CONFORMAL FIELD THEORY
AND
LIE ALGEBRAS

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To my family,

my wife Radmila and daughter Vida,

for all their love and support.

ABSTRACT

Conformal field theories (CFTs) are intimately connected with Lie groups and their Lie algebras. Conformal symmetry is infinite-dimensional and therefore an infinite-dimensional algebra is required to describe it. This is the Virasoro algebra, which must be realized in any CFT. However, there are CFTs whose symmetries are even larger then Virasoro symmetry. We are particularly interested in a class of CFTs called Wess-Zumino-Witten (WZW) models. They have affine Lie algebras as their symmetry algebras. Each WZW model is based on a simple Lie group, whose simple Lie algebra is a subalgebra of its affine symmetry algebra.

This allows us to discuss the dominant weight multiplicities of simple Lie algebras in light of WZW theory. They are expressed in terms of the modular matrices of WZW models, and related objects. Symmetries of the modular matrices give rise to new relations among multiplicities. At least for some Lie algebras, these new relations are strong enough to completely fix all multiplicities.

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Contents

1. Introduction

It is said that most physical systems possess some kind of *symmetry*. What does it mean and how does it help us solve problems involving those systems? To answer this question let us first discuss the meaning of the word "symmetry" in everyday life. When we say that a sphere is a symmetric object we mean that if it is subjected to a rotation around any axis passing through its center, it does not change. In short, the symmetry of an object implies its invariance under some transformations, which in our example are rotations.

A symmetry can be discrete or continuous. As an example of discrete symmetry, we can consider the rotations of a regular n-sided polygon through an angle $2\pi l/n$ ($l = 0, 1, \ldots, n-1$) about an axis through the centre and normal to the plane of the polygon. Any such rotation transforms the polygon into itself. The angle of rotation takes values from a discrete set and the number of possible rotations is finite. However, the rotation angles that leave a circle unchanged have continuous values and the number of independent rotations is infinite. This symmetry is therefore a continuous symmetry.

These different types of transformations have the common property that they form a ***group***. This means that two successive transformations give another one, called the product transformation. Furthermore, there is an identity transformation, which is simply to do nothing. Finally, each transformation has its inverse, which undoes the transformation, i.e., the product of a transformation and its inverse is the identity. Groups describing discrete symmetries are called discrete groups and groups describing continuous symmetries are called continuous (Lie) groups. The continuous group of rotations described above is called compact. That is because the group parameters are rotation angles which have values in a compact range (i.e., one that is both closed and bounded).

Symmetries, and therefore group theory, are widely used in theoretical physics especially because of the structure of quantum mechanics. They have proven to be an elegant and very powerful tool in describing physical phenomena [1][2][3]. For example, if the underlying dynamics of a physical system exhibits some kind of symmetry, conservation laws follow, and conservation laws give us constants of motion that can be measured in experiments.

Some symmetries are obvious but others are hidden in actual physical phenomena. It is one objective of theoretical physics to uncover symmetries. This is not always an easy task because symmetries can be of very different nature. In the paragraphs below we discuss some examples of symmetries in physics that illustrate the importance and variety of groups in physics. Some of these will also provide background for the symmetry that is our main topic.

The group of continuous rotations in 3-dimensional space is called $SO(3)$. Any element of this group can be represented by a matrix that is orthogonal (its transpose equals its inverse) and special (its determinant is one). Those matrices must obey a group multiplication law which is simply matrix multiplication. So instead of doing actual transformations we can simply do matrix multiplication and get the final result of multiple transformations. The set of these matrices together with their multiplication form a *representation* of the group $SO(3)$.

The example above introduces us to the theory of group representations (see [1][4][5], for example). Group theory is an abstract mathematical structure; general features of the group are relevant to all its representations. However, physical systems are only described by fixed representations. Consequently, to describe a real physical system whose symmetry transformations in some sense represent a group, representation theory must be used. It gives us exact mathematical expressions (matrices) describing given transformations in a chosen space. For example, representation theory is the dominant mathematical framework in describing transformations of a quantum mechanical systems. This theory enables us to classify energy levels, their degeneracy, and how this is changed when the symmetry is reduced. It also gives selection rules which tell us when certain matrix elements are zero, and in general defines relations between matrix elements, thus reducing to a minimum the number of independent quantities which have to be calculated. In short it enables us to exploit to the fullest the invariance of the underlying dynamics.

A water molecule is a good example of a physical system possessing some symmetries. Firstly, it is symmetric under exchange of its two hydrogen ions: this transformation doesn't affect the energy of the system. Secondly, there is a translation symmetry: the interaction of two ions depends only on their relative distance and not on their absolute position. In addition, the energy of the system is independent of its orientation in space. Therefore the Hamiltonian of the system is invariant under a set of coordinate transformations including reflections, translations and rotations.

The physical meaning of these symmetries is the following: the outcome of the experiment conducted on the water molecule is not affected by exchange of two hydrogen ions or by overall translations. Furthermore, the rotation invariance implies that a given experiment and its rotated version give the same result.

All the symmetries described above include some kind of spatial transformations on physical systems. In a relativistic theory 3-dimensional space is replaced by 4-dimensional space-time. Invariant transformations in 4-dimensional space-time frame form so called space-time symmetries. To describe those symmetries the Lorentz and Poincaré groups are used instead of $SO(3)$. These groups are not compact because boost velocities can have any magnitude less than *bat not equal* to the speed of light. This group parameter therefore, does not have values in a compact range.

Besides space-time symmetries there are also *internal* symmetries studied in nuclear and particle physics [2]. The strong interactions that keep particles together in a nucleus are charge-independent. This fact allows us to study particles in multiplets that have degenerate or nearly degenerate masses. For example, the proton and neutron can be described as upper and lower components of a 2-component object $N = (p, n)$ called the nucleon. The nuclear strong interaction is invariant under the interchange $p \leftrightarrow n$ and the small difference between the masses of these two particles is due to the electromagnetic interaction, which distinguishes between proton and neutron because the proton is charged, while the neutron is not. All strongly interacting particles fall into such multiplets. Internal symmetries transform those multiplets in an abstract "internal space" in contrast to real space-time. This symmetry, called *isospin invariance*, is described by the group $SU(2)$. Each isospin multiplet is looked upon as the realization of a representation of isospin $SU(2)$.

However, some of the physical phenomena observed at high energy accelerators could not be explained by $SU(2)$ symmetry. The explanation was that interactions between particles observed at higher energy became more symmetric. As a result global $SU(3)$ (flavour) symmetry as an underlying symmetry of strong interactions was introduced. Beside these two groups, there are higher-rank Lie groups $SU(N)$ $(N > 3)$ used as internal symmetry groups too. The $SU(N)$ groups are compact groups.

Other examples of Lie groups in physics occur in the Yang-Mills gauge theories that describe all the known forces except gravity [6]. It has been possible to unify weak and electromagnetic interactions (described together by the group $SU(2) \times$

$U(1)$). The strong interaction symmetry $SU(3)$ (colour) and $SU(2) \times U(1)$ are both local symmetries (elements of the group vary at each point in space time). To restore symmetry of the underlying dynamics in local field theory, new fields called Yang-Mills gauge fields had to be introduced. Some also believe that all these interactions can be described by a Yang-Mills gauge field theory with a Lie group such as $SU(5)$ or $SO(10)$ as a symmetry group. These theories are called Grand Unified Theories (GUTs) [2][6].

The common feature of the groups introduced above is that they are finite dimensional. However, besides these finite symmetries there are infinite-dimensional symmetries as well. Certain infinite dimensional symmetries occur in statistical physics in the study of second order phase transitions in different physical systems in two dimensions ($2d$) [7] [8]. The correlation length, which measures the distance over which degrees of freedom in a given system significantly interact, rapidly increases close to a critical point at which a phase transition occurs. This increase of the correlation length is limited only by impurities and other defects in the sample. Otherwise, it is supposed to diverge to infinity at the critical point. Therefore it can be very difficult to analyse such systems close to the critical point because so many degrees of freedom are coupled together.

The two most studied examples of such critical phenomena are the liquid-gas phase transition at the critical point and the ferromagnetic phase transition. The ferromagnetic phase transition can be theoretically modelled using the Ising model, in which a real sample is represented by a square lattice of up and down spins (see [9][8], for example).

The macroscopic, measurable, long-distance properties of the system can be described by a field theory. At the critical point the system is invariant under scale transformations because of the infinite correlation length. Mathematically, scale transformations of the complex z-plane correspond to a rescaling of the coordinates $z \rightarrow \lambda z$, where λ is a constant. Furthermore, Polyakov showed that an invariance even larger than scale invariance, known as conformal symmetry, exists at the critical point [10]. The rationale behind Polyakov's assertion was that those properties of systems with short-range interactions (such as the Ising model) which lead to scale invariance should not be sensitive to variations in λ as long as they only vary over distance scales much larger than the lattice spacing. Therefore in the symmetry transformations the rescaling factor λ can be made an analytic function of position: $z \rightarrow \lambda(z)z$. Conformal symmetry follows, since in the complex plane conformal transformations are the mappings $z \rightarrow z' = f(z)$ where $f(z)$ is any analytic function. Any field theory invariant under conformal transformations is called a Conformal Field Theory (CFT). Consequently, critical phenomena in $2d$ should be describable by CFTs. Since any analytic function can be expressed as an infinite Laurent series, any conformal transformation is described by an infinite number of coefficients and so the conformal symmetry is infinite dimensional.

In this thesis we deal more often with the algebras of groups than with the groups themselves. Algebras of groups describe small parameter transformations. In the case of continuous groups where any transformation is continuously connected to the identity, algebras tell us almost everything about groups. Algebras are built up from generators which define small parameter transformations and satisfy certain commutation relations. The most important property of those infinitesimal generators is that the matrices (representations) of all proper transformations (i.e., those continuously connected to the identity) can be obtained by exponentiation. The number of these independent generators is the dimension of the algebra. For example, the algebra of $SO(3)$ group has three generators corresponding to rotations about three axes. The algebra corresponding to the group of conformal transformations has an infinite number of generators and is called the Virasoro algebra (Vir). The physically relevant representations of the algebras we study define representations of a group.

The question is: how is CFT useful in describing real physical systems? In 1984 it was discovered by Belavin, Polyakov and Zamolodchikov that there are CFTs describing critical phenomena in two dimensions with only a finite number of representations [11]. So the problem of an infinite number of degrees of freedom in critical phenomena gets much simpler using CFT.

Other infinite dimensional groups, affine Lie groups, and their algebras are discussed in the preliminary section of the thesis. They are the symmetries that are the main objects of study in this thesis. We introduce them because of their close connection to the compact finite dimensional Lie groups whose weight multiplicities we calculate. Besides, affine Lie algebras are symmetry algebras of a very important class of CFT's called Wess-Zumino-Witten (WZW) models [12][13]. They are important because their existence illustrates that there are CFTs with larger symmetries containing the Virasoro symmetry. It is also believed that all rational CFTs can be obtained from WZW models using so called "coset constructions" (see [14]).

Affine Lie algebras are natural generalisations of simple Lie algebras of any rank. To every simple Lie algebra we can associate an affine Lie algebra by adding an extra node, related to the highest root, to the Coxeter-Dynkin diagram of the simple Lie algebra. The introduction of this particular simple root has the effect of making the root system infinite. Affine Lie algebras have both a classification and a representation theory similar to those of simple Lie algebras. CFT is the most natural subject in which to apply affine Lie algebras because the Sugawara construction associates to any affine Lie algebra a Virasoro algebra, the algebra of two-dimensional conformal invariance. This means that any two-dimensional system with affine Lie algebra symmetry is automatically conformally invariant.

In the paragraphs above several different symmetries and the groups describing them were introduced. Even though affine Lie algebras are generalisations of simple Lie algebras their structures are quite different. The symmetries of WZW models that give rise to affine Lie algebras are infinite-dimensional and local (generators of transformations depend on coordinates). It is not then too surprising that affine Lie algebras are realised in CFT (as symmetry algebras of WZW models), since CFTs have local symmetry transformations. The other relevant algebra describing an infinite-dimensional local symmetry is the Virasoro algebra. The affine Lie algebra and Virasoro algebra are related to each other in the sense that universal enveloping algebra of an affine Lie algebra (the algebra of all products of affine Lie algebra generators) contains the Virasoro algebra [12]. The existence of these different symmetries and groups should enable us to get a more complete understanding of the possible fundamental laws of nature studied in physics.

The WZW models realise a current algebra equivalent to an affine Kac-Moody algebra (see [14]). The fields of the WZW models correspond to the representations of affine Kac-Moody algebras. More precisely, we can define the primary fields of WZW models so that they are in one-to-one correspondence with unitary highest weight representations of affine Kac-Moody algebras [13][15]. Also the energy operator of WZW field theory commutes with the semi-simple Lie algebra which is a subalgebra of the affine Kac-Moody algebra (see [12], for example).

This close connection between WZW field theory and semi-simple Lie algebra enables us to calculate the weight multiplicities of semi-simple Lie algebras (and their inverses) using matrices relevant to WZW models [16]. The weight multiplicities are very important numbers in the representation theory of Lie groups (see [4][5]). Physically, they give us the possible quantum numbers of a quantum system, for example. Computationally, they are useful in decomposing direct products of representations, and in decomposing representations in terms of subgroup representations, for example. We also find new relations among the multiplicities that are consequences of the affine-simple Lie algebra connection.

The remaining chapters are organized as follows. Chapter 2 is devoted to introducing mathematical tools necessary to explain the main results, the multiplicity relations of chapter 5. References [1][2][6][12][17] are mainly used. In chapter 3 the work of Patera and Sharp [18] is reviewed, so that it can be adapted to the use of Kac-Peterson modular matrices in chapter 4. There we write a new expression for the dominant weight multiplicities of semi-simple Lie algebras. The symmetries of the Kac-Peterson modular S matrix, and the even Weyl sums $E^{(n)}$ we introduce in chapter 4, are written down in chapter 5. The relations between the multiplicities that follow are also given. Chapter 5 also contains some simple explicit examples of the new relations among multiplicities, and chapter 6 is our conclusion. In the Appendix explicit values of multiplicity matrices and their inverses are listed for the groups $SU(3)$ and G_2.

Finally, it should be mentioned that some of the results in this thesis have already been published in [16].

2. Notations and preliminaries

2.1. Simple Lie groups and Lie algebras

A Lie group G is a group in which the elements are labeled by a set of continuous parameters with a multiplication law that depends smoothly on the parameters. Any group element which can be obtained from the identity by continuous changes in the parameters can be written as

$$\exp\{\alpha_a T^a\} \tag{2.1}$$

where α_a ($a = 1, \ldots, \dim G$) are real parameters, T^a are linearly independent hermitian operators, and a sum over the repeated index a is implied. The set of all linear combinations $\alpha_a T^a$ is a vector space, and the operators T^a form a basis in this space.

There are two nice things about these operators. The first is simple: they form a vector space. The second is that they satisfy simple commutation relations

$$[T^a, T^b] = f^{ab}_{\ \ c} T^c \ . \tag{2.2}$$

The operators T^a and the commutation relations above define the Lie algebra g associated with the Lie group G. The T^a are called *generators* of the algebra g. It is also required that for any $x, y, z \in g$ the Jacobi identity

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \tag{2.3}$$

and the antisymmetry property

$$[x, y] = -[y, x] \tag{2.4}$$

must be satisfied. The Lie algebra is called simple if it has no nontrivial invariant subalgebra. We will deal mainly with semi-simple Lie algebras. Such algebras are isomorphic to direct products of simple Lie algebras and don't have Abelian invariant subalgebras, i.e., subalgebras whose generators commute with all other generators of the given algebra.

The constants $f^{ab}_{\ c}$ are called structure constants of the algebra. The properties (2.3) and (2.4) expressed through the structure constants read

$$f^{ab}_{\ \ c}f^{cd}_{\ \ e} + f^{da}_{\ \ c}f^{cb}_{\ \ e} + f^{bd}_{\ \ c}f^{ca}_{\ \ e} = 0 \quad , \tag{2.5}$$

$$f^{aa}_{\ \ b} = 0 \quad , \tag{2.6}$$

and

$$f^{ab}_{\ \ c} = -f^{ba}_{\ \ c} \quad , \tag{2.7}$$

respectively.

If we define a set of matrices $\tilde{T}^a$ as

$$(\tilde{T}^a)^b_c = f^{ab}_{\ \ c} \tag{2.8}$$

then (2.5) can be rewritten as

$$[\tilde{T}^a, \tilde{T}^b] = f^{ab}_{\ \ c}\tilde{T}^c \ , \tag{2.9}$$

i.e., (2.2) is recovered. Therefore the structure constants themselves generate a representation of the algebra. The representation generated by the structure constants is called the adjoint representation. The dimension of a representation is the dimension of the vector space on which it acts. The dimension of the adjoint representation is just the number of generators and therefore the dimensionality of the adjoint matrix representaton is equal to the dimension of the algebra.

The Killing form of g is defined as a map

$$(x, y) \mapsto k(x, y) := tr(R_{ad}(x)R_{ad}(y)) \tag{2.10}$$

where $x, y \in g$, $R_{ad}(x)$ denotes element x in the adjoint representation and "tr" denotes the trace of matrices. The metric relating upper and lower indices of the structure constants, called the Cartan metric, is given by the Killing form. For example, structure constants with only upper indices are defined as

$$f^{abc} := f^{ab}{}_{d} k^{dc} \tag{2.11}$$

where

$$k^{dc} := k(T^d, T^c) = f^{da}{}_{b} f^{cb}{}_{a} \ . \tag{2.12}$$

Now we will choose a special type of basis called the Cartan-Weyl basis in which the structure constants can be written in a canonical way. First choose a maximal set of linearly independent generators H^i of g possessing zero brackets among themselves,

$$[H^i, H^j] = 0 \text{ for } i, j = 1, \dots, r \ . \tag{2.13}$$

The set of the generators H^i form a maximal Abelian subalgebra g_0 of g called the Cartan subalgebra. The number r of generators H^i is called the rank of g and it is at the same time the dimension of g_0. For a given Cartan subalgebra g_0, the remaining generators E^α of g can be chosen such that they are eigenvectors of g_0 in the sense that

$$[H^i, E^\alpha] = \alpha^i E^\alpha \text{ for } i = 1, \dots, r \ . \tag{2.14}$$

The r-dimensional vector $(\alpha^i)_{i=1,\dots,r}$ of eigenvalues is called a root of g and the eigenvalues α^i themselves are called the Dynkin labels (introduced below) of the given root. The set of all roots of a semi-simple Lie algebra will be denoted by Φ. Therefore the Cartan-Weyl basis of the semisimple Lie algebra g is given as

$$\{H^i | i = 1, \dots, r\} \cup \{E^\alpha | \alpha \in \Phi\} \ . \tag{2.15}$$

In the chosen basis for the root space, the root α is called a positive root if the first component of α in this basis is positive; otherwise α is called a negative root. The sets of positive and negative roots are denoted respectively by

$$\Phi_+ := \{\alpha \in \Phi | \alpha > 0\} \text{ and } \Phi_- := \Phi \setminus \Phi_+ \ . \tag{2.16}$$

Two fundamental properties of the root system Φ are the following: the roots are not degenerate, and if α is a root, then $-\alpha$ is also a root, but no other multiple of α is a root. For a given Cartan subalgebra g_0 of g, the subalgebras of g generated by the operators E^α for α any positive or negative root are denoted by g_+ and g_-, respectively. Therefore, g can be written as the (non-direct) sum

$$g \ = \ g_+ \oplus g_0 \oplus g_- \ . \tag{2.17}$$

This is called the *Cartan decomposition* of g.

