

THE DISTRIBUTION OF THE CLASSICAL ERROR TERMS OF PRIME NUMBER THEORY

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ABSTRACT

The Distribution of the Classical Error Terms of Prime Number Theory

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A function $g : \mathbb{R} \rightarrow \mathbb{R}$ is said to have a limiting distribution μ on \mathbb{R} , if μ is a measure on \mathbb{R} satisfying

•

$$\mu(\emptyset) = 0, \quad \mu(\mathbb{R}) = 1.$$

• For all bounded continuous functions $f : \mathbb{R} \rightarrow \mathbb{R}$,

$$\lim_{Y \rightarrow \infty} \frac{1}{Y} \int_0^Y f(g(y)) dy = \int_{-\infty}^{\infty} f(t) d\mu(t).$$

In this thesis the following results are studied.

(i) Let C be a real constant, $\{\lambda_n\}$ an increasing sequence of positive numbers which tends to infinity. Let $\{r_n\}$ be a sequence of complex numbers. Let $\phi(y) : \mathbb{R} \rightarrow \mathbb{R}$ be a function which is defined for any $X > 0$ by

$$\phi(y) = C + \operatorname{Re} \left(\sum_{\lambda_n \leq X} r_n e^{iy\lambda_n} \right) + E(y, X),$$

where $E(y, X)$ is a real valued function satisfying

$$\int_1^Y |E(y, e^Y)|^2 dy \ll 1.$$

Let $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$ be real constants such that $\theta_4 > -1$ and $2\theta_1 < \theta_4$, and set

$$\mathbf{m} = \begin{cases} (2\theta_4 - 1)/4 & \text{if } -1 < \theta_4 \leq 1 \\ \theta_4/4 & \text{if } 1 \leq \theta_4 \leq 3 \\ 3/4 & \text{if } \theta_4 \geq 3. \end{cases}$$

Suppose that

1.

$$\mathbf{m} > \theta_1;$$

2.

$$\sum_{\lambda_n \leq T} |r_n| \ll T^{\theta_1} (\log T)^{\theta_2};$$

3.

$$\sum_{\lambda_n > T} |r_n|^2 \ll \frac{(\log T)^{\theta_3}}{T^{\theta_4}};$$

4.

$$\sum_{T \leq \lambda_n < T+1} 1 \ll (\log T)^{\theta_5}.$$

Under the assumptions 1 to 4, we prove that $\phi(y)$ has a limiting distribution. As a corollary, we prove the following. Let $\theta_5, \theta_6, \theta_7$ be real constants such that $\theta_6 < 5/4$.

Assume

$$\sum_{T \leq \lambda_n < T+1} 1 \ll (\log T)^{\theta_5}$$

and

$$\sum_{\lambda_n \leq T} \lambda_n^2 |r_n|^2 \ll T^{\theta_6} (\log T)^{\theta_7}.$$

Then $\phi(y)$ has a limiting distribution.

(ii) As a consequence of our general limiting distribution theorems, we deduce the following results.

- For $x > 1$, let

$$\pi(x) := \text{card}(\{p \leq x \mid p \text{ is a prime}\}),$$

and

$$\text{Li}(x) := \int_2^x \frac{dt}{\log t}.$$

Under the assumption of the Riemann hypothesis, $ye^{-y/2}(\pi(e^y) - \text{Li}(e^y))$ has a limiting distribution μ .

This result recovers a theorem of Wintner [30].

- For $n \in \mathbb{N}$, let

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n = p_1 \dots p_k, p_i \text{'s are distinct primes,} \\ 0 & \text{if otherwise,} \end{cases}$$

be the Möbius function. For fixed $\alpha \in [0, 1]$, let

$$M_\alpha(x) = \sum_{n \leq x} \frac{\mu(n)}{n^\alpha},$$

and define

$$\phi_\alpha(y) := \begin{cases} e^{y(-1/2+\alpha)} M_\alpha(e^y) & \text{if } 0 \leq \alpha \leq 1/2 \text{ or } \alpha = 1, \\ e^{y(-1/2+\alpha)} (M_\alpha(e^y) - 1/\zeta(\alpha)) & \text{if } 1/2 < \alpha < 1. \end{cases}$$

Let $\theta < 5/4$ be fixed. We prove that under the assumptions of the Riemann hypothesis and $\sum_{0 < \gamma \leq T} |\zeta(2\rho)/\zeta'(\rho)|^2 \ll T^\theta$, $\phi_\alpha(y)$ has a limiting distribution ν_α .

- For $n \in \mathbb{N}$, let

$$\lambda(n) = (-1)^{\Omega(n)} \tag{1}$$

be the Liouville function, where $\Omega(n)$ is the total number of divisors of n counted with multiplicity. For fixed $\alpha \in [0, 1]$, let

$$L_\alpha(x) = \sum_{n \leq x} \frac{\lambda(n)}{n^\alpha},$$

and define

$$\psi_\alpha(y) := \begin{cases} e^{y(-1/2+\alpha)} L_\alpha(e^y) & \text{if } 0 \leq \alpha < 1/2 \text{ or } \alpha = 1, \\ e^{y(-1/2+\alpha)} (L_\alpha(e^y) - y/2\zeta(1/2)) & \text{if } \alpha = 1/2, \\ e^{y(-1/2+\alpha)} (L_\alpha(e^y) - \zeta(2\alpha)/\zeta(\alpha)) & \text{if } 1/2 < \alpha < 1. \end{cases}$$

Let $\theta < 5/4$ be fixed. We prove that under the assumptions of the Riemann hypothesis and $\sum_{0 < \gamma \leq T} |\zeta'(\rho)|^{-2} \ll T^\theta$, $\psi_\alpha(y)$ has a limiting distribution μ_α .

- Let $q > 1$ and $a \geq 1$, and $\gcd(q, a) = 1$. Define

$$M(x, q, a) := \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} \mu(n),$$

where μ is the Möbius function. Let $\theta < 5/4$ be fixed. We prove that under the assumptions of the Riemann hypothesis for Dirichlet L -functions $L(s, \chi)$ and

$$\sum_{0 < \gamma_\chi \leq T} |L(2\rho_\chi, \chi)/L'(\rho_\chi, \chi)|^2 \ll T^\theta,$$

for characters $\chi \pmod{q}$, $e^{-y/2} M(e^y, q, a)$ has a limiting distribution $\mu_{q, \alpha}$.

(iii) Let $\{\lambda_n\}$ be an increasing sequence of positive real numbers that tends to infinity. Let $\{r_n\}$ be a decreasing sequence of nonnegative real numbers, and set

$$\tilde{N}(x) := \sum_{1/r_n \leq x} 1.$$

For $(\theta_1, \theta_2, \dots) \in \mathbb{T}^\infty$, let

$$X(\theta_1, \theta_2, \dots) = \sum_{n=1}^{\infty} r_n \sin 2\pi\theta_n.$$

By Kolmogorov's three series theorem, if $\sum_{n=1}^{\infty} r_n^2 < \infty$, the above sum converges almost everywhere. Let $c_1, c_2, d_1, d_2, \tilde{c}, \tilde{d} \in \mathbb{R}$ be fixed where $d_j > -1$, and $\tilde{d} > -1$. We

prove that if

$$\sum_{k=1}^{\infty} r_n^2 < \infty,$$

$$\sum_{\substack{n \\ \lambda_n \leq T}} \lambda_n r_n = c_1 T (\log T)^{d_1} + o(T (\log T)^{d_1}),$$

$$\sum_{\substack{n \\ \lambda_n \leq T}} (\lambda_n r_n)^2 = c_2 T (\log T)^{d_2} + o(T (\log T)^{d_2}),$$

and

$$\tilde{N}(x) = \tilde{c} x (\log x)^{\tilde{d}} + o(x (\log x)^{\tilde{d}}),$$

then for any $\epsilon > 0$ and for large $V > 0$

$$P(X(\theta_1, \theta_2, \dots) \geq V) \leq \exp \left(-CV^{2-\frac{d_2}{d_1+1}} \exp \left(\left(\frac{d_1+1}{c_1} V \right)^{\frac{1}{d_1+1}} (1+o(1)) \right) \right),$$

and

$$P(X(\theta_1, \theta_2, \dots) \geq V) \geq \frac{1}{2} \exp \left(-C'V \exp \left(\left(\frac{\tilde{d}+1}{\tilde{c}} V \right)^{\frac{1}{\tilde{d}+1}} (1+o(1)) \right) \right),$$

where P is the canonical probability measure on \mathbb{T}^∞ ,

$$C = \frac{3\epsilon^2(1-\epsilon)}{4c_2(1+\epsilon)^2} \left(\frac{c_1(1+\epsilon)^2}{d_1+1} \right)^{-\frac{d_2}{d_1+1}},$$

and

$$C' = \frac{(\tilde{d}+1)(1+\epsilon)}{(1-\epsilon)}.$$

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Chapter 1

Introduction and Statements of the Results

1.1 Introduction

Determining the size of arithmetic functions is a central problem in number theory. A function $f : \mathbb{N} \rightarrow \mathbb{C}$ is called an *arithmetic function*. Here we consider three frequently used arithmetic functions.

i) The *Möbius function* is defined by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n = p_1 \dots p_k, p_i\text{'s are distinct primes,} \\ 0 & \text{if otherwise.} \end{cases}$$

The first few values of the Möbius function are listed in Table 1.1.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\mu(n)$	1	-1	-1	0	-1	1	-1	0	0	1	-1	0	-1	1	1
$M(n)$	1	0	-1	-1	-2	-1	-2	-2	-2	-1	-2	-2	-3	-2	-1

Table 1.1: Some values of the Möbius function and its summatory function

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\lambda(n)$	1	-1	-1	1	-1	1	-1	-1	1	1	-1	-1	-1	1	1
$L_0(n)$	1	0	-1	0	-1	0	-1	-2	-1	0	-1	-2	-3	-2	-1

Table 1.2: Some values of the Liouville function and its summatory function

ii) The *Liouville function* is defined by

$$\lambda(n) = (-1)^{\Omega(n)},$$

where $\Omega(n)$ is the total number of divisors of n counted with multiplicity. The first few values of the Liouville function are shown in Table 1.2.

iii) The *von Mangoldt function* is defined by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k, k \geq 1, p \text{ prime,} \\ 0 & \text{otherwise} \end{cases}$$

where \log is the natural logarithm.

The *summatory function* of an arithmetic function $f(n)$ is defined by

$$F(x) = \sum_{n \leq x} f(n).$$

The summatory functions of $\mu(n)$, $\lambda(n)$, $\lambda(n)/n$, and $\Lambda(n)$ are denoted, respectively, by $M(x)$, $L_0(x)$, $L_1(x)$, and $\psi(x)$. Some small values for $M(x)$ and $L_0(x)$ are recorded in Tables 1.1 and 1.2.

Over the years, several problems have been proposed about the size of these summatory functions. Among these functions, $\psi(x)$ derives its particular importance due to its connection with the prime numbers. Figure 1.1 illustrates the coincidence of the prime powers with the jump discontinuities of $\psi(x)$. Moreover, the importance of the primes in number theory arises from the “fundamental theorem of arithmetic.” This theorem asserts that any natural number other than one, is a unique product of prime powers. In other words, the prime numbers are the building blocks of the natural numbers.

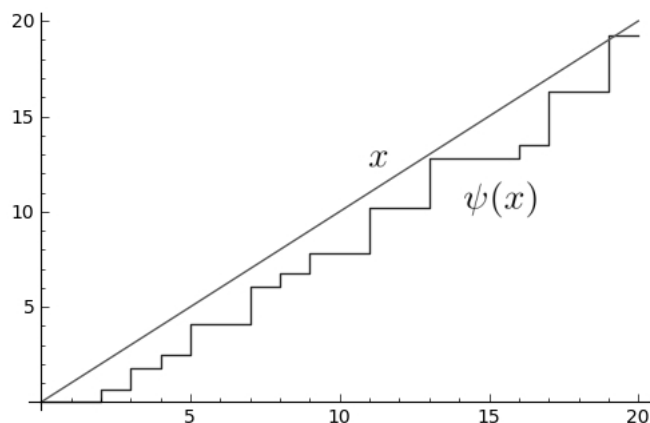


Figure 1.1: Values of $\psi(x)$ for $0 \leq x \leq 20$

Figure 1.1 demonstrates that $y = x$ approximates $y = \psi(x)$ for small values of x . The prime number theorem (PNT) asserts the truth of this observation in general. In other words, the PNT states that

$$\lim_{x \rightarrow \infty} \frac{\psi(x)}{x} = 1.$$

This theorem was proven by Hadamard and de la Vallée-Poussin in 1896.

A stronger version of the PNT provides more precise information about the size of $\psi(x)$. For convenience, we first introduce the following notation.

Definition 1.1. Let $f(x)$ and $g(x)$ be defined for real $x \geq x_0 > 0$, and let $g(x) > 0$ for all $x \geq x_0$. The notation $f(x) = O(g(x))$ (or $f(x) \ll g(x)$) means that there exists a constant $C > 0$ such that

$$|f(x)| \leq Cg(x),$$

for all $x \geq x_0$. $f(x) = o(g(x))$, as $x \rightarrow \infty$, means that

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 0.$$

Moreover, $f(x) \asymp g(x)$, means

$$f(x) \ll g(x) \quad \text{and} \quad g(x) \ll f(x).$$

Using the above notation we can restate the PNT as

$$\psi(x) = x + \mathcal{E}(x),$$

where

$$\mathcal{E}(x) = o(x), \quad \text{as } x \rightarrow \infty.$$

$\mathcal{E}(x)$ is called the *remainder term* of the PNT. Table 1.3 demonstrates the truth of (1.1) for small values of x .

The proof of the PNT is based on the analytic properties of the Riemann zeta function. For a complex number $s = \sigma + it$ in the half-plane $\sigma > 1$, the Riemann zeta function is defined by

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots .$$

Riemann showed that $\zeta(s)$ has a meromorphic continuation, with a simple pole at $s = 1$, to the whole complex plane. Moreover $\zeta(s) = 0$ at $s = -2, -4, -6, \dots$. Let

$$\mathcal{Z} := \{\rho = \beta + i\gamma \mid \zeta(\rho) = 0, \rho \neq -2n \text{ for all } n \in \mathbb{N}\}. \quad (1.1)$$

The elements of \mathcal{Z} are called the *nontrivial zeros* of $\zeta(s)$. It is known that

- if $\beta + i\gamma \in \mathcal{Z}$, then $0 < \beta < 1$;
- \mathcal{Z} is countably infinite;
- for any $T > 0$, there are finitely many $\beta + i\gamma \in \mathcal{Z}$ with $|\gamma| \leq T$;
- for all $\beta + i\gamma \in \mathcal{Z}$,

$$\zeta(\beta + i\gamma) = 0 \quad \iff \quad \zeta(\beta - i\gamma) = 0$$

and

$$\zeta(\beta + i\gamma) = 0 \quad \iff \quad \zeta(1 - \beta + i\gamma) = 0.$$

n	$\Lambda(n)$	$\psi(n)$ (approx.)	$\mathcal{E}(n)$ (approx.)	$\mathcal{E}(n)/n$ (approx.)
1	0	0	-1	-1
2	$\log 2$	0.6932	-1.3068	-0.6534
3	$\log 3$	1.7918	-1.2082	-0.4027
4	$\log 2$	2.4849	-1.5151	-0.3788
5	$\log 5$	4.0943	-0.9057	-0.1811
6	0	4.0943	-1.9057	-0.3176
7	$\log 7$	6.0403	-0.9597	-0.1371
8	$\log 2$	6.7334	-1.2666	-0.1583
9	$\log 3$	7.8399	-1.1601	-0.1289
10	0	7.8399	-2.1601	-0.2160
11	$\log 11$	10.2378	-0.7622	-0.0693
12	0	10.2378	-1.7622	-0.0058
13	$\log 13$	12.8028	-0.1972	-0.0152
14	0	12.8028	-1.1972	-0.0855
15	0	12.8028	-2.1972	-0.1465
16	$\log 2$	13.4959	-2.5041	-0.1565
17	$\log 17$	16.3291	-0.6709	-0.0395
18	0	16.3291	-1.6709	-0.0928
19	$\log 19$	19.2735	0.2735	0.0143
20	0	19.2736	-0.7264	-0.0363

Table 1.3: Some values of $\Lambda(n)$, $\psi(n)$, $\mathcal{E}(n)$, and $\mathcal{E}(n)/n$

Riemann conjectured that if $\beta + i\gamma \in \mathcal{Z}$, then $\beta = 1/2$. This conjecture is the celebrated ‘‘Riemann hypothesis.’’

The following theorem establishes a close connection between $\mathcal{E}(x)$ and the non-trivial zeros of $\zeta(s)$. This assertion was first stated by Riemann and later proved by von Mangoldt.

Theorem 1.2. *Suppose that $\psi_0(x)$ is defined as follows*

$$\psi_0(x) := \begin{cases} \psi(x) & \text{if } x \text{ is not a prime power,} \\ \psi(x) - \Lambda(x)/2 & \text{if } x \text{ is a prime power.} \end{cases}$$

Then for $x \geq 2$ we have

$$\psi_0(x) = x - \sum_{\rho \in \mathcal{Z}} \frac{x^\rho}{\rho} - \log(2\pi(1 - x^{-2})^{1/2}), \quad (1.2)$$

where \mathcal{Z} is defined in (1.1).

Note that in (1.2), the infinite sum $\sum_{\rho \in \mathcal{Z}} x^\rho/\rho$ is interpreted as

$$\sum_{\rho \in \mathcal{Z}} \frac{x^\rho}{\rho} = \lim_{T \rightarrow \infty} \sum_{|\gamma| \leq T} \frac{x^\rho}{\rho}.$$

From (1.2) we deduce that

$$\mathcal{E}(x) = \begin{cases} - \sum_{\rho \in \mathcal{Z}} \frac{x^\rho}{\rho} - \log(2\pi(1 - x^{-2})^{1/2}) & \text{if } x \text{ is not a prime power,} \\ \Lambda(x)/2 - \sum_{\rho \in \mathcal{Z}} \frac{x^\rho}{\rho} - \log(2\pi(1 - x^{-2})^{1/2}) & \text{if } x \text{ is a prime power.} \end{cases}$$

In 1897, by establishing a zero-free region for $\zeta(s)$, de la Vallée Poussin [4] proved that there exists a constant $a > 0$ such that

$$\mathcal{E}(x) = O(xe^{-a\sqrt{\log x}}).$$

(A zero-free region for a complex function $f(z)$ is a subset of \mathbb{C} in which $f(z) \neq 0$.) Moreover, von Koch [29] showed that if the Riemann hypothesis holds, then

$$\mathcal{E}(x) = O(\sqrt{x}(\log x)^2).$$

Despite these investigations, the question of the true order of $\mathcal{E}(x)$ still remains an open problem.

Several problems similar to determining the true order of $\mathcal{E}(x)$ have been proposed during the years. In 1897, Mertens conjectured that

$$\forall x \geq 1, \quad |M(x)| \leq \sqrt{x}.$$

It is known that, the truth of the Mertens conjecture implies the Riemann hypothesis. Similarly, Pólya [22] and Turán [28] conjectured that

$$\forall x \geq 2, \quad L_0(x) \leq 0$$

and

$$\forall x \geq 1, \quad L_1(x) \geq 1,$$

and proved that their conjectures imply the Riemann hypothesis.

Kotnik and Van de Lune [14] verified that $M(n) \leq \sqrt{n}$ holds for all $n \leq 10^{14}$. In 1958, Haselgrove [6] disproved both Pólya and Turán conjectures, and in 1985, Odlyzko and te Riele [21] disproved the Mertens conjecture. Tanaka [26] showed that the first value of n for which $L_0(n) > 0$ is 906105257. Borwein, Ferguson, and Mossinghoff [2] proved that the smallest value of n for which $L_1(n) < 0$ is 72185376951205. Nevertheless, the questions on the true size of $M(x)$, $L_0(x)$, and $L_1(x)$ remain unsolved.

The above problems can be investigated from a different point of view. Let

$$\mathcal{M} = \{x \geq 1 \mid M(x) > \sqrt{x}\}, \quad \mathcal{P} = \{x \geq 1 \mid L_0(x) > 0\}, \quad \mathcal{T} = \{x \geq 1 \mid L_1(x) \leq 0\}.$$

We may ask what one can say about the densities (natural and logarithmic) of these subsets of $[1, \infty)$?

A strategy to approach this question is to use the probability theory. Probability theory was introduced to number theory by Erdős, Kac, and Wintner in the 1930's. The main reason for employing probability theory is that many number theoretical functions behave similarly to random variables. One way to study number theoretical functions via probability theory is by limiting distributions. We say that a real function $f(x)$ has a limiting distribution μ , if there is a measure μ on \mathbb{R} which satisfies

$$\mu(\emptyset) = 0, \quad \mu(\mathbb{R}) = 1, \quad (1.3)$$

and

$$\lim_{X \rightarrow \infty} \frac{1}{X} \text{meas}\{x \in [0, X] \mid f(x) \geq V\} = \mu([V, \infty)), \quad (1.4)$$

where “meas” is the Lebesgue measure on \mathbb{R} . The condition (1.3) shows that μ is a “probability measure.” Note that the left-hand side of (1.4) is the natural density of $\{x \mid f(x) \geq V\}$.

The existence of a limiting distribution for certain arithmetical functions dates back to 1935. Wintner [13] proved that under the assumption of the Riemann hypothesis, $e^{-y/2}\mathcal{E}(e^y)$ has a limiting distribution.

More recently, limiting distribution was employed to study the so-called “Chebyshev’s bias” in prime number races. Given a number $q > 1$ and a set of pairwise relatively prime numbers a_1, \dots, a_r modulo q , let

$$\pi(x, q, a_i) := \text{card}(\{p \leq x \mid p \text{ is prime and } p \equiv a_i \pmod{q}\}).$$

Rubinstein and Sarnak [24] showed that under the assumption of the Riemann hypothesis for all Dirichlet L -functions $L(s, \chi) \pmod{q}$,

$$ye^{-y/2} \times (\varphi(q)\pi(e^y, q, a_1) - \pi(e^y), \dots, \varphi(q)\pi(e^y, q, a_r) - \pi(e^y)) \quad (1.5)$$

has a limiting distribution.

In [20] Ng showed that if the Riemann hypothesis holds and

$$\sum_{0 < \gamma \leq T} |\zeta'(\rho)|^{-2} \ll T, \quad (1.6)$$

then $e^{-y/2}M(e^y)$ has a limiting distribution ν . Ng [20] pointed out that by a method similar to his article on $e^{-y/2}M(e^y)$, we can prove that $e^{-y/2}L_0(e^y)$ has a limiting distribution. Humphries in [10] studied

$$L_\alpha(x) = \sum_{n \leq x} \frac{\lambda(n)}{n^\alpha},$$

for a fixed α ($0 \leq \alpha < 1/2$). By following the method of Ng [20], Humphries [10] proved that under the assumptions of the Riemann hypothesis and (1.6), $e^{-y/2}L_\alpha(e^y)$ has a limiting distribution for $0 \leq \alpha < 1/2$.

In all of the above results, the limiting distribution measures are not explicitly determined. However, under the “linear independence conjecture,” or the “grand linear independence conjecture,” the Fourier transform of these limiting distributions can be explicitly calculated. For example, let μ be the limiting distribution of $e^{-y/2}\mathcal{E}(e^y)$. It is shown that under the extra assumption of the linear independence conjecture (see Definition 1.3,) the Fourier transform $\hat{\mu}(\xi) = \int_{-\infty}^{\infty} e^{-i\xi t} d\mu(t)$ of μ is equal to

$$\hat{\mu}(\xi) = \prod_{n=1}^{\infty} J_0\left(\frac{2\xi}{|\rho_n|}\right), \quad (1.7)$$

where $J_0(z)$ is the Bessel function

$$J_0(z) := \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m}}{(m!)^2}. \quad (1.8)$$

Definition 1.3 (Linear Independence Conjecture). The set

$$\{\gamma \geq 0 \mid \zeta(\beta + i\gamma) = 0, \beta \geq 1/2\}$$

is linearly independent over \mathbb{Q} .

Definition 1.4 (Grand Linear Independence Conjecture). The set

$$\bigcup_{q=2}^{\infty} \bigcup_{\text{primitive } \chi \pmod{q}} \{\gamma \geq 0 \mid L(\beta + i\gamma, \chi) = 0, \beta \geq 1/2\}$$

is linearly independent over \mathbb{Q} .

In [19], Montgomery studied the tail of the limiting distribution of $e^{-y/2}\mathcal{E}(e^y)$ by a probabilistic method. His main observation is that the limiting distribution measure μ of $e^{-y/2}\mathcal{E}(e^y)$ is identical to the probability measure of the random variable

$$X(\theta_1, \theta_2, \dots) = \sum_{n=1}^{\infty} \frac{2}{|\rho_n|} \sin(2\pi\theta_n)$$

on the infinite torus \mathbb{T}^∞ . (Here $\rho_n = 1/2 + i\gamma_n$, under the assumption of the Riemann hypothesis, denotes the nontrivial zeros of $\zeta(s)$ in the upper half plane.) So he can use the theorems on the large deviations of the sum of independent random variables to study the tail of the limiting distributions.

A theorem of Montgomery [19, Section 3, Theorem 2] states that for any $N \geq 1$

$$P\left(X(\theta_1, \theta_2, \dots) \geq 2 \sum_{n=1}^N \frac{2}{|\rho_n|}\right) \leq \exp\left(-\frac{3}{4} \left(\sum_{n=1}^N \frac{2}{|\rho_n|}\right)^2 \left(\sum_{n>N} \frac{4}{|\rho_n|^2}\right)^{-1}\right)$$

and

$$P\left(X(\theta_1, \theta_2, \dots) \geq \frac{1}{2} \sum_{n=1}^N \frac{2}{|\rho_n|}\right) \geq 2^{-40} \exp\left(-100 \left(\sum_{n=1}^N \frac{2}{|\rho_n|}\right)^2 \left(\sum_{n>N} \frac{4}{|\rho_n|^2}\right)^{-1}\right).$$

By employing this theorem, one can achieve upper and lower bounds for $\mu([V, \infty))$. A more precise lower bound for μ can be obtained by applying Theorem 3 of Montgomery [19, Section 3]. It asserts that if $\delta > 0$ and $V > 0$ satisfy

$$\sum_{2/|\rho_n| > \delta} (2/|\rho_n| - \delta) \geq V,$$

then

$$P(X(\theta_1, \theta_2, \dots) \geq V) \geq \frac{1}{2} \exp\left(-\frac{1}{2} \sum_{2/|\rho_n| > \delta} \log\left(\frac{\pi^2 r_k}{\delta |\rho_n|}\right)\right).$$

Using these inequalities, Montgomery [19, Section 3] stated that there exists constants $c_1, c_2 > 0$ such that for large $V > 0$

$$\exp(-c_1\sqrt{V}\exp(\sqrt{2\pi V})) \leq P(X(\theta_1, \theta_2, \dots) \geq V) \leq \exp(-c_2\sqrt{V}\exp(\sqrt{2\pi V})).$$

Similar methods have been employed by Ng [20] and Humphries [10] in studying the tails of the limiting distributions of $e^{-y/2}M(e^y)$ and $e^{-y/2}L_\alpha(e^y)$ (for $0 \leq \alpha < 1/2$.)

We end this section by reviewing the works of Monach and Lamzouri in finding explicit formulas for the tails of the limiting distributions of $e^{-y/2}\mathcal{E}(e^y)$ and (1.5). Monach [17] in his Ph.D. thesis showed that, under the assumptions of the Riemann hypothesis and the linear independence conjecture, there is an explicitly defined constant $C > 0$ such that for large $V > 0$

$$P(X(\theta_1, \theta_2, \dots) \geq V) = \exp\left(-e^{-C}\sqrt{2\pi V}\exp(\sqrt{2\pi V})(1+o(1))\right).$$

Unfortunately, the work for his Ph.D. thesis was never published.

In [16], Lamzouri found explicit formulas for the tail of the limiting distribution $\mu_{q;a_1,\dots,a_r}$ of (1.5). Lamzouri proved [16, Theorem 4] that under the assumptions of the Riemann hypothesis for Dirichlet L -functions $L(s, \chi)$ and the grand linear independence conjecture, if $V/(\varphi(q)\log^2 q) \rightarrow \infty$ as $q \rightarrow \infty$, then there is an explicitly determined function $L(q)$ such that for large $q > 0$

$$\begin{aligned} &\mu_{q;a_1,\dots,a_r}(\|(x_1, \dots, x_\ell)\|_\infty > V) = \\ &\exp\left(-e^{-L(q)}\sqrt{\frac{2(\varphi(q)-1)V}{\pi}}\exp\left(\sqrt{L^2(q) + \frac{2\phi V}{\varphi(q)-1}}\right)\left(1+O\left(\left(\frac{\varphi(q)\log^2 q}{V}\right)^{1/4}\right)\right)\right), \end{aligned}$$

where $\varphi(q)$ is Euler's totient function, and $\|\cdot\|_\infty$ is the "sup" norm.

1.2 Statements of the Results

In this thesis the following problems are studied.

Let $\{\lambda_n\}$ be an increasing sequence of positive numbers which tends to infinity, and $\{r_n\}$ be a complex sequence. Consider the function $\phi(y) : \mathbb{R} \rightarrow \mathbb{R}$ such that for any

$X > 0$ we may write

$$\phi(y) = C + \operatorname{Re} \left(\sum_{\lambda_n \leq X} r_n e^{iy\lambda_n} \right) + E(y, X), \quad (1.9)$$

where C is a real constant and $E(y, X)$ is a function satisfying

$$\int_1^Y |E(y, e^Y)|^2 dy \ll 1.$$

In Chapter 2, we prove the following result.

Theorem 1.5. *Let $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$ be real constants such that $\theta_4 > -1$ and $2\theta_1 < \theta_4$, and set*

$$\mathfrak{m} = \begin{cases} (2\theta_4 - 1)/4 & \text{if } -1 < \theta_4 \leq 1 \\ \theta_4/4 & \text{if } 1 \leq \theta_4 \leq 3 \\ 3/4 & \text{if } \theta_4 \geq 3. \end{cases}$$

Suppose that $\mathfrak{m} > \theta_1$, and that r_n and λ_n satisfy the following conditions

$$\sum_{\lambda_n \leq T} |r_n| \ll T^{\theta_1} (\log T)^{\theta_2},$$

$$\sum_{\lambda_n > T} |r_n|^2 \ll \frac{(\log T)^{\theta_3}}{T^{\theta_4}},$$

and

$$\sum_{T \leq \lambda_n < T+1} 1 \ll (\log T)^{\theta_5}.$$

Moreover, assume

$$\int_1^Y |E(y, e^Y)|^2 dy \ll 1.$$

Then there is a probability measure μ on \mathbb{R} such that for all bounded Lipschitz functions f on \mathbb{R}

$$\lim_{Y \rightarrow \infty} \frac{1}{Y} \int_0^Y f(\phi(y)) dy = \int_{-\infty}^{\infty} f(x) d\mu(x).$$

The Fourier transform $\hat{\mu}$ of a measure μ on \mathbb{R} is given for $\xi \in \mathbb{R}$ by

$$\hat{\mu}(\xi) = \int_{-\infty}^{\infty} e^{-i\xi t} d\mu(t).$$

Assuming the linear independence for $\{\lambda_n\}$ over \mathbb{Q} we will give the following explicit formula for the Fourier transform of the measure μ .

Theorem 1.6. *Assume that the conditions in Theorem 1.5 hold. Moreover, assume that the numbers λ_n are linearly independent over \mathbb{Q} (i.e. for any finite set*

$$\{\lambda_{n_1}, \dots, \lambda_{n_k}\}$$

the relation $\sum_{i=1}^k c_i \lambda_{n_i} = 0$ for $c_i \in \mathbb{Q}$ implies all $c_i = 0$.) Then the Fourier transform $\hat{\mu}$ of μ exists and equals

$$\hat{\mu}(\xi) = e^{-iC\xi} \prod_{n=1}^{\infty} J_0(|r_n|\xi)$$

where $J_0(z)$ is defined in (1.8).

We also prove generalization of the above theorems to a vector-valued function $\vec{\phi}(y) : \mathbb{R} \rightarrow \mathbb{R}^\ell$. For fixed k ($1 \leq k \leq \ell$) let $\{\lambda_{k,n}\}_{n=1}^{\infty}$ be an increasing sequence of positive numbers which tends to infinity and $\{r_{k,n}\}_{n=1}^{\infty} \subseteq \mathbb{C}$. For each k ($1 \leq k \leq \ell$), define

$$\phi_k(y) = C_k + \operatorname{Re} \left(\sum_{\lambda_{k,n} \leq X} r_n e^{iy\lambda_{k,n}} \right) + E_k(y, X),$$

where $E_k(y, X)$ satisfies

$$\int_1^Y |E_k(y, e^Y)|^2 dy \ll 1$$

and let

$$\vec{\phi}(y) = (\phi_1(y), \dots, \phi_\ell(y)).$$

For $\vec{\phi}(y)$ we prove the following theorem.

