EQUIVARIANT RESOLUTIONS OF SINGULARITIES FOR ORBITS IN GENERALIZED QUIVER VARIETIES ARISING IN THE LOCAL LANGLANDS PROGRAM FOR P-ADIC GROUPS

JOEL BENESH
Bachelor of Science, University of Saskatchewan, 2020

A thesis submitted
in partial fulfilment of the requirements for the degree of

MASTER OF SCIENCE

in

MATHEMATICS

Department of Mathematics and Computer Science
University of Lethbridge
LETHBRIDGE, ALBERTA, CANADA

EQUIVARIANT RESOLUTIONS OF SINGULARITIES FOR ORBITS IN
GENERALIZED QUIVER VARIETIES ARISING IN THE LOCAL LANGLANDS
PROGRAM FOR P-ADIC GROUPS

JOEL BENESH

Date of Defence: December 14, 2022

Dr. A. Fiori Thesis Supervisor	Associate Professor	Ph.D.
Dr. A. Akbary Thesis Examination Committee Member	Professor	Ph.D.
Dr. D. Morris Thesis Examination Committee Member	Professor	Ph.D.
Dr. J. Sheriff Chair, Thesis Examination Committee	Assistant Professor	Ph.D.

Abstract

In this thesis, we investigate geometric aspects of the Langlands parameters arising in the local Langlands program for p-adic groups. This work was inspired by David Vogan's "The Local Langlands Conjecture" [19], and we built off of the work of Clifton Cunningham, Andrew Fiori, Ahmed Moussaoui, James Mracek, and Bin Xu in their book "Arthur Packets for p-Adic Groups by Way of Microlocal Vanishing Cycles of Perverse Sheaves, with Examples" [6]. This thesis follows the construction detailed in Part II of their book, and we give more concrete examples to demonstrate their construction. Additionally, we provide an exposition on the local Langlands program for p-adic groups and give an algorithm for computing the resolutions of singularities arising from the study of the orbits of certain generalized quiver varieties for their respective Langlands parameters in the groups GL_n, SO_n, and Sp_{2n}.

Acknowledgments

I'd like to acknowledge Dr. Andrew Fiori, my thesis advisor, not only for his unending support and dedication to the project, but also for providing me with support and the opportunity to get to know both Lethbridge and the surrounding area. I'd also like to thank him for giving me the chance to meet the other departmental staff and students, especially during The Time of Covid, which would have otherwise been very difficult.

A special thanks goes out to my friends and family, for all of their love and support. In particular, I'd like to recognize my parents, John and JoAnne Benesh, without whose support I would not have been able to pursue my love of mathematics and have taken on the challenge of pursuing a master's degree, for which I am eternally grateful.

Last but certainly not least, I would like to express my gratitude to my mentors at the University of Saskatchewan: Drs. Steven Rayan, Jazec Smigielski, Cameron Frank, and Lawrence Chang. Without their support and encouragement, I would never have been able to cultivate such a passion for the subject and also develop my mathematical writing to a point where I could feel pride in my work. Thank you.

Contents

List of Tables

List of Figures

Chapter 1

Introduction

In this chapter, we lay out the foundations and structure of this thesis.

In this thesis, we will be studying various aspects of a certain class of varieties arising in the study of the p-adic local Langlands program. Among other things, we will be detailing the construction of this class of objects, what motivates their study, and where they appear in other areas of research, along with various aspects of their geometry.

The primary focus of this thesis is the application of a theory used to construct resolutions of singularities for orbit closures arising in (symmetric) quiver representation varieties (see Definition 3.5). The secondary goal of this thesis is to apply these resolutions to some of the fundamental theories and hypotheses found in Langlands' correspondence. In particular, their construction and applications in the Kazhdan-Lusztig conjecture (detailed in Section 2.2), which asserts a bijection between certain representation theoretic and geometric objects.

The geometric objects that will be the focus of our study are a type of quiver variety. A quiver is a directed graph, and the "quiver representation variety" is a collection of maps between vector spaces of certain dimensions attached to the nodes. They are a common object found in representation theory. The main resource that we will use to study these objects will be "An Introduction to Quiver Representations" by Harm Derksen and Jerzy Weyman [10], who have written much on the geometry of these varieties. In this thesis, the quiver representations on which we focus are the linear maps between the eigenspaces of semi-simple elements found in reductive algebraic groups. The reductive groups we will

concentrate on are the classical examples GL_n, SO_n, and Sp_{2n}, along with their respective Lie algebras $\mathfrak{gl}_n$, $\mathfrak{so}_n$, and $\mathfrak{sp}_{2n}$, respectively.

In Chapter 2, we explain how David Vogan's theory simplifies our study of the p-adic local Langland's correspondence to the study of his Vogan varieties V_λ (and their H_λ-orbits). We detail the usefulness of constructing smooth algebraic covers for the H_λ-orbits in order to work with the p-adic Kazhdan-Lusztig conjecture (see Conjecture 2.13). This conjecture is known in the real and complex Langlands correspondence but only for some p-adic contexts[1].

In Chapter 3, we make everything more concrete by giving plenty of examples and by studying explicit aspects of the geometry of the Vogan varieties, their orbits, and the resulting equivariant fundamental groups for GL_n, SO_n, and Sp_{2n}.

In Chapter 4, we give an explicit construction of the resolutions of singularities for the orbit closures. In Section 4.2, we do this by showing some of the classical methods that can be applied to our varieties. These classical methods, however, are not well tuned to our setting; they are inefficient, we therefore provide a much more refined algorithm, given by Definition 4.16. In Section 4.3.2, we prove this construction for GL_n via the statement of Theorem 4.20; this is the main result of our thesis, and in Section 4.3.3, we provide many motivating examples. Then in Section 4.3.4, we describe its extension to SO_n and Sp_{2n}. In the final part of this chapter, Section 4.4, we apply our constructions to an explicit computation of the multiplicity matrix described in Section 2.3.

Finally, in Chapter 5, we discuss future work arising from this research. We laid much of the foundation by exploring these objects, and in so doing, we discovered that there remains much work to be accomplished in the p-adic local Langlands program.

---

[1]Though it is not well documented, this conjecture is understood to have been established in the contexts of $\hat{G} = GL_n$, Sp_{2n} and SO_{2n+1}, and possibly others.

Chapter 2

Background and Motivation

2.1 Local Langlands Program for p-adic Groups

Let G be a connected reductive algebraic group over a p-adic field F. In representation theory, we are naturally concerned with the irreducible admissible representations of $G(F)$; in number theory, we are concerned with the representations of the absolute Galois group of F, denoted by $\mathrm{Gal}(F)$. The Langlands program says that these two things are related. More precisely, the local Langlands program for p-adic fields suggests that there is a relationship between the equivalence classes of irreducible admissible representations of $G(F)$, denoted by $\Pi(G(F))$, and the representations of $\mathrm{Gal}(F)$ into $\hat{G}$, where $\hat{G}$ is the Langlands dual group (a topological group over $\mathbb{C}$).

Remark 2.1. We have the following common examples of Langlands dual groups for reductive algebraic groups

$$
\begin{array}{ccc}
\widehat{\mathrm{GL}_n} = \mathrm{GL}_n & \widehat{\mathrm{SL}_n} = \mathrm{PSL}_n & \widehat{\mathrm{PSL}_n} = \mathrm{SL}_n \\
\widehat{\mathrm{Sp}_{2n}} = \mathrm{SO}_{2n+1} & \widehat{\mathrm{SO}_{2n}} = \mathrm{SO}_{2n} & \widehat{\mathrm{SO}_{2n+1}} = \mathrm{Sp}_{2n}.
\end{array}
$$

In this thesis, these are the only cases we consider; in particular, we will be focusing on the classical groups $\mathrm{GL}_n, \mathrm{SO}_n$, and Sp_{2n}. See [6, Part 1, Section 3] for the general construction of the Langlands Dual Group.

To obtain a more well-behaved correspondence, we enhance representations of $\mathrm{Gal}(F)$ into $\hat{G}$ by replacing both $\mathrm{Gal}(F)$ and $\hat{G}$ with different groups. We replace $\mathrm{Gal}(F)$ by considering a dense subgroup, the Weil-Deligne group, W_F, of the absolute Galois group, and

with this we define the local Langlands group of F to be $L_F := W_F \times \mathrm{SL}_2(\mathbb{C})$. Similarly, we augment the definition of the Langlands dual group to define the Langlands group as the following semi-direct product: $^L G := W_F \ltimes \hat{G}$ (where $\ltimes$ is induced from the action of $\mathrm{Gal}(F)$ on the root data of G), see [6, Part 1, Section 3] for the full construction. We introduce Langlands parameters (L-parameters) as a substitute for Galois representations in the correspondence. These are continuous homomorphisms

$$\phi : L_F \to {}^L G,$$

satisfying a few technical hypotheses, namely

1. ϕ commutes with the projections $L_F \to W_F$ and $^L G \to W_F$;

2. the map $\mathrm{SL}_2(\mathbb{C}) \longrightarrow \hat{G}$ is induced from a morphism of algebraic groups; and

3. the image of $\phi \mid_{W_F}$ consists of semisimple elements in $^L G$.

When G is split, the L-group becomes $^L G = W_F \times \hat{G}$ and one can without any loss of generality proceed as though $^L G = \hat{G}$. This is the context in which we will be working on the thesis.

The following is a key conjecture in the Langlands Correspondence:

Conjecture 2.2. [6, Section 3.8] There exists a map

$$\mathrm{LLC} : \Pi^{\mathrm{pure}}(G/F) \longrightarrow \left\{ L\text{-parameters}, \phi : L_F \to {}^L G \right\}\Big/_{\sim},$$

which is surjective with finite fibres, where $\sim$ is equivalence under conjugation. We will henceforth refer to the collection of L-parameters ϕ under this relation as the set $\Phi(G/F)$.[2]

It is useful to augment L-parameters further so that the map of Conjecture 2.2 can be given as a bijection; it is classically done via the following: In order to enhance our set of

---

[2]We write G/F rather than $G(F)$ because our parameters depend only on a class of pure inner forms, and not on one particular form, see [6, Part 1, Section 3.8] or [19] for details. That being said, one can safely ignore this technicality for the purpose of this thesis, and going forward we will write $\Pi(G/F)$ in place of $\Pi^{\mathrm{pure}}(G/F)$.

L-parameters to define a more well-behaved correspondence, we introduce the equivariant fundamental group (see Proposition 2.9 for justification of terminology) associated to an L-parameter ϕ, given by

$$A_\phi := \pi_0(Z_{\hat{G}}(\phi)) = Z_{\hat{G}}(\phi) \big/ (Z_{\hat{G}}(\phi))^\circ,$$

where $(Z_{\hat{G}}(\phi))^\circ$ is the identity component of the centralizer $Z_{\hat{G}}(\phi)$ of ϕ under conjugation by $\hat{G}$. This group is used to define the set of enhanced L-parameters $\Xi(\hat{G})$, that is, the set of equivalence classes of the pairs $\xi = (\phi, \rho)$, where ρ is an irreducible representation of A_ϕ. Then we obtain a refinement of Conjecture 2.2, that there exists a bijection

$$\Pi(G/F) \longleftrightarrow \Xi(\hat{G}),$$

satisfying a list of properties, based mainly on compatibility with class field theory and the principle of functoriality, along with certain normalising choices (see [5] for details). Fixing such a bijection, we write $\pi(\phi, \rho)$ for an irreducible admissible representation of $G(F)$ corresponding to a pair (ϕ, ρ) above.

Rather than studying the whole sets of $\Pi(G/F)$ and $\Xi(\hat{G})$ at once, it is natural to collect "similar" L-parameters and representations. This is done through the introduction of infinitesimal parameters λ, which are continuous homomorphisms between groups

$$\lambda : W_F \to \hat{G},$$

satisfying technical hypotheses analogous to those of L-parameters, such as the image of λ consisting of semisimple elements in $^L G$. There is a natural map from $\lambda \to \phi_\lambda$, given by taking the trivial element in SL_2, $\phi_\lambda(w, x) = \lambda(w)$. To go the other way, we associate to

each ϕ the infinitesimal parameter λ_ϕ, given by the map

$$\lambda_\phi(w) := \phi\left(w, \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix}\right), \ w \in W_F.^3$$

The map $\phi \mapsto \lambda_\phi$ is not injective, but the preimage of any infinitesimal parameter falls into finitely many equivalence classes of Langlands parameters under $\hat{G}$-conjugation.

Recall that in a p-adic local field F (for example $\mathbb{Q}_p$) the unique prime ideal $\mathfrak{p}$ gives rise to the inertial subgroup I_F of $\mathrm{Gal}(F)$ through the exact sequence

$$1 \to I_F \to \mathrm{Gal}(F) \to \mathrm{Gal}(\mathbb{F}_q) \to 1, \tag{2.1}$$

where q is a p-th power and $\mathbb{F}_q$ is the residue field of F. For a local field of characteristic $p > 0$, the Weil group, W_F, is the subgroup of the absolute Galois group of elements that act as a power of the Frobenius automorphism on the constant field (see [2] for further details). We say a representation (or L-parameter) is unramified if it is trivial on inertia. Using that the group $\langle \mathrm{Frob}_p \rangle^4$ is dense in $\mathrm{Gal}(\mathbb{F}_q)$ (in the profinite topology), for unramified parameters we may replace W_F by $\langle \mathrm{Frob}_p \rangle$ since they are dense in the part of the group where the parameter is non-trivial. The restriction to the unramified case is a simplifying assumption that we will make going forward.

Remark 2.3. [6, Section 5.1] details this simplification. The authors of [6] are optimistic that things proved in the unramified case can be lifted to the ramified case, though this may still end up being quite difficult (see [8]).

With this assumption, studying where the generating element Frob_p gets sent tells us everything we need about the maps of W_F into $\hat{G}$. Hence, when considering an unramified infinitesimal parameter, we may, by abuse of notation, write $\lambda = \lambda(\mathrm{Frob}_p)$ so that the map

---

[3]The norm $|\cdot|$, is defined using the map $\mathrm{Gal}(F) \to \mathrm{Gal}(\mathbb{F}_q)$, see Equation (2.1), and follows from specifying the value on Frob_p.

