

## Proposal for testing quantum gravity in the lab

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Attempts to formulate a quantum theory of gravitation are collectively known as quantum gravity. Various approaches to quantum gravity such as string theory and loop quantum gravity, as well as black hole physics and doubly special relativity theories predict a minimum measurable length, or a maximum observable momentum, and related modifications of the Heisenberg Uncertainty Principle to a so-called generalized uncertainty principle (GUP). We have proposed a GUP consistent with string theory, black hole physics, and doubly special relativity theories and have showed that this modifies all quantum mechanical Hamiltonians. When applied to an elementary particle, it suggests that the space that confines it must be quantized, and in fact that all measurable lengths are quantized in units of a fundamental length (which can be the Planck length). On the one hand, this may signal the breakdown of the spacetime continuum picture near that scale, and on the other hand, it can predict an upper bound on the quantum gravity parameter in the GUP, from current observations. Furthermore, such fundamental discreteness of space may have observable consequences at length scales much larger than the Planck scale. Because this influences all the quantum Hamiltonians in an universal way, it predicts quantum gravity corrections to various quantum phenomena. Therefore, in the present work we compute these corrections to the Lamb shift, simple harmonic oscillator, Landau levels, and the tunneling current in a scanning tunneling microscope.

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### I. INTRODUCTION

An intriguing prediction of various theories of quantum gravity (such as string theory) and black hole physics is the existence of a minimum measurable length. This has given rise to the so-called generalized uncertainty principle (GUP) or equivalently, modified commutation relations between position coordinates and momenta. The recently proposed doubly special relativity (DSR) theories on the other hand, also suggest a similar modification of commutators. The commutators that are consistent with string theory, black holes physics, DSR, and which ensure  $[x_i, x_j] = 0 = [p_i, p_j]$  (via the Jacobi identity) have the following form [1] (see Appendix)<sup>1</sup>

$$[x_i, p_j] = i\hbar \left( \delta_{ij} - \alpha \left( p \delta_{ij} + \frac{p_i p_j}{p} \right) + \alpha^2 (p^2 \delta_{ij} + 3p_i p_j) \right), \quad (1)$$

where  $\alpha = \alpha_0/M_{\text{Pl}}c = \alpha_0 \ell_{\text{Pl}}/\hbar$ ,  $M_{\text{Pl}}$  = Planck mass,  $\ell_{\text{Pl}} \approx 10^{-35}$  m = Planck length, and  $M_{\text{Pl}}c^2 =$  Planck energy  $\approx 10^{19}$  GeV.

In one dimension, Eq. (1) gives to  $\mathcal{O}(\alpha^2)$

$$\begin{aligned} \Delta x \Delta p &\geq \frac{\hbar}{2} [1 - 2\alpha \langle p \rangle + 4\alpha^2 \langle p^2 \rangle] \\ &\geq \frac{\hbar}{2} \left[ 1 + \left( \frac{\alpha}{\sqrt{\langle p^2 \rangle}} + 4\alpha^2 \right) \Delta p^2 \right. \\ &\quad \left. + 4\alpha^2 \langle p \rangle^2 - 2\alpha \sqrt{\langle p^2 \rangle} \right]. \end{aligned} \quad (2)$$

Commutators and inequalities similar to (1) and (2) were proposed and derived, respectively, in [2–9]. These in turn imply a minimum measurable length and a maximum measurable momentum—the latter following from the assumption that  $\Delta p$  characterizes the maximum momentum of a particle as well [10], and also from the fact that DSR predicts such a maximum (to the best of our knowledge, (1) and (2) and (2) are the only forms which imply both

$$\Delta x \geq (\Delta x)_{\min} \approx \alpha_0 \ell_{\text{Pl}}, \quad (3)$$

$$\Delta p \leq (\Delta p)_{\max} \approx \frac{M_{\text{Pl}}c}{\alpha_0}. \quad (4)$$

Next, defining (see Appendix)

$$x_i = x_{0i}, \quad p_i = p_{0i}(1 - \alpha p_0 + 2\alpha^2 p_0^2), \quad (5)$$

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<sup>1</sup>The results of this article do not depend on this particular form of GUP chosen and continue to hold for a large class of variants, so long as an  $\mathcal{O}(\alpha)$  term is present in the right-hand side of Eq. (1).

with  $x_{0i}$ ,  $p_{0j}$  satisfying the canonical commutation relations  $[x_{0i}, p_{0j}] = i\hbar\delta_{ij}$ , it can be shown that Eq. (1) is satisfied. Here,  $p_{0i}$  can be interpreted as the momentum at low energies (having the standard representation in position space, i.e.  $p_{0i} = -i\hbar\partial/\partial x_{0i}$ ) and  $p_i$  as that at higher energies.

It is normally assumed that the dimensionless parameter  $\alpha_0$  is of the order of unity, in which case the  $\alpha$  dependent terms are important only when energies (momenta) are comparable to the Planck energy (momentum), and lengths are comparable to the Planck length. However, if we do not impose this condition *a priori*, then this may signal the existence of a new physical length scale of the order of  $\alpha\hbar = \alpha_0\ell_{\text{Pl}}$ . Evidently, such an intermediate length scale cannot exceed the electroweak length scale  $\sim 10^{17}\ell_{\text{Pl}}$  (as otherwise it would have been observed) and this implies that  $\alpha_0 \leq 10^{17}$ .

Using Eq. (5), a Hamiltonian of the form

$$H = \frac{p^2}{2m} + V(\vec{r}) \quad (6)$$

can be written as

$$H = H_0 + H_1 + \mathcal{O}(\alpha^3), \quad (7)$$

where

$$H_0 = \frac{p_0^2}{2m} + V(\vec{r}), \quad (8)$$

and

$$H_1 = -\frac{\alpha}{m}p_0^3 + \frac{5\alpha^2}{2m}p_0^4. \quad (9)$$

Thus, we see that *any* system with a well-defined quantum (or even classical) Hamiltonian  $H_0$  is perturbed by  $H_1$ , defined above, near the Planck scale. Such corrections extend to relativistic systems as well [11], and given the robust nature of GUP, will continue to play a role irrespective of what other quantum gravity corrections one may consider. In other words, they are in some sense universal.

The relativistic Dirac equation is modified in a similar way and confirms the main results of our paper [11]. In this paper, we first study the effects of the above GUP-corrected Hamiltonian to a particle in a box, to  $\mathcal{O}(\alpha)$  in Sec. IA, and to  $\mathcal{O}(\alpha^2)$  in Sec. IB, and show that they lead to virtually identical conclusions. In Sec. II, we study the effects of GUP-corrected Hamiltonian to the Landau levels. In Sec. III, we calculate the corrections due to GUP in the context of a simple harmonic oscillator. In Sec. IV, we study the effects of GUP on the Lamb shift. Furthermore, we compute the GUP corrections on the tunneling current in a scanning tunneling microscope for a step potential in Sec. V and for a potential barrier in Sec. VI. Finally, we summarize our results in the concluding section.