For a given set of positive roots we can choose a basis of the root system, i.e., a set of linearly independent roots that span the whole of Φ. Those roots we will call simple roots $\alpha^{(i)}$. Generically they are not orthonormal and this feature is encoded in the so-called Cartan matrix defined as an $r \times r$ - matrix with elements

$$A^{ij}(g) := 2\frac{(\alpha^{(i)}, \alpha^{(j)})}{(\alpha^{(j)}, \alpha^{(j)})} \ . \tag{2.18}$$

The scalar product $(*, *)^\dagger$ above is related to the Killing form (2.10) by

$$k(H^i, H^j) = 2\sum_k \frac{(\alpha^{(k)}, \alpha^{(i)})(\alpha^{(k)}, \alpha^{(j)})}{(\alpha^{(i)}, \alpha^{(i)})(\alpha^{(j)}, \alpha^{(j)})} \ . \tag{2.19}$$

We now define the dual root or coroot of α by

$$\alpha^\vee := \frac{2\alpha}{(\alpha, \alpha)} \ , \tag{2.20}$$

† For notational convenience $* \cdot *$ is also used to denote a scalar product.

so that the Cartan matrix elements can be written as

$$A^{ij} = (\alpha^{(i)}, \alpha^{(j)\vee}) \ . \tag{2.21}$$

If we choose simple co-roots as a basis of the root space

$$\{\alpha^{(i)\vee} | i = 1, \dots, r\} \tag{2.22}$$

then the fundamental weights $\Lambda_{(i)}$ are defined as the r 1-forms obeying

$$(\Lambda_{(i)}, \alpha^{(j)\vee}) = \delta_i^j \ . \tag{2.23}$$

The fundamental weights form the basis of the weight space

$$\{\Lambda_{(i)} | i = 1, \dots, r\} \tag{2.24}$$

which is dual to (2.22). This basis is called the Dynkin basis and the components of a weight in the Dynkin basis are called Dynkin labels. Therefore any weights λ can be decomposed into its components with respect to (2.22) and (2.24) as

$$\lambda = \lambda_i \alpha^{(i)\vee} = \lambda^j \Lambda_{(j)} \ . \tag{2.25}$$

The matrices which raise and lower Dynkin indices

$$\lambda_i = G_{ij} \lambda^j, \ \lambda^i = G^{ij} \lambda_j, \ G_{ij} G^{jk} = \delta_i^k \tag{2.26}$$

are given by

$$G_{ij} = (\Lambda_{(i)}, \Lambda_{(j)}) \ , \tag{2.27}$$

and

$$G^{ij} = (\alpha^{(i)\vee}, \alpha^{(j)\vee}) = \frac{2}{(\alpha^{(i)}, \alpha^{(i)})} A^{ij} \ . \tag{2.28}$$

As a result, the Dynkin labels of the simple roots are given by the rows of the Cartan matrix

$$(\alpha^{(i)})^j = A^{ij} \ . \tag{2.29}$$

In the case of simple Lie algebras there is a unique root θ such that $\theta - \alpha$ is a positive root for any $\alpha \in \Phi\backslash\{\theta\}$. The root θ is called the highest root of g and its expansions in the bases of simple roots and simple coroots,

$$\theta := \sum_{i=1}^{r} a_i \alpha^{(i)}, \quad \theta^\vee := \sum_{i=1}^{r} a_i^\vee \alpha^{(i)\vee} \ . \tag{2.30}$$

define Coxeter labels (a_i) and dual Coxeter labels $(a_i^\vee)$ of g. The Coxeter number and dual Coxeter number of g are the sums

$$h := 1 + \sum_{i=1}^{r} a_i, \quad h^\vee := 1 + \sum_{i=1}^{r} a_i^\vee \ . \tag{2.31}$$

If we restrict to the simple Lie algebras it can be shown that the Cartan matrix possesses the following properties:

$$\begin{aligned}
&a)\ A^{ii} = 2,\\
&b)\ A^{ij} = 0 \Leftrightarrow A^{ji} = 0,\\
&c)\ A^{ij} \in \mathbb{Z}_{\leq 0}\ for\ i \neq j,\\
&d)\ \det A > 0,\\
&e)\ indecomposability\ .
\end{aligned} \tag{2.32}$$

All possible solutions of the set of equations above give us the classification of simple Lie algebras.

Another way to classify the simple Lie algebras is through the Coxeter-Dynkin diagrams. Each simple Cartan matrix can be represented by a diagram consisting of nodes and lines connecting them. Each node of the diagram represents a simple root or a simple weight and they are connected by $\max\{|A^{ij}|, |A^{ji}|\}$ lines. Because only two different lengths are allowed for the roots of any given simple Lie algebra we can distinguish them if we denote short roots by filled dots ($\bullet$) and long roots by open dots ($\circ$). Figure 1 shows the Coxeter-Dynkin diagrams of all the simple Lie algebras.

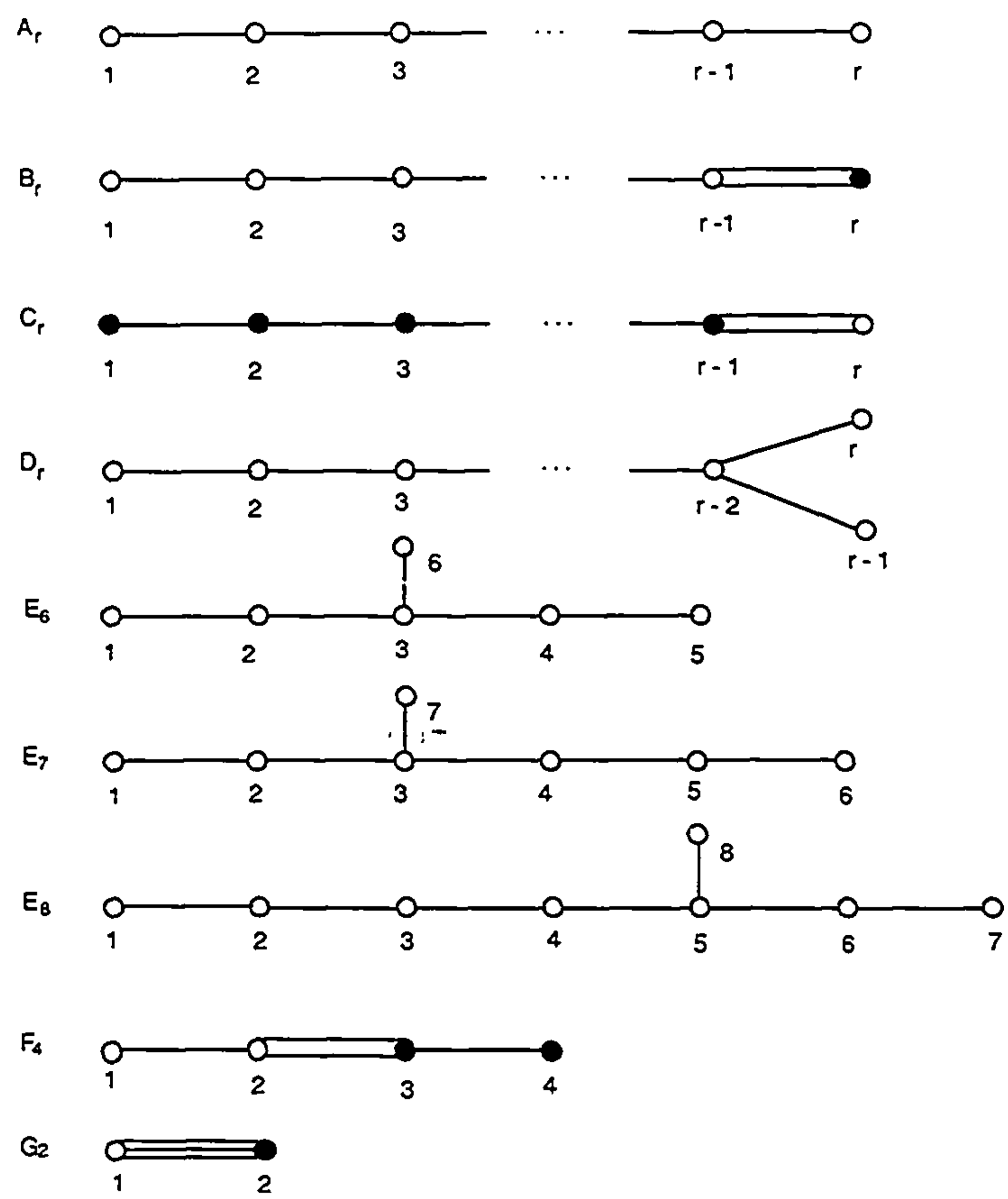

Figure 1. The Coxeter-Dynkin diagrams of simple Lie algebras.

2.2. The Weyl group and the characters of simple Lie algebras

Suppose Φ is a set of roots of a simple Lie algebra g and α, β are an arbitrary pair of roots from Φ. Now we can define a reflection w_α, that takes roots into roots as

$$w_\alpha : \begin{array}{c} \Phi \to \Phi \\ w_\alpha(\beta) = \beta - (\beta \cdot \alpha)\alpha \end{array} \quad . \tag{2.33}$$

The operation w_α geometrically means just a reflection of β across the hyperplane in weight space (through zero) perpendicular to α. Any of these reflections possesses an inverse (namely itself), and the composition of reflections leads to reflections as well as rotations. As a consequence, the maps w_α, $\alpha \in \Phi$, generate a discrete group; this group is called the Weyl group of g and will be denoted by W. Any element of W can be written as a composition of the fundamental reflections

$$w_{(i)} := w_{\alpha^{(i)}} \tag{2.34}$$

where $\alpha^{(i)}$ are simple roots and $i = 1, ..., r$ ($r := \text{rank}(g)$). The most important application of the Weyl group is a calculation of the characters of g. For any highest weight representation R_Λ of a simple Lie algebra g, the character $\chi_\Lambda := \chi_{R_\Lambda}$ is by definition the map

$$\chi_\Lambda : \begin{array}{c} g_0 \to \mathbf{C} \\ h \mapsto \chi_\Lambda(h) := \text{tr}\ \exp(R_\Lambda(h)) \end{array} \quad . \tag{2.35}$$

Here g_0 is a Cartan subalgebra of g and $h \in g_0$. The Cartan subalgebra g_0 can be identified with a weight space, i.e., any element h of g_0 can be written as $h = \mu_i H^i$, where μ_i are the Dynkin labels of a weight μ. If we define $\chi_\Lambda(\mu) := \chi_\Lambda(\mu_i H^i)$, then (2.35) yields:

$$\chi_\Lambda(\mu) = \sum_{\lambda \in P(\Lambda)} \text{mult}_\Lambda(\lambda) \exp(\lambda \cdot \mu) \quad . \tag{2.36}$$

Here $P(\Lambda)$ is the set of weights of the irreducible representation R_Λ of highest weight Λ and $\text{mult}_\Lambda(\lambda)$ is multiplicity of weight λ in this highest weight representation. Using the behavior of the weights λ under Weyl reflections we can derive the Weyl character formula (see [4], for example)

$$\chi_\Lambda(\mu) = \frac{\sum_{w \in W} \det(w) \exp[w(\Lambda + \rho) \cdot \mu]}{\exp(\rho \cdot \mu) \prod_{\alpha > 0} [1 - \exp(-(\alpha \cdot \mu))]} \tag{2.37}$$

Here ρ is the Weyl vector

$$\rho = 1/2 \sum_{\alpha \in \Phi_+} \alpha = \sum_{i=1}^{r} \Lambda_{(i)} \ , \tag{2.38}$$

and $\det(w)$ is sign of the w reflection given by $\det(w) = (-1)^{l(w)}$; $l(w)$ is the length of w in terms of fundamental reflections, i.e., the lowest number of $w_{(i)}$ so that w can be written as a composition of them. In the equation above Φ_+ is a set of positive roots and $\Lambda_{(i)}$ are fundamental weights. The denominator of (2.37) can also be written in the form

$$\exp(\rho \cdot \mu) \prod_{\alpha > 0} [1 - \exp(-(\alpha \cdot \mu))] = \sum_{w \in W} \det(w) \exp[w(\rho) \cdot \mu] \tag{2.39}$$

because $\chi_0 := 1$. This is known as the denominator identity. The definition (2.35), applied to direct sums and Kronecker products of irreducible representations, leads to

$$\chi_{\oplus_i R_{\Lambda_i}} = \sum_i \chi_{\Lambda_i} \quad and \qquad \chi_{R_\Lambda \times R_{\Lambda'}} = \chi_\Lambda \cdot \chi_{\Lambda'} \ . \tag{2.40}$$

2.3. Affine Lie algebras

Simple Lie algebras are finite dimensional. However many interesting systems possess infinitely many independent symmetries and therefore infinite-dimensional Lie algebras are also important.

We are here particularly interested in a class of infinite-dimensional Lie algebras called affine Kac-Moody Lie algebras. They are also termed affine Lie algebras, or simply affine algebras. Affine Kac-Moody algebras are algebras of smooth mappings of the circle S^1 into a finite-dimensional Lie algebra that also allow for a so-called central extension of the algebra.

A finite-dimensional simple Lie algebra is completely characterised by $3r$ generators $\{E^i_\pm, H^i | i = 1, ..., r\}$ obeying the Jacobi identity (2.3):

$$[E_+, [E_-, H]] + [E_-, [H, E_+]] + [H, [E_+, E_-]] = 0 \tag{2.41}$$

for any $H \in g_0$, $E_+ \in g_+$, $E_- \in g_-$, and the relations

$$
\begin{aligned}
&[H^i, H^j] = 0 \ , \\
&[H^i, E^j_\pm] = \pm A^{ji} E^j_\pm \ , \\
&[E^i_+, E^j_-] = \delta^{ij} H^j \ , \\
&(ad_{E^i_\pm})^{1-A^{ji}} E^j_\pm = 0 \ \text{ for } \ i \neq j \ .
\end{aligned}
\tag{2.42}
$$

Here $(ad_x)^n$ is a shorthand notation for $\underbrace{ad_x \circ ad_x \circ \ldots \circ ad_x}_{n \text{times}}$ so that, e.g., $(ad_x)^2(y) := [x,[x,y]]$. The matrix A above is an irreducible Cartan matrix that obeys

$$
\begin{aligned}
&A^{ii} = 2 \ , \\
&A^{ij} \leq 0 \ \text{ for } \ i \neq j \ , \\
&A^{ij} = 0 \ \Leftrightarrow \ A^{ji} = 0 \ , \\
&A^{ij} \in \mathbb{Z} \ ,
\end{aligned}
\tag{2.43}
$$

and

$$
\det(A) \ > \ 0 \ . \tag{2.44}
$$

If we remove the condition $\det(A) > 0$ on the matrix A of simple Lie algebras we get the general class of Kac-Moody algebras. The most important subclass of Kac-Moody algebras is obtained if the constraint $\det(A) > 0$ is replaced by

$$
\det(A_{\{i\}}) > 0 \text{ for all } i = 0, .., r \ , \tag{2.45}
$$

where $A_{\{i\}}$ are the matrices obtained from A by deleting the ith row and column. An irreducible Cartan matrix that satisfies (2.43) and (2.45) is called an affine Cartan matrix. The Lie algebras defined by generators and relations as in (2.42), with A an affine Cartan matrix, are the affine Lie algebras or affine Kac-Moody algebras. All possible solutions of the constraints for an irreducible affine Cartan matrix give us the classification of affine Lie algebras.

The affine Lie algebras we are interested in (called *untwisted* affine Lie algebras) can be obtained as a generalization of simple Lie algebras which allow for a nontrivial central extension. Let us consider the space of analytic maps from the circle S^1 to a simple Lie algebra g. If $\{T^a|a = 1,...,d\}$ is a basis of g and S^1 is the unit circle in the complex plane with coordinate z, then a basis of this vector space is

$$\{T^a_n|a = 1,...,d; n \in \mathbb{Z}\} \ , \tag{2.46}$$

where $T^a_n := T^a \otimes z^n$ with $\otimes$ a formal multiplication. The commutator of g,

$$[T^a, T^b] = f^{ab}_{\ \ c}T^c \ \ , \tag{2.47}$$

applied on this new space leads to

$$[T^a_m, T^b_n] = [T^a \otimes z^m, T^b \otimes z^n] := [T^a, T^b] \otimes (z^m \cdot z^n) \ \ , \tag{2.48}$$

i.e.,

$$[T^a_m, T^b_n] = f^{ab}_{\ \ c}T^c \otimes z^{m+n} = f^{ab}_{\ \ c}T^c_{m+n} \ \ , \tag{2.49}$$

where $f^{ab}_{\ \ c}$ are the structure constants of g. The space of analytic maps from S^1 to g with this commutation rule is the infinite-dimensional Lie algebra called the loop algebra g_{loop} over g.

There is a non-trivial central extension $\tilde{g}$ of g_{loop} characterised by the commutators

$$\begin{aligned} &[T^a_m, T^b_n] = f^{ab}_{\ \ c}T^c_{m+n} + m\delta_{m+n,0}k^{ab}K \\ &[K, T^a_n] = 0 \end{aligned} \tag{2.50}$$

among the generators T^a_n and K. Here k^{ab} is the Cartan metric defined by (2.12).