[4]$\langle \mathrm{Frob}_p \rangle$ is the group generated by the Frobenious automorphism, $x \mapsto x^q$, for each element $x \in \mathbb{F}_q$.

from L-parameters to infinitesimal parameters becomes:

$$\lambda_\phi := \phi\left(\text{Frob}_p, \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}\right).$$

One can show that the function $\phi \mapsto \lambda_\phi$ is surjective and finite. If we write $\Lambda(G/F)$ for the set of $\hat{G}-$conjugacy classes of infinitesimal parameters, then we can restate the local Langlands correspondence, Conjecture 2.2, in terms of these surjective maps with finite fibres as:

$$\Pi(G/F) \xrightarrow{\text{finite}} \Phi(G/F) \xrightarrow{\text{finite}} \Lambda(G/F).$$

This gives us a concrete strategy for studying the Langlands correspondence. Given a fixed λ, we want to study all equivalence classes of enhanced Langlands parameter that map to it. We denote by $P_\lambda(\hat{G})$ the set of L-parameters whose infinitesimal parameter is exactly λ and by $\Phi_\lambda(G/F)$ the isomorphism classes of these L-parameters under $Z_{\hat{G}}(\lambda)$-conjugation. More specifically, fixing $\lambda = \lambda_\phi(\text{Frob}_p)$, we want to study ϕ such that $\phi_\lambda = \lambda$. Since λ is diagonalizable, and we only consider ϕ up to conjugacy, without loss of generality, we may assume that λ is diagonal moving forward.

Given that the local Langlands correspondence suggests a bijection between these sets, it is reasonable that one would like to find an identification between $\Pi(G/F)$, the set of equivalence classes of simple objects in the category of admissible representations of $G(F)$, and $\Phi(G/F)$), the set of equivalence classes of simple objects in another category as well. This might allow us to find a relation between these categories that would recover the bijection above on its simple objects.

In his 1993 paper [19], David Vogan observed that there is a bijection between irreducible representations of G, with infinitesimal parameters λ and simple equivariant (perverse) sheaves[5] on a certain variety X_λ, which one thinks of as the moduli space of these

---

[5]We will not be providing an explicit definition of "simple perverse sheaves". For this, see [6, Part 1, Section 4.4] or [9] for further reading. However, we will be providing examples of how we can use key properties of the category and functorialities to compute with them; see Section 2.3.

L-parameters. From this perspective, the local Langlands correspondence posits a bijection between the set $\Xi_\lambda(\hat{G})$ of equivalence classes of objects in the category of perverse sheaves $\mathrm{Per}_{\hat{G}}(X_\lambda)$ on X_λ [6] and the set of simple objects $\Pi_\lambda(G/F)$ in the category $\mathrm{Rep}_\lambda(G/F)$, with matching "infinitesimal parameter" λ.

We define the Vogan variety V_λ as the q-eigenspace of λ acting on $\hat{\mathfrak{g}} = \mathrm{Lie}(\hat{G})$, the Lie algebra associated to the Lie group $\hat{G}$, and subsequently introduce the centralizer group H_λ, of our infinitesimal parameter λ in $\hat{G}$ by

$$H_\lambda := Z_{\hat{G}}(\lambda) = \{h \in \hat{G} \mid h\lambda = \lambda h\},$$

which acts via the adjoint action on the variety V_λ.

Proposition 2.4. *[6, p.22] For each infinitesimal parameter λ, we obtain the map*

$$P_\lambda(\hat{G}) \to V_\lambda(\hat{G}), \ \ by \ using \ the \ differential$$

$$\phi \mapsto x_\phi := d\phi \left(1, \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right),$$

which is surjective[7]. Then the induced map between the sets of H_λ-orbits

$$\Phi_\lambda(\hat{G}) \to {}^{V_\lambda(\hat{G})}\!\big/\!_{H_\lambda}$$

$$[\phi] \mapsto C_\phi,$$

is a bijection.

This proposition gives a relation between the geometry of P_λ and V_λ. The geometry of V_λ is much simpler than that of P_λ and will hence be the focus of our study.

---

[6]Vogan defines $X_\lambda = {}^{V_\lambda \times \hat{G}}\!\big/\!_{Z_{\hat{G}}(\lambda)}$. The categories of perverse sheaves on X_λ and V_λ are equivalent (see [19, (4.4)(e)] or [6, Part 1, Section 4.3-4.5]), as V_λ is much simpler to study it will be the object we ultimately work with in this thesis.

[7]The differential of ϕ induces a map on Lie algebras $\mathfrak{sl}_2 \to \mathfrak{g}$ and is the evaluation of that map at $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

2.1.1 The Equivariant Fundamental Group A_C

We will reframe our understanding of the local Langlands correspondence, as a bijection between $\Xi(\hat{G})$, the equivalence classes of enhanced Langlands parameters and $\Pi(G/F)$, the set of representations of $\mathrm{Gal}(F)$ into $\hat{G}$ in terms of Vogan's description[8]. Every simple object in $\mathrm{Per}_{H_\lambda}(V_\lambda)$ takes the form $IC(C, \mathcal{L})$[9], where C is an H_λ-orbit in V_λ and $\mathcal{L}$ is a simple equivariant local system on C. In turn, simple equivariant local systems are in bijection with irreducible representations of A_C, the equivariant fundamental group of C.

Definition 2.5 (Local System). A sheaf $\mathcal{L}$ is called a local system on a topological space X if, for every $x \in X$, there is a neighbourhood U containing x such that $\mathcal{L}|_U$ is a constant sheaf [18].

Further, we provide the following definition for an equivariant local system $\mathcal{L}$:

Definition 2.6 (Equivariant Local System). An equivariant local system is a local system $\mathcal{L}$, together with a compatible[10] family of isomorphisms:

$$\alpha_h : h^* \mathcal{L} \to \mathcal{L},$$

for each $h \in H$. i.e., $h^* \mathcal{L} \simeq \mathcal{L}$, for every $h \in H$.

If a local system is to be equivariant for a group action, it follows that the fibres must be constant on entire orbits. As a consequence, an equivariant local system for the étale topology is essentially a local system of representations of the equivariant fundamental group. Ultimately, this is what allows one to realise a bijection between isomorphism classes of local systems and isomorphism classes of representations of the equivariant fundamental group. In this thesis, we will be implicitly using this bijection without making it explicit.

---

[8]This construction is formally described in [6, Section 10.2.3-5], which we specialise to our setup.

[9]The "Intersection Cohomology Sheaf". The precise definition of which is highly technical and not needed in this thesis, we give some information about how to compute them in Section 2.3.

[10]For our purposes, we need only know that there is a compatibility condition with respect to our group. For further reading, see [6, Section 4.4] or [20, Example 3.2.7] for precise definitions.

In order to explain why A_C is the equivariant fundamental group, we recall some definitions and results from algebraic geometry:

Definition 2.7. Suppose that $\widetilde{X}, X$ are connected topological spaces. Then a map $\pi : \widetilde{X} \to X$ is called a covering if the map is

- Finite: if the morphism π is a proper (see Definition 4.8) with finite fibres; and

- Étale: π is both smooth and unramified in the sense that $|\pi^{-1}(\{x\})|$ is locally constant.

Over $\mathbb{C}$ this definition agrees with the familiar definition given in algebraic topology. One notes that a cover $\pi : \widetilde{X} \to X$ is said to be equivariant if there is a group G which acts on both $\widetilde{X}$ and X such that for every $g \in G$:

$$g.\pi(x) = \pi(g.x), \quad \text{for every } x \in \widetilde{X}.$$

Further, one may define the universal cover:

Definition 2.8. A map $\pi : \widehat{X} \to X$ is said to be the universal equivariant cover if for any other connected equivariant cover $\pi' : \widetilde{X} \to X$ there exists a unique map $\pi'' : \widehat{X} \to \widetilde{X}$ such that the following diagram commutes:

$$
\begin{array}{ccc}
\widehat{X} & \xrightarrow{\quad \pi'' \quad} & \widetilde{X} \\
& \searrow{\pi} \qquad \swarrow{\pi'} & \\
& X &
\end{array}
$$

With these definitions in place, the following proposition will be used to explain how the universal equivariant fundamental group is derived:

Proposition 2.9. *Suppose that C is an H_λ-orbit and $c \in C$. Then, by the orbit stabilizer theorem, we may write $C = H_\lambda.c \cong H_\lambda \big/ \mathrm{Stab}_c(H_\lambda)$. Now, we have that*

1. *Every connected cover $\pi : \widehat{C} \to C$ is a single orbit, in that*

$$\widehat{C} = H_\lambda.\hat{c} \text{ for some } \hat{c} \in \pi^{-1}(c)$$

and hence, $\widehat{C} = H_\lambda \big/ \mathrm{Stab}_{\hat{c}}(H_\lambda)$ and $\mathrm{Stab}_{\hat{c}}(H_\lambda) \subseteq \mathrm{Stab}_c(H_\lambda)$ is of finite index.

2. Consequently, the universal equivariant cover is $H_\lambda \big/ \mathrm{Stab}_c(H_\lambda)^\circ$, where $(\mathrm{Stab}_c(H_\lambda))^\circ$ denotes the connected component of $\mathrm{Stab}_c(H_\lambda)$ containing the identity.

3. Hence, the universal equivariant fundamental group is given by

$$A_C = \mathrm{Stab}_c(H_\lambda) \big/ \mathrm{Stab}_c(H_\lambda)^\circ.$$

As a consequence, the simple objects in $\mathrm{Per}_{H_\lambda}(V_\lambda)$ may be parametrized by pairs (C, ρ) where C is an H_λ-orbit in V_λ and ρ is an isomorphism class of irreducible representations of the equivariant fundamental group A_C[11] of C.

This gives us a canonical bijection:

$$\mathrm{Per}_{H_\lambda}(V_\lambda)^{\mathrm{simple}}_{\mathrm{iso}} \Longleftrightarrow \{(C, \rho) \mid H_\lambda-\text{orbit } C \subseteq V_\lambda,\ \rho \in \mathrm{Irrep}(A_C)\}. \tag{2.2}$$

Lemma 2.10. *[6, Lemma 4.4] For any Langlands parameter $\phi : L_F \to {}^L G$,*

$$A_{C_\phi} = A_\phi,$$

where C_ϕ is the H_{λ_ϕ}-orbit of c_ϕ in V_{λ_ϕ}.

Similarly to Proposition 2.4, this lemma relates the geometry of P_λ to that of V_λ, and provides justification that enhanced Langlands parameters are in bijection with the set in Equation (2.2). This lemma shows that A_ϕ is both the equivariant fundamental group of ϕ and the equivariant fundamental group of C_ϕ.

The local Langlands correspondence as formulated in Conjecture 2.2 is equivalent to the following, which was one of the fundamental ideas in Vogan's original paper [19]:

---

[11]This group is commonly denoted by A_C, since there is an identification up to inner automorphism $A_{c_1} \simeq A_{c_2}$ for any $c_1, c_2 \in C$ given by $A_{c_2} = hA_{c_1}h^{-1}$ for any $h \in H_\lambda$ with $hc_1 = c_2$. So when A_c is abelian, as it often is, this identification is canonical.

Conjecture 2.11. Let G be a quasisplit algebraic group. The local Langlands correspondence for pure rational forms determines a bijection between the set of isomorphism classes of simple objects in $\mathrm{Per}_{H_\lambda}(V_\lambda)$ and those of $\Pi_\lambda(G/F)$

$$\mathrm{Per}_{H_\lambda}(V_\lambda)_{\mathrm{iso}}^{\mathrm{simple}} \Longleftrightarrow \Pi_\lambda(G/F).$$

The relationship between the categories $\mathrm{Rep}_\lambda(G/F)$ and $\mathrm{Per}_{\hat{G}}(V_\lambda)$ seems to be more than a bijection between their simple objects. This evidence motivates a closer study of V_λ and the introduction of the category of perverse sheaves into the Langlands program.

2.2 The Kazhdan-Lusztig Conjecture

The following section describes the Kazhdan-Lusztig conjecture, some of the simplifications made along the way, and how this conjecture motivates further investigation and hence our research.

Definition 2.12. Given any abelian category $\mathcal{C}$ we define the Grothendieck group $K\mathcal{C}$ to be the group of all formal $\mathbb{Z}$-linear combinations of objects from the category modulo the equivalence relation generated by exact sequences, that is,

$$[A] + [B] = [C] \quad \text{whenever} \quad 0 \to A \to C \to B \to 0 \quad \text{is exact.}$$

For the categories of representations, $\mathrm{KRep}_\lambda(G/F)$ and perverse sheaves, $\mathrm{KPer}_{H_\lambda}(V_\lambda)$, respectively. The local Langlands correspondence says that there is a natural isomorphism of $\mathbb{Z}$-modules

$$\mathrm{KRep}_\lambda(G/F) \overset{LLC}{\Longleftrightarrow} \mathrm{KPer}_{H_\lambda}(V_\lambda).$$

The Grothendieck group for the category of representations, $\mathrm{KRep}_\lambda(G/F)$, has two natural bases, a natural basis of "irreducible representations" $\pi(\phi,\rho)$, and a basis of "standard modules" $M(\phi,\rho)$[12]. Similarly, the Grothendieck group for the category of perverse sheaves,

---

[12]This is an induced module for which $\pi(\phi,\rho)$ is the unique irreducible quotient, though it is not canonical, its image in KRep_λ is.

$\text{KPer}_{H_\lambda}(V_\lambda)$, has two natural bases, a natural basis of "standard sheaves" $\mathcal{S}(C,\mathcal{L})$, and a basis of "simple perverse sheaves" $I\mathcal{C}(C,\mathcal{L})$. The Kazhdan-Lusztig conjecture suggests that there is a connection between the corresponding change of basis matrices; we make this more precise below.