## A. Solution to order $\alpha$

In this subsection, we briefly review our work in [1]. The wave function of the particle satisfies the following GUP-corrected Schrödinger equation inside the box of length  $L$  (with boundaries at  $x = 0$  and  $x = L$ ), where  $V(\vec{r}) = 0$  (outside,  $V = \infty$  and  $\psi = 0$ )

$$H\psi = E\psi, \quad (10)$$

which is now written, to order  $\alpha$ , as

$$d^2\psi + k^2\psi + 2i\alpha\hbar d^3\psi = 0, \quad (11)$$

where  $d^n$  stands for  $d^n/dx^n$  and  $k = \sqrt{2mE/\hbar^2}$ . A trial solution of the form  $\psi = e^{mx}$  yields

$$m^2 + k^2 + 2i\alpha\hbar m^3 = 0 \quad (12)$$

with the following solution set to leading order in  $\alpha$ :  $m = \{ik', -ik'', i/2\alpha\hbar\}$ , where  $k' = k(1 + k\alpha\hbar)$  and  $k'' = k(1 - k\alpha\hbar)$ . Thus, the general wave function to leading order in  $\ell_{\text{Pl}}$  and  $\alpha$  is of the form

$$\psi = Ae^{ik'x} + Be^{-ik''x} + Ce^{ix/2\alpha\hbar}. \quad (13)$$

Although the first two terms can be considered as perturbative corrections over the standard solutions, the appearance of the new oscillatory third term is noteworthy here, with characteristic wavelength  $4\pi\alpha\hbar$  and momentum  $1/4\alpha = M_{\text{Pl}}c/4\alpha_0$  [which is Planckian for  $\alpha_0 = \mathcal{O}(1)$ ]. This can be termed a nonperturbative solution as the exponent contains  $1/\alpha$  and results in the new quantization mentioned above. Note that, however, as explained in [1,11],  $C$  scales as a power of  $\alpha$ , and the new solution disappears in the  $\alpha \rightarrow 0$  limit.

Imposing the appropriate boundary conditions, i.e.  $\psi = 0$  at  $x = 0, L$ , with  $A$  assumed real without loss of generality, we get, to leading order, the following two series of solutions ( $C = |C|e^{-i\theta_C}$ ):

$$\frac{L}{2\alpha\hbar} = \frac{L}{2\alpha_0\ell_{\text{Pl}}} = n\pi + 2q\pi + 2\theta_C \equiv p\pi + 2\theta_C, \quad (14)$$

$$\frac{L}{2\alpha\hbar} = \frac{L}{2\alpha_0\ell_{\text{Pl}}} = -n\pi + 2q\pi \equiv p\pi, \quad (15)$$

$$p \equiv 2q \pm n \in \mathbb{N}.$$

These show that there cannot be even be a single particle in the box, unless its length is quantized as above. For other lengths, there is no way to probe or measure the box, even if it exists. Hence, effectively all measurable lengths are quantized in units of  $\alpha_0\ell_{\text{Pl}}$ . We interpret this as space essentially having a discrete nature. Note that the above conclusion holds for any unknown but *fixed*  $\theta_C$ , which, however, determines the minimum measurable length, if any. It is hoped that additional physically motivated or consistency conditions will eventually allow one to either determine or at least put reasonable bounds on it.

The minimum length is  $\approx \alpha_0 \ell_{\text{Pl}}$  in each case. Once again, if  $\alpha_0 \approx 1$ , this fundamental unit is the Planck length. However, current experiments do not rule out discreteness smaller than about a thousandth of a Fermi, thus predicting the previously mentioned bound on  $\alpha_0$ . Note that similar quantization of length was shown in the context of loop quantum gravity in [12].

### B. Solution to order $\alpha^2$

We extend the previous solution to include the  $\alpha^2$  term in one dimension. Working to  $\mathcal{O}(\alpha^2)$ , the magnitude of the momentum at high energies as given by Eq. (5) reads

$$p = p_0(1 - \alpha p_0 + 2\alpha^2 p_0^2). \quad (16)$$

The wave function satisfies the following GUP-corrected Schrödinger equation:

$$d^2\psi + k^2\psi + 2i\hbar\alpha d^3\psi - 5\hbar^2\alpha^2 d^4\psi = 0, \quad (17)$$

where  $k = \sqrt{2mE/\hbar^2}$  and  $d^n \equiv d^n/dx^n$ . Substituting  $\psi(x) = e^{mx}$ , we obtain

$$m^2 + k^2 + 2i\alpha\hbar m^3 - 5(\alpha\hbar)^2 m^4 = 0, \quad (18)$$

with the following solution set to leading order in  $\alpha^2$ :  $m = \{ik', -ik'', \frac{2+i}{5\alpha\hbar}, \frac{-2+i}{5\alpha\hbar}\}$ , where  $k' = k(1 + k\alpha\hbar)$  and  $k'' = k(1 - k\alpha\hbar)$ . Thus, the most general solution to leading order in  $\ell_{\text{Pl}}^2$  and  $\alpha^2$  is of the form

$$\psi(x) = Ae^{ik'x} + Be^{-ik''x} + Ce^{(2+i)x/5\alpha\hbar} + De^{(-2+i)x/5\alpha\hbar}. \quad (19)$$

Note again the appearance of new oscillatory terms, with characteristic wavelength  $10\pi\alpha\hbar$ , which as before, by virtue of  $C$  and  $D$  scaling as a power of  $\alpha$ , disappear in the  $\alpha \rightarrow 0$  limit. In addition, we absorb any phase of  $A$  in  $\psi$  so as  $A$  to be real. The boundary condition

$$\psi(0) = 0 \quad (20)$$

implies

$$A + B + C + D = 0 \quad (21)$$

and hence the general solution given in Eq. (19) becomes

$$\psi(x) = 2iA \sin(kx)e^{i\alpha k^2 \hbar x} - (C + D)e^{-ik''x} + e^{ix/5\alpha\hbar}[Ce^{2x/5\alpha\hbar} + De^{(-2x)/5\alpha\hbar}]. \quad (22)$$

If we now combine Eq. (22) and the remaining boundary condition

$$\psi(L) = 0, \quad (23)$$

we get

$$2iA \sin(kL) = (C + D)e^{-i[\alpha k^2 \hbar L + k''L]} - [Ce^{2L/5\alpha\hbar} + De^{(-2L)/5\alpha\hbar}]e^{iL/5\alpha\hbar}e^{-i\alpha k^2 \hbar L}. \quad (24)$$