Theorem 1.7. *For $k = 1, \dots, \ell$ assume that*

$$\sum_{\substack{n \geq 1 \\ \lambda_{k,n} \leq T}} |r_{k,n}| \ll T^{\theta_{1,k}} (\log T)^{\theta_{2,k}},$$

$$\sum_{\substack{n \geq 1 \\ \lambda_{k,n} > T}} |r_{k,n}|^2 \ll \frac{(\log T)^{\theta_{3,k}}}{T^{\theta_{4,k}}},$$

$$\sum_{\substack{n \geq 1 \\ T \leq \lambda_{k,n} < T+1}} 1 \ll (\log T)^{\theta_{5,k}},$$

and

$$\int_1^Y |E_k(y, e^Y)|^2 dy \ll 1.$$

Moreover, let

$$\Theta_1 = \max_{1 \leq k \leq \ell} \theta_{1,k}, \quad \Theta_2 = \max_{1 \leq k \leq \ell} \theta_{2,k}, \quad \Theta_3 = \max_{1 \leq k \leq \ell} \theta_{3,k}, \quad \Theta_4 = \min_{1 \leq k \leq \ell} \theta_{4,k},$$

$$\Theta_5 = \max_{1 \leq k \leq \ell} \theta_{5,k},$$

and

$$\mathfrak{M} = \begin{cases} (2\Theta_4 - 1)/4 & \text{if } -1 < \Theta_4 \leq 1 \\ \Theta_4/4 & \text{if } 1 \leq \Theta_4 \leq 3 \\ 3/4 & \text{if } \Theta_4 \geq 3. \end{cases}$$

Assume that $\Theta_4 > -1/2$, $2\Theta_1 < \Theta_4$, and $\mathfrak{M} > \Theta_1$. Then there is a probability measure μ on \mathbb{R}^ℓ such that

$$\lim_{Y \rightarrow \infty} \frac{1}{Y} \int_0^Y f(\vec{\phi}(y)) dy = \int_{\mathbb{R}^\ell} f(x_1, \dots, x_\ell) d\mu(x_1, \dots, x_\ell)$$

for all bounded Lipschitz functions f on \mathbb{R}^ℓ .

Next, by making the linear independence assumption for $\{\lambda_{k,n}\}_{k,n}$ we prove the following result.

Theorem 1.8. *Assume that the conditions in Theorem 1.7 hold. Moreover, assume that $\{\lambda_{k,n}\}_{k,n}$ is linearly independent over \mathbb{Q} . Then the Fourier transform*

$$\hat{\mu}(\xi_1, \dots, \xi_\ell) = \int_{\mathbb{R}^\ell} \exp\left(-i \sum_{k=1}^{\ell} \xi_k t_k\right) d\mu(t_1, \dots, t_\ell)$$

of μ exists and equals

$$\hat{\mu}(\xi) = \exp\left(-i \sum_{k=1}^{\ell} c_k \xi_k\right) \times \prod_{k=1}^{\ell} \prod_{n=1}^{\infty} J_0(|r_{k,n}| \xi_k),$$

where $J_0(z)$ is defined in (1.8).

In Chapter 3, we will discuss applications of our general limiting distribution theorem. We do this by proving explicit formulas of the form (1.9) for some arithmetical functions and employing Theorems 1.5 and 1.6. Our examples include the normalized error term of the prime number theorem, the weighted sums of the Möbius and the Liouville functions, and the summatory function of the Möbius function in arithmetic progressions. Here we review some of these results.

In 1935, Wintner [30] showed that under the Riemann hypothesis, $e^{-y/2} \mathcal{E}(e^y)$ has a limiting distribution. We recover Wintner's result as a straightforward consequence of Theorem 1.5. More precisely, for $x > 1$, let

$$\pi(x) := \text{card}(\{p \leq x \mid p \text{ is a prime}\}),$$

and

$$\text{Li}(x) := \int_2^x \frac{dt}{\log t}.$$

Theorem 1.9 (Error Term of the PNT). *Under the assumption of the Riemann hypothesis, $ye^{-y/2}(\pi(e^y) - \text{Li}(e^y))$ has a limiting distribution μ . Under the extra assumption of the linear independence conjecture, the Fourier transform $\hat{\mu}$ of μ is equal to*

$$\hat{\mu}(\xi) = e^{i\xi} \prod_{\gamma > 0} J_0\left(\frac{2\xi}{\sqrt{1/4 + \gamma^2}}\right).$$

Our next application establishes a limiting distribution for the weighted sums of the Möbius function $M_\alpha(x)$, $0 \leq \alpha \leq 1$. Ng [20] studied this problem for $\alpha = 0$. For $0 \leq \alpha \leq 1$, let

$$M_\alpha(x) = \sum_{n \leq x} \frac{\mu(n)}{n^\alpha},$$

where $\mu(n)$ is the Möbius function. Let

$$\phi_\alpha(y) := \begin{cases} e^{y(-1/2+\alpha)} M_\alpha(e^y) & \text{if } 0 \leq \alpha \leq 1/2 \text{ or } \alpha = 1, \\ e^{y(-1/2+\alpha)} (M_\alpha(e^y) - 1/\zeta(\alpha)) & \text{if } 1/2 < \alpha < 1. \end{cases}$$

Theorem 1.10 (Weighted Sums of the Möbius Function). *Let $\theta < 5/4$ be fixed. Under the assumptions of the Riemann hypothesis and $\sum_{0 < \gamma \leq T} |\zeta'(\rho)|^{-2} \ll T^\theta$, $\phi_\alpha(y)$ has a limiting distribution ν_α . Under the extra assumption of the linear independence conjecture, the Fourier transform $\hat{\nu}_\alpha$ of ν_α exists and is equal to*

$$\hat{\nu}_\alpha(\xi) = \prod_{\gamma > 0} J_0 \left(\frac{2\xi}{|(\rho - \alpha)\zeta'(\rho)|} \right).$$

Observe that this theorem improves the assumption $\sum_{0 < \gamma \leq T} |\zeta'(\rho)|^{-2} \ll T$ in Ng's article [20, Theorem 2] to $\sum_{0 < \gamma \leq T} |\zeta'(\rho)|^{-2} \ll T^\theta$ for any $\theta < 5/4$.

The next application is related to the weighted sums of the Liouville function $L_\alpha(x)$, $0 \leq \alpha \leq 1$. This problem has been studied by Humphries [10] for $0 \leq \alpha < 1/2$. We recover Humphries' result and moreover extend it to $0 \leq \alpha \leq 1$. For $0 \leq \alpha \leq 1$, let

$$L_\alpha(x) = \sum_{n \leq x} \frac{\lambda(n)}{n^\alpha},$$

where $\lambda(n)$ is the Liouville function. Let

$$\psi_\alpha(y) := \begin{cases} e^{y(-1/2+\alpha)} L_\alpha(e^y) & \text{if } 0 \leq \alpha < 1/2 \text{ or } \alpha = 1, \\ e^{y(-1/2+\alpha)} (L_\alpha(e^y) - y/2\zeta(1/2)) & \text{if } \alpha = 1/2, \\ e^{y(-1/2+\alpha)} (L_\alpha(e^y) - \zeta(2\alpha)/\zeta(\alpha)) & \text{if } 1/2 < \alpha < 1. \end{cases}$$

Theorem 1.11 (Weighted Sums of the Liouville Function). *Let $\theta < 5/4$ be fixed. Under the assumptions of the Riemann hypothesis and $\sum_{0 < \gamma \leq T} |\zeta(2\rho)/\zeta'(\rho)|^2 \ll T^\theta$, $\psi_\alpha(y)$ has a limiting distribution μ_α . Under the extra assumption of the linear independence conjecture, the Fourier transform $\hat{\mu}_\alpha$ of μ_α exists and is equal to*

$$\hat{\mu}_\alpha(\xi) = e^{-iC_\alpha \xi} \prod_{\gamma > 0} J_0 \left(\frac{2|\zeta(2\rho)|\xi}{|(\rho - \alpha)\zeta'(\rho)|} \right),$$

where

$$C_\alpha = \begin{cases} 1/((1-2\alpha)\zeta(1/2)) & \text{if } 0 \leq \alpha < 1/2 \text{ or } 1/2 < \alpha \leq 1, \\ \gamma/\zeta(1/2) & \text{if } \alpha = 1/2, \end{cases}$$

γ being Euler's constant.

Our last application is for the summatory function of the Möbius function in arithmetic progressions. Let $q > 1$ and $a \geq 1$ be such that $(q, a) = 1$ and define

$$M(x, q, a) := \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} \mu(n)$$

where μ is the Möbius function.

Theorem 1.12 (Möbius Function in Arithmetic Progressions). *Let $\theta < 5/4$ be fixed. For each $\chi \pmod{q}$, assume that for the Dirichlet L -function $L(s, \chi)$ the Riemann hypothesis holds and*

$$\sum_{0 < \gamma_\chi \leq T} |L(2\rho_\chi, \chi)/L'(\rho_\chi, \chi)|^2 \ll T^\theta.$$

Then $e^{-y/2}M(e^y, q, a)$ has a limiting distribution $\mu_{q;\alpha}$. Moreover, if we further assume the grand linear independence conjecture, then the Fourier transform $\hat{\mu}_{q;\alpha}$ of $\mu_{q;\alpha}$ exists and is equal to

$$\hat{\mu}_{q;\alpha}(\xi) = \prod_{\chi \pmod{q}} \prod_{\gamma_\chi > 0} J_0 \left(\frac{2\xi |\chi(a)|}{\varphi(q) |\rho_\chi L'(\rho_\chi, \chi)|} \right).$$

We note that the existence of a limiting distribution associated to the Shanks-Rényi prime number race, which was studied by Rubinstein and Sarnak [24] also follows from our general limiting distribution theorem.

The last section of Chapter 3 is devoted to the connections of $x^{-1/2}M_\alpha(x)$ and $x^{-1/2}L_\alpha(x)$ with the Riemann hypothesis. It is well-known that the truth of any of Mertens, Pólya, or Turán conjectures would imply the Riemann hypothesis. In fact one can prove more generally that for a fixed positive constant K , the truth of either of the four inequalities

$$x^{-1/2}M(x) < K, \quad x^{-1/2}M(x) > -K, \quad x^{-1/2}L_0(x) < K, \quad x^{-1/2}L_0(x) > -K$$

for large values of x , implies the Riemann hypothesis. Moreover, Ingham [12, Theorem 2] proved that any of the above inequalities imply the simplicity of the zeros of $\zeta(s)$ and more surprisingly that the zeros are linearly dependent. In the last section of Chapter 3 we prove a generalization of Ingham's theorem as follows.

Theorem 1.13. *Let $\alpha \in [0, 1]$. Set*

$$m_\alpha = \begin{cases} 0 & \text{if } 0 \leq \alpha \leq 1/2, \\ 1/\zeta(\alpha) & \text{if } 1/2 < \alpha < 1, \\ 0 & \text{if } \alpha = 1, \end{cases}$$

and

$$l_\alpha(x) = \begin{cases} 0 & \text{if } 0 \leq \alpha < \frac{1}{2}, \\ \log x / 2\zeta(1/2) & \text{if } \alpha = \frac{1}{2}, \\ \zeta(2\alpha)/\zeta(\alpha) & \text{if } \frac{1}{2} < \alpha < 1, \\ 0 & \text{if } \alpha = 1. \end{cases}$$

If there is a constant $K > 0$ such that either of the following is true

$$M_\alpha(x) - m_\alpha > -Kx^{1/2-\alpha}, \quad M_\alpha(x) - m_\alpha < Kx^{1/2-\alpha},$$

$$L_\alpha(x) - l_\alpha(x) > -Kx^{1/2-\alpha}, \quad L_\alpha(x) - l_\alpha(x) < Kx^{1/2-\alpha},$$

for sufficiently large x , then the Riemann hypothesis holds and all the zeros of $\zeta(s)$ are simple. Moreover, either of the above inequalities will disprove the linear independence conjecture.

In the final chapter, we investigate the tails of limiting distributions associated to certain infinite sums of random variables. We show that the limiting distribution measures obtained in Theorem 2.16 and Corollary 2.17 are identical to the probability measure of the random variable X defined on the infinite torus \mathbb{T}^∞ by

$$X(\theta_1, \theta_2, \dots) = \sum_{n=1}^{\infty} r_n \sin(2\pi\theta_n).$$

Here $r_n \in \mathbb{R}$ are nonnegative numbers. Set

$$\tilde{N}(x) := \sum_{1/r_n \leq x} 1.$$

Let $\{\lambda_n\}$ be an increasing sequence of positive real numbers which tends to infinity. In this setting, we will prove the following results.

Theorem 1.14. *Let $\{r_n\} \subseteq \mathbb{R}$ be decreasing. Assume*

$$\sum_{k=1}^{\infty} r_n^2 < \infty,$$

$$\sum_{\substack{n \\ \lambda_n \leq T}} \lambda_n r_n = c_1 T (\log T)^{d_1} + o(T (\log T)^{d_1}),$$

$$\sum_{\substack{n \\ \lambda_n \leq T}} (\lambda_n r_n)^2 = c_2 T (\log T)^{d_2} + o(T (\log T)^{d_2}),$$

and

$$\tilde{N}(x) = \tilde{c} x (\log x)^{\tilde{d}} + o(x (\log x)^{\tilde{d}}),$$

where $c_i, d_j \in \mathbb{R}$, $d_j > -1$, and $\tilde{d} > -1$. Then for any $\epsilon > 0$ and for large V

$$P(X(\theta_1, \theta_2, \dots) \geq V) \leq \exp \left(-CV^{2-\frac{d_2}{d_1+1}} \exp \left(\left(\frac{d_1+1}{c_1} V \right)^{\frac{1}{d_1+1}} (1+o(1)) \right) \right),$$

and

$$P(X(\theta_1, \theta_2, \dots) \geq V) \geq \frac{1}{2} \exp \left(-C'V \exp \left(\left(\frac{\tilde{d}+1}{\tilde{c}} V \right)^{\frac{1}{\tilde{d}+1}} (1+o(1)) \right) \right),$$

where

$$C = \frac{3\epsilon^2(1-\epsilon)}{4c_2(1+\epsilon)^2} \left(\frac{c_1(1+\epsilon)^2}{d_1+1} \right)^{-\frac{d_2}{d_1+1}}.$$

and

$$C' = \frac{(\tilde{d}+1)(1+\epsilon)}{(1-\epsilon)}.$$

Our final result is an explicit estimate for the tail of the limiting distribution which is derived by the saddle point method. A key point in this estimation is a precise asymptotic assumption for $\tilde{N}(x) = \sum_{1/r_n \leq x} 1$. We finish this section by asserting our last result.

Theorem 1.15. *Let $\tilde{c}, \tilde{d}, \tilde{k}, c_3, c_4, d_3$ be constants such that $0 \leq \tilde{d} < 2$. Assume*

$$\sum_{k=1}^{\infty} r_n^2 < \infty,$$

$$\tilde{N}(x) = \tilde{c}x(\log x)^{\tilde{d}} + \tilde{k}x(\log x)^{\tilde{d}-1} + O\left(x(\log x)^{\tilde{d}-2}\right);$$

and

$$\exp\left(-\exp\left(c_3 V^{d_3}\right)\right) \ll P(X(\theta_1, \theta_2, \dots) \geq V) \ll \exp\left(-\exp\left(c_4 V^{d_3}\right)\right).$$

and that $\{\lambda_n\}$ is linearly independent over \mathbb{Q} . Then there exists a constant $A > 0$ and functions $\epsilon_1(V), \epsilon_2(V)$ such that $\epsilon_i(V) \rightarrow 0$, as $V \rightarrow \infty$, for $i = 1, 2$, and

$$P(X(\theta_1, \theta_2, \dots) \geq V) \geq \exp\left(-\tilde{c}\left(\frac{\tilde{d}+1}{\tilde{c}}V\right)^{\frac{\tilde{d}}{\tilde{d}+1}} \exp\left(-\frac{A+\tilde{c}}{\tilde{c}} + \left(\frac{\tilde{d}+1}{\tilde{c}}V\right)^{\frac{1}{\tilde{d}+1}}(1+\epsilon_1(V))\right)\right),$$

$$P(X(\theta_1, \theta_2, \dots) \geq V) \leq \exp\left(-\tilde{c}\left(\frac{\tilde{d}+1}{\tilde{c}}V\right)^{\frac{\tilde{d}}{\tilde{d}+1}} \exp\left(-\frac{A+\tilde{c}}{\tilde{c}} + \left(\frac{\tilde{d}+1}{\tilde{c}}V\right)^{\frac{1}{\tilde{d}+1}}(1+\epsilon_2(V))\right)\right).$$

Chapter 2

A General Limiting Distribution Theorem

2.1 Introduction

In this chapter we establish the existence of limiting distributions of certain functions $\phi(y) : \mathbb{R} \rightarrow \mathbb{R}$ and more generally $\vec{\phi}(y) : \mathbb{R} \rightarrow \mathbb{R}^\ell$ where $\ell \in \mathbb{N}$. Roughly speaking, a limiting distribution $G(x)$ of ϕ tells us how often ϕ takes on certain values. For instance, how often does $\phi(y)$ lie in the interval $[A, B] \subset \mathbb{R}$ or more precisely, what is the value of

$$\lim_{Y \rightarrow \infty} \frac{1}{Y} \text{meas}\{y \in [0, Y] \mid A \leq \phi(y) \leq B\}.$$

One interpretation of a limiting distribution of $\phi(y)$ is that there exists a Riemann-Stieltjes integrable function $G(x)$ such that the above limit exists and

$$\lim_{Y \rightarrow \infty} \frac{1}{Y} \text{meas}\{y \in [0, Y] \mid A \leq \phi(y) \leq B\} = \int_A^B dG(x). \quad (2.1)$$

In order to further investigate limiting distributions, we require a number of notions from probability. We review these facts here. For complete details the standard probability text books Billingsley [1] and Shiryaev [25] are good references.

Let Ω be a nonempty set. A class \mathcal{F} of subsets of Ω is a *field* if

- (i) $\Omega \in \mathcal{F}$;

- (ii) $A \in \mathcal{F}$ implies $A^c \in \mathcal{F}$;
- (iii) $A, B \in \mathcal{F}$ implies $A \cup B \in \mathcal{F}$.

\mathcal{F} is a σ -field if it is a field and if $A_1, A_2, \dots \in \mathcal{F}$ implies $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$. For a collection \mathcal{C} of subsets of Ω , the σ -field generated by \mathcal{C} is the smallest σ -field that contains \mathcal{C} . In the case of $\Omega = \mathbb{R}^\ell$, the *Borel σ -field* denoted by \mathcal{B}^ℓ is the σ -field generated by the collection of all sets of the form $(-\infty, a_1] \times \dots \times (-\infty, a_\ell]$, $a_i \in \mathbb{R}$. An ordered pair (Ω, \mathcal{F}) is a *measurable space* if Ω is a nonempty set and \mathcal{F} is a σ -field of subsets of Ω . If (Ω, \mathcal{F}) is a measurable space, then a function μ is a *measure* on (Ω, \mathcal{F}) if it satisfies the following conditions:

- (i) $\mu(A) \in [0, \infty]$ for any $A \in \mathcal{F}$;
- (ii) $\mu(\emptyset) = 0$;
- (iii) if $\{A_n\}$ is a disjoint sequence of sets in \mathcal{F} then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

A *probability measure* P on a measurable space (Ω, \mathcal{F}) is a measure on (Ω, \mathcal{F}) that satisfies the extra properties

- (i') $0 \leq P(A) \leq 1$ for $A \in \mathcal{F}$;
- (ii') $P(\Omega) = 1$.

An ordered triple (Ω, \mathcal{F}, P) is a *probability space* if (Ω, \mathcal{F}) is a measurable space and P is a probability measure on (Ω, \mathcal{F}) . Let (Ω, \mathcal{F}) and (Ω', \mathcal{F}') be measurable spaces and consider a function $X : \Omega \rightarrow \Omega'$. Then X is a $(\mathcal{F}, \mathcal{F}')$ -*measurable function* if for any $A' \in \mathcal{F}'$ we have

$$X^{-1}(A') := \{\omega \in \Omega \mid X(\omega) \in A'\} \in \mathcal{F}.$$

Definition 2.1. A *generalized distribution function* on \mathbb{R}^ℓ is a function $F : \mathbb{R}^\ell \rightarrow \mathbb{R}$ which is non-decreasing and right-continuous in each of its variables.

Definition 2.2. Let $\ell \in \mathbb{N}$ be fixed. A *distribution function* on \mathbb{R}^ℓ is a function

$$F : (\mathbb{R} \cup \{\pm\infty\})^\ell \rightarrow \mathbb{R}$$

that satisfies the following properties:

- F is non-decreasing in each of its variables;
- F is right-continuous in each of its variables;
- $F(+\infty, \dots, +\infty) = 1$;
- $F(x_1, \dots, x_\ell) = 0$, if at least one x_i is $-\infty$.

It is known (see [25, pp. 159–160]) that if P is a probability measure on $(\mathbb{R}^\ell, \mathcal{R}^\ell)$, then

$$F_P(x_1, \dots, x_\ell) := P((-\infty, x_1] \times \cdots \times (-\infty, x_\ell])$$

is a distribution function on \mathbb{R}^ℓ . Conversely, given a distribution function F on \mathbb{R}^ℓ , there exists a unique probability measure on \mathbb{R}^ℓ such that $F = F_P$. Therefore, probability measures and distribution functions are two closely interrelated concepts. In this thesis, we will use distribution functions and distribution measures alternately.

For a sequence of distribution functions $\{F_n\}$ and a distribution function F defined on \mathbb{R}^ℓ we say that F_n *converges to F weakly* if

$$\lim_{n \rightarrow \infty} F_n(x_1, \dots, x_\ell) = F(x_1, \dots, x_\ell)$$

at any continuity point (x_1, \dots, x_ℓ) of F .

It turns out that it will be technically more convenient to give a slightly different definition for limiting distribution.

Definition 2.3. A function $\phi(y) : \mathbb{R} \rightarrow \mathbb{R}$ has a limiting distribution function $G(x)$ on \mathbb{R} if $G(x)$ is a distribution function on \mathbb{R} and

$$\lim_{Y \rightarrow \infty} \frac{1}{Y} \int_0^Y f(\phi(y)) dy = \int_{-\infty}^{\infty} f(x) dG(x), \quad (2.2)$$

for all bounded continuous real functions f on \mathbb{R} .

Observe that (2.1) follows by setting $f(t) = 1_{[A,B]}(t)$, the indicator function of $[A, B]$, in (2.2).

Our definition of limiting distribution can be generalized to a vector function $\vec{\phi}(y) : \mathbb{R} \rightarrow \mathbb{R}^\ell$.

Definition 2.4. A function $\vec{\phi}(y) : \mathbb{R} \rightarrow \mathbb{R}^\ell$ is said to have a limiting distribution $G : \mathbb{R}^\ell \rightarrow \mathbb{R}$ if G is a distribution function on \mathbb{R}^ℓ and

$$\lim_{Y \rightarrow \infty} \frac{1}{Y} \int_0^Y f(\vec{\phi}(y)) dy = \int_{\mathbb{R}^\ell} f(x_1, \dots, x_\ell) dG(x_1, \dots, x_\ell),$$

for all bounded continuous real functions f on \mathbb{R}^ℓ .

In this chapter, we shall study limiting distributions of functions $\phi(y)$ with a particular shape. These functions will depend on two sequences $\{\lambda_n\}$ and $\{r_n\}$. Throughout this chapter we shall assume

- λ_n is a positive, non-decreasing sequence that increases to infinity,
- r_n is a complex sequence.

Our functions $\phi(y)$ may be expressed as

$$\phi(y) = c + \operatorname{Re} \left(\sum_{\lambda_n \leq X} r_n e^{i\lambda_n y} \right) + E(y, X), \quad (2.3)$$

where $E(y, X)$ satisfies the condition

$$\int_1^Y |E(y, e^Y)|^2 dy \ll 1. \quad (2.4)$$

For further investigation we write

$$\phi(y) = c + \operatorname{Re} \left(\sum_{\lambda_n \leq T} r_n e^{i\lambda_n y} \right) + \epsilon^{(T)}(y)$$

where

$$\epsilon^{(T)}(y) = \operatorname{Re} \left(\sum_{T < \lambda_n \leq e^Y} r_n e^{i\lambda_n y} \right) + E(y, e^Y).$$

It should be noted that many of the classical functions in prime number theory have an expression in the form given in (2.3). For instance, under the Riemann hypothesis the explicit formula for $e^{-y/2}(\psi(e^y) - e^y)$ is

$$e^{-y/2}(\psi(e^y) - e^y) = \operatorname{Re} \left(\sum_{0 < \gamma < T} \frac{-2}{\rho} e^{i\gamma y} \right) + R(y, T)$$

for a suitable error term $R(y, T)$. See [3, pp. 104–110] for more explanation.

Finally, we define the Fourier transform of a probability measure.

Definition 2.5. The *Fourier transform* of a probability measure μ on \mathbb{R}^ℓ is defined for $\vec{\xi} = (\xi_1, \dots, \xi_\ell) \in \mathbb{R}^\ell$ by

$$\hat{\mu}(\vec{\xi}) = \int_{\mathbb{R}^\ell} \exp \left(-i \sum_{k=1}^{\ell} \xi_k t_k \right) d\mu.$$

2.2 Limiting Distributions of Real Functions With “Explicit Formula”

In this section we prove the existence of a limiting distribution for the function $\phi(y)$ with explicit formula of the form (2.3).

A key point in proving that $\phi(y)$ possesses a limiting distribution is to demonstrate that

$$\int_1^Y |\epsilon^{(T)}(y)|^2 dy \ll Y \frac{(\log T)^\alpha}{T^\beta} + 1 \tag{2.5}$$

where $\epsilon^{(T)}(y)$ is given in (2.1), and α, β are positive constants.

In order to prove the existence of a limiting distribution for $\phi(y)$ we will need to assume certain bounds on average for the sequence $|r_n|$ and their powers. More precisely, our conditions are the following:

Let $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$ be real constants such that $\theta_4 > -1/2$ and $2\theta_1 < \theta_4$. We shall assume

$$\sum_{\lambda_n \leq T} |r_n| \ll T^{\theta_1} (\log T)^{\theta_2},$$

$$\sum_{\lambda_n > T} |r_n|^2 \ll \frac{(\log T)^{\theta_3}}{T^{\theta_4}},$$

and

$$\sum_{T \leq \lambda_n < T+1} 1 \ll (\log T)^{\theta_5}.$$

We begin with some standard results from probability theory.

Lemma 2.6. (i) (Helly's theorem) *Let $\{F_n\}$ be a sequence of distribution functions on \mathbb{R}^ℓ . There exist a subsequence $\{F_n\}$ and a generalized distribution function F on \mathbb{R}^ℓ such that F_n converges to F weakly.*

(ii) *Let $\{F_n\}$ be a sequence of distribution functions and F be a generalized distribution function on \mathbb{R}^ℓ . Then F_n converges to F weakly if and only if*

$$\int_{\mathbb{R}^\ell} f(x_1, \dots, x_\ell) dF_n(x_1, \dots, x_\ell) \rightarrow \int_{\mathbb{R}^\ell} f(x_1, \dots, x_\ell) dF(x_1, \dots, x_\ell)$$

for all bounded continuous real-valued functions f on \mathcal{R}^ℓ .

(iii) (Levy's theorem) *Let $\{F_n\}$ and F be distribution functions on \mathbb{R}^ℓ with Fourier transforms $\{\widehat{F}_n\}$ and \widehat{F} , respectively. Then F_n converges to F weakly if and only if \widehat{F}_n converges to \widehat{F} pointwise.*

Proof. (i) See [25, p. 321, Problem 1].

(ii) See [1, pp. 344–346].

(iii) See [1, pp. 359–360].

□

Our next lemma is the change of variable formula for measurable spaces.

Lemma 2.7 (Change of Variable Formula). *Let (Ω, \mathcal{F}) and (Ω', \mathcal{F}') be measurable spaces, and μ be a measure on (Ω, \mathcal{F}) . Suppose that $X : \Omega \rightarrow \Omega'$ is $(\mathcal{F}, \mathcal{F}')$ -measurable and define a measure μX^{-1} on (Ω', \mathcal{F}') by*

$$\mu X^{-1}(A') = \mu(X^{-1}(A')), \quad A' \in \mathcal{F}'.$$

Then a function $g : \Omega' \rightarrow \mathbb{R}$ is integrable with respect to μX^{-1} if and only if $g \circ X : \Omega \rightarrow \mathbb{R}$ is integrable with respect to μ , and in this case we have

$$\int_{X^{-1}(A')} (g \circ X) d\mu = \int_{A'} g d(\mu X^{-1}), \quad (2.6)$$

for any $A' \in \mathcal{F}'$. Moreover, if g is nonnegative, then (2.6) is always true.

Proof. See [1, Theorem 16.13]. □

The next lemma is the standard partial summation formula.

Lemma 2.8. *Let $\{\alpha_n\}$ be an increasing sequence of positive numbers which increases to infinity. Let $\{c_n\}$ be a complex sequence, and set*

$$C(t) = \sum_{\alpha_n \leq t} c_n.$$

If $T \geq \alpha_1$ and $\phi(t)$ is a real function with continuous derivative, then

$$\sum_{\alpha_n \leq T} c_n \phi(\alpha_n) = - \int_{\alpha_1}^T C(t) \phi'(t) dt + C(T) \phi(T).$$

Proof. See [11, Theorem A]. □

The next lemma establishes upper bounds for certain averages of r_n .

Lemma 2.9. *Let $\{\lambda_n\}$ be a positive, non-decreasing sequence that increases to infinity. Let $u, v, a, b \in \mathbb{R}$ and assume $b > u$. If $\sum_{\lambda_n \leq T} |r_n| \ll T^u (\log T)^v$, then*

$$\sum_{\lambda_n > T} |r_n| \frac{(\log \lambda_n)^a}{\lambda_n^b} \ll \frac{(\log T)^{a+v}}{T^{b-u}}.$$

Proof. In Lemma 2.8 put $\alpha_n = \lambda_n$, $c_n = |r_n|$, and $\phi(t) = (\log t)^a / t^b$. Then ϕ is continuously differentiable on $\mathbb{R} \setminus \{0\}$ and

$$\phi'(t) = \frac{a(\log t)^{a-1} - b(\log t)^a}{t^{b+1}}.$$

Hence

$$\sum_{\lambda_n > T} |r_n| \frac{(\log \lambda_n)^a}{\lambda_n^b} = \left[\phi(t) \left(\sum_{\lambda_n \leq t} |r_n| \right) \right]_T^\infty - \int_T^\infty \phi'(t) \left(\sum_{\lambda_n \leq t} |r_n| \right) dt.$$

We have

$$\phi(t) \left(\sum_{\lambda_n \leq t} |r_n| \right) \ll \frac{(\log t)^{a+v}}{t^{b-u}} \rightarrow 0,$$

as $t \rightarrow \infty$, since $b - u > 0$. Hence

$$\left[\phi(t) \left(\sum_{\lambda_n \leq t} |r_n| \right) \right]_T^\infty \ll \frac{(\log T)^{a+v}}{T^{b-u}}.$$

Also

$$\int_T^\infty \phi'(t) \left(\sum_{\lambda_n \leq t} |r_n| \right) dt \ll \int_T^\infty \frac{(\log t)^{a+v}}{t^{b-u+1}} dt \ll \frac{(\log T)^{a+v}}{T^{b-u}}.$$

Hence

$$\sum_{\lambda_n > T} |r_n| \frac{(\log \lambda_n)^a}{\lambda_n^b} \ll \frac{(\log T)^{a+v}}{T^{b-u}}$$

which completes the proof. \square

The next lemma is crucial in proving the existence of a limiting distribution for $\phi(y)$. In fact, this lemma provides reasonable conditions on moments of r_n in order to establish (2.5).

Lemma 2.10. *Let $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$ be real constants such that $\theta_4 > -1$ and $2\theta_1 < \theta_4$. Assume that*

$$\sum_{\lambda_n \leq T} |r_n| \ll T^{\theta_1} (\log T)^{\theta_2}, \quad (2.7)$$

$$\sum_{\lambda_n > T} |r_n|^2 \ll \frac{(\log T)^{\theta_3}}{T^{\theta_4}}, \quad (2.8)$$

and

$$\sum_{T \leq \lambda_n < T+1} 1 \ll (\log T)^{\theta_5}. \quad (2.9)$$

Then

$$\int_V^{V+1} \left| \sum_{T < \lambda_n \leq X} r_n e^{iy\lambda_n} \right|^2 dy \ll \frac{(\log T)^{\theta_2 + (\theta_3 + \theta_5)/2}}{T^{\mathfrak{m} - \theta_1}},$$

for $V \in \mathbb{R}$ and $1 \leq T < X$, where

$$\mathfrak{m} = \begin{cases} (2\theta_4 - 1)/4 & \text{if } -1 < \theta_4 \leq 1 \\ \theta_4/4 & \text{if } 1 \leq \theta_4 \leq 3 \\ 3/4 & \text{if } \theta_4 \geq 3. \end{cases}$$

Proof. Before proceeding note that by (2.9)

$$\sum_{\lambda_n \leq T} 1 \leq \sum_{k=1}^{\lfloor T \rfloor + 1} \sum_{k-1 < \lambda_n \leq k} 1 \ll \sum_{k=1}^{\lfloor T \rfloor + 1} (\log k)^{\theta_5} \ll T(\log T)^{\theta_5}. \quad (2.10)$$

Furthermore, by (2.7) and Lemma 2.9 we have

$$\sum_{\lambda_n > T} |r_n| \frac{(\log \lambda_n)^\alpha}{\lambda_n^\beta} \ll \frac{(\log T)^{\alpha + \theta_2}}{T^{\beta - \theta_1}}, \quad (2.11)$$

provided that $\beta > \theta_1$.