2.2.1 Change of Basis Matrix for Representations with Infinitesimal parameter λ

To describe the change of basis matrix for the representations with the infinitesimal parameter λ, following [6, p.86] we define the 'representation multiplicity matrix', $m_{\text{rep}}((\phi,\rho),(\phi',\rho'))$ of $\pi(\phi,\rho)$ in the standard module $M(\phi,\rho)$ so that in the Grothendieck group of admissible representations generated by $\Pi_\phi(G/F)$ we have

$$M(\phi',\rho') = \sum_{(\phi,\rho)} m_{\text{rep}}((\phi,\rho),(\phi',\rho'))\, \pi(\phi,\rho),$$

where the sum is taken over all $\phi \in P_\lambda(^LG)$, and all $\rho \in \text{Irrep}(A_\phi)$. We can think of m_{rep} as the multiplicities of irreducible modules, $\pi(\phi,\rho)$, in the Jordan-Hölder decomposition for the standard module, $M(\phi,\rho)$.

Working now on the geometric side, for every H_λ-orbit $C \subseteq V_\lambda$, and every H_λ-equivariant local system $\mathcal{L}$ on V_λ, we consider the shifted standard sheaf

$$\mathcal{S}(C,\mathcal{L}) := j_{C!}\mathcal{L}[\dim C], \tag{2.3}$$

where $j_C : C \hookrightarrow V_\lambda$ is inclusion. Then, in $\text{KPer}_{H_\lambda}(V_\lambda)$, we have

$$I\mathcal{C}(C,\mathcal{L}) \equiv \sum_{(C',\mathcal{L}')} m_{\text{geo}}((C',\mathcal{L}'),(C,\mathcal{L}))\, \mathcal{S}(C',\mathcal{L}').$$

We refer to the matrix m_{geo} as the geometric multiplicity matrix. This describes the multiplicities of the standard sheaves when we express a perverse sheaf in this basis for the Grothendieck group, $\text{KPer}_{H_\lambda}(V_\lambda)$. We define

$$m^*_{\text{geo}}((C',\mathcal{L}'),(C,\mathcal{L})) := (-1)^{\dim(C)-\dim(C')} m_{\text{geo}}((C',\mathcal{L}'),(C,\mathcal{L})), \tag{2.4}$$

as the normalised geometric multiplicity matrix. The sign normalisation $(-1)^{\dim(C)-\dim(C')}$ relates to the relative degree of the contributing cohomology and is similar to the fact that an Euler characteristic is an alternating sum based on the degree.

Conjecture 2.13. Given an infinitesimal parameter $\lambda \in \hat{G}$. The Kazhdan–Lusztig Conjecture for p-adic groups can be understood as the claim that

$$m_{\text{rep}}^{T} = m_{\text{geo}}^{*},$$

see [6, §10.3.3, p.97] for further details.

This deep conjecture is known in many cases where the local Langlands correspondence is also known, for instance in the case of GL_n and several other families of classical groups. One outcome of my thesis will be to provide a mechanism to compute the geometric multiplicities, m_{geo}, for various families of infinitesimal parameters for classical groups GL_n, SO_n, and Sp_{2n}. We give an overview of the general ideas in the next section.

2.3 Explicit Computations of Geometric Multiplicities

The simple H_λ-equivariant perverse sheaves on the Vogan variety V_λ are completely parametrized by the pairs of local systems $\mathcal{L}$ on the orbits C. Though we will be avoiding a precise definition of perverse sheaves, which are technically not sheaves, but complexes of sheaves, we have several important results for computing them. If Y is smooth, a local system $\mathcal{L}$ shifted to degree, $\dim(Y)$, on Y will be a perverse sheaf (see [3, p.102]).

We provide some basic results concerning these perverse sheaves which one can freely use to simplify computations involving perverse sheaves:

Proposition 2.14. *[6] Suppose that C is an H_λ-orbit and $\mathcal{L}$ is an equivariant local system on C. Then, we have the following results for equivariant perverse sheaves:*

1. *$IC(C, \mathcal{L})\mid_C = \mathcal{L}[\dim C]$.*

2. *$IC(C, \mathcal{L})\mid_D = 0$ if $D \nsubseteq \overline{C}$.*

3. *If $\overline{C}$ is smooth, then*

$$IC(C, \mathbb{1}_C)\,|_D = \begin{cases} \mathbb{1}_D[\dim C] & D \subseteq \overline{C}, \\ \\ 0 & \text{otherwise}, \end{cases}$$

where by $\mathbb{1}_C$ we mean the sheaf for the trivial representation of A_C.

We now give the following definition regarding the size of our resolution of singularities which will be used for further computations regarding perverse sheaves:

Definition 2.15. Let C be an H_λ-orbit. Then the map $p : \widetilde{C} \to \overline{C}$ is said to be semi-small if, for all orbits $D \subseteq \overline{C}$, and $d \in D$, we have that

$$2\dim(p^{-1}(d)) \leq \dim(C) - \dim(D).$$

Further the map is said to be small if the inequality is strict for $D \subset \overline{C}$ (and $D \neq C$).

This definition is essentially a statement about the boundedness of the dimensions of the fibres on C.

Proposition 2.16. *Suppose that the morphism $p : Y \to X$ is proper and semi-small. Then, if L is any perverse sheaf on Y, the push-forward sheaf, p_*L is perverse.*

This is an immediate corollary of the decomposition theorem (for details see the formal statement of the theorem in [9]). The decomposition theorem gives us an effective method for computing the coefficients $m(C', L')$ in

$$p_*L = \sum_{(C',L')} m(C', L')\, IC(C', L').$$

We know that many of these coefficients will vanish. In particular, when the map p is small, we have that $m(C', L') = 0$, except for a single orbit.

When the map p is a proper morphism, the principle of proper base change also provides

an effective strategy to compute the coefficients $n(C', \mathcal{L}')$ in

$$p_* \mathcal{L} = \sum_{(C', \mathcal{L}')} n(C', \mathcal{L}') \, S(C', \mathcal{L}').$$

Since, we know that for standard sheaves

$$S(C', \mathcal{L}')_c = \begin{cases} \mathcal{L}'_c & \text{for } c \in C', \\ \\ 0 & \text{otherwise}, \end{cases}$$

where $\mathcal{L}_c$ is a vector space equipped with a representation ρ of A_c. Additionally, we know that

$$(p_* \mathcal{L})_c = \sum_{(C', \mathcal{L}')} n(C', \mathcal{L}') \mathcal{L}'|_c \quad \text{and} \quad (p_* \mathcal{L})_c = \sum_j H^j(p^{-1}(c), \mathcal{L}|_{p^{-1}(c)})[-j], \quad (2.5)$$

this latter expression is the statement of proper base change. When $\mathcal{L}$ is the constant sheaf associated with the trivial representation, $H^j(p^{-1}(c), \mathcal{L}|_{p^{-1}(c)})$ is the usual cohomology of the variety $p^{-1}(c)$.

Using this, the idea to construct the change of basis matrix m_{geo} is then to construct a third basis of the form $(p_* \mathcal{L})_i$ (where i indexes over all of the H_λ-orbits of V_λ). Using the above ideas, we can compute the change of basis for both bases.

To find a basis of the form $p_* \mathcal{L}$, for each orbit $C \subseteq V_\lambda$ and each local system $\mathcal{L}$ on C (parametrized by an irreducible representation $\rho \in \mathrm{Irrep}(A_c)$) we must construct an equivariant resolution of singularities of $\overline{C}$, that is, a proper cover $p : \widetilde{C} \to V_\lambda$ such that $\widetilde{C}$ is smooth and $I\mathcal{C}(C, \mathcal{L})$ appears in

$$p_* \mathbb{1}_{\widetilde{C}} = I\mathcal{C}(C, \mathcal{L}) + \sum_{(C', \mathcal{L}')} m(C', \mathcal{L}') \, I\mathcal{C}(C', \mathcal{L}').$$

These then give us a new basis for the Grothendieck group, which we can then use to

compute the change of basis matrix m_{geo}.

Studying these equivariant resolutions of singularities also gives us an effective method for doing computations with perverse sheaves, which arise from exact functors satisfying base change properties. For example, in [6] a functor Ev is introduced and used to conjecturally generalize Arthur's A-packets. The primary tool used to compute with this functor is precisely the same equivariant resolutions of singularities mentioned above (see, for example [6, Part 2], [5], [7]).

One of the main goals of our thesis is to provide the exposition and methods for computing equivariant resolutions of singularities $\widetilde{C}$ for each orbit $C \subseteq V_\lambda$, in the case of $\hat{G} = \text{GL}_n$, and explain its extension to SO_n and Sp_{2n}. It is generally understood how to do this for GL_n (see examples in [6, Part 2] and [5]), though a systematic approach is only ever alluded to. In Section 4.3.2 we provide a clear exposition, which will make the method precise and prove a systematic approach. For the cases of SO_n and Sp_{2n} only sporadic examples have been worked out in [6]. In this thesis, we provide the exposition on infinitesimal parameters arising in SO_n and Sp_{2n}, and explain new intricacies that can arise, such as non-trivial equivariant fundamental groups. The fact that one will find non-trivial equivariant fundamental groups in SO_n and Sp_{2n}, requires one to additionally find resolutions for certain finite covers of orbit closures. Lastly, we give an explicit computation of the multiplicity matrix for an infinite family of infinitesimal parameters arising in $\hat{G} = \text{GL}_n$, this will be done in Section 4.4.

Chapter 3

Geometry of the Langlands Parameters for Classical Groups

3.1 Geometry of the Moduli Space of Langlands Parameters in GL_n

In this section, we will be studying aspects of the geometry of the moduli space of Langlands parameters for GL_n and further down, we will relay the same construction in the subgroups SO_n and Sp_{2n} with the introduction of a symmetric respectively alternating bilinear pairing J. We will discuss how the Vogan variety V_λ and the centralizer H_λ are constructed, how they decompose, a description of their orbits, and (equivariant) fundamental groups of these orbits, where everything will be made more concrete.

3.1.1 Definition of the Vogan Variety V_λ

Recall that the Vogan variety, V_λ is the q-eigenspace of λ acting on the Lie algebra of the Lie group $\hat{G}$, which we denote by $\hat{\mathfrak{g}}$. This description can be concretely described as the vector space

$$V_\lambda := \{v \in \hat{\mathfrak{g}} \mid \mathrm{Ad}(\lambda(\mathrm{Frob}_p))v = qv\},$$

where Ad is the adjoint representation (or adjoint action) of the Lie group $\hat{G}$, is a way of representing elements of the group as linear transformations of the group's Lie algebra $\hat{\mathfrak{g}}$, considered as a vector space [11]. In the case of $\hat{G} = \mathrm{GL}_n$, the adjoint representation is the group homomorphism that sends an invertible $n \times n$ matrix $\lambda(\mathrm{Frob}_p)$, to an endomorphism of the vector space of all linear transformations, here defined by

$$v \mapsto \lambda v \lambda^{-1}.$$

The semi-simple element $\lambda(\mathrm{Frob_p}) \in \hat{G}$ defines the diagonalizable matrix λ, whence specifying a value on $\mathrm{Frob_p}$, hence we will assume that λ is diagonal without any loss of generality.

3.1.2 Quiver Varieties in the General Linear Group

We begin by describing the Vogan variety as they appear in the general linear group $\hat{G} = GL_n$, this will be important in understanding their structure as we pass to the subgroups $\hat{G}' = SO_n$ or Sp_{2n}. Since $\lambda \in \hat{G}$ is diagonalizable, we can simplify the study of these varieties by reordering our eigenvalues. The eigenvalues in λ will be reordered in such a way that the study of the larger Vogan varieties can be reduced to the study of individual (non-interacting) blocks, which consist of maps between the eigenspaces of λ detailed below in the following lemma.

Lemma 3.1. *Suppose that the infinitesimal parameter $\lambda \in \hat{G} = GL_n$. Then, the Vogan variety V_λ, can be decomposed as a finite direct product of blocks of varieties*

$$V_\lambda = \prod_{j=1}^{m} V_\lambda^{(j)}$$

where each block consists of chains of maps between the eigenspaces of λ. That is, each $V_\lambda^{(j)}$ is a variety of the form:

$$V_\lambda^{(j)} \cong \hom(E_2, E_1) \times \hom(E_3, E_2) \times \cdots \times \hom(E_k, E_{k-1}),$$

where each E_i is an eigenspace with eigenvalue $\lambda_i = q^{a_i}$ such that $a_{i+1} = a_i - 1$.

I.e., under this presentation, the sequential eigenvalues of λ will differ by a power of q.

Proof. Let E denote the vector space on which $\hat{G}$ is acting, and let λ be a semi-simple element of $\hat{G}$ such that $\lambda : E \rightarrow E$. Then we can write E in terms of the following direct sum decomposition as the subspace spanned by the eigenvectors of λ as

$$E \cong E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_n},$$

where E_{λ_i} denotes the eigenspace corresponding to the eigenvalue λ_i. With this decomposition, we can identify the entire Lie algebra in terms of a collection of maps between these eigenspaces,

$$\hat{\mathfrak{g}} = \hom(E, E) = \bigoplus_{i,j} \hom(E_{\lambda_i}, E_{\lambda_j}).$$

The Vogan variety $V_\lambda := \{v \in \hat{\mathfrak{g}} \mid \lambda v \lambda^{-1} = qv\}$ is a subspace of this Lie algebra. Since $\lambda v \lambda^{-1} = (v_{ij}\lambda_i\lambda_j^{-1})_{1 \leq i,j \leq n}$, we have that v_{ij} is a q-eigenvector when $\lambda_i\lambda_j^{-1} = q$. We are free to reorder and group the eigenvalues of λ in such a way that consecutive eigenvalues form maximal chains with respect to the following condition:

$$\lambda_i\lambda_{i+1}^{-1} = q^{a_i}q^{-a_{i+1}} = q^{a_i - a_{i+1}} = q \implies a_i - a_{i+1} = 1 \iff a_{i+1} = a_i - 1.^{13}$$

These maximal chains will correspond to blocks $V_\lambda^{(j)}$ in V_λ. Now, we partition the collection of eigenvalues of λ into a disjoint union of subsets satisfying that for $\lambda_i = q^{a_i}$ we have that $a_{i+1} = a_i - 1$, and re-write the set of eigenvalues of λ as

$$\left\{ \lambda_1^{(1)}, \lambda_2^{(1)}, \ldots, \lambda_{k_1}^{(1)}, \lambda_1^{(2)}, \lambda_2^{(2)}, \ldots, \lambda_{k_2}^{(2)}, \ldots \lambda_1^{(m)}, \lambda_2^{(m)}, \ldots, \lambda_{k_m}^{(m)} \right\},$$

where each subset $\{\lambda_1^{(j)}, \lambda_2^{(j)}, \ldots, \lambda_{k_j}^{(j)}\}_{1 \leq j \leq m}$ forms a maximal chain.