The affine algebra $\hat{g}$ is obtained from $\tilde{g}$ by adding one further generator D and new commutators

$$\begin{aligned} &[D, T^a_m] = -[T^a_m, D] = mT^a_m \\ &[D, K] = 0 \ . \end{aligned} \tag{2.51}$$

From the equations (2.50) and (2.51) it can be shown that a maximal abelian subalgebra of $\hat{g}$, $\hat{g}_0$, is generated by

$$\{K, D, H_0^i | i = 1, 2, ..., r\} \quad . \tag{2.52}$$

The roots of $\hat{g}_0$ can be found from the following commutators:

$$[H_0^i, E_n^\alpha] = \alpha^i E_n^\alpha, \ [K, E_n^\alpha] = 0, \ [D, E_n^\alpha] = n E_n^\alpha \ , \tag{2.53}$$

and

$$[H_0^i, H_n^j] = [K, H_n^j] = 0, \ [D, H_n^j] = n H_n^j \ , \tag{2.54}$$

where α is any root of g and $n \in \mathbb{Z}$. Therefore the roots with respect to (H_0, K, D) are

$$\hat{\alpha} = (\alpha, 0, n), \ \alpha \in \Phi(g), \ n \in \mathbb{Z} \ , \tag{2.55}$$

and

$$\hat{\alpha} = (0, 0, n), \ n \in \mathbb{Z} \setminus \{0\} \quad , \tag{2.56}$$

corresponding to the generators E_n^α and H_n^j, $n \neq 0$, respectively†. The roots (2.55) are non-degenerate while the roots (2.56) are r-fold degenerate because they do not depend on the label j of H_n^j. These non-degenerate roots are called *real* roots whereas the degenerate ones are called *imaginary* (and sometimes *light-like*) roots and their sets are denoted by $\hat{\Phi}_r$ and $\hat{\Phi}_i$ respectively.

The next thing to do is to define positive roots and identify a set of simple roots. The set of positive roots can be defined by

$$\hat{\Phi}_+ := \{\hat{\alpha} = (\alpha, 0, n) \in \hat{\Phi} | n > 0 \text{ or } (n = 0, \alpha \in \Phi_+)\} \quad . \tag{2.57}$$

For this choice of positive roots the simple roots are:

$$\alpha^i = (\alpha^{(i)}, 0, 0) = \alpha^{(i)} \text{ for } i = 1, ..., r \ , \tag{2.58}$$

† Notice that $\hat{\alpha}$ denotes a root of $\hat{g}$ while α is a root of g.

and

$$\alpha^0 = (-\theta, 0, 1) = \delta - \theta \ . \tag{2.59}$$

Here θ is the highest root of g, and $\delta = (0,0,1)$. Notice the difference in notation: the affine simple roots are $\{\alpha^i\}$, and the finite simple roots are $\{\alpha^{(i)}\}$!

Analogously to the simple Lie algebras, the classification of affine Lie algebras can be done through affine Coxeter-Dynkin diagrams (see Fig. 2). The numbering of the nodes of the affine Dynkin diagram would be the same as for Dynkin diagrams of simple Lie algebras, with the additional node representing α^0 or Λ_0. Removal of the additional node reduces the Dynkin diagram of affine Lie algebra to that of simple Lie algebra.

It can be shown that the simple roots provide a basis for the root space; i.e., they are linearly independent and span the whole of $\hat{\Phi}(\hat{g})$. But instead of simple roots as a basis of the root space, we can choose simple coroots

$$\{\alpha^{i\vee} | i = 0, 1, \ldots, r\} \tag{2.60}$$

where

$$\alpha^{i\vee} := \frac{2\alpha^i}{(\alpha^i, \alpha^i)} \ . \tag{2.61}$$

Then we can also define the matrix

$$A^{ij} := (\alpha^i, \alpha^{j\vee}), \ i, j = 0, \ldots, r \ . \tag{2.62}$$

This is the same affine Cartan matrix introduced at the beginning of this chapter. Analogously to the simple Lie algebra the fundamental weights Λ_i of the affine Lie algebra are defined as

$$(\Lambda_i, \alpha^{j\vee}) = \delta_i^j \tag{2.63}$$

and therefore the basis of the weight space which is dual to (2.60) is

$$\{\Lambda_i | i = 0, 1, \ldots, r\} \ . \tag{2.64}$$

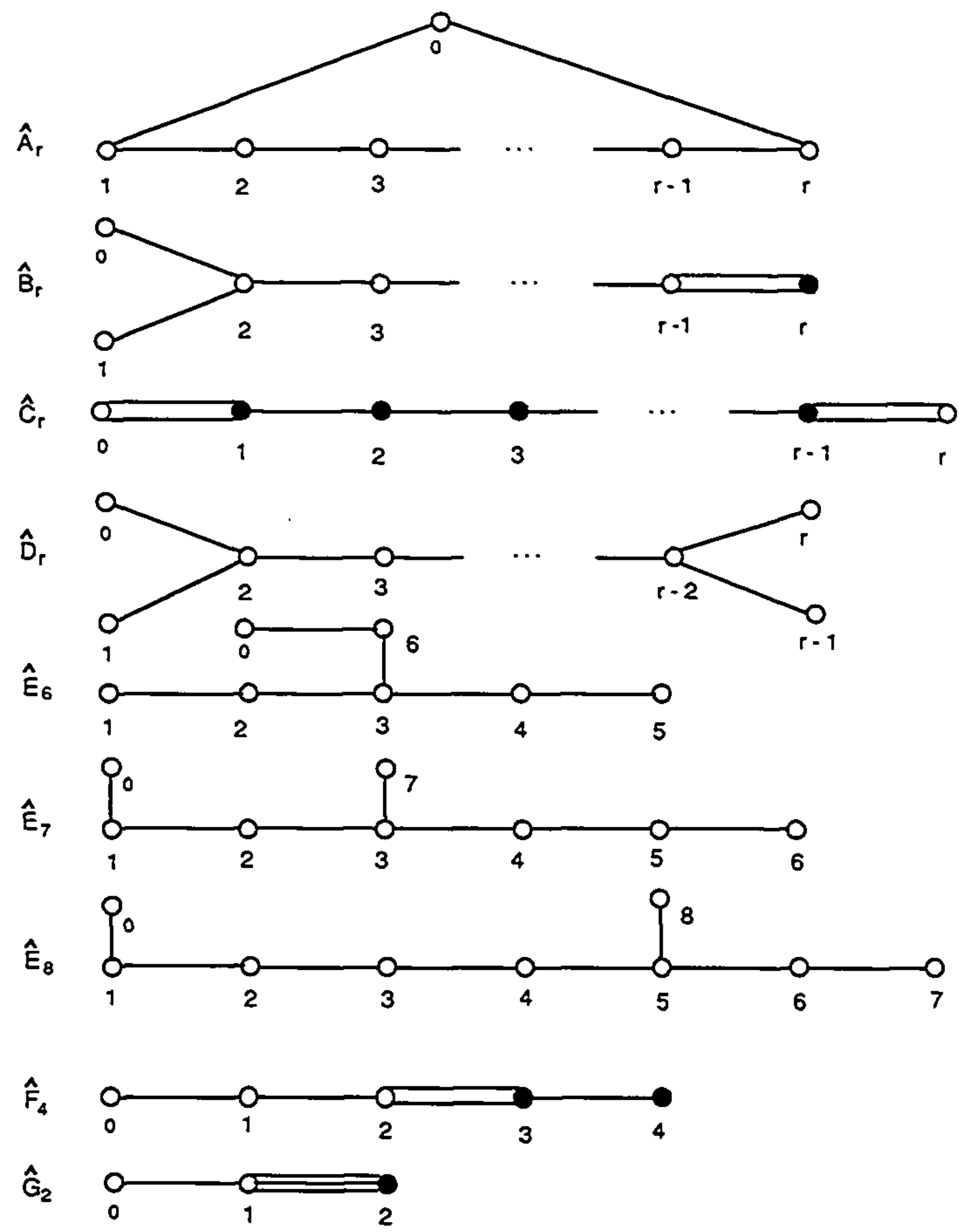

Figure 2. The Coxeter-Dynkin diagrams of untwisted affine Lie algebras.

The basis (2.64) is called the Dynkin basis and the components of a weight in the Dynkin basis are called Dynkin labels.

From the equation (2.63) it is obvious that both bases (2.60) and (2.64) span weight space. The transformation of the root (weight) components from one basis to the other is done by a metric

$$\hat{G}^{ij} = \begin{pmatrix} G^{ij} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.65}$$

where G^{ij} is defined by (2.28). Therefore the scalar product of the affine roots is equal to the scalar product of their g-components,

$$(\hat{\alpha}, \hat{\alpha}') = ((\alpha, 0, n), (\alpha', 0, n')) = (\alpha, \alpha') \ . \tag{2.66}$$

So for the non-degenerate roots

$$(\hat{\alpha}, \hat{\alpha}) > 0 \text{ for } \alpha \in \Phi \ , \tag{2.67}$$

while for the degenerate roots

$$(\hat{\alpha}, \hat{\alpha}) = 0 \text{ for } \hat{\alpha} = n\delta, \ n \neq 0 \ . \tag{2.68}$$

To avoid confusion in notation, Dynkin labels of an affine weight $\hat{\lambda}$ will be given as $[\lambda^0, \lambda^1, \dots, \lambda^r]$ while Dynkin labels for a finite weight λ will be written as $(\lambda^1, \lambda^2, \dots, \lambda^r)$. Notice that if $\Lambda_{(i)}$ and Λ_i are identified for $i = 1, \dots, r$, then we can write $\hat{\lambda} = \lambda^0 \Lambda_0 + \lambda$.

2.4. The Weyl group and the characters of affine Lie algebras

In analogy to simple Lie algebras the Weyl reflection $\hat{w}_{\hat{\alpha}}$ of an affine algebra is defined by

$$\hat{w}_{\hat{\alpha}}(\hat{\lambda}) := \hat{\lambda} - (\hat{\lambda}, \hat{\alpha}^\vee)\hat{\alpha} \ . \tag{2.69}$$

Here, $\hat{\alpha}$ is any real root ($\hat{\alpha}^\vee = \frac{2}{(\alpha,\alpha)}\hat{\alpha}$ and hence $\hat{\alpha}^\vee$ is not defined for imaginary roots). Since

$$(\hat{w}_{\hat{\alpha}}(\hat{\lambda}), \hat{\alpha}^\vee) = -(\hat{\lambda}, \hat{\alpha}^\vee) \quad , \tag{2.70}$$

$\hat{w}_{\hat{\alpha}}$ is a reflection across the hyperplane perpendicular to $\hat{\alpha}$. Therefore affine Weyl reflections, together with the identity map, generate a group under composition called the affine Weyl group.

Although most characteristics of the infinite-dimensional affine Weyl groups and finite-dimensional Weyl groups of simple Lie algebras are similar, there are some differences too. They are mainly caused by the existence of the imaginary roots. Since $(\hat{\alpha}, \delta) = 0$ for any real root $\hat{\alpha}$ then

$$\hat{w}_{\hat{\alpha}}(\delta) := \delta - (\delta, \hat{\alpha})\hat{\alpha}^\vee = \delta \tag{2.71}$$

and therefore the set of imaginary roots $\hat{\Phi}_i$ is not changed under the action of the Weyl group, i.e.,

$$\hat{w}_{\hat{\alpha}}|_{\hat{\Phi}_i} = id_{\hat{\Phi}_i} \quad . \tag{2.72}$$

Any reflection is an automorphism of the root lattice, so we have

$$\widehat{W}(\hat{\Phi}_r) = \hat{\Phi}_r \quad . \tag{2.73}$$

The affine Weyl group is generated by a finite set of fundamental transformations

$$w_i := w_{\alpha^i}, \ i = 0, 1, ..., r \tag{2.74}$$

which now include the reflection with respect to α^0.

Analogously to the case of simple Lie algebras the characters of affine Lie algebras are defined by

$$\hat{\chi}_{\hat{\Lambda}}(\hat{\mu}) := \sum_{\hat{\lambda}} \mathrm{mult}_{\hat{\Lambda}}(\hat{\lambda}) \exp(\hat{\lambda}, \hat{\mu}) \quad . \tag{2.75}$$

Since the relevant modules of affine Lie algebras are infinite-dimensional, the sum above is an infinite sum. Nevertheless it is possible to determine the characters more explicitly because the Weyl character formula (2.37) can be generalized for arbitrary Kac-Moody algebras. Consequently for any irreducible integrable highest weight module $R_{\hat{\Lambda}}$ the characters obey the Weyl-Kac character formula [19]

$$\hat{\chi}_{\hat{\Lambda}}(\hat{\mu}) = \frac{\sum_{\hat{w}\in\widehat{W}} \det(\hat{w})\exp[\hat{w}(\hat{\Lambda}+\hat{\rho})\cdot\hat{\mu}]}{\sum_{\hat{w}\in\widehat{W}} \det(\hat{w})\exp[\hat{w}(\hat{\rho})\cdot\hat{\mu}]} \tag{2.76}$$

with $\hat{\rho} = \sum_{i=0}^{r} \Lambda_i$, the Weyl vector of $\hat{g}$. The infinite sums appearing in this formula converge for appropriately chosen weights $\hat{\mu}$. There is also an analogue of the denominator identity:

$$\sum_{\hat{w}\in\widehat{W}} \det(\hat{w})\exp[\hat{w}(\hat{\rho})\cdot\hat{\mu}] = \exp(\hat{\rho}\cdot\hat{\mu}) \prod_{\hat{\alpha}>0} [1-\exp(-(\hat{\alpha}\cdot\hat{\mu}))]^{\mathrm{mult}(\hat{\alpha})} \quad . \tag{2.77}$$

The exponent $\mathrm{mult}(\hat{\alpha})$ appearing here takes into account the fact that the multiplicity of a root can now be larger than one.

The Weyl-Kac formula is valid for arbitrary Kac-Moody algebras but for affine algebras, where the multiplicities of the roots and the structure of the Weyl group are known explicitly, it can be exploited further.

2.5. Modular transformations

The characters of affine Lie algebras possess a simple transformation property with respect to the modular group $PSL_2(\mathbb{Z})$. This is a group of 2×2 matrices with integer entries and determinant one. In addition any such matrix can be identified with its negative. The modular group is generated by two elements S and T which satisfy

$$S^2 = 1, \quad (ST)^3 = 1 \quad . \tag{2.78}$$

In the defining two-dimensional representation, they can be written as

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad . \tag{2.79}$$

Thus the generating modular transformations of a given parameter τ are defined by

$$\begin{aligned} S &: \tau \longmapsto -1/\tau \\ T &: \tau \longmapsto \tau + 1 \quad . \end{aligned} \tag{2.80}$$

The action of an arbitrary element of the modular group

$$\tau \longmapsto \frac{a\tau + b}{c\tau + d} , \tag{2.81}$$

where $ad - bc = 1$ and $a, b, c, d \in \mathbb{Z}$, on the weight $\hat{\lambda} = 2\pi i(\zeta, \tau, t)$ is defined as

$$(\zeta, \tau, t) \longmapsto \left(\frac{\zeta}{c\tau + d}, \frac{a\tau + b}{c\tau + d}, t + \frac{c(\zeta, \zeta)}{2(c\tau + d)} \right) \quad . \tag{2.82}$$

The modular transformation properties of the affine characters take the following form

$$\begin{aligned} \hat{\chi}_{\hat{\lambda}}(\zeta, \tau + 1, t) &= \sum_{\hat{\mu} \in P_+^k} T_{\hat{\lambda}\hat{\mu}} \hat{\chi}_{\hat{\mu}}(\zeta, \tau, t) \\ \hat{\chi}_{\hat{\lambda}}(\zeta/\tau, -1/\tau, t + (\zeta, \zeta)/2\tau) &= \sum_{\hat{\mu} \in P_+^k} S_{\hat{\lambda}\hat{\mu}} \hat{\chi}_{\hat{\mu}}(\zeta, \tau, t) \quad . \end{aligned} \tag{2.83}$$

Here the characters of the set of integrable representations at a given level transform into each other under the action of the modular group, i.e., they form a module of the modular group while the corresponding matrices T and S generate a representation of the modular group. The explicit form of the matrix $S_{\hat{\lambda}\hat{\mu}}$ is [20]

$$S_{\hat{\lambda}\hat{\mu}} = i^{|\Phi_+|} |M/M^*|^{-1/2} (k + h^\vee)^{-r/2} \sum_{w \in W} \det(w) \exp[\frac{-2\pi i}{k + h^\vee}(w(\lambda + \rho) \cdot (\mu + \rho))] \, . \tag{2.84}$$

Notice that if $\hat{\mu}$ has level k ($\sum_{i=0}^r a_i^\vee \mu^i = k$) then $\hat{\mu} + \hat{\rho}$ has level $k + h^\vee$. In the equation above, $|M/M^*|$ is the number of points of the weight lattice M lying in an elementary cell of the coroot lattice M^*. This number equals the determinant of the matrix whose rows are the Dynkin labels of the coroots:

$$|M/M^*| = \det[(\alpha^{i\vee}, \alpha^{j\vee})] = \det[(\alpha^{i\vee})^j] \quad . \tag{2.85}$$

In a case of simply-laced algebras that have only single lines in the Coxeter-Dynkin diagrams ($\alpha^i = \alpha^{i\vee}; i = 0, 1, \cdots r$), this is the determinant of the Cartan matrix:

$$|M/M^*| = \det A^{ij} \quad . \tag{2.86}$$

$|\Phi_+|$ is the number of positive roots in the finite Lie algebra g. Using the basic properties of the Weyl group W it can be shown that the matrix S is unitary and symmetric [20]

$$S^{-1} = S^\dagger = S^* \quad . \tag{2.87}$$

Therefore the representation of the modular group on the set of characters of integrable modules is a unitary one.

Here are some simple examples of the modular matrix S. For $\hat{g} = \widehat{su}(2)$ and $k = 2$, since $|\Phi_+| = 1$, $|M/M^*| = 2$, $h = h^\vee = 2$ and $|\Lambda_1|^2 = 1/2$, the S matrix becomes

$$S_{\hat{\lambda}\hat{\mu}} = \left[\frac{2}{k+2}\right]^{\frac{1}{2}} \sin\left[\frac{\pi(\lambda+1)\cdot(\mu+1)}{(k+2)}\right] \quad . \tag{2.88}$$

On the other hand, the S matrix for $\hat{g} = \widehat{su}(3)$ and $k = 1$ is

$$S = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & t & t^2 \\ 1 & t^2 & t \end{pmatrix} \quad t = \exp[2\pi i/3] \ , \tag{2.89}$$

where the fields are ordered as $[1,0,0]$, $[0,1,0]$, $[0,0,1]$.

