectrum.

A major field in theoretical physics is quantum field theory (QFT). The aim of quantum field theory is to describe the physics of quantum particles by providing mathematical methods. Standard QFT examines the physics of quantum fields in Minkowski spacetime which is a four-dimensional flat spacetime. However, for the curved spacetimes or equivalently for accelerating observers the physical laws for quantum fields would be different. Chapter four covers the basic elements of quantum field theory in Minkowski spacetime for different classes of fields. Fields are categorized to different types based on their spins. We introduce real scalar fields, complex scalar fields and fermionic fields. We shall describe the quantization of fields and canonical commutations and will explain the equations of motion for each of them. We will see that equations of motion are obtained by generalizing the Lagrangian method that is used in classical mechanics. At the end of this section we define a crucial concept in QFT, i.e, vacuum.

A challenging subject in contemporary physics is finding a theory for quantizing gravity. However, no acceptable theory has been found yet. Consequently, there is no viable theory to study the effects of a quantized background gravitational field on

matter fields which are also quantized. Hence, one has to concentrate on the case where gravity is a classical field and matter fields are quantized. This approach is known as semiclassical gravity or quantum field theory in curved spacetime. The rest of chapter four is about the physics of quantum fields in curved spacetime. We examine the physics of different types of fields in curved spacetime and discuss the concept of vacuum in curved spacetime which is different from Minkowski spacetime vacuum.

In chapter five we introduce the concept of vacuum polarization which is used to explain the emission of the Hawking radiation in the vicinity of the black hole horizon. In modern quantum field theory, vacuum is defined as the state with the lowest energy which is full of virtual particle-antiparticle pairs. However, before this accepted model for the vacuum, the Dirac sea was the available description for the vacuum. In this chapter we will talk about Dirac sea as a sea of negative energy electrons and describe its deficiencies and then will review the modern view of vacuum polarization. We will explain how to compute the rate of the vacuum decay by defining the S matrix of the interaction. Our objective is to justify the Hawking radiation as due to the vacuum polarization close to the black hole horizon. Since we are interested in computing the rate of the vacuum decay in the Schwarzschild spacetime we will define the possible vacua for this background spacetime. Based on our selection for the vacuum we will try to find the rate of particle production in the vicinity of the horizon. This rate gives us the spectrum of the Hawking radiation which was predicted by Hawking.

Our method to obtain Hawking radiation from the concept of vacuum polarization is a new approach. Our idea is to consider a cylindrical quasi-local region around the horizon which has two boundaries, one inside the horizon and the other one outside the horizon. The interaction term of the scalar field action in the presence of gravity is the key term for our computations. We will show that the vacuum expectation

value of the interaction term of the action in the defined quasi-local region gives us the spectrum of Hawking radiation.

Hawking radiation as an emission from black holes is an outcome of quantum field theory in curved spacetime which is a semi-classical quantum gravity. This theory is an approximation to semiclassical gravity and in order to achieve a full quantum theory of gravity one has to consider gravity as a semi-classical or if possible as a quantum field. QFT in curved spacetime and Hawking radiation as one of its predictions is an active research area and more research has to be done to present a viable quantum theory of gravity.

Chapter 2

An Introduction to Einstein's Theory of General Relativity

General relativity (GR) is a theory which describes gravity and was announced by Albert Einstein in 1916. According to the general relativity, gravity is considered as the manifestation of curvature of spacetime. Before the advent of GR, gravity was viewed as an attractive force between two massive bodies that obeys Newton's law of gravity. For two masses m_1 and m_2 which are separated a distance r, the gravitational force is directly proportional to the product of their masses and inversely proportional to the square of the distance between them, so,

$$\vec{F} = \frac{Gm_1m_2}{r^2}\vec{r} \tag{2.1}$$

with $G = 6.67 \times 10^{-11} m^3 kg^{-1} s^{-2}$ is the gravitational constant (Fig. 2.1). According to GR, the distribution of matter in spacetime defines the geometry and curvature of spacetime which is interpreted as gravity. GR is generalization of the special theory of relativity (SR) which is valid in flat spacetime or equivalently for

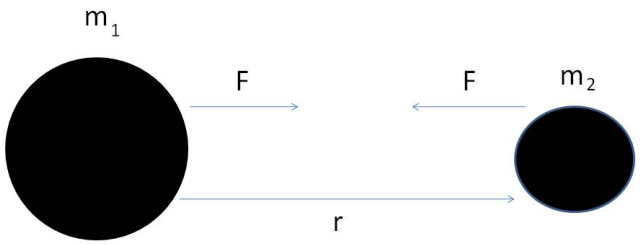

Figure 2.1: Gravitational force between two massive bodies m_1 and m_2 is $F = \frac{Gm_1m_2}{r^2}$.

non-accelerating observers. SR was proposed by Einstein in 1905 in order to solve the problem of inertial frames of Newtonian mechanics. The Galilean principle of relativity (or Newtonian relativity) says

- all laws of physics should have the same form in all inertial frames.

Because the observers move with constant velocity, they can be considered as inertial observers. Inertial frames are those classes of frames with zero acceleration with respect to a preferred frame of reference. We can have an infinite number of inertial observers. The problem is whether there exists a way to distinguish between different inertial frames? Is there any preferred inertial frame among them? To answer this question, Newton had to propose the concept of absolute space. Absolute space is an independent entity which has no interaction with anything else and always remains the same. Absolute space is like a background for the physical events that can affect them. The absolute reference frame is the one which is not accelerated with respect to the absolute space. From Newton's third law we know that all the physical interactions must be mutual. Therefore, it does not make sense that absolute space can not be influenced by anything else while it influences all the physical events. So,

absolute space seemed to be trivial.

Einstein's special relativity solved this problem by assuming that all the inertial frames are equivalent for all physical experiments, therefore, the concept of absolute space is not required anymore.

2.1 Special relativity

Special relativity is a generalization of Newtonian relativity and has the same statement but it includes the laws of electrodynamics besides the laws of mechanics. According to the SR all motions are relative and we don't have any absolute motion. The basic assumptions of SR are as follows:

1. The speed of light is the same in all inertial frames.
2. All laws of physics are the same in all inertial frames.

In Newtonian mechanics inertial frames are related to each other by the so called 'Galilean transformations' which will be described later. SR needed an alternative for Galilean transformations in order to have the speed of light invariant in all inertial frames. Lorentz transformations are the desired transformations which will be explained in the following.

2.1.1 Lorentz transformations

Before the announcement of SR, the Newtonian description of spacetime was the most accepted pattern and the Galilean transformations were the required transformations to relate inertial observers.

Suppose that we have two inertial frames S_1 and S_2, where S_2 is moving with constant velocity v along the x-axis relative to S_1 (Fig. 2.2).

If these two frames measure the same event in spacetime, their measurements can

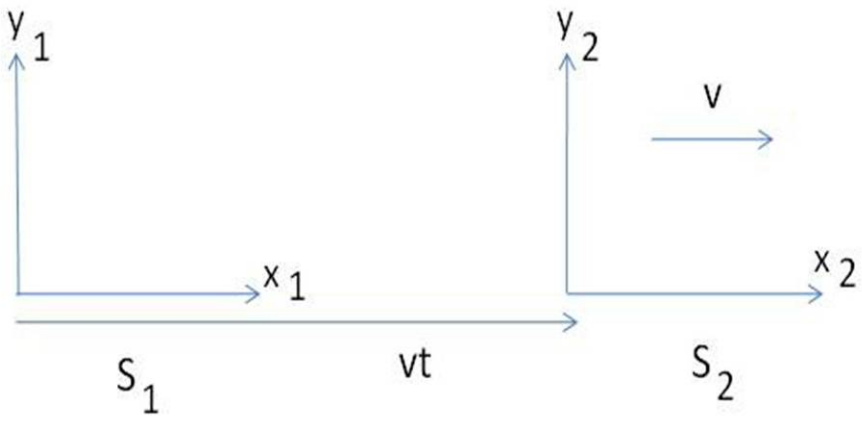

Figure 2.2: Two inertial frames moving in x direction with constant velocity v with respect to each other.

be related to each other by the Galilean transformations such as

$$t_2 = t_1, \tag{2.2}$$

$$x_2 = x_1 - vt, \tag{2.3}$$

$$y_2 = y_1, \tag{2.4}$$

$$z_2 = z_1. \tag{2.5}$$

A basic presumption in Galilean transformation is that there is a universal time. So, time does not change from one inertial frame to another inertial frame. But Einstein believed that time is a frame-dependent concept and different observers in different frames have different time coordinates. One famous thought experiment to understand the frame-dependence of time is as follows:

assume that there are two observers A and B. A is standing outside a train which is travelling with constant velocity v along a straight line relative to his frame of reference. B is standing inside the train in the middle of a carriage. Two flashes of light are coming towards B from the two ends of the carriage which are equidistant from A. B will see the two flashes of light at the same time, so, according to his frame of reference these two events are simultaneous. However, A is moving towards one of the signals and away from the other one, hence, according to A's frame of reference one signal comes before the other one. A concludes that these two events are not simultaneous (Fig. 2.3).

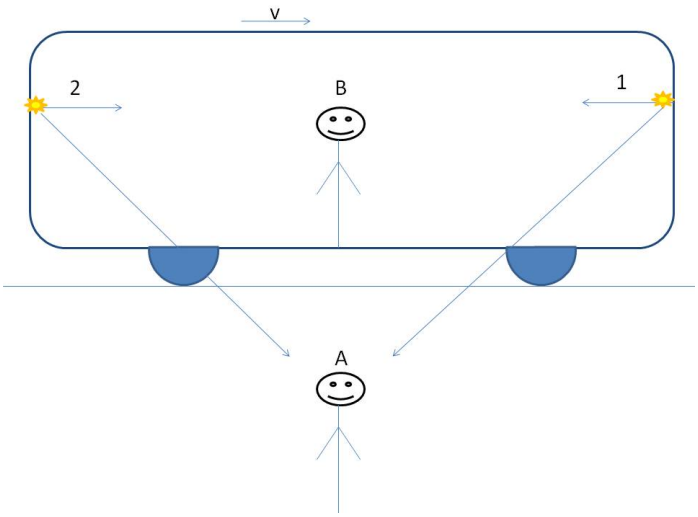

Figure 2.3: Two light sources at two ends of a carriage. B will receive the two light signals at the same time while A receives the light signal from source 2 before the one from source 1.

This simple thought experiment shows that time is not a universal concept. Whether two events occur at the same time depends on the observer. So, Galilean transformations are not the proper transformations to relate the inertial frames.

Einstein introduced Lorentz transformations to relate inertial frames. Special relativity is formulated in a four-dimensional spacetime with three spacelike coordinates and one timelike coordinate known as Minkowski spacetime. If an event occurs at

$P_1 = (ct, x, y, z)$ and another event occurs at $P_2 = (ct + cdt, x + dx, y + dy, z + dz)$, the interval between these two events is

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -c^2 dt^2 + dx^2 + dy^2 + dz^2, \tag{2.6}$$

with $\eta = \text{diag}(-1, 1, 1, 1)$ the metric of Minkowski spacetime and $\mu, \nu = (0, 1, 2, 3)$. c is a constant factor with the units of velocity which is known as the speed of light. The factor of c fixes the units of cdt to be meters. Lorentz transformations are defined as the transformations that leave this interval invariant, i.e.,

$$ds^2 = ds'^2 = -c^2 dt'^2 + dx'^2 + dy'^2 + dz'^2. \tag{2.7}$$

If we denote our coordinate system by $x^\mu = (t, x, y, z)$, with $\mu = (0, 1, 2, 3)$, then

$$x'^\mu = \Lambda^\mu_\nu x^\nu, \tag{2.8}$$

where Λ^μ_ν are homogeneous Lorentz transformations and x'^μ is the transformed coordinate system. For the case of two inertial frames S_1 and S_2, where S_2 is moving with velocity v along the x-axis with respect to S_1, the Lorentz transformations are

$$t_2 = \frac{t_1 - (v/c^2)x_1}{\sqrt{1 - v^2/c^2}}, \tag{2.9}$$

$$x_2 = \frac{x_1 - vt_1}{\sqrt{1 - v^2/c^2}}, \tag{2.10}$$

$$y_2 = y_1, \tag{2.11}$$

$$z_2 = z_1. \tag{2.12}$$

In this case Λ^{μ}_{ν} is the following matrix

$$\Lambda^{\mu}_{\nu} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.13}$$

where

$$\beta = \frac{v}{c}, \tag{2.14}$$

and

$$\gamma = \frac{1}{\sqrt{1-(v/c)^2}}. \tag{2.15}$$

A set of linear Lorentz transformations form a group called the Lorentz group. The group of transformations of a space with coordinates $(y_1, ..., y_m, x_1, ..., x_n)$ which leaves invariant the quadratic form $(y_1^2 + ... + y_m^2) - (x_1^2 + ... + x_n^2)$ are called the orthogonal group $O(m, n)$ so, Lorentz group is $SO(3, 1)$ since as we will see later the determinant of the Lorentz transformation is one, [2]. Lorentz group is a group, since

1) The identity element does exist such that

$$\text{if } \ \Lambda^{\mu}_{\nu} = \delta^{\mu}_{\nu} \rightarrow x'^{\mu} = \delta^{\mu}_{\nu} x^{\nu} = x^{\mu}, \tag{2.16}$$

this is the case where the two inertial frames are identical.

2) Inverse of Λ^{μ}_{ν} is defined as

$$(\Lambda^{-1})^{\mu}_{\nu} = \Lambda^{\nu}_{\mu}, \tag{2.17}$$

where $\Lambda^{\nu}_{\mu}\Lambda^{\mu}_{\sigma} = \delta^{\nu}_{\sigma}$.

3) If Λ^{μ}_{ν} is a Lorentz transformation which takes x^{μ} into $x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu}$ and $\overline{\Lambda^{\mu}_{\rho}}$

is another Lorentz transformation which takes x'^{μ} into $x''^{\mu} = \overline{\Lambda^{\mu}_{\rho}} x'^{\rho}$, then we have $x''^{\mu} = \overline{\Lambda^{\mu}_{\rho}} \Lambda^{\rho}_{\nu} x^{\nu}$, which means that $\overline{\Lambda^{\mu}_{\rho}} \Lambda^{\rho}_{\nu}$ is itself a Lorentz transformation, so the group is closed.

By using Lorentz transformation we have:

$$\eta_{\mu\nu} x'^{\mu} x'^{\nu} = \eta_{\mu\nu} (\Lambda^{\mu}_{\sigma} x^{\sigma})(\Lambda^{\nu}_{\rho} x^{\rho}) = \eta_{\sigma\rho} x^{\sigma} x^{\rho} \tag{2.18}$$

since it must hold for any x^{μ}, we conclude that

$$\eta_{\sigma\rho} = \eta_{\mu\nu} \Lambda^{\mu}_{\sigma} \Lambda^{\nu}_{\rho}. \tag{2.19}$$

In matrix notation x^{μ} can be represented by a column vector and $\eta_{\mu\nu}$ and Λ^{μ}_{ν} by matrices, therefore equation (2.19) will change to

$$\eta = \Lambda^{T} \eta \Lambda \tag{2.20}$$

T is the symbol for the transpose of a matrix which is obtained by writing the rows of the original matrix as the columns of the transpose matrix. Taking the determinant of both sides, we get

$$1 = \det \Lambda^{T} \det \Lambda \tag{2.21}$$

since $\det \Lambda^{T} = \det \Lambda$, we get

$$\det \Lambda = \pm 1. \tag{2.22}$$

Lorentz transformations with $\det \Lambda = +1$ are proper Lorentz transformations and those with $\det \Lambda = -1$ are improper Lorentz transformations, [2], [3].

The most well known types of Lorentz transformations are boosts and rotations.

1. Boosts

A boost along the x-axis is the following transformation

$$t' = t\cosh\eta - x\sinh\eta \tag{2.23}$$

$$x' = -t\sinh\eta + x\cosh\eta \tag{2.24}$$

$$y' = y \tag{2.25}$$

$$z' = z \tag{2.26}$$

where

$$\eta = \tanh^{-1}\left(\frac{v}{c}\right) \tag{2.27}$$

is called rapidity.

2.Rotations

The transformations

$$t' = t \tag{2.28}$$

$$X'^i = a^{ij}X^j \tag{2.29}$$

describe a rotation, where a^{ij} is an orthogonal matrix with $det(a) = 1$, [3]. For example, the rotation matrix a^{ij} for a rotation around the z axis in 3-dimension is

$$a^{ij} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.30}$$

Infinitesimal Lorentz transformation

An infinitesimal Lorentz transformation has the form

$$\Lambda^{\mu}_{\nu} = \delta^{\mu}_{\nu} + \omega^{\mu}_{\nu} \tag{2.31}$$

if we substitute Λ^{μ}_{ν} into (2.19), we find that $\omega^{\mu}_{\nu} = -\omega^{\nu}_{\mu}$. The Λ^{μ}_{ν} for the infinitesimal Lorentz transformation is

$$\Lambda^{\mu}_{\nu} = \begin{pmatrix} 1 & -\beta & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.32}$$

Infinitesimal Lorentz transformations occur when $\gamma \approx 1$.