We can consider the exponentials  $e^{-i\alpha k^2 \hbar L} \approx 1$ , otherwise, since they are multiplied with  $C$  or  $D$ , terms of higher order

in  $\alpha$  will appear. Therefore, we have ( $C = |C|e^{-i\theta_C}$ ,  $D = |D|e^{-i\theta_D}$ )

$$2iA \sin(kL) = [|C|e^{-i\theta_C} + |D|e^{-i\theta_D}]e^{-ikL} - [|C|e^{-i\theta_C}e^{2L/5\alpha\hbar} + |D|e^{-i\theta_D}e^{(-2L)/5\alpha\hbar}]e^{iL/5\alpha\hbar}. \quad (25)$$

Now, equating the real parts of Eq. (25) (remembering that  $A \in \mathbb{R}$ ), we have

$$0 = |C| \cos(\theta_C + kL) + |D| \cos(\theta_D + kL) - e^{2L/5\alpha\hbar}|C| \cos\left(\theta_C - \frac{L}{5\alpha\hbar}\right) - e^{(-2L)/5\alpha\hbar}|D| \cos\left(\theta_D - \frac{L}{5\alpha\hbar}\right). \quad (26)$$

Note that the third term in the right-hand side dominates over the other terms in the limit  $\alpha \rightarrow 0$ . Thus we arrive at the following equation to leading order:

$$\cos(L/5\alpha\hbar - \theta_C) = 0. \quad (27)$$

This implies the quantization of the space by the following equation:

$$\frac{L}{5\alpha\hbar} = (2p + 1)\frac{\pi}{2} + \theta_C, \quad p \in \mathbb{N}. \quad (28)$$

Once again, even though the  $\alpha^2$  term has been included, the space quantization given in Eq. (28) suggests that the dimension of the box, and hence all measurable lengths are quantized in units of  $\alpha_0 \ell_{\text{Pl}}$ , and if  $\alpha_0 \approx 1$ , this fundamental unit is of the order of Planck length. And as before, the yet undetermined constant  $\theta_C$  determines the minimum measurable length.

## II. THE LANDAU LEVELS

Consider a particle of mass  $m$  and charge  $e$  in a constant magnetic field  $\vec{B} = B\hat{z}$ , described by the vector potential  $\vec{A} = Bx\hat{y}$  and the Hamiltonian

$$H_0 = \frac{1}{2m}(\vec{p}_0 - e\vec{A})^2 \quad (29)$$

$$= \frac{p_{0x}^2}{2m} + \frac{p_{0y}^2}{2m} - \frac{eB}{m}xp_{0y} + \frac{e^2B^2}{2m}x^2. \quad (30)$$

Since  $p_{0y}$  commutes with  $H$ , replacing it with its eigenvalue  $\hbar k$ , we get

$$H_0 = \frac{p_{0x}^2}{2m} + \frac{1}{2}m\omega_c^2\left(x - \frac{\hbar k}{m\omega_c}\right)^2, \quad (31)$$

where  $\omega_c = eB/m$  is the cyclotron frequency. This is nothing but the Hamiltonian of a harmonic oscillator in the  $x$  direction, with its equilibrium position given by  $x_0 \equiv \hbar k/m\omega_c$ . Consequently, the eigenfunctions and eigenvalues are given, respectively, by

$$\psi_{k,n}(x, y) = e^{iky} \phi_n(x - x_0), \quad (32)$$

$$E_n = \hbar\omega_c(n + \frac{1}{2}), \quad n \in \mathbb{N}, \quad (33)$$

where  $\phi_n$  are the harmonic oscillator wave functions.

The GUP-corrected Hamiltonian assumes the form [9]

$$\begin{aligned} H &= \frac{1}{2m}(\vec{p}_0 - e\vec{A})^2 - \frac{\alpha}{m}(\vec{p}_0 - e\vec{A})^3 + \frac{5\alpha^2}{2m}(\vec{p}_0 - e\vec{A})^4 \\ &= H_0 - \sqrt{8m\alpha}H_0^{3/2} + 10\alpha^2mH_0^2, \end{aligned} \quad (34)$$

where in the last step we have used Eq. (29). Evidently, the eigenfunctions remain unchanged. However, the eigenvalues are shifted by

$$\begin{aligned} \Delta E_{n(\text{GUP})} &= \langle \phi_n | -\sqrt{8m\alpha}H_0^{3/2} + 6\alpha^2mH_0^2 | \phi_n \rangle \\ &= -\sqrt{8m\alpha}(\hbar\omega_c)^{3/2}(n + \frac{1}{2})^{3/2} \\ &\quad + 10m\alpha^2(\hbar\omega_c)^2(n + \frac{1}{2})^2, \end{aligned} \quad (35)$$

which can be written as

$$\begin{aligned} \frac{\Delta E_{n(\text{GUP})}}{E_n^{(0)}} &= -\sqrt{8m\alpha}(\hbar\omega_c)^{1/2}\left(n + \frac{1}{2}\right)^{1/2} \\ &\quad + 10m\alpha^2(\hbar\omega_c)\left(n + \frac{1}{2}\right). \end{aligned} \quad (36)$$

For  $n = 1$ , we obtain the following relation

$$\frac{\Delta E_{1(\text{GUP})}}{E_1^{(0)}} = -\frac{\sqrt{12m}(\hbar\omega_c)^{1/2}}{M_{\text{Pl}}c}\alpha_0 + \frac{15m(\hbar\omega_c)}{M_{\text{Pl}}^2c^2}\alpha_0^2. \quad (37)$$

For an electron in a magnetic field of 10 T,  $\omega_c \approx 10^3$  GHz

$$\frac{\Delta E_{1(\text{GUP})}}{E_1^{(0)}} \approx -10^{-26}\alpha_0 + 10^{-52}\alpha_0^2. \quad (38)$$

Thus, quantum gravity/GUP does affect the Landau levels. However, once again, assuming  $\alpha_0 \sim 1$  renders the correction too small to be measured. Without this assumption, due to an accuracy of one part in  $10^3$  in direct measurements of Landau levels using a scanning tunnel microscope (STM) (which is somewhat optimistic) [13], the upper bound on  $\alpha_0$  becomes

$$\alpha_0 < 10^{23}. \quad (39)$$

Note that this is more stringent than the one derived in previous works [9].