---

[13]We can write $\lambda_i = q^{a_i}$, by taking the principle branch of the complex logarithm.

Then, after selecting a basis where λ is diagonal, we may write

$$\lambda = \begin{pmatrix} \lambda^{(1)} & 0 & \cdots & 0 \\ 0 & \lambda^{(2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda^{(m)} \end{pmatrix}, \quad \text{where } \lambda^{(j)} = \begin{pmatrix} \lambda_1^{(j)}\mathrm{Id}_{n_1^{(j)}} & 0 & \cdots & 0 \\ 0 & \lambda_2^{(j)}\mathrm{Id}_{n_2^{(j)}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{k_j}^{(j)}\mathrm{Id}_{n_{k_j}^{(j)}} \end{pmatrix},$$

and $n_i^{(j)}$ are the multiplicities of $\lambda_i^{(j)}$ respectively. A direct computation following this construction for V_λ will result in a block decomposition of its elements in the form:

$$v^{(j)} = \begin{pmatrix} 0 & v_1^{(j)} & 0 & \cdots & 0 \\ 0 & 0 & v_2^{(j)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & v_{k_j-1}^{(j)} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \cong v_1^{(j)} \oplus v_2^{(j)} \oplus \cdots \oplus v_{k_j-1}^{(j)},$$

where $v_i^{(j)} \in \hom(E_{\lambda_{i+1}^{(j)}}, E_{\lambda_i^{(j)}})$.

Thus, if we write E_i for the eigenspace corresponding to the eigenvalue $\lambda_i^{(j)}$ of λ, the Vogan variety V_λ is naturally isomorphic to the desired product of varieties of the form:

$$V_\lambda^{(j)} \cong \hom(E_2, E_1) \times \hom(E_3, E_2) \times \cdots \times \hom(E_k, E_{k-1}). \qquad \square$$

We split the Vogan variety V_λ in terms of these blocks $V_\lambda^{(j)}$ and will only be studying one block at a time, since each V_λ will look like a direct product of $V_\lambda^{(j)}$ and since the geometry and categories we are interested in are compatible with this direct product structure.

Remark 3.2. In this thesis, we have chosen by convention that $a_{i+1} = a_i - 1$ be the condition defining the partitions of the eigenvalues of λ. On the other hand, one could have chosen by convention that the eigenvalues satisfy $a_{i+1} = a_i + 1$, and our maps v_i would appear as $v_i : E_i \to E_{i+1}$. This essentially comes down to a relabeling, running from $1 \to k$ (yielding the former condition) or $k \to 1$ (yielding the latter condition).

In this thesis, we consider $a_i \in (\frac{1}{2})\mathbb{Z}$, since integral and half-integral powers of q will allow for the study of symmetric quiver varieties, a necessity when passing to $\lambda \in \hat{G}' = SO_n$ or Sp_{2n}. However, in principle, the eigenspaces may be skewed by a factor of ξ, for any $\xi \in \mathbb{C}$, this would result in eigenvalues of the form $\lambda_i = \xi q^{a_i} = q^{a_i + \log_q(\xi)}$. When working with a single block, we may proceed without any loss in generality as though $\xi = 1$, since the geometry of the block is not impacted by the skew factor. The possibility of different skew factors is essentially what allows for multiple pieces in the direct product decomposition.

Example 3.3. Consider the Vogan variety defined by $\lambda = (q^3, q^2, q^1)$. The geometry defined by this Vogan variety is the same as that of $\lambda' = (q^1, q^0, q^{-1})$, since they are equal up to a skew factor, $\lambda = q^2\lambda'$ (here $\xi = q^2$). However, the latter is symmetric, which would allow us to pass into $\hat{G}' = SO_n$ or Sp_{2n}.

In principle, this skew factor could be any complex number, for example $\xi = \sqrt{2} + \pi i$. However, without any loss in generality, we consider the "reduced" varieties where $\lambda_i = q^{a_i}$, for $a_i \in (\frac{1}{2})\mathbb{Z}$.

Remark 3.4. The elements of V_λ, are known as quiver representations, and its elements will be denoted by $v = (v_1, v_2, \ldots, v_{k-1})$ for $v_i \in \hom(E_{i+1}, E_i)$, as

$$\overset{E_1}{\bullet} \xleftarrow{ v_1 } \overset{E_2}{\bullet} \xleftarrow{ v_2 } \cdots \xleftarrow{ v_{k-2} } \overset{E_{k-1}}{\bullet} \xleftarrow{ v_{k-1} } \overset{E_k}{\bullet}.$$

This is a unidirectional quiver of type A.

Definition 3.5 (Quiver Variety). A quiver is a directed graph where loops and multiple arrows between two vertices are allowed. A representation V of a quiver assigns a vector space $V(x)$ to each vertex x of the quiver and a linear map $V(a)$ to each arrow a. A quiver representation variety is the variety of all representations for a fixed set of vector spaces $V(x)$.

The Vogan variety is equipped with the natural action by conjugation with the centralizer

$$H_\lambda := Z_{\hat{G}}(\lambda(\mathrm{Frob_p})) := \{h \in \hat{G} \mid h\lambda = \lambda h\},$$

which may be decomposed as

$$H_\lambda \cong \mathrm{GL}(E_1) \times \mathrm{GL}(E_2) \times \cdots \times \mathrm{GL}(E_k).$$

We denote the elements of H_λ by $h = (h_1, h_2, \ldots, h_k)$ for $h_i \in \mathrm{GL}(E_i)$.

Lemma 3.6. *With notation as above, the action of H_λ on V_λ is given by the conjugation*

$$(h \cdot v)_i := h_i v_i h_{i+1}^{-1},$$

for $h = (h_1, h_2, \ldots, h_k) \in H_\lambda$ and $v = (v_1, v_2, \ldots, v_{k-1}) \in V_\lambda$.

Proof. Indeed, with our preferred choice of basis, the Vogan variety V_λ and the centralizer H_λ which acts on V_λ are of the form

$$V_\lambda = \left\{ \begin{pmatrix} 0 & v_1 & 0 & \cdots & 0 \\ 0 & 0 & v_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & v_{k-1} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \right\} \quad \text{and} \quad H_\lambda = \left\{ \begin{pmatrix} h_1 & 0 & 0 & \cdots & 0 \\ 0 & h_2 & 0 & \cdots & 0 \\ 0 & 0 & h_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & h_k \end{pmatrix} \right\},$$

where $v_i \in \mathrm{hom}(E_{i+1}, E_i)$ and $h_i \in \mathrm{GL}(E_i)$. Then when we conjugate V_λ by H_λ, we find that

$$h.v = \begin{pmatrix} h_1 & 0 & 0 & \cdots & 0 \\ 0 & h_2 & 0 & \cdots & 0 \\ 0 & 0 & h_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & h_k \end{pmatrix} \begin{pmatrix} 0 & v_1 & 0 & \cdots & 0 \\ 0 & 0 & v_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & v_{k-1} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} h_1^{-1} & 0 & 0 & \cdots & 0 \\ 0 & h_2^{-1} & 0 & \cdots & 0 \\ 0 & 0 & h_3^{-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & h_k^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & h_1 v_1 h_2^{-1} & 0 & \cdots & 0 \\ 0 & 0 & h_2 v_2 h_3^{-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & h_{k-1} v_{k-1} h_k^{-1} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

23

So, denoting by $v = (v_1, v_2, \ldots, v_{k-1})$ for the elements of the Vogan variety V_λ and by $h = (h_1, h_2, \ldots, h_k)$, the elements of the centralizer H_λ, then the action of H_λ on V_λ is given by

$$(h.v)_i = (h_i v_i h_{i+1}^{-1}). \qquad \qquad \square$$

Example 3.7. Consider the infinitesimal parameter

$$\lambda = \text{diag}(\underbrace{q^{1/2}, \ldots, q^{1/2}}_{k-\text{times}}, \underbrace{q^{-1/2}, \ldots, q^{-1/2}}_{k-\text{times}}).$$

The Vogan variety will consist of a collection of maps between the q-eigenspaces of λ. Here we see that there are two eigenspaces, $E_{q^{1/2}}$ and $E_{q^{-1/2}}$, each of which has dimension k and they differ by a power of q. The Vogan variety will then consist of the map

$$v : E_{q^{-1/2}} \to E_{q^{1/2}} \quad \text{where } v \in \text{Mat}_{k \times k}.$$

The corresponding Vogan variety V_λ and centralizer H_λ are of the form:

$$V_\lambda = \left\{ \begin{pmatrix} 0 & v_1 \\ 0 & 0 \end{pmatrix} \middle| v_1 \in \text{Mat}_{k \times k} \right\} \quad \text{and} \quad H_\lambda = \left\{ \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} \middle| \begin{array}{l} h_1 \in \text{GL}(E_{q^{1/2}}) \\ h_2 \in \text{GL}(E_{q^{-1/2}}) \end{array} \right\}.$$

The action of H_λ on V_λ is given by

$$h.v := h_1 \, v_1 \, h_2^{-1} \quad \text{for } h = (h_1, h_2) \in H_\lambda \text{ and } v = (v_1) \in V_\lambda.$$

We can consider the orbits under this action, which are determined by the rank of the v, since the rank of these orbits is invariant under simultaneous base change. Let r be an integer such that $0 \le r \le k$, then the orbits are described by

$$C_r = \{v \in V_\lambda \mid \text{rank}(v) = r\}.$$

Thus, C_r consists of all $k \times k$ matrices v of rank r. The closure $\overline{C_r}$ will consist of the set of all matrices of rank at most r, that is

$$\overline{C_r} = \bigcup_{i \leq r} C_i = \{v \in V_\lambda \mid \text{rank}(v) \leq r\}. \tag{3.1}$$

The spaces of matrices with a given upper bound on their ranks are well understood in the literature and are known as determinantal varieties. The equations defining the orbits closure $\overline{C_r}$ is generated by the $(r+1) \times (r+1)$ minors of $v \in V_\lambda$ of rank r and is naturally an algebraic variety as the condition that a matrix have rank at most r is defined by the vanishing of all of its $(r+1) \times (r+1)$ minors.

In the next section, we explore how we can extend this construction to "generalized determinantal varieties" since the structure of our Vogan variety can consist of many of these determinant conditions that form determinantal varieties. The orbit classification will then depend on the ranks of the compositions of matrices mapping between the eigenspaces of λ.

3.1.3 Classification of Orbits

The setup in [6, Section 4.3] gives a bijection between Langlands parameters with infinitesimal parameter λ and the H_λ-orbits in V_λ. Since H_λ acts on V_λ by simultaneous change of basis, we can use this as a means to classify the H_λ-orbits, C in V_λ. The H_λ-orbits in V_λ are classified by the ranks of all possible combinations of the matrices $v_1, v_2, \ldots, v_{k-1}$ appearing in $v \in V_\lambda$. More precisely, this idea may be captured by the following proposition.

Proposition 3.8. *The H_λ-orbits of $v = (v_1, v_2, \ldots, v_{k-1}) \in V_\lambda$ are fully characterised by the ranks*

$$r_{ij} := \text{rank}(v_i \cdots v_j), \text{ for } 0 < i \leq j < k.$$

As an immediate consequence, V_λ is stratified into finitely many orbits.[14]

---

[14]We may sometimes refer to the product $v_i \cdots v_j$ as v_{ij}, then the map v_i may sometimes be written as v_{ii}.

We will be giving a sketch of the proof for this proposition, for a formal proof see Connor Riddlesden's thesis [14, Proposition 2.2.2].

Sketch of Proof for Proposition 3.8. We begin by gathering some facts about quiver representations from Derksen-Weyman's "An Introduction to Quiver Representations" [10].

- The Krull-Remak-Schmidt Theorem [10, §1.7, Theorem 1.7.4] says every quiver representation is uniquely a sum of indecomposables.[15]

- Gabriel's Theorem [10, §4.4, Theorem 4.4.13] classifies indecomposable quiver representations.[16]

- By [10, §1.3, Corollary 1.3.1], we have that there is a bijection between quiver representations and orbits.

Lastly, an observation that the ranks distinguish the indecomposables completes the argument. □

Remark 3.9. The classification for these orbits descends to symmetric quiver representations; this is a fact that will be made precise in Proposition 3.22, when applied to our study of the subgroups SO$_n$ and Sp$_{2n}$ of GL$_n$.