Newtonian mechanics is valid to study dynamics of an object when the velocity of object is not very high compared to the speed of light and when the object is not close to a very strong gravitational field. In the limit of high speeds, special relativity is the appropriate theory to study the dynamics of a system. Dynamics of an object can be understood by having information about its mass, momentum and energy. Mass is a central concept in mechanics. According to Newtonian theory two masses can be defined which can indicate different properties of an object. These are inertial mass and gravitational mass.

1)Inertial mass: is represented by m_I and appears in the second law of Newton. Inertial mass is a measure of a body's resistance to change in its state of motion. If we apply a force on an object, its inertial mass determines its acceleration. In the second law of Newton force is directly proportional to the acceleration and the constant of

proportionality is the inertial mass,

$$\mathbf{F} = m_I \mathbf{a}. \tag{2.33}$$

2)Gravitational mass: indicates the magnitude of the gravitational force that a body can exert on other objects or can experience because of the presence of the other objects. Gravitational mass is divided into two different types, passive gravitational mass and active gravitational mass.

Active gravitational mass, m_A, is the one that produces a gravitational field. Objects with small active mass produce weak gravitational fields. If we place an object with mass m_A at the origin, the gravitational potential at distance r from the origin is

$$\phi = \frac{-Gm_A}{r}, \tag{2.34}$$

therefore, the gravitational force that it applies on another object with mass m_P located at distance r from m_A would be

$$\mathbf{F} = -\frac{Gm_P m_A}{r^2}\hat{r}. \tag{2.35}$$

Passive gravitational mass, m_P, is a measure of a body's reaction to a gravitational field. Within the same gravitational field, an object with a smaller passive gravitational mass experiences a smaller force than an object with a larger passive gravitational mass. This force is called the weight of the object.

$$\mathbf{F} = -m_P \nabla \phi, \tag{2.36}$$

where, $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$, in cartesian coordinates and ϕ is the gravitational potential at some point of space.

However, one can easily show that there is a unique mass in Newtonian mechanics and there is no distinction between different definitions of mass, [4]. Thus,

$$\boxed{m = m_I = m_P = m_A.} \tag{2.37}$$

On the other hand, special relativity introduces two different ways to define mass

I) **rest mass**, m_0, is equivalent to the Newtonian mass. Rest mass is the same in all reference frames which are related by Lorentz transformations. It is measured by observers that are moving with the object, i.e, observers in the comoving frame.

II) **relativistic mass**, m, is the mass which is observer-dependent. This mass is not the same as measured by different observers. It is related to the rest mass as

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}, \tag{2.38}$$

v is the velocity of the mass with respect to the rest frame. Before we talk about relativistic momentum and energy, we should introduce four-vectors. In special relativity, vectors live in a four-dimensional real space which is called Minkowski space. Any vector in Minkowski space is called a four-vector. An arbitrary four-vector can be written as

$$A^\mu = (A^0, A^1, A^2, A^3). \tag{2.39}$$

The squared norm of the four-vector A^μ is defined as the square of its length

$$A^2 = g_{\mu\nu} A^\mu A^\nu = A_\mu A^\mu, \tag{2.40}$$

so, the four-vector is

$$\begin{cases} \text{null} & \text{if } A^2 = 0, \\ \text{timelike} & \text{if } A^2 > 0, \\ \text{spacelike} & \text{if } A^2 < 0. \end{cases}$$

The four-momentum for a massive object moving with velocity $\vec{u}$ is defined as

$$p^\mu = (\frac{E}{c}, \vec{p}) \tag{2.41}$$

$\vec{p}$ is the relativistic three-momentum defined as

$$\vec{p} = \frac{m_0 \vec{u}}{\sqrt{1 - v^2/c^2}}, \tag{2.42}$$

and E is the total energy of the object. Einstein proposed that mass and energy are equivalent and mass of a body is a measure of its energy. Energy and mass are related by the following famous equation

$$E = mc^2 = \gamma m_0 c^2. \tag{2.43}$$

Special relativity with Lorentz transformations as the proper transformation group is successful theory when one just considers inertial frames. In the next section we will see that a generalized theory is required to study physics in non-inertial frames. We will study the motivations that helped Einstein to develop the general theory of relativity.

2.2 Appearance of general relativity

Special relativity deals with flat spacetime and therefore does not include gravity and only applies to the inertial frames. If one is interested in studying curved spacetimes or equivalently studying accelerated observers, then a generalization of SR is required. General relativity is the result of efforts to find a theory which also includes gravity or be appropriate to study accelerated frames of reference. General relativity which was published by Einstein in 1916, can describe gravity as the geometric property of spacetime. There were some guiding principles that motivated Einstein to develop his general theory of relativity. These principles are:

1)Mach's principle

2)Principle of equivalence

3)Principle of general covariance.

In the following, I will explain them briefly.

2.2.1 Mach's principle

There has been debate within the physics community about the nature of space and time for a long time. There are two different viewpoints about the nature of space, the absolute view and the relative view. The advocates of absolute space believe that space is like a container that includes all the material objects, in this view space is an independent entity which is not influenced by its contents. Absolute view of space introduces space as a background that material objects can move in. On the other hand, relative view of space states that space has no meaning without matter in it. So, if there is matter then there is space. One of the most well known opponents of absolute space is Ernst Mach. He believed that the absolute view of space and time that stems from Newton's concept of absolute space and time should be modified.

One famous observation that motivated Newton to introduce the notion of absolute space is the bucket experiment:

if there is a bucket at rest which is filled with water and another bucket with same shape and size and same amount of water in rotation, the surface of water in the bucket at rest would be flat while for the bucket in rotation the surface of water will be curved (Fig. 2.4).

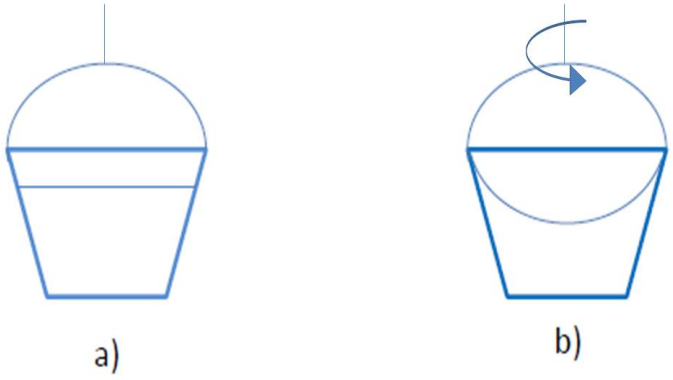

Figure 2.4: a) Flat surface of water in a bucket at rest, b) curved surface of water in a bucket in absolute rotation.

In order to justify this observation, Newton assumed that the rotation with respect to the absolute space is the reason for the difference between these two cases. However, Mach opposed Newton's description for the bucket experiment. He stated that in a universe which is filled with matter, the curved surface of water for the rotating bucket is due to its rotation with respect to the frame of fixed stars. It implies that if one tries to do the bucket experiment in an otherwise empty universe there must be no difference between the two states of the bucket. According to Mach's opinion, one should consider motion of a body with respect to the masses of the universe rather than the absolute space. One consequence of Machian viewpoint is that the concept of inertia is also relative. For a sample mass, inertia is the tendency to maintain its

state of motion and velocity. Hence, if we have a collection of masses the inertia for each mass in the collection is its tendency to preserve the state of motion and velocity with respect to the other masses in the collection. Consequently, for a body in an empty universe, inertia has no meaning. The essence of Mach's principle is:

- all the motions in nature are relative motions.

Mach's principle has two versions, strong Mach's principle and weak Mach's principle, [4]. The weak Mach's principle says

- The distribution of matter determines the geometry.

The strong Mach's principle states

- If there is no matter then there is no geometry.

2.2.2 Principle of equivalence

The principle of equivalence played an important role in the formation of general relativity. Newtonian mechanics proves that inertial mass is identical to the gravitational mass, Einstein promoted this concept to the Principle of Equivalence which is one of the fundamental motivations of formation of the general relativity.

There are two different versions of the principle of equivalence. One is the weak form of the principle of equivalence and the other one is the strong form of the equivalence principle. The strong form of the principle of equivalence says

- Motion of a gravitational test particle (a particle that just feels the gravitational field and does not change the gravitational field) that moves in a gravitational field, is independent of its mass.

The weak form of the principle of equivalence states

- For all massive objects, the inertial mass and gravitational mass are equal.

One consequence of the principle of equivalence is that the motion of a freely falling body in a gravitational field and an accelerating frame is indistinguishable if the acceleration due to the gravity is equal to the acceleration of the accelerating frame.

We can conclude that one can remove gravity locally by going to a freely falling frame of reference. But, note that this statement is true if we consider a 'small enough region of spacetime'. In a big region of spacetime, the gravitational field changes from one place to another place. If we place a big box in a gravitational field, we observe that two freely falling bodies inside the box will move towards each other while they are falling (Fig. 2.5).

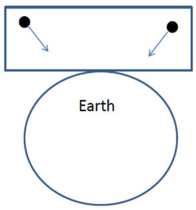

Figure 2.5: Motion of two freely falling bodies in a big box in a gravitational field.

Imagine that we have two elevators, one is at rest in a gravitational field with gravitational acceleration g and another one is accelerating with a constant acceleration $a = g$ with respect to an inertial frame of reference. If we let two balls fall freely in these two elevators we will observe that the motions of the two balls are the same. In other words, one can not distinguish between a gravitational field and an accelerating frame by just studying the motion of a freely falling body. So, one can produce gravity by going to an accelerated frame (Fig. 2.6).

Now consider that we have an elevator which is at rest in a location where there is no gravitational field and another elevator is falling freely in a gravitational field which produces gravitational acceleration equal to g. Again, the motions of two freely

falling balls in these two elevators would be the same. Therefore, one can eliminate gravity by going to a freely falling frame (Fig. 2.7).

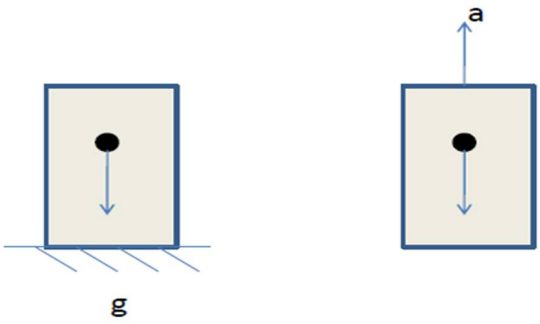

Figure 2.6: The motions of two freely falling bodies are identical in an accelerating frame and a frame at rest in a gravitational field if a=g.

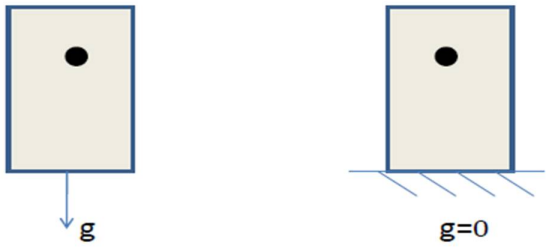

Figure 2.7: The motions of two freely falling bodies are the same in a freely falling frame of reference and a frame at rest in a location where there is no gravitational field around.

After Einstein proposed the theory of special relativity, he tried to find something more inclusive than the principle of equivalence. Finally, he found that

- All the physical experiments should have the same results in a gravitational field and in a uniformly accelerated frame not just the free fall experiment.

One outcome of this principle is that light must bend in a gravitational field. One can justify this as follows:

based on the special theory of relativity mass and energy are equivalent, so, although light has zero rest mass, it has energy. A flash of light in an accelerated frame will have the same motion as a massive particle falling in the accelerated frame. If one can not distinguish between a gravitational field and an accelerating frame, therefore it must have the same trajectory in a gravitational field.

Light rays that are propagating in a strong gravitational field are bent more than the light rays in a weaker gravitational field (Fig. 2.8), [4], [5], [6].

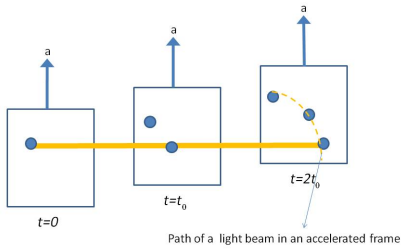

Figure 2.8: Path of a light beam in an accelerated frame.

2.2.3 Principle of general covariance

General covariance is also known as diffeomorphism covariance or general invariance. It states that

- physical laws must have the same form in all frames of reference.

The main core of this principle is that laws of physics exist independent of the coordinate systems. Special theory of relativity satisfies covariance but, special relativity is just valid for inertial frames. According to the principle of equivalence accelerated frames are indistinguishable from the frames in a gravitational field. Therefore, if

there is a gravitational field it is the same as being in a non-inertial frame. Thus, in the presence of a gravitational field a more comprehensive theory is required that can explain physics in non-inertial frames as well. Einstein's idea to solve this problem was:

- the effects of gravity can be removed locally by going to a freely falling frame, therefore, one can reduce the general metric of spacetime to the Minkowski space time metric in a freely falling frame of reference.

Physical laws must be written in a form which would be the same in all frames of reference. We will explain tensors later and will see that tensorial equations have the same form in all coordinate systems. Einstein expressed the principle of general covariance as

- all the equations of physics should have tensorial form, [4].

2.3 Einstein's equation

Einstein's field equations are a set of equations in general relativity to describe gravitational interaction which is a result of curvature of spacetime. In order to introduce these equations and understand them, first we should study some basic concepts. Since gravity depends on the structure of spacetime under consideration, we have to know how to characterize the spacetime. In the following, I will briefly explain some required concepts to understand field equations.

Manifolds

A manifold M, is defined in differential geometry as a geometric entity which locally looks like an n-dimensional Euclidean space $\mathbf{R}^n$. So, it means that if we have a

manifold with any complicated shape, a small enough part of the manifold resembles the flat Euclidean space with the appropriate dimension. We can mention lots of examples for manifolds. For instance, the plane (R^2) , the 3-sphere (S^3), the n-sphere (S^n) are all manifolds since all of them locally resemble the Euclidean space, their dimensions are 2, 3 and n respectively.

Topology of a manifold are properties that are unchanged under continuous deformations of the manifold such as stretching the manifold without tearing or gluing. Topology talks about the global properties of manifolds and has nothing to do with local properties. Therefore, different manifolds can have complicated topology or can be curved, but all of them will locally look like Euclidean space.

Tensors and metric tensor

Tensors are geometric objects which live on manifolds. Lets consider an arbitrary tensor $A^{\mu\nu}_{\rho\lambda}$, the upper indices are called contravariant indices and the lower indices are called covariant indices. A tensor with the covariant rank p and contravariant rank m has type (m, p). So, $A^{\mu\nu}_{\rho\lambda}$ is a tensor of type $(2, 2)$.

An arbitrary contravariant tensor A^μ (rank 1) is a set of geometric objects at point P in the x^μ-coordinate which transform as follows under the change of coordinates $x^\mu \to x'^\mu$

$$A^\mu = \frac{\partial x^\mu}{\partial x'^\rho} A'^\rho \tag{2.44}$$

where

$$\frac{\partial x^\mu}{\partial x'^\rho} = \begin{bmatrix} \frac{\partial x^1}{\partial x'^1} & \frac{\partial x^1}{\partial x'^2} & \cdots & \frac{\partial x^1}{\partial x'^n} \\ \frac{\partial x^2}{\partial x'^1} & \frac{\partial x^2}{\partial x'^2} & \cdots & \frac{\partial x^2}{\partial x'^n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial x^n}{\partial x'^1} & \frac{\partial x^n}{\partial x'^2} & \cdots & \frac{\partial x^n}{\partial x'^n} \end{bmatrix}. \tag{2.45}$$

In four-dimensional spacetime, this matrix is a 4×4 matrix. The determinant of the

matrix J, is called the Jacobian of the transformation

$$J = \left| \frac{\partial x^\mu}{\partial x'^\rho} \right|. \tag{2.46}$$

For a given tensor with one contravariant index the Lorentz transformation is

$$A^\mu \to \Lambda^\mu_\rho A'^\rho, \tag{2.47}$$

By comparing (2.44) and (2.47), one concludes that

$$\Lambda^\mu_\rho = \frac{\partial x^\mu}{\partial x'^\rho}, \tag{2.48}$$

Λ^μ_ρ is a 4×4 matrix. For an arbitrary covariant tensor A_μ (rank 1) the transformation rule is

$$A_\mu = \frac{\partial x'^\rho}{\partial x^\mu} A'_\rho = \Lambda^\rho_\mu A'_\rho, \tag{2.49}$$

in this case, the Jacobian is,

$$J' = \left| \frac{\partial x'^\rho}{\partial x^\mu} \right|, \tag{2.50}$$

J and J' satisfy,

$$J = \frac{1}{J'}. \tag{2.51}$$

The interesting property of a tensor is that if you have a tensorial equation in one coordinate system, it will remain the same in any other coordinate system. This is the reason that the principle of general covariance implies that the laws of physics must have tensorial form.

For example, if in a coordinate system we have $A_\mu = B_\mu$, after that we multiply

$\partial x^\mu / \partial x'^\rho$ and sum over the indices we get

$$\frac{\partial x^\mu}{\partial x'^\rho} A_\mu = \frac{\partial x^\mu}{\partial x'^\rho} B_\mu \tag{2.52}$$

since A_μ and B_μ are covariant tensors with the same rank we conclude that $A'_\mu = B'_\mu$. This proves that tensorial equations are observer-independent and remain the same in all frames of reference, [4].