2.6. Conformal Field Theory

A conformal field theory is a field theory which is invariant under the group of conformal transformations. By definition, conformal transformations are the restricted general coordinate transformations, $x \longrightarrow \tilde{x}$, which preserve the angles between any two vectors, or equivalently, for which the metric is invariant up to a scale factor [12],

$$g_{\mu\nu}(x) \longrightarrow \tilde{g}_{\mu\nu}(\tilde{x}) = \exp[\omega(x)] g_{\mu\nu}(x) \quad . \tag{2.90}$$

The metric $g_{\mu\nu}(x)$ defines an invariant line element $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$, and it transforms under a finite transformation $x^\mu \longrightarrow \tilde{x}^\mu$ as a rank-2 symmetric tensor

$$g_{\mu\nu}(x) \longrightarrow \tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\lambda}{\partial \tilde{x}^\mu} \frac{\partial x^\rho}{\partial \tilde{x}^\nu} g_{\lambda\rho}(x) \quad . \tag{2.91}$$

For an infinitesimal reparametrisation

$$x^\mu \longrightarrow x^\mu + \epsilon f^\mu(x) \ , \quad (\epsilon \ll 1) \ , \tag{2.92}$$

the conservation of angles means that

$$d(\partial_\mu f_\nu + \partial_\nu f_\mu) - 2\delta_{\mu\nu} \sum_{\rho=1}^{d} \partial_\rho f_\rho = 0 \quad . \tag{2.93}$$

Here the space-time is Euclidean, and d is its dimension. For $d = 2$, the constraints (2.93) become the Cauchy-Riemann equations

$$\begin{aligned} \partial_0 f_0 &= \partial_1 f_1 \\ \partial_0 f_1 &= -\partial_1 f_0 \quad . \end{aligned} \tag{2.94}$$

In complex notation $(z, \bar{z} = x_0 \pm i x_1;\ \partial, \bar{\partial} = (\partial_0 \mp i\partial_1)/2;\ f, \bar{f} = (f_0 \pm i f_1))$, this means

$$\bar{\partial} f(z, \bar{z}) \ = \ 0 \ = \ \partial \bar{f}(z, \bar{z}) \ , \tag{2.95}$$

i.e., $f = f(z)$ and $\bar{f} = \bar{f}(\bar{z})$ are analytic functions of z and $\bar{z}$, respectively. The algebra generated by these transformations is infinite-dimensional. Therefore in a two-dimensional space-time the conformal transformations form an infinite-dimensional Lie group which is the direct product of the group of holomorphic coordinate transformations with that of antiholomorphic coordinate transformations, i.e.,

$$z \longrightarrow w = f(z), \quad \bar{z} \longrightarrow \bar{w} = \bar{f}(\bar{z}) \ , \tag{2.96}$$

with f and $\bar{f}$ independent analytic functions. The Lie algebra of each factor of this group is isomorphic to the so-called Witt algebra. This is the Lie algebra of smooth vector fields on the unit circle S^1, with generators $L_n^{(c)} = -z^{n+1}\frac{d}{dz}$ and commutators

$$[L_m^{(c)}, L_n^{(c)}] = (m-n)L_{m+n}^{(c)} \ . \tag{2.97}$$

However the conformal symmetries as encoded in (2.97) must be modified to describe physical systems. This modification gives an additional term on the right hand side of the commutator above. The Witt algebra is replaced by the Virasoro algebra:

$$[L_n, L_m] \ = \ (n-m)L_{n+m} + \frac{c}{12}n(n+1)(n-1)\delta_{n+m,0} \ . \tag{2.98}$$

The $\bar{L}'s$ obey the same algebra as the $L's$ and the $L's$ and $\bar{L}'s$ commute. The additional term involving the *central charge* c, can be physically understood in terms of a Casimir energy (see [21][22], for example). The complex plane can be conformally transformed into a cylinder of finite circumference (as indicated in Fig. 4 below). The circumference introduces a scale and it can be shown [23] [24] that the Casimir energy induced by the periodic boundary condition around the cylinder is proportional to c.

Now we introduce the generating functions for the operators L_n and $\bar{L}_n$ of the Virasoro algebra as fields $T(z) = \sum_{n\in \mathbf{Z}} z^{-n-2}L_n$ and $\bar{T}(\bar{z}) = \sum_{n\in \mathbf{Z}} \bar{z}^{-n-2}\bar{L}_n$ so that the operators L_n and $\bar{L}_n$ are obtainable as Fourier-Laurent coefficients of them

$$L_n = \frac{1}{2\pi i}\oint dz\; z^{n+1}T(z)$$
$$\bar{L}_n = \frac{1}{2\pi i}\oint d\bar{z}\; \bar{z}^{n+1}\bar{T}(\bar{z})\;. \tag{2.99}$$

This gives the following hermiticity properties [25]:

$$L_n^+ = L_{-n},\; \bar{L}_n^+ = \bar{L}_{-n}\;\;. \tag{2.100}$$

To prove that the operators (2.99) generate the Virasoro algebra we have to introduce the *operator product* of the fields. For two arbitrary fields $F(z)$ and $M(w)$ the operator product is given by

$$F(z)M(w)\;:=\;\begin{cases} F(z)M(w), & \text{for } |z|>|w| \\ M(w)F(z), & \text{for } |z|<|w|\;. \end{cases} \tag{2.101}$$

The ordering above, called *radial ordering*, is forced on us by the aim to obtain convergent power series when z and w are interpreted as complex numbers. When in physical applications the radial direction corresponds to the direction of proper time, the radial ordering requirement reproduces the time ordering prescription of relativistic quantum field theory. In a general conformal field theory the operator product of two T fields takes the form

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{(z-w)}\partial_w T + \dots \tag{2.102}$$

where c is a fixed number.

To show that coefficients (2.99) are the Virasoro operators we compute their commutators

$$[L_n, L_m]\;=\;\left[\oint\frac{dz}{2\pi i}\oint\frac{dw}{2\pi i}\;-\;\oint\frac{dw}{2\pi i}\oint\frac{dz}{2\pi i}\right]z^{n+1}T(z)w^{m+1}T(w)\;. \tag{2.103}$$

We can now insert the expression (2.102) into the above commutator which gives

$$[L_n, L_m] = \oint\frac{dw}{2\pi i}w^{n+1}\oint\frac{dz}{2\pi i}z^{n+1}\times\left[\frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{(z-w)}\partial_w T\right]\;. \tag{2.104}$$

The non singular part of the expansion (2.102) has vanishing contribution after integration. Drawing the z contour tightly about the point w (see Fig. 3) gives

$$[L_n, L_m] = \oint \frac{dw}{2\pi i} w^{m+1} \{(n+1)w^n \cdot 2T(w) + w^{n+1}\partial_w T + (c/2)\cdot\frac{w^{n-2}}{3!}(n+1)n(n-1)\} . \tag{2.105}$$

After another integration we finally get

$$\begin{aligned}[L_n, L_m] = \oint \frac{dw}{2\pi i} \{&(2n+2)w^{n+m+1}T(w) - (m+n+2)w^{n+m+1}T(w) \\ &+ \frac{c}{12}n(n+1)(n-1)w^{m+n-1}\} .\end{aligned} \tag{2.106}$$

This is just the Virasoro algebra (2.98). The central charge c of the Virasoro algebra can be interpreted as an eigenvalue of some operator C. This operator commutes with all L_n components and therefore furnishes a one-dimensional center of the Virasoro algebra. Hence the Virasoro algebra can be recognised as a central extension of the Witt algebra.

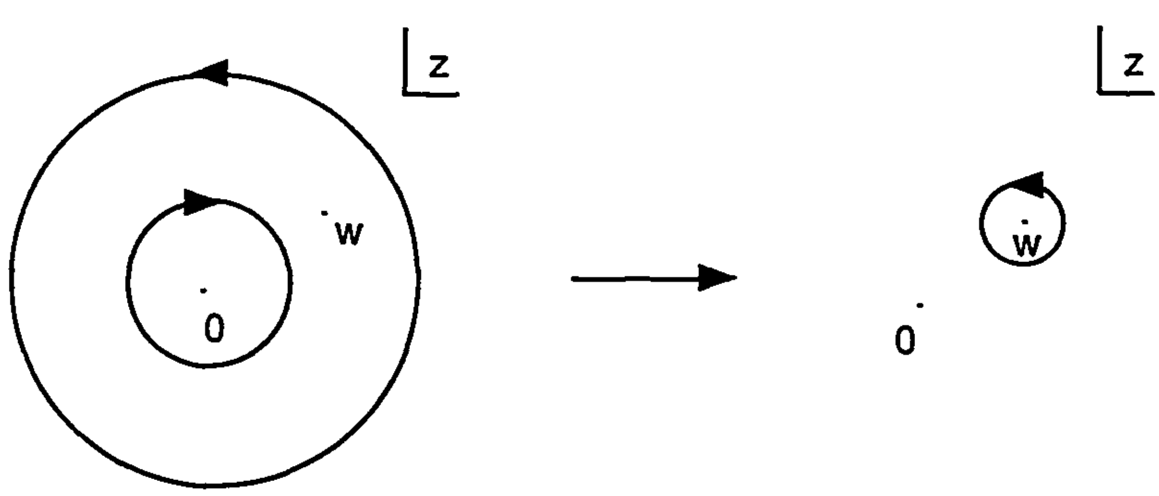

Figure 3. The integration contour is deformed into a contour encircling w in the negative sense.

The fields $T(z)$ and $\bar{T}(\bar{z})$ can be used as generators of conformal transformations, instead of their Laurent components L_n and $\bar{L}_n$. These fields represent the holomorphic and antiholomorphic energy-momentum tensors obtained by varying the action S with respect to the metric $g^{\mu\nu}$:

$$T_{\mu\nu} \sim \delta S/\delta g^{\mu\nu} \quad . \tag{2.107}$$

The energy-momentum tensor is symmetric, $T_{\mu\nu} = T_{\nu\mu}$, owing to the symmetry of the metric $g^{\mu\nu}$; it is conserved, $\sum_\mu \partial T_{\mu\nu}(x)/\partial x^\mu = 0$, owing to translation invariance; and it is traceless, $T_{\mu\mu} = 0$, owing to dilatation invariance. As a consequence, the energy-momentum tensor of a two-dimensional conformal field theory has only two independent components $T = T(z)$ and $\bar{T} = \bar{T}(\bar{z})$ which are purely holomorphic and antiholomorphic respectively, with $z = x_0 + ix_1$, $\bar{z} = x_0 - ix_1$.

We see that scale invariance and a local conserved stress-energy tensor imply the existence of an infinite dimensional algebra that acts on the state space of the theory. The state space will decompose into irreducible representations (irreps) of this algebra which will be very large (of infinite dimension). The properties of all states in an irrep (of the Virasoro algebra) are related to each other purely by the Virasoro algebra.

To see what these representations look like we will diagonalize the hermitian operators L_0 and $\bar{L}_0$:

$$L_0|h\rangle \;=\; h|h\rangle,\; \bar{L}_0|\bar{h}\rangle \;=\; \bar{h}|\bar{h}\rangle \quad h, \bar{h} \text{ real} \quad . \tag{2.108}$$

Equation (2.98) shows that for $n > 0$, L_{-n} raises L_0 by n units and L_n lowers L_0 by n units. We can now construct highest-weight representations in the standard fashion: choose an eigenstate of L_0, $|h\rangle$, that is annihilated by all the L_n. Then the set of states formed by applying products of the L_{-n} is a representation space for the algebra. These states can be organised into grades by the eigenvalues of L_0. $L_{-1}|h\rangle$ is at grade 1; $L_{-1}^2|h\rangle$, $L_{-2}|h\rangle$ are at grade 2; and so on. This set of states is called a Verma module: h is the weight of the irrep, $|h\rangle$ is a highest weight state [26].

Such states exist in a conformal field theory. The hamiltonian, $L_0 + \bar{L}_0$, is lowered by L_n. The lowest energy state, the vacuum $|0\rangle$, must be annihilated by all the L_{+n}. It is a highest-weight state. Scale invariance implies that L_0 annihilates $|0\rangle$ as well: $L_0|0\rangle = 0$. This implies translation invariance, $L_{-1}|0\rangle = 0$, since

$$||L_{-1}|0\rangle||^2 \;=\; \langle 0|L_{+1}L_{-1}|0\rangle \;=\; \langle 0|2L_0|0\rangle \;=\; 0 \;, \qquad (2.109)$$

using (2.98).

From (2.98) we can see that the modes $L_0, L_{\pm 1}$ play a special role. These generate a finite-dimensional subalgebra of the Virasoro algebra which is nothing but the simple Lie algebra sl_2 that we have shown annihilates the vacuum. There are in general an infinite number of sl_2 irreps in each Virasoro irrep; the Virasoro algebra ties together their behavior.

There is more in a field theory than states; there are local field operators as well. In a $2d$ conformally invariant quantum field theory there are special fields called primary fields characterized by their operator products with $T(z)$

$$T(z)\phi(w,\bar{w}) \sim \frac{h\phi(w)}{(z-w)^2} + \frac{1}{(z-w)}\frac{\partial\phi}{\partial w} \;\; . \qquad (2.110)$$

The absence of higher-order poles distinguishes primary from other scaling fields. The coefficients are fixed by the requirement that T generate dilatations and translations.

The equation above gives the following commutation relations

$$[L_n, \phi(w)] = h(n+1)w^n\phi + w^{n+1}\partial_w\phi \;\; . \qquad (2.111)$$

Observing that $\bar{L}_n$ will obey a similar formula with h replaced by $\bar{h}$ and specializing to $n = 0$ serves to identify $h + \bar{h}$ as the scaling dimension of ϕ ($L_0 + \bar{L}_0$ generates dilatations) and $h - \bar{h}$ as its Euclidean spin ($L_0 - \bar{L}_0$ generates rotations) [27].

From (2.111) and the highest-weight properties of the vacuum, we find, for $n > 0$,

$$L_n\phi(0)|0\rangle \; = \; 0; \;\; L_0\phi(0)|0\rangle \; = \; h\phi(0)|0\rangle \tag{2.112}$$

so $\phi(0)|0\rangle = |h\rangle$ is also a highest-weight state with weight h. Primary fields are those that create highest-weight states from the vacuum and therefore primary fields correspond to highest-weight representations.

The fields corresponding to non-highest weight vectors are called secondary fields or descendants of the corresponding primary field. It is easy to see from (2.99) that

$$L_{k_n}\ldots L_{k_1}\phi(0)|0\rangle \tag{2.113}$$

is created by a field $\phi^{(k)}(z)$ at $z = 0$ where

$$\phi^{(k)}(z) = L_{k_n}(z)\ldots L_{k_1}(z)\phi(z) \tag{2.114}$$

and

$$L_k(z) = \frac{1}{2\pi i}\oint dw(w-z)^{k+1}T(w) \; . \tag{2.115}$$

It is clear that for $n = 0, \pm 1$ the central term in (2.98) vanishes so that $L_0, L_\pm$ can be identified with their classical analogues:

$$L_{-1} = -\frac{d}{dz},\; L_0 = -z\frac{d}{dz},\; \text{and}\; L_1 = -z^2\frac{d}{dz} \; . \tag{2.116}$$

Other descendant fields are composites of the stress-energy tensor with ϕ. The properties of all these fields and not just the derivatives above are organised by the Virasoro algebra. Invariance of a field theory under conformal transformation thus means that an action of T and $\bar{T}$ is defined on its Hilbert space of states, and therefore on the fields of the theory. That is, the conformal symmetry of the theory implies that its states will fill out representations of the conformal algebra.

Physical examples where conformal symmetry is particularly relevant are strings in high-energy physics (see [28] [29]), and critical phenomena at the second-order phase transitions in statistical physics. Both classical critical phenomena in two dimensions and quantum critical phenomena in one dimension can be treated. In a statistical system at the critical point, the correlation length diverges and the effective field theory becomes scale invariant. Together with the assumption that the interactions of the theory are local, this implies conformal invariance in 2d [10]. The space on which the effective field theory lives is the complex plane as described above.

Another two-dimensional space that is conformally equivalent to the complex plane is encountered in the theory of closed relativistic strings; the two-dimensional space-time "world sheet" traced out by a free closed string has the topology of a cylinder, and the cylinder described by the compact space coordinate σ and the time coordinate τ can be mapped to the complex plane via

$$z = ae^{\frac{\tau+i\sigma}{a}}, \quad \bar{z} = z^* = ae^{\frac{\tau-i\sigma}{a}} \quad , \tag{2.117}$$

where $\tau \in \mathrm{R}$, $0 < \sigma \leq 2\pi a$ and a is the cylinder radius (see Fig. 4 below).

In particular, the time direction on the cylinder is the radial direction on the plane. Equal-time slices become the circles of fixed radius on the plane, whereas equal-space slices become the lines radiating from origin. Therefore infinite past and future, $\tau = \mp\infty$, on the cylinder are mapped into the points $z = 0, \infty$, respectively, on the plane. The *time translations* $\tau \longrightarrow \tau + \lambda$ on the cylinder are the *dilatations* $z \longrightarrow e^{\lambda}z = z + \lambda z + \ldots$, on the complex plane whereas the *space translations* $\sigma \longrightarrow \sigma + \theta$ on the cylinder are the *rotations* $z \longrightarrow e^{i\theta}z$ on the complex plane. Hence, the Hamiltonian of the system, $L_0 + \bar{L}_0$, can be identified with the dilatation generator on the plane, while the Hilbert space of states comprises surfaces of constant radius.

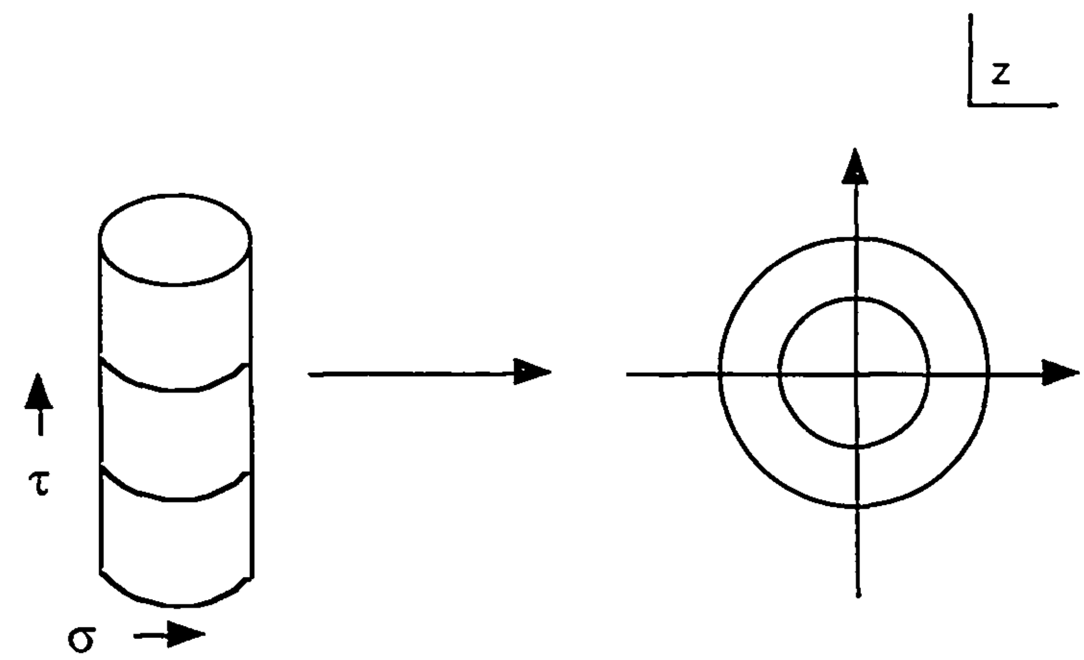

Figure 4. The conformal map of a cylinder to a plane.