### III. SIMPLE HARMONIC OSCILLATOR

We now consider a particle of mass  $m$ . The Hamiltonian of the simple harmonic oscillator with the GUP-corrected Hamiltonian assumes the form

$$H = H_0 + H_1 = \frac{p_0^2}{2m} + \frac{1}{2}m\omega^2x^2 - \frac{\alpha}{m}p_0^3 + \frac{5\alpha^2}{2}p_0^4. \quad (40)$$

Employing time-independent perturbation theory, the eigenvalues are shifted up to the first order of  $\alpha$  by

$$\Delta E_{\text{GUP}} = \langle \psi_n | H_1 | \psi_n \rangle, \quad (41)$$

where  $\psi_n$  are the eigenfunctions of the simple harmonic oscillator and are given by

$$\psi_n(x) = \left(\frac{1}{2^n n!}\right)^{1/2} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-(m\omega x^2/2\hbar)} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right), \quad (42)$$

where

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \quad (43)$$

are the Hermite polynomials.

The  $p_0^3$  term will not make any contribution to first order because it is an odd function and thus, it gives a zero by integrating over a Gaussian integral. On the other hand, the  $p_0^4$  term will make a nonzero contribution to first order. The contribution of the  $p_0^4$  term to first order is given by

$$\Delta E_{0(\text{GUP})}^{(1)} = \frac{5\alpha^2}{2m} \langle \psi_0 | \hbar^4 \frac{d^4}{dx^4} | \psi_0 \rangle, \quad (44)$$

and thus we get

$$\begin{aligned} \Delta E_{0(\text{GUP})}^{(1)} &= \frac{5\alpha^2 \hbar^4}{2m} \left(\frac{\gamma}{\pi}\right)^{1/2} \gamma^2 \\ &\quad \times \int dx e^{-\gamma x^2} (3 - 6\gamma x^2 + \gamma^2 x^4), \end{aligned} \quad (45)$$

where  $\gamma$  is equal to  $\frac{m\omega}{\hbar}$ .

By integrating, we get the shift of the energy to first order of perturbation as follows:

$$\Delta E_0^{(1)} = \frac{15}{8} \hbar^2 \omega^2 m \alpha^2, \quad (46)$$

or, equivalently,

$$\frac{\Delta E_0^{(1)}}{E_0^{(0)}} = \frac{15}{4} \hbar \omega m \alpha^2. \quad (47)$$

We now compute the contribution of the  $p^3$  term to second order of perturbation

$$\Delta E_n^{(2)} = \sum_{k \neq n} \frac{|\langle \psi_k | V_1 | \psi_n \rangle|^2}{E_n^{(0)} - E_k^{(0)}}, \quad (48)$$

where

$$V_1 = i \frac{\alpha}{m} \hbar^3 \frac{d^3}{dx^3}. \quad (49)$$

In particular, we are interested in computing the shift in the ground state energy to second order

$$\Delta E_0^{(2)} = \sum_{k \neq n} \frac{|\langle \psi_k | V_1 | \psi_0 \rangle|^2}{E_0^{(0)} - E_k^{(0)}}, \quad (50)$$

and for this reason we employ the following properties of the harmonic oscillator eigenfunctions

$$\langle \psi_m | x | \psi_n \rangle = \begin{cases} 0, & m \neq n \pm 1 \\ \sqrt{\frac{n+1}{2\gamma}}, & m = n + 1 \\ \sqrt{\frac{n}{2\gamma}}, & m = n - 1 \end{cases} \quad (51)$$

and

$$\langle \psi_m | x^3 | \psi_0 \rangle = \sum_{k,l} \langle \psi_m | x | \psi_k \rangle \langle \psi_k | x | \psi_l \rangle \langle \psi_l | x | \psi_0 \rangle, \quad (52)$$

which is nonvanishing for the  $(l, k, m)$  triplets:  $(1, 0, 1)$ ,  $(1, 2, 1)$ , and  $(1, 2, 3)$ .

Thus, the ground state energy is shifted by

$$\Delta E_0^{(2)} = \frac{\alpha^2 \hbar^6}{m^2} \sum_{m \neq 0} \frac{|\langle \psi_m | \frac{d^3}{dx^3} | \psi_0 \rangle|^2}{E_0^{(0)} - E_m^{(0)}}. \quad (53)$$

Since the eigenfunction  $|\psi_0\rangle = (\frac{m\omega}{\pi\hbar})^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$ , we have  $\frac{d^3}{dx^3} |\psi_0\rangle = (3\gamma^2 x - \gamma^3 x^3) |\psi_0\rangle$ . By employing these into Eq. (50), we get

$$\Delta E_0^{(2)} = \frac{\alpha^2 \hbar^6}{m^2} \gamma^4 \sum_{m \neq 0} \frac{|\langle \psi_m | (3x - \gamma x^3) | \psi_0 \rangle|^2}{E_0^{(0)} - E_m^{(0)}}. \quad (54)$$

Using Eqs. (51) and (52), the energy shift finally takes the form

$$\Delta E_0^{(2)} = -\frac{11}{2} \alpha^2 m (E_0^{(0)})^2, \quad (55)$$

or, equivalently,

$$\frac{\Delta E_0^{(2)}}{E_0^{(0)}} = -\frac{11}{2} \alpha^2 m E_0^{(0)} = -\frac{11}{4} \alpha^2 m \hbar \omega. \quad (56)$$

It is noteworthy that there are some systems that can be represented by the Harmonic oscillator such as heavy meson systems like charmonium [14]. The charm mass is  $m_c \approx 1.3 \text{ GeV}/c^2$  and the binding energy  $\omega$  of the system is roughly equal to the energy gap separating adjacent levels and is given by  $\hbar\omega \approx 0.3 \text{ GeV}$ . The correction due to GUP can be calculated at the second order of  $\alpha$ . Using Eqs. (47) and (56), we found the shift in energy is given by

$$\frac{\Delta E_0^{(2)}}{E_0^{(0)}} = \alpha_0^2 \frac{m \hbar \omega}{M_{\text{Pl}}^2 c^2} \approx 2.7 \times 10^{-39} \alpha_0^2. \quad (57)$$

Once again, assuming  $\alpha_0 \sim 1$  renders the correction too small to be measured. On the other hand, if such an assumption is not made, the current accuracy of precision measurement in the case of  $J/\psi$  [15] is at the level of  $10^{-5}$ . This sets the upper bound on  $\alpha_0$  to be

$$\alpha_0 < 10^{17}. \quad (58)$$

It should be stressed that this bound is in fact consistent with that set by the electroweak scale. Therefore, it could signal a new and intermediate length scale between the electroweak and the Planck scale.