One crucial tensor in general relativity is the metric tensor $g_{\mu\nu}(x)$. The metric tensor is a covariant tensor of rank 2 which includes information about the geometry of the manifold and varies from point to point on the manifold. A line element on the manifold, which is defined as the interval between two points with coordinates x^μ and $x^\mu + dx^\mu$, can be written in terms of the metric tensor as

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu. \tag{2.53}$$

The metric tensor is a symmetric tensor, i.e.,

$$g_{\mu\nu}(x) = g_{\nu\mu}(x). \tag{2.54}$$

We can put the metric in its canonical form, which means that the metric would be a diagonal matrix and its components are $0, 1, -1$. Therefore, the canonical metric is

$$g_{\mu\nu} = diag(+1, +1, +1, ..., -1, -1, -1, ..., 0, 0, ..., 0). \tag{2.55}$$

If n is the dimension of the manifold and t is the number of -1's and s is the number of $+1$'s in the canonical form of the metric, then, rank of the metric is $s + t$ and the signature of the metric is $s - t$. In the case where $t = 0$, the positive definite metric is called Euclidean or Riemannian, also when $t = 1$ the metric is called Lorentzian or

pseudo-Riemannian. An indefinite metric occurs when we have some +1's and some −1's in the canonical form of the metric, [5].

A manifold that is equipped with a Riemannian metric is called Riemannian manifold and a manifold with a pseudo-Riemannian metric on it is called a pseudo-Riemannian manifold.

For example, a line element in the Minkowski spacetime is

$$ds^2 = \eta_{ab} dx^a dx^b = -dt^2 + dx^2 + dy^2 + dz^2, \tag{2.56}$$

where $\eta_{ab} = \text{diag}(-1, +1, +1, +1)$ is the Minkowski metric and we set c=1. The signature of the Minkowski metric is +2 and rank of the metric is 4.

Since the metric is a covariant tensor, its transformation is like

$$g'_{\mu\nu}(x') = \frac{\partial x^\mu}{\partial x^{\rho'}} \frac{\partial x^\nu}{\partial x^{\lambda'}} g_{\mu\nu}(x). \tag{2.57}$$

A transformation that leaves the metric invariant is called an isometry. Suppose that we have an infinitesimal coordinate transformation

$$x^\mu \to x'^\mu = x^\mu + \delta\zeta^\mu(x), \tag{2.58}$$

δ is small and ζ^μ is a vector field. If this transformation is isometry, the metric $g_{\mu\nu}$ must satisfy

$$g'_{\mu\nu}(x) = g_{\mu\nu}(x), \tag{2.59}$$

the condition to have isometry is

$$L_\zeta g_{\mu\nu} = \nabla_\nu \zeta_\mu + \nabla_\mu \zeta_\nu = 0, \tag{2.60}$$

where L is the Lie derivative (directional derivative) which is defined as

$$L_x Y_a = X^b \partial_b Y_a + Y_b \partial_a X^b, \tag{2.61}$$

and ∇ is known as covariant derivative which is defined as

$$\nabla_\mu X^\nu = \partial_\mu X^\nu + \Gamma^\nu_{\mu\lambda} X^\lambda. \tag{2.62}$$

Equation (2.60) is called Killing equation and any solution of it is called Killing vector field ζ^a. Killing vectors identify the symmetries of the spacetime. If the spacetime coordinates are $x^\mu = (x^0, x^1, x^2, x^3)$ and if the metric of the spacetime is independent of one of the coordinates, lets say x^0, then $X^0 = \delta^0_a$ would be a Killing vector.

The determinant of the metric is given by

$$g = \det(g_{\mu\nu}). \tag{2.63}$$

If $g \neq 0$, then the metric is said to be non-singular and therefore the inverse of the metric can be defined as,

$$g_{\mu\nu} g^{\nu\lambda} = \delta^\lambda_\mu. \tag{2.64}$$

Einstein's field equations

In order to study GR, we need to know more about the structure of the manifold that models the spacetime. We can define a connection which is a function of the metric and its first derivatives. The resultant connection, $\Gamma^\sigma_{\mu\nu}$, is called the metric connection or Christoffel connection. It can be derived as the function of the metric by using $\nabla_\mu g_{\nu\lambda} = 0$

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}). \tag{2.65}$$

We mentioned before that the metric is symmetric, from the definition of the metric connection one can easily see that the metric connection is also symmetric

$$\Gamma^{\sigma}_{\mu\nu} = \Gamma^{\sigma}_{\nu\mu}. \tag{2.66}$$

Now, we have enough information to introduce other important tensors in general relativity.

The curvature tensor or Riemann-Christoffel tensor that includes information about the curvature of the manifold is defined as a function of the metric connection and its first derivatives or equivalently as a function of the metric and its first and second derivatives, namely,

$$R^{\rho}_{\sigma\mu\nu} = \partial_{\mu}\Gamma^{\rho}_{\nu\sigma} - \partial_{\nu}\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}. \tag{2.67}$$

The curvature tensor is antisymmetric in its last pair of indices,

$$R^{\rho}_{\sigma\mu\nu} = -R^{\rho}_{\sigma\nu\mu}, \tag{2.68}$$

furthermore, it satisfies what are called Bianchi identities

$$\nabla_{\lambda}R_{\rho\sigma\mu\nu} + \nabla_{\rho}R_{\sigma\lambda\mu\nu} + \nabla_{\sigma}R_{\lambda\rho\mu\nu} \equiv 0. \tag{2.69}$$

Another important tensor which will be used to write down Einstein's tensor is the Ricci tensor. The Ricci tensor is defined as

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu} = g^{\lambda\rho}R_{\rho\mu\lambda\nu}. \tag{2.70}$$

The curvature scalar or Ricci scalar is given by the following contraction,

$$R = g^{\mu\nu} R_{\mu\nu}. \tag{2.71}$$

Now, we are ready to define Einstein's tensor $G_{\mu\nu}$,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \tag{2.72}$$

Einstein's tensor also satisfies the Bianchi identities,(as $T_{\mu\nu}$ is conserved)

$$\nabla_\mu G^\nu_\mu \equiv 0. \tag{2.73}$$

Einstein field equations are:

$$\boxed{G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}} \tag{2.74}$$

where, $T_{\mu\nu}$ is the stress-energy tensor and we set $G = c = 1$. The stress-energy tensor is the source of the gravitational field in general relativity, while mass is the source of gravity in Newtonian theory. Einstein's tensor depends on the geometry of the spacetime, therefore, Einstein field equations relate the distribution of matter in spacetime to the geometry of spacetime. One can describe field equations such that given a stress-energy tensor, one can find the related metric tensor. This is similar to Mach's ideas since it states that if we have a distribution of matter we can specify the proper metric and geometry of the manifold.

Also, if we know the metric of the spacetime which identifies the geometry, we can find the stress-energy tensor by solving the field equations.

Geodesics

In a flat spacetime, a geodesic is a straight line that connects two points of spacetime. A straight line is the shortest path between two points that parallel transports its tangent vector. However, in the presence of gravity the spacetime is no longer flat and the notion of geodesic would be different from flat spacetime. Geodesics represent the trajectory or world line of free particles moving in curved spacetime. There are three different types of geodesics in curved spacetime

I) *Timelike geodesics*: are classes of curves whose tangent vectors are timelike vectors. Given a curve $x^\mu(\lambda)$ (λ is the parameter along the curve), the vector tangent to it is $dx^\mu/d\lambda$. If the geodesic is defined as the path which parallel transports its tangent, one can write the geodesic equation as

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\nu}\frac{dx^\rho}{d\lambda}\frac{dx^\nu}{d\lambda} = 0, \tag{2.75}$$

$\Gamma^\mu_{\rho\nu}$ are Christoffel symbols which are an indication of being in curved spacetime. In the case of flat spacetime the Christoffel symbols vanish and the geodesic equation would be

$$\frac{d^2x^\mu}{d\lambda^2} = 0, \tag{2.76}$$

which is the equation for a straight line. Physical free particles travel on timelike geodesics.

II) *Null geodesics*: are curves which have null tangent vectors. Light rays (or photons) travel on null geodesics. Null geodesics describe the situation where the distance between two points is zero. The distance between two points P and P' is

$$s = \int_P^{P'} \frac{ds}{d\lambda}d\lambda = \int_P^{P'} (g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda})^{1/2}d\lambda, \tag{2.77}$$

in the case of the null geodesics, the geodesic equation is given by (2.77), where

$$g_{\mu\nu}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda}=0, \tag{2.78}$$

which is the consequence of the fact that the distance between the points is zero.

III) *Spacelike geodesics*: are geodesics with spacelike tangent vectors. Spacelike geodesics do not explain the path of physical particles, [5].

2.4 Summary

In this chapter we reviewed Newton's law of gravity and its dependence on inertial frames. We observed that GR was proposed as an alternative to Newton's explanation of gravity. Finally, by mentioning some basic concepts about manifolds and introducing essential tensors in GR we wrote down Einstein's field equations. The notations of GR tensors are similar to [5].

Chapter 3

Black Holes

3.1 Black hole solution

Einstein published his field equations of general relativity in 1915. He postulated that gravity is a result of the curvature of spacetime due to the presence of matter and energy in spacetime. Karl Schwarzschild was the first person who would find an exact vacuum solution for the field equations which is known as the 'Schwarzschild solution'. In GR, the Schwarzschild solution describes the gravitational field outside a spherical, uncharged, non-rotating mass. Einstein equation in vacuum is for the case where $T_{\mu\nu} = 0$, therefore according to Einstein field equations one gets

$$R_{\mu\nu} = 0. \tag{3.1}$$

Schwarzschild made some assumptions to simplify the field equations. He tried to solve the field equations in spherical coordinates (t, r, θ, ϕ).

In a spherical coordinate system, we can consider a general line element as

$$ds^2 = -D(t,r)dt^2 + 2F(t,r)dtdr + C(t,r)dr^2 + K(t,r)(d\theta^2 + \sin^2\theta d\phi^2), \tag{3.2}$$

where D, F, C, K are functions of t and r. As we mentioned before, GR has no dependence on coordinate systems. Therefore, we can have a coordinate transformation which will not change the spherical symmetry of the line element. The desired coordinate transformation is

$$F(t,r) = 0, \qquad K(t,r) = r^2. \tag{3.3}$$

The resulting metric is

$$ds^2 = -D(t,r)dt^2 + C(t,r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{3.4}$$

it is obvious that the above spherically symmetric metric is invariant under the transformation

$$\theta \to \theta' = -\theta, \qquad \phi \to \phi' = -\phi, \qquad r \to r' = -r. \tag{3.5}$$

The Schwarzschild solution can be obtained if one tries to solve Einstein field equations in the vacuum for the (3.4) metric, [4], [5].

The Schwarzschild line element is

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{3.6}$$

for the following metric

$$g_{\mu\nu} = \begin{pmatrix} -(1 - \frac{2GM}{r}) & 0 & 0 & 0 \\ 0 & (1 - \frac{2GM}{r})^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \tag{3.7}$$

M is mass of the object and G is the universal gravitational constant and we have set

$c = 1$. The Schwarzschild metric is time-independent and therefore is time-symmetric, so, $\frac{\partial}{\partial t}$ is a timelike Killing vector.

From the form of the Schwarzschild solution one can conclude:

- the metric is invariant under three-dimensional spatial rotations, so, the Schwarzschild solution is spherically symmetric;

- the Schwarzschild metric satisfies

$$\frac{\partial g_{\mu\nu}}{\partial t} = 0, \tag{3.8}$$

  hence, it is time-independent and therefore stationary.

- the metric is invariant under time reversal, $(t \to t' = -t)$, and there is no cross term of the form $dx^0 dx^\alpha$ where $\alpha = 1, 2, 3$; thus, the Schwarzschild solution is static. The timelike Killing vector is orthogonal to the hypersurface $t = const.$

- in the limit $r \to \infty$ the solution is the same as the flat spacetime metric which can be justified as due to the elimination of gravity at large distances from the source of gravity. Therefore, it is asymptotically flat.

If one tries to find the solution of the vacuum field equations in any other coordinate system, the solution would be the same as the Schwarzschild solution. There is a theorem in GR called **Birkhoff's theorem** which says:

- any spherically symmetric solution of the vacuum field equations must be static and asymptotically flat. It follows that the Schwarzschild solution is the unique solution of the vacuum field equations.

Singularities

In general, a singularity is a point at which an equation, surface, etc., blows up or becomes degenerate. We can write the metric components of the Schwarzschild metric as

$$g_{00} = -\left(1 - \frac{2GM}{r}\right),\ g_{11} = \left(1 - \frac{2GM}{r}\right)^{-1},\ g_{22} = r^2,\ g_{33} = r^2 \sin^2\theta. \qquad (3.9)$$

At $r = 2GM$ which is called the Schwarzschild radius, the g_{11} component of the metric is infinity and g_{00} component vanishes. However, at $r = 0$, g_{00} is infinity and g_{11} is zero. We can define two different regions in Schwarzschild spacetime (Fig 3.1),

I. $0 < r < 2GM$

II. $2GM < r < \infty$.

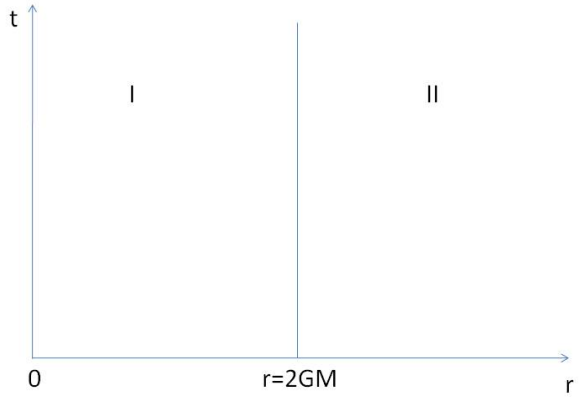

Figure 3.1: Two different regions of the Schwarzschild spacetime.

These two regions have different characteristics. If we denote the coordinate system by x^μ, the surface $x^\mu = const$ is timelike if $g^{\mu\mu} > 0$ and is spacelike if $g^{\mu\mu} < 0$. Hence in region I, t is spacelike and r is timelike, while in region II, t is timelike and

r is spacelike. The Killing vector $\frac{\partial}{\partial t}$ becomes a null vector at $r = 2GM$, therefore, one concludes that $r = 2GM$ is a Killing horizon. $r = 2GM$ is a null horizon which has a null Killing vector on it.

Some singularities are coordinate singularities that reveal the deficiencies of the coordinate system to describe the manifold. One can remove coordinate singularities by choosing an appropriate coordinate transformation. Other singularities are geometric singularities which are not related to the choice of coordinate systems. The geometric singularities are essential singularities that are not removable by coordinate transformations. We should look for a way to distinguish between coordinate singularities and singularities which are related to the geometry of the manifold.

When the curvature of the manifold is infinite it can not be removed by just a coordinate transformation. Curvature scalar of the manifold is coordinate-independent, so, if it is infinite in one coordinate system it will be infinite in any other coordinate system. The best quantity that includes information about the curvature of the manifold is the Ricci scalar. If the Ricci scalar is infinite for a value of r then it is infinite in any coordinate system and the geometry of the manifold is degenerate. The singularity at $r = 2GM$ which is the Schwarzschild radius is a coordinate singularity, because

$$R_{abcd}R^{abcd} = \frac{12(GM)^2}{r^6}. \tag{3.10}$$

At $r = 2GM$ (3.10) is finite, therefore by a proper choice of coordinate transformation one can remove the singularity at $r = 2GM$. However, at $r = 0$ (3.10) is infinite and therefore this singularity is irremovable (Fig 3.2). $r = 0$ singularity has different names such as intrinsic, curvature or real singularity.

As we mentioned before, the singularity at $r = 2GM$ is a coordinate singularity and is removable, so, if we apply an appropriate coordinate transformation the singularity at $r = 2GM$ will vanish. Now, our task is to find the coordinate transformation

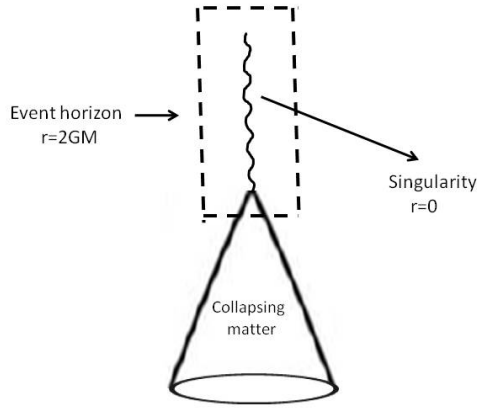

Figure 3.2: Spacetime diagram of collapsing matter with two singularities at $r = 0$ and $r = 2GM$.

to remove this singularity. The trajectory of a light ray is a null curve

$$ds^2 = 0$$

while for a radial null curve

$$ds^2 = 0 \quad \text{and} \quad d\theta = d\phi = 0.$$

For the Schwarzschild metric, radial null curves are

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + (1 - \frac{2GM}{r})^{-1}dr^2 = 0, \tag{3.11}$$

so,

$$\frac{dt}{dr} = \pm(1 - \frac{2GM}{r})^{-1}, \tag{3.12}$$

this is the slope of the light cones on a spacetime diagram of the $t - r$ plane. A light

cone is made of all trajectories of light signals that are emanating from an arbitrary point P in spacetime and propagate in all directions. Emission of light from point P is an occurrence in spacetime therefore it indicates an event (Fig. 3.3).