With this proposition, we have a finite decomposition of the H_λ-orbits of V_λ in terms of their ranks. In particular, we will enumerate the orbits in terms of a rank-triangle in the following way:

An infinitesimal parameter $\lambda : \langle \text{Frob}_p \rangle \to \text{GL}_n(\mathbb{C})$, may be defined as

$$\lambda(\text{Frob}_p) := n_1\lambda_1 \oplus n_2\lambda_2 \oplus \cdots \oplus n_{k-1}\lambda_{k-1} \oplus n_k\lambda_k,$$

where n_i denotes the multiplicity of the eigenvalue $\lambda_i \in \text{Frob}_p$. Since every H_λ-orbit C in V_λ is uniquely determined by the ranks appearing in Proposition 3.8, these equations

---

[15]This is analogous to the existence and uniqueness of a Jordan decomposition.
[16]This is a generalizations of the structure theorem for individual Jordan blocks.

exactly describe C as a generalized determinantal variety, also known as a rank variety in the literature. We arrange these ranks into a triangle to reflect the corresponding maps between the eigenspaces and refer to this as a rank-triangle:

$$
C: \quad
\begin{array}{ccccc}
n_1 & n_2 & \cdots & & n_k \\
\hline
 & r_{11} & r_{22} & \cdots & r_{k-1\,k-1} \\
 & & r_{12} & \vdots & \\
 & & & \ddots & \\
 & & & r_{1\,k-1} &
\end{array}
\quad .
$$

The integer values given in the rank-triangle represent the ranks of the maps between the eigenspaces of λ:

1. The first row encodes the dimensions of the overlying eigenspaces: $\dim(E_i) = n_i$.[17]

2. The second row encodes the ranks r_{ii} of the maps v_i moving between the eigenspaces, that is $v_i : E_{i+1} \to E_i$.

3. The following rows encode the ranks r_{ij} of the maps $v_i \cdots v_j : E_{j+1} \to E_i$.

Remark 3.10. Proposition 3.8 is analogous to the classification of nilpotent matrices in Jordan canonical form. Indeed, since the elements $v \in V_\lambda$ take the form of a super diagonal (nilpotent) matrix (see Lemma 3.1), we may consider taking the natural powers of these elements. Then the rows of the rank-triangle correspond to taking the ranks of the non-zero elements of the Vogan variety's natural powers (see Table 3.1).

---

[17]Later on, we refer to this value as $n_i := r_{i(i-1)}$ so that the rank-triangle will entirely consists of the ranks r_{ij} and Proposition 3.11 will still hold. In particular, the first row corresponds to the ranks of the identity map on E_i, for each $1 \le i \le k$ (see the "0'th power" of $v \in V_\lambda$, in Table 3.1).

Table 3.1: Analogy to the Jordan Decomposition of Nilpotent Matrices

	Quiver Representation	Quiver
v^0	$(\mathrm{Id}_{n_1}, \mathrm{Id}_{n_2}, \ldots, \mathrm{Id}_{n_k})$	
v^1	$(v_1, v_2, \ldots, v_{k-1})$	
v^2	$(v_{12}, v_{23}, \ldots, v_{(k-2)(k-1)})$	
$\vdots$	$\vdots$	
v^{k-1}	$(v_{1(k-1)})$	

Since $v \in V_\lambda$ is nilpotent, we have that $v^k = 0$. Now, if we consider taking the ranks of these maps, we recover the rows of the rank-triangle, which we then use to classify the orbits.

The collection of H_λ-orbits in V_λ is naturally in bijection with the rank-triangles subject to the conditions in the following proposition.

Proposition 3.11. *[5, Section 1.5] The ranks $r_{ij} = \mathrm{rank}\,(v_i \cdots v_j)$ with $i \leq j$ are subject to the following conditions:*

1. *Every rank is less than or equal to the two ranks above it, more succinctly written as*

$$r_{i(j+1)} \leq r_{ij} \leq r_{(i+1)j}.$$

2. *Any sub-diamond in the rank-triangle must satisfy the property:*

$$r_{ik} - r_{ij} \leq r_{\ell k} - r_{\ell j}, \quad \text{with } \ell \leq i, \ k \leq j.$$

These conditions are immediate corollaries of the rank-nullity theorem. These are basic facts from linear algebra that concern the ranks of the compositions of linear maps:

- The first condition is equivalent to

$$\mathrm{rank}(v_i \cdots v_j) \leq \min(\mathrm{rank}(v_i), \ldots, \mathrm{rank}(v_j)), \ \forall \ 0 < i \leq j < k.$$

- The second condition can be seen via a basic application of the rank-nullity theorem or by applying the Frobenius inequality from linear algebra, that

$$\text{rank}(AB) + \text{rank}(BC) \leq \text{rank}(ABC) + \text{rank}(B),$$

where $A \in \text{Mat}_{k \times \ell}$, $B \in \text{Mat}_{\ell \times m}$, $C \in \text{Mat}_{m \times n}$.

Proposition 3.12. *The rank-triangle for every H_λ-orbit C records the equations that define C as a subvariety in V_λ*

$$C = \{v \in V_\lambda \mid \text{rank}(v_i \cdots v_j) = r_{ij}, \ 0 < i \leq j < k\},$$

where r_{ij} refers to the terms appearing in the rank-triangle for C. The Zariski closure of C in V_λ is then given by

$$\overline{C} = \{v \in V_\lambda \mid \text{rank}(v_i \cdots v_j) \leq r_{ij}, \ 0 < i \leq j < k\}.$$

This proposition leads to the following partial ordering.

Definition 3.13. Let $C_0, C_1, \ldots, C_n$ denote the H_λ-orbits in the Vogan variety V_λ. We introduce the partial ordering on the above collection of C_i as $C_i \leq C_j$ if $C_i \subseteq \overline{C_j}$.

This partial ordering is well understood from the above rank-triangle, as follows: If r_{ij}, r'_{ij} are the ranks of C, C' respectively, then $C \leq C'$ if and only if $r_{ij} \leq r'_{ij}$, for all i, j.

In order to illustrate this construction, we provide an example:

Example 3.14. Consider the infinitesimal parameter $\lambda = \text{diag}(q, 1, 1, 1, q^{-1})$.

For $\hat{G} = \text{GL}_5$, we present both the Vogan variety $V_\lambda = \{v \in \hat{\mathfrak{g}} \mid \lambda v \lambda^{-1} = qv\}$, and the centralizer $H_\lambda = \{h \in \hat{G} \mid hvh^{-1} = v\}$, in terms of matrices as maps related to the eigenspace of λ. They are given by

$$V_\lambda = \left\{ \begin{pmatrix} 0 & x_1 & x_2 & x_3 & 0 \\ 0 & 0 & 0 & 0 & y_1 \\ 0 & 0 & 0 & 0 & y_2 \\ 0 & 0 & 0 & 0 & y_3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right\} \text{ and } H_\lambda = \left\{ \begin{pmatrix} h_1 & 0 & 0 & 0 & 0 \\ 0 & a_1 & b_1 & c_1 & 0 \\ 0 & a_2 & b_2 & c_2 & 0 \\ 0 & a_3 & b_3 & c_3 & 0 \\ 0 & 0 & 0 & 0 & h_3 \end{pmatrix} \middle| \begin{matrix} h_1, h_3 \neq 0 \\ \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \in \text{GL}_3 \end{matrix} \right\}.$$

We note that $V_\lambda \cong \text{hom}(E_1, E_q) \times \text{hom}(E_{q^{-1}}, E_1)$ and $H_\lambda \cong \text{GL}(E_q) \times \text{GL}(E_1) \times \text{GL}(E_{q^{-1}})$.

The action of conjugation on V_λ by H_λ is given by:

$$h.v := \left(h_1 \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}^{-1}, \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} h_3^{-1} \right).$$

We stratify V_λ into H_λ-orbits by considering the rank of the compositions of these matrices:

$$C_0 = \{ v \in V_\lambda \mid x, y = 0 \}, \quad C_1 = \{ v \in V_\lambda \mid x = 0, \ y \neq 0 \}, \quad C_2 = \{ v \in V_\lambda \mid x \neq 0, \ y = 0 \}$$

$$C_3 = \{ v \in V_\lambda \mid x, y \neq 0, \ xy = 0 \}, \quad \text{and} \quad C_4 = \{ v \in V_\lambda \mid xy \neq 0 \}.$$

Indeed, observe that we can present each of these orbits in terms of their given rank-triangles characterised by Proposition 3.8 as

$$C_0: \begin{array}{ccc} 1 & 3 & 1 \\ \hline 0 & & 0 \\ & 0 & \end{array}, \quad C_1: \begin{array}{ccc} 1 & 3 & 1 \\ \hline 0 & & 1 \\ & 0 & \end{array}, \quad C_2: \begin{array}{ccc} 1 & 3 & 1 \\ \hline 1 & & 0 \\ & 0 & \end{array},$$

$$C_3: \begin{array}{ccc} 1 & 3 & 1 \\ \hline 1 & & 1 \\ & 0 & \end{array}, \quad \text{and} \quad C_4: \begin{array}{ccc} 1 & 3 & 1 \\ \hline 1 & & 1 \\ & 1 & \end{array}$$

The collection of H_λ-orbits in V_λ carry the following partial ordering with respect to the Zariski topology:

$$C_0 < C_1, C_2 < C_3 < C_4.$$

A partial ordering since, $C_1 \nless C_2$, while they both contain C_0 in their closures and are contained within the closure of C_3, they are distinctly not contained within the other's closure, which is why this is considered a partial ordering. Thus, this partial ordering is more accurately understood in terms of Figure 3.1.
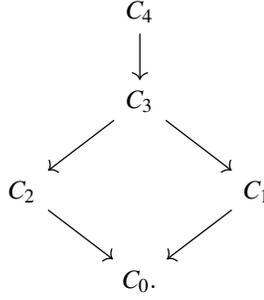
Figure 3.1: Diagram describing the partial ordering on the 5 H_λ-orbits in Example 4.29.

This partial ordering will ultimately allow us to analyse the singularities of the above orbits, since the singularities of $\overline{C}$ must be on its boundary, that is, somewhere on $C' < C$.

In Example 3.14, we may enumerate the simple objects as in Equation (2.2) by computing the component group of the stabilizer, denoted A_C. Upon computing these stabilizers, one would find that their stabilizers are all connected, and hence their equivariant fundamental groups, A_C were trivial, so they each carried only one equivariant local system. Thus, the simple objects in $\mathrm{Per}_{H_\lambda}(V_\lambda)$ are enumerated by

$$\mathrm{Per}_{H_\lambda}^{\mathrm{simple}}(V_\lambda) = \{IC(C_0, \mathcal{L}_{\mathbb{1}_{C_0}}), IC(C_1, \mathcal{L}_{\mathbb{1}_{C_1}}), IC(C_2, \mathcal{L}_{\mathbb{1}_{C_2}}), IC(C_3, \mathcal{L}_{\mathbb{1}_{C_3}}), IC(C_4, \mathcal{L}_{\mathbb{1}_{C_4}})\},$$

where by $\mathcal{L}_{\mathbb{1}_{C_i}}$ we mean the sheaf for the trivial representation on C_i.[18]

Proposition 3.15. *Suppose that $\lambda \in \hat{G} = \mathrm{GL}_n$. Then, the stabilizers $\mathrm{Stab}_c(H_\lambda)$ of the H_λ-orbits C in V_λ are connected. Consequently, the equivariant fundamental groups A_C are trivial.*

This is a well-known and utilised fact in the *p*-adic local Langlands correspondence, as such, we will only be giving a sketch using other well-established results in the field.

---

[18]In other places, such as in Section 4.4, we may write the sheaf for the trivial representation on the orbit as $\mathbb{1}_{C_i} := \mathcal{L}_{\mathbb{1}_{C_i}}$

Sketch of Proof. We may identify the stabilizer of a point $c \in V_\lambda$ with the automorphism group of its associated quiver representation by [4, Lemma 2.1.2]. The structure of this automorphism group can be inferred from [4, Theorem 1.3.4] which states that there is a finite decomposition of vector spaces as

$$\text{End}(M) = I \oplus B,$$

where we may think of M as the vector space V_λ when viewed as a module for $\mathbb{C}[c]$, I is a nilpotent ideal, and B is a subalgebra isomorphic to $\prod_{i=1}^{\ell} \text{Mat}_{m_i \times m_i}$, for some $m_i \in \mathbb{N}$. Then, from [4, Proposition 2.2.1] we know that the automorphism group is an open affine subset of its endomorphism group, and as a consequence the automorphism group (and hence $\text{Stab}_c(H_\lambda)$) is a connected linear algebraic group, since it admits the following decomposition

$$\text{Stab}_c(H_\lambda) = U \rtimes \prod_{i=1}^{\ell} \text{GL}(m_i), \tag{3.2}$$

where U is a closed unipotent subgroup (which is connected). Since this is a product of connected groups, the stabilizers over GL_n are connected, thus A_C is trivial. $\square$

3.2 Geometry of Langlands parameters in SO and Sp

In this section, we describe the transition from the group $\hat{G} = \text{GL}_n$ to the subgroups $\hat{G}' = \text{SO}_n$ or Sp_{2n}. An interesting geometry will arise in the study of these subgroups, namely the identification between the eigenspaces of λ which imposes heavy restrictions on the types of H'_λ-orbits that can arise in V'_λ and furthermore, this identification may give rise to non-trivial equivariant fundamental groups.

When passing from GL_n to one of the subgroups SO_n or Sp_{2n}, we introduce a non-degenerate bilinear pairing J which is used to define the respective subgroups. Upon fixing a basis, J is given by a nonsingular, (skew-)symmetric matrix. This pairing will give rise to a non-degenerate symmetric (symplectic) bilinear form, defined by $B(x, y) = x^T J y$.

Definition 3.16. Let V be a vector space, we say that a form $B(\cdot,\cdot) : V \times V \to \mathbb{C}$ is

- Non-degenerate: If $B(x,y) = 0$ for all $y \in V$ implies that $x = 0$;

- Bilinear: If it is linear in each argument separately; and is

    - Alternating: If $B(x,y) = -B(y,x)$ for all $x,y \in V$, whenever J is skew-symmetric;

    - Symmetric: If $B(x,y) = B(y,x)$ for all $x,y \in V$, whenever J is symmetric.