#### IV. THE LAMB SHIFT

For the Hydrogen atom,  $V(\vec{r}) = -k/r$  ( $k = e^2/4\pi\epsilon_0 = \alpha\hbar c$ ,  $e =$  electronic charge). To first order, the perturbing Hamiltonian  $H_1$ , shifts the wave functions to [16]

$$|\psi_{nlm}\rangle_1 = |\psi_{nlm}\rangle + \sum_{\{n'l'm'\} \neq \{nlm\}} \frac{e_{n'l'm'\{nlm\}}}{E_n^{(0)} - E_{n'}^{(0)}} |\psi_{n'l'm'}\rangle, \quad (59)$$

where  $n, l, m$  have their usual significance, and  $e_{n'l'm'\{nlm\}} = \langle \psi_{n'l'm'} | H_1 | \psi_{nlm} \rangle$ .

Using the expression  $p_0^2 = 2m[H_0 + k/r]$  [8], the perturbing Hamiltonian reads

$$H_1 = -(\alpha\sqrt{8m}) \left[ H_0 + \frac{k}{r} \right] \left[ H_0 + \frac{k}{r} \right]^{1/2}. \quad (60)$$

So for GUP effect to  $\alpha$  order, we have

$$e_{n'l'm'\{nlm\}} = \langle \psi_{n'l'm'} | \left( -\frac{\alpha}{m} \right) p_0^2 p_0 | \psi_{nlm} \rangle. \quad (61)$$

It follows from the orthogonality of spherical harmonics that the above are nonvanishing if and only if  $l' = l$  and  $m' = m$

$$e_{200|100} = 2i\alpha\hbar \langle \psi_{200} | \left[ H_0 + \frac{k}{r} \right] \left( \frac{\partial}{\partial r} \right) | \psi_{100} \rangle. \quad (62)$$

We utilize the following to calculate the shift in the energy:

- (i) the first term in the sum in Eq. (59) ( $n' = 2$ ) dominates, since  $E_n = -E_0/n^2$  ( $E_0 = e^2/8\pi\epsilon_0 a_0 = k/2a_0 = 13.6 \text{ eV}$ ,  $a_0 = 4\pi\epsilon_0 \hbar^2 / me^2 = 5.3 \times 10^{-11}$  metre,  $m =$  electron mass  $= 0.5 \text{ MeV}/c^2$ ),
- (ii)  $\psi_{nlm}(\vec{r}) = R_{nl}(r) Y_{lm}(\theta, \phi)$ ,
- (iii)  $R_{10} = 2a_0^{-3/2} e^{-r/a_0}$ , and  $R_{20} = (2a_0)^{-3/2} \times (2 - r/a_0) e^{-r/2a_0}$ ,
- (iv)  $Y_{00}(\theta, \phi) = 1/(\sqrt{4\pi})$ .

Thus, we derive

$$e_{200|100} = -i \frac{2\alpha\hbar k}{a_0} \langle \psi_{200} | \frac{1}{r} | \psi_{100} \rangle \quad (63)$$

$$= -i \frac{8\sqrt{2}\alpha\hbar k}{27a_0^2}. \quad (64)$$

Therefore, the first order shift in the ground state wave function is given by (in the position representation)

$$\begin{aligned}\Delta\psi_{100}(\vec{r}) &\equiv \psi_{100}^{(1)}(\vec{r}) - \psi_{100}^{(0)}(\vec{r}) = \frac{e^{200|100}}{E_1 - E_2} \psi_{200}(\vec{r}) \\ &= i \frac{32\sqrt{2}\alpha\hbar k}{81a_0^2 E_0} \psi_{200}(\vec{r})\end{aligned}\quad (65)$$

$$= i \frac{64\sqrt{2}\alpha\hbar}{81a_0} \psi_{200}(\vec{r}).\quad (66)$$

Next, we consider the Lamb shift for the  $n$ th level of the hydrogen atom [17]

$$\Delta E_n^{(1)} = \frac{4\alpha^2}{3m^2} \left( \ln \frac{1}{\alpha} \right) |\psi_{nlm}(0)|^2.\quad (67)$$

Varying  $\psi_{nlm}(0)$ , the additional contribution due to GUP in proportion to its original value is given by

$$\frac{\Delta E_{n(\text{GUP})}^{(1)}}{\Delta E_n^{(1)}} = 2 \frac{\Delta |\psi_{nlm}(0)|}{\psi_{nlm}(0)}.\quad (68)$$

Thus, for the ground state, we obtain

$$\frac{\Delta E_{1(\text{GUP})}^{(1)}}{\Delta E_1^{(1)}} = \frac{64\hbar\alpha_0}{81a_0 M_{\text{Pl}} c} \approx 1.2 \times 10^{-22} \alpha_0.\quad (69)$$

The above result may be interpreted in two ways. First, if one assumes  $\alpha_0 \sim 1$ , then it predicts a nonzero but virtually *unmeasurable* effect of GUP and thus of quantum gravity. On the other hand, if such an assumption is not made, the current accuracy of precision measurement of Lamb shift of about one part in  $10^{12}$  [8,18] sets the following upper bound on  $\alpha_0$ :

$$\alpha_0 < 10^{10}.\quad (70)$$

It should be stressed that this bound is more stringent than the ones derived in previous examples [9], and is in fact consistent with that set by the electroweak scale. Therefore, it could signal a new and intermediate length scale between the electroweak and the Planck scale.

## V. POTENTIAL STEP

Next, we study the one-dimensional potential step given by

$$V'(x) = V_0 \theta(x),\quad (71)$$

where  $\theta(x)$  is the usual step function. Assuming  $E < V_0'$ , the Schrödinger equation to the left and right of the barrier are written, respectively, as

$$d^2\psi_{<} + k^2\psi_{<} + 2i\alpha\hbar d\psi_{<} = 0,\quad (72)$$

$$d^2\psi_{>} - k_1^2\psi_{>} + 2i\alpha\hbar d\psi_{>} = 0,\quad (73)$$

where  $k = \sqrt{2mE/\hbar^2}$  and  $k_1 = \sqrt{2m(V_0' - E)/\hbar^2}$ .

Considering solutions of the form  $\psi_{<,>} = e^{mx}$ , we get

$$m^2 + k^2 + 2i\alpha\hbar m^3 = 0,\quad (74)$$

$$m^2 - k_1^2 + 2i\alpha\hbar m^3 = 0,\quad (75)$$

with the following solution sets to leading order in  $\alpha$ , each consisting of three values of  $m$ :

$$x < 0: m = \left\{ ik', -ik'', \frac{i}{2\alpha\hbar} \right\},\quad (76)$$

$$x \geq 0: m = \left\{ k_1', -k_1'', \frac{i}{2\alpha\hbar} \right\},\quad (77)$$

where

$$k' = k(1 + k\alpha\hbar), \quad k'' = k(1 - k\alpha\hbar),\quad (78)$$

$$k_1' = k_1(1 - i\alpha\hbar k_1), \quad k_1'' = k_1(1 + i\alpha\hbar k_1).\quad (79)$$

Therefore, the wave functions take the form

$$\psi_{<} = Ae^{ik'x} + Be^{-ik''x} + Ce^{ix/2\alpha\hbar}, \quad x < 0,\quad (80)$$

$$\psi_{>} = De^{-k_1'x} + Ee^{ix/2\alpha\hbar}, \quad 0 \leq x,\quad (81)$$

where we have omitted the left mover from  $\psi_{>}$ .