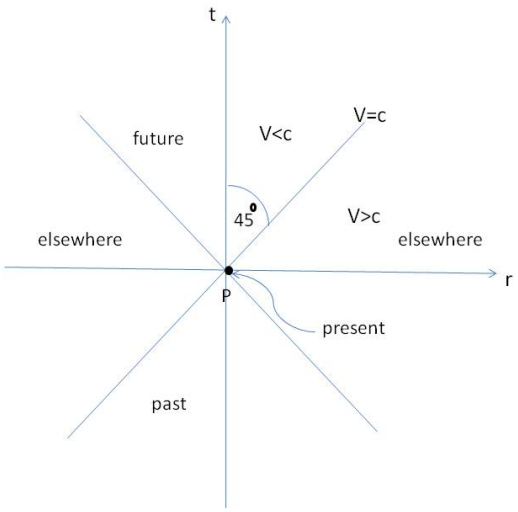

Figure 3.3: Light cone for an event occurring at P. The cone consists of four regions: future, past and two elsewhere regions. All the events in the future light cone occur after P while the events in the past light cone occur before P. There is no causal communication between P and points in elsewhere regions.

At $r \to \infty$ the trajectory of light rays make an angle equal to 45 degree in the $t - r$ plane. However, as we get close to $r = 2GM$, the light cones become narrower. Light rays asymptote to $r = 2GM$ but they can not get to it (Fig. 3.4).

If we select the positive sign for the slope, (3.12) implies

$$r > 2GM \Rightarrow \frac{dr}{dt} > 0, \tag{3.13}$$

this means that r increases as t increases. Taking the integral gives

$$t = r + 2GM \ln \left| \frac{r}{2GM} - 1 \right| + constant. \tag{3.14}$$

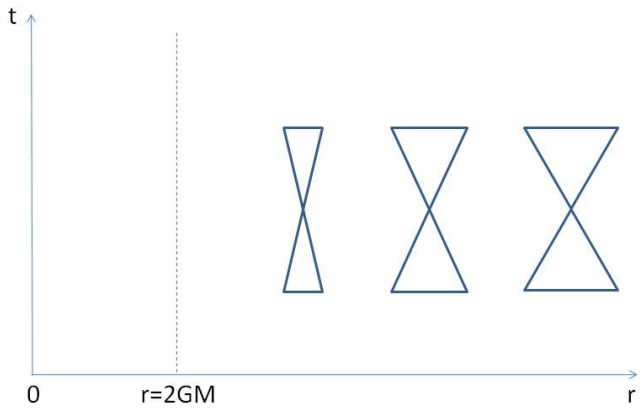

Figure 3.4: Light cones get narrower as they get closer to r=2GM.

These curves denote the outgoing radial null geodesics. We introduce a new coordinate called **'Regge-Wheeler tortoise coordinate'**

$$r^* = r + 2GM \ln(\frac{r}{2GM} - 1), \tag{3.15}$$

it implies

$$\frac{dr^*}{dr} = (1 - \frac{2GM}{r})^{-1}. \tag{3.16}$$

Therefore (3.14) can be written as,

$$t = +r^* + constant. \tag{3.17}$$

Similarly, if we select the negative sign of the slope, we get

$$r > 2GM \Rightarrow \frac{dr}{dt} < 0, \tag{3.18}$$

after taking the integral, in tortoise coordinate we get

$$t = -r^* + constant, \tag{3.19}$$

which are congruence of ingoing radial null geodesics. Ingoing radial null geodesics become outgoing and outgoing ones become ingoing by time reversal transformation, $t \to -t$.

Now, we define a new null coordinate v as

$$v = t + r^*, \tag{3.20}$$

the ingoing radial null geodesics can be obtained by $v = constant$. From (3.15) we know that r^* is a function of r, so, if we write the Schwarzschild metric as a function of v and r we get

$$ds^2 = -(1 - \frac{2GM}{r})dv^2 + (dvdr + drdv) + r^2 d\Omega^2, \tag{3.21}$$

with

$$d\Omega^2 = r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{3.22}$$

On the other hand, we can define another null coordinate

$$u = t - r^*, \tag{3.23}$$

the outgoing radial null geodesics are given by $u = constant$ (Fig. 3.5).

The resultant Schwarzschild line element in terms of u and r is

$$ds^2 = -(1 - \frac{2GM}{r})du^2 - (dudr + drdu) + r^2 d\Omega^2. \tag{3.24}$$

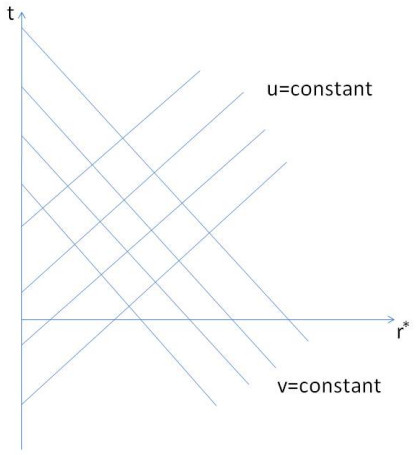

Figure 3.5: Outgoing and ingoing null geodesics.

(3.21) and (3.24) are known as **Eddington-Finkelstein** forms of the Schwarzschild line element. These forms of the Schwarzschild metric are regular at $r = 2GM$, so, we could remove the singularity by a coordinate transformation.

As a function of both v and u the metric has the form

$$ds^2 = (1 - \frac{2GM}{r})(dvdu + dudv) + r^2 d\Omega^2, \tag{3.25}$$

which is again regular at $r = 2GM$. The Schwarzschild solution is regular for $2GM < r < \infty$, however, Eddington-Finkelstein coordinates could extend the solution to $0 < r < \infty$ (Fig. 3.6). The singularity at $r = 0$ which is an intrinsic singularity still exists. One can not remove it by coordinate transformations.

3.1.1 The Kruskal solution

The Eddington-Finkelstein solution in terms of both ingoing and outgoing null parameters is

$$ds^2 = (1 - \frac{2GM}{r})(dvdu + dudv) + r^2 d\Omega^2. \tag{3.26}$$

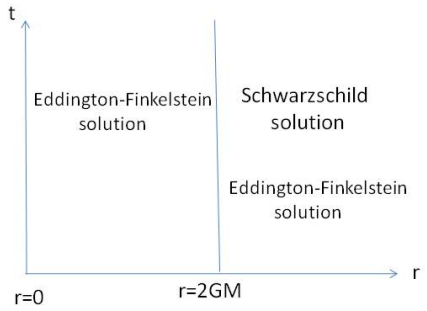

Figure 3.6: Eddington-Finkelstein and Schwarzschild coordinates.

The Kruskal coordinates U and V are defined as a function of u and v such that

$$U = -e^{-u/4GM} = -e^{-(t-r^*)/4GM}, \tag{3.27}$$

$$V = e^{v/4GM} = e^{(t+r^*)/4GM}. \tag{3.28}$$

U and V satisfy

$$UV = (\frac{r}{2GM} - 1)e^{-r/2GM}. \tag{3.29}$$

The metric (3.26) in terms of Kruskal coordinates, (U, V, θ, ϕ), is

$$ds^2 = -\frac{32(GM)^3 e^{-r/2GM}}{r} dUdV + r^2 d\Omega^2.$$

At $r = 2GM$, we have

$$u \to +\infty \Rightarrow U = 0, \tag{3.30}$$

and

$$v \to -\infty \Rightarrow V = 0, \tag{3.31}$$

therefore, the metric is regular at $r = 2GM$. $V = 0$ is called the past horizon and is a surface that just emits radiation. Also, the surface $U = 0$ is known as the future horizon which is a surface that can just absorb particles. Future null infinity surface $\mathcal{I}^+$, is defined as $(r \to \infty, t = +\infty)$ and past null infinity surface $\mathcal{I}^-$ is $(r \to \infty, t = -\infty)$. If one takes

$$\begin{aligned} T &= \frac{(U+V)}{2} \\ X &= \frac{(V-U)}{2}, \end{aligned} \tag{3.32}$$

the Kruskal line element will be

$$ds^2 = \frac{32(GM)^3 e^{-r/2GM}}{r}(-dT^2 + dX^2) + r^2(d\theta^2 + \sin^2\theta d\phi). \tag{3.33}$$

The above form of the Schwarzschild metric was found by Kruskal in 1960 (Fig 3.7).

3.1.2 Penrose diagram

The aim of introducing the Penrose diagram is to study the causal structure of infinite points. The points at infinity can come to finite positions by choosing a proper transformation. Penrose found a way to study the asymptotic infinities of Minkowski spacetime. The Minkowski line element in polar coordinates is given by

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2. \tag{3.34}$$

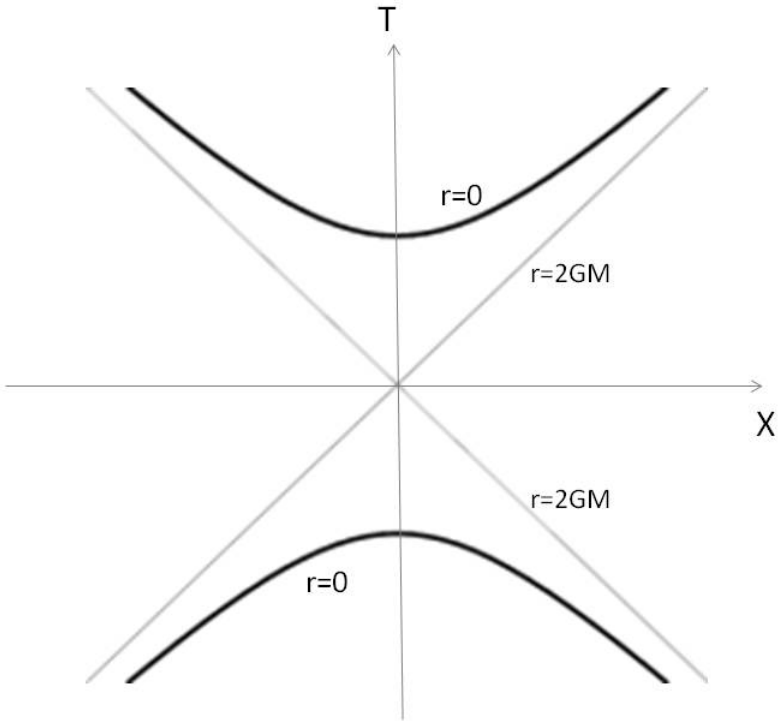

Figure 3.7: The Kruskal diagram.

In terms of the null coordinates u and v the Minkowski line element is

$$ds^2 = dvdu - \frac{1}{4}(v-u)^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{3.35}$$

where u and v are defined as

$$\begin{aligned} v &= t + r \\ u &= t - r, \end{aligned} \tag{3.36}$$

the coordinate range is

$$\begin{aligned} -\infty < v < +\infty \\ -\infty < u < +\infty \end{aligned} \tag{3.37}$$

From (3.36) it is obvious that

$$r = \frac{1}{2}(v - u), \tag{3.38}$$

therefore,

$$\text{if} \quad r \geq 0 \Rightarrow v - u \geq 0 \Rightarrow v \geq u. \tag{3.39}$$

The appropriate transformation that can bring the infinities of u and v coordinates to finite positions is

$$\begin{aligned} p &= \tan^{-1} v \\ q &= \tan^{-1} u, \end{aligned} \tag{3.40}$$

thus from (3.37) we have

$$\begin{aligned} \frac{-1}{2}\pi < p < \frac{1}{2}\pi \\ \frac{1}{2}\pi < q < \frac{1}{2}\pi. \end{aligned} \tag{3.41}$$

The Minkowski line element in terms of p and q is

$$ds^2 = \frac{1}{4}\sec^2 p \sec^2 q \left[4dpdq - \sin^2(p-q)(d\theta^2 + \sin^2\theta d\phi^2)\right]. \tag{3.42}$$

The main idea to study the infinities by using the Penrose diagram is based on conformal transformation of the spacetime metric. If we denote the physical metric

of the Minkowski spacetime by $g_{\mu\nu}$, the conformal transformation of the mentioned metric is

$$\bar{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu}, \tag{3.43}$$

$\Omega(x)$ is a continuous, non-zero function of spacetime which is called conformal factor and $\bar{g}_{\mu\nu}$ is the unphysical metric conformally related to the physical metric of spacetime.

The unphysical line element for (3.42) is

$$d\bar{s}^2 = 4dpdq - \sin^2(p-q)(d\theta^2 + \sin^2\theta d\phi^2), \tag{3.44}$$

with $\Omega^2(x) = \frac{1}{4}\sec^2 p \sec^2 q$. One can compactify the whole Minkowski spacetime by defining new coordinates t' and r' such that

$$t' = p + q, \tag{3.45}$$

and

$$r' = p - q. \tag{3.46}$$

In terms of t' and r', the unphysical line element (3.44) is

$$d\bar{s}^2 = dt'^2 - dr'^2 - \sin^2 r'(d\theta^2 + \sin^2\theta d\phi^2), \tag{3.47}$$

where, according to (3.41), the coordinate ranges are

$$\begin{aligned} -\pi < t' + r' < \pi \\ \pi < t' - r' < \pi, \end{aligned} \tag{3.48}$$

where

$$r' \geq 0. \tag{3.49}$$

Hence, the physical metric that describe the whole Minkowski spacetime is

$$ds^2 = \frac{1}{4}\sec^2\left[\frac{1}{2}(t'+r')\right]\sec^2\left[\frac{1}{2}(t'-r')\right]\left(dt'^2 - dr'^2 - \sin^2 r'(d\theta^2 + \sin^2\theta d\phi^2)\right). \tag{3.50}$$

Therefore, the whole Minkowski spacetime is placed in a finite region which is given by (3.48) and is called compactified Minkowski spacetime. The Penrose diagram is a two-dimensional diagram of the compactified Minkowski spacetime (Fig. 3.8). Time is the vertical axis and space is the horizontal axis in the two-dimensional Penrose diagram. Infinities of the Minkowski spacetime are the boundaries of the compactified Minkowski spacetime. One can define the future and past null infinities in terms of the coordinates of the compactified Minkowski spacetime as the following surfaces

$$\mathcal{I}^+ = \text{future null infinity} \quad \left(p = \frac{1}{2}\pi\right), \tag{3.51}$$

$$\mathcal{I}^- = \text{past null infinity} \quad \left(q = -\frac{1}{2}\pi\right). \tag{3.52}$$

Also, the future timelike infinity i^+ and past timelike infinity i^- are the following points

$$i^+ = \text{future timelike infinity} \quad \left(p = \frac{1}{2}\pi, q = \frac{1}{2}\pi\right), \tag{3.53}$$

$$i^- = \text{past timelike infinity} \quad \left(p = -\frac{1}{2}\pi, q = -\frac{1}{2}\pi\right). \tag{3.54}$$

$$i^0 = \text{Spacelike infinity} \quad \left(p = \frac{1}{2}\pi, q = -\frac{1}{2}\pi\right). \tag{3.55}$$

Kruskal solution can also be conformally compactified by defining new coordinates

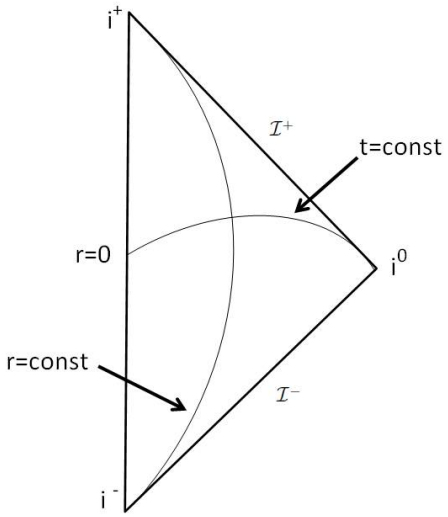

Figure 3.8: Penrose diagram of Minkowski spacetime with one set of asymptotic infinities.

in terms of the Kruskal coordinates U and V as

$$V' = \tan^{-1}\left[V/(2GM)^{1/2}\right]$$
$$U' = \tan^{-1}\left[U/(2GM)^{1/2}\right]. \tag{3.56}$$

The coordinate ranges are

$$-\frac{\pi}{2} < V' < \frac{\pi}{2},$$
$$-\frac{\pi}{2} < U' < \frac{\pi}{2},$$
$$-\pi < U' + V' < \pi.$$

The Penrose diagram consists of four regions. In regions I and I' the ingoing and outgoing radial null geodesics end up at future null infinity $\mathcal{I}^+$. The future singularity at $r = 0$ originates from the future timelike infinity of one asymptotic region and ends

at the future time like infinity of the other one (Fig. 3.9). For past singularity $r = 0$ the end points are the past timelike infinities of the two asymptotic regions.