Particularly interesting to us is a class of conformal field theories called *Wess-Zumino-Witten models* (WZW models). These are by definition those conformal field theories for which the symmetry algebra is given by the semi-direct sum of a holomorphic and an antiholomorphic copy of the affine Kac-Moody algebras and the Virasoro algebras $\hat{g} \oplus \nu$, and the energy-momentum tensor is of the so-called Sugawara form:

$$T(z) = \xi k_{ab} \ : J^a(z) J^b(z) : \ , \tag{2.118}$$

where k_{ab} is the Cartan metric of the underlying simple Lie algebra g of the affine Kac-Moody algebra. The constant ξ above is incorporated to tune the normalization of the field $T(z)$ and the fields $J^a(z)$ are generating functions for the operators T^a_n of affine Kac-Moody algebras

$$J^a(z) := \sum_{n \in Z} T^a_n z^{-n-1} \ . \tag{2.119}$$

The notation : $J^a(z) J^b(z)$: means that we keep only the zero-mode of the operator product of these fields; i.e., for

$$J^a(z)J^b(w) = \sum_{n=-n_0}^{\infty} (z-w)^n f^{ab}{}_c J^c_{(n)}(w) \quad . \tag{2.120}$$

the normal-ordered product is given by

$$: J^a(z)J^b(z) : := f^{ab}{}_c J^c_{(0)}(z) \quad . \tag{2.121}$$

The normal ordering above is forced on us by the aim of avoiding infinite energies and other non-physical results. For example, without normal ordering $< 0|k_{ab}J^a(z)J^b(z)|0 >$ is ill-defined.

2.7. Fusion rules and modular transformations

The aim of this section is to introduce a surprising connection between rules for coupling primary fields of WZW models, called fusion rules, and modular transformations. By definition a conformal field theory must be endowed with an associative algebra of its primary fields ϕ_i

$$\phi_i(z,\bar{z})\phi_j(w,\bar{w}) = \sum_k (z-w)^{h_k-h_i-h_j}(\bar{z}-\bar{w})^{\bar{h}_k-\bar{h}_i-\bar{h}_j}(C_{ij}{}^k\phi_k(w,\bar{w})+\ldots) \tag{2.122}$$

where $C_{ij}{}^k$ are complex constants, called the operator product coefficients†. For many purposes we only need to know which primary fields appear on the right-hand side of an operator product. This information is the context of the fusion rules of the conformal field theory. They read

$$\phi_i * \phi_j = \sum_k N_{ij}{}^k \phi_k \tag{2.123}$$

† Operator products of some other fields besides primary fields were considered in (2.102) and (2.110) above.

where $\{\phi_k\}$ is the set of primary fields of the conformal field theory, and $N_{ij}{}^k$ are non-negative integers. This is analogous to the situation of calculating Kronecker products for representations of simple Lie algebras

$$R_{\Lambda_i} \times R_{\Lambda_j} = \sum_k T_{ij}{}^k R_{\Lambda_k} \quad . \tag{2.124}$$

The reason why the fusion rules have to be compatible with the Kronecker product is that the zero mode subalgebra of the WZW symmetry algebra $\hat{g}$ is just the simple Lie algebra g. But because a much larger symmetry (affine symmetry) is imposed on fusion rules we have

$$N_{\phi_i \phi_j}{}^{\phi_k} \leq T_{\Lambda_i \Lambda_j}{}^{\Lambda_k} \tag{2.125}$$

for $\phi_i := \phi_{\Lambda_i}$ and $\Lambda_i := R_{\Lambda_i}$. In the following we will go through the main properties of the fusion coefficients. The $N_{ij}{}^k$ can be integers larger than one because two primary fields can couple in several distinct ways to a third primary field just as for the T's (for example, the direct product of two representations $R_{(1,1)}$ of $su(3)$ contains two copies of $R_{(1,1)}$). The identity primary field is present in any conformal field theory and it is denoted as $\phi_0 := 1$. Therefore it follows immediately that

$$N_{i0}{}^j = \delta_i^j \tag{2.126}$$

and

$$N_{ij}{}^0 = \delta_{ij+} \tag{2.127}$$

where by definition ϕ_{j+} is the field conjugate to ϕ_j. The fusion rule coefficients with lower indices only are defined as:

$$N_{ijk} := N_{ij}{}^{k+} \ . \tag{2.128}$$

In addition, because the operator products are radially ordered products, the fusion rule coefficients are symmetric in their lower indices:

$$N_{ij}{}^k = N_{ji}{}^k \ . \tag{2.129}$$

Consequently, the fusion rules of a conformal field theory form a commutative algebra. Also, the associativity of the operator product algebra implies that the fusion rule algebra is associative as well:

$$\sum_k N_{ij}{}^k N_{kl}{}^m = \sum_k N_{jl}{}^k N_{ik}{}^m \quad . \tag{2.130}$$

In terms of the matrices N_i with entries $(N_i)_j^k = N_{ij}{}^k$, we have

$$[N_i, N_j] = 0 \quad . \tag{2.131}$$

Using this fact together with (2.127), (2.128), and (2.129), it is easily seen that the N_{ijk} are totally symmetric in i, j, k:

$$N_{ijk} = N_{ikj} = N_{kji} \; . \tag{2.132}$$

Finally, the fusion rules are invariant under the conjugation $\phi_i \longrightarrow \phi_{i^+}$:

$$N_{i^+j^+}{}^{k^+} = N_{ij}{}^k \quad . \tag{2.133}$$

We can interpret the fusion rules of the conformal field theory as an abstract algebra with the primary fields as generators. The structure of fusion rule algebras bears some similarity with the representation theory of finite groups (see [3], for example). This is because the tensor product rules for a finite group define a commutative associative algebra with non-negative integer structure constants. The characters of the irreps of the finite group are the numbers which obey these same rules. It turns out that such an algebra is isomorphic to the fusion rule algebra of a conformal field theory if the modular transformation matrix S on these characters satisfies $S^2 = C$ where C is a conjugation. Therefore in this case, the primary fields correspond to the irreps of the finite group.

As a result, the connection between the fusion rules and the action of the modular matrix S may be less surprising. Remarkably, in the case of a generic conformal field theory, there exists a relation between the fusion rule algebra and the modular transformation matrix S that acts on the characters associated to the primary fields ϕ_i. To get this connection we have to place the conformal field theory on the torus represented as a parallelogram on the complex plane with corners at $0,1,\tau$ and $\tau+1$. Here τ is a complex number known as the modular parameter. Now we can choose a basis a and b of homology cycles on the torus such that the a cycle corresponds to the straight line from 0 to 1, and the b cycle to the straight line from 0 to τ. The fact that the modular transformation $\tau \longrightarrow -1/\tau$ interchanges the cycles a and b is used to show that the modular matrix S diagonalizes the fusion rules

$$N_{ij}{}^{k} = \sum_n S_{in} l_j^{(n)} S_{nk}^{-1} \qquad (2.134)$$

where $l_j^{(n)}$ are N^2 eigenvalues of the fusion rule algebra satisfying

$$l_i^{(n)} l_j^{(n)} = \sum_{k=1}^{\mathrm{N}} N_{ij}{}^{k} l_k^{(n)} \quad . \qquad (2.135)$$

The fusion rule eigenvalues can be expressed in terms of the matrix elements S_{ij} as follows

$$l_i^{(j)} = S_{ij}/S_{0j} \quad . \qquad (2.136)$$

As a result, the fusion coefficients can be expressed through the matrix S directly:

$$N_{ij}{}^{k} = \sum_n \frac{S_{in} S_{jn} S_{nk}^{-1}}{S_{0n}} \quad . \qquad (2.137)$$

This result is known as the Verlinde formula [30].

2.8. Galois transformations

Galois theory says (among other things) that if a polynomial equation with rational coefficients is satisfied by one I-th primitive root of unity it is satisfied by all others primitive I-th roots of unity.

Suppose we have a polynomial equation

$$\sum_i a_i q^i = 0 \ , \tag{2.138}$$

where q is a primitive I-th root of unity, i.e., $q^I = 1$, and a_i are rational numbers. Then

$$\sum_i a_i \tilde{q}^i = 0 \ , \tag{2.139}$$

where $\tilde{q}$ is any other primitive I-th root of unity. Primitive here means that if $q^I = 1$ then $q^{I_1} \neq 1$ for $0 < I_1 < I$. If q is a primitive I-th root of unity, then so is $\tilde{q} = q^a$, for $gcd(a, I) = 1$.

The equation $x^2 + 1 = 0$ is a simple example of a polynomial equation satisfied by a root of unity. This equation is satisfied by $(+i)$ which is a primitive 4-th root of unity, i.e., $(+i)^4 = 1$ and therefore it is satisfied by $(+i)^3$ too, for $gcd(3, 4) = 1$.

These transformations where one primitive root of unity is replaced by another are called Galois transformations. The reason we introduce Galois transformations here is that some equations involving the Kac-Peterson modular matrix, studied in chapters 4 and 5, are invariant under these transformations. As a result new relations between weight multiplicities are obtained.

3. Lie group multiplicities from Lie characters

The dominant weight multiplicities are introduced in this chapter. They are obtained as coefficients of the Lie character's expansion in terms of even Weyl functions. This chapter is mainly a review of the work done by Patera and Sharp in [18].

Define the Weyl orbit sum

$$E_\lambda(\sigma) := \frac{|W(\lambda-\rho)|}{|W|} \sum_{w\in W} \sigma^{w(\lambda-\rho)} \quad . \tag{3.1}$$

Here E stands for an even Weyl sum, $|W|$ is the order of the Weyl group of g, and $|W\mu|$ is the order of the Weyl orbit of $\mu \in P_+ := \{\sum_{i=1}^r \mu^i \Lambda_{(i)} \,|\, \mu^i \in \mathbb{Z}_\geq \}$. In (3.1) $\lambda \in P_{++} := \{\sum_{i=1}^r \lambda^i \Lambda_{(i)} \,|\, \lambda^i \in \mathbb{Z}_> \}$, and ρ is the half-sum of positive roots: $\rho = \sum_{\alpha>0} \alpha/2 = \sum_{i=1}^r \Lambda_{(i)}$, where the $\Lambda_{(i)}$ are the fundamental weights of g. We use the somewhat abusive notation $\sigma^\mu = \exp[-i\mu\cdot\sigma]$, with σ any weight, so that $\sigma^\lambda\sigma^\mu = \sigma^{\lambda+\mu}$.

An odd Weyl sum is the so-called discriminant

$$O_\lambda(\sigma) := \sum_{w\in W} \det(w)\, \sigma^{w\lambda} \quad , \tag{3.2}$$

where $\det(w)$ is the sign of the Weyl group element w. Let H^j be the elements of the Cartan subalgebra of g. Weyl's character formula (2.37) for the trace $\chi_\lambda(\sigma)$ of $\exp\left[-i\sum_{j=1}^r H^j\sigma_j\right]$ in the representation of highest weight $\lambda-\rho$ is

$$\chi_\lambda(\sigma) = O_\lambda(\sigma)/O_\rho(\sigma) \quad . \tag{3.3}$$

The character, being an even Weyl function, can be expanded in terms of the even functions E_μ [31]:

$$\chi_\lambda(\sigma) = \sum_{\mu\in P_{++}} m_\lambda^{\ \mu} E_\mu(\sigma) \quad . \tag{3.4}$$

The non-negative integers $m_\lambda^{\ \mu}$ are the dominant weight multiplicities: $m_\lambda^{\ \mu}$ denotes the multiplicity of the weight $\mu-\rho$ in the representation with highest weight $\lambda-\rho$. Note that $m_\lambda^{\ \lambda}=1$ for all $\lambda \in P_{++}$. We can consider the $m_\lambda^{\ \mu}$ to be the elements of an infinite matrix m. If $\lambda-\mu \notin \mathbb{Z}_{\geq}\{\alpha^{(1)},\ldots,\alpha^{(r)}\}$, where $\alpha^{(i)}$ are the simple roots, then $m_\lambda^{\ \mu}=0$. Therefore m is a lower triangular matrix, provided the weights are ordered appropriately.

Not only can the Weyl character be expanded in terms of the Weyl orbit sums E_λ, but the reverse is also true:

$$E_\lambda(\sigma) \;=\; \sum_{\mu\in P_{++}} \ell_\lambda^{\ \mu}\chi_\mu(\sigma) \quad . \tag{3.5}$$

The coefficients $\ell_\lambda^{\ \mu}$ are easily shown to be integers, but are *not* in general non-negative. However if ℓ is the matrix with elements $\ell_\lambda^{\ \mu}$, then clearly $m=\ell^{-1}$. So, if the triangular matrix ℓ can be calculated, it is a simple matter to invert it to obtain the dominant weight multiplicities [18].

Patera and Sharp [18] also point out that the equations above allow the calculation of the $\ell_\lambda^{\ \mu}$ for fixed λ using the Weyl group. From equation (3.5), the defining relation for the $\ell_\lambda^{\ \mu}$, we get

$$E_\lambda(\sigma)O_\rho(\sigma) \;=\; \sum_{\mu\in P_{++}} \ell_\lambda^{\ \mu}O_\mu(\sigma) \quad . \tag{3.6}$$

Then using the definitions of even and odd Weyl sums we can write:

$$\frac{|W(\lambda-\rho)|}{|W|}\sum_{w,x\in W}\det(x)\sigma^{w(\lambda-\rho)+x\rho} = \sum_{\mu\in P_{++}}\ell_\lambda^{\ \mu}\sum_{r\in W}\sigma^{r\mu}\det(r) \quad . \tag{3.7}$$

The dominant sector is the only sector we need consider, by Weyl symmetry, to which the last equation can be restricted. The Weyl orbit of any dominant weight intersects the dominant sector only at one point, and it happens for the identity

Weyl group element. Therefore upon restriction of the last equation we can simply drop the summation over the Weyl group on the RHS and write:

$$\frac{|W(\lambda-\rho)|}{|W|} \sum_{\substack{w,x\in W \\ w(\lambda-\rho)+x\rho\in P_{++}}} \det(x)\sigma^{w(\lambda-\rho)+x\rho} = \sum_{\mu\in P_{++}} \ell_\lambda^{\ \mu}\sigma^\mu \quad . \tag{3.8}$$

Equating the coefficients of σ^μ, $\mu \in P_{++}$, gives the final result:

$$\boxed{\ell_\lambda^{\ \mu} = \frac{|W(\lambda-\rho)|}{|W|} \sum_{w,x\in W} \det(x)\, \delta^\mu_{x\rho+w(\lambda-\rho)}} \quad . \tag{3.9}$$

4. Lie group multiplicities from WZW modular matrices

The results discussed in the previous chapter are adapted to the use of the Kac-Peterson modular matrix [20] in this section. This leads to the new expression for the dominant-weight multiplicities of simple Lie algebra written below.

Define

$$O_\lambda^{(n)}(\sigma) \;:=\; F_n \sum_{w\in W} \det(w)\,\sigma_{(n)}^{w\lambda} \;=\; S_{\lambda\sigma}^{(n)} \quad , \tag{4.1}$$

and

$$E_\lambda^{(n)}(\sigma) \;:=\; \frac{|W(\lambda-\rho)|}{|W|} \sum_{w\in W} \sigma_{(n)}^{w(\lambda-\rho)} \quad , \tag{4.2}$$

with

$$\sigma_{(n)}^{\mu} \;:=\; \exp[-2\pi i\mu\cdot\sigma/n] \ , \quad F_n \;:=\; \frac{i^{|\Phi_+|}}{n^{r/2}\sqrt{|M^*/M|}} \quad ; \tag{4.3}$$

$|\Phi_+|$ is the number of positive roots of g, and M here is the weight lattice. Because the scalar product of the affine weights is equal to the scalar product of their finite components, the summation is over the finite Weyl group in the equations above. The matrix $S^{(n)}$ in (4.1) is the Kac-Peterson modular matrix of WZW models, corresponding to the affine algebra $\hat{g}$ at level $k = n - h^\vee$, where $h^\vee$ is the dual Coxeter number of g. Define

$$P_+^n \;:=\; \left\{ \sum_{i=0}^{r} \lambda^i \Lambda_i \,|\, \lambda^i \in \mathbb{Z}_\geq \ , \ \sum_{i=0}^{r} a_i^\vee \lambda^i = n \right\} , \tag{4.4}$$

and P_{++}^n similarly, except with $\mathbb{Z}_\geq$ replaced with $\mathbb{Z}_>$. The positive integers $a_i^\vee$ in (4.4) are the dual Coxeter labels [19] of $\hat{g}$. We read from (4.1) that

$$S_{\lambda\sigma}^{(n)}/S_{\rho\sigma}^{(n)} \;=\; O_\lambda^{(n)}(\sigma)/O_\rho^{(n)}(\sigma) \;=:\; \chi_\lambda^{(n)}(\sigma) \ , \tag{4.5}$$

where

$$\chi_\lambda^{(n)}(\sigma) \;=\; \chi_\lambda(2\pi\,\sigma/n) \tag{4.6}$$

for any $\lambda, \sigma \in P^n_{++}$. This was discovered by Kac and Peterson [20], and has been much exploited elsewhere ([32][33][34], for example). We have $S_{\hat\lambda,\hat\sigma} = S^{(n)}_{\lambda+\rho,\sigma+\rho}$ from (2.84) (notice that if $\hat\lambda, \hat\sigma \in P^k_+$, then $\lambda + \rho, \sigma + \rho \in P^n_{++}$).

The relation (4.5) between the WZW modular matrix and semi-simple Lie algebra characters may not be too surprising, if other facts are taken into account. The coupling rules for primary fields are defined by fusion coefficients. Since a WZW model has an affine symmetry algebra, the fusion coefficients are less than or equal to the corresponding tensor product coefficients of the underlying simple Lie algebra [15]. Products of simple Lie algebra characters are defined by these coefficients. It was discovered by Verlinde that products of certain ratios of elements of the modular matrix S decompose into integer linear combinations of these same ratios, with the coefficients being the fusion coefficients [30]. This may only be possible if these ratios coincide with the simple Lie algebra characters, but evaluated at special points.