From the identity $|z|^2 = z\bar{z}$ we obtain

$$\begin{aligned} & \int_V^{V+1} \left| \sum_{T < \lambda_n \leq X} r_n e^{iy\lambda_n} \right|^2 dy \\ &= \int_V^{V+1} \left(\sum_{T < \lambda_n \leq X} r_n e^{iy\lambda_n} \right) \overline{\left(\sum_{T < \lambda_m \leq X} r_m e^{iy\lambda_m} \right)} dy \\ &= \sum_{T < \lambda_n \leq X} \sum_{T < \lambda_m \leq X} r_n \bar{r}_m \int_V^{V+1} e^{iy(\lambda_n - \lambda_m)} dy \\ &\ll \sum_{T < \lambda_n \leq X} \sum_{T < \lambda_m \leq X} |r_n r_m| \min \left(1, \frac{1}{|\lambda_n - \lambda_m|} \right) = \Sigma_1 + \Sigma_2, \end{aligned}$$

where Σ_1 is the sum of those terms for which $|\lambda_n - \lambda_m| \leq 1$, and Σ_2 is the sum over

the complementary set. For Σ_1 , by the Cauchy-Schwarz inequality we have

$$\begin{aligned}\Sigma_1 &\leq \sum_{T < \lambda_n \leq X} |r_n| \sum_{\lambda_n - 1 \leq \lambda_m \leq \lambda_n + 1} |r_m| \\ &\leq \sum_{T < \lambda_n \leq X} |r_n| \left(\sum_{\lambda_n - 1 \leq \lambda_m \leq \lambda_n + 1} |r_m|^2 \right)^{1/2} \left(\sum_{\lambda_n - 1 \leq \lambda_m \leq \lambda_n + 1} 1 \right)^{1/2}.\end{aligned}$$

Hence by (2.8), (2.9), and (2.11) we have

$$\begin{aligned}\Sigma_1 &\ll \sum_{T < \lambda_n \leq X} |r_n| \frac{(\log(\lambda_n - 1))^{\theta_3/2}}{(\lambda_n - 1)^{\theta_4/2}} (\log(\lambda_n - 1))^{\theta_5/2} \\ &\ll \sum_{T < \lambda_n \leq X} |r_n| \frac{(\log \lambda_n)^{(\theta_3 + \theta_5)/2}}{\lambda_n^{\theta_4/2}} \\ &\leq \sum_{\lambda_n > T} |r_n| \frac{(\log \lambda_n)^{(\theta_3 + \theta_5)/2}}{\lambda_n^{\theta_4/2}} \ll \frac{(\log T)^{\theta_2 + (\theta_3 + \theta_5)/2}}{T^{\theta_4/2 - \theta_1}},\end{aligned}$$

since $\theta_4/2 > \theta_1$. To study Σ_2 , we define for any $T \geq 1$

$$S_T(U) = \sum_{\substack{\lambda_m \geq T \\ |U - \lambda_m| \geq 1}} \frac{|r_m|}{|U - \lambda_m|},$$

where $U \geq T$. Then we can write

$$\Sigma_2 = \sum_{T < \lambda_n \leq X} |r_n| \sum_{\substack{T < \lambda_m \leq X \\ |\lambda_n - \lambda_m| \geq 1}} \frac{|r_m|}{|\lambda_n - \lambda_m|} \leq \sum_{T < \lambda_n \leq X} |r_n| S_T(\lambda_n). \quad (2.12)$$

Let $T \geq 1$ be fixed. For any number $U \geq T$ consider the set of numbers \sqrt{U} , $U - \sqrt{U}$, and $U - 1$. Either of the following cases occurs

$$T \leq \sqrt{U}, \quad \sqrt{U} < T \leq U - \sqrt{U}, \quad U - \sqrt{U} < T \leq U - 1,$$

or

$$U - 1 < T \leq U.$$

Suppose that the first case happens, that is, $T \leq \sqrt{U}$. Then

$$S_T(U) = \left(\sum_{T \leq \lambda_m < \sqrt{U}} + \sum_{\sqrt{U} \leq \lambda_m \leq U - \sqrt{U}} + \sum_{U - \sqrt{U} \leq \lambda_m < U - 1} \right. \\ \left. + \sum_{U + 1 \leq \lambda_m < U + \sqrt{U}} + \sum_{U + \sqrt{U} \leq \lambda_m \leq 2U} + \sum_{\lambda_m \geq 2U} \right) \frac{|r_m|}{|U - \lambda_m|} \quad (2.13)$$

Denote these sums by $\sigma_1, \dots, \sigma_6$. Then by repeated application of Cauchy-Schwarz inequality, (2.8), and (2.10) we deduce

$$\sigma_1 \leq \frac{1}{U - \sqrt{U}} \sum_{T \leq \lambda_m < \sqrt{U}} |r_m| \\ \ll \frac{1}{U} \left(\sum_{T \leq \lambda_m < \sqrt{U}} |r_m|^2 \right)^{1/2} \left(\sum_{T \leq \lambda_m < \sqrt{U}} 1 \right)^{1/2} \\ \ll \frac{1}{U} \left(\frac{(\log T)^{\theta_3}}{T^{\theta_4}} \right)^{1/2} \left(\sqrt{U} (\log U)^{\theta_5} \right)^{1/2} \leq \frac{(\log U)^{(\theta_3 + \theta_5)/2}}{U^{3/4}} \quad (2.14)$$

and

$$\sigma_2 \leq \frac{1}{\sqrt{U}} \sum_{\sqrt{U} \leq \lambda_m < U - \sqrt{U}} |r_m| \\ \leq \frac{1}{\sqrt{U}} \left(\sum_{\sqrt{U} \leq \lambda_m < U - \sqrt{U}} |r_m|^2 \right)^{1/2} \left(\sum_{\sqrt{U} \leq \lambda_m < U - \sqrt{U}} 1 \right)^{1/2} \\ \ll \frac{1}{\sqrt{U}} \left(\frac{(\log U)^{\theta_3}}{U^{\theta_4/2}} \right)^{1/2} \left(U (\log U)^{\theta_5} \right)^{1/2} = \frac{(\log U)^{(\theta_3 + \theta_5)/2}}{U^{\theta_4/4}}. \quad (2.15)$$

For σ_3 by using (2.9) instead of (2.10) we have

$$\sigma_3 \leq \left(\sum_{U - \sqrt{U} \leq \lambda_m < U - 1} |r_m|^2 \right)^{1/2} \left(\sum_{U - \sqrt{U} \leq \lambda_m < U - 1} 1 \right)^{1/2} \\ \ll \left(\frac{(\log U)^{\theta_3}}{U^{\theta_4}} \right)^{1/2} \left(\sqrt{U} (\log U)^{\theta_5} \right)^{1/2} = \frac{(\log U)^{(\theta_3 + \theta_5)/2}}{U^{(2\theta_4 - 1)/4}}. \quad (2.16)$$

By similar tricks we can bound σ_4 and σ_5 as

$$\begin{aligned}\sigma_4 &\leq \left(\sum_{U+1 \leq \lambda_m < U+\sqrt{U}} |r_m|^2 \right)^{1/2} \left(\sum_{U+1 \leq \lambda_m < U+\sqrt{U}} 1 \right)^{1/2} \\ &\ll \left(\frac{(\log U)^{\theta_3}}{U^{\theta_4}} \right)^{1/2} \left(\sqrt{U} (\log U)^{\theta_5} \right)^{1/2} = \frac{(\log U)^{(\theta_3+\theta_5)/2}}{U^{(2\theta_4-1)/4}}\end{aligned}\quad (2.17)$$

and

$$\begin{aligned}\sigma_5 &\leq \frac{1}{\sqrt{U}} \sum_{U+\sqrt{U} \leq \lambda_m < 2U} |r_m| \\ &\leq \frac{1}{\sqrt{U}} \left(\sum_{U+\sqrt{U} \leq \lambda_m < 2U} |r_m|^2 \right)^{1/2} \left(\sum_{U+\sqrt{U} \leq \lambda_m < 2U} 1 \right)^{1/2} \\ &\ll \frac{1}{\sqrt{U}} \left(\frac{(\log U)^{\theta_3}}{U^{\theta_4}} \right)^{1/2} \left(U (\log U)^{\theta_5} \right)^{1/2} = \frac{(\log U)^{(\theta_3+\theta_5)/2}}{U^{\theta_4/2}}.\end{aligned}\quad (2.18)$$

Finally, for σ_6 we will divide the interval of summation into subintervals $[2^k U, 2^{k+1} U)$, $k = 1, 2, \dots$, and use the elementary fact that for $A \gg 1$ and $B \gg 1$ we have $A + B \ll AB$. We see that it becomes

$$\begin{aligned}\sigma_6 &\leq \sum_{k=1}^{\infty} \sum_{2^k U \leq \lambda_m < 2^{k+1} U} \frac{|r_m|}{|U - \lambda_m|} \\ &\leq \sum_{k=1}^{\infty} \frac{1}{(2^k - 1)U} \left(\sum_{2^k U \leq \lambda_m < 2^{k+1} U} |r_m|^2 \right)^{1/2} \left(\sum_{2^k U \leq \lambda_m < 2^{k+1} U} 1 \right)^{1/2} \\ &\ll \sum_{k=1}^{\infty} \frac{1}{(2^k - 1)U} \left(\frac{(\log(2^k U))^{\theta_3}}{(2^k U)^{\theta_4}} \right)^{1/2} \left(2^{k+1} U (\log(2^{k+1} U))^{\theta_5} \right)^{1/2} \\ &\ll \sum_{k=1}^{\infty} \frac{k^{(\theta_3+\theta_5)/2}}{2^{k(1+\theta_4)/2}} \frac{(\log U)^{(\theta_3+\theta_5)/2}}{U^{(1+\theta_4)/2}} \ll \frac{(\log U)^{(\theta_3+\theta_5)/2}}{U^{(1+\theta_4)/2}},\end{aligned}\quad (2.19)$$

which is true as long as $\theta_4 > -1$.

If we denote the minimum value of the powers of the denominators in relations

(2.14) to (2.19) by \mathbf{m} then

$$\mathbf{m} = \begin{cases} (2\theta_4 - 1)/4 & \text{if } -1/2 < \theta_4 \leq 1 \\ \theta_4/4 & \text{if } 1 \leq \theta_4 \leq 3 \\ 3/4 & \text{if } \theta_4 \geq 3, \end{cases}$$

and therefore

$$S_T(U) \ll (\log U)^{(\theta_3 + \theta_5)/2} / U^{\mathbf{m}}$$

as long as $T \leq \sqrt{U}$. The same argument applies in the other three cases. Hence by (2.11) we have

$$\Sigma_2 \ll \sum_{\lambda_n > T} |r_n| S_T(\lambda_n) \ll \frac{(\log T)^{\theta_2 + (\theta_3 + \theta_5)/2}}{T^{\mathbf{m} - \theta_1}}.$$

Note that always $\mathbf{m} \leq \theta_4/2$ whenever $\theta_3 > 0$. Hence

$$\Sigma_1 + \Sigma_2 \ll \frac{(\log T)^{\theta_2 + (\theta_3 + \theta_5)/2}}{T^{\mathbf{m} - \theta_1}},$$

and the proof follows. \square

In the next lemma we will reduce the number of conditions in the previous lemma by considering a new average bound assumption for r_n .

Lemma 2.11. *Let $\theta_5, \theta_6, \theta_7$ be real constants such that $\theta_6 < 3/2$. Assume*

$$\sum_{T \leq \lambda_n < T+1} 1 \ll (\log T)^{\theta_5} \tag{2.20}$$

and

$$\sum_{\lambda_n \leq T} \lambda_n^2 |r_n|^2 \ll T^{\theta_6} (\log T)^{\theta_7}. \tag{2.21}$$

Then for $V \in \mathbb{R}$ and $1 \leq T < X$

$$\int_V^{V+1} \left| \sum_{T < \lambda_n \leq X} r_n e^{iy\lambda_n} \right|^2 dy \ll \frac{(\log T)^{(\theta_5 + 3\theta_7)/2}}{T^{\mathbf{m}'}} ,$$

where

$$\mathbf{m}' = \begin{cases} 5/4 - \theta_6 & \text{if } 1 \leq \theta_6 < 3/2 \\ 1 - 3\theta_6/4 & \text{if } -1 \leq \theta_6 \leq 1 \\ 5/4 - \theta_6/2 & \text{if } \theta_6 \leq -1. \end{cases} \quad (2.22)$$

Proof. By the Cauchy-Schwarz inequality

$$\begin{aligned} \sum_{\lambda_n \leq T} \lambda_n |r_n| &\leq \left(\sum_{\lambda_n \leq T} \lambda_n^2 |r_n|^2 \right)^{1/2} \left(\sum_{\lambda_n \leq T} 1 \right)^{1/2} \\ &\ll (T^{\theta_6} (\log T)^{\theta_7})^{1/2} (T (\log T)^{\theta_5})^{1/2} = T^{(1+\theta_6)/2} (\log T)^{(\theta_5+\theta_7)/2}. \end{aligned} \quad (2.23)$$

In Lemma 2.8 we take $\alpha_n = \lambda_n$, $c_n = \lambda_n |r_n|$ and $\phi(t) = t^{-1}$. Then $\phi'(t) = -t^{-2}$ and by (2.23) we have

$$\begin{aligned} \sum_{\lambda_n \leq T} |r_n| &= \int_{\lambda_1}^T \left(\sum_{\lambda_n \leq t} \lambda_n |r_n| \right) \frac{dt}{t^2} + \frac{1}{T} \sum_{\lambda_n \leq T} \lambda_n |r_n| \\ &\ll \int_{\lambda_1}^T t^{\frac{\theta_6-3}{2}} (\log t)^{(\theta_5+\theta_7)/2} dt + T^{\frac{\theta_6-1}{2}} (\log T)^{(\theta_5+\theta_7)/2} \\ &\ll T^{(\theta_6-1)/2} (\log T)^{(\theta_5+\theta_7)/2}. \end{aligned} \quad (2.24)$$

By another application of Lemma 2.8 with $\alpha_n = \lambda_n$, $c_n = \lambda_n^2 |r_n|^2$ and $\phi(t) = t^{-2}$, and using (2.21) we find that

$$\begin{aligned} \sum_{\lambda_n > T} |r_n|^2 &= 2 \int_T^\infty \left(\sum_{\lambda_n \leq t} \lambda_n^2 |r_n|^2 \right) \frac{dt}{t^3} + \\ &\quad \lim_{X \rightarrow \infty} X^{-2} \left(\sum_{\lambda_n \leq X} \lambda_n^2 |r_n|^2 \right) - T^{-2} \left(\sum_{\lambda_n \leq T} \lambda_n^2 |r_n|^2 \right) \\ &\ll \int_T^\infty t^{\theta_6-3} (\log t)^{\theta_7} dt + T^{\theta_6-2} (\log T)^{\theta_7} \ll (\log T)^{\theta_7} / T^{2-\theta_6}, \end{aligned} \quad (2.25)$$

which is true as long as $\theta_6 < 2$. Now we use the bounds (2.24) and (2.25) in Lemma 2.10 to obtain

$$\int_V^{V+1} \left| \sum_{T < \lambda_n \leq X} r_n e^{iy\lambda_n} \right|^2 dy \ll \frac{(\log T)^{(\theta_5+3\theta_7)/2}}{T^{\mathbf{m}'}}$$

subject to $\theta_6 < 3/2$, where \mathbf{m}' is given by (2.22). This completes the proof. \square

As we mentioned earlier, a key point in proving the existence of a limiting distribution for $\phi(y)$ is to demonstrate the relation (2.5) for $\epsilon^{(T)}(y)$. We will do this in the next lemma.

Lemma 2.12. (i) *Assume the conditions of Lemma 2.10 and (2.4). Then*

$$\int_1^Y |\epsilon^{(T)}(y)|^2 dy \ll Y \frac{(\log T)^{\theta_2 + (\theta_3 + \theta_5)/2}}{T^{\mathbf{m} - \theta_1}} + 1.$$

(ii) *Assume the conditions of Lemma 2.11 and (2.4). Then*

$$\int_1^Y |\epsilon^{(T)}(y)|^2 dy \ll Y \frac{(\log T)^{(\theta_5 + 3\theta_7)/2}}{T^{\mathbf{m}'}} + 1.$$

Proof. (i) For complex numbers A, B we have

$$|A + B|^2 \leq 2(|A|^2 + |B|^2).$$

Hence by Lemma 2.10 and (2.4) we have

$$\begin{aligned} \int_1^Y |\epsilon^{(T)}(y)|^2 dy &\ll \int_1^Y \left| \sum_{T < \lambda_n \leq e^Y} r_n e^{iy\lambda_n} \right|^2 dy + \int_1^Y |E(y)|^2 dy \\ &\ll \sum_{j=1}^{\lfloor Y \rfloor} \int_j^{j+1} \left| \sum_{T < \lambda_n \leq e^Y} r_n e^{iy\lambda_n} \right|^2 dy + 1 \\ &\ll Y \frac{(\log T)^{\theta_2 + (\theta_3 + \theta_5)/2}}{T^{\mathbf{m} - \theta_1}} + 1. \end{aligned} \tag{2.26}$$

(ii) The proof is similar to (i), except that here we need to apply Lemma 2.11 instead of Lemma 2.10. \square

For further investigation, it is necessary to state Kronecker-Weyl theorem here.

Let \mathbb{T}^N denote the N -torus, namely,

$$\mathbb{T}^N = \{(\theta_1, \dots, \theta_N) \mid \theta_i \in [0, 1)\}.$$

Then we have the following result.

Lemma 2.13. *Consider a sequence $\{\alpha_1, \dots, \alpha_N\}$ of real numbers and let ω be the normalized Haar measure on \mathbb{T}^N . Then $\{\alpha_1, \dots, \alpha_N\}$ is linearly independent over \mathbb{Q} if and only if for all continuous functions $g : \mathbb{R}^N \rightarrow \mathbb{R}$ of period 1 in each variable we have*

$$\lim_{Y \rightarrow \infty} \frac{1}{Y} \int_0^Y g(y\alpha_1, \dots, y\alpha_N) dy = \int_{\mathbb{T}^N} g(a) d\omega,$$

where ω is the normalized Haar measure on A . In this case the set

$$\{(\{y\alpha_1/2\pi\}, \dots, \{y\alpha_N/2\pi\}) \mid y \in \mathbb{R}\}$$

is dense in \mathbb{T}^N , where $\{x\}$ is the fractional part of $x \in \mathbb{R}$.

Proof. See [8, pp. 1–16.] □

The next lemma is a consequence of the Kronecker-Weyl theorem. This lemma is important in proving the existence of a limiting distribution function for $\phi(y)$.

Lemma 2.14. *Let t_1, \dots, t_N be arbitrary real numbers. Suppose that A is the topological closure of*

$$\{(\{yt_1/2\pi\}, \dots, \{yt_N/2\pi\}) \mid y \in \mathbb{R}\}$$

in \mathbb{T}^N , where $\{x\}$ is the fractional part of $x \in \mathbb{R}$. Let $g : \mathbb{R}^N \rightarrow \mathbb{R}$ be a continuous function of period 1 in each of its variables. Then we have

$$\lim_{Y \rightarrow \infty} \frac{1}{Y} \int_0^Y g(yt_1, \dots, yt_N) dy = \int_A g(a) d\omega$$

where ω is the normalized Haar measure on A .

Proof. Let j be the maximum number of linearly independent elements in $\{t_1, \dots, t_N\}$ and suppose that those elements are t_1, \dots, t_j , say. Then A is the topological closure of $\{(\{yt_1/2\pi\}, \dots, \{yt_j/2\pi\}) \mid y \in \mathbb{R}\}$ in \mathbb{T}^N . Now apply Lemma 2.13 to $\{t_1, \dots, t_j\}$. □

To prove the existence of the limiting distribution of $\phi(y)$, we will construct a sequence of distribution functions μ_T and select a weakly convergent subsequence of it. To do this, we first prove the following lemma.

Lemma 2.15. *Let $T > 0$ and*

$$\phi^{(T)}(y) := c + \operatorname{Re} \left(\sum_{\lambda_n \leq T} r_n e^{iy\lambda_n} \right)$$

where c is a constant real number. For each $T \geq \lambda_1$ there is a probability measure μ_T on \mathbb{R} such that

$$\mu_T(f) := \int_{-\infty}^{\infty} f(x) d\mu_T(x) = \lim_{Y \rightarrow \infty} \frac{1}{Y} \int_0^Y f(\phi^{(T)}(y)) dy$$

for all bounded continuous functions f on \mathbb{R} .

Proof. Let $N = N(T)$ denote the number of numbers $\lambda_n \leq T$. Then we have

$$\phi^{(T)}(y) = c + \operatorname{Re} \left(\sum_{n=1}^N r_n e^{iy\lambda_n} \right).$$

Let $X_T : \mathbb{T}^N \rightarrow \mathbb{R}$ and $g : \mathbb{T}^N \rightarrow \mathbb{R}$ be defined as

$$X_T(\theta_1, \dots, \theta_N) = c + \operatorname{Re} \left(\sum_{n=1}^N r_n e^{2\pi i \theta_n} \right)$$

and

$$g(\theta_1, \dots, \theta_N) = f(X_T(\theta_1, \dots, \theta_N)).$$

By applying Lemma 2.14 to the numbers $\lambda_1/2\pi, \dots, \lambda_N/2\pi$, we have

$$\lim_{Y \rightarrow \infty} \frac{1}{Y} \int_0^Y g \left(\frac{y\lambda_1}{2\pi}, \dots, \frac{y\lambda_N}{2\pi} \right) dy = \int_A g(a) d\omega,$$

where A is the closure of $\{(\{y\lambda_1/2\pi\}, \dots, \{y\lambda_N/2\pi\}) \mid y \in \mathbb{R}\}$ in \mathbb{T}^N , and the measure ω is the normalized Haar measure on A . Define a probability measure μ_T on \mathbb{R} by

$$\mu_T(B) = \omega(X_T|_A^{-1}(B))$$

where B is any Borel set in \mathbb{R} . By change of variable formula (Lemma 2.7) we have

$$\int_A g(a) d\omega = \int_{\mathbb{R}} f(x) d\mu_T(x)$$

whence

$$\lim_{Y \rightarrow \infty} \frac{1}{Y} \int_0^Y f(\phi^{(T)}(y)) dy = \int_{-\infty}^{\infty} f(x) d\mu_T(x)$$

and the proof is complete. \square

We now have all the required material to prove the existence of the limiting distribution for $\phi(y)$.

Theorem 2.16. *Let $\phi(y)$ and $E(y, X)$ be defined as in (2.3). Suppose that r_n and λ_n satisfy the conditions (2.7), (2.8), and (2.9) in Lemma 2.10, and that $\mathfrak{m} > \theta_1$. Moreover, assume*

$$\int_1^Y |E(y, e^Y)|^2 dy \ll 1.$$

Then there is a probability measure μ on \mathbb{R} such that for all bounded Lipschitz functions f on \mathbb{R}

$$\lim_{Y \rightarrow \infty} \frac{1}{Y} \int_0^Y f(\phi(y)) dy = \int_{-\infty}^{\infty} f(x) d\mu(x).$$

Proof. First observe that

$$\int_0^1 f(\phi(y)) dy \ll 1.$$

Hence

$$\lim_{Y \rightarrow \infty} \frac{1}{Y} \int_0^Y f(\phi(y)) dy = \lim_{Y \rightarrow \infty} \frac{1}{Y} \int_1^Y f(\phi(y)) dy.$$

For fixed T let

$$\phi^{(T)}(y) := c + \operatorname{Re} \left(\sum_{\lambda_n \leq T} r_n e^{iy\lambda_n} \right).$$

Assume that f is a bounded Lipschitz function which satisfies

$$|f(x) - f(y)| \leq c_f |x - y|, \quad c_f > 0.$$

By the Cauchy-Schwartz inequality and Lemma 2.12(i) we have

$$\begin{aligned}
\frac{1}{Y} \int_1^Y f(\phi(y)) dy &= \frac{1}{Y} \int_1^Y f(\phi^{(T)}(y)) dy + O\left(\frac{c_f}{Y} \int_1^Y |\epsilon^{(T)}(y)| dy\right) \\
&= \frac{1}{Y} \int_1^Y f(\phi^{(T)}(y)) dy + \\
&\quad + O\left(\frac{c_f}{\sqrt{Y}} \left(\int_1^Y |\epsilon^{(T)}(y)|^2 dy\right)^{1/2}\right) \\
&= \frac{1}{Y} \int_1^Y f(\phi^{(T)}(y)) dy + \\
&\quad + O\left(c_f \left(\frac{(\log T)^{\theta_2 + (\theta_3 + \theta_5)/2}}{T^{m - \theta_1}} + \frac{1}{Y}\right)^{1/2}\right).
\end{aligned}$$

By Lemma 2.15, there is a probability measure μ_T for each $T \geq \lambda_1$ such that

$$\mu_T(f) = \int_{-\infty}^{\infty} f(x) d\mu_T(x) = \lim_{Y \rightarrow \infty} \frac{1}{Y} \int_1^Y f(\phi^{(T)}(y)) dy.$$

Letting $Y \rightarrow \infty$ we see that

$$\begin{aligned}
\mu_T(f) - O\left(\frac{c_f (\log T)^{\theta_2/2 + (\theta_3 + \theta_5)/4}}{T^{(m - \theta_1)/2}}\right) &\leq \liminf_{Y \rightarrow \infty} \frac{1}{Y} \int_1^Y f(\phi(y)) dy \\
&\leq \limsup_{Y \rightarrow \infty} \frac{1}{Y} \int_1^Y f(\phi(y)) dy \\
&\leq \mu_T(f) + O\left(\frac{c_f (\log T)^{\theta_2/2 + (\theta_3 + \theta_5)/4}}{T^{(m - \theta_1)/2}}\right).
\end{aligned} \tag{2.27}$$

By Lemma 2.6(i) we can choose a subsequence $\{\mu_{T_k}\}$ of these probability measures μ_T and a measure μ on \mathbb{R} such that $\mu_{T_k} \rightarrow \mu$ weakly. By Lemma 2.6(ii)

$$\mu_{T_k}(f) = \int_{\mathbb{R}} f(x) d\mu_{T_k}(x) \rightarrow \int_{\mathbb{R}} f(x) d\mu(x) = \mu(f).$$

Replacing T by T_k and letting $k \rightarrow \infty$ in (2.27) we deduce that

$$\lim_{Y \rightarrow \infty} \frac{1}{Y} \int_1^Y f(\phi(y)) dy = \int_{\mathbb{R}} f(x) d\mu(x) = \mu(f).$$

Hence (2.27) becomes

$$\mu_{T_k}(f) - O\left(\frac{c_f(\log T_k)^{\theta_2/2 + (\theta_3 + \theta_5)/4}}{T_k^{(m-\theta_1)/2}}\right) \leq \mu(f) \leq \mu_{T_k}(f) + O\left(\frac{c_f(\log T_k)^{\theta_2/2 + (\theta_3 + \theta_5)/4}}{T_k^{(m-\theta_1)/2}}\right),$$

or equivalently

$$\left| \int_{\mathbb{R}} f(x) d\mu(x) - \int_{\mathbb{R}} f(x) d\mu_{T_k}(x) \right| \ll \frac{c_f(\log T_k)^{\theta_2/2 + (\theta_3 + \theta_5)/4}}{T_k^{(m-\theta_1)/2}}. \quad (2.28)$$

By applying equation (2.28) with $f(x) = 1$ we obtain

$$\left| \int_{\mathbb{R}} d\mu(x) - 1 \right| \ll \frac{(\log T_k)^{\theta_2/2 + (\theta_3 + \theta_5)/4}}{T_k^{(m-\theta_1)/2}}.$$

By letting $k \rightarrow \infty$, this proves that μ is a probability measure. Hence the proof is complete. \square

Corollary 2.17. *Suppose that r_n and λ_n satisfy the conditions (2.20) and (2.21) in Lemma 2.11, and that $\theta_6 < 5/4$. Moreover, assume*

$$\int_1^Y |E(y, e^Y)|^2 dy \ll 1.$$

Then there is a probability measure μ on \mathbb{R} such that for all bounded Lipschitz functions f on \mathbb{R}

$$\lim_{Y \rightarrow \infty} \frac{1}{Y} \int_0^Y f(\phi(y)) dy = \int_{-\infty}^{\infty} f(x) d\mu(x).$$

Proof. The proof is similar to Theorem 2.16, except that here we use Lemma 2.11 instead of Lemma 2.10. \square

In Theorem 2.16 and Corollary 2.17, it is possible to extend the range of f from

Lipschitz functions to all bounded continuous functions. (See Lemmas 2.24 and 2.25 bellow.)

In the next theorem, we find an explicit formula for the Fourier transform of the limiting distribution measures given in Theorem 2.16 and Corollary 2.17.

Theorem 2.18. *Assume that the conditions in Theorem 2.16 or Corollary 2.17 hold and let μ be the limiting distribution of $\phi(y)$. Moreover, assume that $\{\lambda_n\}$ is linearly independent over \mathbb{Q} (i.e., for any finite subset $\{\lambda_{n_1}, \dots, \lambda_{n_k}\} \subseteq \{\lambda_n\}$, any relation of the form $\sum_{i=1}^k c_i \lambda_{n_i} = 0$ ($c_i \in \mathbb{Q}$) implies all $c_i = 0$.) Then the Fourier transform $\hat{\mu}(\xi)$ exists and equals*

$$\hat{\mu}(\xi) = e^{-ic\xi} \prod_{n=1}^{\infty} J_0(|r_n|\xi)$$

where $J_0(z)$ is the Bessel function

$$J_0(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m}}{(m!)^2}. \quad (2.29)$$

Proof. Consider the distribution function μ_T in the proof of Lemma 2.15. For any Borel subset B of \mathbb{R}

$$\mu_T(B) = \omega(X_T|_A^{-1}(B)),$$

where ω is the normalized Haar measure on the topological closure A of the set

$$\{(\{y\lambda_1/2\pi\}, \dots, \{y\lambda_N/2\pi\}) \mid y \in \mathbb{R}\}$$

in \mathbb{T}^N . By Kronecker-Weyl theorem the linear independence assumption on the numbers $\lambda_1, \dots, \lambda_N$ implies that $A = \mathbb{T}^N$. So the normalized Haar measure $d\omega(\theta_1, \dots, \theta_N)$ on A is equal to the Lebesgue measure $d\theta_1 \dots d\theta_N$ on \mathbb{T}^N . Hence by the change of

variable formula and Fubini's theorem

$$\begin{aligned}
\hat{\mu}_T(\xi) &= \int_{\mathbb{R}} e^{-i\xi t} d\mu_T(t) = \int_{\mathbb{T}^N} e^{-i\xi X_T(\theta_1, \dots, \theta_N)} d\omega \\
&= \int_{\mathbb{T}^N} \exp\left(-i c\xi - i\xi \sum_{n=1}^N \operatorname{Re}(r_n e^{2\pi i \theta_n})\right) d\theta_1 \dots d\theta_N \\
&= e^{-i c\xi} \prod_{n=1}^N \int_0^1 \exp(-i\xi \operatorname{Re}(r_n e^{2\pi i \theta})) d\theta.
\end{aligned}$$

By Lemma 2.6(iii) for any $\xi \in \mathbb{R}$

$$\lim_{T \rightarrow \infty} \hat{\mu}_T(\xi) = \hat{\mu}(\xi).$$

Hence

$$\begin{aligned}
\hat{\mu}(\xi) &= \lim_{T \rightarrow \infty} \hat{\mu}_T(\xi) = \lim_{T \rightarrow \infty} e^{-i c\xi} \prod_{n=1}^{N(T)} \int_0^1 \exp(-i\xi \operatorname{Re}(r_n e^{2\pi i \theta})) d\theta \\
&= e^{-i c\xi} \prod_{n=1}^{\infty} \int_0^1 \exp(-i\xi \operatorname{Re}(r_n e^{2\pi i \theta})) d\theta
\end{aligned}$$

since $N(T) \rightarrow \infty$ as $T \rightarrow \infty$. The last integral above equals

$$\int_0^1 \exp(-i\xi \operatorname{Re}(|r_n| e^{i(2\pi\theta + \arg(r_n))})) d\theta = \int_0^1 \exp(-i\xi |r_n| \cos(2\pi\theta + \arg(r_n))) d\theta.$$

By the change of variable $t = \theta + \arg(r_n)/2\pi$ we obtain the integral

$$\int_{\arg(r_n)/2\pi}^{1+\arg(r_n)/2\pi} \exp(-i\xi |r_n| \cos(2\pi t)) dt = \int_0^1 \exp(-i\xi |r_n| \cos(2\pi t)) dt,$$

since the integrand is periodic of period 1. Hence

$$\hat{\mu}(\xi) = \prod_{n=1}^{\infty} \int_0^1 \exp(-i\xi |r_n| \cos(2\pi t)) dt.$$

It is known that the Bessel function $J_0(z)$ equals

$$J_0(z) = \int_0^1 e^{-iz \cos(2\pi t)} dt.$$

From this we conclude that

$$\hat{\mu}(\xi) = e^{-ic\xi} \prod_{n=1}^{\infty} J_0(|r_n|\xi)$$

which completes the proof. □

2.3 Limiting Distributions of Vector-valued Functions With “Explicit Formula”

The results of the previous section can be stated in a general case where we have a finite sequence of random variables. More precisely, let for each $k = 1, \dots, \ell$ the sequence $\{\lambda_{k,n}\}_{n=1}^{\infty}$ be an increasing sequence of positive numbers that tends to infinity. Consider a complex sequence $\{r_{k,n}\}_{n=1}^{\infty}$ for each $k = 1, \dots, \ell$. Define for each $k = 1, \dots, \ell$

$$\phi_k(y) := c_k + \operatorname{Re} \left(\sum_{\substack{n \geq 1 \\ \lambda_{k,n} \leq X}} r_{k,n} e^{iy\lambda_{k,n}} \right) + E_k(y, X),$$

$$\phi_k^{(T)}(y) := c_k + \operatorname{Re} \left(\sum_{\substack{n \geq 1 \\ \lambda_{k,n} \leq T}} r_{k,n} e^{iy\lambda_{k,n}} \right)$$

and

$$\epsilon_k^{(T)}(y) := \operatorname{Re} \left(\sum_{\substack{n \geq 1 \\ T < \lambda_{k,n} \leq e^Y}} r_{k,n} e^{iy\lambda_{k,n}} \right) + E_k(y, e^Y),$$

and let

$$\vec{\phi}(y) := (\phi_1(y), \dots, \phi_\ell(y)), \tag{2.30}$$

$$\vec{\phi}^{(T)}(y) := (\phi_1^{(T)}(y), \dots, \phi_\ell^{(T)}(y)), \tag{2.31}$$

and

$$\vec{\epsilon}^{(T)}(y) := (\epsilon_1^{(T)}(y), \dots, \epsilon_\ell^{(T)}(y)).$$

Then we have the following result.