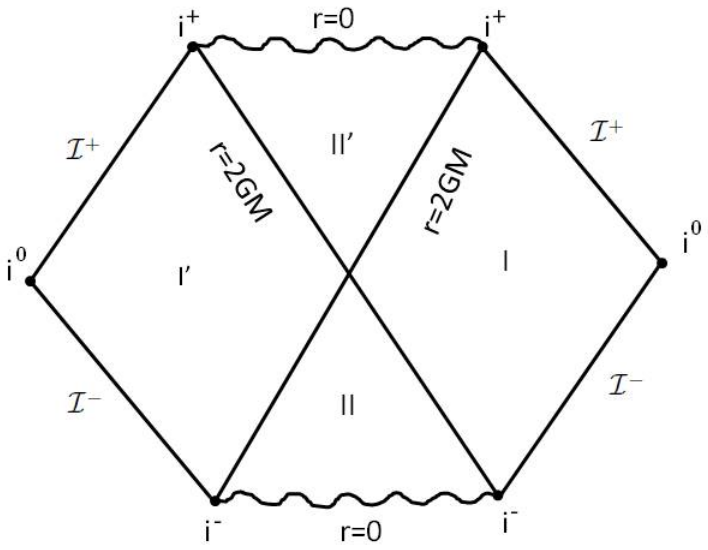

Figure 3.9: Penrose diagram of the Kruskal extension of Schwarzschild solution.

An interesting aspect of this diagram is that the structure of infinity is the same as for the Minkowski spacetime. This proves that the Schwarzschild solution is asymptotically flat.

3.2 Black hole thermodynamics

Black hole thermodynamics emerged when physicists tried to assign the laws of thermodynamics to the black holes. Before we start talking about the thermodynamics of black holes we should introduce what is called 'surface gravity' of a black hole, κ.

Surface gravity is defined for black holes which possess a Killing horizon. A Killing horizon is a null hypersurface that has a null Killing vector on it. Based on Newtonian mechanics, the surface gravity of an astronomical object indicates the gravitational acceleration on its surface. More clearly, for a test particle with negligible mass close to the surface of an astronomical object , surface gravity is its acceleration due to the gravitational field of the astronomical object. For a black hole which has a Killing horizon the surface gravity is defined as the acceleration which is required to keep a test particle at the horizon. If X^a is a Killing vector, we have

$$X^a \nabla_a X^b = \kappa X^b, \tag{3.57}$$

[8], surface gravity at the horizon is obtained by evaluating (3.57) at the horizon. To find the surface gravity for the Schwarzschild black hole, we write (3.57) in another form

$$\kappa^2 = -\frac{1}{2}(\nabla^a X^b)(\nabla_a X_b). \tag{3.58}$$

At the horizon of a Schwarzschild black hole, the Killing vector is $\frac{\partial}{\partial t}$. Since we are evaluating the surface gravity at the horizon, the Killing vector would be $X^a = (1, 0, 0, 0)$. Therefore, $X_a = g_{ab}X^b = [-(1 - \frac{2GM}{r}), 0, 0, 0]$. The Schwarzschild radius is a Killing horizon since X^a is literally a null Killing vector at $r = 2GM$

$$X^a X_a = 1.[-(1 - \frac{2GM}{r})]|_{r=2GM} = 0. \tag{3.59}$$

The only non-zero terms for $\nabla^a X^b$ are $\nabla^r X^t$ and $\nabla^t X^r$. So,

$$\kappa^2 = -\frac{1}{2}(\nabla^t X^r \nabla_t X_r + \nabla^r X^t \nabla_r X_t) = \frac{1}{16G^2M^2}, \tag{3.60}$$

hence, for Schwarzschild black hole

$$\kappa = \frac{1}{4GM}. \tag{3.61}$$

Surface gravity is constant for the Schwarzschild solution which describes a stationary black hole. Therefore, one can imagine that the surface gravity for a stationary black hole is analogous to the temperature of a body in thermal equilibrium.

The zeroth law of thermodynamics says that

- for a system in thermal equilibrium, the temperature is constant throughout the system.

The respective zeroth law for the black hole horizon is

- The surface gravity κ is constant at the horizon of a stationary black hole.

The first law of thermodynamics is a statement of conservation of energy which says

- If the amount of heat supplied to an isolated system is dQ, the change in internal energy of the system is dU and the work done by the system is dW, then they satisfy

$$dQ = dU + dW.$$

The first law of black hole thermodynamics is also about conservation. For a rotating, charged black hole the first law states

- Change of mass of a black hole is related to the change of its area, angular momentum and charge as follows

$$dM = \frac{\kappa}{8\pi} dA + \omega dJ + \phi dq, \tag{3.62}$$

ω is the angular velocity, J is the angular momentum and q is the electric charge of the black hole and ϕ is the electric potential.

In classical thermodynamics, entropy S is a fundamental concept which is defined in terms of the temperature T and the heat supplied to the system dQ such that

$$dQ = TdS. \tag{3.63}$$

Entropy is a measure of the disorder of an isolated system. The second law of thermodynamics states

- the change in entropy is greater than or equal to zero, therefore, entropy of an isolated system always increases

$$\frac{dS}{dt} \geq 0. \tag{3.64}$$

The black hole event horizon has an interesting property that its surface area always increases when the black hole absorbs matter or radiation. The surface area of the horizon can be a measure of the entropy of the black hole. Stephen Hawking and Jacob Bekenstein defined the entropy of a black hole as

$$S_{BH} = \frac{kA}{4l_P^2}, \tag{3.65}$$

A is the surface area of the horizon, $k = 1.380648 \times 10^{-23}(JK^{-1})$ is the Boltzmann constant and $l_P = (\frac{G\hbar}{c^3})^{1/2} = 1.616 \times 10^{-35}m$ is the Planck length. The second law of black hole thermodynamics expresses that

- The surface area of the horizon always increases, i.e,

$$\frac{dA}{dt} \geq 0. \tag{3.66}$$

The third law of black hole thermodynamics says

- It is not possible to have a black hole with zero surface gravity. It means that $\kappa = 0$ is not achievable.

This law is not the exact analogue of the third law of the classical thermodynamics. The third law of thermodynamics says that

- the entropy of a system goes to zero when the temperature goes to zero.

Therefore, the exact analogue of the third law implies that the surface area of the black hole goes to zero when the surface gravity is zero which does not make sense.

3.3 Hawking radiation

There is an important theorem about black holes known as the 'no hair theorem' [4] which states that

- A black hole solution of Einstein field equations can be specified by three properties which are: mass, angular momentum and electric charge of the black hole.

It implies that a group of black holes with the same mass, angular momentum and electric charge are identical and one can not distinguish between them. If a distribution of matter is undergoing a collapse to form a black hole all information about the distribution will be lost behind the event horizon. After the formation of the black hole the only observable characteristics will be mass, angular momentum and charge. The rest of the properties are hidden behind the event horizon. Therefore, a black hole with specific mass, charge and angular momentum could have been formed from the collapse of any one of the different distributions of matter. In statistical thermodynamics the logarithm of the number of microstates that correspond to a single macrostate is proportional to the entropy of the system. If N denotes the number of microstates, entropy is defined as

$$S = k \ln N, \tag{3.67}$$

where k is the Boltzmann's constant. Since the number of microstates is finite, entropy is finite and therefore the black hole must have a finite temperature. From thermodynamics we know that any object with temperature higher than absolute zero must radiate, hence, black holes should radiate. Based on the classical theory of black holes, nothing can escape the strong gravitational field of a black hole. However, when we take into account quantum effects, black holes can radiate. So, black holes are no longer black!

In 1974, Stephen Hawking proved that black holes radiate what is called 'Hawking radiation', [7]. The temperature of the Hawking radiation in natural units where G, c and $\hbar$ are equal to 1, is

$$T_H = \frac{1}{8\pi M} \approx 10^{-8}(\frac{M}{M_\odot})^{-1}K, \tag{3.68}$$

where

$$M_\odot \approx 10^{33}g, \tag{3.69}$$

is the solar mass. In terms of the surface gravity of the black hole κ, the temperature of the Hawking radiation is

$$T_H = \frac{\kappa}{2\pi}. \tag{3.70}$$

From (3.68) one can deduce that the temperature of the Hawking radiation emitted from small mass black holes is higher than those emitted from large black holes. The temperature of the Hawking radiation that is emitted from a black hole with the solar mass is much less than the temperature of the cosmic microwave background (CMB) which is 3°K. The rate of radiation absorption for these black holes is much higher than the radiation emission rate, hence, their masses will increase gradually. Small mass black holes radiate more than whatever they absorb, so, they will become smaller and consequently hotter. When the temperature is above $10^{12}K$, the mass

of the black hole is approximately about $10^{14}g$, at this point, black hole will emit a great amount of different species of particles and radiate away its whole mass in about 10^{-23} seconds, [7]. In the next chapter we will study how one can justify Hawking radiation as a result of vacuum polarization in the vicinity of the black hole horizon.

There is a famous paradox associated with the loss of information in black hole physics. As we mentioned, the no hair theorem states that a specific black hole could have been formed from the collapse of many possible distributions of matter. If we consider a black hole as a physical final state and some distributions of matter (before they collapse) as initial state, according to the no hair theorem one can conclude that many physical states can evolve into the same final physical states. Quantum mechanics describes physical states by wave functions which are functions of time. The evolution of quantum states is explained by unitary operators. If we denote the initial state of a physical system by $|\psi(0)>$, it evolves into the final state $|\psi(t)>$ as

$$|\psi(t)>= e^{-iHt}|\psi(0)>, \tag{3.71}$$

where we set $\hbar = 1$ and H is the Hamiltonian (energy operator) of the system. The operator that denotes the evolution of the system is e^{-iHt} which is a unitary operator

$$e^{-iHt}e^{iHt} = 1. \tag{3.72}$$

Equation (3.71) shows that by having information about the initial state of a system one can find the final state by the action of the unitary operator e^{-iHt} on the initial state. Inversely, if we know about the final state of a system the unitary operator gives us information about the initial state of the system. These two statements imply the uniqueness of the initial and final states of a physical system. Therefore, it is not in agreement with the no hair theorem since information must be preserved

if a unitary operator describes the evolution of a system and if a physical system is a result of unitary evolution of just one state.

The density matrix in quantum mechanics is defined as

$$\rho = |\psi\rangle\langle\psi|, \tag{3.73}$$

a state is called a pure state if it satisfies

$$\rho^2 = \rho. \tag{3.74}$$

If $|\psi(0)\rangle \to e^{-iHt}|\psi(0)\rangle$ one can write

$$\rho = e^{-iHt}|\psi(0)\rangle\langle\psi(0)|e^{+iHt} = |\psi(t)\rangle\langle\psi(t)|. \tag{3.75}$$

One can conclude from (3.75) and (3.76) that a pure state remains a pure state under unitary transformations. Therefore, if a matter field which is described by a pure state falls into the black hole it must remain pure. Thus, one should be able to get information about the matter field since its evolution is described by a unitary operator. Hence, if information loss is valid then non-unitary operators must explain the evolution of states. Hawking radiation is a result of a semiclassical quantum gravity approach which considers gravity as a classical field and matter fields are quantum fields, however, based on the above discussion one has to invent a new theory to justify Hawking radiation.

3.3.1 Summary

In this chapter we obtained a solution of Einstein field equations in the vacuum called the Schwarzschild solution. The Schwarzschild solution has two singularities. One of the singularities is a coordinate singularity and is removable by choosing a proper coordinate transformation. We mentioned the procedure to remove the coordinate singularity of the Schwarzschild solution. We introduced the Kruskal coordinate and then a method to compactify the Minkowski spacetime and the Kruskal solution known as the Penrose diagram. We finished the chapter by reviewing black hole thermodynamics and Hawking radiation. Hawking radiation can be explained by studying the interaction of a scalar field with the gravitational field of the black hole close to the horizon. Since the scalar field is a quantum field, one has to know quantum field theory to understand the procedure of Hawking radiation emission. In the following chapter we will describe the physics of quantum fields in Minkowski and in curved spacetime. By understanding the physics of quantum fields in curved spacetime we will have enough information to realize the process of Hawking radiation emission. We mainly followed references [4], [5], [6], [7], [8],[9], [10] , [11], [12], [13], [14], [15].

Chapter 4

Quantum Fields in Minkowski and in Curved Spacetime

4.1 Quantum field theory in Minkowski spacetime

As we mentioned before, the special theory of relativity assumes that all inertial observers are equivalent. Since inertial observers are related to each other by Lorentz transformations, it means that all laws of physics are invariant under Lorentz transformations or in other words they have Lorentz symmetry. However, quantum field theory requires the invariance of physical laws under spacetime translations as well as the Lorentz transformations. The transformation that is the result of both spacetime translations and Lorentz transformations is called the Poincaré transformation. Poincaré transformations are defined as

$$x^\mu = \Lambda^\mu_\nu x^\nu + a^\mu, \tag{4.1}$$

and

$$a^\mu \in \mathbb{R}. \tag{4.2}$$

$\mathbb{R}$ is a symbol for real numbers including integers and fractions. a^μ are four parameters(for four-dimensional spacetime) that identify the spacetime translations.

The basic constituents of QFT are quantum fields which are functions of space and time and obey appropriate commutation relations. As we said, quantum field theory is invariant under Poincaré transformations. Since quantum fields are defined in different points of spacetime, the interactions of these fields must be local. Hence, the equations of motion and commutation relations that explain the evolution of a given quantum field at a given point in spacetime and depend only on the behavior of the field and its derivatives at that point.

In this section we study the action principle and the way to find equations of motion for classical fields. Then we review scalar field theory which includes real scalar fields and complex scalar fields. Then we continue the study of Minkowski spacetime field theory by talking about fermionic fields. Finally, we will describe the concept of vacuum in flat spacetime.

4.1.1 The action principle

One important subject in classical mechanics is the Lagrangian formalism. In order to study a classical system with N degrees of freedom, one can use a set of generalized coordinates $x_i(t)$, with $i = 1, 2, ..., N$. The Lagrangian L is a function of x_i's and their first derivatives with respect to the time $\dot{x}_i$'s, and of course a function of time, i.e.,

$$L = L(x_i, \dot{x}_i, t). \tag{4.3}$$

The Lagrangian is defined as kinetic energy minus the potential energy,

$$L(x_i, \dot{x}_i, t) = \sum_i (m_i/2)\dot{x}_i^2 - V(x_i), \tag{4.4}$$

where the potential energy $V(x)$ is just a function of coordinates. The action S is given by

$$S = \int L(x_i, \dot{x}_i, t)dt. \tag{4.5}$$

There is an important principle related to the action called the action principle which states:

- if one considers two points A and B, a classical particle at A can take several different paths to go to B. However, the correct physical path is the one that extremizes the action.

If the particle is at A at initial time t_A and at B at final time t_B, the extremum of the action can be found by setting the variation of the action equal to zero

$$\delta S = \delta \int_{t_A}^{t_B} L(x, \dot{x}, t)dt = 0, \tag{4.6}$$

x_i's are denoted collectively by x and the boundary conditions are fixed while we are performing the variation. Solutions of (4.6) are the Euler-Lagrange equations

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0. \tag{4.7}$$

Since the action principle talks about paths and not about coordinate systems, one can conclude that the Euler-Lagrange equations hold in any coordinate system.

In quantum field theory we deal with fields which are functions of spacetime. To describe the classical dynamics of a generic field $\phi(x)$ where x refers to the spacetime, one can generalize the Lagrangian method. The Lagrangian for a given field is a function of spacetime coordinates.

In the case of a classical field, the Lagrangian is a function of the field and its first derivative,

$$L = \int d^3x \mathcal{L}(\phi, \partial_\mu \phi), \tag{4.8}$$

where $\mathcal{L}$ is called the Lagrangian density and $\partial_\mu = \partial/\partial x^\mu$. The action is defined as

$$S = \int L dt = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi). \tag{4.9}$$

The same as for point particles, the action is stationary for the fields, i.e.,

$$\delta S = \delta \int d^4x \mathcal{L}(\phi, \partial_\mu \phi) = 0. \tag{4.10}$$

Therefore, the equations of motion which are found by equating zero the variation of the action are

$$\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} = 0, \tag{4.11}$$

these equations are Euler-Lagrange equations for a classical field.

4.1.2 Real scalar fields

In QFT fields are categorized based on their spins. The spin of an object comes from how it transforms under spatial rotations. Scalar fields are spin zero fields; if $\phi(x)$ is a real scalar field, it assigns a real numerical value to each point in spacetime.

Since, QFT must be invariant under Lorentz transformations, scalar fields should be the same for all inertial observers. If we have a scalar field $\phi(x)$ in coordinates x^μ, and $\tilde{\phi}(\tilde{x})$ in coordinates $\tilde{x}^\mu$ which is related to x^μ by

$$\tilde{x}^\mu = \Lambda^\mu_\nu x^\nu, \tag{4.12}$$

we must have

$$\phi(x) = \tilde{\phi}(\tilde{x}). \tag{4.13}$$

It means that the numerical value of a scalar field at a point is Poincaré invariant.