The matrices m and ℓ are lower triangular. Therefore, whenever $\lambda \in P^n_{++}$, the relation (4.5) implies

$$\chi^{(n)}_\lambda(\sigma) = \sum_{\mu \in P^n_{++}} m_\lambda{}^\mu E^{(n)}_\mu(\sigma) \ , \tag{4.7}$$

and

$$E^{(n)}_\lambda(\sigma) = \sum_{\mu \in P^n_{++}} \ell_\lambda{}^\mu \chi^{(n)}_\mu(\sigma) \ . \tag{4.8}$$

Equation (4.8) also holds for $\lambda \in P_{++} \cap P^n_+$. Using the unitarity of the modular $S^{(n)}$ matrix

$$\sum_{\nu \in P^n_{++}} O^{(n)}_\lambda(\nu)\, O^{(n)*}_\mu(\nu) = \sum_{\nu \in P^n_{++}} S^{(n)}_{\lambda\nu}\, S^{(n)*}_{\nu\mu} = \delta^\mu_\lambda \ , \tag{4.9}$$

we arrive at

$$\boxed{\ell_\lambda^{\ \mu} = \sum_{\sigma\in P^n_{++}} E^{(n)}_\lambda(\sigma)\, O^{(n)}_\rho(\sigma)\, O^{(n)*}_\mu(\sigma) = \sum_{\sigma\in P^n_{++}} E^{(n)}_\lambda(\sigma)\, S^{(n)}_{\rho\sigma}\, S^{(n)*}_{\mu\sigma}} \ , \tag{4.10}$$

valid whenever both $\lambda \in P_{++} \cap P^n_+$ and $\mu \in P^n_{++}$. Equation (4.10) can be generalised:

$$\sum_{\sigma\in P^n_{++}} \ell_\lambda^{\ \sigma} N^{(n)\nu}_{\mu\sigma} = \sum_{\sigma\in P^n_{++}} E^{(n)}_\lambda(\sigma)\, S^{(n)}_{\mu\sigma}\, S^{(n)*}_{\nu\sigma} \ , \tag{4.11}$$

where $N^{(n)\nu}_{\mu\sigma}$ are the WZW fusion rules, which we may take to be defined by (2.137):

$$N^{(n)\nu}_{\lambda\mu} = \sum_{\sigma\in P^n_{++}} \chi^{(n)}_\lambda(\sigma)\, S^{(n)}_{\mu\sigma}\, S^{(n)*}_{\nu\sigma} \ . \tag{4.12}$$

Of course, we also get directly from (4.7) that

$$N^{(n)\nu}_{\lambda\mu} = \sum_{\sigma,\gamma\in P^n_{++}} m_\lambda^{\ \gamma} E^{(n)}_\gamma(\sigma)\, S^{(n)}_{\mu\sigma}\, S^{(n)*}_{\nu\sigma} \ . \tag{4.13}$$

Another way to calculate the fusion coefficients is by manipulation of the Weyl character formula. As a result we can get:

$$N^{(n)\nu}_{\lambda\mu} = \sum_{\hat{w}\in\widehat{W}} \det(\hat{w})\ \mathrm{mult}_\mu(\hat{w}\nu - \lambda) \ . \tag{4.14}$$

Here, $\mathrm{mult}_\mu(\sigma)$ is the multiplicity of weight σ (dominant or not) in R_λ. Therefore

$$\mathrm{mult}_\mu(\sigma) = m_\mu^{\ \sigma} \ \text{ if } \ \sigma \in P_{++} \tag{4.15}$$

and

$$\mathrm{mult}_\lambda(\kappa) = \mathrm{mult}_\lambda(\sigma) \ \text{ if } \ \kappa = \hat{w}(\sigma - \rho) + \rho \tag{4.16}$$

for others. The general procedure to calculate fusion coefficients using (4.14) is to add to $\lambda + \rho$ the weights P_λ, then transform the results into the dominant sector

P^n_{++} using the affine Weyl group $\widehat{W}$. Any weight that cannot be so transformed is ignored, while the rest contribute according to the sign of the transforming Weyl group element.

Because ℓ is lower triangular, an easy argument gives

$$m_\lambda{}^\mu = (\ell^{(n)-1})_\lambda{}^\mu \tag{4.17}$$

for all $\lambda, \mu \in P^n_{++}$, where $\ell^{(n)}$ is defined to be the sublattice of ℓ obtained by restricting it to the set P^n_{++}. Thus equations like (4.10) provide a simple method of calculating dominant weight multiplicities $m_\lambda{}^\mu$. As examples, the ℓ and m matrices of A_2 and G_2 algebras, for different values of a level k, are given in the Appendix. Moreover, if we find a permutation π of P^n_{++} which commutes with both $S^{(n)}$ and $E^{(n)}$, then it will be an exact symmetry of both ℓ and m. More generally, if $S^{(n)}$ and $E^{(n)}$ both transform "nicely" under a permutation π of P^n_{++}, then we can expect to derive new relations for ℓ and m. This is the motivation for the following section.

5. New relations between multiplicities

In this chapter, we will show that symmetries of the Kac-Peterson modular matrices $S^{(n)}$ of WZW models give rise to new relations between semi-simple Lie algebra multiplicities.

5.1. *Affine Weyl group symmetries*

The most obvious symmetry concerns the affine Weyl group $\widehat{W}$ of $\hat{g}$. We know [19] that it is isomorphic to the semi-direct product of the (finite) Weyl group W with the group of translations in the coroot lattice M^*. We also know that the $\widehat{W}$-orbit of any weight intersects P_+^n in exactly one point. More precisely, let $\lambda \in M$ be some weight. Then there exists an element α in the coroot lattice of g, and some $w \in W$, such that

$$[\lambda] \;:=\; w(\lambda + n\alpha) \;\in\; P_+^n \;. \tag{5.1}$$

We will use this observation throughout this section. $[\lambda]$ is uniquely determined by λ (and n), but w will be only if $[\lambda] \in P_{++}^n$. Define $\epsilon(\lambda) := 0$ if $[\lambda] \notin P_{++}^n$, and $\epsilon(\lambda) := \det(w)$ otherwise, where $w \in W$ satisfies (5.1) .

We can replace λ with $[\lambda]$ in (4.1) so that

$$S_{[\lambda]\sigma}^{(n)} = F_n \sum_{w\in W} \det(w)\sigma^{w(v(\lambda+n\alpha))} \tag{5.2}$$

where $v, w \in W$. Since α is an element of the coroot lattice M^* of g, $u\alpha$ is an element of the coroot lattice M^* too, for any $u \in W$. Therefore $\sigma^{nw(v\alpha)} = 1$ and we get

$$S_{[\lambda]\sigma}^{(n)} = F_n \sum_{w\in W} \det(w)\sigma^{wv\lambda} \;. \tag{5.3}$$

Using the substitution $wv = u \in W$ we obtain the final result

$$S_{\lambda\sigma}^{(n)} = \epsilon(\lambda) S_{[\lambda]\sigma}^{(n)} \;. \tag{5.4}$$

The scalar product of the weights is invariant under the Weyl group of transformations: $w\lambda \cdot w\mu = \lambda \cdot \mu$. Adding the above statement to the arguments that gave us the last eq., we get

$$S^{(n)}_{\lambda\sigma} = \epsilon(\sigma) S^{(n)}_{\lambda[\sigma]} \quad . \tag{5.5}$$

The expression for the even Weyl sum (4.2) does not depend on the sign of the Weyl reflections. Using simple manipulations we arrive at the next relation:

$$E^{(n)}_{\lambda}(\sigma) = E^{(n)}_{\lambda}([\sigma]) = \frac{|W(\lambda-\rho)|}{|W([\lambda-\rho])|} E^{(n)}_{[\lambda-\rho]+\rho}(\sigma) \, . \tag{5.6}$$

Equation (3.5) is valid for all $\lambda \in P_{++}$. Using the particular values, $\sigma = (2\pi/n)\alpha$, in eq. (3.5) we can write:

$$E_{\lambda}(\frac{2\pi}{n}\alpha) \; = \; E^{(n)}_{\lambda}(\alpha) = \sum_{\mu \in P_{++}} \ell_{\lambda}^{\;\mu} \chi_{\mu}(\frac{2\pi}{n}\alpha) \quad . \tag{5.7}$$

The unitarity of the $S^{(n)}$ matrix only involves weights in P^n_{++}. Therefore, to be able to use the same argument as we did for (4.10) we have to convert μ into P^n_{++}. We get from (4.5) and (5.4) that

$$E^{(n)}_{\lambda}(\alpha) \; = \; \sum_{\mu \in P_{++}} \ell_{\lambda}^{\;\mu} \epsilon(\mu) \chi^{(n)}_{[\mu]}(\alpha) \quad . \tag{5.8}$$

Now the unitarity of the $S^{(n)}$ matrix can be used and as a final result we get that for any $\lambda \in P_{++}$, $\mu \in P^n_{++}$,

$$\sum_{\substack{\nu \in P_{++} \\ [\nu]=\mu}} \epsilon(\nu) \, \ell_{\lambda}^{\;\nu} = \sum_{\sigma \in P^n_{++}} E^{(n)}_{\lambda}(\sigma) \, S^{(n)}_{\rho\sigma} \, S^{(n)*}_{\mu\sigma} \, . \tag{5.9}$$

On the LHS of the last equation we have summed over all weights from P_{++} which are reflected into μ in P^n_{++}. Thus for any $\mu \in P^n_{++}$, and any $\lambda \in P_{++}$ with $[\lambda-\rho]+\rho \in P^n_{+}$, we get the truncation

$$\boxed{\ell_{[\lambda-\rho]+\rho}^{\;\mu} = \frac{|W([\lambda-\rho])|}{|W(\lambda-\rho)|} \sum_{\substack{\nu \in P_{++} \\ [\nu]=\mu}} \epsilon(\nu) \, \ell_{\lambda}^{\;\nu}} \quad . \tag{5.10}$$

Roughly speaking, (5.10) says that if we know the ℓ's for "large" weights, then we know them for "small" ones. Incidently, if $[\lambda - \rho] + \rho \not\in P_+^n$, then (5.10) holds if we replace its LHS with a sum similar to that of its RHS. Similar comments hold below if $\pi_A(\lambda) \not\in P_+^n$ in (5.12), or $\pi_a(\lambda) \not\in P_+^n$ in (5.19).

5.2. Outer automorphism symmetries

Next, consider the symmetries involving the outer automorphisms of affine Lie algebras $\hat{g}$, or equivalently, the automorphisms of the extended Dynkin diagrams of g. If an outer automorphism is also a symmetry of the unextended Dynkin diagram of g, i.e., it fixes the extended node, then it is well known to be an exact symmetry C (a *conjugation*) of both the ℓ's and m's:

$$m_{C\lambda}^{C\mu} = m_\lambda^\mu \ , \qquad \ell_{C\lambda}^{C\mu} = \ell_\lambda^\mu \ . \tag{5.11}$$

We are interested here instead in those automorphisms which are not conjugations. Denote such an automorphism by A, and recall that the fundamental weights of $\hat{g}$ are written as Λ_i, with $i = 0, 1, 2, \ldots, r$. There is one of these automorphisms $A = A_{(i)}$ for every node of the extended diagram with mark $a_i = 1$; $A_{(i)}$ will send Λ_0 to Λ_i. The automorphisms A can be expressed through the Weyl group elements v_A as

$$A(\lambda - n\Lambda_0) \ = \ v_A(\lambda - n\Lambda_0) \ , \tag{5.12}$$

for all $\lambda \in P_+^n$. As a result we have

$$S_{A\lambda\sigma}^{(n)} = \sum_{w \in W} \det(w) \sigma_{(n)}^{w(v_A(\lambda - n\Lambda_0) + nA\Lambda_0)} \ . \tag{5.13}$$

The w and v_A are elements of the Weyl group and so is wv_A. The fundamental weight Λ_0 is invariant under the finite Weyl group. So we have

$$\sigma_{(n)}^{wv_A n\Lambda_0} = 1 \ . \tag{5.14}$$

We know, from the definition of the Weyl group, that

$$w(A\Lambda_0) = A\Lambda_0 + \alpha \tag{5.15}$$

where $w \in W$ and α is an element of the coroot lattice M^* of g. M^* is invariant under the Weyl group and consequently we have

$$\sigma_{(n)}^{wnA\Lambda_0} = \exp[-2\pi i(A\Lambda_0)\cdot\sigma] \tag{5.16}$$

for any $\sigma \in P_+^n$ integral weight. Finally, using results of (5.14) and (5.16) we can rewrite (5.13) as

$$S^{(n)}_{A\lambda\sigma} = S^{(n)}_{\lambda\sigma} \exp[-2\pi i(A\Lambda_0)\cdot\sigma] \det(v_A) \ . \tag{5.17}$$

Similarly, we have

$$\begin{aligned} E^{(n)}_{\lambda}(A\sigma) &= E^{(n)}_{\lambda}(\sigma) \exp[-2\pi i(A\Lambda_0)\cdot(\lambda-\rho)] \ , \\ E^{(n)}_{A\lambda}(\sigma) &= E^{(n)}_{\pi_A\lambda}(\sigma) \frac{|W(A\lambda-\rho)|}{|W(\pi_A(\lambda)-\rho)|} \exp[-2\pi i(A\Lambda_0)\cdot\sigma] \ , \end{aligned} \tag{5.18}$$

where π_A denotes the one-to-one map from P^n_{++} to P_{++} given by

$$\boxed{\pi_A(\lambda) := [\lambda - w_A^{-1}\rho] + \rho} \ . \tag{5.19}$$

For fixed g, w_A and hence the map π_A are readily obtained from (5.12) – π_A will be the identity only when A is. For example, the automorphisms of the $su(r+1) = A_r$, B_r, and C_r algebras are given below:

1. for $g = A_r = su(r+1)$ and $A = A_j$ satisfying $A\Lambda_i = \Lambda_{i+j}$, we get

$$\left(w_A\lambda\right)^i = \begin{cases} \lambda^{i-j} & \text{if } i \neq j \\ -\lambda^1 - \cdots - \lambda^r & \text{if } i = j \end{cases} , \tag{5.20}$$

$\det(w_A) = (-1)^{rj}$, and $\pi_A(\lambda) = \lambda + (r+1)\,\Lambda_{r+1-j}$,

2. for $g = B_r$ the only possibility is $A = A_1$, and we get

$$\left(w_A\lambda\right)^i = \begin{cases} \lambda^i & \text{if } i \neq 1 \\ -\lambda^1 - \cdots - \lambda^r & \text{if } i = 1 \end{cases}, \tag{5.21}$$

$\det(w_A) = -1$, and $\pi_A(\lambda) = \lambda + (r+1)\,\Lambda_1$

3. for $g = C_r$ and $A = A_r$ satisfying $A\Lambda_i = \Lambda_{r-i}$, we get

$$\left(w_A\lambda\right)^i = \begin{cases} \lambda^{r-i} & \text{if} i \neq r \\ -\lambda^1 - \cdots - \lambda^r & \text{if } i = r \end{cases}, \tag{5.22}$$

$\det(w_A) = (-1)^{\frac{1}{2}r(1+r)}$, and $\pi_A(\lambda) = \lambda + (r+1)\,\Lambda_r$.

From our main result (4.10), we immediately find

$$\boxed{\ell_{A\lambda}^{\,A\mu} = \ell_{\pi_A(\lambda)}^{\,\mu}\,\frac{|W(A\lambda-\rho)|}{|W(\pi_A\lambda-\rho)|}\,\det(w_A)}\quad, \tag{5.23}$$

for any $\lambda, \mu \in P^n_{++}$, provided $\pi_A(\lambda) \in P^n_{+}$. Unfortunately π_A will only be a permutation (the identity permutation) of P^n_{++} in the trivial case when $A = id$, so it is not easy to see what (5.23) directly implies for the $m_\lambda^{\ \mu}$. As an example of (5.23) consider the dominant sector P^6_{++} of $su(3)$,

$$P^6_{++} = \{(1,1),(2,1),(1,2),(3,1),(2,2),(1,3),(4,1),(3,2),(2,3),(1,4)\}\ . \tag{5.24}$$

After acting with automorphism $A = A_1$ on the dominant sector above we get

$$A(P^6_{++}) = \{(4,1),(3,2),(3,1),(2,3),(2,2),(2,1),(1,4),(1,3),(1,2),(1,1)\}\ . \tag{5.25}$$

Therefore, the left-hand side of the matrix ℓ in equation (5.23) is read as ($\bar{1} := -1$, etc)

	(4,1)	(3,2)	(3,1)	(2,3)	(2,2)	(2,1)	(1,4)	(1,3)	(1,2)	(1,1)
(4,1)	1	0	0	0	$\bar{1}$	0	0	0	0	1
(3,2)	0	1	0	0	0	$\bar{1}$	0	$\bar{1}$	0	0
(3,1)	0	0	1	0	0	0	0	0	$\bar{1}$	0
(2,3)	0	0	$\bar{1}$	1	0	0	0	0	$\bar{1}$	0
(2,2)	0	0	0	0	1	0	0	0	0	$\bar{2}$
(2,1)	0	0	0	0	0	1	0	0	0	0
(1,4)	0	0	0	0	$\bar{1}$	0	1	0	0	1
(1,3)	0	0	0	0	0	$\bar{1}$	0	1	0	0
(1,2)	0	0	0	0	0	0	0	0	1	0
(1,1)	0	0	0	0	0	0	0	0	0	1

In this case, the transformation π_A is given by

$$\pi_A(\lambda) = \lambda + 3\Lambda_2 \tag{5.26}$$

and P^6_{++} goes into

$$\pi_A(P^6_{++}) = \{(1,4),(2,4),(1,5),(3,4),(2,5),(1,6),(4,4),(3,5),(2,6),(1,7)\}\ . \tag{5.27}$$

Clearly, only weights (1,4), (2,4), and (1,5) are in P^6_+, and as a result the matrix $\ell^{\mu}_{\pi_A(\lambda)}$ shrinks to

	(1,1)	(2,1)	(1,2)	(3,1)	(2,2)	(1,3)	(4,1)	(3,2)	(2,3)	(1,4)
(1,4)	1	0	0	0	$\bar{1}$	0	0	0	0	1
(2,4)	0	1	0	0	0	$\bar{1}$	0	$\bar{1}$	0	0
(1,5)	0	0	1	0	0	0	0	0	$\bar{1}$	0

There are also Galois symmetries of the Kac-Peterson modular matrix $S^{(n)}$, first discovered in [35] [36] (and generalized to all rational conformal field theories in [37]). The $S^{(n)}_{\lambda\mu}/F_n$ and $E^{(n)}_{\lambda}(\sigma)$ are polynomials with rational coefficients in a primitive (nN)-th root of unity, where $N = |M^*/M|^{\frac{1}{2}}$, M here being the weight lattice of g. So, any polynomial relation involving them and rational numbers only, will also be satisfied if this primitive (nN)-th root of unity is replaced by another.

Let a be an integer coprime to nN, and let $g_a(S^{(n)})$ denote the Kac-Peterson matrix after the primitive (nN)-th root of unity is replaced by its a-th power (ignoring here the irrelevant factor F_n). For such a, and for $\lambda \in P^n_{++}$, recall the quantities $[a\lambda] \in P^n_{++}$ and $\epsilon(a\lambda) \in \{\pm 1\}$ defined around (5.1). For each a coprime to nN, the map $\lambda \mapsto [a\lambda]$ is a permutation of P^n_{++}. As an example, the Galois transformations of the dominant sector P^5_{++} of $g = su(3)$ for $a = 2$ and $a = 7$ are shown on the next page (Fig. 5).