Lemma 2.19. *Let $\vec{\phi}^{(T)}(y)$ be given by (2.31). For each $T \geq \lambda_1$ there is a probability measure μ_T on \mathbb{R}^ℓ such that*

$$\mu_T(f) := \int_{\mathbb{R}^\ell} f(x_1, \dots, x_\ell) d\mu_T(x_1, \dots, x_\ell) = \lim_{Y \rightarrow \infty} \frac{1}{Y} \int_0^Y f(\vec{\phi}^{(T)}(y)) dy$$

for all bounded continuous functions f on \mathbb{R}^k .

Proof. Proof is similar to Lemma 2.15 by taking

$$\vec{c} = (c_1, \dots, c_\ell)$$

and

$$\vec{r}_n = (r_{1,n}, \dots, r_{\ell,n})$$

for $n \geq 1$ and considering \mathbb{R}^ℓ instead of \mathbb{R} in that proof. \square

The next theorem proves the existence of a limiting distribution for $\vec{\phi}(y)$ under certain conditions.

Theorem 2.20. *Let $\vec{\phi}(y)$ be given by (2.30). Suppose that for each $k = 1, \dots, \ell$ the following conditions hold*

$$\sum_{\substack{n \geq 1 \\ \lambda_{k,n} \leq T}} |r_{k,n}| \ll T^{\theta_{1,k}} (\log T)^{\theta_{2,k}}, \quad (2.32)$$

$$\sum_{\substack{n \geq 1 \\ \lambda_{k,n} > T}} |r_{k,n}|^2 \ll \frac{(\log T)^{\theta_{3,k}}}{T^{\theta_{4,k}}}, \quad (2.33)$$

$$\sum_{\substack{n \geq 1 \\ T \leq \lambda_{k,n} < T+1}} 1 \ll (\log T)^{\theta_{5,k}}, \quad (2.34)$$

and

$$\int_1^Y |E_k(y, e^Y)|^2 dy \ll 1.$$

Moreover, let

$$\Theta_1 = \max_{1 \leq k \leq \ell} \theta_{1,k}, \quad \Theta_2 = \max_{1 \leq k \leq \ell} \theta_{2,k}, \quad \Theta_3 = \max_{1 \leq k \leq \ell} \theta_{3,k}, \quad \Theta_4 = \min_{1 \leq k \leq \ell} \theta_{4,k},$$

$$\Theta_5 = \max_{1 \leq k \leq \ell} \theta_{5,k},$$

$$\mathfrak{M} = \begin{cases} (2\Theta_4 - 1)/4 & \text{if } -1 < \Theta_4 \leq 1 \\ \Theta_4/4 & \text{if } 1 \leq \Theta_4 \leq 3 \\ 3/4 & \text{if } \Theta_4 \geq 3, \end{cases}$$

and assume that $\Theta_4 > -1/2$, $2\Theta_1 < \Theta_4$, and $\mathfrak{M} > \Theta_1$. Then there is a probability measure μ on \mathbb{R}^ℓ such that

$$\int_{\mathbb{R}^\ell} f(x_1, \dots, x_\ell) d\mu(x_1, \dots, x_\ell) = \lim_{Y \rightarrow \infty} \frac{1}{Y} \int_0^Y f(\vec{\phi}(y)) dy$$

for all bounded Lipschitz functions f on \mathbb{R}^ℓ .

Proof. The proof follows an ℓ -dimensional adaptation of the proof of Theorem 2.16. \square

In the next corollary we will reduce the conditions of the previous theorem.

Corollary 2.21. *Let $\Theta_5, \Theta_6, \Theta_7$ be real numbers such that $\Theta_6 < 5/4$. Suppose that in Theorem 2.20 the assumptions (2.32), (2.33), and (2.34) are replaced by the following conditions*

$$\sum_{\substack{n \geq 1 \\ T \leq \lambda_{k,n} < T+1}} 1 \ll (\log T)^{\Theta_5},$$

$$\sum_{\substack{n \geq 1 \\ \lambda_{k,n} \leq T}} \lambda_{k,n}^2 |r_{k,n}|^2 \ll T^{\Theta_6} (\log T)^{\Theta_7}.$$

Then there is a probability measure μ on \mathbb{R}^ℓ such that

$$\int_{\mathbb{R}^\ell} f(x_1, \dots, x_\ell) d\mu(x_1, \dots, x_\ell) = \lim_{Y \rightarrow \infty} \frac{1}{Y} \int_0^Y f(\vec{\phi}(y)) dy,$$

for all bounded Lipschitz functions f on \mathbb{R}^ℓ .

Proof. Let

$$\mathfrak{M}' = \begin{cases} 5/4 - \Theta_6 & \text{if } 1 \leq \Theta_6 < 3/2 \\ 1 - 3\Theta_6/4 & \text{if } -1 \leq \Theta_6 \leq 1 \\ 5/4 - \Theta_6/2 & \text{if } \Theta_6 \leq -1. \end{cases}$$

Then the proof follows an ℓ -dimensional adaptation of the proof of Theorem 2.16. \square

Theorem 2.22. *Assume that the conditions in Theorem 2.20 or Corollary 2.21 hold and let μ be the limiting distribution of $\vec{\phi}(y)$. Moreover, assume that $\{\lambda_{k,n}\}_{k,n}$ is linearly independent over \mathbb{Q} . Then the Fourier transform*

$$\hat{\mu}(\xi_1, \dots, \xi_\ell) = \int_{\mathbb{R}^\ell} \exp\left(-i \sum_{k=1}^{\ell} \xi_k t_k\right) d\mu(t_1, \dots, t_\ell)$$

exists and equals

$$\hat{\mu}(\xi) = \exp\left(-i \sum_{k=1}^{\ell} c_k \xi_k\right) \times \prod_{k=1}^{\ell} \prod_{n=1}^{\infty} J_0(|r_{k,n}| \xi_k),$$

where $J_0(z)$ is the Bessel function

$$J_0(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m}}{(m!)^2}.$$

Proof. By Fubini's theorem we have

$$\hat{\mu}(\xi) = \prod_{k=1}^{\ell} \int_{-\infty}^{\infty} e^{-i\xi_k t} d\tilde{\mu}_k(t),$$

where, for $k = 1, \dots, \ell$, $\tilde{\mu}_k$ is the probability measure defined on \mathbb{R} by

$$\tilde{\mu}_k(B) = \mu(B)$$

for any Borel set $B \subseteq \mathbb{R}$. Each integral in this product is by Theorem 2.18 equal to

$$\int_{-\infty}^{\infty} e^{-i\xi_k t} d\tilde{\mu}_k(t) = e^{-ic_k \xi_k} \prod_{n=1}^{\infty} J_0(|r_{k,n}| \xi_k).$$

Hence

$$\hat{\mu}(\xi) = \exp\left(-i \sum_{k=1}^{\ell} c_k \xi_k\right) \times \prod_{k=1}^{\ell} \prod_{n=1}^{\infty} J_0(|r_{k,n}| \xi_k),$$

which completes the proof. \square

2.4 Appendix

Here we prove that in Theorem 2.16 and Corollary 2.17 we can extend the range of f from Lipschitz functions to all bounded continuous functions. We first introduce a concept from functional analysis.

Definition 2.23. A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is said to *vanish at infinity* if for any $\epsilon > 0$ there exists a compact $K \subseteq \mathbb{R}$ such that for all $x \in \mathbb{R} \setminus K$, we have $|f(x)| < \epsilon$. The set of all continuous functions $f : \mathbb{R} \rightarrow \mathbb{R}$ that vanish at infinity is denoted by $C_0(\mathbb{R}, \mathbb{R})$.

In the next lemma, we extend the results of Theorem 2.16 and Corollary 2.17 to elements of $C_0(\mathbb{R}, \mathbb{R})$.

Lemma 2.24. *Let μ be a probability measure on \mathbb{R} . Let $\phi(y)$ be defined by (2.3). Assume that*

$$\lim_{Y \rightarrow \infty} \frac{1}{Y} \int_0^Y f(\phi(y)) dy = \int_{-\infty}^{\infty} f(x) d\mu(x) \quad (2.35)$$

holds for all bounded Lipschitz functions f on \mathbb{R} . Then (2.35) holds for all $f \in C_0(\mathbb{R}, \mathbb{R})$.

Proof. The space of all Lipschitz functions in $C_0(\mathbb{R}, \mathbb{R})$ is dense in $C_0(\mathbb{R}, \mathbb{R})$, with respect to the “sup” norm. (This is a version of the Stone-Weierstrass theorem. See [7, Exercise 7.37(b)]). Let $f \in C_0(\mathbb{R}, \mathbb{R})$. Let $\epsilon > 0$ be given. There exists a sequence $\{f_n\} \subseteq C_0(\mathbb{R}, \mathbb{R})$ of Lipschitz functions such that $f_n \rightarrow f$ uniformly on \mathbb{R} . Hence there

is an $N = N(\epsilon)$ such that

$$\frac{1}{Y} \int_0^Y |f(\phi(y)) - f_N(\phi(y))| dy < \epsilon/3.$$

Moreover, since $\int_{-\infty}^{\infty} d\mu(t) = 1$,

$$\int_{-\infty}^{\infty} |f_N(t) - f(t)| d\mu(t) < \epsilon/3.$$

Also since (2.35) is true for f_N , there exists $Y_0 = Y_0(\epsilon)$ such that for all $Y \geq Y_0$

$$\left| \frac{1}{Y} \int_0^Y f_N(\phi(y)) dy - \int_{-\infty}^{\infty} f_N(t) d\mu(t) \right| < \epsilon/3.$$

These all together imply that for all $Y \geq Y_0$

$$\left| \frac{1}{Y} \int_0^Y f(\phi(y)) dy - \int_{-\infty}^{\infty} f(t) d\mu(t) \right| < \epsilon.$$

Hence (2.35) is true for f and this completes the proof. \square

In the next lemma we show that the result holds for all bounded continuous functions f on \mathbb{R} .

Lemma 2.25. *Let μ be a probability measure on \mathbb{R} . Let $\phi(y)$ be defined by (2.3). Assume that*

$$\lim_{Y \rightarrow \infty} \frac{1}{Y} \int_0^Y f(\phi(y)) dy = \int_{-\infty}^{\infty} f(x) d\mu(x) \quad (2.36)$$

holds for all bounded Lipschitz functions f on \mathbb{R} and for all $f \in C_0(\mathbb{R}, \mathbb{R})$. Then (2.36) holds for all bounded continuous functions f on \mathbb{R} .

Proof. Let $f(x)$ be bounded and continuous on \mathbb{R} . Note that

$$f(x) = f^+(x) + f^-(x),$$

where

$$f^+(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0, \\ 0 & \text{if otherwise,} \end{cases}$$

and

$$f^-(x) = \begin{cases} f(x) & \text{if } f(x) \leq 0, \\ 0 & \text{if otherwise.} \end{cases}$$

The functions $f^+(x)$ and $f^-(x)$ are continuous on \mathbb{R} and we have

$$\forall x \in \mathbb{R}, \quad 0 \leq f^+(x) \leq |f(x)| \quad \text{and} \quad -|f(x)| \leq f^-(x) \leq 0.$$

If we can prove that (2.36) holds for $f^+(x)$, then applying it to $-f^-(x)$ proves that (2.36) holds for $f^-(x)$. Hence

$$\begin{aligned} & \lim_{Y \rightarrow \infty} \frac{1}{Y} \int_0^Y f(\phi(y)) dy \\ &= \lim_{Y \rightarrow \infty} \frac{1}{Y} \int_0^Y f^+(\phi(y)) dy + \lim_{Y \rightarrow \infty} \frac{1}{Y} \int_0^Y f^-(\phi(y)) dy \\ &= \int_{-\infty}^{\infty} f^+(x) d\mu(x) + \int_{-\infty}^{\infty} f^-(x) d\mu(x) \\ &= \int_{-\infty}^{\infty} f(x) d\mu(x). \end{aligned}$$

Thus we assume $f(x) \geq 0$, for all $x \in \mathbb{R}$.

Let $M > 0$ be a fixed number such that $0 \leq f(x) \leq M$. Consider a sequence of continuous functions $f_n : \mathbb{R} \rightarrow \mathbb{R}$ that satisfies the following properties:

1. $\forall x \in [-n, n], f_n(x) = f(x)$;
2. $\forall x \in (-\infty, -n - 1] \cup [n + 1, \infty), f_n(x) = 0$;
3. $\forall x \in [-n - 1, -n] \cup [n, n + 1], 0 \leq f_n(x) \leq f(x)$.

Then for any $n \in \mathbb{N}$, $f_n(x) \in C_0(\mathbb{R}, \mathbb{R})$. Let $\epsilon > 0$ be given. From $\int_{-\infty}^{\infty} d\mu = 1$, it follows that there is an $N = N(\epsilon)$ such that

$$\int_{-\infty}^{-N} d\mu + \int_N^{\infty} d\mu < \epsilon/M.$$

Thus

$$\begin{aligned} \int_{-\infty}^{\infty} (f(x) - f_N(x))d\mu(x) &< M\epsilon/M + \int_{-N}^N (f(x) - f_N(x))d\mu(x) \\ &= \epsilon + \int_{-N}^N 0 d\mu(x) = \epsilon. \end{aligned}$$

This proves that

$$\int_{-\infty}^{\infty} f_n(x)d\mu(x) \rightarrow \int_{-\infty}^{\infty} f(x)d\mu(x), \quad (2.37)$$

as $n \rightarrow \infty$.

For any $n \in \mathbb{N}$, we have $0 \leq f_n \leq f$. Thus for fixed $Y > 0$

$$\frac{1}{Y} \int_0^Y f_n(\phi(y))dy \leq \frac{1}{Y} \int_0^Y f(\phi(y))dy.$$

Letting $Y \rightarrow \infty$ and applying (2.36), we obtain

$$\int_{-\infty}^{\infty} f_n(x)d\mu(x) = \lim_{Y \rightarrow \infty} \frac{1}{Y} \int_0^Y f_n(\phi(y))dy \leq \liminf_{Y \rightarrow \infty} \frac{1}{Y} \int_0^Y f(\phi(y))dy.$$

Letting $n \rightarrow \infty$ and applying (2.37), we deduce that

$$\int_{-\infty}^{\infty} f(x)d\mu(x) \leq \liminf_{Y \rightarrow \infty} \frac{1}{Y} \int_0^Y f(\phi(y))dy. \quad (2.38)$$

Next we prove that

$$\limsup_{Y \rightarrow \infty} \frac{1}{Y} \int_0^Y f(\phi(y))dy \leq \int_{-\infty}^{\infty} f(x)d\mu(x).$$

Consider

$$\frac{1}{Y} \int_0^Y f(\phi(y))dy = \frac{1}{Y} \int_0^Y f_n(\phi(y))dy + \frac{1}{Y} \int_0^Y [f(\phi(y)) - f_n(\phi(y))]dy. \quad (2.39)$$

For $A \subseteq \mathbb{R}$, let $I_A(x)$ be the indicator function of A defined by

$$I_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Let $h_n(x)$ be a Lipschitz function that satisfies

$$\forall x \in \mathbb{R}, I_{[n, \infty)}(x) \leq h_n(x) \leq I_{[n-1, \infty)}(x).$$

For all $x \in \mathbb{R}$, $0 \leq f(x) - f_n(x) \leq M$. For fixed $Y > 0$ and $n \in \mathbb{N}$ we have

$$\begin{aligned} \frac{1}{Y} \int_0^Y [f(\phi(y)) - f_n(\phi(y))] dy &\leq \frac{M}{Y} \int_0^Y I_{[n, \infty)}(\phi(y)) dy \\ &\leq \frac{M}{Y} \int_0^Y h_n(\phi(y)) dy. \end{aligned}$$

Letting $Y \rightarrow \infty$ and applying (2.36), we obtain

$$\begin{aligned} \limsup_{Y \rightarrow \infty} \frac{1}{Y} \int_0^Y [f(\phi(y)) - f_n(\phi(y))] dy &\leq M \lim_{Y \rightarrow \infty} \frac{1}{Y} \int_0^Y h_n(\phi(y)) dy \\ &= M \int_{-\infty}^{\infty} h_n(x) d\mu(x) \\ &\leq M \int_{-\infty}^{\infty} I_{[n-1, \infty)}(x) d\mu(x) \\ &= M \int_{n-1}^{\infty} d\mu(x). \end{aligned}$$

Now let $\epsilon > 0$ be given. From $\int_{-\infty}^{\infty} d\mu = 1$, it follows that there is an $N = N(\epsilon)$ such that, $n \geq N$ implies

$$\int_{n-1}^{\infty} d\mu(x) < \epsilon/M.$$

Hence for all $n \geq N$

$$\limsup_{Y \rightarrow \infty} \frac{1}{Y} \int_0^Y [f(\phi(y)) - f_n(\phi(y))] dy < M\epsilon/M = \epsilon.$$

Applying this in (2.39) implies that for all $n \geq N$

$$\begin{aligned} \limsup_{Y \rightarrow \infty} \frac{1}{Y} \int_0^Y f(\phi(y)) dy &\leq \lim_{Y \rightarrow \infty} \frac{1}{Y} \int_0^Y f_n(\phi(y)) dy + \epsilon \\ &= \int_{-\infty}^{\infty} f_n(x) d\mu(x) + \epsilon. \end{aligned}$$

Letting $n \rightarrow \infty$ and applying (2.37) we get

$$\limsup_{Y \rightarrow \infty} \frac{1}{Y} \int_0^Y f(\phi(y)) dy \leq \int_{-\infty}^{\infty} f(x) d\mu(x) + \epsilon.$$

Since this is true for any $\epsilon > 0$, we deduce that

$$\limsup_{Y \rightarrow \infty} \frac{1}{Y} \int_0^Y f(\phi(y)) dy \leq \int_{-\infty}^{\infty} f(x) d\mu(x).$$

This together with (2.38) imply

$$\lim_{Y \rightarrow \infty} \frac{1}{Y} \int_0^Y f(\phi(y)) dy = \int_{-\infty}^{\infty} f(x) d\mu(x).$$

The proof is complete. □

Corollary 2.26. *Under the assumptions of Theorem 2.20 or Corollary 2.21 there is a probability measure μ on \mathbb{R}^ℓ such that for all continuous functions f on \mathbb{R}^ℓ*

$$\int_{\mathbb{R}^\ell} f(x_1, \dots, x_\ell) d\mu(x_1, \dots, x_\ell) = \lim_{Y \rightarrow \infty} \frac{1}{Y} \int_0^Y f(\vec{\phi}(y)) dy.$$

In other words, $\vec{\phi}(y)$ has a limiting distribution.

Proof. The proof follows an ℓ -dimensional adaptation of the proofs of Lemmas 2.24 and 2.25. □

Chapter 3

Applications of the General Limiting Distribution Theorem

3.1 Preliminaries

In this chapter, we will present applications for the general limiting distribution theorem proved in Chapter 2. Some of the examples in this chapter are well-known.

We start by reviewing some basic theorems and definitions in analytic number theory.

Lemma 3.1. (*Perron's formula*) *Let*

$$f(s) = \sum_1^{\infty} \frac{a_n}{n^s} \quad (\sigma > 1),$$

where $a_n = O(\psi(n))$, $\psi(n)$ being positive and non-decreasing and

$$\sum_1^{\infty} \frac{|a_n|}{n^{\sigma}} = O\left(\frac{1}{(\sigma - 1)^k}\right)$$

as $\sigma \rightarrow 1^+$. If $w = u + iv$, $c > 0$, $u + c > 1$ and $T > 0$, then for all $x \geq 1$ we have

$$\sum_{n \leq x} \frac{a_n}{n^w} = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} f(w+s) \frac{x^s}{s} ds + O\left(\frac{x^c}{T(u+c-1)^k} + \frac{\psi(2x)x^{1-u} \log(2x)}{T} + \psi(2x)x^{-u}\right).$$

Proof. See [23, pp. 376–379] for a proof. □

We recall some facts regarding the Riemann-zeta function.

Definition 3.2. Let $s = \sigma + it \in \mathbb{C}$ and $\sigma > 1$. Then the *Riemann-zeta function* $\zeta(s)$ is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}. \quad (3.1)$$

Note that $\sigma > 1$ guarantees the absolute convergence of (3.1).

We highlight some properties of $\zeta(s)$. (See [3, Chapter 8].)

Properties of $\zeta(s)$:

- $\zeta(s)$ has a meromorphic continuation to the whole complex plane with only a simple pole of residue 1 at $s = 1$.
- $\zeta(s)$ satisfies the functional equation

$$\pi^{-s/2} \Gamma(s/2) \zeta(s) = \pi^{-(1-s)/2} \Gamma((1-s)/2) \zeta(1-s). \quad (3.2)$$

- The set of zeros of $\zeta(s)$ consists of two disjoint sets, one of which being the set of negative even integers; the other one being a countable infinite set of elements $\rho = \beta + i\gamma$ that satisfy $0 < \beta < 1$. Any element of the former set is called a *trivial zero* and any element of the latter set is called a *non-trivial zero*. For non-trivial zeros we have

$$\begin{aligned} \zeta(\beta + i\gamma) = 0 &\iff \zeta(\beta - i\gamma) = 0, \\ \zeta(\beta + i\gamma) = 0 &\iff \zeta(1 - \beta + i\gamma) = 0. \end{aligned}$$

Moreover, if $N(T)$ denotes the number of the nontrivial zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ with $0 < \gamma \leq T$, then

$$N(T) = (T/2\pi) \log(T/2\pi e) + O(\log T). \quad (3.3)$$

- *Euler's Identity:*

In the region $\sigma > 1$ we have

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}.$$

This identity gives a close relationship between the Riemann zeta function and prime numbers.

- *The Riemann Hypothesis:*

The nontrivial zeros of $\zeta(s)$ lie on the line $\sigma = 1/2$.

Next we recall some facts regarding the Gamma function.

Definition 3.3. Let $s = \sigma + it \in \mathbb{C}$ and $\sigma > 0$. Then the *Gamma function* $\Gamma(s)$ is defined by

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt. \quad (3.4)$$

By the comparison test for improper integrals, the infinite integral in (3.4) converges if $\sigma > 0$.

The following are some properties of $\Gamma(s)$. (See [3, Chapter 10].)

Properties of the $\Gamma(s)$:

- $\Gamma(s)$ has a meromorphic continuation to the whole complex plane with simple poles at $s = 0, -1, -2, \dots$.
- $\Gamma(s)$ satisfies

$$\Gamma(s+1) = s\Gamma(s), \quad \Gamma(s)\Gamma(1-s) = \pi / \sin(\pi s),$$

and

$$\Gamma(s)\Gamma(s + 1/2) = 2^{1-2s}\pi^{1/2}\Gamma(2s).$$

- *Stirling's asymptotic formula:*

$$\log \Gamma(s) = (s - 1/2) \log s - s + 1/2 \log(2\pi) + O(|s|^{-1}),$$

uniformly in $-\pi + \delta < \arg s < \pi - \delta$, for any fixed $\delta > 0$, as $|s| \rightarrow \infty$.

The next lemma is a consequence of (3.2).

Lemma 3.4. *Let $A > 0$ be fixed. Then*

$$|\zeta(s)| \asymp |t|^{1/2-\sigma} |\zeta(1-s)|$$

uniformly in $|\sigma| \leq A$, $|t| \geq 1$.

Proof. See [18, Corollary 10.5]. □

The next lemma gives an estimate for the orders of $\zeta(s)$ and $1/\zeta(s)$ when σ is close to the line $\sigma = 1/2$.

Lemma 3.5. *Assume that the Riemann hypothesis holds. Then for any $\delta > 0$ there exist constants $C, C' > 0$ such that*

$$|\zeta(s)| < \exp\left(\frac{C \log(|t| + 4)}{\log \log(|t| + 4)}\right)$$

and

$$\left|\frac{1}{\zeta(s)}\right| < \exp\left(\frac{C' \log(|t| + 4)}{\log \log(|t| + 4)}\right)$$

uniformly in $\sigma \geq 1/2 + \delta$, $|t| \geq 1$.

Proof. See [18, Theorems 13.18 and 13.23]. □

The following is an immediate consequence of the previous lemma.

Corollary 3.6. *Assume that the Riemann hypothesis holds. Then for any $\epsilon, \delta > 0$*

$$\zeta(s) \ll |t|^\epsilon$$

and

$$\frac{1}{\zeta(s)} \ll |t|^\epsilon$$

uniformly for $\sigma \geq 1/2 + \delta$, $|t| \geq 1$.

In the next lemma we establish an order of magnitude for $1/\zeta(s)$ that is uniform in the region $-1 \leq \sigma \leq 2$.

Lemma 3.7. *Assume the Riemann hypothesis. For any $\epsilon > 0$ there exists a sequence $\mathcal{T} = \{T_n\}_{n=0}^\infty$ which satisfies $n - 1 \leq T_n < n$ and*

$$\frac{1}{\zeta(\sigma + iT_n)} = O(T_n^\epsilon)$$

as $n \rightarrow \infty$, uniformly in the region $-1 \leq \sigma \leq 2$.

Proof. This is a consequence of [18, Theorem 13.22]. □

In the last section of this chapter, we will study the Möbius function in arithmetic progressions. In order to do that, we need to introduce Dirichlet L -functions.

Definition 3.8. Let $q > 1$ be an integer. A *Dirichlet character* $\chi \bmod q$ is a function from \mathbb{N} to \mathbb{C} that satisfies the following properties:

- i) $\chi(n) = \chi(n + q)$, for all $n \in \mathbb{N}$;
- ii) $\chi(n) \neq 0$ if and only if $\gcd(n, q) = 1$;
- iii) $\chi(mn) = \chi(m)\chi(n)$, for all $m, n \in \mathbb{N}$.

The *principal character* $\chi_0 \bmod q$ is the character that takes the value 1 for any n with $\gcd(n, q) = 1$.

Let χ be a character modulo q and suppose that $d > 1$. We say that d is a *quasiperiod* of χ if $\chi(m) = \chi(n)$ whenever $m \equiv n \pmod{d}$ and $\gcd(mn, q) = 1$.

Definition 3.9. A character χ modulo q is *primitive* if q is the smallest quasiperiod of χ .

Next we associate Dirichlet series to Dirichlet characters.

Definition 3.10. Let $q > 1$ be an integer, and χ be a Dirichlet character mod q . Then the *Dirichlet L-function* $L(s, \chi)$ associated to χ is defined for $\sigma > 1$ by

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n)n^{-s}.$$

$L(s, \chi)$ satisfies properties analogous to $\zeta(s)$.

Properties of the Dirichlet L-Functions:

- For any nonprincipal character χ mod q , the function $L(s, \chi)$ has an analytic continuation to the whole complex plane. Also, $L(s, \chi_0)$ has a meromorphic continuation to the whole plane with a simple pole at $s = 1$, with residue $\varphi(q)$.
- $L(s, \chi)$ satisfies the functional equation

$$\pi^{-(1-s)/2} q^{(1-s)/2} \Gamma((1-s)/2) L(1-s, \bar{\chi}) = \frac{q^{1/2}}{\tau(\chi)} \pi^{-s/2} q^{s/2} \Gamma(s/2) L(s, \chi), \quad (3.5)$$

where the function $\bar{\chi}$ is given by $\bar{\chi}(n) = \overline{\chi(n)}$, the complex conjugate of $\chi(n)$, and $\tau(\chi)$ is

$$\tau(\chi) = \sum_{m=1}^q \chi(m) e^{2\pi i m/q}.$$

- The set of zeros of $L(s, \chi)$ consists of two disjoint sets, one of which being the set of negative odd integers; the other one being a countable infinite set of elements $\rho_\chi = \beta_\chi + i\gamma_\chi$ that satisfy $0 < \beta_\chi < 1$. Any element of the former set is called a *trivial zero* and any element of the latter set is called a *non-trivial zero* of $L(s, \chi)$. For non-trivial zeros we have

$$L(\beta_\chi + i\gamma_\chi, \chi) = 0 \iff L(\beta_\chi - i\gamma_\chi, \chi) = 0,$$

$$L(\beta_\chi + i\gamma_\chi, \chi) = 0 \iff L(1 - \beta_\chi + i\gamma_\chi, \bar{\chi}) = 0.$$

Moreover, if $N(T, \chi)$ denotes the number of nontrivial zeros $\rho_\chi = \beta_\chi + i\gamma_\chi$ with

$0 \leq \gamma_\chi \leq T$, then

$$N(T, \chi) = (T/2\pi) \log(qT/2\pi e) + O(\log qT). \quad (3.6)$$

The next lemma is a consequence of (3.5).

Lemma 3.11. *Let χ be a primitive character modulo q . For each $A > 0$ we have*

$$|L(s, \chi)| \asymp (q(|t| + 4))^{1/2-\sigma} |L(1-s, \bar{\chi})|$$

uniformly for $|\sigma| \leq A$.

Proof. See [18, Corollary 10.10]. □

The next lemma gives information on the order of magnitude of $L(s, \chi)$ and $1/L(s, \chi)$ on the half-plane $\sigma > 1/2$. This lemma is analogous to Lemma 3.5 and is stated in [18, p. 445, Exercises 8 and 10].

Lemma 3.12. *Let χ be a primitive character modulo q , $q > 1$, and suppose that $L(s, \chi) \neq 0$ for $\sigma > 1/2$. Then for any $\delta > 0$ there exist constants $C, C' > 0$ such that*

$$|L(s, \chi)| < \exp\left(\frac{C \log(q(|t| + 4))}{\log \log(q(|t| + 4))}\right)$$

and

$$\left|\frac{1}{L(s, \chi)}\right| < \exp\left(\frac{C' \log(q(|t| + 4))}{\log \log(q(|t| + 4))}\right)$$

uniformly in $\sigma \geq 1/2 + \delta$.

As a corollary we have the following.

Corollary 3.13. *Let χ be a character modulo q , $q > 1$, and suppose that $L(s, \chi) \neq 0$ for $\sigma > 1/2$. Then for any $\epsilon, \delta > 0$*

$$L(s, \chi) \ll |t|^\epsilon$$

and

$$\frac{1}{L(s, \chi)} \ll |t|^\epsilon$$

uniformly for $\sigma \geq 1/2 + \delta$.

Proof. If χ is primitive, then the result follows from Lemma 3.11.

Suppose that χ is not primitive. Then there is a primitive character χ^* modulo d , for some $d \mid q$, such that

$$L(s, \chi) = L(s, \chi^*) \prod_{\substack{p \text{ prime} \\ p \mid q}} \left(1 - \frac{\chi^*(p)}{p^s}\right) \quad (3.7)$$

(see [18, p. 282]). Since the above product is bounded for $\sigma \geq 1/2 + \delta$, the corollary holds for $L(s, \chi)$. \square

The next lemma establishes lower bounds for $L(s, \chi)$ on certain horizontal lines.

Lemma 3.14. *Let χ be a primitive character modulo q and suppose that $L(s, \chi) \neq 0$ for $\sigma > 1/2$. Then there is an absolute constant $C > 0$ such that each interval $[T, T+1]$ contains a value of t such that*

$$|L(s, \chi)| \geq \exp\left(-\frac{C \log q(T+4)}{\log \log q(T+4)}\right) \quad (3.8)$$

uniformly for $-1 \leq \sigma \leq 2$.