Consider a set of scalar fields $\phi_i(x)$ and a Lagrangian density $\mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x))$, if we make an infinitesimal change to ϕ field

$$\phi_i(x) \to \phi_i(x) + \delta\phi_i(x)$$

the Lagrangian density will also change

$$\mathcal{L}(x) \to \mathcal{L}(x) + \delta\mathcal{L}(x)$$

where,

$$\delta\mathcal{L}(x) = \frac{\partial \mathcal{L}}{\partial \phi_i(x)} \delta\phi_i(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i(x))} \partial_\mu \delta\phi_i(x). \tag{4.14}$$

If the equations of motion are satisfied we get

$$\frac{\delta S}{\delta \phi_i(x)} = 0, \tag{4.15}$$

therefore (4.14) would be

$$0 = \frac{\partial \mathcal{L}(x)}{\partial \phi_i(x)} - \partial_\mu \frac{\partial \mathcal{L}(x)}{\partial(\partial_\mu \phi_i(x))} \tag{4.16}$$

now, the Noether current is defined as

$$j^\mu(x) = \frac{\partial \mathcal{L}(x)}{\partial(\partial_\mu \phi_i(x))} \delta\phi_i(x) \tag{4.17}$$

from (4.16) it is manifest that the Noether current is conserved for a set of infinitesimal

transformations that leave the Lagrangian invariant, so,

$$\delta\mathcal{L}(x) = 0 \longrightarrow \partial_\mu j^\mu(x) = 0. \tag{4.18}$$

The action that explain dynamics of a field must contain $\partial_\mu\phi$ which shows the change in $\phi(x)$. In order to have a Lorentz invariant action the index μ must be saturated and, for a scalar field, the only possibility is to contract it with another factor $\partial^\mu\phi$, [2]. The action for a free real scalar field in a 4-dimensional Minkowski spacetime is,

$$S = \int d^4x \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2\right] \tag{4.19}$$

with

$$\partial^\mu = \eta^{\mu\nu}\partial_\nu, \tag{4.20}$$

and m is mass of the field quanta.

The Lagrangian density for a real scalar field is given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2. \tag{4.21}$$

Equations of motion are obtained by taking the variation of the action with respect to the ϕ field and setting it equal to zero; the resultant equation is called Klein-Gordon equation

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\phi + m^2\phi = (\Box + m^2)\phi = 0, \tag{4.22}$$

where $\Box = \partial_\mu\partial^\mu$.

One set of normal modes that are solutions of the Klein-Gordon equation are plane waves $u_i(x)$, where x denotes the spacetime coordinates.

$$u_i(x) \propto e^{i(\mathbf{k}\cdot x - \omega t)}, \tag{4.23}$$

$\mathbf{k}$ is an arbitrary real vector [17] and in the Cartesian coordinates its magnitude is

$$k = |\mathbf{k}| = (k_x^2 + k_y^2 + k_z^2)^{1/2}. \tag{4.24}$$

ω is the frequency of the wave which is given by

$$\omega = (k^2 + m^2)^{1/2} > 0, \tag{4.25}$$

The solutions (4.23) are positive frequency modes with respect to t since they satisfy

$$\frac{\partial u_i(x)}{\partial t} = -i\omega u_i(x). \tag{4.26}$$

Another set of solutions are the complex conjugates of the u_i modes, i.e.,

$$u_i^*(x) \propto e^{-i(\mathbf{k}\cdot x - \omega t)}, \tag{4.27}$$

these are negative frequency modes with respect to t, since

$$\frac{\partial u_i^*(x)}{\partial t} = +i\omega u_i^*(x). \tag{4.28}$$

The complete set of solutions, u_i and their complex conjugates u_i^* satisfy the orthonormality condition

$$\begin{aligned} &< u_i, u_j >= \delta_{ij} \\ &< u_i^*, u_j >= 0, \end{aligned} \tag{4.29}$$

if the scalar product is defined as,

$$< u_i, u_j >= -i \int \left[u_i(x) \partial_t u_j^*(x) - [\partial_t u_i(x)] u_j^*(x) \right] d^3x. \tag{4.30}$$

One can expand the ϕ field in terms of the $u_i(x)$ modes and their complex conjugates, $u_i^*(x)$ such that

$$\phi(x) = \Sigma_i \left[a_i u_i(x) + a_i^\dagger u_i^*(x) \right], \tag{4.31}$$

In order to quantize the ϕ field, one has to promote it to operator and apply the following equal time canonical commutation relations

$$[\phi(t, \mathbf{x}), \phi(t, \mathbf{x}')] = 0 \quad [\pi(t, \mathbf{x}), \pi(t, \mathbf{x}')] = 0 \quad [\phi(t, \mathbf{x}), \pi(t, \mathbf{x}')] = i\delta^3(\mathbf{x} - \mathbf{x}'). \tag{4.32}$$

$\pi(t, \mathbf{x})$ is the conjugate momentum for the ϕ field which is defined as

$$\pi_i(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_i(x))}, \tag{4.33}$$

and in the last commutation relation $\delta(\mathbf{x} - \mathbf{x}')$ is the Dirac delta function which is,

$$\delta(\mathbf{x} - \mathbf{x}') = \begin{cases} \infty & \text{if } \mathbf{x} = \mathbf{x}' \\ 0 & \text{if } \mathbf{x} \neq \mathbf{x}.' \end{cases}$$

If we substitute $\phi(x)$ from (4.31) into (4.32), we get that a_i and $a_i^\dagger$ satisfy the following commutation relations,

$$[a_i, a_j] = 0 \quad [a_i^\dagger, a_j^\dagger] = 0 \quad [a_i, a_j^\dagger] = \delta_{ij}. \tag{4.34}$$

These are commutation relations for creation and annihilation operators, hence,

$a_i^\dagger$ and a_i are creation and annihilation operators respectively.

If we look at the expansion of $\phi(x)$ in (4.31), we see that $\phi(x)$ is equal to its Hermitian conjugate. Hermitian conjugation occurs when

$$a_i \leftrightarrow a_i^*, \quad \text{and} \quad u_i(x) \leftrightarrow u_i^*(x). \tag{4.35}$$

It implies that scalar fields describe particles that are equal to their antiparticles.

In classical mechanics the Hamiltonian of a system of point particles is a central quantity that one should know in order to study the dynamics of the system. The Hamiltonian of a system corresponds to the total amount of energy that the system includes. Since in quantum field theory we are interested in dynamics of quantum fields, we should know about the Hamiltonian of the ϕ fields.

The Hamiltonian of a ϕ field is defined in terms of the stress-energy tensor of the field which is given by

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}\eta^{\lambda\rho}\partial_\lambda\phi\partial_\rho\phi + \frac{1}{2}m^2\phi^2\eta_{\mu\nu}, \tag{4.36}$$

for $\mu, \nu = 0, 1, 2, 3$. The above equation can be obtained by using $T_{\mu\nu} = \frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}}$, and for the Minkowski spacetime we have $g_{\mu\nu} = \eta_{\mu\nu}$. One can write the Hamiltonian density $\mathcal{H}(x)$ as,

$$\mathcal{H}(x) = T_{00} = \frac{1}{2}\left[(\partial_0\phi)^2 + \Sigma_{i=1}^{n-1}(\partial_i\phi)^2 + m^2\phi^2\right]. \tag{4.37}$$

The total Hamiltonian is

$$H = \int \mathcal{H}(x)d^3x, \tag{4.38}$$

by using (4.31) and taking the integral over all space, we get

$$H = \frac{1}{2}\Sigma_i \left(a_i^\dagger a_i + a_i a_i^\dagger\right)\omega. \tag{4.39}$$

From the commutation relations of a_i and $a_i^\dagger$, the Hamiltonian can be written as [17],

$$H = \Sigma_i \left(a_i^\dagger a_i + \frac{1}{2}\right)\omega. \tag{4.40}$$

In QFT the operator that counts the number of particles in a specific quantum state is the number operator N_i defined in terms of the creation and annihilation operators such that

$$N_i = a_i^\dagger a_i, \tag{4.41}$$

as we can see, it is the first term in (4.40), hence

$$H = \Sigma_i \left(N_i + \frac{1}{2}\right)\omega, \tag{4.42}$$

using (4.42) one can show that H and N_i commute, i.e.,

$$[N_i, H] = 0. \tag{4.43}$$

4.1.3 Complex scalar fields

Complex scalar fields are functions of spacetime coordinates that assign complex numbers to each point of spacetime. A combination of two real scalar fields ϕ_1 and ϕ_2 with the same mass m can form a complex scalar field such that

$$\phi = \frac{(\phi_1 + i\phi_2)}{\sqrt{2}}. \tag{4.44}$$

The action for the complex field is the sum of the actions of the two real scalar fields. In terms of the $\phi(x)$, the action is given by

$$S = \int d^4x(\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi), \tag{4.45}$$

and the Lagrangian density is

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi. \tag{4.46}$$

The equations of motion are obtained by taking the variation of the action with respect to the ϕ^* and setting it equal to zero while ϕ is fixed. Since ϕ_1 and ϕ_2 satisfy the Klein-Gordon equation separately, ϕ also satisfies Klein-Gordon equation.

ϕ can be expanded in terms of the orthonormal modes which are solutions of Klein-Gordon equation

$$\phi(x) = \Sigma_i \left[b_i u_i(x) + c_i^\dagger u_i^*(x) \right]. \tag{4.47}$$

The same as real scalar field case, $u_i(x)$ are plane waves and $u_i^*(x)$ are their complex conjugates. b_i and c_i obey the following commutation relations

$$[b_i, b_j] = [c_i, c_j] = [b_i, c_j] = [b_i, c_j^\dagger] = 0. \tag{4.48}$$

$b_i^\dagger$ and $c_i^\dagger$ are two independent creation operators. They create two different types of spin zero particles which possess the same mass. They are known as particles and antiparticles. Thus, a complex scalar field is not its own Hermitian conjugate.

The current for the complex scalar field is defined as

$$j_\mu = -i \left(\phi \partial_\mu \phi^* - \phi^* \partial_\mu \phi \right) = i \phi^* \overleftrightarrow{\partial_\mu} \phi, \tag{4.49}$$

and the scalar product of two complex scalar fields is given by

$$< \phi_1 | \phi_2 >= i \int d^3x \phi_1^* \overleftrightarrow{\partial_0} \phi_2. \tag{4.50}$$

4.1.4 Fermionic fields

Fermionic fields are spin $\frac{1}{2}$ fields. Quanta of a fermionic field is a fermion. Electrons and protons are examples of fermions. One famous fermionic field is the Dirac field $\psi(x)$. Dirac spinor is represented as a four-component spinor

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}, \tag{4.51}$$

or in terms of the Weyl spinors, ψ_L and ψ_R

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \tag{4.52}$$

this representation of the Dirac field is called chiral representation. ψ_L is the left-handed Weyl spinor which is $(0, \frac{1}{2})$ representation. The right-handed Weyl spinor ψ_R, is $(\frac{1}{2}, 0)$ representation, (see [2] for more details).

The Lagrangian density for the Dirac field is defined as

$$\mathcal{L}_D = i\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L + i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R - m(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L), \tag{4.53}$$

$\sigma^\mu = (1, \sigma^i)$ and $\bar{\sigma}^\mu = (1, -\sigma^i)$, 1 is the 2×2 identity matrix and σ^i are the Pauli

matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4.54}$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{4.55}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.56}$$

$\bar{\sigma}_\mu$ are the Hermitian conjugates of the Pauli matrices. The Pauli matrices are used to define Dirac gamma matrices in three dimensions as

$$\gamma^i = i\sigma^i. \tag{4.57}$$

By using the chiral representation for the Dirac field in the Lagrangian, the equations of motion are obtained as

$$(i\gamma^\mu \partial_\mu - m)\psi = 0. \tag{4.58}$$

The γ^μ matrices are Dirac gamma matrices in four dimensions

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{4.59}$$

for $i = 1, 2, 3$. Or, briefly

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \tag{4.60}$$

Dirac gamma matrices obey the Clifford algebra,

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}. \tag{4.61}$$

4.1.5 Vacuum in Minkowski spacetime

In quantum field theory vacuum state or zero-particle state $|0>$, is the state with the lowest possible energy and zero physical particles in it. Vacuum is the state that can be annihilated by the annihilation operator

$$a_i|0>=0, \qquad \forall i, \tag{4.62}$$

or equivalently

$$<0|a_i^\dagger = 0, \qquad \forall i. \tag{4.63}$$

In QFT, the space of quantum states is called Fock space and a basis in this space is called the Fock representation. Fock space can be constructed by operating on the vacuum with the creation operator $a^\dagger$

$$a_i^\dagger|0>=|1_i> \tag{4.64}$$

where the state $|1>$ is called the one-particle state. As we mentioned before, vacuum is the state that includes zero physical particles. One can prove this by finding the vacuum expectation value of the number operator

$$<0|N_i|0>=<0|a_i^\dagger a_i|0>, \tag{4.65}$$

from (4.62) and (4.63), one can conclude that

$$<0|N_i|0>=0. \tag{4.66}$$

It means that the number of particles in the vacuum state is zero which is simply the definition of the vacuum state. If the number of particles in the vacuum state is zero,

the energy of the state must be zero as well. For example, we can find the vacuum expectation value of the Hamiltonian operator (4.42) as

$$< 0|H|0 >=< 0|\Sigma_i(N_i + \frac{1}{2})\omega|0 >=< 0|\Sigma_i\frac{1}{2}\omega|0 >, \tag{4.67}$$

and orthonormality condition requires

$$< 0|0 >= 1, \tag{4.68}$$

the vacuum expectation value of the Hamiltonian would be

$$< 0|H|0 >= \frac{1}{2}\Sigma_i\omega. \tag{4.69}$$

So, the energy of the vacuum state is non-zero though there are no physical particles in the state. The term $\frac{1}{2}\omega$ is the zero-point energy of the harmonic oscillator modes of the ϕ field, [17]. Since there is no limit for the value of the ω, it can be very large or even infinite. It implies that the energy of the vacuum state can be infinite. One way to solve this problem of the vacuum is to define *normal ordering* operation. If one has an operator which is a mixture of creation and annihilation operators, the normal ordered version of the operator is obtained by placing all creation operators to the left of the annihilation operators. So, the normal ordering implies

$$: a_i a_i^\dagger := a_i^\dagger a_i \rightarrow : H := \Sigma_i a_i^\dagger a_i \omega. \tag{4.70}$$

The vacuum expectation value of the normal ordered Hamiltonian is therefore

$$< 0| : H : |0 >=< 0|\Sigma_i a_i^\dagger a_i \omega|0 >= 0, \tag{4.71}$$

which is consistent with the zero amount of the energy of the vacuum state.

4.2 Quantum field theory in curved spacetime

At present, gravity is described by Einstein's field equations. In the presence of gravity, the spacetime is no longer the flat Minkowski spacetime. Thus, the theory of quantum fields in Minkowski spacetime should be modified. However, gravity is a classical field though matter fields are quantized. So, the problem is that if there is no quantum theory of gravity, how can one study the effects of a classical gravitational field on quantum matter fields? One can consider a classical background gravitational field and examine its effects on quantum matter fields. It can be an approximation of a more generalized theory in which spacetime is quantized. This approximation works very well except in spacetime singularities. This approximation is valid for our universe after the timescale 10^{-43} s, [7].

In the following, we will study the physics of scalar fields and fermionic fields in curved spacetime and we show that vacuum is not unique in curved spacetime and is observer dependent. We introduce Bogoliubov transformations which are used to relate different mode expansions of the quantum fields.

4.2.1 Real scalar fields

The Lagrangian for a real scalar field in curved spacetime is given by

$$L(x) = \frac{1}{2}[-g(x)]^{1/2}\left[g^{\mu\nu}(x)\partial_\mu\phi(x)\partial_\nu\phi(x) - [m^2 + \xi R(x)]\phi^2(x)\right], \tag{4.72}$$

where, m is the mass of the field, $R(x)$ is the Ricci scalar which is an indication of the curved spacetime. ξ is a constant that shows the coupling between the scalar field and curvature of spacetime. There are two important cases based on different values

of ξ,

1) Conformally coupled case:

it occurs when $\xi = \frac{1}{4}[(n-2)/(n-1)] = \xi(n)$, where n is the dimension of spacetime. For example, for a four-dimensional spacetime $\xi = \frac{1}{6}$.

Under a conformal transformation

$$g_{\mu\nu} \to \bar{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu}(x) \tag{4.73}$$

$\phi(x)$ will transform as

$$\bar{\phi}(x) = \Omega^{(2-n)/2}(x)\phi(x). \tag{4.74}$$

If $m = 0$, the action and the equations of motion will be invariant under conformal transformations, [27]. Equations of motion for the conformally coupled case are

$$\left(\Box + m^2 + \frac{1}{4}[(n-2)/(n-1)]R(x)\right)\phi(x) = 0, \tag{4.75}$$

with

$$\Box\phi = g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = (-g)^{-1/2}\partial_\mu[(-g)^{1/2}g^{\mu\nu}\partial_\nu\phi]. \tag{4.76}$$

2) $\xi = 0$ or minimally coupled case:

for a two-dimensional spacetime, the minimally coupled case and conformally coupled case would be the same since for both of them $\xi = 0$.

For minimally coupled case the Lagrangian is,

$$L(x) = \frac{1}{2}[-g(x)]^{1/2}\left\{g^{\mu\nu}(x)\partial_\mu\phi\partial_\nu\phi - m^2\phi^2\right\}. \tag{4.77}$$

We get the equations of motion as follow,

$$\delta S = \delta \int L(x) d^4x = 0 \rightarrow [\Box + m^2]\phi(x) = 0, \tag{4.78}$$

with $\Box$ is the same as (4.76). If we consider our coordinate system as $x = x^\mu$, there exists a complete set of orthonormal modes $u_i(x)$ which are solutions of the equations of motion and satisfy

$$< u_i, u_j >= \delta_{ij} \qquad < u_i^*, u_j >= 0 \qquad < u_i^*, u_j^* >= -\delta_{ij}. \tag{4.79}$$

With the inner product defined as

$$< u_1, u_2 >= -i \int_\Sigma u_1(x) \partial_\mu u_2^*(x) [-g(x)]^{1/2} d\Sigma^\mu, \tag{4.80}$$

where Σ is a spacelike Cauchy surface, [15]. The ϕ field can be expanded in terms of these normal modes

$$\phi(x) = \Sigma_i \left(a_i u_i(x) + a_i^\dagger u_i^*(x) \right). \tag{4.81}$$

ϕ fields satisfy the same equal time canonical commutation relations as the Minkowski spacetime case and from them one can get

$$[a_i, a_i^\dagger] = \delta_{ij} \qquad [a_i, a_j] = 0 \qquad [a_i^\dagger, a_j^\dagger] = 0. \tag{4.82}$$

The above commutation relations are the same as for the creation and annihilation operators in Minkowski spacetime, therefore we can conclude that they are creation and annihilation operators.