From (4.1) we can find the Galois transformation of the $S^{(n)}$ matrix replacing q by q^a. The Galois transformation of the weight μ can be written as

$$a\mu = r[a\mu] + n\alpha \tag{5.28}$$

where $r \in W$ and $\alpha \in M^*$. So, we have

$$g_a(S^{(n)}_{\lambda\mu}) = S^{(n)}_{\lambda,a\mu} = F_n \sum_{v\in W} det(v)\lambda^{v(r[a\mu]+n\alpha)} \tag{5.29}$$

Since the coroot lattice M^* is invariant under the Weyl group we get

$$S^{(n)}_{\lambda,a\mu} = \epsilon(a\mu) S^{(n)}_{\lambda,[a\mu]} \quad , \tag{5.30}$$

and using the symmetry of the $S^{(n)}$ matrix we arrive at

$$g_a(S^{(n)}_{\lambda\mu}) = S^{(n)}_{a\lambda,\mu} = \epsilon(a\lambda) S^{(n)}_{[a\lambda],\mu} \quad . \tag{5.31}$$

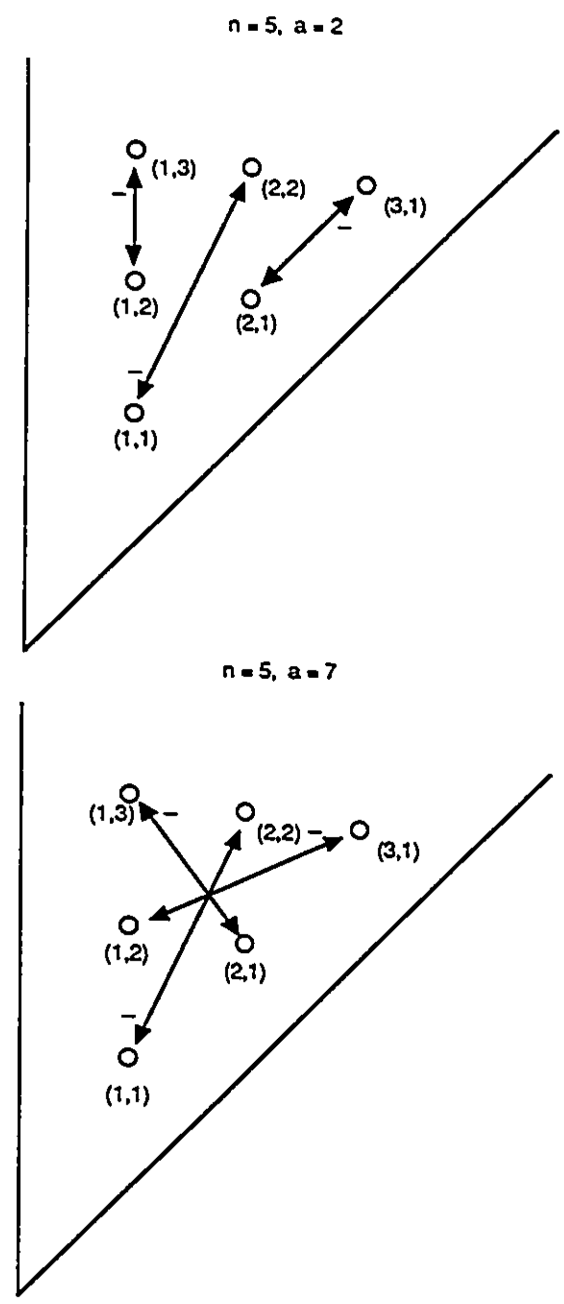

Figure 5. The Galois transformations of the dominant sector of $g = su(3)$ algebra for $n = 5$ and $a = 2, 7$. A minus sign means that the sign of the Weyl reflection is (-1).

In a similar fashion, we find

$$g_a\left(E_\lambda^{(n)}(\sigma)\right) = \frac{|W(\lambda-\rho)|}{|W(\pi_a\lambda-\rho)|}\,E^{(n)}_{\pi_a(\lambda)}(\sigma) = E_\lambda^{(n)}([a\sigma]) \ , \tag{5.32}$$

where π_a denotes the one-to-one map from P^n_{++} to P_{++} defined by

$$\boxed{\pi_a(\lambda) := [a\lambda - a\rho] + \rho} \ . \tag{5.33}$$

Fig. 6 shows the π_7 transformation of P^5_{++} for $su(3)$.

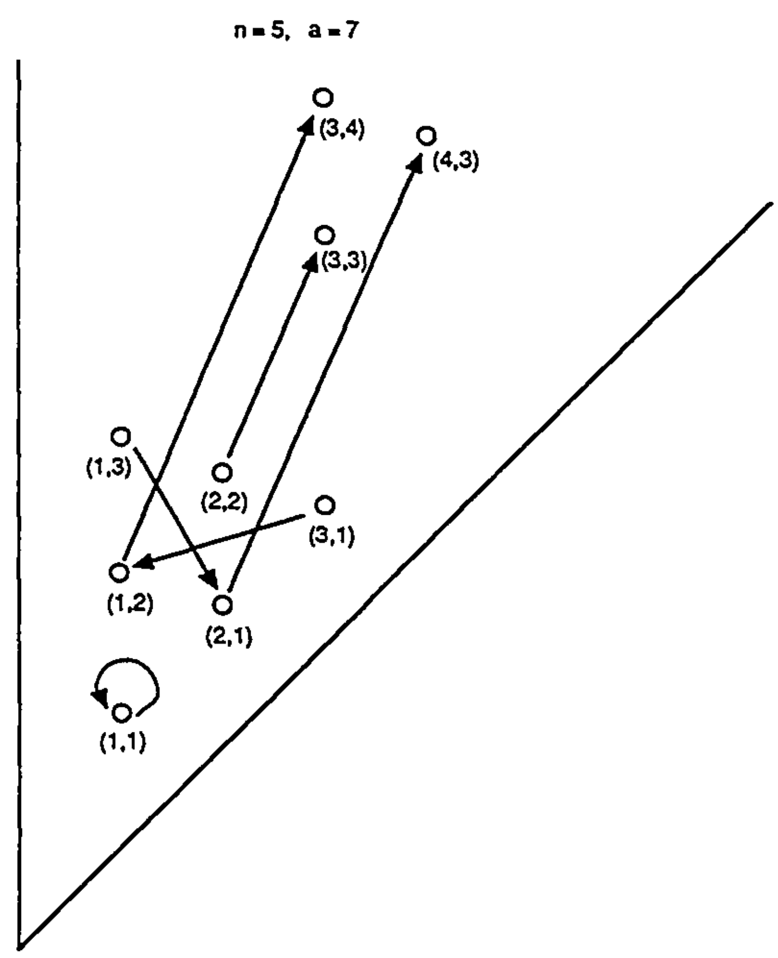

Figure 6. The π_a transformation of the dominant sector of $su(3)$ algebra for $n = 5$ and $a = 7$. The weights (3,4),(3,3), and (4,3) are not in P^5_{++}. Clearly, this is a mapping $P^5_{++} \longrightarrow P_{++}$.

A little work yields

$$\boxed{\ell_\lambda^{\ \mu} \;=\; \epsilon(a\mu)\,\epsilon(a\rho)\,\frac{|W(\lambda-\rho)|}{|W(\pi_a\lambda-\rho)|}\sum_{\nu\in P^n_{++}} \ell_{\pi_a(\lambda)}^{\ \nu}\, N^{(n)\ \ [a\mu]}_{\nu[a\rho]}}\ , \tag{5.34}$$

or

$$\ell_{\pi_a(\lambda)}^{\ \beta} \;=\; \sum_{[a\mu]}\epsilon(a\mu)\,\epsilon(a\rho)\,\frac{|W(\pi_a\lambda-\rho)|}{|W(\lambda-\rho)|}\,\ell_\lambda^{\ \mu}\, N^{-1\ \ \beta}_{[a\rho][a\mu]}\ , \tag{5.35}$$

whenever $\mu\in P^n_{++}$ and $\lambda,\pi_a(\lambda)\in P_{++}\cap P^n_{+}$. Here we have used the Verlinde formula [30] (4.12) for the fusion coefficients $N^{(n)\ \nu}_{\lambda\mu}$. Let $N^{(n)}_\lambda$ denote the *fusion matrix* defined by $(N^{(n)}_\lambda)^\nu_\mu := N^{(n)\ \nu}_{\lambda\mu}$. The matrix $N^{(n)}_{[a\rho]}$ will always be invertible [37], so (5.34) tells us that for any fixed $\lambda,\sigma\in P_{++}$, the values $\ell_{\pi_a(\lambda)}^{\ \sigma}$ will be known once the $\ell_\lambda^{\ \sigma}$ are known, and conversely, provided a and n satisfy the usual conditions. Equation (5.34) can also be interpreted as an expression for the fusion matrices $N^{(n)}_{[a\rho]}$ in terms of the ℓ's and m's as it is given below:

$$N^{(n)\ \ [a\mu]}_{[a\rho]\beta} \;=\; \epsilon(a\mu)\,\epsilon(a\rho)\sum_{\gamma\in P_{++}\cap P^n_{+}}\frac{|W(\gamma-\rho)|}{|W(\pi_a^{-1}\gamma-\rho)|}\,m_\beta^{\ \gamma}\,\ell_{\pi_a^{-1}(\gamma)}^{\ \mu}\ . \tag{5.36}$$

For example, in a case of $\hat{g}=su(3)_n$ for $n=5$ and $a=7$, the Galois transformation of the dominant sector

$$P^5_{++}=\{[3,1,1],[2,2,1],[2,1,2],[1,3,1],[1,2,2],[1,1,3]\} \tag{5.37}$$

reads

$$[7(P^5_{++})]=\{[1,2,2],(-)[1,1,3],(-)[1,3,1],[2,1,2],(-)[3,1,1],[2,2,1]\}\ . \tag{5.38}$$

Here a minus sign in front of a weight means that the sign of the Weyl reflection is (-1). Let's choose $\mu=\beta=(21)$ so that $\epsilon(a\mu)=-1$ and $\epsilon(a\rho)=1$. Using the formula (4.14) on the LHS we get

$$N^{(5)\ \ (1,3)}_{(2,2)(2,1)}=1\ . \tag{5.39}$$

On the RHS, we have a summation over weights γ from the set

$$P_{++} \cap P^5_+ = P^5_{++} \cup \{[0,1,4],[0,4,1],[0,3,2],[0,2,3]\} \quad . \tag{5.40}$$

However, the multiplicity $m_{(2,1)}{}^{\gamma}$ is equal to one for $\gamma = (2,1)$ and zero for all other values of γ. As a result, the RHS is given by the value of

$$-\ell_{\pi_a^{-1}(2,1)}{}^{(2,1)} = -\ell_{(1,3)}{}^{(2,1)} = 1 \quad . \tag{5.41}$$

Using the unitarity of the $S^{(n)}$ matrix, the Verlinde formula (4.12) can be rewritten as

$$S^{(n)}_{\lambda\sigma} S^{(n)}_{\mu\sigma} = S^{(n)}_{\rho\sigma} \sum_{\nu \in P^n_{++}} N^{(n)\nu}_{\lambda\mu} S^{(n)}_{\nu\sigma} \quad . \tag{5.42}$$

This is a polynomial equation with rational coefficients satisfied by a primitive (nN) – th root of unity. Therefore we can apply the Galois transformation on it and as a result we get

$$\epsilon(a\lambda) S^{(n)}_{[a\lambda]\sigma} \epsilon(a\mu) S^{(n)}_{[a\mu]\sigma} = \epsilon(a\rho) S^{(n)}_{[a\rho]\sigma} \sum_{\nu \in P^n_{++}} N^{(n)\nu}_{\lambda\mu} \epsilon(a\nu) S^{(n)}_{[a\nu]\sigma} \quad . \tag{5.43}$$

Now if we use (5.42) on the LHS of the last equation we can write it as

$$\sum_{\beta \in P^n_{++}} N^{(n)}_{[a\lambda][a\mu]}{}^{\beta} S^{(n)}_{\beta\sigma} = \sum_{\nu \in P^n_{++}} \frac{\epsilon(a\rho)\epsilon(a\nu)}{\epsilon(a\lambda)\epsilon(a\mu)} N^{(n)\nu}_{\lambda\mu} \frac{S^{(n)}_{[a\rho]\sigma} S^{(n)}_{[a\nu]\sigma}}{S^{(n)}_{\rho\sigma}} \quad . \tag{5.44}$$

After using unitarity of $S^{(n)}$ matrix we arrive at

$$N^{(n)}_{[a\lambda][a\mu]}{}^{\beta} = \sum_{\nu \in P^n_{++}} \frac{\epsilon(a\rho)\epsilon(a\nu)}{\epsilon(a\lambda)\epsilon(a\mu)} N^{(n)}_{\lambda\mu}{}^{\nu} N^{(n)}_{[a\nu][a\rho]}{}^{\beta} \quad . \tag{5.45}$$

This is another way to find $N's$ if we already know some of them [37].

Let us do one explicit example of the formula above. Consider $n = 6$, and $a = 7$. The Galois transformation of the dominant sector

$$\begin{aligned} P^6_{++} = \{&[4,1,1],[3,2,1],[2,3,1],[1,4,1],[3,1,2], \\ &[2,2,2],[1,3,2],[2,1,3],[1,2,3],[1,1,4]\} \end{aligned} \tag{5.46}$$

is given by

$$[7(P^6_{++})] = \{[4,1,1],[1,3,2],[3,1,2],[1,4,1],[1,2,3],$$
$$[2,2,2],[2,1,3],[3,2,1],[2,3,1],[1,1,4]\} \tag{5.47}$$

with $\epsilon(a\lambda) = 1$ for all $\lambda \in P^6_{++}$. The entries of the $N^{(6)}$ matrices on both sides of (5.45) are calculated using formula (4.14). For example if $\lambda = (2,2)$ and $\mu = (2,2)$ the only values of $N_{(2,2)(2,2)}{}^{\sigma}$ (for all $\sigma \in P^{++}$) different from zero are

$$N^{(6)}_{(2,2)(2,2)}{}^{(2,2)} = 2,$$
$$N^{(6)}_{(2,2)(2,2)}{}^{(4,1)} = N^{(6)}_{(2,2)(2,2)}{}^{(1,4)} = N^{(6)}_{(2,2)(2,2)}{}^{(1,1)} = 1 \ . \tag{5.48}$$

At the same time, the RHS of (5.45) reads

$$\sum_{\nu \in P^6_{++}} N^{(6)}_{(2,2)(2,2)}{}^{\nu} N^{(6)}_{(1,1)[a\nu]}{}^{\beta} \ . \tag{5.49}$$

If $\beta = (2,2)$, the only value of $N^{(6)}_{(1,1)[a\nu]}{}^{(2,2)}$ different from 0 is the one for $\nu = (2,2)$. It is straightforward to verify that for the other values of β, the LHS is equal to the RHS.

It is again difficult to express this Galois symmetry directly at the level of the multiplicities $m_\lambda{}^\mu$ themselves. But if π_a *is* a permutation of P^n_{++}, we get from (5.34) :

$$\boxed{m_\lambda{}^\mu = \epsilon(a\mu)\,\epsilon(a\rho)\,\frac{|W(\pi_a\lambda - \rho)|}{|W(\lambda - \rho)|} \sum_{\nu \in P^n_{++}} (N^{(n)-1}_{[a\rho]})^{\nu}{}_{[a\lambda]}\, m_\nu{}^{\pi_a(\mu)}} \ . \tag{5.50}$$

A special case of (5.34) occurs when $[a\rho] = \rho$. Then $\epsilon(a\rho) = \epsilon(a\lambda)$ for all $\lambda \in P^n_{++}$ (apply (5.31) with $\mu = \rho$, together with the fact that $S^{(n)}_{\rho\nu} > 0$ for all $\nu \in P^n_{++}$). Equation (5.34) reduces to

$$\ell_\lambda{}^\mu = \frac{|W(\lambda - \rho)|}{|W(\pi_a\lambda - \rho)|}\, \ell_{\pi_a(\lambda)}{}^{[a\mu]} \ . \tag{5.51}$$

For example, this happens whenever $a = -1$, and we get an example of (5.11).

More generally, a similar simplification happens whenever $[a\rho] = A\rho$ for some outer automorphism A. To find out how $\ell_\lambda^{\ \mu}$ is changed by such an automorphism we can use the relations for the Galois transformations of $S^{(n)}$ matrix, (5.30) and (5.31), and the Verlinde formula (4.12). If we replace λ by ρ in (5.30) and (5.31) we get

$$\epsilon(a\rho)\exp[-2\pi i(A\omega^0)\cdot(\mu-\rho)]S^{(n)}_{\rho\mu} = \epsilon(a\mu)S^{(n)}_{\rho[a\mu]} \ . \tag{5.52}$$

$S^{(n)}_{\rho\mu}$ is positive for any $\mu \in P^n_{++}$ and so both sides have to be the same sign

$$\epsilon(a\mu) = \epsilon(a\rho)\exp[-2\pi i(A\omega^0)\cdot(\mu-\rho)] \ . \tag{5.53}$$

From the definition of $S^{(n)}$, (4.1) and the Verlinde formula (4.12) it is easy to see that

$$N^{(n)}_{A\lambda A^{-1}\mu}{}^{\nu} = N^{(n)}_{\lambda\mu}{}^{\nu} \tag{5.54}$$

and therefore

$$N^{(n)}_{A\lambda\mu}{}^{\nu} = N^{(n)}_{\lambda A\mu}{}^{\nu} \ . \tag{5.55}$$

This is useful because after replacing $[a\mu]$ by $A\rho$ in (5.34) we get the fusion coefficients

$$N^{(n)}_{\nu A\rho}{}^{[a\mu]} = N^{(n)}_{A\nu\rho}{}^{[a\mu]} = \delta_{A\nu}{}^{[a\mu]} \ . \tag{5.56}$$

Therefore the RHS summation is reduced to one term, and after substituting (5.53) we arrive at

$$\ell_\lambda^{\ \mu} = \frac{|W(\lambda-\rho)|}{|W(\pi_a\lambda-\rho)|}\ell_{\pi_a(\lambda)}{}^{A^{-1}[a\mu]}\exp[-2\pi i\,(A\omega^0)\cdot(\mu-\rho)] \ . \tag{5.57}$$

Similarly, suppose $[a\mu] = A\rho$, for some outer automorphism A. Then provided $\lambda, \pi_a(\lambda) \in P_{++} \cap P^n_+$, (5.34) reduces to

$$\ell_\lambda^{\ \mu} = \epsilon(a\mu)\,\epsilon(a\rho)\frac{|W(\lambda-\rho)|}{|W(\pi_a\lambda-\rho)|}\,\ell_{\pi_a\lambda}^{\ A[a\rho]} \ . \tag{5.58}$$

To illustrate the above relation we do a simple example for the $\hat{g} = su(3)$ algebra. Let us choose $n = 7$, $a = 4$, and $A = A_1$. The Galois and π_4 transformations of the dominant sector

$$P_{++}^{7} = \{[5,1,1],[4,2,1],[3,3,1],[2,4,1],[1,5,1],[4,2,1],[3,2,2],[2,3,2],$$
$$[1,4,2],[3,1,3],[2,2,3],[1,3,3],[2,1,4],[1,2,4],[1,1,5]\} \tag{5.59}$$

read

$$[4(P_{++}^{7})] = \{(-)[1,3,3],[4,2,1],(-)[2,2,3],[2,4,1],(-)[3,1,3],[4,2,1],[5,1,1],[1,4,2],$$
$$(-)[2,3,2],[1,1,5],(-)[3,2,2],[2,1,4],[1,2,4],(-)[3,3,1]\} \tag{5.60}$$

and

$$[\pi_4(P_{++}^{7})] = \{(1,1),(5,1),(7,2),(3,6),(1,6),(1,5),(-)(4,4),(3,2),$$
$$(-)(3,4),(2,7),(2,3),(2,2),(6,3),(-)(4,3),(6,1)\} \tag{5.61}$$

For a given A it is easy to see that

$$A\rho = (5,1)$$
$$A[a\rho] = A(3,3) = (1,3) \tag{5.62}$$

so that $\mu = (3,2)$ satisfies $[a\mu] = (5,1)$. Finally if we choose the weight λ to be $(5,1)$ we get

$$\ell_{(5,1)}^{\;(3,2)} = -\ell_{(1,6)}^{(1,3)} \tag{5.63}$$

Using the values of ℓ's entries given in Appendix, we can check that the equation above is valid.