Proof. We will follow the proof of [27, Theorem 14.16] and employ [18, pp. 444–445, Exercises 6 and 10]. By Lemma 3.11, for $-1 \leq \sigma \leq 1/2$, we have

$$|L(s, \chi)| \gg q^{1/2-\sigma} |L(1-s, \bar{\chi})|.$$

Thus we only have to prove the result for $1/2 \leq \sigma \leq 2$. Put

$$\delta = 1/\log \log q(T+4)$$

and $s_1 = 1/2 + \delta + it$. It is known that (3.8) holds for $1/2 + \delta \leq \sigma \leq 2$ (see [18, pp. 444–445, Exercise 6]). Let $s = \sigma + it$. For $1/2 \leq \sigma \leq 1/2 + \delta$ we have

$$\frac{L'}{L}(s, \chi) = \sum_{|\gamma_\chi - t| \leq \delta} \frac{1}{s - \rho_\chi} + O(\log qT), \quad (3.9)$$

where $\rho_\chi = 1/2 + i\gamma_\chi$ denotes a zero of $L(s, \chi)$ (see [18, p. 445, Exercise 10]). Integrating (3.9) with respect to σ from σ to $1/2 + \delta$ gives

$$\log \frac{L(s, \chi)}{L(s_1, \chi)} = \sum_{|\gamma_\chi - t| \leq \delta} \log \frac{s - \rho_\chi}{s_1 - \rho_\chi} + O\left(\frac{\log qT}{\log \log qT}\right).$$

Also, we have

$$|L(s, \chi)| \leq \log \frac{1}{1 - \sigma} + O\left(\frac{(\log q(|t| + 4))^{2-2\sigma}}{(1 - \sigma) \log \log(q(|t| + 4))}\right)$$

uniformly for $1/2 + 1/\log \log(q(|t| + 4)) \leq \sigma \leq 1 - 1/\log \log(q(|t| + 4))$ (see [18, p. 445, Exercise 6(c)]). This implies that

$$\log L(s_1, \chi) = O\left(\frac{\log qT}{\log \log qT}\right).$$

Since $|s - \rho_\chi| \geq |t - \gamma_\chi|$ and $|s_1 - \rho_\chi| \leq 2\delta$, by taking real parts, we deduce

$$\begin{aligned} \log |L(s, \chi)| &= \sum_{|\gamma_\chi - t| \leq \delta} \log \left| \frac{s - \rho_\chi}{s_1 - \rho_\chi} \right| + O\left(\frac{\log qT}{\log \log qT}\right) \\ &\geq \sum_{|\gamma_\chi - t| \leq \delta} \log \frac{|t - \gamma_\chi|}{2\delta} + O\left(\frac{\log qT}{\log \log qT}\right), \end{aligned}$$

Note that

$$\begin{aligned} \int_T^{T+1} \sum_{|\gamma_\chi - t| \leq \delta} \log \frac{|t - \gamma_\chi|}{2\delta} dt &= \int_T^{T+1} \sum_{t-\delta \leq \gamma_\chi \leq t+\delta} \log \frac{|t - \gamma_\chi|}{2\delta} dt \\ &= \sum_{T-\delta \leq \gamma_\chi \leq T+1+\delta} \int_{\max\{\gamma_\chi - \delta, T\}}^{\min\{\gamma_\chi + \delta, T+1\}} \log \frac{|t - \gamma_\chi|}{2\delta} dt \\ &\geq \sum_{T-\delta \leq \gamma_\chi \leq T+1+\delta} \int_{\gamma_\chi - \delta}^{\gamma_\chi + \delta} \log \frac{|t - \gamma_\chi|}{2\delta} dt \\ &= \sum_{T-\delta \leq \gamma_\chi \leq T+1+\delta} (-2\delta - 2\delta \log 2) \\ &\gg -\delta \log qT, \end{aligned}$$

where the last step is derived from (3.6). Hence there is a $T \leq t \leq T + 1$ such that

$$\sum_{|\gamma_\chi - t| \leq \delta} \log \frac{|t - \gamma_\chi|}{2\delta} \gg -\delta \log qT.$$

It follows that for this t

$$\log |L(s, \chi)| \geq -\frac{C \log q(T + 4)}{\log \log q(T + 4)}$$

for any $1/2 \leq \sigma \leq 1/2 + \delta$ which completes the proof. \square

The following is an immediate consequence of Lemma 3.14.

Corollary 3.15. *Let χ be a character modulo q , $q > 1$, and suppose that $L(s, \chi) \neq 0$ for $\sigma > 1/2$. Then for any $\epsilon > 0$ there exists a sequence $\mathcal{T}_\chi = \{T_{n, \chi}\}_{n=0}^\infty$ which satisfies $n - 1 \leq T_{n, \chi} < n$ and*

$$\frac{1}{L(\sigma + iT_{n, \chi}, \chi)} = O(T_{n, \chi}^\epsilon)$$

as $n \rightarrow \infty$, uniformly in the region $-1 \leq \sigma \leq 2$.

Proof. If χ is primitive, then the result follows from Lemma 3.14. If χ is not primitive, then we apply the corollary to $L(s, \chi^*)$ in (3.7). \square

3.2 The Error Term of the Prime Number Theorem

The first application for our general limiting distribution theorem given in Chapter 1 is for the error term of the prime number theorem. Recall that for $x > 0$ the function $\pi(x)$ is defined by

$$\pi(x) := \text{card}(\{p \leq x \mid p \text{ prime}\}).$$

For a number $x > 1$ the *logarithmic integral* $\text{Li}(x)$ is defined by

$$\text{Li}(x) := \int_2^x \frac{dt}{\log t}.$$

The *prime number theorem* (PNT) can be written in the form

$$\pi(x) = \text{Li}(x) + \mathcal{E}(x), \quad (3.10)$$

where $\mathcal{E}(x) = o(\text{Li}(x))$ as $x \rightarrow \infty$. The term $\mathcal{E}(x)$ in relation (3.10) is called the error term of the prime number theorem.

For $T \geq 2$ and $x \geq 2$, the error term can be written in the form

$$\mathcal{E}(x) = -\frac{1}{\log x} \sum_{|\gamma| < T} \frac{x^\rho}{\rho} - \frac{\sqrt{x}}{\log x} + O\left(\frac{x \log^2(xT)}{T \log x} + \frac{\sqrt{x}}{\log^2 x}\right), \quad (3.11)$$

where $\rho = \beta + i\gamma$ denote a nontrivial zero of $\zeta(s)$ (see [24, (2.12)]). Assume that the Riemann hypothesis is true. Then (3.11) implies

$$(\pi(x) - \text{Li}(x)) \frac{\log x}{\sqrt{x}} = -1 - \sum_{|\gamma| \leq T} \frac{x^{i\gamma}}{1/2 + i\gamma} + O\left(\frac{\sqrt{x} \log^2(xT)}{T} + \frac{1}{\log x}\right). \quad (3.12)$$

Let $\phi_1(y) := (\pi(e^y) - \text{Li}(e^y))ye^{-y/2}$. Suppose that we index the zeros $1/2 + i\gamma$ ($\gamma > 0$) of $\zeta(s)$, as $\{1/2 + i\gamma_n\}_{n=1}^\infty$ so that $\gamma_1 \leq \gamma_2 \leq \dots$. Then from (3.12) we have

$$\begin{aligned} \phi_1(y) &= -1 - \sum_{\gamma_n \leq T} \frac{e^{i\gamma_n y}}{1/2 + i\gamma_n} - \sum_{\gamma_n \leq T} \frac{e^{-i\gamma_n y}}{1/2 - i\gamma_n} + E_1(e^y, e^Y) \\ &= -1 + \text{Re} \left(\sum_{\gamma_n \leq T} \frac{-2}{1/2 + i\gamma_n} e^{i\gamma_n y} \right) + E_1(e^y, e^Y), \end{aligned}$$

where

$$E_1(e^y, e^Y) = O\left(\frac{yY e^{y/2}}{e^Y} + \frac{1}{y}\right).$$

We have

$$\begin{aligned} \int_1^Y |E_1(e^y, e^Y)|^2 dy &\ll \int_1^Y \left(\frac{yY e^{y/2}}{e^Y} + \frac{1}{y}\right)^2 dy \\ &\ll \int_1^Y \left(\frac{y^2 Y^2 e^y}{e^{2Y}} + \frac{1}{y^2}\right) dy \ll 1. \end{aligned}$$

If we take in Corollary 2.17, $\lambda_n = \gamma_n$ and $r_n = -2/(1/2 + i\gamma_n)$, then we will have

$$\sum_{\lambda_n \leq T} \lambda_n^2 |r_n|^2 = \sum_{\gamma_n \leq T} \frac{4\gamma_n^2}{1/4 + \gamma_n^2} \ll \sum_{\gamma_n \leq T} 1 = N(T) \ll T \log T.$$

Hence under the assumption of the Riemann hypothesis, by Corollary 2.17, the function $\phi_1(y) = (\pi(e^y) - \text{Li}(e^y))ye^{-y/2}$ has a limiting distribution μ on \mathbb{R} .

This result was originally proved by Wintner [30], in 1935.

If further we assume the linear independence hypothesis (i.e., $\{\gamma_n\}_{n=1}^{\infty}$ is linearly independent over \mathbb{Q}), then by Theorem 2.18 the Fourier transform of μ is of the form

$$\hat{\mu}(\xi) = e^{i\xi} \prod_{n=1}^{\infty} J_0 \left(\frac{2\xi}{\sqrt{1/4 + \gamma_n^2}} \right),$$

where $J_0(z)$ is the Bessel function of order 0.

3.3 Weighted Sums of the Möbius Function

The second application of our general limiting distribution theorem is for weighted sums of the Möbius function. Recall that the *Möbius function* is defined for $n \in \mathbb{N}$ by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n = p_1 \dots p_k, \text{ where } p_i\text{'s are different,} \\ 0 & \text{otherwise.} \end{cases}$$

Ng, in [20], proves that under a certain condition for average of $1/|\zeta'(s)|$, the summatory function

$$M(x) = \sum_{n \leq x} \mu(n),$$

has a limiting distribution ν . Moreover, under the linear independence assumption, he gives explicit formulas for the Fourier transform of ν . In this section, we will consider the weighted sums of the Möbius function and investigate conditions under which these functions have limiting distributions.

Definition 3.16. For $\alpha \in [0, 1]$ and $x > 0$, let

$$M_\alpha(x) = \sum_{n \leq x} \frac{\mu(n)}{n^\alpha}.$$

Let $\theta < 2$ be a fixed real number. For now, we assume the Riemann hypothesis and

$$\sum_{0 < \gamma \leq T} |\zeta'(\rho)|^{-2} \ll T^\theta.$$

In the next lemma we establish an explicit formula for $M_\alpha(x)$.

Lemma 3.17. *Assume the Riemann hypothesis. Let $b < 1/2$ be a real number, which is less than α if $\alpha < 1/2$ and let ϵ be a number in $(0, 1/2 - b)$. Then for any $x > 1$ and $T_n \in \mathcal{T}$ (the sequence in Lemma 3.7), we have*

$$M_\alpha(x) = 1/\zeta(\alpha) + \sum_{|\gamma| < T_n} \frac{x^{\rho-\alpha}}{(\rho-\alpha)\zeta'(\rho)} + O\left(\frac{x^{1-\alpha} \log x}{T_n} + \frac{x^{1-\alpha}}{T_n^{1-\epsilon} \log x} + x^{b-\alpha}\right).$$

Proof. In Lemma 3.1 we take $w = \alpha$, $c = 1 - \alpha + 1/\log x$, $a_n = \mu(n)$, $\psi(n) = 1$, and $F(s) = 1/\zeta(s)$ to obtain

$$M_\alpha(x) = \sum_{n \leq x} \frac{\mu(n)}{n^\alpha} = \frac{1}{2\pi i} \int_{c-iT_n}^{c+iT_n} \frac{1}{\zeta(s+\alpha)} \frac{x^s}{s} ds + O\left(\frac{x^{1-\alpha} \log x}{T_n} + x^{-\alpha}\right).$$

On changing s by $s - \alpha$ in the above integral we have

$$\frac{1}{2\pi i} \int_{c+\alpha-iT_n}^{c+\alpha+iT_n} \frac{1}{\zeta(s)} \frac{x^{s-\alpha}}{s-\alpha} ds.$$

Let $b < 1/2$ be a real number, which is less than α if $\alpha < 1/2$. Consider a rectangle with vertices $c+\alpha+iT_n$, $b+iT_n$, $b-iT_n$, $c+\alpha-iT_n$. Then by applying Cauchy's residue

theorem we obtain

$$\begin{aligned}
M_\alpha(x) &= 1/\zeta(\alpha) + \sum_{|\gamma| < T_n} \frac{x^{\rho-\alpha}}{(\rho-\alpha)\zeta'(\rho)} \\
&\quad - \frac{1}{2\pi i} \left(\int_{c+\alpha-iT_n}^{b-iT_n} + \int_{b-iT_n}^{b+iT_n} + \int_{b+iT_n}^{c+\alpha+iT_n} \right) \frac{1}{\zeta(s)} \frac{x^{s-\alpha}}{s-\alpha} ds \\
&\quad + O\left(\frac{x^{1-\alpha} \log x}{T_n} + x^{-\alpha}\right). \quad (3.13)
\end{aligned}$$

Note that the term $1/\zeta(\alpha)$ appears since the integrand has a simple pole inside the rectangle at $s = \alpha$ with residue $1/\zeta(\alpha)$. Also, the sum over the zeros of $\zeta(s)$ shows up because of the residues of the integrand at those zeros inside the rectangle. Corollary 3.6 together with Lemma 3.4 imply that for any $\epsilon > 0$ we have

$$\frac{1}{\zeta(b+it)} \ll |t|^{-1/2+b+\epsilon}$$

for all $|t| \geq 1$. Hence if $\epsilon < 1/2 - b$ the contribution of the middle integral in (3.13) is

$$\begin{aligned}
\frac{1}{2\pi i} \int_{b-iT_n}^{b+iT_n} \frac{1}{\zeta(s)} \frac{x^{s-\alpha}}{s-\alpha} ds &\ll x^{b-\alpha} \left(\int_{-1}^1 \left| \frac{1}{\zeta(b+it)} \right| \frac{dt}{\sqrt{(b-\alpha)^2 + t^2}} + \int_1^{T_n} t^{-1/2+b+\epsilon} \frac{dt}{t} \right) \\
&\ll x^{b-\alpha} + x^{b-\alpha} (1 + T_n^{-1/2+b+\epsilon}) \\
&\ll x^{b-\alpha}. \quad (3.14)
\end{aligned}$$

Moreover, by Lemma 3.7, we have for any $\epsilon > 0$

$$\frac{1}{\zeta(\sigma+iT_n)} \ll T_n^\epsilon$$

for all $b \leq \sigma \leq c + \alpha$. Hence the contribution of the last integral in (3.13) is

$$\frac{1}{2\pi i} \int_{b+iT_n}^{c+\alpha+iT_n} \frac{1}{\zeta(s)} \frac{x^{s-\alpha}}{s-\alpha} ds \ll \frac{x^{-\alpha}}{T_n^{1-\epsilon}} \int_b^{c+\alpha} x^\sigma d\sigma \ll \frac{x^c}{T_n^{1-\epsilon} \log x} \ll \frac{x^{1-\alpha}}{T_n^{1-\epsilon} \log x}. \quad (3.15)$$

The first integral can be handled in a way similar to (3.15). Hence from (3.13), (3.14), and (3.15) the lemma follows. \square

The next lemma establishes upper bound estimates for certain sums involving $|\zeta'(s)|^{-1}$.

Lemma 3.18. *Let $\theta < 2$ be fixed. Assume for $T > 0$ that*

$$\sum_{0 < \gamma \leq T} |\zeta'(\rho)|^{-2} \ll T^\theta. \quad (3.16)$$

Then

(i)

$$\sum_{\gamma \geq T} \frac{1}{|(\rho - \alpha)\zeta'(\rho)|^2} \ll T^{\theta-2};$$

(ii)

$$\sum_{0 < \gamma \leq T} \frac{1}{|(\rho - \alpha)\zeta'(\rho)|} \ll T^{(\theta-1)/2} (\log T)^{3/2}.$$

Proof. (i) In Lemma 2.8, take $c_n = |\zeta'(\rho_n)|^{-2}$, $\phi(t) = 1/t^2$, and $\alpha_n = |\rho_n - \alpha|$, where $\rho_n = 1/2 + i\gamma_n$ ($\gamma_1 \leq \gamma_2 \leq \dots$) is the sequence of nontrivial zeros of $\zeta(s)$ in the upper half plane. Note that if $|\rho_n - \alpha| \leq T$ then $\gamma_n \leq |\rho_n| \leq T + \alpha$, so

$$\sum_{|\rho_n - \alpha| \leq T} |\zeta'(\rho)|^{-2} \leq \sum_{0 < \gamma_n \leq T + \alpha} |\zeta'(\rho_n)|^{-2} \ll (T + \alpha)^\theta \ll T^\theta.$$

Thus we have

$$\begin{aligned} \sum_{|\rho_n - \alpha| \geq T} \frac{1}{|(\rho_n - \alpha)\zeta'(\rho_n)|^2} &= 2 \int_T^\infty \frac{1}{t^3} \left(\sum_{|\rho_n - \alpha| \leq t} |\zeta'(\rho)|^{-2} \right) dt + \frac{1}{T^2} \sum_{|\rho_n - \alpha| \leq T} |\zeta'(\rho)|^{-2} \\ &\ll \int_T^\infty \frac{t^\theta}{t^3} dt + \frac{T^\theta}{T^2} \ll T^{\theta-2}, \end{aligned}$$

where we use the condition $\theta < 2$ to bound the last improper integral. Now observe that if $\gamma_n \geq T$ then $|\rho_n - \alpha| \geq |\rho_n| - \alpha \geq \gamma_n - \alpha \geq T - \alpha$. Therefore,

$$\sum_{\gamma_n \geq T} \frac{1}{|(\rho_n - \alpha)\zeta'(\rho_n)|^2} \leq \sum_{|\rho_n - \alpha| \geq T - \alpha} \frac{1}{|(\rho_n - \alpha)\zeta'(\rho_n)|^2} \ll (T - \alpha)^{\theta-2} \ll T^{\theta-2}.$$

This proves (i).

(ii) By the Cauchy-Schwarz inequality we have

$$\sum_{0 < \gamma \leq T} \frac{1}{|\zeta'(\rho)|} \leq \left(\sum_{0 < \gamma \leq T} \frac{1}{|\zeta'(\rho)|^2} \right)^{1/2} \left(\sum_{0 < \gamma \leq T} 1 \right)^{1/2}.$$

From (3.3) we have

$$\sum_{0 < \gamma \leq T} 1 \ll T \log T.$$

Hence by the assumption (3.16) we obtain

$$\sum_{0 < \gamma \leq T} \frac{1}{|\zeta'(\rho)|} \ll T^{\theta/2} (T \log T)^{1/2} = T^{(\theta+1)/2} (\log T)^{1/2}. \quad (3.17)$$

In Lemma 2.8, take $c_n = |\zeta'(\rho_n)|^{-1}$, $\phi(t) = 1/t$, and $\alpha_n = |\rho_n - \alpha|$. From (3.17) we have

$$\begin{aligned} \sum_{|\rho_n - \alpha| \leq T} |\zeta'(\rho)|^{-1} &\leq \sum_{0 < \gamma_n \leq T + \alpha} |\zeta'(\rho_n)|^{-1} \ll (T + \alpha)^{(\theta+1)/2} (\log(T + \alpha))^{1/2} \\ &\ll T^{(\theta+1)/2} (\log T)^{1/2}. \end{aligned}$$

Thus

$$\begin{aligned} \sum_{|\rho_n - \alpha| \leq T} \frac{1}{|(\rho_n - \alpha)\zeta'(\rho_n)|} &= \int_{|\rho_1 - \alpha|}^T \frac{1}{t^2} \left(\sum_{|\rho_n - \alpha| \leq t} |\zeta'(\rho)|^{-1} \right) dt + \frac{1}{T} \sum_{|\rho_n - \alpha| \leq T} |\zeta'(\rho)|^{-1} \\ &\ll \int_{|\rho_1 - \alpha|}^T \frac{t^{(\theta+1)/2} (\log t)^{1/2}}{t^2} dt + \frac{T^{(\theta+1)/2} (\log T)^{1/2}}{T} \\ &\ll T^{(\theta-1)/2} (\log T)^{3/2}. \end{aligned}$$

Thus

$$\begin{aligned} \sum_{\gamma_n \leq T} \frac{1}{|(\rho_n - \alpha)\zeta'(\rho_n)|} &\leq \sum_{|\rho_n - \alpha| \leq 4T} \frac{1}{|(\rho_n - \alpha)\zeta'(\rho_n)|} \ll (4T)^{(\theta-1)/2} (\log 4T)^{3/2} \\ &\ll T^{(\theta-1)/2} (\log T)^{3/2}. \end{aligned}$$

□

In the next lemma we generalize the result of Lemma 3.17 to all $T > 0$.

Lemma 3.19. *Let $\theta < 2$. Assume the Riemann hypothesis, all the zeros of $\zeta(s)$ are simple, and $\sum_{0 < \gamma \leq T} |\zeta'(\rho)|^{-2} \ll T^\theta$. Let $0 < b < 1/2$ and $0 < \epsilon < 1/2 - b$. Then for $x > 1$ and $T > 0$,*

$$M_\alpha(x) = 1/\zeta(\alpha) + \sum_{|\gamma| < T} \frac{x^{\rho-\alpha}}{(\rho-\alpha)\zeta'(\rho)} + O\left(\frac{x^{1-\alpha} \log x}{T} + \frac{x^{1-\alpha}}{T^{1-\epsilon} \log x} + x^{1/2-\alpha} (T^{\theta-2} \log T)^{1/2} + x^{b-\alpha}\right).$$

Proof. Let $T > 0$, and $n \in \mathbb{N}$ be such that $n-1 \leq T < n$. Then we have either $n-1 \leq T_n \leq T < n$ or $n-1 \leq T \leq T_n < n$, where $T_n \in \mathcal{T}$. Without loss of generality, suppose that the first case occurs. Then by Lemma 3.17

$$\begin{aligned} M_\alpha(x) &= 1/\zeta(\alpha) + \left(\sum_{|\gamma| < T} - \sum_{T_n \leq |\gamma| < T} \right) \frac{x^{\rho-\alpha}}{(\rho-\alpha)\zeta'(\rho)} \\ &\quad + O\left(\frac{x^{1-\alpha}}{T_n^{1-\epsilon} \log x} + \frac{x^{1-\alpha} \log x}{T_n} + x^{-\alpha}\right) \\ &= 1/\zeta(\alpha) + \left(\sum_{|\gamma| < T} - \sum_{T_n \leq |\gamma| < T} \right) \frac{x^{\rho-\alpha}}{(\rho-\alpha)\zeta'(\rho)} \\ &\quad + O\left(\frac{x^{1-\alpha}}{T^{1-\epsilon} \log x} + \frac{x^{1-\alpha} \log x}{T} + x^{-\alpha}\right). \end{aligned}$$

By the Cauchy-Schwarz inequality we have

$$\left| \sum_{T_n \leq \gamma < T} \frac{x^{\rho-\alpha}}{(\rho-\alpha)\zeta'(\rho)} \right| \leq x^{1/2-\alpha} \left(\sum_{T_n \leq \gamma < T} \frac{1}{|(\rho-\alpha)\zeta'(\rho)|^2} \right)^{1/2} \left(\sum_{T_n \leq \gamma < T} 1 \right)^{1/2}.$$

By Lemma 3.18(i), the first sum on the right is $\ll T^{\theta-2}$. Now (3.3) implies that

$$\begin{aligned} \sum_{T_n \leq \gamma < T} 1 &\leq \sum_{T-1 \leq \gamma < T} 1 = N(T) - N(T-1) \\ &= \frac{T}{2\pi} \log \frac{T}{T-1} + \frac{1}{2\pi} \log \frac{T-1}{2\pi e} + O(\log T) \\ &\ll \log T. \end{aligned} \tag{3.18}$$

From these estimates we conclude that

$$\left| \sum_{T_n \leq \gamma < T} \frac{x^{\rho-\alpha}}{(\rho-\alpha)\zeta'(\rho)} \right| \ll x^{1/2-\alpha} (T^{\theta-2})^{1/2} (\log T)^{1/2} = x^{1/2-\alpha} (T^{\theta-2} \log T)^{1/2}.$$

This proves the result when $n-1 \leq T_n \leq T < n$.

The proof is similar if $n-1 \leq T \leq T_n < n$. \square

As a direct consequence of the previous lemma we have the following result that generalizes a Theorem of Ng ([20, Theorem 1(i).])

Proposition 3.20. *Let $0 < \theta < 2$. Assume the Riemann hypothesis and $\sum_{0 < \gamma \leq T} |\zeta'(\rho)|^{-2} \ll T^\theta$. Then we have*

$$M_\alpha(x) - 1/\zeta(\alpha) \ll x^{\theta/2-\alpha} (\log x)^{3/2}.$$

Proof. By Lemma 3.19

$$\begin{aligned} M_\alpha(x) - 1/\zeta(\alpha) &\ll x^{1/2-\alpha} \sum_{|\gamma| \leq T} \frac{1}{|(\rho-\alpha)\zeta'(\rho)|} + \frac{x^{1-\alpha}}{T^{1-\epsilon} \log x} + \frac{x^{1-\alpha} \log x}{T} \\ &\quad + x^{1/2-\alpha} (T^{\theta-2} \log T)^{1/2} + x^{b-\alpha} \end{aligned}$$

where $0 < b < 1/2$ and $0 < \epsilon < 1/2 - b$. The sum term is by Lemma 3.18(ii)

$$O(x^{1/2-\alpha} T^{(\theta-1)/2} (\log T)^{3/2}).$$

Therefore choosing $T = x$ and $\epsilon < b < \theta/2$ implies

$$\begin{aligned} M_\alpha(x) - 1/\zeta(\alpha) &\ll x^{\theta/2-\alpha}(\log x)^{3/2} + \frac{x^{\epsilon-\alpha}}{\log x} + x^{-\alpha} \log x + x^{b-\alpha} \\ &\ll x^{\theta/2-\alpha}(\log x)^{3/2}. \end{aligned}$$

□

We will now apply the general limiting distribution theorem of Chapter 2 to the following function. Let

$$\phi_\alpha(y) := \begin{cases} e^{y(-1/2+\alpha)} M_\alpha(e^y) & \text{if } 0 \leq \alpha \leq 1/2 \text{ or } \alpha = 1, \\ e^{y(-1/2+\alpha)} (M_\alpha(e^y) - 1/\zeta(\alpha)) & \text{if } 1/2 < \alpha < 1. \end{cases}$$

Let $\theta < 5/4$. Assume the Riemann hypothesis and $\sum_{0 < \gamma \leq T} |\zeta'(\rho)|^{-2} \ll T^\theta$. For $1/2 < \alpha < 1$, Lemma 3.19 asserts that for any $X > 0$ and for any $y > 0$ we have

$$\begin{aligned} \phi_\alpha(y) &= e^{y(-1/2+\alpha)} \sum_{|\gamma| \leq X} \frac{e^{y(\rho-\alpha)}}{(\rho-\alpha)\zeta'(\rho)} + E_\alpha(y, X) \\ &= \sum_{|\gamma| \leq X} \frac{e^{iy\gamma}}{(\rho-\alpha)\zeta'(\rho)} + E_\alpha(y, X) \\ &= \operatorname{Re} \left(\sum_{0 < \gamma \leq X} \frac{2e^{iy\gamma}}{(\rho-\alpha)\zeta'(\rho)} \right) + E_\alpha(y, X), \end{aligned} \tag{3.19}$$

where for any $0 < b < 1/2$ and $0 < \epsilon < 1/2 - b$

$$E_\alpha(y, X) = O \left(\frac{ye^{y/2}}{X} + \frac{e^{y/2}}{yX^{1-\epsilon}} + (X^{\theta-2} \log X)^{1/2} + \frac{1}{e^{y(1/2-b)}} \right). \tag{3.20}$$

Observe that for $0 \leq \alpha < 1/2$ or $\alpha = 1$, (3.19) holds and in $E_\alpha(y, X)$, $e^{-y(1/2-b)}$ is changed to $e^{-y(1/2-\alpha)}$. For $\alpha = 1/2$ we set

$$\phi_{1/2}(y) = 1/\zeta(1/2) + \operatorname{Re} \left(\sum_{0 < \gamma \leq X} \frac{2e^{iy\gamma}}{(\rho-1/2)\zeta'(\rho)} \right) + E_{1/2}(y, X),$$

where $E_{1/2}(y, X)$ satisfies (3.20).

In order to apply Corollary 2.17 we need to check its conditions first. Take $\lambda_n = \gamma_n$ and $r_n = 2/|(\rho_n - \alpha)\zeta'(\rho_n)|$. Then we can rewrite (3.19) as

$$\phi_\alpha(y) = \operatorname{Re}\left(\sum_{\gamma_n \leq X} \frac{2e^{iy\gamma_n}}{(\rho_n - \alpha)\zeta'(\rho_n)}\right) + E_\alpha(y, X).$$

Also we get

$$\phi_{1/2}(y) = 1/\zeta(1/2) + \operatorname{Re}\left(\sum_{\gamma_n \leq X} \frac{2e^{iy\gamma_n}}{(\rho_n - \alpha)\zeta'(\rho_n)}\right) + E_\alpha(y, X).$$

The relation (3.18) asserts that

$$\sum_{T \leq \gamma_n < T+1} 1 \ll \log T.$$

Also the assumption $\sum_{\gamma \leq T} |\zeta'(\rho)|^{-2} \ll T$ implies that

$$\sum_{\gamma_n \leq T} \lambda_n^2 |r_n|^2 = \sum_{\gamma_n \leq T} \frac{4\gamma_n^2}{|(\rho_n - \alpha)\zeta'(\rho_n)|^2} \ll \sum_{\gamma_n \leq T} |\zeta'(\rho_n)|^{-2} \ll T.$$

Moreover, we have

$$\begin{aligned} \int_1^Y |E_\alpha(y, e^Y)|^2 dy &\ll \int_1^Y \left(\frac{ye^{y/2}}{e^Y} + \frac{e^{y/2}}{ye^{(1-\epsilon)Y}} + \left(\frac{Y}{e^{Y(2-\theta)}} \right)^{1/2} + \frac{1}{e^{y(1/2-b)}} \right)^2 dy \\ &\ll \int_1^Y \left(\frac{y^2 e^y}{e^{2Y}} + \frac{e^y}{y^2 e^{2(1-\epsilon)Y}} + \frac{Y}{e^{Y(2-\theta)}} + \frac{1}{e^{y(1-2b)}} \right) dy \ll 1. \end{aligned}$$

Hence by applying Corollary 2.17 it turns out that $\phi_\alpha(y)$ has a limiting distribution ν_α . The further assumption of the linear independence for the zeros of $\zeta(s)$ implies that the Fourier transform $\hat{\nu}_\alpha$ of ν_α is given by

$$\hat{\nu}_\alpha(\xi) = \prod_{n=1}^{\infty} J_0\left(\frac{2\xi}{|(\rho_n - \alpha)\zeta'(\rho_n)|}\right),$$

(see Theorem 2.18 in Chapter 2).

3.4 Weighted Sums of the Liouville Function

This section is concerned with the Liouville function $\lambda(n)$ defined for $n \in \mathbb{N}$ by $\lambda(n) = (-1)^{\Omega(n)}$, where $\Omega(n)$ is the total number of prime divisors of n . For its summatory function

$$L_0(x) = \sum_{n \leq x} \lambda(n)$$

Fawaz [5, Theorem 2] proved the following explicit formula

$$L_0(x) = \frac{x^{1/2}}{\zeta(1/2)} + \sum_{\rho} \frac{\zeta(2\rho)}{\rho \zeta'(\rho)} x^{\rho} + I(x), \quad (x > 1, x \notin \mathbb{N}),$$

where $I(x)$ is a certain function of x satisfying $I(x) = 1 + O(x^{-1/2})$. In this section, we will investigate the weighted sums of the Liouville function.

Definition 3.21. For $\alpha \in [0, 1]$ and $x > 0$ the weighted sum of the Liouville function of weight α is defined by

$$L_{\alpha}(x) = \sum_{n \leq x} \frac{\lambda(n)}{n^{\alpha}}.$$

Similar to the case of $M_{\alpha}(x)$, we first try to establish an explicit formula for $L_{\alpha}(x)$. We then apply our main theorems from Chapter 2.

Lemma 3.22. *Assume the Riemann hypothesis and that all zeros of $\zeta(s)$ are simple. For $x > 1$, $T_n \in \mathcal{T}$ (the sequence in Lemma 3.7), and for any $0 < \epsilon < 2/9$ we have*

$$L_{\alpha}(x) = R_{\alpha}(x) + \sum_{|\gamma| < T_n} \frac{x^{\rho - \alpha}}{\rho - \alpha} \frac{\zeta(2\rho)}{\zeta'(\rho)} + O\left(\frac{x^{1-\alpha} \log x}{T_n} + \frac{x^{1-\alpha}}{T_n^{1-\epsilon} \log x} + x^{1/4-\epsilon/8-\alpha}\right),$$

where

$$R_{\alpha}(x) = \begin{cases} x^{1/2-\alpha}/((1-2\alpha)\zeta(1/2)) + \zeta(2\alpha)/\zeta(\alpha) & \text{if } \alpha \neq 1/2, \\ \gamma/\zeta(1/2) - \zeta'(1/2)/2\zeta(1/2)^2 + \log x/2\zeta(1/2) & \text{if } \alpha = 1/2, \end{cases} \quad (3.21)$$

where γ is Euler's constant.