The $u_i(x)$ normal modes are not the only set of normal modes in terms of which ϕ field can be expanded. Since we can solve the Klein-Gordon equation in different

coordinate systems, we can find different normal modes as the solution. If we select another coordinate system $\bar{x} = \bar{x}^{\mu}(x)$, we can take another set of normal modes $\bar{u}_j(x)$ and write ϕ in terms of them, so,

$$\phi(x) = \Sigma_j \left(\bar{a}_j \bar{u}_j(x) + \bar{a}_j^{\dagger} \bar{u}_j^*(x) \right). \tag{4.83}$$

Since both u_i and $\bar{u}_j$ are complete sets, one can write

$$\bar{u}_j = \Sigma_i \left(\alpha_{ji} u_i + \beta_{ji} u_i^* \right), \tag{4.84}$$

and

$$u_i = \Sigma_j \left(\alpha_{ji}^* \bar{u}_j - \beta_{ji} \bar{u}_j^* \right). \tag{4.85}$$

These relations are called Bogoliubov transformations and the matrices α_{ij} and β_{ij} are Bogoliubov coefficients. The Bogoliubov coefficients are defined as

$$\alpha_{ij} = < \bar{u}_i, u_j > \tag{4.86}$$

$$-\beta_{ij} = < \bar{u}_i, u_j^* > . \tag{4.87}$$

The definition for the Noether current is the same as the Minkowski spacetime,

$$j^{\mu}(x) = \frac{\partial \mathcal{L}(x)}{\partial(\partial_{\mu}\phi_a(x))} \delta\phi_a(x). \tag{4.88}$$

Since the Lagrangian density $\mathcal{L}(x)$, for the ϕ field in the curved spacetime is not the same as the Minkowski spacetime, the Noether current would be different.

4.2.2 Complex scalar fields

The definition of the complex scalar field is the same as what we mentioned for the Minkowski spacetime. The action for a complex scalar field in a four-dimensional curved spacetime is given by,

$$S = \int \sqrt{-g}(\partial_\mu \phi^* \partial_\nu \phi g^{\mu\nu} - \frac{1}{6} R\phi^* \phi) d^4x, \tag{4.89}$$

with R the Ricci scalar [16]. For the ϕ and ϕ^* fields, the conjugate momenta π and π^* are defined as

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \sqrt{-g} g^{0\mu} \partial_\mu \phi^* \tag{4.90}$$

and

$$\pi^* = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi^*)} = \sqrt{-g} g^{0\mu} \partial_\mu \phi. \tag{4.91}$$

The equations that describe the complex ϕ field are

$$\nabla_\nu \left[(\partial_\mu \phi) g^{\mu\nu}\right] + \frac{1}{6} R\phi = 0. \tag{4.92}$$

A complex scalar field can be expanded in terms of the solutions of (4.92), [14].

4.2.3 Fermionic fields

For a manifold M, the set of all possible vectors which are tangent to the manifold at a point P will form the Tangent space T_P at point P and the set of all possible tangent spaces of a manifold is called a tangent bundle $T(M)$. The geometry of the manifold and the tangent space are different. Also, the basis vectors on the manifold and on the tangent space are not the same. The basis vectors of the tangent space T_P at point P are

$$\hat{e}_{(a)} = \frac{\partial}{\partial x^a}, \tag{4.93}$$

where, x^a denotes the coordinate system on the tangent space. If the basis vectors on the manifold are $\hat{e}_{(\mu)}$, we have

$$\hat{e}_{(\mu)} = e^a_\mu \hat{e}_{(a)}, \tag{4.94}$$

where e^a_μ are $n \times n$ invertible matrices which are known as tetrads. Tetrads are crucial to relate the tangent space variables to the variables on the manifold. Tetrads are used to relate the metric of the flat spacetime $\eta_{\mu\nu}$(can be the tangent space), and the metric of the curved spacetime $g_{\mu\nu}$(or the manifold), as

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}. \tag{4.95}$$

We will need the tetrads in order to study fermionic fields in curved spacetime. They will be required to find the proper expression for the Dirac gamma matrices in curved spacetime.

The action for a spin $\frac{1}{2}$ field in the curved spacetime is

$$S = \int i\sqrt{-g}(\bar{\psi} g^{\mu\nu} \gamma_\nu \nabla_\mu \psi) d^4x, \tag{4.96}$$

where $\bar{\psi}$ is defined as, [16]

$$\bar{\psi} = \psi^\dagger \alpha, \tag{4.97}$$

and α obeys

$$\alpha\gamma^\mu - \gamma^{\mu\dagger}\alpha = 0. \tag{4.98}$$

The Dirac gamma matrices satisfy the generalized Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \tag{4.99}$$

Since we are in curved spacetime we have to write $g_{\mu\nu}$, the metric of the curved spacetime instead of the Minkowski metric $\eta_{\mu\nu}$.

The equations of motion for the Dirac field are

$$\gamma^\mu(\partial_\mu - \omega_\mu)\psi = 0. \tag{4.100}$$

The spin connection components ω_μ, are

$$\omega_\mu = \omega_{\mu ab}\sigma^{ab} \tag{4.101}$$

where

$$\omega^a_{\mu b} = e^a_\nu e^\lambda_b \Gamma^\nu_{\mu\lambda} - e^\lambda_b \partial_\mu e^a_\lambda \tag{4.102}$$

and

$$\sigma^{ab} = \frac{i}{2}[\gamma^a, \gamma^b]. \tag{4.103}$$

Spin connection is a connection which is defined on the tangent bundle of a manifold. γ^μ are Dirac gamma matrices in curved spacetime

$$\gamma^\mu = e^\mu_a \gamma^a. \tag{4.104}$$

Therefore, to solve the equations of motion for fermionic fields in curved spacetime one has to find the Dirac gamma matrices in curved spacetime and the spin connection components and substitute them in (4.100) and solve for ψ.

4.2.4 Vacuum in curved spacetime

The procedure to construct the Fock space from the vacuum state is the same as the Minkowski spacetime. The main difference is that the vacuum is not unique in curved spacetime since each creation and annihilation operator defines a new vacuum. So, we have

$$a_i|0>=0, \qquad \forall i, \tag{4.105}$$

and

$$\bar{a}_j|\bar{0}>=0 \qquad \forall j. \tag{4.106}$$

$|0>$ and $|\bar{0}>$ are two different vacua. By acting on different vacuum states by respective creation operators one can get one-particle states

$$a_i^\dagger|0>=|1_i> \tag{4.107}$$

and

$$\bar{a}_j^\dagger|\bar{0}>=|\bar{1}_j> . \tag{4.108}$$

The orthonormality conditions give the two annihilation operators a_i and $\bar{a}_j$ as

$$a_i =<\phi, u_i>=\Sigma_j \left[\bar{a}_j <\bar{u}_j, u_i> +\bar{a}_j^\dagger <\bar{u}_j^*, u_i>\right], \tag{4.109}$$

and

$$\bar{a}_j =<\bar{u}_j, \phi>=\Sigma_i \left[a_i <\bar{u}_j, u_i> +a^\dagger <\bar{u}_j, u_i^*>\right]. \tag{4.110}$$

The Bogoliubov transformations give

$$a_i = \Sigma_j \left(\alpha_{ji}\bar{a}_j + \beta_{ji}^*\bar{a}_j^\dagger\right), \tag{4.111}$$

and

$$\bar{a}_j = \Sigma_i \left(\alpha^*_{ji} a_i - \beta^*_{ji} a_i^\dagger \right). \tag{4.112}$$

It is evident that a vacuum that can be annihilated by $\bar{a}_j$ will not be annihilated by a_i, since

$$a_i|\bar{0}>= \Sigma_j \left(\alpha_{ji} \bar{a}_j + \beta^*_{ji} \bar{a}_j^\dagger \right) |\bar{0}>= \Sigma_j |\beta^*_{ji}| \bar{1}_j > \neq 0. \tag{4.113}$$

Therefore, the two Fock spaces based on the two different modes u_i and $\bar{u}_j$ would be different as long as $\beta_{ji} \neq 0$. The vacuum expectation value of the operator $N = a_i^\dagger a_i$ that gives the number of u_i-mode particles in the state $|\bar{0}>$ is

$$< \bar{0}|N_i|\bar{0}>= \Sigma_j |\beta_{ij}|^2, \tag{4.114}$$

it states that vacuum of the $\bar{u}_j$ contains $\Sigma_j |\beta_{ij}|^2$ particles in the u_i mode. Therefore, the vacuum depends on the observer. If an observer observes a state as being a vacuum state, another observer can see that the observed state contains some particles.

In the case of the Minkowski spacetime, it was easy to determine the positive and negative frequency modes. In Minkowski spacetime, we could identify positive frequency modes and negative frequency modes by simply taking the partial derivatives of the modes with respect to the time. However, in the case of curved spacetime we do not have a unique time coordinate. For the curved spacetime, one should find the Lie derivative of the modes with respect to a timelike Killing vector $\frac{\partial}{\partial t}$, so,

$$L_{\partial t} u_i = -i\omega u_i, \qquad \omega > 0, \tag{4.115}$$

are positive frequency modes. Negative frequency modes are

$$L_{\partial_t} u_i^* = i\omega u_i^*, \qquad \omega > 0. \tag{4.116}$$

4.2.5 Summary

This chapter was devoted to study the main aspects of quantum field theory in Minkowski spacetime and curved spacetime. Concepts such as the equations of motion for different classes of fields, and the vacuum in flat and curved spacetime were explained. In this chapter we mainly followed the following references: [2], [3], [11], [15], [16], [18], [21], [22], [23] and [27].

Chapter 5

Vacuum Decay in a Quasi-Local Region

5.1 Vacuum polarization

Vacuum polarization is an outcome of definition of the vacuum in modern quantum field theory. One of the first suggested models of the vacuum was proposed by Dirac. However, his model had some problems. In this section we will review the Dirac model and then we will talk about vacuum polarization in modern quantum field theory. We will see that Hawking radiation can be described as a result of vacuum polarization in the Schwarzschild spacetime and in the vicinity of the black hole horizon.

5.1.1 Dirac sea and negative energy states

The equation that relates energy, momentum and mass of a particle in special relativity is

$$E^2 = p^2c^2 + m^2c^4 \tag{5.1}$$

E is the energy, p is the momentum and m is the mass of the particle. For massless particles like photons, the equation changes to

$$E^2 = p^2c^2 \tag{5.2}$$

and for particles at rest it will be

$$E^2 = m^2c^4. \tag{5.3}$$

If we take the square root of (5.1), we get

$$E = \pm\sqrt{p^2c^2 + m^2c^4}, \tag{5.4}$$

this shows that if there is a state with energy $+E$, there exists a state with energy $-E$. Quantum mechanics assigns operators to physical observables such as energy, momentum, angular momentum and so on. For example, the energy operator is

$$\hat{E} = i\hbar\frac{\partial}{\partial t}, \tag{5.5}$$

and the momentum operator is

$$\hat{p} = -i\hbar\nabla = -i\hbar(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}). \tag{5.6}$$

Furthermore, particles are described by wave functions which are functions of space and time, $\psi(r, t)$. The Schrodinger equation for a massive free particle is

$$-\frac{\hbar^2}{2m}\nabla^2\psi = i\hbar\frac{\partial}{\partial t}\psi, \tag{5.7}$$

by using (5.5) and (5.6) one writes

$$\frac{\hat{p}^2}{2m}\psi = \hat{E}\psi. \tag{5.8}$$

Energy, momentum and mass satisfy (5.4), so, if we substitute the operators of energy and momentum in (5.4) we get

$$(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2)\psi = \frac{m^2c^2}{\hbar^2}\psi, \tag{5.9}$$

which is the relativistic Klein-Gordon equation. This equation could explain the electron dynamics pretty well. So, (5.4) implies that negative energy electrons are allowed. The Pauli exclusion principle says

- no two fermions can occupy a single energy state within an atom.

Electrons are spin 1/2 particles and therefore they obey Pauli exclusion principle. In 1930, Paul Dirac postulated what is known as 'Dirac sea' to explain the negative-energy quantum states. Dirac postulated that the state in which all the negative energy states are filled is the vacuum state. This definition of the vacuum is known as the 'Dirac sea'. So, a single electron must be in positive energy state since all the negative energy states are filled. In the case that all the negative energy states are filled except one, there would be a hole in the sea. This occurs when a negative energy electron gets energy by absorbing a photon and become positive energy (Fig. 5.1).

The total charge of the Dirac sea is the charge of the vacuum minus the charge of the electron,

$$Q = Q_{vac} - (-e) = Q_{vac} + |e|, \tag{5.10}$$

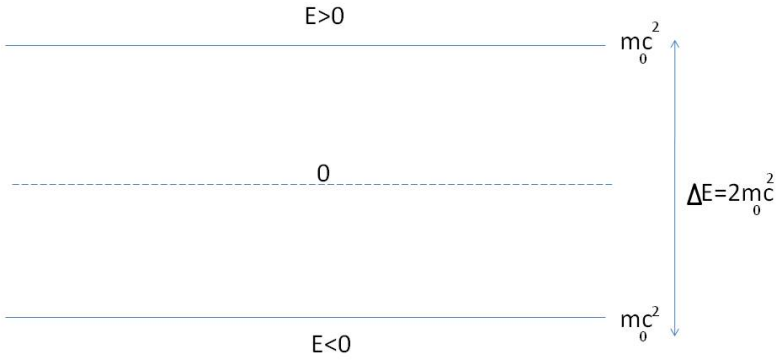

Figure 5.1: A negative energy electron can absorb a photon and become a positive energy electron. The required energy for this procedure is $2m_0c^2$.

thus, the charge of the hole is

$$Q_{hole} = Q - Q_{vac} = +|e|. \tag{5.11}$$

This hole can be interpreted as a positive energy particle which possesses the same mass as the electron. The Dirac hole has same properties as the positron which is the antiparticle of the electron.

The Dirac sea was not accepted by several physicists due to infinite negative charge of the vacuum. Infinite positive charge is required to form a neutral vacuum. As we mentioned in previous chapter, modern QFT describes the vacuum as the state with zero particles. The positron is a real particle in modern QFT not just the absence of a real particle.

5.1.2 S-matrix and vacuum polarization

In quantum field theory, the S-matrix is an operator that determines the evolution of a state. States are functions of time in the Schrodinger picture of the quantum mechanics. A state $A|t>$ evolves as

$$|A>(t) = e^{-iH(t-T_i)/\hbar}|A>, \tag{5.12}$$

where $|A>=|A>(T_i)$ and H is the Hamiltonian. If $|A>(t)$ at final time T_f is denoted by $|A'>$, we can write

$$|A'>= e^{-iH(T_f-T_i)/\hbar}|A> . \tag{5.13}$$

The process of evolution of the initial state $|A>$ to the final state $|A'>$ has the following amplitude

$$< A'|e^{-iH(T_f-T_i)/\hbar}|A> . \tag{5.14}$$

In the limit $T_f - T_i \to \infty$ the evolution operator $e^{-iH(T_f-T_i)/\hbar}$ is called the S-matrix, [2]. S is a unitary operator, since

$$SS^\dagger = S^\dagger S = 1. \tag{5.15}$$

According to modern QFT, the virtual particle-antiparticle pairs are created out of the vacuum spontaneously and annihilate each other very shortly thereafter. This process shows that the vacuum is active. The decay rate of the vacuum can be computed from the vacuum expectation value of the S matrix of the interaction

$$|<0|S|0>|^2 = \exp(-\int d^4x\ w(x)) = \exp(-W), \tag{5.16}$$

$w(x)$ is the rate of pair creation per unit time per unit volume and $W = \int d^4x\ w(x)$ is the net rate of the decay of the vacuum. If W is non-zero and positive it manifests a 'decay'.