Now the boundary conditions at  $x = 0$  consist of three equations (instead of the usual two)

$$d^n \psi_{<}|_0 = d^n \psi_{>}|_0, \quad n = 0, 1, 2.\quad (82)$$

This leads to the following conditions:

$$A + B + C = D + E,\quad (83)$$

$$i \left( k'A - k''B + \frac{C}{2\alpha\hbar} \right) = -k_1'D + \frac{iE}{2\alpha\hbar},\quad (84)$$

$$k'^2A + k_1'^2B + \frac{C}{(2\alpha\hbar)^2} = \frac{E}{(2\alpha\hbar)^2} - k_1''^2D.\quad (85)$$

Assuming  $C \sim E \sim \mathcal{O}(\alpha^2)$ , we have the following solutions to leading order in  $\alpha$

$$\frac{B}{A} = \frac{ik' + k_1''}{ik'' - k_1''},\quad (86)$$

$$\frac{D}{A} = \frac{2ik}{ik'' - k_1''},\quad (87)$$

$$\frac{E - C}{(2\alpha\hbar)^2 A} = \frac{k'^2(ik'' - k_1'') + k_1'^2(ik' + k_1'') + k_1''^2(2ik)}{ik'' - k_1''}.\quad (88)$$

It can be easily shown that the GUP-corrected time-dependent Schrödinger equation admits the following modified conserved current density, charge density, and conservation law, respectively, [9]

$$J = \frac{\hbar}{2mi} \left( \psi^* \frac{d\psi}{dx} - \psi \frac{d\psi^*}{dx} \right) + \frac{\alpha\hbar^2}{m} \left( \frac{d^2|\psi|^2}{dx^2} - 3 \frac{d\psi}{dx} \frac{d\psi^*}{dx} \right), \quad (89)$$

$$\rho = |\psi|^2, \quad \frac{\partial J}{\partial x} + \frac{\partial \rho}{\partial t} = 0. \quad (90)$$

The conserved current is given as

$$J = J_0 + J_1 = \frac{\hbar k}{m} (|A|^2 - |B|^2) + \frac{2\alpha\hbar^2 k^2}{m} (|A|^2 + |B|^2) + \frac{|C|^2}{\alpha m}. \quad (91)$$

The reflection and transmission coefficients are given by

$$R = \left| \frac{B}{A} \right|^2 \frac{1 - 2\alpha\hbar k}{1 + 2\alpha\hbar k} = \left| \frac{ik' + k_1''}{ik'' - k_1'} \right|^2 \frac{1 - 2\alpha\hbar k}{1 + 2\alpha\hbar k} = \frac{(k^2 + k_1^2)^2}{(k_1^2 + k^2)^2 (1 - 4\alpha\hbar k)} \frac{1 - 2\alpha\hbar k}{1 + 2\alpha\hbar k} = 1, \quad (92)$$

$$T = \frac{-\frac{\alpha\hbar^2 k_1^2}{m} |D|^2 e^{-2k_1 x} + \frac{\alpha\hbar^2 k^2}{m} |D|^2 e^{-2kx}}{\frac{\hbar k}{m} |A|^2 (1 + 2\alpha\hbar k)} \quad (93)$$

$$= 0, \quad (94)$$

$$R + T = 1. \quad (95)$$

At this point we should note that GUP did not affect  $R$  and  $T$  up to  $\mathcal{O}(\alpha)$ .

## VI. POTENTIAL BARRIER

In this section we apply the above formalism to an STM and show that in an optimistic scenario, the effect of the GUP-induced term may be measurable. In an STM, free electrons of energy  $E$  (close to the Fermi energy) from a metal tip at  $x = 0$ , tunnel quantum mechanically to a sample surface a small distance away at  $x = a$ . This gap (across which a bias voltage may be applied) is associated with a potential barrier of height  $V_0'' > E$  [19]. Thus

$$V''(x) = V_0''[\theta(x) - \theta(x - a)], \quad (96)$$

where  $\theta(x)$  is the usual step function. The wave functions for the three regions, namely,  $x \leq 0$ ,  $0 \leq x \leq a$ , and  $x \geq a$ , are  $\psi_1$ ,  $\psi_2$ , and  $\psi_3$ , respectively, and satisfy the GUP-corrected time-independent Schrödinger equation

$$d^2\psi_{1,3} + k^2\psi_{1,3} + 2i\alpha\hbar d^3\psi_{1,3} = 0,$$

$$d^2\psi_2 - k_1^2\psi_2 + 2i\alpha\hbar d^3\psi_2 = 0,$$

where  $k = \sqrt{2mE/\hbar^2}$  and  $k_1 = \sqrt{2m(V_0'' - E)/\hbar^2}$ . The solutions to the aforementioned equations to leading order in  $\alpha$  are

$$\psi_1 = Ae^{ik'x} + Be^{-ik''x} + Pe^{ix/2\alpha\hbar}, \quad (97)$$

$$\psi_2 = Fe^{k_1'x} + Ge^{-k_1''x} + Qe^{ix/2\alpha\hbar}, \quad (98)$$

$$\psi_3 = Ce^{ik'x} + Re^{ix/2\alpha\hbar}, \quad (99)$$

where  $k' = k(1 + \alpha\hbar k)$ ,  $k'' = k(1 - \alpha\hbar k)$ ,  $k_1' = k_1(1 - i\alpha\hbar k_1)$ ,  $k_1'' = k_1(1 + i\alpha\hbar k_1)$  and  $A, B, C, F, G, P, Q, R$  are constants of integration. In the above, we have omitted the left mover from  $\psi_3$ . Note the appearance of the new oscillatory terms with characteristic wavelengths  $\sim \alpha\hbar$ , due to the third order modification of the Schrödinger equation. The boundary conditions at  $x = 0, a$  are given by

$$d^n \psi_1|_{x=0} = d^n \psi_2|_{x=0}, \quad n = 0, 1, 2, \quad (100)$$

$$d^n \psi_2|_{x=a} = d^n \psi_3|_{x=a}, \quad n = 0, 1, 2. \quad (101)$$

If we assume that  $P \sim Q \sim R \sim \mathcal{O}(\alpha^2)$ , we get the following solutions:

$$\frac{C}{A} = \frac{i(k'k_1'' + k''k_1' + k'k_1' + k''k_1'')e^{-ik'a + k_1''a}}{e^{(k_1' + k_1'')a}(k' + ik_1')(k'' + ik_1'') - (k' - ik_1')(k'' - ik_1'')}, \quad (102)$$

$$\frac{B}{A} = \frac{k_1'' + ik'}{k_1'' - ik''} \left[ e^{ik'a - k_1''a} \frac{C}{A} - 1 \right], \quad (103)$$

$$\frac{F}{A} = \frac{(1 + i\frac{k'}{k_1''})e^{ik'a - k_1''a} \frac{C}{A}}{1 + \frac{k'}{k_1''}}, \quad (104)$$

$$\frac{G}{A} = \frac{(1 - i\frac{k'}{k_1''})e^{ik'a + k_1''a} \frac{C}{A}}{1 + \frac{k'}{k_1''}}. \quad (105)$$

From Eq. (89), it follows that the transmission coefficient of the STM, given by the ratio of the right moving currents to the right and left of the barrier, namely,  $J_R$  and  $J_L$ , respectively, is to  $\mathcal{O}(\alpha)$

$$T = \frac{J_R}{J_L} = \left| \frac{C}{A} \right|^2 - 2\alpha\hbar k \left| \frac{B}{A} \right|^2, \quad (106)$$

which gives using the solutions in Eqs. (102) and (103) the following final expression:

$$T = T_0[1 + 2\alpha\hbar k(1 - T_0^{-1})], \quad (107)$$

$$T_0 = \frac{16E(V_0'' - E)}{V_0''^2} e^{-2k_1 a}, \quad (108)$$

where  $T_0$  is the standard STM transmission coefficient. The measured tunneling current is proportional to  $T$  (usually

magnified by a factor  $\mathcal{G}$ ), and using the following approximate (but realistic) values [19]

$$m = m_e = 0.5 \text{ MeV}/c^2, \quad E \approx V_0'' = 10 \text{ eV},$$

$$a = 10^{-10} \text{ m}, \quad I_0 = 10^{-9} \text{ A}, \quad \mathcal{G} = 10^9,$$

we get

$$\frac{\delta I}{I_0} = \frac{\delta T}{T_0} = 10^{-26}, \quad \delta I \equiv \mathcal{G} \delta I = 10^{-26} \text{ A}, \quad (109)$$

where we have chosen  $\alpha_0 = 1$  and  $T_0 = 10^{-3}$ , also a fairly typical value. Thus, for the GUP-induced excess current  $\delta I$  to give the difference of the charge of just one electron,  $e \approx 10^{-19} \text{ C}$ , one would have to wait for a time

$$\tau = \frac{e}{\delta I} = 10^7 \text{ s}, \quad (110)$$

or, equivalently, about 4 months, which can perhaps be argued to be not that long. In fact, higher values of  $\alpha_0$  and a more accurate estimate will likely reduce this time, and conversely, current studies may already be able to put an upper bound on  $\alpha_0$ .

What is perhaps more interesting is the following relation between the *apparent barrier height*  $\Phi_A \equiv V_0'' - E$  and the (logarithmic) rate of increase of current with the gap, which follows from Eq. (107)

$$\sqrt{\Phi_A} = \frac{\hbar}{\sqrt{8m}} \left| \frac{d \ln I}{da} \right| - \frac{\alpha \hbar^2 (k^2 + k_1^2)^2}{8m(kk_1)} e^{2k_1 a}. \quad (111)$$

Note the GUP-induced deviation from the usual linear  $\sqrt{\Phi_A}$  vs  $|d \ln I / da|$  curve. The exponential factor makes this particularly sensitive to changes in the tip-sample distance  $a$ , and hence amenable to observations. Any such observed deviation may signal the existence of GUP and, thus, in turn an underlying theory of quantum gravity.

## VII. CONCLUSIONS

In this work we have investigated the consequences of quantum gravitational corrections to various quantum phenomena such as the Landau levels, simple harmonic oscillator, the Lamb shift, and the tunneling current in a scanning tunneling microscope and have found that the upper bounds on  $\alpha_0$  to be  $10^{23}$ ,  $10^{17}$ , and  $10^{10}$  from the first three, respectively. The first one gives a length scale bigger than electroweak length that is not right experimentally. It should be stressed that the last three bounds are more stringent than the ones derived in the previous study [9], and might be consistent with that set by the electroweak scale. Therefore, it could signal a new and intermediate length scale between the electroweak and the Planck scale.

On the other side, we have found that even if  $\alpha_0 \sim 1$ , we still might measure quantum gravitational corrections in a scanning tunneling microscopic case as was shown in Eq. (110). This is in fact an improvement over the general conclusion of [9], where it was shown that quantum gravitational effects are virtually negligible if the GUP parameter  $\beta_0 \sim 1$ , and appears to be a new and interesting result. It would also be interesting to apply our formalism to other areas including cosmology, black hole physics and Hawking radiation, selection rules in quantum mechanics, statistical mechanical systems, etc. We hope to report on these in the future.

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## APPENDIX

### 1. Proof for Eq. (1)

Since black hole physics and string theory suggest a modified Heisenberg algebra (which is consistent with GUP) quadratic in the momenta (see e.g. Ref. [1]) while DSR theories suggest one that is linear in the momenta (see e.g. Ref. [2]), we try to incorporate both of the above, and start with the most general algebra with linear and quadratic terms

$$[x_i, p_j] = i\hbar \left( \delta_{ij} + \delta_{ij} \alpha_1 p + \alpha_2 \frac{p_i p_j}{p} + \beta_1 \delta_{ij} p^2 + \beta_2 p_i p_j \right). \quad (A1)$$

Assuming that the coordinates commute among themselves, as do the momenta, it follows from the Jacobi identity that

$$-[[x_i, x_j], p_k] = [[x_j, p_k], x_i] + [[p_k, x_i], x_j] = 0. \quad (A2)$$

Employing Eq. (A1) and the commutator identities, and expanding the right-hand side, we get (summation convention assumed)

$$\begin{aligned} 0 &= [[x_j, p_k], x_i] + [[p_k, x_i], x_j] \\ &= i\hbar (-\alpha_1 \delta_{jk} [x_i, p] - \alpha_2 [x_i, p_j p_k p^{-1}] \\ &\quad - \beta_1 \delta_{jk} [x_i, p_l p_l] - \beta_2 [x_i, p_j p_k]) - (i \leftrightarrow j) \\ &= i\hbar (-\alpha_1 \delta_{jk} [x_i, p] - \alpha_2 ([x_i, p_j] p_k p^{-1} \\ &\quad + p_j [x_i, p_k] p^{-1} + p_j p_k [x_i, p^{-1}]) \\ &\quad - \beta_1 \delta_{jk} ([x_i, p_l] p_l + p_l [x_i, p_l]) \\ &\quad - \beta_2 ([x_i, p_j] p_k + p_j [x_i, p_k])) - (i \leftrightarrow j). \end{aligned} \quad (A3)$$