Proof. Before proceeding, note that

$$\zeta(s) = \frac{s}{s-1} - s \int_1^\infty \frac{\{u\}}{u^{s+1}} du$$

for $\sigma > 0$, where $\{x\}$ is the fractional part of $x \in \mathbb{R}$. From this we have $\zeta(s) \neq 0$ for any real number $0 < s < 1$. This ensures that (3.21) makes sense.

We take in Lemma 3.1, $w = \alpha$, $c = 1 - \alpha + 1/\log x$, $a_n = \lambda(n)$, $\psi(n) = 1$, and

$$f(s) = \zeta(2s)/\zeta(s),$$

to obtain

$$L_\alpha(x) = \frac{1}{2\pi i} \int_{c-iT_n}^{c+iT_n} \frac{\zeta(2(s+\alpha))}{\zeta(s+\alpha)} \frac{x^s}{s} ds + O\left(\frac{x^{1-\alpha} \log x}{T_n} + x^{-\alpha}\right). \quad (3.22)$$

In the integral on the right of (3.22) we change s by $s - \alpha$ to obtain

$$\frac{1}{2\pi i} \int_{c+\alpha-iT_n}^{c+\alpha+iT_n} \frac{\zeta(2s)}{\zeta(s)} \frac{x^{s-\alpha}}{s-\alpha} ds. \quad (3.23)$$

Next, we will shift the path of integration to the left of $\sigma = 1/2$. Let $0 < b < 1/4$ be a real number which is less than α if $0 < \alpha < 1/4$. Consider the rectangle with vertices $c + \alpha + iT_n, b + iT_n, b - iT_n, c + \alpha - iT_n$. By Cauchy's residue theorem, the integral (3.23) equals

$$R_\alpha(x) + \sum_{|\gamma| < T_n} \frac{x^{\rho-\alpha}}{(\rho-\alpha)} \frac{\zeta(2\rho)}{\zeta'(\rho)} - \frac{1}{2\pi i} \left(\int_{c+\alpha-iT_n}^{b-iT_n} + \int_{b-iT_n}^{b+iT_n} + \int_{b+iT_n}^{c+\alpha+iT_n} \right) \frac{\zeta(2s)}{\zeta(s)} \frac{x^{s-\alpha}}{s-\alpha} ds, \quad (3.24)$$

where $R_\alpha(x)$ is defined in (3.21). Note that $R_\alpha(x)$ is the residue of the integrand at $s = \alpha$, where the integrand has a simple pole when $\alpha \neq 1/2$ and has a double pole if $\alpha = 1/2$. We will study the integrals in (3.24) as follows. By Corollary 3.6 and Lemma 3.4, for all $\epsilon > 0$

$$\zeta(2(b+it)) \ll |t|^{1/2-2b+\epsilon/2}$$

and

$$\frac{1}{\zeta(b+it)} \ll |t|^{-1/2+b+\epsilon/2}$$

for all $|t| \geq 1$. Hence

$$\frac{\zeta(2(b+it))}{\zeta(b+it)} \ll |t|^{-b+\epsilon}$$

for all $|t| \geq 1$. Hence if we choose $\epsilon < b$ the middle integral in (3.24) is

$$\begin{aligned} &= \frac{x^{-\alpha}}{2\pi i} \int_{b-iT_n}^{b+iT_n} \frac{\zeta(2s)}{\zeta(s)} \frac{x^s}{s-\alpha} ds \\ &\ll x^{b-\alpha} \left(\int_{-1}^1 \left| \frac{\zeta(2(b+it))}{\zeta(b+it)} \right| \frac{dt}{\sqrt{(b-\alpha)^2+t^2}} + \int_1^{T_n} t^{-b+\epsilon} \frac{dt}{t} \right) \\ &\ll x^{b-\alpha} + x^{b-\alpha} (1 + T_n^{-b+\epsilon}) \ll x^{b-\alpha}, \end{aligned} \tag{3.25}$$

where the last inequality holds since $-b+\epsilon < 0$. For other two integrals in (3.24), we need to apply Lemma 3.7. By Lemma 3.7 for any $\epsilon > 0$ we have

$$\frac{1}{\zeta(\sigma+iT_n)} \ll T_n^{\epsilon/2} \tag{3.26}$$

for all $b \leq \sigma \leq c+\alpha$. Moreover, by Corollary 3.6 we have

$$\zeta(2(\sigma+iT_n)) \ll T_n^{\epsilon/4} \tag{3.27}$$

for all $\sigma \geq 1/4$. Hence by Lemma 3.4 for $b \leq \sigma \leq 1/4$

$$\zeta(2(\sigma+iT_n)) \ll T_n^{1/2-2\sigma+\epsilon/4}.$$

If we choose b very close to $1/4$ we can make the power of T_n in the above inequality small. We choose $b = 1/4 - \epsilon/8$. With this choice of b and the condition $\epsilon < b$, we have $\epsilon < 2/9$. For this choice of b we deduce that if $b \leq \sigma \leq 1/4$ then

$$\zeta(2(\sigma+iT_n)) \ll T_n^{1/2-2\sigma+\epsilon/4} \leq T_n^{\epsilon/2}. \tag{3.28}$$

The inequalities (3.27) and (3.28) imply that for this choice of b we have

$$\zeta(2(\sigma + iT_n)) \ll T_n^{\epsilon/2}$$

for $b \leq \sigma \leq c + \alpha$. Combining this with (3.26) gives

$$\frac{\zeta(2(\sigma + iT_n))}{\zeta(\sigma + iT_n)} \ll T_n^\epsilon$$

for $b \leq \sigma \leq c + \alpha$. Hence the first and the last integrals in (3.24) are

$$\ll T_n^{\epsilon-1} \int_b^{c+\alpha} x^{\sigma-\alpha} d\sigma \ll \frac{x^c}{T_n^{1-\epsilon} \log x} \ll \frac{x^{1-\alpha}}{T_n^{1-\epsilon} \log x}. \quad (3.29)$$

Now the proof follows from (3.24), (3.25), and (3.29). \square

In studying $M_\alpha(x)$, we assumed an estimate of the form

$$\sum_{0 < \gamma \leq T} |\zeta'(\rho)|^{-2} \ll T^\theta,$$

where $\theta < 2$. In the case of $L_\alpha(x)$ we consider a similar upper bound for the sums of the form $\sum_{0 < \gamma \leq T} |\zeta(2\rho)/\zeta'(\rho)|^2$.

Let $\theta < 2$ be fixed. For now we assume the Riemann hypothesis and

$$\sum_{0 < \gamma \leq T} \left| \frac{\zeta(2\rho)}{\zeta'(\rho)} \right|^2 \ll T^\theta. \quad (3.30)$$

The next lemma asserts estimates for certain sums involving $|\zeta(2s)/\zeta'(s)|$.

Lemma 3.23. *Under the assumptions of the Riemann hypothesis and (3.30) we have*

(i)

$$\sum_{\gamma \geq T} \left| \frac{\zeta(2\rho)}{(\rho - \alpha)\zeta'(\rho)} \right|^2 \ll T^{\theta-2}.$$

(ii)

$$\sum_{0 < \gamma \leq T} \left| \frac{\zeta(2\rho)}{(\rho - \alpha)\zeta'(\rho)} \right| \ll T^{(\theta-1)/2} (\log T)^{3/2}.$$

Proof. This lemma is analogous to Lemma 3.18 and can be proved in the same way. \square

In the next lemma, we will establish a more general explicit formula for $L_\alpha(x)$.

Lemma 3.24. *Assume the Riemann hypothesis and (3.30). Then for all $x > 1$, $T > 0$, and $0 < \epsilon < 2/9$, we have*

$$L_\alpha(x) = R_\alpha(x) + \sum_{|\gamma| < T} \frac{x^{\rho-\alpha} \zeta(2\rho)}{\rho - \alpha \zeta'(\rho)} + O\left(\frac{x^{1-\alpha}}{T^{1-\epsilon} \log x} + \frac{x^{1-\alpha} \log x}{T} + x^{1/2-\alpha} (T^{\theta-2} \log T)^{1/2} + x^{1/4-\epsilon/8-\alpha}\right),$$

where $R_\alpha(x)$ is given by (3.21).

Proof. This lemma is analogous to Lemma 3.19 and can be proved in the same way by using Lemma 3.19(i) instead of Lemma 3.18(i). \square

As a direct consequence of the previous lemma we have the following result which is analogous to Proposition 3.20.

Proposition 3.25. *Assume the Riemann hypothesis and (3.30). Then we have*

$$L_\alpha(x) - R_\alpha(x) \ll x^{\theta/2-\alpha} (\log x)^{3/2}.$$

Proof. This proposition is analogous to Proposition 3.20 and can be proved in the same fashion by applying Lemmas 3.24 and 3.23(ii) instead of Lemmas 3.19 and 3.18(ii). \square

Now, we show the existence of a limiting distribution for the function

$$\psi_\alpha(y) := \begin{cases} e^{y(-1/2+\alpha)} L_\alpha(e^y) & \text{if } 0 \leq \alpha < 1/2 \text{ or } \alpha = 1, \\ e^{y(-1/2+\alpha)} (L_\alpha(e^y) - y/2\zeta(1/2)) & \text{if } \alpha = 1/2, \\ e^{y(-1/2+\alpha)} (L_\alpha(e^y) - \zeta(2\alpha)/\zeta(\alpha)) & \text{if } 1/2 < \alpha < 1. \end{cases}$$

Assume the Riemann hypothesis and (3.30) with $\theta < 5/4$. Let

$$C_\alpha = \begin{cases} 1/((1-2\alpha)\zeta(1/2)) & \text{if } 0 \leq \alpha < 1/2 \text{ or } 1/2 < \alpha \leq 1, \\ \gamma/\zeta(1/2) & \text{if } \alpha = 1/2, \end{cases} \quad (3.31)$$

where γ is Euler's constant. By Lemma 3.24 for any $y > 0$ and $X > 0$ we can write

$$\begin{aligned}\psi_\alpha(y) &= C_\alpha + e^{y(-1/2+\alpha)} \sum_{|\gamma| \leq X} \frac{\zeta(2\rho)e^{y(\rho-\alpha)}}{(\rho-\alpha)\zeta'(\rho)} + F_\alpha(y, X) \\ &= C_\alpha + \sum_{|\gamma| \leq X} \frac{\zeta(2\rho)e^{iy\gamma}}{(\rho-\alpha)\zeta'(\rho)} + F_\alpha(y, X) \\ &= C_\alpha + \operatorname{Re} \left(\sum_{0 < \gamma \leq X} \frac{2\zeta(2\rho)e^{iy\gamma}}{(\rho-\alpha)\zeta'(\rho)} \right) + F_\alpha(y, X),\end{aligned}$$

where, for any $0 < \epsilon < 2/9$,

$$F_\alpha(y, X) \ll \left(\frac{ye^{y/2}}{X} + \frac{e^{y/2}}{yX^{1-\epsilon}} + (X^{\theta-2} \log X)^{1/2} + \frac{1}{e^{y(1/4+\epsilon/8)}} \right).$$

Let ρ_n and γ_n have the same meaning as in the case of $\phi_\alpha(y)$. Take $\lambda_n = \gamma_n$ and $r_n = 2\zeta(2\rho_n)/(\rho_n - \alpha)\zeta'(\rho_n)$. Then $\psi_\alpha(y)$ can be written in the form of

$$\psi_\alpha(y) = C_\alpha + \operatorname{Re} \left(\sum_{\gamma_n \leq X} \frac{2\zeta(2\rho_n)}{(\rho_n - \alpha)\zeta'(\rho_n)} e^{iy\gamma_n} \right) + F_\alpha(y, X).$$

Note that by (3.30) we have

$$\sum_{\lambda_n \leq T} \lambda_n^2 |r_n|^2 = \sum_{\gamma_n \leq T} \frac{4\gamma_n^2 |\zeta(2\rho_n)|^2}{|(\rho_n - \alpha)\zeta'(\rho_n)|^2} \ll \sum_{\gamma_n \leq T} \left| \frac{\zeta(2\rho_n)}{\zeta'(\rho_n)} \right|^2 \ll T^\theta.$$

Furthermore, for $F_\alpha(y, X)$ we have

$$\begin{aligned}\int_1^Y |F_\alpha(y, e^Y)|^2 dy &\ll \int_1^Y \left(\frac{ye^{y/2}}{e^Y} + \frac{e^{y/2}}{ye^{(1-\epsilon)Y}} + \left(\frac{Y}{e^{Y(\theta-2)}} \right)^{1/2} + \frac{1}{e^{y(1/4+\epsilon/8)}} \right)^2 dy \\ &\ll \int_1^Y \left(\frac{y^2 e^y}{e^{2Y}} + \frac{e^y}{y^2 e^{2(1-\epsilon)Y}} + \frac{Y}{e^{Y(\theta-2)}} + \frac{1}{e^{y(1/2+\epsilon/4)}} \right) dy \ll 1.\end{aligned}$$

So the conditions of Corollary 2.17 are satisfied. Hence $\psi_\alpha(y)$ has a limiting distribution μ_α . Moreover, if one assumes the linear independence of the zeros of $\zeta(s)$, then by

Theorem 2.18 the Fourier transform of μ_α can be calculated by the following formula

$$\hat{\mu}_\alpha(\xi) = e^{-iC_\alpha\xi} \prod_{n=1}^{\infty} J_0\left(\frac{2|\zeta(2\rho_n)|\xi}{|(\rho_n - \alpha)\zeta'(\rho_n)|}\right).$$

3.5 The Summatory Function of the Möbius Function in Arithmetic Progressions

In this section, we will investigate the summatory function of the Möbius function $\mu(n)$ when n belongs to an arithmetic progression. Our goal is to prove the existence of a limiting distribution for this function.

Definition 3.26. Let $q > 1$ and $a \geq 1$ be such that $(q, a) = 1$. Define

$$M(x, q, a) := \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} \mu(n)$$

where μ is the Möbius function.

In the following lemma we will establish an explicit formula for $M(x, q, a)$.

Lemma 3.27. *Let $q > 1$ and $a \geq 1$ be relatively prime and $0 < b < 1/2$ arbitrary. Assume that for any character χ modulo q we have $L(s, \chi) \neq 0$ for $\sigma > 1/2$ and that all zeros of $L(s, \chi)$ are simple. Then for $x \geq 2$ and $T_{n, \chi} \in \mathcal{T}_\chi$ (the sequence in Corollary 3.15), we have*

$$M(x, q, a) = \frac{1}{\varphi(q)} \sum_{\chi \pmod{q}} \bar{\chi}(a) \sum_{|\gamma_\chi| < T_{n, \chi}} \frac{x^{\rho_\chi}}{\rho_\chi L'(\rho_\chi, \chi)} + O\left(\frac{x \log x}{T_{n, \chi_0}} + \frac{x}{T_{n, \chi_0}^{1-\epsilon} \log x} + x^b\right),$$

where $\rho_\chi = 1/2 + i\gamma_\chi$ runs over the zeros of $L(s, \chi)$, χ_0 is the principal character modulo q , and φ is the Euler totient function.

Proof. For positive integers a and q , it is known that if $(a, q) = 1$, then

$$\frac{1}{\varphi(q)} \sum_{\chi \pmod{q}} \bar{\chi}(a) \chi(n) = \begin{cases} 1 & \text{if } n \equiv a \pmod{q}, \\ 0 & \text{otherwise} \end{cases}$$

(see [18, p. 122]). Thus

$$M(x, q, a) = \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} \mu(n) = \sum_{n \leq x} \frac{\mu(n)}{\varphi(q)} \sum_{\chi \pmod{q}} \bar{\chi}(a) \chi(n)$$

If for each character χ modulo q we put

$$a_{n, \chi} = \mu(n) \chi(n),$$

then

$$M(x, q, a) = \frac{1}{\varphi(q)} \sum_{\chi \pmod{q}} \bar{\chi}(a) \sum_{n \leq x} a_{n, \chi}. \quad (3.32)$$

For each summand by applying Perron's formula (Lemma 3.1) with $w = 0$, and $c = 1 + 1/\log x$ we obtain

$$\sum_{n \leq x} a_{n, \chi} = \frac{1}{2\pi i} \int_{c-iT_{n, \chi}}^{c+iT_{n, \chi}} L(s, \chi)^{-1} \frac{x^s}{s} ds + O\left(\frac{x \log x}{T_{n, \chi}} + 1\right),$$

where $\{T_{n, \chi}\}$ is the sequence in Corollary 3.15. Consider a number $0 < b < 1/2$ and a rectangle with vertices $c + iT_{n, \chi}, b + iT_{n, \chi}, b - iT_{n, \chi}, c - iT_{n, \chi}$. By Cauchy's residue theorem we have

$$\begin{aligned} \frac{1}{2\pi i} \int_{c-iT_{n, \chi}}^{c+iT_{n, \chi}} L(s, \chi)^{-1} \frac{x^s}{s} ds &= \sum_{|\gamma_\chi| < T_{n, \chi}} \frac{x^{\rho_\chi}}{\rho_\chi L'(\rho_\chi, \chi)} \\ &- \frac{1}{2\pi i} \left(\int_{c-iT_{n, \chi}}^{b-iT_{n, \chi}} + \int_{b-iT_{n, \chi}}^{b+iT_{n, \chi}} + \int_{b+iT_{n, \chi}}^{c+iT_{n, \chi}} \right) L(s, \chi)^{-1} \frac{x^s}{s} ds. \end{aligned} \quad (3.33)$$

By Lemma 3.11 and Corollary 3.13 for any $\epsilon > 0$

$$L(b + it, \chi)^{-1} \ll |t|^{-1/2+b+\epsilon}.$$

Hence if $\epsilon < 1/2 - b$ then we have

$$\begin{aligned} \frac{1}{2\pi i} \int_{b-iT_{n,\chi}}^{b+iT_{n,\chi}} L(s, \chi)^{-1} \frac{x^s}{s} ds &\ll x^b \int_0^{T_{n,\chi}} \frac{dt}{(|t|+1)^{3/2-b-\epsilon}} \\ &\ll x^b (1 + T_{n,\chi}^{-1/2+b+\epsilon}) \ll x^b. \end{aligned}$$

Next by Corollary 3.15, for any $\epsilon > 0$,

$$L(\sigma + iT_{n,\chi}, \chi)^{-1} \ll T_{n,\chi}^\epsilon$$

for all $b \leq \sigma \leq c$. If we denote the principal character by χ_0 , then the first and the last integrals on the RHS of (3.33) are

$$\ll \frac{1}{T_{n,\chi}^{1-\epsilon}} \int_b^c x^\sigma d\sigma \ll \frac{x^c}{T_{n,\chi}^{1-\epsilon} \log x} \ll \frac{x}{T_{n,\chi}^{1-\epsilon} \log x} \ll \frac{x}{T_{n,\chi_0}^{1-\epsilon} \log x},$$

since $T_{n,\chi}, T_{n,\chi_0} \in [n, n+1]$. This proves that

$$\sum_{n \leq x} a_{n,\chi} = \sum_{|\gamma_\chi| < T_{n,\chi}} \frac{x^{\rho_\chi}}{\rho_\chi L'(\rho_\chi, \chi)} + O\left(\frac{x \log x}{T_{n,\chi_0}} + \frac{x}{T_{n,\chi_0}^{1-\epsilon} \log x} + x^b\right).$$

Substituting this back into (3.32) implies the result. \square

As in the case of $M_\alpha(x)$, in order to establish an explicit formula for $M(x, q, a)$, which is true for all $T > 0$, we need to make some extra assumption. Unfortunately, no upper bounds for sums of the form

$$\sum_{0 < \gamma_\chi < T} |L'(\rho_\chi, \chi)|^{-2}$$

are known. However, it seems likely that this sum is bounded by T^θ , where $\theta = O(1)$. By work of Hughes, Keating, and O'Connell [9], a random matrix model suggests that $\theta = 1$. If such bound exists with $\theta < 5/4$, then we can apply theorems of Chapter 2 to

prove the existence of a limiting distribution for $M(x, q, a)$. For now we assume that

$$\sum_{0 < \gamma_\chi < T} |L'(\rho_\chi, \chi)|^{-2} \ll T^\theta \quad (3.34)$$

for all characters $\chi \pmod q$, where $\theta < 2$ is a fixed number. We also assume the generalized Riemann hypothesis for the zeros of each $L(s, \chi)$ (i.e., $L(s, \chi) \neq 0$ for $\sigma > 1/2$), as well as the simplicity of the zeros of each $L(s, \chi)$.

Lemma 3.28. *In addition to assumptions of Lemma 3.27 suppose that for each character χ modulo q we have*

$$\sum_{0 < \gamma_\chi < T} |L'(\rho_\chi, \chi)|^{-2} \ll T^\theta,$$

where $\theta < 2$. Then for all $0 < b < 1/2$, $x > 1$, and $T > 0$, we have

$$M(x, q, a) = \frac{1}{\varphi(q)} \sum_{\chi \pmod q} \bar{\chi}(a) \sum_{|\gamma_\chi| < T} \frac{x^{\rho_\chi}}{\rho_\chi L'(\rho_\chi, \chi)} + O\left(\frac{x \log x}{T} + \left(\frac{x \log T}{T^{2-\theta}}\right)^{1/2} + \frac{x}{T^{1-\epsilon} \log x} + x^b\right).$$

Proof. By the partial summation formula, for each character χ modulo q , we have

$$\begin{aligned} \sum_{|\gamma_\chi| < T} \frac{1}{|\rho_\chi L'(\rho_\chi, \chi)|^2} &= 4 \int_{\gamma_{1,x}}^T \frac{1}{t^3} \left(\sum_{|\gamma_\chi| < t} |L'(\rho_\chi, \chi)|^{-2} \right) dt + \frac{2}{T^2} \sum_{|\gamma_\chi| < T} |L'(\rho_\chi, \chi)|^{-2} \\ &\ll \int_{\gamma_{1,x}}^T \frac{1}{t^3} t^\theta dt + \frac{1}{T^2} T^\theta \ll T^{\theta-2}. \end{aligned}$$

Now let $T \geq 2$ satisfy $n \leq T \leq n+1$. Without loss of generality we assume that $n \leq T_{n,x} \leq T \leq n+1$. Then by the Cauchy-Schwarz inequality we have

$$\begin{aligned} \left| \sum_{T_{n,x} \leq |\gamma_\chi| < T} \frac{x^{\rho_\chi}}{\rho_\chi L'(\rho_\chi, \chi)} \right| &\ll x^{1/2} \left(\sum_{T_{n,x} \leq |\gamma_\chi| < T} \frac{1}{|\rho_\chi L'(\rho_\chi, \chi)|^2} \right)^{1/2} \left(\sum_{T_{n,x} \leq |\gamma_\chi| < T} 1 \right)^{1/2} \\ &\leq x^{1/2} T^{\theta/2-1} (\log T)^{1/2} = \left(\frac{x \log T}{T^{2-\theta}} \right)^{1/2}. \end{aligned}$$

Hence

$$\sum_{|\gamma_\chi| < T_{n,x}} \frac{x^{\rho_\chi}}{\rho_\chi L'(\rho_\chi, \chi)} = \sum_{|\gamma_\chi| < T} \frac{x^{\rho_\chi}}{\rho_\chi L'(\rho_\chi, \chi)} + O\left(\left(\frac{x \log T}{T^{2-\theta}}\right)^{1/2}\right).$$

Inserting this into the explanation for $M(x, q, a)$ in Lemma 3.27 yields the desired result. \square

Now assume that the generalized Riemann hypothesis is true for all Dirichlet L -functions $L(s, \chi)$ modulo q and let

$$\phi(y, q, a) := e^{-y/2} M(e^y, q, a).$$

By Lemma 3.28 we have

$$\phi(y, q, a) = \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \bar{\chi}(a) \sum_{|\gamma_\chi| < X} \frac{e^{iy\gamma_\chi}}{\rho_\chi L'(\rho_\chi, \chi)} + E(y, X, q, a)$$

where

$$E(y, X, q, a) \ll \frac{ye^{y/2}}{X} + \frac{(\log X)^{1/2}}{X^{1-\theta/2}} + \frac{e^{y/2}}{yX^{1-\epsilon}} + \frac{1}{e^{y(1/2-b)}}.$$

For each character χ modulo q , let S_χ be the set of positive ordinates γ_χ of the zeros of $L(s, \chi)$ over the line $\sigma = 1/2$. Let $\{\lambda_n\}_{n=1}^\infty$ be a sequence which consists of the union of all S_χ such that $0 \leq \lambda_1 \leq \lambda_2 \leq \dots$, and define a sequence $\{r_n\}_{n=1}^\infty$ by

$$r_n = \frac{2 \bar{\chi}_{\lambda_n}(a)}{\varphi(q)(1/2 + i\lambda_n)L'(1/2 + i\lambda_n, \chi_{\lambda_n})},$$

where χ_{λ_n} is the corresponded character to λ_n . Now we observe that for real χ we have $L(1/2 + i\gamma, \chi) = 0$ if and only if $L(1/2 - i\gamma, \chi) = 0$, however, for complex χ we have $L(1/2 + i\gamma, \chi) = 0$ if and only if $L(1/2 - i\gamma, \bar{\chi}) = 0$. Hence we can write $\phi(y, q, a)$ in the form

$$\phi(y, q, a) = \operatorname{Re}\left(\sum_{\lambda_n < e^y} r_n e^{iy\lambda_n}\right) + E(e^y, e^y, q, a).$$

From the inequality

$$|A + B|^2 \leq 2(|A|^2 + |B|^2),$$

valid for any pair of complex numbers A, B , it follows that

$$\begin{aligned} \int_1^Y |E(y, e^Y, q, a)|^2 dy &\ll \int_1^Y \left(\frac{ye^{y/2}}{e^Y} + \frac{Y^{1/2}}{e^{Y(1-\theta/2)}} + \frac{y^{-1}e^{y/2}}{e^{Y(1-\epsilon)}} + e^{y(b-1/2)} \right)^2 dy \\ &\ll \int_1^Y \left(\frac{y^2 e^y}{e^{2Y}} + \frac{Y}{e^{Y(2-\theta)}} + \frac{y^{-2}e^y}{e^{2Y(1-\epsilon)}} + e^{y(2b-1)} \right) dy \ll 1. \end{aligned}$$

By (3.34) we have

$$\sum_{\lambda_n < T} \lambda_n^2 |r_n|^2 \ll \sum_{\chi} \sum_{0 < \gamma_\chi < T} \frac{4\gamma_\chi^2 |\bar{\chi}(a)|^2}{\varphi^2(q) |\rho_\chi L(\rho_\chi, \chi)|^2} \ll \sum_{\chi} \sum_{0 < \gamma_\chi < T} |L'(\rho_\chi, \chi)|^{-2} \ll T^\theta.$$

Hence by Corollary 2.17, the assumptions of the Riemann hypothesis for each $L(s, \chi)$ and (3.34) with $\theta < 5/4$ imply that the function $\phi(y, q, a)$ has a limiting distribution $\mu_{q;a}$ on \mathbb{R} . If further one assumes the grand linear independence conjecture (see Definition 1.3) then by Theorem 2.18 we can deduce the following formula for the Fourier transform of $\mu_{q;a}$:

$$\begin{aligned} \hat{\mu}_{q;a}(\xi) &= \prod_{n=1}^{\infty} J_0 \left(\left| \frac{2 \bar{\chi}_{\lambda_n}(a)}{\varphi(q)(1/2 + i\lambda_n)L'(1/2 + i\lambda_n, \chi_{\lambda_n})} \right| \xi \right) \\ &= \prod_{\substack{\chi \pmod q \\ \gamma_\chi \geq 0}} J_0 \left(\frac{2\xi |\chi(a)|}{\varphi(q) |\rho_\chi L'(\rho_\chi, \chi)|} \right). \end{aligned}$$

3.6 Connections With the Riemann Hypothesis

In this section we will discuss the connections of the functions that we investigated in the previous sections with the nontrivial zeros of $\zeta(s)$. The main ideas of this study come from the well-known theorem of Ingham [12, Theorem A] and the well-known fact that any of the assumptions

$$M(x) > -Kx^{1/2}, \quad M(x) < Kx^{1/2}, \quad L(x) > -Kx^{1/2}, \quad L(x) < Kx^{1/2},$$

for a fixed $K > 0$ and for all sufficiently large x , would imply the Riemann hypothesis. We start by reviewing some basic results.

Theorem 3.29. (*Landau's lemma*) (i) Let

$$f(s) = \sum_1^{\infty} \frac{a_n}{n^s}$$

be a Dirichlet series that has a finite abscissa of convergence, say σ_c . If $a_n \geq 0$ for all n then $f(s)$ has a singularity at the point $s = \sigma_c$.

(ii) Let

$$f(s) = \int_1^{\infty} \frac{a(x)}{x^s} dx$$

be a Dirichlet integral whose abscissa of convergence is $\sigma_c < \infty$. If $a(x) \geq 0$ for all $x \geq 1$ then $f(s)$ has a singularity at the point $s = \sigma_c$.

Proof. The first part is Theorem 1.7 and the second part is Lemma 15.1 of [18]. \square

Theorem 3.30. (*Ingham*) Let

$$F(s) = \int_0^{\infty} \frac{A(u)}{e^{su}} du,$$

where $A(u)$ is absolutely integrable over every finite interval $[0, U]$ and the integral is convergent in some half-plane $\sigma > \sigma_1 \geq 0$. Let $\{\lambda_n\}_{n \geq 0}$ and $\{r_n\}_{n \geq 0}$ be real and complex sequences, respectively, and put $\lambda_{-n} = -\lambda_n$, $r_{-n} = \bar{r}_n$. Let

$$F^*(s) = \sum_{n=-N}^N \frac{r_n}{s - i\lambda_n},$$

for $\sigma > 0$, and

$$A_T^*(u) = \sum_{|\lambda_n| < T} \left(1 - \frac{|\lambda_n|}{T}\right) r_n e^{i\lambda_n u}.$$

Suppose that $F(s) - F^*(s)$ is analytic in the region $\sigma \geq 0$, $-T \leq t \leq T$, for some $T > 0$. Then for this T we have

$$\liminf_{u \rightarrow \infty} A(u) \leq \liminf_{u \rightarrow \infty} A_T^*(u)$$

and

$$\limsup_{u \rightarrow \infty} A_T^*(u) \leq \limsup_{u \rightarrow \infty} A(u).$$

Proof. See [12, Theorem 1]. □

Assume the Riemann hypothesis is true and all the zeros of $\zeta(s)$ are simple. Then the next theorem asserts that the sequence of residues of the functions $1/\zeta(s)$ and $\zeta(2s)/\zeta(s)$ at the nontrivial zeros of $\zeta(s)$ are divergent series.

Theorem 3.31. (*Ingham*) *Assume the Riemann hypothesis and that all the zeros of $\zeta(s)$ are simple. Let $\{\rho_n = 1/2 + i\gamma_n\}$ denote the sequence of all nontrivial zeros of $\zeta(s)$ with $0 < \gamma_1 < \gamma_2 < \dots$. Then we have*

$$\sum_{n=1}^{\infty} \frac{1}{|\rho_n \zeta'(\rho_n)|} = \infty,$$

and

$$\sum_{n=1}^{\infty} \left| \frac{\zeta(2\rho_n)}{\rho_n \zeta'(\rho_n)} \right| = \infty.$$

Proof. See [12, Theorem 2]. □

The next lemma states analogous assertions to those of Theorem 3.31 for the functions $1/\zeta(s + \alpha)$ and $\zeta(2(s + \alpha))/\zeta(s + \alpha)$.

Lemma 3.32. *Assume the Riemann hypothesis is true and that all the zeros of $\zeta(s)$ are simple. Let the nontrivial zeros of $\zeta(s)$ above the real line be denoted by $\rho_n = 1/2 + i\gamma_n$ ($0 < \gamma_1 < \gamma_2 < \dots$). Then*

$$\sum_{n=1}^{\infty} \frac{1}{|(\rho_n - \alpha) \zeta'(\rho_n)|} = \infty,$$

and

$$\sum_{n=1}^{\infty} \left| \frac{\zeta(2\rho_n)}{(\rho_n - \alpha) \zeta'(\rho_n)} \right| = \infty.$$

Proof. For all $n \geq 1$ we have

$$|\rho_n - \alpha| \leq |\rho_n| + \alpha \leq 2|\rho_n|.$$

Hence

$$\sum_{n=1}^{\infty} \frac{1}{|(\rho_n - \alpha)\zeta'(\rho_n)|} \geq \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{|\rho_n \zeta'(\rho_n)|},$$

and

$$\sum_{n=1}^{\infty} \left| \frac{\zeta(2\rho_n)}{(\rho_n - \alpha)\zeta'(\rho_n)} \right| \geq \frac{1}{2} \sum_{n=1}^{\infty} \left| \frac{\zeta(2\rho_n)}{\rho_n \zeta'(\rho_n)} \right|,$$

so, by applying Lemma 3.31 on the above inequalities, the lemma follows. \square

Let

$$m_\alpha = \begin{cases} 0 & \text{if } 0 \leq \alpha \leq 1/2, \\ 1/\zeta(\alpha) & \text{if } 1/2 < \alpha < 1, \\ 0 & \text{if } \alpha = 1. \end{cases}$$

In the next theorem we prove that the Riemann hypothesis is a consequence of any of the assumptions

$$M_\alpha(x) - m_\alpha > Kx^{\frac{1}{2}-\alpha},$$

$$M_\alpha(x) - m_\alpha < -Kx^{\frac{1}{2}-\alpha},$$

for a fixed $K > 0$ and for all sufficiently large x .