When J is skew-symmetric, and hence induces an alternating bilinear form, we define the group of J-symplectic matrices as

$$\mathrm{Sp} = \mathrm{Sp}_J = \left\{ M \in \mathrm{GL}(2n) \mid M^T J M = J \right\}.$$

Remark 3.17. Since J is a skew-symmetric matrix, we must have that matrices in $\mathrm{Sp}(n)$ are even dimensional, observe:

$$J = -J^T \implies \det(J) = \det(-J^T) \implies \det(J) = (-1)^{2n} \det(J).$$

Whenever J is symmetric, we analogously define the group of J-orthogonal matrices as

$$O_J = \left\{ M \in \mathrm{GL}_n \mid M^T J M = J \right\}.$$

The group of J-orthogonal matrices is disconnected[19], since $M^T J M = J$, we find that the matrices M must have two disconnected components, observe:

$$\implies \det(M^T J M) = \det(J) \iff \det(M^T)\det(J)\det(M) = \det(J) \iff \det(M)^2 = 1.$$

One denotes by $\mathrm{SO} := \mathrm{SO}_J$, for the special J-orthogonal group, the group of J-orthogonal matrices with determinant 1.

Remark 3.18. When $J = \mathrm{Id}_n$, we recover the familiar notion of the orthogonal and special orthogonal groups, $O(n)$ and $\mathrm{SO}(n)$ respectively.

---

[19]It is this fact which may give rise to the disconnected equivariant fundamental groups.

However, the standard choice of the identity matrix for this bilinear pairing, which one would choose in the familiar setting of SO_n does not work here. Since $\lambda \in \hat{G}$ is chosen to be a diagonal matrix and we want λ to preserve J, this means that J cannot also be chosen to be diagonal, or else $\lambda \in SO_n$ defined with respect to a diagonal matrix would imply that all of λ's eigenvalues are limited to ± 1.

Thus, whenever $\lambda \in \hat{G}' = SO$ or Sp, we may choose a non-degenerate (skew-)symmetric bilinear pairing, J, which will satisfy the following proposition:

Proposition 3.19. *If $\lambda \in \hat{G}'$ and λ preserves J, then J must induce isomorphisms $E_{\lambda_i} \cong E^*_{\lambda_i^{-1}}$[20] between the eigenspaces of λ, this implies*

1. *that the eigenvalues of λ must be symmetric, in that if λ_i is an eigenvalue, then λ_i^{-1} is also an eigenvalue with the same multiplicity; and*

2. *that in our notation, if the first condition is satisfied, and $\lambda_i, \lambda_i^{-1}$ occur in the same maximal chain (described in Lemma 3.1), we have*

$$E_i \cong E^*_{k-i}, \quad for \quad 0 \leq i \leq k.$$

Proof. A non-degenerate (skew-)symmetric matrix J induces a non-degenerate symmetric (symplectic) bilinear pairing $B(\cdot, \cdot)$. Then λ is said to preserve $J \leftrightarrow B(\cdot, \cdot)$ if for all x, y,

$$B(\lambda x, \lambda y) = x^T \lambda^T J \lambda y = x^T J y = B(x, y).$$

We conclude that for eigenvectors v_i and v_j[21] we must have that $B(v_i, v_j) = 0$, if $\lambda_i \neq \lambda_j^{-1}$. That is, if λ_i and λ_j do not come from complementary eigenspaces. Observe,

$B(\lambda v_i, \lambda v_j) = B(v_i, v_j)$ Since λ preserves J.

$B(\lambda_i v_i, \lambda_j v_j) = B(v_i, v_j)$ Since λ_i, λ_j are eigenvalues for v_i, v_j eigenvectors, resp.

$\lambda_i \lambda_j B(v_i, v_j) = B(v_i, v_j)$ Since B is a bilinear form.

---

[20]Where $*$ here denotes the dual space.

[21]We can decompose any vector $v \in V$ as a linear sum of eigenvectors in our vector space, since $B(\cdot, \cdot)$ is a bilinear pairing, knowing what happens to the eigenvectors informs us about any other vector in the space.

Then this implies that either $B(v_i, v_j) \equiv 0$ or $\lambda_j = \lambda_i^{-1}$. Since J defines a non-degenerate pairing, $B(\cdot, \cdot)$ cannot be identically zero, and hence we must have that λ_i and λ_i^{-1} must be complementary.

Now, if λ_i and λ_i^{-1} occur in the same maximal chain, described in Lemma 3.1, then in our notation, we will have that

$$E_i \cong E_{k-i}^*, \quad \text{for } 1 \leq i \leq k.$$

Otherwise, no two eigenspaces $E_1, E_2, \ldots, E_k$ in a given chain are dual, in which case the chain itself will be dual to some other chain arising in the total Vogan variety. In this case V_λ is isomorphic to one arising from an inclusion of a subgroup of type A,[22] i.e., as an inclusion of GL_n into SO_n or Sp_{2n}.

Since this result did not rely on J being a symmetric or symplectic form, the result extends to both SO and Sp. $\qquad\square$

Remark 3.20. A standard choice for J is the anti-diagonal identity matrix, where, in the case of Sp, we negate the first $n/2$ entries. Since all non-degenerate (skew-)symmetric bilinear forms are isomorphic over $\mathbb{C}$, it is convenient to choose one that is well adapted to our setting. We describe our choice of J below, which will be defined using the dimensions of the overlying eigenspaces; this will give us "nice" relations on the maps between these eigenspaces.

3.2.1 Symmetric Quiver Varieties in $\hat{G}' = \mathrm{SO}$ or Sp

When $\lambda \in \hat{G}$, the Vogan variety was decomposable as a finite direct product of maps between the q-eigenspaces of λ, that is, as a finite collection of quiver representations

$$V_\lambda \cong \hom(E_2, E_1) \times \hom(E_3, E_2) \times \cdots \times \hom(E_k, E_{k-1})$$

and the centralizer which acts on the Vogan variety was given by

$$H_\lambda \cong \mathrm{GL}(E_1) \times \mathrm{GL}(E_2) \times \cdots \times \mathrm{GL}(E_k).$$

---

[22]That is, Dynkin type A_n, or type A. Often used to describe the structure of semi-simple Lie Algebras.

When we descend to $\hat{G}' = $ SO or Sp, there are two cases to consider: either $E_i \cong E^*_{k-i}$, or no two eigenspaces $E_1, E_2, \ldots, E_k$ are dual. In the latter case, V_λ is isomorphic to one arising from an inclusion of a subgroup of type A, and one can freely study the variety by passing to this subgroup. In the former case, there are four sub-cases to consider depending on whether we are inside the orthogonal or symplectic group and if we have integral or half-integral powers of q as eigenvalues; such values will allow for $\lambda \in \hat{G}' = $ SO or Sp. We can classify the Vogan varieties since they are given by the intersection of the Lie sub-algebras with the Vogan variety associated with the classical groups over the complex numbers. In terms of this setup, we find that

Proposition 3.21. *[6, §10.2.1, p.90]*

Suppose that k denotes the number of distinct eigenvalues in λ and let $\ell = \lceil k/2 \rceil$.

1. *When λ consists of integral powers of q,[23] k is odd and the identification of eigenspaces allows us to decompose the Vogan variety in terms of the following maps between the eigenspaces of λ as:*

$$V'_\lambda \cong \hom(E_2, E_1) \times \hom(E_3, E_2) \times \cdots \times \hom(E_\ell, E_{\ell-1}),$$

   with the group H'_λ is acting on V'_λ by a factor of $\mathrm{GL}(E_i)$ at every factor except at E_ℓ, where the group is:

   (a) *$\mathrm{SO}(E_\ell)$, for $\lambda \in$ SO; and*

   (b) *$\mathrm{Sp}(E_\ell)$, for $\lambda \in$ Sp.*

2. *When λ consists of half-integral powers of q,[24], k is even and the identification of eigenspaces allows us to decompose the Vogan variety in terms of these maps between the eigenspaces of*

---

[23]There will be an <u>odd</u> number of eigenspaces, since our quiver varieties have been skewed to be symmetric and we have a q^0-eigenspace.

[24]There will be an <u>even</u> number of eigenspaces, since our quiver varieties have been skewed to be symmetric and we have no q^0-eigenspace.

(a) $\lambda \in$ SO *as:*

$$V'_\lambda \cong \hom(E_2, E_1) \times \hom(E_3, E_2) \times \cdots \times \hom(E_\ell, E_{\ell-1}) \times \text{Alt}^2(E_\ell); \text{ and}$$

(b) for $\lambda \in$ Sp *as:*

$$V'_\lambda \cong \hom(E_2, E_1) \times \hom(E_3, E_2) \times \cdots \times \hom(E_\ell, E_{\ell-1}) \times \text{Sym}^2(E_\ell);$$

with the group H'_λ *is acting on* V'_λ *by a factor of* $\text{GL}(E_i)$ *at every factor.*

We will choose J with respect to the overlying dimensions, n_i of the eigenspaces, E_i of λ in the following way:

1. When λ consists of integral powers of q, the corresponding top row of the rank-triangle will encode this collection as

$$\underline{n_1 \quad n_2 \quad \cdots \quad n_{\ell-1} \quad n_\ell \quad n_{\ell-1} \quad \cdots \quad n_2 \quad n_1}.$$

   (a) For $\lambda \in$ SO, we choose J as a matrix with identity blocks on the anti-diagonal, where the dimensions of each block corresponds to the dimensions of the overlying eigenspace.

   (b) For $\lambda \in$ Sp, we choose J as above in 1.(a), except that the first half of the columns have been negated, and the middle block corresponding to $n_\ell = \dim(E_\ell)$ is the standard n_ℓ-dimensional symplectic matrix.[25]

2. When λ consist of half-integral powers of q, the corresponding top row of the rank-triangle is similarly given as:

$$\underline{n_1 \quad n_2 \quad \cdots n_\ell \quad n_\ell \quad \cdots \quad n_2 \quad n_1}$$

   (a) For $\lambda \in$ SO, we choose J as above in 1.(a);

   (b) For $\lambda \in$ Sp, we choose J as above in 1.(a), except that the first half of the columns have been negated.

---

[25]For shorthand we will simple refer to this n_ℓ-dimensional symplectic matrix as J.

The above decompositions shown in Proposition 3.21 are demonstrated by the following computations: We note that the identification of $E_i \cong E_{k-i}^*$, for $1 \leq i \leq k$ will be demonstrated below, where the identification only becomes interesting/different whenever $E_\ell \cong E_\ell^*$ in the integral case, which will change the relation in the centralizer, and whenever the map $v_\ell : E_\ell^* \to E_\ell$ in the half-integral case, which will lead to the identification of v_ℓ with itself.

As such, we demonstrate the properties that arise from the fewest number of quiver representations and centralizing elements. Thus, we treat the middle quivers as individual elements, the outer quivers as blocks of elements, and the 1's and 0's as compatible identity and zero matrices, respectively. Observe,

1. Suppose that the eigenvalues of λ consist of integral powers of q, then we can look at the case for

   (a) $\lambda \in SO$; The Vogan variety is given by $V_\lambda' = \{v \in V_\lambda \mid Jv + v^T J = 0\}$, thus

   $$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & v_1 & 0 & 0 & 0 \\ 0 & 0 & v_2 & 0 & 0 \\ 0 & 0 & 0 & v_2^* & 0 \\ 0 & 0 & 0 & 0 & v_1^* \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & v_1 & 0 & 0 & 0 \\ 0 & 0 & v_2 & 0 & 0 \\ 0 & 0 & 0 & v_2^* & 0 \\ 0 & 0 & 0 & 0 & v_1^* \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} = 0$$

   $$\implies \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & v_1^* + v_1^T \\ 0 & 0 & 0 & v_2^* + v_2^T & 0 \\ 0 & 0 & v_2 + (v_2^*)^T & 0 & 0 \\ 0 & v_1 + (v_1^*)^T & 0 & 0 & 0 \end{pmatrix} = 0$$

   $$\implies v_1^* = -v_1^T \quad \text{and} \quad v_2^* = -v_2^T.$$

   Thus, we decompose the Vogan variety in terms of these maps between the eigenspaces of λ as:

   $$V_\lambda' \cong \underbrace{\prod_{i=1}^{\ell-2} \hom(E_{i+1}, E_i)}_{v_1} \times \underbrace{\hom(E_\ell, E_{\ell-1})}_{v_2};$$

The centralizer is given by $H'_\lambda = \{h \in H_\lambda \mid h^T J h = J\}$, thus

$$
\begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_1^* \end{pmatrix}^T
\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}
\begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_1^* \end{pmatrix}
=
\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}
$$

$$
\implies
\begin{pmatrix} 0 & 0 & h_1^T h_1^* \\ 0 & h_2^T h_2 & 0 \\ (h_1^*)^T h_1 & 0 & 0 \end{pmatrix}
=
\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}
$$

$$\implies h_1^* = h_1^{-T} \quad \text{and} \quad h_2 \in \mathrm{SO}(E_2).$$

This gives us the following decomposition for the centralizer,

$$H'_\lambda \cong \underbrace{\prod_{i=1}^{\ell-1} \mathrm{GL}(E_i)}_{h_1} \times \underbrace{\mathrm{SO}(E_\ell)}_{h_2};$$

(b) $\lambda \in \mathrm{Sp}$[26]; The Vogan variety is given by $V'_\lambda = \{v \in V_\lambda \mid Jv + v^T J = 0\}$, thus

$$
\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & J & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}
\begin{pmatrix} 0 & v_1 & 0 & 0 & 0 \\ 0 & 0 & v_2 & 0 & 0 \\ 0 & 0 & 0 & v_2^* & 0 \\ 0 & 0 & 0 & 0 & v_1^* \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}
+
\begin{pmatrix} 0 & v_1 & 0 & 0 & 0 \\ 0 & 0 & v_2 & 0 & 0 \\ 0 & 0 & 0 & v_2^* & 0 \\ 0 & 0 & 0 & 0 & v_1^* \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T
\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & J & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}
= 0
$$

$$
\implies
\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & v_1^* \\ 0 & 0 & 0 & Jv_2^* & 0 \\ 0 & 0 & -v_2 & 0 & 0 \\ 0 & -v_1 & 0 & 0 & 0 \end{pmatrix}
+
\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & v_1` \\ 0 & 0 & 0 & v_2^T & 0 \\ 0 & 0 & J(v_2^*)^T & 0 & 0 \\ 0 & -(v_1^*)^T & 0 & 0 & 0 \end{pmatrix}
= 0
$$

$$\implies v_1^* = -v_1^T, \text{ and } Jv_2^* = v_2^T \text{ or } v_2^* = -Jv_2^T.$$

Thus, we decompose the Vogan variety in terms of these maps between the eigenspaces of λ as:

$$V'_\lambda \cong \underbrace{\prod_{i=1}^{\ell-2} \hom(E_{i+1}, E_i)}_{v_1} \times \underbrace{\hom(E_\ell, E_{\ell-1})}_{v_2};$$

---

[26]This case requires that "the middle block, or the q^0 eigenspace" be even dimensional.