To simplify the right-hand side of Eq. (A3), we now evaluate the following commutators

(i)  $[x_i, p]$  to  $\mathcal{O}(p)$

Note that

$$\begin{aligned} [x_i, p^2] &= [x_i, p \cdot p] = [x_i, p]p + p[x_i, p] \\ &= [x_i, p_k p_k] = [x_i, p_k]p_k + p_k[x_i, p_k] \\ &= i\hbar(\delta_{ik} + \alpha_1 p \delta_{ik} + \alpha_2 p_i p_k p^{-1})p_k \\ &\quad + i\hbar p_k(\delta_{ik} + \alpha_1 p \delta_{ik} + \alpha_2 p_i p_k p^{-1}) \end{aligned} \quad (\text{A4})$$

[to  $\mathcal{O}(p)$  using Eq. (A1)]

$$= 2i\hbar p_i [1 + (\alpha_1 + \alpha_2)p]. \quad (\text{A5})$$

Comparing (A4) and (A5), we get

$$[x_i, p] = i\hbar(p_i p^{-1} + (\alpha_1 + \alpha_2)p). \quad (\text{A6})$$

(ii)  $[x_i, p^{-1}]$  to  $\mathcal{O}(p)$

Using

$$0 = [x_i, I] = [x_i, p \cdot p^{-1}] = [x_i, p]p^{-1} + p[x_i, p^{-1}], \quad (\text{A7})$$

it follows that

$$\begin{aligned} [x_i, p^{-1}] &= -p_i^{-1}[x_i, p]p^{-1} \\ &= -i\hbar p^{-1}(p_i p^{-1} + (\alpha_1 + \alpha_2)p_i)p^{-1} \\ &= -i\hbar p_i p^{-3}(1 + (\alpha_1 + \alpha_2)p). \end{aligned} \quad (\text{A8})$$

Substituting (A6) and (A8) in (A3) and simplifying, we get

$$\begin{aligned} 0 &= [[x_j, p_k], x_i] + [[p_k, x_i], x_j] \\ &= ((\alpha_1 - \alpha_2)p^{-1} + (\alpha_1^2 + 2\beta_1 - \beta_2))\Delta_{jki}, \end{aligned} \quad (\text{A9})$$

where  $\Delta_{jki} = p_i \delta_{jk} - p_j \delta_{ik}$ . Thus one must have  $\alpha_1 = \alpha_2 \equiv -\alpha$  (with  $\alpha > 0$ ; the negative sign follows from Ref. [3] of our paper), and  $\beta_2 = 2\beta_1 + \alpha_1^2$ . Since from dimensional grounds it follows that  $\beta \sim \alpha^2$ , for simplicity, we assume  $\beta_1 = \alpha^2$ . Hence  $\beta_2 = 3\alpha^2$ , and we get Eq. (1) of this paper, namely,

$$\begin{aligned} [x_i, p_j] &= i\hbar \left( \delta_{ij} - \alpha \left( p \delta_{ij} + \frac{p_i p_j}{p} \right) \right. \\ &\quad \left. + \alpha^2 (p^2 \delta_{ij} + 3p_i p_j) \right). \end{aligned} \quad (\text{A10})$$

## 2. Proof for Eq. (5)

We would like to express the momentum  $p_j$  in terms of the *low energy momentum*  $p_{0j}$  (such that  $[x_i, p_{0j}] = i\hbar \delta_{ij}$ ). Since Eq. (A10) is quadratic in  $p_j$ , the latter can at most be a cubic function of the  $p_{0i}$ . We start with the most general form consistent with the index structure

$$p_j = p_{0j} + a p_0 p_{0j} + b p_0^2 p_{0j}, \quad (\text{A11})$$

where  $a \sim \alpha$  and  $b \sim a^2$ . From Eq. (A11) it follows that

$$\begin{aligned} [x_i, p_j] &= [x_i, p_{0j} + a p_0 p_{0j} + b p_0^2 p_{0j}] \\ &= i\hbar \delta_{ij} + a([x_i, p_0]p_{0j} + p_0[x_i, p_{0j}]) \\ &\quad + b([x_i, p_0]p_0 p_{0j} + p_0[x_i, p_0]p_{0j} + p_0^2[x_i, p_{0j}]). \end{aligned} \quad (\text{A12})$$

Next, we use the following four results to  $\mathcal{O}(a)$  and  $[x_i, p_{0j}] = i\hbar$  in Eq. (A12):

(i)  $[x_i, p_0] = i\hbar p_{0i} p_0^{-1}$ , which follows from Eq. (A6) when  $\alpha_i = 0$ , as well from the corresponding Poisson bracket.

(ii)  $p_j = p_{0j}(1 + a p_0) + \mathcal{O}(a^2) \simeq p_{0j}(1 + a p)$  [from Eq. (A11)]. Therefore,  $p_{0j} \simeq \frac{p_j}{1 + a p} \simeq (1 - a p)p_j$ .

(iii)  $p_0 = (p_{0j} p_{0j})^{1/2} = ((1 - a p)^2 p_j p_j)^{1/2} = (1 - a p)p$ .

(iv)  $p_{0i} p_0^{-1} p_{0j} = (1 - a p)p_i (1 - a p)^{-1} p^{-1} (1 - a p)p_j = (1 - a p)p_i p_j p^{-1}$ .

Thus, Eq. (A2) yields

$$\begin{aligned} [x_i, p_j] &= i\hbar \delta_{ij} + i a \hbar (p \delta_{ij} + p_i p_j p^{-1}) \\ &\quad + i\hbar (2b - a^2)p_i p_j + i\hbar (b - a^2)p^2 \delta_{ij}. \end{aligned} \quad (\text{A13})$$

Comparing with Eq. (A10), it follows that  $a = -\alpha$  and  $b = 2\alpha^2$ . In other words

$$\begin{aligned} p_j &= p_{0j} - \alpha p_0 p_{0j} + 2\alpha^2 p_0^2 p_{0j} \\ &= p_{0j}(1 - \alpha p_0 + 2\alpha^2 p_0^2), \end{aligned} \quad (\text{A14})$$

which is Eq. (5) in this paper.

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