Some of the virtual pairs which are created out of the vacuum are charged. In the presence of an external electromagnetic field, these pairs will be displaced from their original positions. As a result of the displacement of the pairs, the electromagnetic field will be different from the one before the displacement. In this case it is said that the vacuum is polarized. Vacuum polarization is defined as the reorientation of the short lived particle-antiparticle pairs due to the presence of an external field. In the case of an external electromagnetic field the S-matrix of the interaction is

$$S = \exp\left(\frac{i}{\hbar}\int d^4x j^\mu A_\mu\right), \tag{5.17}$$

A_μ is the vector potential which is defined by

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{5.18}$$

$F^{\mu\nu}$ is known as the electromagnetic field tensor which is defined in terms of electric field E and magnetic induction B as

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -Ez & B_y & -B_x & 0 \end{pmatrix} \tag{5.19}$$

j^μ is the operator scalar/fermion matter field current given by

$$\partial_\nu F^{\mu\nu} = j^\mu. \tag{5.20}$$

After expanding (5.16) we get

$$W = \frac{2\mathrm{Im}}{\hbar} < 0| \int d^4x j^\mu A_\mu |0> \tag{5.21}$$

which gives the net rate of vacuum decay in the presence of an external electromagnetic field. This was computed in [26] and it was crucial that the vacuum expectation of the interaction term had an imaginary term for the vacuum to decay

Gravity can also force particle-antiparticle pairs to be separated. In the presence of a background gravitational field, particle-antiparticle pairs will be separated under the influence of gravity and if the gravitational field is strong enough real particles can emerge. General relativity suggests the stress-energy tensor $T_{\mu\nu}$, as the source of gravity. Therefore, the coupling between matter fields and gravity is described by the stress-energy tensor. The S matrix of the interaction is

$$S = \int d^4x \sqrt{-g} g^{\mu\nu} T_{\mu\nu}. \tag{5.22}$$

$g^{\mu\nu}$ is the metric of the spacetime and g is its determinant. So, the exact decay rate of the vacuum is given by

$$W_{grav} = \frac{2\mathrm{Im}}{\hbar} < 0| \int d^4x \sqrt{-g} g^{\mu\nu} T_{\mu\nu} |0> . \tag{5.23}$$

5.1.3 Vacua of the Schwarzschild spacetime

In the presence of a background gravitational field or equivalently for an observer in an accelerated reference frame there is not a unique Fock space vacuum. As we mentioned before, vacuum would be different from the point of view of different observers in curved spacetime. It implies different vacuum expectation values for different vacua.

The Schwarzschild spacetime consists of two regions

$$\begin{cases} \text{interior region} & \text{if } r < 2GM, \\ \text{exterior region} & \text{if } r > 2GM. \end{cases}$$

The spacetime at $r > 2GM$ is described by the Schwarzschild metric

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + (1 - \frac{2GM}{r})^{-1}dr^2 + r^2 d\Omega^2. \tag{5.24}$$

In terms of the tortoise coordinate r^* the metric is

$$ds^2 = -(1 - \frac{2GM}{r})[dt^2 - dr^{*2}] + r^2 d\Omega^2. \tag{5.25}$$

However, $r < 2GM$ region is accessible using the Kruskal coordinates

$$ds^2 = -\frac{32(GM)^3}{r} e^{-r/2GM} dU dV + r^2 d\Omega^2, \tag{5.26}$$

where

$$UV = (\frac{r}{2GM} - 1)e^{-r/2GM}. \tag{5.27}$$

The vacua in the interior and exterior regions are not the same. Therefore, in order to expand a scalar field in terms of normal modes in these two regions one has to find the orthonormal modes in both regions. To do so, one has to solve (4.78) in Schwarzschild spacetime. In the region $r > 2GM$, the spacetime is described by the Schwarzschild metric while in the region $r < 2GM$ the spacetime is characterized by the Kruskal metric. One has to substitute the metric coefficients in (4.76) for the two regions inside and outside the horizon and then solve (4.78). As we mentioned before, positive frequency modes are not obtained like the Minkowski spacetime by

taking the partial derivatives of the normal modes with respect to time. One has to find the appropriate parameter in curved spacetime to define the positive frequency modes with respect to it. Based on the different ways to define positive frequency modes, one can define different vacua in curved spacetime.

There are three well defined vacua for the Schwarzschild spacetime, [24], as follows

(i) The Boulware Vacuum $|B>$

This vacuum is defined by requiring normal modes that are incoming from $\mathcal{I}^-$ to be positive frequency with respect to the Killing vector $\frac{\partial}{\partial t}$,

$$\text{modes} \propto e^{-i\omega t}. \tag{5.28}$$

(ii) The Hartle-Hawking Vacuum $|H>$

For the Hartle-Hawking vacuum, modes incoming from future horizon are positive frequency with respect to the Kruskal coordinates V, the canonical affine parameter on the future horizon,

$$\text{modes} \propto e^{-i\omega V}. \tag{5.29}$$

Also, outgoing modes from past horizon are taken to be positive frequency with respect to U the affine parameter on the past horizon,

$$\text{modes} \propto e^{-i\omega U}. \tag{5.30}$$

(iii) The Unruh Vacuum $|U>$

The Unruh vacuum is defined by taking modes that are incoming from $\mathcal{I}^-$ to be positive frequency with respect to $\frac{\partial}{\partial t}$,

$$\text{modes} \propto e^{-i\omega t}, \tag{5.31}$$

furthermore, modes that emanate from the past horizon are taken to be positive frequency with respect to U, the canonical affine parameter on the past horizon

$$\text{modes} \propto e^{-i\omega U}. \tag{5.32}$$

If we consider a scalar field in Schwarzschild spacetime, based on the choice of the vacuum the mode expansion of the scalar field would be different. We will select the Unruh vacuum in this thesis since it is the most appropriate one for our computations, [30].

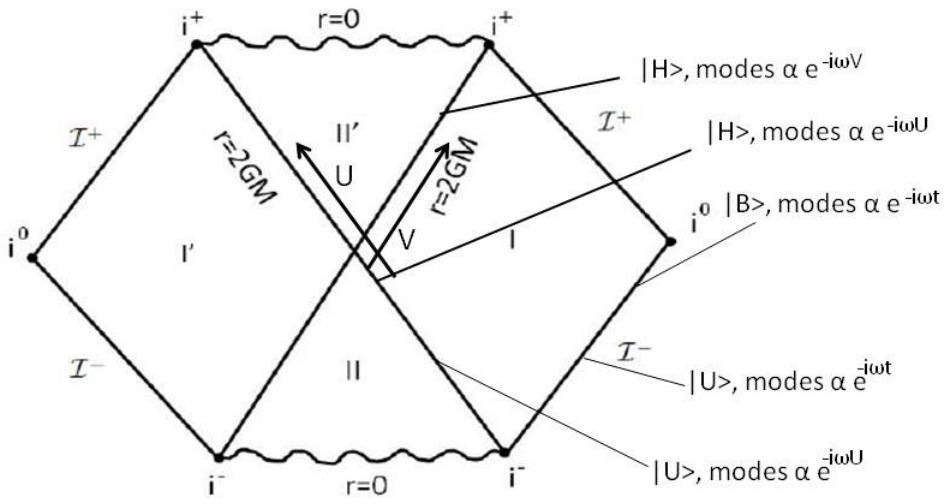

Figure 5.2: Three different vacuua of Schwarzschild spacetime.

5.1.4 Vacuum polarization in the vicinity of the horizon

We introduced Hawking radiation in chapter three as radiation from the black hole horizon, however, it has not been detected in the lab yet. In the initial derivation of Hawking [7] , a geometric approximation at the time of horizon formation in a collapsing situation was used to show a net flux of particles emergent at asymptotic future, [30].

Hawking radiation can also be interpreted as the result of the vacuum fluctuations in the vicinity of the horizon. Vacuum fluctuation refers to the spontaneous pair creation in the vacuum.

If a particle-antiparticle pair is created near the horizon, one member of the pair may fall into the hole while the other member escapes into infinity. The flux of outgoing particles is detected as Hawking radiation. The member that escapes into infinity carries positive energy, therefore, in order to satisfy the conservation of energy law the other member must have negative energy or equivalently negative mass. Also, if we consider the particle-antiparticle pair and the hole as an isolated system, one can say that the hole loses mass.

In this part our aim is to find the net rate of the vacuum decay in the background geometry of a black hole. The rate of vacuum decay close to the black hole horizon can give us the rate of particle production which is interpreted as Hawking radiation.

The (5.23) amplitude has been computed by several physicists before [17], however, we compute a modified version of that. We consider a quasi-local region that includes the horizon. Previous computations were based on computing the vacuum expectation value of the scalar energy-momentum tensor and the answers were infinite, [24]. It was then renormalized using a particular prescription of subtracting the expectation value in Boulware vacuum from say the Unruh vacuum, and in the asymptotic limit it yielded the Hawking flux, [30]. The desired quasi-local region for our computation

is a hollow cylinder that has two boundaries. One boundary is inside the horizon at $r_A = 2GM - \delta$ and the other boundary is outside the horizon at $r_B = 2GM + \delta$.

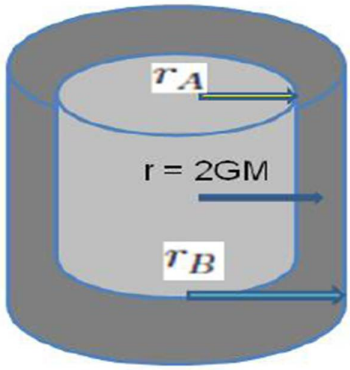

Figure 5.3: A hollow cylinder with two boundaries at $r_A = 2GM - \delta$ and $r_B = 2GM + \delta$.

We like to study the interaction of a massless scalar field with the background gravitational field in this quasi-local region. The action for a scalar field is found over the quasi-local volume that we assumed around the horizon. The action for the massless scalar field is

$$S = \int d^4x\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi. \tag{5.33}$$

The boundary term of the action that can be found by doing partial integration is

$$S_{quasi} = \int_{r_A}^{r_B} d^3x\sqrt{-g}g^{\mu\nu}\phi\partial_\mu\phi\eta_\nu = \int_{r_B} d^3x\sqrt{-g}g^{\mu\nu}j_\mu\eta_\nu - \int_{r_A} d^3x\sqrt{-g}g^{\mu\nu}j_\mu\eta_\nu \tag{5.34}$$

η_ν is the unit vector normal to the surface of the cylinder and $j_\mu = \phi\partial_\mu\phi$ is the current. Hence, the vacuum decay would be

$$W = 2\text{Im} < 0|S_{quasi}|0 > . \tag{5.35}$$

We are interested in computing the vacuum expectation value of the quasi-local action for the scalar field in an appropriate vacuum. For our purpose the Unruh vacuum is the most appropriate vacuum. We should find the appropriate expansion of the scalar field in terms of the normal modes inside and outside the horizon. After that, we can find the current j_μ in these two regions and then we can compute the quasi-local action.

At r_B, the Schwarzschild metric is the proper metric to study the spacetime. The scalar field equations are

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu)\phi = 0 \tag{5.36}$$

$$\partial_t^2\phi - \partial_{r^*}^2\phi + (1 - \frac{2GM}{r})L_{\theta,\phi}\phi = 0 \tag{5.37}$$

$L_{\theta,\phi}$ is the angular momentum operator with eigenfunctions being the spherical harmonics $Y_{lm}(\theta,\phi)$. At r_B according to the definition of the Unruh vacuum, the modes incoming from $\mathcal{I}^-$ are positive frequency with respect to the Killing vector $\frac{\partial}{\partial t}$. These modes are named $u_{\omega lm}$ [11], where ω is the frequency of the mode and l and m are angular momentum quantum numbers. However, since our boundary conditions are different from [11], we redefine the normal modes, so

$$u_{\omega lm} = \frac{1}{(4\pi\omega)^{1/2}r}e^{-i\omega t}e^{+i\omega r^*}Y_{lm}. \tag{5.38}$$

The scalar field can be expanded in terms of these normal modes and their complex conjugates as

$$\phi(r_B) = \Sigma_{\omega lm}\left[a_{r_B}u_{\omega lm} + a^\dagger_{r_B}u^*_{\omega lm}\right], \tag{5.39}$$

$u_{\omega lm}$ shows the flux of outgoing particles at r_B.

For the boundary at r_A, we take the vacuum corresponding to modes of the scalar

field which are positive frequency with respect to the null generators of the past horizon, as these are the modes which emerge from behind the horizon even in a collapsing case [16].

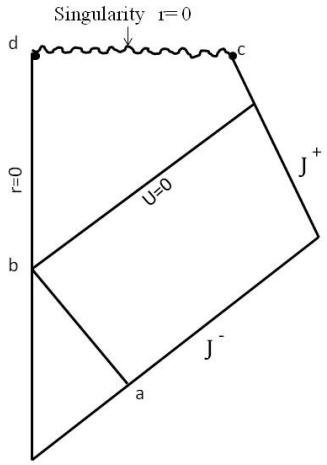

Figure 5.4: Penrose diagram of a gravitational collapse.

These modes are named as $p_{\omega lm}$

$$p_{\omega lm} = \frac{1}{[2\sinh(4\pi M\omega)]^{1/2}} \left[e^{2\pi M\omega} u_{\omega lm} + c.c\right], \tag{5.40}$$

these modes are the Bogoliubov transformations of $u_{\omega lm}$ modes and satisfy

$$\frac{\partial p_{\omega lm}}{\partial U} p_{\omega lm} = -i\omega p_{\omega lm}. \tag{5.41}$$

The scalar field inside the horizon can be expanded in terms of $p_{\omega lm}$ modes and their complex conjugates as

$$\phi(r_A) = \Sigma_{\omega lm} \left[a_{r_A} p_{\omega lm} + a^{\dagger}_{r_A} p^{*}_{\omega lm}\right]. \tag{5.42}$$

Thus, our purpose is to find

$$< U|S_{quasi}|U > \tag{5.43}$$

for the desired quasi-local region. So, in the limit $r_A = r_B \to 2GM$ or $\delta \to 0$ one gets

$$W = Lim_{\delta\to 0} 2Im \left[\int_{r_B} d^3x\sqrt{-g}g^{rr}\Sigma_{\omega lm}u_{\omega lm}\partial_r u^*_{\omega lm} - \int_{r_A} d^3x\sqrt{-g}g^{rr}\Sigma_{\omega lm}p_{\omega lm}\partial_r p^*_{\omega lm} \right]. \tag{5.44}$$

Hence, the net rate of the decay of the vacuum could be found as

$$W = \int d^3x \sin\theta \frac{1}{2\pi}\Sigma_{\omega lm}\frac{1}{e^{\omega/T_H}-1}|Y_{lm}|^2, \tag{5.45}$$

where $T_H = \frac{1}{8\pi M}$ is the Hawking temperature. This is precisely the rate of particle emission expected and predicted in [7]. Thus the flux naturally emerges as in a Bose-Einstein spectrum and is of the form as expected at the horizon. A fraction of this will emerge at infinity due to scattering from the exterior geometry of the black hole. This computation of vacuum polarization amplitude is a new insight in the method of particle creation at the horizon, as we show a finite result by using a quasi-local volume.

5.1.5 Summary

In the previous section we considered a quasi-local volume which is enclosed by two cylindrical membranes, one inside the horizon and the other one outside the horizon. The net rate of the decay of the vacuum due to the interaction of the scalar field with the background classical gravitational field in this volume will give us the Hawking radiation rate. The references for this chapter are: [7], [16], [11], [24], [25], [26], [17] and [30].

Chapter 6

Conclusions

The aim of this thesis was to obtain the flux of Hawking radiation using the concept of vacuum polarization in a quasi-local region. Our approach was slightly different from previous calculations. We presented a chapter to introduce general relativity that assigns gravity to geometric properties of the spacetime. In order to comprehend the Hawking radiation emission procedure one has to know about quantum field theory in curved spacetime besides general relativity. Therefore, we devoted a chapter to study the physics of quantum fields.

We defined a quasi-local volume which is enclosed by two cylindrical membranes, one inside the horizon and the other one outside the horizon. The interaction of the scalar field in the background geometry of the black hole is the desired term for us. The interaction just includes the boundary term of the scalar field action. We showed how the expectation value of a quasi-local action at the horizon of a black hole shows that the scalar vacuum decays into a flux of particles at a thermal temperature. The net rate of the decay of the vacuum due to the interaction of the scalar field with the background classical gravitational field in this volume will give us the Hawking radiation rate. We could get finite answers by simply finding the difference of the net

rate of the flux at the inner surface and the outer surface of the timelike cylinder. However, this derivation still does not answer the question of how gravity reacts to the escaping flux.

This method is applicable for fermionic fields. We computed the interaction term of the action for a fermionic field in the background geometry of a black hole. The action for a fermion in the presence of a gravitational field is

$$S = \int d^4x\sqrt{-g}\bar{\psi}e^{\mu}_{\nu}\gamma^{a}(\partial_{\mu} - \omega_{\mu})\psi.$$

The boundary term of the action is given by

$$S_{quasi} = \int_{r_A, r_B} d^3x\sqrt{-g}\bar{\psi}\gamma^{\mu}\psi\eta_{\mu}.$$

In order to find ψ inside and outside the horizon one has to solve the Dirac equation, $\gamma^{\mu}(\partial_{\mu} - \omega_{\mu})\psi = 0$, inside and outside the horizon. We defined γ^{μ} and ω_{μ} in chapter four, one has to find these quantities outside the horizon which is described by the Schwarzschild metric and inside the horizon which is explained by the Kruskal metric. After that one can compute the interaction term of the action at r_B and r_A. The vacuum expectation value of the interaction term of the action in the quasi-local volume around the horizon should give us the flux of Hawking radiation, however, this work is not complete yet.

As we mentioned before, Hawking radiation has not been detected in the lab yet. However, some physicists hope they can simulate similar effects in the lab. One of the recent suggestions is to simulate the horizon using a curved Graphene sheet. Graphene is a sheet of carbon atoms. By simulating the horizon one can see Hawking radiation emerging from the graphene sheet. However, there is still debate among physicists whether it can be done practically. These are some of the projects in progress based

on this thesis, [31], [32].

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