The centralizer is given by $H'_\lambda = \{h \in H_\lambda \mid h^T Jh = J\}$, thus

$$\begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_1^* \end{pmatrix}^T \begin{pmatrix} 0 & 0 & 1 \\ 0 & J & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_1^* \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & J & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$\implies \begin{pmatrix} 0 & 0 & h_1^T h_1^* \\ 0 & h_2^T J h_2 & 0 \\ -(h_1^*)^T h_1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & J & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$\implies h_1^* = h_1^{-T} \quad \text{and} \quad h_2 \in \mathrm{Sp}(E_2).$$

This gives us the following decomposition for the centralizer,

$$H'_\lambda \cong \underbrace{\prod_{i=1}^{\ell-1} \mathrm{GL}(E_i)}_{h_1} \times \underbrace{\mathrm{Sp}(E_\ell)}_{h_2};$$

2. The eigenvalues of λ consist of half-integral powers of q, then we can look at the case for

   (a) $\lambda \in \mathrm{SO}$; The Vogan variety is given by $V'_\lambda = \{v \in V_\lambda \mid Jv + v^T J = 0\}$, thus

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & v_1 & 0 & 0 \\ 0 & 0 & v_2 & 0 \\ 0 & 0 & 0 & v_1^* \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & v_1 & 0 & 0 \\ 0 & 0 & v_2 & 0 \\ 0 & 0 & 0 & v_1^* \\ 0 & 0 & 0 & 0 \end{pmatrix}^T \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = 0$$

$$\implies \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & v_1^* + v_1^T \\ 0 & 0 & v_2 + v_2^T & 0 \\ 0 & v_1 + (v_1^*)^T & 0 & 0 \end{pmatrix} = 0$$

$$\implies v_1^* = -v_1^T \quad \text{and} \quad v_2 \in \mathrm{Alt}^2(E_2).$$

Thus, we decompose the Vogan variety in terms of these maps between the eigenspaces of λ as:

$$V'_\lambda \cong \underbrace{\prod_{i=1}^{\ell-2} \mathrm{hom}(E_{i+1}, E_i)}_{v_1} \times \underbrace{\mathrm{Alt}^2(E_\ell)}_{v_2};$$

The centralizer is given by $H'_\lambda = \{h \in H_\lambda \mid h^T Jh = J\}$, thus

$$\begin{pmatrix} h_1 & 0 & 0 & 0 \\ 0 & h_2 & 0 & 0 \\ 0 & 0 & h_2^* & 0 \\ 0 & 0 & 0 & h_1^* \end{pmatrix}^T \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} h_1 & 0 & 0 & 0 \\ 0 & h_2 & 0 & 0 \\ 0 & 0 & h_2^* & 0 \\ 0 & 0 & 0 & h_1^* \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\implies \begin{pmatrix} 0 & 0 & 0 & h_1^T h_1^* \\ 0 & 0 & h_2^T h_2^* & 0 \\ 0 & (h_2^*)^T h_2 & 0 & 0 \\ (h_1^*)^T h_1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\implies h_1^* = h_1^{-T} \quad \text{and} \quad h_2^* = h_2^{-T}.$$

This gives us the following decomposition for the centralizer,

$$H_\lambda' \cong \underbrace{\prod_{i=1}^{\ell-1} \mathrm{GL}(E_i)}_{h_1} \times \underbrace{\mathrm{GL}(E_\ell)}_{h_2};$$

(b) $\lambda \in \mathrm{Sp}$; The Vogan variety is given by $V_\lambda' = \{v \in V_\lambda \mid Jv + v^T J = 0\}$, thus

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & v_1 & 0 & 0 \\ 0 & 0 & v_2 & 0 \\ 0 & 0 & 0 & v_1^* \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & v_1 & 0 & 0 \\ 0 & 0 & v_2 & 0 \\ 0 & 0 & 0 & v_1^* \\ 0 & 0 & 0 & 0 \end{pmatrix}^T \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} = 0$$

$$\implies \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & v_1^* + v_1^T \\ 0 & 0 & v_2 - v_2^T & 0 \\ 0 & -(v_1 + (v_1^*)^T) & 0 & 0 \end{pmatrix} = 0$$

$$\implies v_1^* = -v_1^T \quad \text{and} \quad v_2 \in \mathrm{Sym}^2(E_2).$$

Thus, we decompose the Vogan variety in terms of these maps between the eigenspaces of λ as:

$$V_\lambda' \cong \underbrace{\prod_{i=1}^{\ell-2} \hom(E_{i+1}, E_i)}_{v_1} \times \underbrace{\mathrm{Sym}^2(E_\ell)}_{v_2};$$

The centralizer is given by $H_\lambda' = \{h \in H_\lambda \mid h^T J h = J\}$, thus

$$
\begin{pmatrix} h_1 & 0 & 0 & 0 \\ 0 & h_2 & 0 & 0 \\ 0 & 0 & h_2^* & 0 \\ 0 & 0 & 0 & h_1^* \end{pmatrix}^T
\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}
\begin{pmatrix} h_1 & 0 & 0 & 0 \\ 0 & h_2 & 0 & 0 \\ 0 & 0 & h_2^* & 0 \\ 0 & 0 & 0 & h_1^* \end{pmatrix} =
\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}
$$

$$
\implies
\begin{pmatrix} 0 & 0 & 0 & h_1^T h_1^* \\ 0 & 0 & h_2^T h_2^* & 0 \\ 0 & -(h_2^*)^T h_2 & 0 & 0 \\ -(h_1^*)^T h_1 & 0 & 0 & 0 \end{pmatrix} =
\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}
$$

$$
\implies h_1^* = h_1^{-T} \quad \text{and} \quad h_2^* = h_2^{-T}.
$$

This gives us the following decomposition for the centralizer,

$$
H_\lambda' \cong \underbrace{\prod_{i=1}^{\ell-1} \mathrm{GL}(E_i)}_{h_1} \times \underbrace{\mathrm{GL}(E_\ell)}_{h_2};
$$

Now that we've illustrated the structure of the Vogan varieties and centralizers arising for $\lambda \in \hat{G}'$, we may move our discussion towards the classification of orbits, how this will affect the stabilizers, and subsequently the equivariant fundamental groups.

Classification for the Orbits Arising in $\hat{G}' = \mathrm{SO}$ or Sp

The classification for these orbits (Proposition 3.8) descends to symmetric quiver representations in the SO and Sp. The following proposition describes the conditions on the ranks that can appear in the rank-triangles arising for $\lambda \in \hat{G}' = \mathrm{SO}$ or Sp.

Proposition 3.22. *The ranks $r_{ij} = \mathrm{rank}(v_i \cdots v_j)$ with $i \leq j$ are subject to the conditions in Proposition 3.11, together with the following symmetry and induced parity conditions:*

1. *The rank-triangles must be completely symmetric about the middle, since the identification $E_i \cong E_{k-i}^*$, for $1 \leq i \leq k$ fixes a relation $v_i^* \cong v_{k-i}$.*

2. *Additionally, the ranks appearing down the middle of the triangle must be even for*

   (a) *$\lambda \in \mathrm{SO}$, when λ consist of half-integral powers of q; and for*

   (b) *$\lambda \in \mathrm{Sp}$, when λ consist of integral powers of q.*

That is whenever the ranks appearing down the middle of the rank-triangle corre-spond to skew-symmetric matrices.

This first part of the proposition is an immediate corollary of Proposition 3.21, and the second part is proved with the following series of Lemmata:

Lemma 3.23. *Suppose that A is an $n \times n$ skew-symmetric matrix, then* $\mathrm{rank}(A) = 2k$, *for some $k \in \mathbb{N}$.*

Proof. Observe that if A is skew-symmetric, then its eigenvalues will come in conjugate pairs i.e. if λ_i is an eigenvalue then so is λ_i^{-1}, or be 0. Then, there exists an invertible $n \times n$ matrix P for which

$$P^{-1}AP = \mathrm{diag}(\lambda_1, \lambda_1^{-1}, \lambda_2, \lambda_2^{-1}, \ldots, \lambda_k, \lambda_k^{-1}, 0, \ldots, 0).$$

Since the rank of a matrix is preserved under conjugation, it is clear that $\mathrm{rank}(A) = 2k$, for some $k \leq n \in \mathbb{N}$. $\square$

This will naturally extend to whenever the compositions must be skew symmetric.

Now, for $\lambda \in \mathrm{SO}$, when the eigenvalues of λ consist of half-integral powers of q, its corresponding Vogan variety decomposes as:

$$V_\lambda' \cong \prod_{i=1}^{\ell-1} \hom(E_{i+1}, E_i) \times \mathrm{Alt}^2(E_\ell).$$

Which leads to the following lemma, where the possible quiver representations in V_λ' must satisfy a parity condition.

Lemma 3.24. *For $\lambda \in \mathrm{SO}$, with half-integral powers of q, if we let $v = v_i v_{i+1} \cdots v_{\ell-1}$ and let $A \in \mathrm{Alt}^2(E_\ell)$. Then, vAv^* will be a square skew-symmetric matrix, where $v^* = -v^T$, such that* $\mathrm{rank}(vAv^*) = 2k$, *for some $k \in \mathbb{N}$.*
Proof. Given that $v^* = -v^T$, we have that vAv^* is skew-symmetric, observe:

$$-(vAv^*)^T = -(vA(-v^T))^T = -(v(-A)v^T)^T = -v(-A^T)v^T = v(-A^T)(-v^T) = vAv^*.$$

Since vAv^* is skew-symmetric, the result follows from the previous lemma. $\square$

Now, for $\lambda \in \mathrm{Sp}$, when the eigenvalues of λ consist of half-integral powers of q, its corresponding Vogan variety decomposes as:

$$V_\lambda' \cong \prod_{i=1}^{\ell-1} \hom(E_{i+1}, E_i) \times \mathrm{Sym}^2(E_\ell),$$

Which leads to the following lemma, where the quiver representations in V_λ' evade the parity condition.

Lemma 3.25. *For $\lambda \in \mathrm{Sp}$, where the eigenvalues of λ consist of half-integral powers of q, vAv^* will be a square symmetric matrix, where $v^* = -v^T$.*

Proof. Given that $v^* = -v^T$, and that $A \in Sym^2(E_\ell)$ is symmetric, the matrix vAv^* will also be symmetric. Observe:

$$(vAv^*)^T = (vA(-v^T))^T = (-vAv^T)^T = vA^T(-v^T) = vAv^*. \qquad \square$$

In the case where the eigenvalues of λ consist of integral powers of q, for $\lambda \in \hat{G}'$, we have an even number of quiver representations in the Vogan variety, and these will lead to the following lemma:

Lemma 3.26. *For $\hat{G}' = \mathrm{SO}$, where the eigenvalues of λ consist of integral powers of q, the $n \times n$ matrix vv^* is symmetric, where $v^* = -v^T$.*

Proof. Suppose that $v \in \mathrm{Mat}_{n \times m}$, then $vv^* \in \mathrm{Mat}_{n \times n}$ and

$$(vv^*)^T = (v(-v^T))^T = (-vv^T)^T = v(-v^T) = vv^*. \qquad \square$$

Lemma 3.27. *For $\hat{G}' = \mathrm{Sp}$, where the eigenvalues of λ consist of integral powers of q, the $n \times n$ matrix vv^* is skew-symmetric, where v is an $n \times m$ matrix such that $v^* = -Jv^T$.*

Proof. Suppose that v is an $n \times m$ matrix, then $vv^* \in \mathrm{Mat}_{n \times n}$ for $v^* = -Jv^T$

$$-(vv^*)^T = -(v(-Jv^T))^T = (vJv^T)^T = vJ^Tv^T = v(-J)v^T = v(-Jv^T) = vv^*.$$

Hence, the rank-triangles for $\lambda \in \hat{G}' = \mathrm{Sp}$, for integral powers of q also carry the parity condition that the ranks down the centre of the rank-triangle must be even. □

We demonstrate these new rank-triangle conditions as we descend from $\hat{G} = \mathrm{GL}_n$ to $\hat{G}' = \mathrm{SO}$ or Sp. These new conditions will significantly reduce (by less than half) the number of H'_λ-orbits C' in V'_λ, and it will alter the centre of the group H_λ. Observe the following decomposition for $\lambda \in \mathrm{SO}$,

Example 3.28. Consider $\lambda = \mathrm{diag}(q^2, q, q, q^0, q^0, q^0, q^{-1}, q^{-1}, q^{-2})$, in the case of $\hat{G} = \mathrm{GL}(9)$, in terms of matrices as maps related to the eigenspace of λ, the structure of our Vogan variety $V_\lambda = \{v \in \hat{\mathfrak{g}} \mid \lambda v \lambda^{-1} = qv\}$ is of the form:

$$V_\lambda = \left\{ \begin{pmatrix} 0 & v_1 & 0 & 0 & 0 \\ 0 & 0 & v_2 & 0 & 0 \\ 0 & 0 & 0 & v_3 & 0 \\ 0 & 0 & 0 & 0 & v_4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \left| \begin{array}{l} v_1 \in \mathrm{Mat}_{1\times2}(\mathbb{C}) \\ v_2 \in \mathrm{Mat}_{2\times3}(\mathbb{C}) \\ v_3 \in \mathrm{Mat}_{3\times2}(\mathbb{C}) \\ v_4 \in \mathrm{Mat}_{2\times1}(\mathbb{C}) \end{array} \right. \right\}.$$

We can label the matrices as:

$$V_\lambda = \{v_1, v_2, v_3, v_4\} = \left\{ \begin{pmatrix} w_1 & w_2 \end{pmatrix}, \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{pmatrix}, \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \\ y_{31} & y_{32} \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right\}.$$

The quiver representations of V_λ capture the maps between the eigenspaces in the following way:

$$\overset{E_1}{\bullet} \overset{v_1}{\longleftarrow} \overset{E_2}{\bullet} \overset{v_2}{\longleftarrow} \overset{E_3}{\bullet} \overset{v_3}{\longleftarrow} \overset{E_4}{\bullet} \overset{v_4}{\longleftarrow} \overset{E_5}{\bullet} .$$

Where $E_1 = E_{q^2}$, $E_2 = E_{q^1}$, $E_3 = E_{q^0}$, $E_4 = E_{q^{-1}}$, and $E_5 = E_{q^{-2}}$.