Another noteworthy special case of (5.34) involves those weights λ' with the property that $\ell_{\lambda'}^{\mu} = \delta_{\lambda'}^{\mu}$ for all $\mu \in P_{++}$, i.e., those λ' for which $\lambda' - \rho$ is a miniscule weight. For $su(r+1)$, they are the fundamental weights $\lambda' = \Lambda_{(i)} + \rho$. Then for any $\lambda \in P_{++}^{(n)}$ with $\pi_a(\lambda) = \lambda'$,

$$\ell_{\lambda}^{\;\mu} = \epsilon(a\mu)\,\epsilon(a\rho)\,\frac{|W(\lambda-\rho)|}{|W(\lambda'-\rho)|}\,N_{\lambda'[a\rho]}^{(n)\;\;[a\mu]}\ , \tag{5.64}$$

if, as usual, a is coprime to nN and $\lambda', \mu \in P^n_{++}$. The fusions involving the fundamental weights $\Lambda_{(i)} + \rho$ are easy to compute, so the RHS of (5.64) can be explicitly evaluated in all cases. For example, consider $k = 4$, $a = 4$, $\lambda = (3,3)$, and $\mu = (1,4)$. For the chosen values eq. (5.64) reads

$$\ell_{(3,3)}{}^{(1,4)} = -N^{(7)}_{(2,2)(3,3)}{}^{(1,4)} \quad . \tag{5.65}$$

The RHS can be calculated using the formula (4.14) and the LHS is given in the Appendix.

One of the reasons equations (5.34) - (5.64) could be interesting is because they suggest the rank-level duality that WZW fusions satisfy [38] could appear in some way in the ℓ's and m's.

6. Conclusion

The goal of physics as a science is to describe laws of nature. The most fundamental laws are usually the simplest ones, too. The simplicity of fundamental physics is shown through fundamental symmetry principles. Symmetry leads to group theory which is today widely used as one of the dominant methods describing the laws of physics. It is believed that the diversity of nature should be matched by a diversity of symmetries and therefore it should give rise to many different groups as mathematical tools describing them. The main purpose of this thesis was to examine a connection between simple Lie algebras and higher symmetry algebras such as affine Kac-Moody and Virasoro algebras.

The long preliminary chapter at the begining of the thesis was written to introduce everything we thought was necessary for a reader who is not expert in this field to understand the main subject of our research: the connection between CFT and Lie algebras of different kinds. The simple Lie algebras are described first and then affine Lie algebras are developed as a natural generalization of simple Lie algebras.

The characters of representations are very important in representation theory. They are intrinsic properties of representations, i.e., equivalent representations have the same characters. They obey the same rules in a direct product of representations as the representations themselves. Lie characters and multiplicities are directly related; they determine each other. The Weyl group can be used as a tool to calculate characters of both simple and affine Lie algebras. The results are the Weyl character formula for characters of simple Lie algebras and the Weyl-Kac character formula for affine characters, introduced in sections (2.2) and (2.4).

The characters of affine Lie algebras transform among themselves under the action of a the modular group. The starting point for this work is that the characters of the underlying simple Lie algebra, evaluated at special points, are equal to ratios of elements of the Kac-Peterson modular matrix [20]. Our main result, the expression (4.10) for the inverse of the matrix of dominant weight multiplicities in terms of the Kac-Peterson modular matrix $S^{(n)}$ and the even Weyl orbit sums $E_{\lambda}^{(n)}(\sigma)$, is a direct consequence of this discovery. Surely the even Weyl orbit sums will find other uses in the study of WZW models and affine Kac-Moody algebras.

Also obtained were relations (5.10), (5.23) and (5.34) among the "inverse multiplicities" $\ell_{\lambda}^{\ \mu}$ that are consequences of the symmetries of the Kac-Peterson matrices and of the $E_{\lambda}^{(n)}(\sigma)$. Relations between the dominant weight multiplicities $m_{\lambda}^{\ \mu}$ follow in certain cases, as eq. (5.50) shows. Most of these relations are exemplified in this thesis by explicit calculations. The new relations would be difficult to understand from the point of view of Lie groups and their simple Lie algebras only, but are quite natural in WZW models, or in their affine Kac-Moody current algebras. The new relations arise because the simple Lie algebras are special subalgebras of these affine current algebras. The affine Weyl and outer automorphism symmetries are especially natural in the Kac-Moody context.

Simple Lie algebras and their representations are already well understood. But there are still many unanswered questions about affine Lie and conformal symmetries, their representations, and especially the CFTs that realize them. The close connection between simple Lie algebras and affine Lie algebras studied in this thesis, may prove useful in answering some of these questions.

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APPENDIX

The $SU(3)$ and G_2 groups' inverse weight multiplicity matrices ℓ and weight multiplicity matrices m are listed below for different values of the level k. The equation (4.10) was used to calculate elements of the matrix ℓ. The coefficients ℓ_λ^μ are integers but not necessarily non-negative. The matrix m is obtained as the inverse of ℓ. In both cases, the matrices of lower levels are contained as submatrices in the higher-level ones. The matrices ℓ and m are both lower triangular. The matrix element m_λ^μ is the multiplicity of the weight $\mu - \rho$ in the representation with the highest weight $\lambda - \rho$, and so is a non-negative integer. The order of the columns in the matrices listed below is the same as the order of the rows.

- List of the ℓ matrices for $SU(3)$ Lie group for $k = 1, 2, 3, 4, 5$:

$$\begin{matrix}(0,0)\\(1,0)\\(0,1)\end{matrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$\begin{matrix}(0,0)\\(1,0)\\(0,1)\\(2,0)\\(1,1)\\(0,2)\end{matrix}\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&\bar{1}&1&0&0\\\bar{2}&0&0&0&1&0\\0&\bar{1}&0&0&0&1\end{pmatrix}$$

$$
\begin{array}{c}
(0,0)\\(1,0)\\(0,1)\\(2,0)\\(1,1)\\(0,2)\\(3,0)\\(2,1)\\(1,2)\\(0,3)
\end{array}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \bar{1} & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
\bar{2} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & \bar{1} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & \bar{1} & 0 & 1 & 0 & 0 & 0\\
0 & \bar{1} & 0 & 0 & 0 & \bar{1} & 0 & 1 & 0 & 0\\
0 & 0 & \bar{1} & \bar{1} & 0 & 0 & 0 & 0 & 1 & 0\\
1 & 0 & 0 & 0 & \bar{1} & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

$$
\begin{array}{c}
(0,0)\\ (1,0)\\ (0,1)\\ (2,0)\\ (1,1)\\ (0,2)\\ (3,0)\\ (2,1)\\ (1,2)\\ (0,3)\\ (4,0)\\ (3,1)\\ (2,2)\\ (1,3)\\ (0,4)
\end{array}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \bar{1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\bar{2} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \bar{1} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & \bar{1} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \bar{1} & 0 & 0 & 0 & \bar{1} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \bar{1} & \bar{1} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & \bar{1} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & \bar{1} & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & \bar{1} & 0 & 0 & 0 & 0 & \bar{1} & 0 & 0 & 1 & 0 & 0 & 0\\
\bar{1} & 0 & 0 & 0 & 0 & 0 & \bar{1} & 0 & 0 & \bar{1} & 0 & 0 & 1 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & \bar{1} & 0 & \bar{1} & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \bar{1} & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

$$
\begin{array}{c}
(0,0)\\ (1,0)\\ (0,1)\\ (2,0)\\ (1,1)\\ (0,2)\\ (3,0)\\ (2,1)\\ (1,2)\\ (0,3)\\ (4,0)\\ (3,1)\\ (2,2)\\ (1,3)\\ (0,4)\\ (5,0)\\ (4,1)\\ (3,2)\\ (2,3)\\ (1,4)\\ (0,5)
\end{array}
\left(\begin{array}{ccccccccccccccccccccc}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&\bar{1}&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
\bar{2}&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&\bar{1}&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&\bar{1}&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&\bar{1}&0&0&0&\bar{1}&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&\bar{1}&\bar{1}&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&\bar{1}&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&\bar{1}&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&1&\bar{1}&0&0&0&0&\bar{1}&0&0&1&0&0&0&0&0&0&0&0&0\\
\bar{1}&0&0&0&0&0&\bar{1}&0&0&\bar{1}&0&0&1&0&0&0&0&0&0&0&0\\
0&1&0&0&0&\bar{1}&0&\bar{1}&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&\bar{1}&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&\bar{1}&0&0&0&1&0&0&0&0&0\\
0&0&0&0&1&0&\bar{1}&0&0&0&0&0&\bar{1}&0&0&0&1&0&0&0&0\\
0&\bar{1}&0&0&0&1&0&0&0&0&\bar{1}&0&0&\bar{1}&0&0&0&1&0&0&0\\
0&0&\bar{1}&1&0&0&0&0&0&0&0&\bar{1}&0&0&\bar{1}&0&0&0&1&0&0\\
0&0&0&0&1&0&0&0&0&\bar{1}&0&0&\bar{1}&0&0&0&0&0&0&1&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&\bar{1}&0&0&0&0&0&0&1
\end{array}\right).
$$

- List of the m matrices for $SU(3)$ Lie group for $k = 1, 2, 3, 4, 5$:

$$
\begin{array}{c}
(0,0) \\ (1,0) \\ (0,1)
\end{array}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
$$

$$
\begin{array}{c}
(0,0) \\ (1,0) \\ (0,1) \\ (2,0) \\ (1,1) \\ (0,2)
\end{array}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
2 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

$$
\begin{array}{c}
(0,0) \\ (1,0) \\ (0,1) \\ (2,0) \\ (1,1) \\ (0,2) \\ (3,0) \\ (2,1) \\ (1,2) \\ (0,3)
\end{array}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

$$
\begin{array}{c}
(0,0)\\ (1,0)\\ (0,1)\\ (2,0)\\ (1,1)\\ (0,2)\\ (3,0)\\ (2,1)\\ (1,2)\\ (0,3)\\ (4,0)\\ (3,1)\\ (2,2)\\ (1,3)\\ (0,4)
\end{array}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 2 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0\\
3 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0\\
0 & 2 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

$$
\begin{matrix}
(0,0)\\ (1,0)\\ (0,1)\\ (2,0)\\ (1,1)\\ (0,2)\\ (3,0)\\ (2,1)\\ (1,2)\\ (0,3)\\ (4,0)\\ (3,1)\\ (2,2)\\ (1,3)\\ (0,4)\\ (5,0)\\ (4,1)\\ (3,2)\\ (2,3)\\ (1,4)\\ (0,5)
\end{matrix}
\begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
2&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&2&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&2&1&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&1&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&1&0&1&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&2&2&0&0&0&0&1&0&0&1&0&0&0&0&0&0&0&0&0\\
3&0&0&0&2&0&1&0&0&1&0&0&1&0&0&0&0&0&0&0&0\\
0&2&0&0&0&2&0&1&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&1&1&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&1&1&0&0&0&0&1&0&0&1&0&0&0&1&0&0&0&0&0\\
2&0&0&0&2&0&2&0&0&1&0&0&1&0&0&0&1&0&0&0&0\\
0&3&0&0&0&2&0&2&0&0&1&0&0&1&0&0&0&1&0&0&0\\
0&0&3&2&0&0&0&0&2&0&0&1&0&0&1&0&0&0&1&0&0\\
2&0&0&0&2&0&1&0&0&2&0&0&1&0&0&0&0&0&0&1&0\\
0&1&0&0&0&1&0&1&0&0&0&0&0&1&0&0&0&0&0&0&1
\end{pmatrix}.
$$

- List of the ℓ matrices for G_2 Lie group for $k = 1, 2, 3, 4, 5$:

$$\begin{matrix}(0,0)\\(0,1)\end{matrix}\begin{pmatrix}1 & 0\\ \bar{1} & 1\end{pmatrix}$$

$$\begin{matrix}(0,0)\\(0,1)\\(1,0)\\(0,2)\end{matrix}\begin{pmatrix}1 & 0 & 0 & 0\\ \bar{1} & 1 & 0 & 0\\ \bar{1} & \bar{1} & 1 & 0\\ 0 & \bar{1} & \bar{1} & 1\end{pmatrix}$$

$$\begin{matrix}(0,0)\\(0,1)\\(1,0)\\(0,2)\\(1,1)\\(0,3)\end{matrix}\begin{pmatrix}1 & 0 & 0 & 0 & 0 & 0\\ \bar{1} & 1 & 0 & 0 & 0 & 0\\ \bar{1} & \bar{1} & 1 & 0 & 0 & 0\\ 0 & \bar{1} & \bar{1} & 1 & 0 & 0\\ 2 & 0 & 0 & \bar{2} & 1 & 0\\ 0 & 1 & \bar{1} & 0 & \bar{1} & 1\end{pmatrix}$$

$$
\begin{array}{c}
(0,0)\\(0,1)\\(1,0)\\(0,2)\\(1,1)\\(0,3)\\(2,0)\\(1,2)\\(0,4)
\end{array}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\bar{1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\bar{1} & \bar{1} & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \bar{1} & \bar{1} & 1 & 0 & 0 & 0 & 0 & 0\\
2 & 0 & 0 & \bar{2} & 1 & 0 & 0 & 0 & 0\\
0 & 1 & \bar{1} & 0 & \bar{1} & 1 & 0 & 0 & 0\\
\bar{1} & 1 & 0 & 0 & 0 & \bar{1} & 1 & 0 & 0\\
0 & 0 & 1 & 1 & \bar{1} & \bar{1} & \bar{1} & 1 & 0\\
\bar{1} & 0 & 1 & 0 & 0 & 0 & 0 & \bar{1} & 1
\end{pmatrix}
$$

$$
\begin{array}{c}
(0,0)\\(0,1)\\(1,0)\\(0,2)\\(1,1)\\(0,3)\\(2,0)\\(1,2)\\(0,4)\\(2,1)\\(1,3)\\(0,5)
\end{array}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\bar{1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\bar{1} & \bar{1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \bar{1} & \bar{1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
2 & 0 & 0 & \bar{2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & \bar{1} & 0 & \bar{1} & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
\bar{1} & 1 & 0 & 0 & 0 & \bar{1} & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 1 & \bar{1} & \bar{1} & \bar{1} & 1 & 0 & 0 & 0 & 0\\
\bar{1} & 0 & 1 & 0 & 0 & 0 & 0 & \bar{1} & 1 & 0 & 0 & 0\\
0 & \bar{1} & 0 & 1 & 0 & 1 & 0 & \bar{1} & \bar{1} & 1 & 0 & 0\\
2 & \bar{1} & \bar{1} & 0 & 2 & 0 & \bar{1} & 0 & \bar{1} & \bar{1} & 1 & 0\\
\bar{1} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \bar{1} & 1
\end{pmatrix}.
$$

- List of the m matrices for G_2 Lie group for $k = 1, 2, 3, 4, 5$:

$$\begin{matrix}(0,0)\\(0,1)\end{matrix}\begin{pmatrix}1 & 0\\1 & 1\end{pmatrix}$$

$$\begin{matrix}(0,0)\\(0,1)\\(1,0)\\(02)\end{matrix}\begin{pmatrix}1 & 0 & 0 & 0\\1 & 1 & 0 & 0\\2 & 1 & 1 & 0\\3 & 2 & 1 & 1\end{pmatrix}$$

$$\begin{matrix}(0,0)\\(0,1)\\(1,0)\\(0,2)\\(1,1)\\(0,3)\end{matrix}\begin{pmatrix}1 & 0 & 0 & 0 & 0 & 0\\1 & 1 & 0 & 0 & 0 & 0\\2 & 1 & 1 & 0 & 0 & 0\\3 & 2 & 1 & 1 & 0 & 0\\4 & 4 & 2 & 2 & 1 & 0\\5 & 4 & 3 & 2 & 1 & 1\end{pmatrix}$$

$$
\begin{array}{c}
(0,0)\\ (0,1)\\ (1,0)\\ (0,2)\\ (1,1)\\ (0,3)\\ (2,0)\\ (1,2)\\ (0,4)
\end{array}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
3 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0\\
4 & 4 & 2 & 2 & 1 & 0 & 0 & 0 & 0\\
5 & 4 & 3 & 2 & 1 & 1 & 0 & 0 & 0\\
5 & 3 & 3 & 2 & 1 & 1 & 1 & 0 & 0\\
9 & 8 & 6 & 5 & 3 & 2 & 1 & 1 & 0\\
8 & 7 & 5 & 5 & 3 & 2 & 1 & 1 & 1
\end{pmatrix}
$$

$$
\begin{array}{c}
(0,0)\\ (0,1)\\ (1,0)\\ (0,2)\\ (1,1)\\ (0,3)\\ (2,0)\\ (1,2)\\ (0,4)\\ (2,1)\\ (1,3)\\ (0,5)
\end{array}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
3 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
4 & 4 & 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
5 & 4 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
5 & 3 & 3 & 2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0\\
9 & 8 & 6 & 5 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & 0\\
8 & 7 & 5 & 5 & 3 & 2 & 1 & 1 & 1 & 0 & 0 & 0\\
10 & 10 & 7 & 7 & 5 & 3 & 2 & 2 & 1 & 1 & 0 & 0\\
16 & 14 & 12 & 10 & 7 & 6 & 4 & 3 & 2 & 1 & 1 & 0\\
12 & 11 & 9 & 8 & 6 & 5 & 3 & 3 & 2 & 1 & 1 & 1
\end{pmatrix}.
$$