Theorem 3.33. *For $\alpha \in [0, 1]$ and $x > 0$, let*

$$M_\alpha(x) = \sum_{n \leq x} \frac{\mu(n)}{n^\alpha}$$

and

$$g_\alpha(x) = \frac{M_\alpha(x) - m_\alpha}{x^{\frac{1}{2}-\alpha}}.$$

If there is a constant $K > 0$ such that either $g_\alpha(x) > -K$ or $g_\alpha(x) < K$, for sufficiently large x , then the Riemann hypothesis holds and all the zeros of $\zeta(s)$ are simple.

Proof. Suppose that the case $g_\alpha(x) > -K$ holds for $x > x_0 \geq 1$. Thus we have

$$M_\alpha(x) - m_\alpha + Kx^{\frac{1}{2}-\alpha} > 0$$

for $x \geq x_0$. By the partial summation formula one has

$$\int_1^\infty \frac{M_\alpha(x)}{x^{s+1}} dx = \frac{1}{s\zeta(s+\alpha)},$$

for $\sigma > 0$. Let

$$G(s) = \int_1^\infty \frac{M_\alpha(x) - m_\alpha + Kx^{\frac{1}{2}-\alpha}}{x^{s+1}} dx$$

the integral being convergent inside the half-plane $\sigma > 0$. For $1/2 \leq \alpha < 1$ the relation

$$G(s) = \frac{1}{s\zeta(s+\alpha)} - \frac{1}{s\zeta(\alpha)} - \frac{K}{\frac{1}{2} - \alpha - s} \quad (3.35)$$

holds for $\sigma > 1/2 - \alpha$ and $G(s)$ is analytic on the real axis for $s > 1/2 - \alpha$ since in the Laurent expansion of $1/s\zeta(s+\alpha) - 1/s\zeta(\alpha)$ around $s = 0$ the terms with singularity cancel each other. Hence by Lemma 3.29(ii) both sides of (3.35) are analytic in the half-plane $\sigma > 1/2 - \alpha$. It follows that the function $\zeta(s+\alpha)$ has no zero in the half-plane $\sigma > 1/2 - \alpha$. This proves the Riemann hypothesis. Moreover, since $M_\alpha(x) - m_\alpha + Kx^{1/2-\alpha} > 0$ for $x \geq x_0$ we have

$$\begin{aligned} |G(s)| &= \left| \int_1^\infty \frac{M_\alpha(x) - m_\alpha + Kx^{\frac{1}{2}-\alpha}}{x^{s+1}} dx \right| \\ &\leq O(1) + \int_{x_0}^\infty \frac{M_\alpha(x) - m_\alpha + Kx^{\frac{1}{2}-\alpha}}{x^{\sigma+1}} dx \\ &\leq O(1) + G(\sigma) = O\left(\frac{1}{\sigma - (1/2 - \alpha)}\right) \end{aligned} \quad (3.36)$$

as $\sigma \rightarrow (1/2 - \alpha)^+$. The last inequality holds since the function $1/\sigma\zeta(\sigma+\alpha) - 1/\sigma\zeta(\alpha)$ is bounded for values of σ in some neighborhood $(1/2 - \alpha, 1/2 - \alpha + \epsilon)$. Now if we assume that $\zeta(s)$ (respectively, $\zeta(s+\alpha)$) has a multiple zero say of order $k \geq 2$ at some point $\rho_0 = 1/2 + i\gamma_0$ (respectively $s_0 = 1/2 - \alpha + i\gamma_0$), then

$$\lim_{s \rightarrow s_0} (s - s_0)^k G(s) = \ell \neq 0.$$

This implies that

$$\lim_{s \rightarrow s_0} |(s - s_0)^k G(s)| = |\ell| \neq 0$$

so there is a neighborhood of s_0 such that

$$|s - s_0|^k |G(s)| \geq 1/2 |\ell|,$$

for any s in this neighborhood. Hence

$$|G(s)| \geq \frac{|\ell|}{2|s - s_0|^k} \gg \frac{1}{(\sigma - (1/2 - \alpha))^k} \gg \frac{1}{(\sigma - (1/2 - \alpha))^2}$$

as $\sigma \rightarrow (1/2 - \alpha)^+$. This contradicts (3.36). Thus we conclude that all the zeros of $\zeta(s)$ are simple.

If $0 \leq \alpha < \frac{1}{2}$ or $\alpha = 1$, we eliminate the middle terms on both sides of (3.35), then a similar reasoning would work for these cases.

The proof will be similar if we assume the case $g_\alpha(x) < K$ by considering

$$G(s) = \int_1^\infty \frac{Kx^{\frac{1}{2}-\alpha} - M_\alpha(x) + m_\alpha}{x^{s+1}} dx$$

instead of (3.35). □

The next theorem asserts that the assumptions of the previous theorem will imply a linear dependence for the zeros of $\zeta(s)$.

Theorem 3.34. *Let $g_\alpha(x)$ be defined as in Theorem 3.33 and let $\gamma_1, \gamma_2, \dots$ be the imaginary parts of the distinct zeros of $\zeta(s)$ above the real line. If $\{\gamma_n\}$ is linearly independent over \mathbb{Q} , that is, there is no relation of type*

$$\sum_{n=1}^N c_n \gamma_n = 0 \tag{3.37}$$

with $c_n \in \mathbb{Q}$, $c_j \neq 0$ for some j , or there are only a finite number of relations of type (3.37), then

$$\liminf_{x \rightarrow \infty} g_\alpha(x) = -\infty \quad \text{and} \quad \limsup_{x \rightarrow \infty} g_\alpha(x) = +\infty. \tag{3.38}$$

Proof. We will follow the proof of [12, Theorem A]. Without loss of generality, assume in the contrary that there exists a constant $K > 0$ such that $g_\alpha(x) > -K$. Therefore by Theorem 3.33, RH is true and all zeros of $\zeta(s)$ are simple. We may suppose then that the result of Lemma 3.32 holds. Let

$$r_n = \frac{1}{(1/2 - \alpha + i\gamma_n)\zeta'(1/2 + i\gamma_n)}$$

for $n \geq 1$, $\gamma_0 = 0$, $r_0 = 0$, $\gamma_{-n} = -\gamma_n$, and $r_{-n} = \bar{r}_n$. Given any $T > 0$, choose $N = N(T)$ such that $T \leq \gamma_N$. Define

$$A^*(u) = \sum_{n=-N}^N r_n e^{i\gamma_n u}$$

and

$$A_T^*(u) = \sum_{|\gamma_n| < T} \left(1 - \frac{|\gamma_n|}{T}\right) r_n e^{i\gamma_n u}.$$

Hence by Theorem 3.30 with $A(u) = g_\alpha(e^u)$ we have

$$\liminf_{u \rightarrow \infty} g_\alpha(e^u) \leq \liminf_{u \rightarrow \infty} A_T^*(u)$$

and

$$\limsup_{u \rightarrow \infty} A_T^*(u) \leq \limsup_{u \rightarrow \infty} g_\alpha(e^u).$$

Suppose that γ_n ($0 < \gamma_n < T$) are linearly independent. Then by the Kronecker theorem for any $\epsilon > 0$ we can find a value of u for which we have

$$|\gamma_n u - (-\arg(\pm r_n))| < \epsilon,$$

for all n such that $0 < \gamma_n < T$. Thus

$$\limsup_{u \rightarrow \infty} A_T^*(u) = r_0 + 2 \sum_{0 < \gamma_n < T} \left(1 - \frac{\gamma_n}{T}\right) |r_n|,$$

$$\liminf_{u \rightarrow \infty} A_T^*(u) = r_0 - 2 \sum_{0 < \gamma_n < T} \left(1 - \frac{\gamma_n}{T}\right) |r_n|.$$

However we have

$$2 \sum_{0 < \gamma_n < T} \left(1 - \frac{\gamma_n}{T}\right) |r_n| \geq \sum_{0 < \gamma_n < T/2} |r_n|$$

which, according to Lemma 3.32, gives

$$\begin{aligned} \limsup_{u \rightarrow \infty} g_\alpha(e^u) &\geq \lim_{T \rightarrow \infty} \limsup_{u \rightarrow \infty} A_T^*(u) \\ &\geq \sum_{n=1}^{\infty} |r_n| = \infty \end{aligned}$$

and

$$\begin{aligned} \liminf_{u \rightarrow \infty} g_\alpha(e^u) &\leq \lim_{T \rightarrow \infty} \liminf_{u \rightarrow \infty} A_T^*(u) \\ &\leq - \sum_{n=1}^{\infty} |r_n| = -\infty. \end{aligned}$$

This contradiction proves the theorem when there is no relation of type $\sum_{n=1}^N c_n \gamma_n = 0$. If there is a finite number of relations of this type, then we can consider the greatest γ_n , say γ_M , which involves such a relation, and apply the Kronecker theorem to those γ_n in the range $\gamma_M < \gamma_n < T$ for some large enough T . \square

The next theorem asserts analogous results to those of Theorem 3.33 for $L_\alpha(x) - l_\alpha(x)$.

Theorem 3.35. *For $\alpha \in [0, 1]$ and $x \geq 1$, let*

$$L_\alpha(x) = \sum_{n \leq x} \frac{\lambda(n)}{n^\alpha},$$

$$l_\alpha(x) = \begin{cases} 0 & \text{if } 0 \leq \alpha < \frac{1}{2}, \\ \log x / 2\zeta(1/2) & \text{if } \alpha = \frac{1}{2}, \\ \zeta(2\alpha) / \zeta(\alpha) & \text{if } \frac{1}{2} < \alpha < 1, \\ 0 & \text{if } \alpha = 1, \end{cases}$$

and

$$h_\alpha(x) = \frac{L_\alpha(x) - l_\alpha(x)}{x^{\frac{1}{2}-\alpha}}.$$

If there is a constant $K > 0$ such that either $h_\alpha(x) > -K$ or $h_\alpha(x) < K$, for sufficiently large x , then the Riemann hypothesis holds and all the zeros of $\zeta(s)$ are simple.

Proof. Let us assume that $h_\alpha(x) > -K$ for $x \geq x_0$. Then $L_\alpha(x) - l_\alpha(x) + Kx^{\frac{1}{2}-\alpha} > 0$ for $x \geq x_0$. By applying the partial summation formula we obtain

$$s \int_1^\infty \frac{L_\alpha(x)}{x^{s+1}} dx = \sum_{n=1}^\infty \frac{\lambda(n)}{n^{s+\alpha}} = \frac{\zeta(2(s+\alpha))}{\zeta(s+\alpha)}$$

for $\sigma > 1/2$. Suppose that $0 \leq \alpha < 1/2$. Then we have the relation

$$\int_1^\infty \frac{L_\alpha(x) - l_\alpha(x) + Kx^{\frac{1}{2}-\alpha}}{x^{s+1}} dx = \frac{\zeta(2(s+\alpha))}{s\zeta(s+\alpha)} - \frac{K}{\frac{1}{2} - \alpha - s} \quad (3.39)$$

valid for $\sigma > 1/2 - \alpha$. Both sides of (3.39) are analytic on the real axis for $s > 1/2 - \alpha$. Thus by Lemma 3.29(ii) the integral in (3.39) and therefore the right-hand side is analytic in the half-plane $\sigma > 1/2 - \alpha$. This proves that $\zeta(s + \alpha)$ has no zeros in the half-plane $\sigma > 1/2 - \alpha$, that is the Riemann hypothesis. To prove the simplicity of zeros note that

$$\left| \frac{\zeta(2(s+\alpha))}{s\zeta(s+\alpha)} - \frac{K}{\frac{1}{2} - \alpha - s} \right| = O\left(\frac{1}{\sigma - (1/2 - \alpha)}\right)$$

as $\sigma \rightarrow (1/2 - \alpha)^+$. From here we can deduce similar to Theorem 3.33 that $\zeta(s + \alpha)$ has no multiple zero. For $\alpha = 1/2$ and $1/2 < \alpha < 1$ we should consider

$$\int_1^\infty \frac{L_\alpha(x) - l_\alpha(x) + Kx^{\frac{1}{2}-\alpha}}{x^{s+1}} dx = \frac{\zeta(2s+1)}{s\zeta(s+1/2)} - \frac{1}{2s^2\zeta(1/2)} + \frac{K}{s}$$

and

$$\int_1^\infty \frac{L_\alpha(x) - l_\alpha(x) + Kx^{\frac{1}{2}-\alpha}}{x^{s+1}} dx = \frac{\zeta(2s+1)}{s\zeta(s+1/2)} - \frac{\zeta(2\alpha)}{s\zeta(\alpha)} - \frac{K}{\frac{1}{2} - \alpha - s}$$

for $\sigma > 0$, respectively, and follow similar argument as the previous case. If we have $h_\alpha(x) < K$ then we will consider the integral

$$\int_1^\infty \frac{Kx^{\frac{1}{2}-\alpha} - L_\alpha(x) + l_\alpha(x)}{x^{s+1}} dx$$

and we can adapt the proof of the previous case to this new situation line by line. \square

As the last result of this chapter, we state an analogous result to that of Theorem 3.34 for $h_\alpha(x)$.

Theorem 3.36. *Let $h_\alpha(x)$ be defined as in Theorem 3.35 and let $0 < \gamma_1 < \gamma_2 < \dots$ be the imaginary parts of the distinct zeros of $\zeta(s)$. If there is no relation of type*

$$\sum_{n=1}^N c_n \gamma_n = 0$$

or there are only a finite number of such relations, then

$$\liminf_{x \rightarrow \infty} h_\alpha(x) = -\infty \quad \text{and} \quad \limsup_{x \rightarrow \infty} h_\alpha(x) = +\infty. \quad (3.40)$$

Proof. Repeat the proof of Theorem 3.34 with $A(u) = h_\alpha(e^u)$ and

$$r_n = \frac{\zeta(1 + 2i\gamma_n)}{(1/2 - \alpha + i\gamma_n)\zeta'(1/2 + i\gamma_n)},$$

by employing Lemma 3.32. \square

Chapter 4

Large Deviations of Infinite Sums of Random Variables

4.1 Introduction

The limiting distribution μ of

$$\phi(y) = c + \operatorname{Re}\left(\sum_{\lambda_n \leq X} r_n e^{iy\lambda_n}\right) + E(y, X),$$

studied in Chapter 2, can be viewed from a probabilistic point of view. We start by reviewing some facts from probability.

Let (E, \mathcal{E}) be a measurable space and (Ω, \mathcal{F}, P) be a probability space.

Definition 4.1. A function $X : \Omega \rightarrow E$ is a *random variable* if for every subset $B \in \mathcal{E}$ we have

$$X^{-1}(B) := \{\omega \in \Omega \mid X(\omega) \in B\} \in \mathcal{F}.$$

The *expected value* of X is defined by

$$\mathbb{E}[X] := \int_{\Omega} X(\omega) dP(\omega).$$

If X is a real random variable (i.e., $E = \mathbb{R}$), then we denote the value

$$P(\{\omega \in \Omega \mid X(\omega) \leq r\})$$

by $P(X \leq r)$.

Fourier transforms in analysis are analogous to characteristic functions in probability theory which will be defined next.

Definition 4.2. The *characteristic function* of a probability measure μ on \mathbb{R} is defined, for $t \in \mathbb{R}$, by

$$\varphi(t) = \int_{-\infty}^{\infty} e^{itx} d\mu(x).$$

Note that $\varphi(t) = \mathbb{E}[e^{itX}]$.

For each $n \in \mathbb{N}$, $X_n : \mathbb{T}^1 \rightarrow \mathbb{R}$, defined by $X_n(\theta_n) = \sin 2\pi\theta_n$, is a random variable on the 1-dimensional torus \mathbb{T}^1 . Let

$$\mathbb{T}^\infty := \{(\theta_1, \theta_2, \dots) \in \mathbb{R}^\infty \mid \forall n \in \mathbb{N}, 0 \leq \theta_n < 1\}$$

denote the infinite torus. For $\underline{\theta} = (\theta_1, \theta_2, \dots) \in \mathbb{T}^\infty$, let

$$X(\underline{\theta}) = \sum_{n=1}^{\infty} r_n X_n(\theta_n) = \sum_{n=1}^{\infty} r_n \sin 2\pi\theta_n, \quad (4.1)$$

where $r_n \geq 0$. From this point on, we assume that r_n is real and non-negative. By Kolmogorov's three series theorem [1, Theorem 22.8] (see also [1, Example 22.3]), X converges in probability almost everywhere on \mathbb{T}^∞ , so (4.1) defines a random variable on \mathbb{T}^∞ .

Recall that Bessel function $J_0(z)$ is given for $z \in \mathbb{C}$ by

$$J_0(z) = \int_0^1 e^{iz \sin 2\pi\theta} d\theta.$$

In Theorem 2.18 we proved that if $\{\lambda_n\}$ is linearly independent over \mathbb{Q} , then the Fourier transform of μ is equal to

$$\hat{\mu}(\xi) = \prod_{n=1}^{\infty} J_0(r_n \xi),$$

where $J_0(z)$ is Bessel function of order 0. $X(\underline{\theta})$ defines a probability measure μ_X on \mathbb{R} which is given for any $V \in \mathbb{R}$ by

$$\mu_X((-\infty, V]) = P(X \leq V),$$

where P is the canonical probability measure on \mathbb{T}^∞ .

The random variables $\{X_n\}$ are independent. The independence of $\{X_n\}$ and the convergence of X in probability imply (see [1, p. 273, and pp. 345–346]) that the characteristic function of $X(\underline{\theta})$ is equal to

$$\varphi_X(t) = \prod_{n=1}^{\infty} \mathbb{E}[e^{itX_n}] = \prod_{n=1}^{\infty} \int_0^1 e^{ir_n t \sin 2\pi\theta_n} d\theta_n = \prod_{n=1}^{\infty} J_0(r_n t) = \hat{\mu}(t).$$

From the uniqueness of the Fourier transform of the distribution measures, we deduce that μ is equal to μ_X . This correspondence allows us to treat our limiting distribution measure μ like the probability measure μ_X .

In this chapter we will study the large deviations $P(X \geq V)$ of sums X of independent random variables of the shape (4.1). Throughout this chapter, we assume that

$$\sum_{n=1}^{\infty} r_n^2 < \infty.$$

4.2 Explicit Bounds for Sums of Independent Random Variables

In this section, we find explicit upper and lower bounds for the probability $P(X \geq V) := \mu_X([V, \infty))$, where X is the random variable defined by (4.1).

Theorem 4.3. *Let $X(\underline{\theta}) = \sum_{k=1}^{\infty} r_k \sin 2\pi\theta_k$, where $\sum_{k=1}^{\infty} r_k^2 < \infty$. Let $V > 0$ be fixed. Then we have*

(i) *For fixed $\epsilon > 0$ if K is such that*

$$\sum_{k \leq K} r_k \leq \frac{V}{1 + \epsilon}$$

then

$$P(X \geq V) \leq \exp \left(-\frac{3\epsilon^2}{4(1+\epsilon)^2} V^2 \left(\sum_{k>K} r_k^2 \right)^{-1} \right).$$

(ii) If $\delta > 0$ satisfies

$$\sum_{r_k > \delta} (r_k - \delta) \geq V,$$

then

$$P(X \geq V) \geq \frac{1}{2} \exp \left(-\frac{1}{2} \sum_{r_k > \delta} \log \left(\frac{\pi^2 r_k}{2\delta} \right) \right).$$

Proof. Part (ii) is [19, Section 3, Theorem 2]. We will prove part (i) by following the proof of [19, Section 3, Theorem 1]. Chernoff's inequality asserts that for any $\lambda > 0$ and $V \geq 0$ we have

$$P(X \geq V) \leq e^{-\lambda V} \mathbb{E}[e^{\lambda X}],$$

(see [19, p. 19]). Also, we have

$$\mathbb{E}[e^{\lambda X}] \leq \exp \left(\lambda \sum_{k \leq K} r_k + \frac{\lambda^2}{4} \sum_{k > K} r_k^2 \right),$$

(see the proof of [19, Section 3, Theorem 1]). Hence

$$\begin{aligned} P(X \geq V) &\leq e^{-\lambda V} \exp \left(\lambda \sum_{k \leq K} r_k + \frac{\lambda^2}{4} \sum_{k > K} r_k^2 \right) \\ &\leq \exp \left(-\lambda \left(V - \sum_{k \leq K} r_k \right) + \frac{\lambda^2}{4} \sum_{k > K} r_k^2 \right). \end{aligned}$$

By the assumption we have

$$V - \sum_{k \leq K} r_k \geq V - V/(1+\epsilon) = \epsilon V/(1+\epsilon).$$

Thus we obtain

$$P(X \geq V) \leq \exp\left(-\lambda\epsilon V/(1+\epsilon) + \frac{\lambda^2}{4} \sum_{k>K} r_k^2\right).$$

Now we take $\lambda = \epsilon V \left((1+\epsilon) \sum_{k>K} r_k^2\right)^{-1}$ to get

$$\begin{aligned} P(X \geq V) &\leq \exp\left(-\frac{\epsilon^2}{(1+\epsilon)^2} V^2 \left(\sum_{k>K} r_k^2\right)^{-1} + \frac{\epsilon^2}{4(1+\epsilon)^2} V^2 \left(\sum_{k>K} r_k^2\right)^{-1}\right) \\ &= \exp\left(-\frac{3\epsilon^2}{4(1+\epsilon)^2} V^2 \left(\sum_{k>K} r_k^2\right)^{-1}\right), \end{aligned}$$

which is the desired inequality. \square

The goal of this chapter is to establish sharp upper and lower bounds for the tail of the probability measure that $X(\underline{\theta})$ determines on \mathbb{T}^∞ . Furthermore, we will establish an asymptotic relation for the probability $P(X(\underline{\theta}) \geq V)$, under specific assumptions. For the former goal we will apply Theorem 4.3 below, while for the latter we will use the saddle point method. In fact, the saddle point method suggests that the bounds that are found from Theorem 4.3 are sharp. It is necessary to mention that we became familiar with the methods out of ideas of Ng (the author's supervisor) and a preprint of Lamzouri [15].

Next we state a result which is a direct consequence of Lemma 2.8.

Lemma 4.4. *Let $\{\alpha_n\}$ be a real sequence which increases and has the limit infinity. Let $\{c_n\}$ be a complex sequence and*

$$C(t) = \sum_{\alpha_n \leq t} c_n.$$

We have

(i)

$$\sum_{\alpha_n \leq X} \frac{c_n}{\alpha_n} = \int_{\alpha_1}^X \frac{C(t)}{t^2} dt + \frac{C(X)}{X},$$

(ii)

$$\sum_{\alpha_n > X} \frac{c_n^2}{\alpha_n^2} = \lim_{T \rightarrow \infty} \frac{C(T)}{T^2} - \frac{C(X)}{X^2} + 2 \int_X^\infty \frac{C(t)}{t^3} dt.$$

Proof. Apply Lemma 2.8 with $\phi(x) = 1/x$ for (i) and with $\phi(x) = 1/x^2$ for (ii). \square

Lemma 4.5. *Suppose $f(x) = o(g(x))$, where $f(x)$ and $g(x)$ are integrable on compact subsets of $[a, \infty)$, and $g(x) \geq 0$ for all $x \in [a, \infty)$. Then we have*

(i)

$$\int_x^\infty f(t) dt = o\left(\int_x^\infty g(t) dt\right),$$

as $x \rightarrow \infty$.

(ii) If

$$\lim_{x \rightarrow \infty} \int_a^x g(t) dt = \infty$$

then

$$\int_a^x f(t) dt = o\left(\int_a^x g(t) dt\right),$$

as $x \rightarrow \infty$.

Proof. (i) By the hypothesis, given $\epsilon > 0$, we can find $x_0 = x_0(\epsilon)$ such that $x \geq x_0$ implies

$$|f(x)| < \epsilon g(x).$$

So we have

$$\left| \int_x^\infty f(t) dt \right| \leq \int_x^\infty |f(t)| dt < \epsilon \int_x^\infty g(t) dt.$$

Thus for any $x \geq x_0$ we have

$$\frac{\left| \int_x^\infty f(t) dt \right|}{\int_x^\infty g(t) dt} < \epsilon.$$

This proves (i).

(ii) By assumption, given $\epsilon > 0$, there is a $t_0 = t_0(\epsilon) \geq a$ such that

$$|f(t)| < \frac{\epsilon}{2} g(t)$$

for all $t \geq t_0$. This gives

$$\left| \int_{t_0}^x f(t) dt \right| \leq \int_{t_0}^x |f(t)| dt < \frac{\epsilon}{2} \int_{t_0}^x g(t) dt.$$

Consequently,

$$\frac{\left| \int_{t_0}^x f(t) dt \right|}{\int_{t_0}^x g(t) dt} < \frac{\epsilon}{2}$$

for all $x \geq t_0$. On the other hand since $\int_a^x g(t) dt \rightarrow \infty$ there is $x_0 \geq t_0$ such that

$$\frac{\left| \int_a^{t_0} f(t) dt \right|}{\int_{t_0}^x g(t) dt} < \frac{\epsilon}{2}$$

for all $x > x_0$. Hence for $x > x_0$ we have

$$\begin{aligned} \left| \frac{\int_a^x f(t) dt}{\int_a^x g(t) dt} \right| &\leq \frac{\left| \int_a^{t_0} f(t) dt \right|}{\int_a^x g(t) dt} + \frac{\left| \int_{t_0}^x f(t) dt \right|}{\int_a^x g(t) dt} \\ &\leq \frac{\left| \int_a^{t_0} f(t) dt \right|}{\int_{t_0}^x g(t) dt} + \frac{\left| \int_{t_0}^x f(t) dt \right|}{\int_{t_0}^x g(t) dt} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

since $g(x) \geq 0$ for $x \geq a$. This proves (ii). \square

If in Theorem 4.3 we consider certain estimates for averages of r_n , then we can find an upper bound for the probability $P(X(\underline{\theta}) \geq V)$. Therefore, we can state the following theorem.

Theorem 4.6. *Let $\{\lambda_n\}$ be an increasing sequence of positive real numbers that tends to infinity, and let $\{r_n\}$ be a positive real sequence. Let c_1, c_2, d_1, d_2 be real constants such that $d_1 \neq -1$. Assume*

$$\sum_{\substack{n \\ \lambda_n \leq T}} \lambda_n r_n = c_1 T (\log T)^{d_1} + o(T (\log T)^{d_1}), \quad (4.2)$$

and

$$\sum_{\substack{n \\ \lambda_n \leq T}} (\lambda_n r_n)^2 = c_2 T (\log T)^{d_2} + o(T (\log T)^{d_2}), \quad (4.3)$$

as $T \rightarrow \infty$. Then for any $\epsilon > 0$ and for large $V > 0$

$$P(X(\underline{\theta}) \geq V) \leq \exp \left(-C V^{2 - \frac{d_2}{d_1+1}} \exp \left(\left(\frac{d_1+1}{c_1} V \right)^{\frac{1}{d_1+1}} (1 + o(1)) \right) \right),$$

where

$$C = \frac{3\epsilon^2(1-\epsilon)}{4c_2(1+\epsilon)^2} \left(\frac{c_1(1+\epsilon)^2}{d_1+1} \right)^{-\frac{d_2}{d_1+1}}.$$

Proof. In Lemma 4.4(i) take $\alpha_n = \lambda_n$ and $c_n = \lambda_n r_n$. Then by (4.2) and Lemma 4.5(i) we have

$$\begin{aligned} \sum_{\substack{n \\ \lambda_n \leq T}} r_n &= \int_{\lambda_1}^T \frac{1}{t^2} \left(\sum_{\substack{n \\ \lambda_n \leq t}} \lambda_n r_n \right) dt + \frac{1}{T} \sum_{\substack{n \\ \lambda_n \leq T}} \lambda_n r_n \\ &= \int_{\lambda_1}^T \frac{1}{t^2} (c_1 t (\log t)^{d_1} + o(t (\log t)^{d_1})) dt \\ &\quad + \frac{1}{T} (c_1 T (\log T)^{d_1} + o(T (\log T)^{d_1})) \\ &= c_1 \int_{\lambda_1}^T \frac{(\log t)^{d_1}}{t} dt + o((\log T)^{d_1+1}) \\ &= \frac{c_1}{d_1+1} (\log T)^{d_1+1} + o((\log T)^{d_1+1}). \end{aligned}$$

Thus we have

$$\sum_{\substack{n \\ \lambda_n \leq T}} r_n \sim \frac{c_1}{d_1+1} (\log T)^{d_1+1}$$

as $T \rightarrow \infty$. On the other hand, In Lemma 4.4(ii), take $\alpha_n = \lambda_n$ and $c_n = (\lambda_n r_n)^2$.

Then by (4.3) and Lemma 4.5(ii) we have

$$\begin{aligned}
\sum_{\substack{n \\ \lambda_n > T}} r_n^2 &= 2 \int_T^\infty \frac{1}{t^3} \left(\sum_{\substack{n \\ \lambda_n \leq t}} (\lambda_n r_n)^2 \right) dt - \frac{1}{T^2} \sum_{\substack{n \\ \lambda_n \leq T}} (\lambda_n r_n)^2 \\
&= 2 \int_T^\infty \frac{1}{t^3} (c_2 t (\log t)^{d_2} + o(t (\log t)^{d_2})) dt \\
&\quad - \frac{1}{T^2} (c_2 T (\log T)^{d_2} + o(T (\log T)^{d_2})) \\
&= 2c_2 \int_T^\infty \frac{(\log t)^{d_2}}{t^2} dt - \frac{c_2 (\log T)^{d_2}}{T} + o\left(\frac{(\log T)^{d_2}}{T}\right) \\
&= c_2 \frac{(\log T)^{d_2}}{T} + o\left(\frac{(\log T)^{d_2}}{T}\right).
\end{aligned}$$

Thus

$$\sum_{\substack{n \\ \lambda_n > T}} r_n^2 \sim c_2 \frac{(\log T)^{d_2}}{T}, \tag{4.4}$$

as $T \rightarrow \infty$.

Now for given $V > 0$ and $\epsilon > 0$ choose $T = T(V)$ such that $\frac{c_1}{d_1+1} (\log T)^{d_1+1} = V/(1+\epsilon)^2$. So

$$\begin{aligned}
\sum_{\substack{n \\ \lambda_n \leq T}} r_n &= \frac{c_1}{d_1+1} (\log T)^{d_1+1} (1+o(1)) \\
&\leq \frac{c_1}{d_1+1} (\log T)^{d_1+1} (1+\epsilon) \\
&\leq \frac{V}{1+\epsilon}.
\end{aligned}$$

Then we have

$$(\log T)^{d_2} = \left(\frac{d_1+1}{c_1(1+\epsilon)^2} V \right)^{\frac{d_2}{d_1+1}}$$

and

$$\frac{1}{T} = \exp \left(- \left(\frac{d_1+1}{c_1(1+\epsilon)^2} V \right)^{\frac{1}{d_1+1}} \right) = \exp \left(- \left(\frac{d_1+1}{c_1} V \right)^{\frac{1}{d_1+1}} (1+o(1)) \right).$$

Thus (4.4) implies that

$$\sum_{\substack{n \\ \lambda_n > T}} r_n^2 = c_2 \left(\frac{d_1 + 1}{c_1(1 + \epsilon)^2} V \right)^{\frac{d_2}{d_1+1}} \exp \left(- \left(\frac{d_1 + 1}{c_1} V \right)^{\frac{1}{d_1+1}} (1 + o(1)) \right),$$

as $V \rightarrow \infty$. Hence we can apply Theorem 4.3(i) to get

$$P(X(\underline{\theta}) \geq V) \leq \exp \left(- C V^{2 - \frac{d_2}{d_1+1}} \exp \left(\left(\frac{d_1 + 1}{c_1} V \right)^{\frac{1}{d_1+1}} (1 + o(1)) \right) \right),$$

where

$$C = \frac{3\epsilon^2(1 - \epsilon)}{4c_2(1 + \epsilon)^2} \left(\frac{c_1(1 + \epsilon)^2}{d_1 + 1} \right)^{-\frac{d_2}{d_1+1}}.$$

This completes the proof. \square

In the previous theorem in order to apply Theorem 4.3 we needed to consider the estimates (4.2) and (4.3) for the averages $\sum_{\lambda_n \leq T} \lambda_n r_n$ and $\sum_{\lambda_n \leq T} (\lambda_n r_n)^2$. For obtaining a lower bound for $P(X(\underline{\theta}) \geq V)$ we will need to make suitable assumptions on the counting function

$$\tilde{N}(x) := \sum_{\substack{n \\ 1/r_n \leq x}} 1.$$

More precisely, we have the following theorem.

Theorem 4.7. *Let $\{r_n\}$ be a decreasing sequence of positive real numbers with the limit 0. Let \tilde{c}, \tilde{d} be real constants such that $\tilde{d} \neq -1$. Define*

$$\tilde{N}(x) := \sum_{\substack{n \\ 1/r_n \leq x}} 1.$$

Assume

$$\tilde{N}(x) = \tilde{c}x(\log x)^{\tilde{d}} + o\left(x(\log x)^{\tilde{d}}\right). \quad (4.5)$$

Then for any $\epsilon > 0$ and for large enough $V > 0$

$$P(X(\underline{\theta}) \geq V) \geq \frac{1}{2} \exp \left(- C' V \exp \left(\left(\frac{\tilde{d} + 1}{\tilde{c}} V \right)^{\frac{1}{\tilde{d}+1}} (1 + o(1)) \right) \right),$$

where

$$C' = \frac{(\tilde{d} + 1)(1 + \epsilon)}{(1 - \epsilon)}.$$

Proof. In Lemma 4.4(i) take $\alpha_n = r_n^{-1}$ and $c_n = 1$. Then by (4.5) and Lemma 4.5(i) we have

$$\begin{aligned} \sum_{\substack{n \\ 1/r_n \leq x}} r_n &= \int_{1/r_1}^x \frac{\tilde{N}(t)}{t^2} dt + \frac{\tilde{N}(x)}{x} \\ &= \int_{1/r_1}^x \left(\frac{\tilde{c}(\log t)^{\tilde{d}}}{t} + o\left(\frac{(\log t)^{\tilde{d}}}{t}\right) \right) dt \\ &\quad + \tilde{c}(\log x)^{\tilde{d}} + o\left((\log x)^{\tilde{d}}\right) \\ &= \frac{\tilde{c}}{\tilde{d} + 1} (\log x)^{\tilde{d}+1} + o\left((\log x)^{\tilde{d}+1}\right). \end{aligned} \tag{4.6}$$

Hence from (4.5) and (4.6) we see that

$$\begin{aligned} \sum_{\substack{n \\ r_n > \delta}} (r_n - \delta) &= \sum_{\substack{n \\ \lambda_n < 1/\delta}} r_n - \delta \tilde{N}(1/\delta) \\ &= \frac{\tilde{c}}{\tilde{d} + 1} \left(\log \frac{1}{\delta} \right)^{\tilde{d}+1} + o\left(\left(\log \frac{1}{\delta} \right)^{\tilde{d}+1} \right). \end{aligned}$$

This is equivalent to

$$\sum_{\substack{n \\ r_n > \delta}} (r_n - \delta) \sim \frac{\tilde{c}}{\tilde{d} + 1} \left(\log \frac{1}{\delta} \right)^{\tilde{d}+1}$$

as $\delta \rightarrow 0^+$. Hence given $\epsilon > 0$ we can choose δ small enough such that

$$\sum_{\substack{n \\ r_n > \delta}} (r_n - \delta) \geq \frac{\tilde{c}}{\tilde{d} + 1} \left(\log \frac{1}{\delta} \right)^{\tilde{d}+1} (1 - \epsilon). \tag{4.7}$$

For $V > 0$ we choose $\delta = \delta(V, \epsilon)$ such that

$$\frac{\tilde{c}}{\tilde{d} + 1} \left(\log \frac{1}{\delta} \right)^{\tilde{d}+1} (1 - \epsilon) = V,$$

or equivalently,

$$\delta = \exp \left(- \left(\frac{\tilde{d} + 1}{\tilde{c}(1 - \epsilon)} V \right)^{\frac{1}{\tilde{d}+1}} \right).$$

For such δ by (4.7) we have

$$\sum_{\substack{n \\ r_n > \delta}} (r_n - \delta) \geq V.$$

The condition $r_n \rightarrow 0$ implies that there exists a constant $c_3 > 0$ such that $r_n \leq c_3$ for all n . Thus by Theorem 4.3(ii) we have

$$\begin{aligned} P(X(\underline{\theta}) \geq V) &\geq \frac{1}{2} \exp \left(- \frac{1}{2} \sum_{\substack{n \\ r_n > \delta}} \log \left(\frac{\pi^2 r_n}{2\delta} \right) \right) \\ &\geq \frac{1}{2} \exp \left(- \frac{1}{2} \sum_{1/r_n < 1/\delta} \log \left(\frac{\pi^2 c_3}{2\delta} \right) \right) \\ &= \frac{1}{2} \exp \left(- \frac{1}{2} \left(\log \frac{\pi^2 c_3}{2\delta} \right) \tilde{N} \left(\frac{1}{\delta} \right) \right) \\ &= \frac{1}{2} \exp \left(- \frac{\tilde{c}}{2} \left(\log \frac{\pi^2 c_3}{2\delta} \right) \left(\log \frac{1}{\delta} \right)^{\tilde{d}} \frac{1}{\delta} (1 + o(1)) \right) \\ &\geq \frac{1}{2} \exp \left(- \tilde{c} \left(\frac{\tilde{d} + 1}{\tilde{c}(1 - \epsilon)} V \right)^{\frac{1}{\tilde{d}+1}} \left(\frac{\tilde{d} + 1}{\tilde{c}(1 - \epsilon)} V \right)^{\frac{\tilde{d}}{\tilde{d}+1}} \exp \left(\left(\frac{\tilde{d} + 1}{\tilde{c}(1 - \epsilon)} V \right)^{\frac{1}{\tilde{d}+1}} \right) (1 + o(1)) \right) \\ &\geq \frac{1}{2} \exp \left(- C' V \exp \left(\left(\frac{\tilde{d} + 1}{\tilde{c}(1 - \epsilon)} V \right)^{\frac{1}{\tilde{d}+1}} \right) \right), \end{aligned}$$

where

$$C' = \frac{(\tilde{d} + 1)(1 + \epsilon)}{(1 - \epsilon)}.$$

Therefore we have

$$P(X(\underline{\theta}) \geq V) \geq \frac{1}{2} \exp \left(- C' V \exp \left(\left(\frac{\tilde{d} + 1}{\tilde{c}} V \right)^{\frac{1}{\tilde{d}+1}} \right) (1 + o(1)) \right).$$

This completes the proof. □

Combining Theorem 4.6 with Theorem 4.7 gives the following result.

Corollary 4.8. *Suppose that the assumptions of Theorem 4.6 and Theorem 4.7 for r_n and λ_n hold. Then for any $\epsilon > 0$ and for large $V > 0$ we have*

$$\begin{aligned} \frac{1}{2} \exp \left(- C' V \exp \left(\left(\frac{\tilde{d} + 1}{\tilde{c}} V \right)^{\frac{1}{\tilde{d}+1}} (1 + o(1)) \right) \right) \leq \\ P(X(\theta) \geq V) \leq \\ \exp \left(- C V^{2 - \frac{d_2}{d_1+1}} \exp \left(\left(\frac{d_1 + 1}{c_1} V \right)^{\frac{1}{d_1+1}} (1 + o(1)) \right) \right), \end{aligned}$$

where

$$C = \frac{3\epsilon^2(1-\epsilon)}{4c_2(1+\epsilon)^2} \left(\frac{c_1(1+\epsilon)^2}{d_1+1} \right)^{-\frac{d_2}{d_1+1}}$$

and

$$C' = \frac{(\tilde{d} + 1)(1 + \epsilon)}{(1 - \epsilon)}.$$

Proof. This is a direct consequence of Theorems 4.6 and 4.7. □

4.3 An Asymptotic Formula

In this section under a finer asymptotic assumption for the counting function

$$\tilde{N}(x) := \sum_{\substack{n \\ 1/r_n \leq x}} 1,$$

we will derive an asymptotic relation for $P(X \geq V)$. This work is based on the Laplace transform of the measure μ_X defined by

$$\mu_X([V, \infty)) := P(X \geq V).$$

The Laplace transform of μ_X is given by

$$L(s) := \int_{-\infty}^{\infty} e^{st} d\mu_X(t). \tag{4.8}$$

The idea is to first establish an asymptotic estimate for $L(s)$ for real s and then approximate $\mu_X([V, \infty))$. Throughout this section, we suppose the following assumptions for r_n .

1. The sequence $\{r_n\}$ is decreasing and approaches 0 and

$$\sum_{k=1}^{\infty} r_n^2 < \infty; \quad (4.9)$$

- 2.

$$\tilde{N}(x) = \tilde{c}x(\log x)^{\tilde{d}} + \tilde{k}x(\log x)^{\tilde{d}-1} + O\left(x(\log x)^{\tilde{d}-2}\right); \quad (4.10)$$

3. There exists positive constants c_4, c_5, d_3 such that

$$\exp\left(-\exp\left(c_4V^{d_3}\right)\right) \ll P\left(X(\underline{\theta}) \geq V\right) \ll \exp\left(-\exp\left(c_5V^{d_3}\right)\right). \quad (4.11)$$

We follow [16, Section 4].

For $t \in \mathbb{R}$, let

$$\Phi(t) := P\left(X(\underline{\theta}) \geq t\right).$$

Define

$$L(s) := \int_{-\infty}^{\infty} se^{st}\Phi(t)dt.$$

Note that by (4.11) this integral converges. Moreover, (4.11) implies that this definition for $L(s)$ matches up with (4.8), since $\Phi'(t) = -d\mu(t)$ and therefore we can write

$$\int_{-\infty}^{\infty} e^{st}d\mu(t) = -\lim_{x \rightarrow \infty} e^{sx}\Phi(x) + \lim_{x \rightarrow -\infty} e^{sx}\Phi(x) + \int_{-\infty}^{\infty} se^{st}\Phi(t)dt = \int_{-\infty}^{\infty} se^{st}\Phi(t)dt.$$

Let

$$I_0(z) = \sum_{n=0}^{\infty} \frac{(z/2)^{2n}}{(n!)^2} = \int_0^1 e^{z \cos(2\pi\theta)} d\theta$$

be the “modified Bessel function” of order 0. For $t \in \mathbb{R}$ define $h(t) = \log I_0(t)$ and

$$f(t) := \begin{cases} h(t) & \text{if } 0 \leq t < 1, \\ h(t) - t & \text{if } t \geq 1. \end{cases} \quad (4.12)$$

For $f(t)$ we have the following theorem.

Lemma 4.9. *The function h is a Lipschitz smooth function with Lipschitz constant 1. Moreover, we have*

$$f(t) = \begin{cases} O(t^2) & \text{if } 0 \leq t < 1, \\ O(\log t) & \text{if } t \geq 1. \end{cases}$$

Proof. Since $I_0(x)$ is smooth and does not vanish, $h(t)$ is also smooth and

$$|h'(t)| = \left| \frac{I_0'(t)}{I_0(t)} \right| = \left| \frac{\int_0^1 \cos(2\pi\theta) e^{t \cos(2\pi\theta)} d\theta}{\int_0^1 e^{t \cos(2\pi\theta)} d\theta} \right| \leq 1. \quad (4.13)$$

To prove the second assertion, note that

$$I_0(t) = \sum_{n=0}^{\infty} \frac{(t/2)^{2n}}{(n!)^2}.$$

Hence $I_0(t) = 1 + O(t^2)$ for $0 \leq t \leq 1$ and therefore $f(t) = O(t^2)$. When $t \geq 1$ we write

$$\frac{1}{10\pi t} \leq \frac{e^{t(\cos(1/t)-1)}}{2\pi t} \leq \int_0^{1/2\pi t} e^{t(\cos 2\pi\theta-1)} d\theta \leq \int_0^1 e^{t(\cos 2\pi\theta-1)} d\theta = \frac{I_0(t)}{e^t}$$

and

$$\frac{I_0(t)}{e^t} = \int_0^1 e^{t(\cos 2\pi\theta-1)} d\theta \leq 1.$$

Hence

$$-\log 10\pi t \leq \log I_0(t) - t \leq 0$$

which implies

$$f(t) = \log I_0(t) - t = O(\log t).$$

□

In Theorem 2.18, under the assumption of the linear independence of $\{\lambda_n\}$ over \mathbb{Q} , we found an explicit formula for the Fourier transform of μ_X . More precisely,

$$\hat{\mu}_X(\xi) = \prod_{n=1}^{\infty} J_0(r_n \xi), \quad (4.14)$$

where $J_0(z)$ is the Bessel function of order 0 given in (2.29). Our idea to estimate the tail of μ is based on the preceding fact that the Fourier transform $\hat{\mu}_X(\xi)$ of μ_X coincides with $L(i\xi)$. More precisely, we have the following result.

Proposition 4.10. *For all $s \in \mathbb{C}$ we have*

$$L(s) = \prod_{n=1}^{\infty} I_0(r_n s).$$

Proof. By (4.8), the definition of $\hat{\mu}_X$, and (4.14), for real ξ we have

$$L(i\xi) = \int_{-\infty}^{\infty} e^{i\xi t} d\mu(t) = \hat{\mu}(-\xi) = \prod_{n=1}^{\infty} J_0(-\xi r_n).$$

Hence since $J_0(\xi) = I_0(-i\xi)$ we get

$$L(i\xi) = \prod_{n=1}^{\infty} I_0(i\xi r_n). \tag{4.15}$$

Thus the infinite product $\prod_{n=1}^{\infty} I_0(sr_n)$ equals to $L(s)$ on the imaginary axis. Both sides of (4.15) are holomorphic functions, so by the identity theorem for holomorphic functions they are equal over the whole complex plane. This proves the lemma. \square

We will now find an asymptotic formula for $L(s)$ when s is real. This is a crucial step in the saddle point method.

Proposition 4.11. *Assume that $\tilde{d} < 2$. Then for $s \in \mathbb{R}$ with $|s| \geq 2$ we have*

$$L(s) = \exp \left(\frac{\tilde{c}}{\tilde{d} + 1} |s| (\log |s|)^{\tilde{d}+1} + A |s| (\log |s|)^{\tilde{d}} + O(|s|) \right), \tag{4.16}$$

where

$$A = \tilde{c} \left(1 + \int_0^{\infty} \frac{f(u)}{u^2} du \right) + \frac{\tilde{k}}{\tilde{d}}.$$

Proof. Since I_0 is an even function we may suppose that $s \geq 2$. By Proposition 4.10

and (4.12) we have

$$\begin{aligned}
\log L(s) &= \sum_{n=1}^{\infty} h(sr_n) \\
&= \sum_{\substack{n \\ 1/r_n \leq s}} sr_n + \sum_{n=1}^{\infty} f(sr_n) \\
&= S_1 + S_2,
\end{aligned} \tag{4.17}$$

where

$$S_1 = \sum_{\substack{n \\ 1/r_n \leq s}} sr_n, \quad S_2 = \sum_{n=1}^{\infty} f(sr_n).$$

For S_1 by Lemma 4.4(i) we have

$$\begin{aligned}
S_1 &= s \left(\frac{\tilde{N}(s)}{s} + \int_{1/r_1}^s \frac{\tilde{N}(t)}{t^2} dt \right) \\
&= \tilde{c}s(\log s)^{\tilde{d}} + \tilde{k}s(\log s)^{\tilde{d}-1} + \tilde{c}s \int_{1/r_1}^s \frac{(\log t)^{\tilde{d}}}{t} dt \\
&\quad + \tilde{k}s \int_{1/r_1}^s \frac{(\log t)^{\tilde{d}-1}}{t} dt + O\left(s(\log s)^{\tilde{d}-2}\right) \\
&= \frac{\tilde{c}}{\tilde{d}+1} s(\log s)^{\tilde{d}+1} + \left(\tilde{c} + \frac{\tilde{k}}{\tilde{d}} \right) s(\log s)^{\tilde{d}} + O(s).
\end{aligned} \tag{4.18}$$

Note that the error term is $O(s)$ since $\tilde{d} < 2$. For S_2 we have

$$\begin{aligned}
S_2 &= \lim_{t \rightarrow \infty} f\left(\frac{s}{t}\right) \tilde{N}(t) - \int_{1/r_1}^{\infty} \left(f\left(\frac{s}{t}\right)\right)' \tilde{N}(t) dt \\
&= - \int_{1/r_1}^{\infty} \left(f\left(\frac{s}{t}\right)\right)' \tilde{N}(t) dt,
\end{aligned}$$

the limit being equal to 0 by Lemma 4.9. Hence

$$\begin{aligned}
S_2 &= - \int_{1/r_1}^{\infty} \left(f\left(\frac{s}{t}\right) \right)' \left[\tilde{c}t(\log t)^{\tilde{d}} + \tilde{k}t(\log t)^{\tilde{d}-1} + O\left(t(\log t)^{\tilde{d}-2}\right) \right] dt \\
&= - \lim_{X \rightarrow \infty} \left[\tilde{c}f\left(\frac{s}{t}\right) t(\log t)^{\tilde{d}} \right]_{1/r_1}^X + \tilde{c} \int_{1/r_1}^{\infty} f\left(\frac{s}{t}\right) (\log t)^{\tilde{d}} dt \\
&\quad + O\left(s + \int_{1/r_1}^{\infty} \frac{s}{t^2} \left| f'\left(\frac{s}{t}\right) \right| (\log t)^{\tilde{d}-1} dt\right) \\
&= \tilde{c} \int_{1/r_1}^{\infty} f\left(\frac{s}{t}\right) (\log t)^{\tilde{d}} dt + O(s),
\end{aligned}$$

where we have used the fact that $|f'(t)| \leq 2$, which follows from (4.12) and (4.13). In the last integral we make the change of variable $u = s/t$, so

$$\begin{aligned}
S_2 &= \tilde{c}s \int_0^{sr_1} \frac{f(u)}{u^2} \left(\log \frac{s}{u} \right)^{\tilde{d}} du + O(s) \\
&= \tilde{c}s(\log s)^{\tilde{d}} \int_0^{\infty} \frac{f(u)}{u^2} du + O(s).
\end{aligned} \tag{4.19}$$

By combining (4.17), (4.18), and (4.19), we deduce the result. \square

We can now state the main theorem of this section which gives an asymptotic formula for $\Phi(V) = P(X \geq V)$.

Theorem 4.12. *Assume that (4.9), (4.10), and (4.11), hold ($0 \leq \tilde{d} < 2$), and that $\{\lambda_n\}$ is linearly independent over \mathbb{Q} . Then there exist functions $\epsilon_1(V), \epsilon_2(V)$ such that $\epsilon_i(V) \rightarrow 0$, as $V \rightarrow \infty$, for $i = 1, 2$, and*

$$\begin{aligned}
\Phi(V) &\geq \exp\left(-\tilde{c}\left(\frac{\tilde{d}+1}{\tilde{c}}V\right)^{\frac{\tilde{d}}{\tilde{d}+1}} \exp\left(-\frac{A+\tilde{c}}{\tilde{c}} + \left(\frac{\tilde{d}+1}{\tilde{c}}V\right)^{\frac{1}{\tilde{d}+1}}(1+\epsilon_1(V))\right)\right), \\
\Phi(V) &\leq \exp\left(-\tilde{c}\left(\frac{\tilde{d}+1}{\tilde{c}}V\right)^{\frac{\tilde{d}}{\tilde{d}+1}} \exp\left(-\frac{A+\tilde{c}}{\tilde{c}} + \left(\frac{\tilde{d}+1}{\tilde{c}}V\right)^{\frac{1}{\tilde{d}+1}}(1+\epsilon_2(V))\right)\right).
\end{aligned}$$

Proof. Put $x = \log s$, $s \geq 2$, and let s be such that

$$V = \frac{\tilde{c}}{\tilde{d}+1}x^{\tilde{d}+1} + (A+\tilde{c})x^{\tilde{d}}, \tag{4.20}$$

or equivalently,

$$x^{\tilde{d}+1} + \frac{(A + \tilde{c})(\tilde{d} + 1)}{\tilde{c}} x^{\tilde{d}} = \frac{V(\tilde{d} + 1)}{\tilde{c}}.$$

By the binomial theorem we have

$$\begin{aligned} \left(x + \frac{A + \tilde{c}}{\tilde{c}}\right)^{\tilde{d}+1} &= x^{\tilde{d}+1} + \frac{(A + \tilde{c})(\tilde{d} + 1)}{\tilde{c}} x^{\tilde{d}} + \frac{(A + \tilde{c})^2(\tilde{d} + 1)\tilde{d}}{2!\tilde{c}^2} x^{\tilde{d}-1} + \dots \\ &= \frac{V(\tilde{d} + 1)}{\tilde{c}} + \frac{(A + \tilde{c})^2(\tilde{d} + 1)\tilde{d}}{\tilde{c}^2} \left(\frac{x^{\tilde{d}-1}}{2!} + \frac{(A + \tilde{c})(\tilde{d} - 1)}{3!\tilde{c}} x^{\tilde{d}-2} + \dots \right). \end{aligned}$$

By using the binomial theorem we can show that

$$\frac{x^{\tilde{d}-1}}{2!} + \frac{(A + \tilde{c})(\tilde{d} - 1)}{3!\tilde{c}} x^{\tilde{d}-2} + \dots = O\left(x^{\tilde{d}-1}\right).$$

Hence

$$\left(x + \frac{A + \tilde{c}}{\tilde{c}}\right)^{\tilde{d}+1} = \frac{V(\tilde{d} + 1)}{\tilde{c}} + O\left(x^{\tilde{d}-1}\right).$$

It follows from (4.20) that $x^{\tilde{d}-1} \asymp V^{\frac{\tilde{d}-1}{\tilde{d}+1}}$ (see Definition 1.1 for meaning of “ \asymp ”). Hence

$$\left(x + \frac{A + \tilde{c}}{\tilde{c}}\right)^{\tilde{d}+1} = \frac{V(\tilde{d} + 1)}{\tilde{c}} + O\left(V^{\frac{\tilde{d}-1}{\tilde{d}+1}}\right).$$

This gives

$$x = \left(\frac{V(\tilde{d} + 1)}{\tilde{c}} + O\left(V^{\frac{\tilde{d}-1}{\tilde{d}+1}}\right) \right)^{\frac{1}{\tilde{d}+1}} - \frac{A + \tilde{c}}{\tilde{c}},$$

and therefore

$$\begin{aligned} s &= \exp \left(\left(\frac{V(\tilde{d} + 1)}{\tilde{c}} + O\left(V^{\frac{\tilde{d}-1}{\tilde{d}+1}}\right) \right)^{\frac{1}{\tilde{d}+1}} - \frac{A + \tilde{c}}{\tilde{c}} \right) \\ &= \exp \left(-\frac{A + \tilde{c}}{\tilde{c}} + \left(\frac{\tilde{d} + 1}{\tilde{c}} V \right)^{\frac{1}{\tilde{d}+1}} \left(1 + O\left(V^{\frac{-2}{\tilde{d}+1}}\right) \right) \right). \end{aligned} \quad (4.21)$$

Let s be given by (4.21) and consider $S_1 = s(1 + \epsilon)$, $S_2 = s(1 - \epsilon)$, $V_1 = V(1 + \delta)$, and

$V_2 = V(1 - \delta)$, where $\epsilon, \delta > 0$ will be chosen later. Define

$$I_1 := \int_{V_1}^{\infty} se^{st}\Phi(t)dt \quad \text{and} \quad I_2 := \int_{-\infty}^{V_2} se^{st}\Phi(t)dt.$$

Then

$$\begin{aligned} \frac{I_1}{L(s)} &= \frac{1}{(1 + \epsilon)L(s)} \int_{V_1}^{\infty} e^{-\epsilon st} S_1 e^{S_1 t} \Phi(t) dt \\ &\leq \exp(-\epsilon s V_1) \frac{L(S_1)}{(1 + \epsilon)L(s)} \\ &= \exp\left(-\epsilon s V - \epsilon s \delta V + \frac{\tilde{c}}{\tilde{d} + 1} \left(S_1 (\log S_1)^{\tilde{d}+1} - s (\log s)^{\tilde{d}+1} \right) \right. \\ &\quad \left. + A \left(S_1 (\log S_1)^{\tilde{d}} - s (\log s)^{\tilde{d}} \right) + O(s) \right). \end{aligned}$$

By our choice of V we can show, using the binomial theorem, that

$$\begin{aligned} \frac{\tilde{c}}{\tilde{d} + 1} \left(S_1 (\log S_1)^{\tilde{d}+1} - s (\log s)^{\tilde{d}+1} \right) \\ + A \left(S_1 (\log S_1)^{\tilde{d}} - s (\log s)^{\tilde{d}} \right) = \\ \epsilon s V + \tilde{c} s (\log s)^{\tilde{d}} \left((1 + \epsilon) \log(1 + \epsilon) - \epsilon \right) + O(s). \end{aligned}$$

Since $(1 + \epsilon) \log(1 + \epsilon) - \epsilon \leq \epsilon^2$, we conclude that

$$\frac{I_1}{L(s)} \leq \exp\left(-\epsilon s \delta V + \tilde{c} \epsilon^2 s (\log s)^{\tilde{d}} + O(s)\right).$$

We choose $\delta = 2\epsilon(\tilde{d} + 1)/\log s$ and $\epsilon = K(\log s)^{-\tilde{d}/2}$, for a sufficiently large constant

$K > 0$, so that

$$\begin{aligned} \frac{I_1}{L(s)} &\leq \exp\left(\frac{-2\epsilon^2 s(\tilde{d}+1)}{\log s}\left(\frac{\tilde{c}}{\tilde{d}+1}(\log s)^{\tilde{d}+1} + (A+\tilde{c})(\log s)^{\tilde{d}}\right) + \tilde{c}\epsilon^2 s(\log s)^{\tilde{d}} + O(s)\right) \\ &= \exp\left(-\tilde{c}\epsilon^2 s(\log s)^{\tilde{d}} - 2\epsilon^2 s(\tilde{d}+1)(A+\tilde{c})(\log s)^{\tilde{d}-1} + O(s)\right) \\ &= \exp\left(-\tilde{c}K^2 s - \frac{2K^2 s(\tilde{d}+1)(A+\tilde{c})}{\log s} + O(s)\right). \end{aligned}$$

Hence

$$\frac{I_1}{L(s)} \leq \exp(-Ms + Ns/\log s),$$

where $M > 0$ and N are constants. In a similar way we can show

$$\frac{I_2}{L(s)} \leq \exp(-M's + N's/\log s),$$

where $M' > 0$ and N' are constants. This implies that

$$\frac{I_1 + I_2}{L(s)} = o(s),$$

as $s \rightarrow \infty$. Now if we write

$$1 = \frac{I_1 + \int_{V_2}^{V_1} se^{st}\Phi(t)dt + I_2}{L(s)} = \frac{\int_{V_2}^{V_1} se^{st}\Phi(t)dt}{L(s)} + o(s),$$

then we conclude that

$$\int_{V_2}^{V_1} se^{st}\Phi(t)dt = L(s)(1 + o(s)), \tag{4.22}$$

as $s \rightarrow \infty$. By combining (4.16) and (4.22) we obtain

$$\begin{aligned}
\int_{V_2}^{V_1} se^{st}\Phi(t)dt &= \exp\left(\frac{\tilde{c}}{\tilde{d}+1}s(\log s)^{\tilde{d}+1} + As(\log s)^{\tilde{d}} + O(s)\right)(1+o(s)) \\
&= \exp\left(\frac{\tilde{c}}{\tilde{d}+1}s(\log s)^{\tilde{d}+1} + As(\log s)^{\tilde{d}} + O(s) + \log(1+o(s))\right) \\
&= \exp\left(\frac{\tilde{c}}{\tilde{d}+1}s(\log s)^{\tilde{d}+1} + As(\log s)^{\tilde{d}} + O(s) + o(s)\right) \\
&= \exp\left(\frac{\tilde{c}}{\tilde{d}+1}s(\log s)^{\tilde{d}+1} + As(\log s)^{\tilde{d}} + O(s)\right). \tag{4.23}
\end{aligned}$$

Since $\Phi(t)$ is non-increasing, we deduce that the left-hand side of (4.23) is

$$\geq \Phi(V_1) \int_{V_2}^{V_1} se^{st}\Phi(t)dt \quad \text{and} \quad \leq \Phi(V_2) \int_{V_2}^{V_1} se^{st}\Phi(t)dt. \tag{4.24}$$

The integrals in (4.24) are

$$\begin{aligned}
&= \exp(sV(1+\delta)) - \exp(sV(1-\delta)) \\
&= \exp(sV + O(s\delta V)) \\
&= \exp\left(sV + O\left(sV^{\frac{\tilde{d}}{2(\tilde{d}+1)}}\right)\right),
\end{aligned}$$

since $\delta \asymp (\log s)^{(-2-\tilde{d})/2} \asymp V^{(-2-\tilde{d})/2(\tilde{d}+1)}$. Hence by (4.23), (4.20), and (4.24), we deduce that

$$\Phi(V(1+\delta)) \leq \exp\left(-\tilde{c}s\left((\log s)^{\tilde{d}} + O\left(V^{\frac{\tilde{d}}{2(\tilde{d}+1)}}\right)\right)\right) \leq \Phi(V(1-\delta)). \tag{4.25}$$

Observe that

$$\begin{aligned}
(\log s)^{\tilde{d}} &= \left(-\frac{A+\tilde{c}}{\tilde{c}} + \left(\frac{\tilde{d}+1}{\tilde{c}} V + O(V^{\frac{\tilde{d}-1}{\tilde{d}+1}}) \right)^{\frac{1}{\tilde{d}+1}} \right)^{\tilde{d}} \\
&= \left(-\frac{A+\tilde{c}}{\tilde{c}} + \left(\frac{\tilde{d}+1}{\tilde{c}} V \right)^{\frac{1}{\tilde{d}+1}} \left(1 + O(V^{\frac{-2}{\tilde{d}+1}}) \right) \right)^{\tilde{d}} \\
&= \left(\frac{\tilde{d}+1}{\tilde{c}} V \right)^{\frac{\tilde{d}}{\tilde{d}+1}} \left(1 + O(V^{\frac{-1}{\tilde{d}+1}}) \right)^{\tilde{d}} \\
&= \left(\frac{\tilde{d}+1}{\tilde{c}} V \right)^{\frac{\tilde{d}}{\tilde{d}+1}} \left(1 + O(V^{\frac{-1}{\tilde{d}+1}}) \right),
\end{aligned}$$

and therefore

$$\begin{aligned}
(\log s)^{\tilde{d}} + O\left(V^{\frac{\tilde{d}}{2(\tilde{d}+1)}}\right) &= \left(\frac{\tilde{d}+1}{\tilde{c}} V \right)^{\frac{\tilde{d}}{\tilde{d}+1}} \left(1 + O(V^{\frac{-1}{\tilde{d}+1}}) + O(V^{\frac{-\tilde{d}}{2(\tilde{d}+1)}}) \right) \\
&= \left(\frac{\tilde{d}+1}{\tilde{c}} V \right)^{\frac{\tilde{d}}{\tilde{d}+1}} \left(1 + O(V^{\frac{-1}{\tilde{d}+1}} + V^{\frac{-\tilde{d}}{2(\tilde{d}+1)}}) \right).
\end{aligned}$$

Thus by (4.21)

$$\begin{aligned}
-\tilde{c}s \left((\log s)^{\tilde{d}} + O\left(V^{\frac{\tilde{d}}{2(\tilde{d}+1)}}\right) \right) &= \\
&= -\tilde{c} \left(\frac{\tilde{d}+1}{\tilde{c}} V \right)^{\frac{\tilde{d}}{\tilde{d}+1}} \left(1 + O(V^{\frac{-1}{\tilde{d}+1}} + V^{\frac{-\tilde{d}}{2(\tilde{d}+1)}}) \right) \times \\
&\quad \exp \left(-\frac{A+\tilde{c}}{\tilde{c}} + \left(\frac{\tilde{d}+1}{\tilde{c}} V \right)^{\frac{1}{\tilde{d}+1}} \left(1 + O\left(V^{\frac{-2}{\tilde{d}+1}}\right) \right) \right) = (\star).
\end{aligned}$$

By substituting the estimate (\star) in (4.25) we obtain

$$\Phi(V(1+\delta)) \leq \exp(\star) \leq \Phi(V(1-\delta)). \tag{4.26}$$

We now replace $V(1+\delta)$ by V in the lower inequality and $V(1-\delta)$ by V in the upper

inequality of (4.26) and note that

$$\frac{V}{1+\delta} = V(1+O(\delta)) = V\left(1+O\left(V^{\frac{-2-\tilde{d}}{2(\tilde{d}+1)}}\right)\right)$$

and

$$\frac{V}{1-\delta} = V(1+O(\delta)) = V\left(1+O\left(V^{\frac{-2-\tilde{d}}{2(\tilde{d}+1)}}\right)\right).$$

It follows that there exist functions $\epsilon_1(V), \epsilon_2(V), \epsilon_3(V), \epsilon_4(V)$ such that $\epsilon_i(V) \rightarrow 0$, as $V \rightarrow \infty$, for $i = 1, \dots, 4$, and

$$\Phi(V) \geq \exp\left(-\tilde{c}\left(\frac{\tilde{d}+1}{\tilde{c}}V\right)^{\frac{\tilde{d}}{\tilde{d}+1}}(1+\epsilon_3(V)) \exp\left(-\frac{A+\tilde{c}}{\tilde{c}} + \left(\frac{\tilde{d}+1}{\tilde{c}}V\right)^{\frac{1}{\tilde{d}+1}}(1+\epsilon_1(V))\right)\right),$$

$$\Phi(V) \leq \exp\left(-\tilde{c}\left(\frac{\tilde{d}+1}{\tilde{c}}V\right)^{\frac{\tilde{d}}{\tilde{d}+1}}(1+\epsilon_4(V)) \exp\left(-\frac{A+\tilde{c}}{\tilde{c}} + \left(\frac{\tilde{d}+1}{\tilde{c}}V\right)^{\frac{1}{\tilde{d}+1}}(1+\epsilon_2(V))\right)\right).$$

Finally, observe that

$$1 + o(1) = e^{\log(1+o(1))} = e^{o(1)},$$

as $V \rightarrow \infty$. Thus we can ignore the factors $(1 + \epsilon_3(V))$ and $(1 + \epsilon_4(V))$ to deduce the result. \square

It is worth noting that Theorem 4.12 gives evidence that the lower bound that we found in Corollary 4.8 is perhaps sharp. In fact, in both problems the coefficients and the powers of V in the double exponential factor came from the asymptotic relation that we considered for $\tilde{N}(x)$.

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