The centralizer $H_\lambda = \{h \in \hat{G} \mid hvh^{-1} = v\}$ is of the form:

$$H_\lambda = \left\{ \begin{pmatrix} h_1 & 0 & 0 & 0 & 0 \\ 0 & h_2 & 0 & 0 & 0 \\ 0 & 0 & h_3 & 0 & 0 \\ 0 & 0 & 0 & h_4 & 0 \\ 0 & 0 & 0 & 0 & h_5 \end{pmatrix} \left| \begin{array}{c} h_1, h_5 \neq 0 \\ h_2, h_4 \in \mathrm{GL}(2) \\ h_3 \in \mathrm{GL}(3) \end{array} \right. \right\},$$

with

$$h_2 := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad h_3 := \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \quad h_4 := \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}.$$

We see that

$$V_\lambda \cong \hom(E_{q^1}, E_{q^2}) \times \hom(E_{q^0}, E_{q^1}) \times \hom(E_{q^{-1}}, E_{q^0}) \times \hom(E_{q^{-2}}, E_{q^{-1}})$$

and

$$H_\lambda = \mathrm{GL}(E_{q^2}) \times \mathrm{GL}(E_{q^1}) \times \mathrm{GL}(E_{q^0}) \times \mathrm{GL}(E_{q^{-1}}) \times \mathrm{GL}(E_{q^{-2}}).$$

The action of H_λ on V_λ is of the form

$$h.v := \left(h_1 \, v_1 \, h_2^{-1}, \; h_2 \, v_2 \, h_3^{-1}, \; h_3 \, v_3 \, h_4^{-1}, \; h_4 \, v_4 \, h_5^{-1} \right),$$

this group would have <u>86 orbits</u>, where each of their associated equivariant fundamental groups would have been trivial by Proposition 3.15.

To demonstrate more interesting decomposition, we pass to $\hat{G}' = \mathrm{SO}(9)$. To do this, we introduce the symmetric non-degenerate bilinear form

$$J = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \mathrm{Id}_2 & 0 \\ 0 & 0 & \mathrm{Id}_3 & 0 & 0 \\ 0 & \mathrm{Id}_2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Where the sizes of the identity blocks in J correspond to the dimensions of the overlying eigenspace. This choice of J allows for $\lambda \in \hat{G}'$, and will induce an isomorphism between the eigenspaces of λ in an appropriate way.

The structure of our Vogan variety $V'_\lambda = \{v \in V_\lambda \mid Jv + v^T J = 0\}$[27] is of the form:

$$
V'_\lambda = \left\{ \begin{pmatrix} 0 & v_1 & 0 & 0 & 0 \\ 0 & 0 & v_2 & 0 & 0 \\ 0 & 0 & 0 & v_2^* & 0 \\ 0 & 0 & 0 & 0 & v_1^* \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| \begin{array}{l} v_1 \in \mathrm{Mat}_{1\times2}(\mathbb{C}) \\ v_2 \in \mathrm{Mat}_{2\times3}(\mathbb{C}) \\ v_2^* \in \mathrm{Mat}_{3\times2}(\mathbb{C}) \\ v_1^* \in \mathrm{Mat}_{2\times1}(\mathbb{C}) \end{array} \right\}.
$$

Where v_i^* denotes $-v_i^T$. We can label the matrices as:

$$
V'_\lambda = \{v_1, v_2, v_2^*, v_1^*\} = \left\{ \begin{pmatrix} x_1 & x_2 \end{pmatrix}, \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \end{pmatrix}, \begin{pmatrix} -y_{11} & -y_{21} \\ -y_{12} & -y_{22} \\ -y_{13} & -y_{23} \end{pmatrix}, \begin{pmatrix} -x_1 \\ -x_2 \end{pmatrix} \right\}.
$$

The centralizer $H'_\lambda = \{h \in H_\lambda \mid h^T J h = J\}$ is of the form:

$$
H'_\lambda = \left\{ \begin{pmatrix} h_1 & 0 & 0 & 0 & 0 \\ 0 & h_2 & 0 & 0 & 0 \\ 0 & 0 & h_3 & 0 & 0 \\ 0 & 0 & 0 & h_2^* & 0 \\ 0 & 0 & 0 & 0 & h_1^{-1} \end{pmatrix} \middle| \begin{array}{l} h_1 \neq 0 \\ h_2^T h_2^* = \mathrm{Id}_2 \\ h_3 \in \mathrm{SO}(3) \end{array} \right\}.
$$

Here we see that $(h_2^*)^{-1} = h_2^T$, we can write

$$
h_2 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad h_3 = \begin{pmatrix} c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \\ e_1 & e_2 & e_3 \end{pmatrix}, \quad \text{and} \quad h_2^* = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}^{-1}.
$$

Notice that

$$
H_\lambda = \mathrm{GL}(E_{q^2}) \times \mathrm{GL}(E_{q^1}) \times \mathrm{SO}(E_{q^0}).
$$

Since $H'_\lambda \in \mathrm{SO}$, the bilinear pairing J induces an isomorphism between E_{λ_i} and $E_{\lambda_i^{-1}}$, the block decomposition of H'_λ resembles that of H_λ, yet with fewer GL spaces after our identification has been made.

The action of H'_λ on V'_λ is of the form

$$
h'.v' := \left(h_1 \, v_1 \, h_2^{-1}, \; h_2 \, v_2 \, h_3^{-1}, \; h_3 \, v_2^* \, (h_2^*)^{-1}, \; h_2^* \, v_1^* \, (h_1^{-1})^{-1} \right).
$$

---

[27]This equation follows from evaluating the tangent space at the identity.
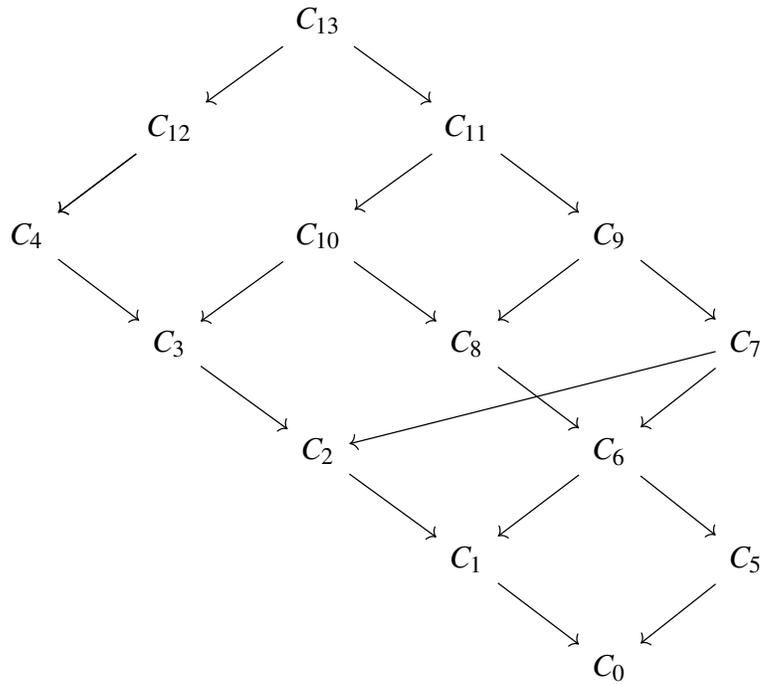
Figure 3.2: Partial Ordering of the 14 H'_λ-orbits.

There are 14 H'_λ-orbits, which in principle may be enumerated using some Python code. This collection may be described as a directed graph (see Figure 3.2). Where our H'_λ-orbits have the implied partial orderings, $C_i \to C_j \implies C_j \subset \overline{C_i}$.

While we will not be running through the stabilizer calculations explicitly, this example was designed to describe the decomposition in larger spaces since this descent generalizes quite nicely. In this example, we saw just how drastically the imposition of symmetry on the rank-triangles can decrease the number of H'_λ-orbits, which didn't even include the parity condition on the middle of the rank-triangle.

Consider now performing the same process on the flushed out example in GL(5), where we demonstrate the case where the products of quiver representations in the Vogan variety are symmetric. This will tend to have a significant impact on the equivariant fundamental groups, and hence the set of equivariant local systems that must be considered.

Example 3.29. Consider $\lambda = \mathrm{diag}(q,1,1,1,q^{-1})$ as in Example 3.14. For $\lambda \in \hat{G}' = \mathrm{SO}(5)$, we introduce the non-degenerate symmetric bilinear form:

$$
J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \mathrm{Id}_3 & 0 \\ 1 & 0 & 0 \end{pmatrix},
$$

this choice of J allows for $\lambda \in \hat{G}'$.

We compute $V'_\lambda = \{v \in V_\lambda \mid Jv + v^T J = 0\}$ and $H'_\lambda = \{h \in H_\lambda \mid h^T J h = J\}$, to find that

$$
V'_\lambda = \left\{ \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & x^* \\ 0 & 0 & 0 \end{pmatrix} \;\middle|\; \begin{array}{l} x \in \mathrm{Mat}_{1\times 3} \\ x^* = -x^T \end{array} \right\}
\quad \text{and} \quad
H'_\lambda = \left\{ \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_1^{-1} \end{pmatrix} \;\middle|\; \begin{array}{l} h_1 \neq 0 \\ h_2 \in \mathrm{SO}(3) \end{array} \right\}.
$$

The action of H'_λ on V'_λ is of the form:

$$
h'.v' := \left(h_1 \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}, \; - \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} h_1 \right),
$$

where $h' \in H'_\lambda$ and $v' \in V'_\lambda$. We stratify V'_λ into H'_λ-orbits by considering the following rank-triangles:

$$
C'_0 : \begin{array}{ccc} \overline{1 \quad 3 \quad 1} \\ 0 \quad\; 0 \\ 0 \end{array}, \quad
C'_1 : \begin{array}{ccc} \overline{1 \quad 3 \quad 1} \\ 1 \quad\; 1 \\ 0 \end{array}, \quad \text{and} \quad
C'_2 : \begin{array}{ccc} \overline{1 \quad 3 \quad 1} \\ 1 \quad\; 1 \\ 1 \end{array}.
$$

That is

$$
C'_0 = \{v' \in V_\lambda \mid x = 0\}, \quad C'_1 = \{v' \in V_\lambda \mid x \neq 0, \; x \cdot x = 0\}, \quad C'_2 = \{v' \in V_\lambda \mid x \cdot x \neq 0\}.
$$

They have the partial ordering $C'_0 < C'_1 < C'_2$. Then we have that $\mathrm{Stab}_c(H'_\lambda)$:

- The stabilizer for C'_0 is all of H'_λ, since $h'.0 = 0, \forall h' \in H'_\lambda$.

- For C'_1, the defining condition for the equations is $x_1^2 + x_2^2 + x_3^2 = 0$. Thus choosing as

representatives $x_1 = 1$, $x_2 = i$, and $x_3 = 0$, we find that

$$\mathrm{Stab}_{c_1'}(H_\lambda') = \left\{ h' \in H_\lambda' \,\middle|\, h_1 \begin{pmatrix} a_1 + b_1 i & a_2 + b_2 i & a_3 + b_3 i \end{pmatrix} = \begin{pmatrix} 1 & i & 0 \end{pmatrix} \right\},$$

with all other variables free. We note that this group is connected.

- For C_2', any choice of x_1, x_2, x_3 such that $x_1^2 + x_2^2 + x_3^2 \neq 0$ for representatives will suffice. Thus, choosing $x_1 = 1$, $x_2 = x_3 = 0$, we find that

$$\mathrm{Stab}_c(H_\lambda') = \left\{ \begin{pmatrix} h_1 & & & \\ & \begin{pmatrix} h_1 & 0 & 0 \\ 0 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{pmatrix} & \\ & & h_1^{-1} \end{pmatrix} \;\middle|\; \begin{matrix} \begin{pmatrix} b_2 & c_2 \\ b_3 & c_3 \end{pmatrix} \in O(2) \\ h_1(b_2 c_3 - c_2 b_3) = 1 \end{matrix} \right\}.$$

Note that our two conditions imply that $h_1 = \pm 1$. Thus, $\mathrm{Stab}_{c'}(H_\lambda')$ is disconnected, and so the fundamental group

$$A_{C'} = \mathrm{Stab}_{c'}(H_\lambda') \big/ \mathrm{Stab}_{c'}(H_\lambda')^o = \{\pm 1\} \quad \text{is non-trivial!}$$

Thus, the simple objects in $\mathrm{Per}_{H_\lambda'}(V_\lambda')$ belongs to

$$\mathrm{Per}_{H_\lambda'}^{\mathrm{simple}}(V_\lambda') = \{ IC(C_0, \mathcal{L}_{\mathbb{1}_{C_0}}), IC(C_1, \mathcal{L}_{\mathbb{1}_{C_1}}), IC(C_2, \mathcal{L}_{\mathbb{1}_{C_2}}), IC(C_2, \mathcal{L}_{\chi_{C_2}}) \},$$

where $\mathcal{L}_{\chi_{C_2}}$ is the unique non-trivial irreducible representation of the group $\mathbb{Z}/2\mathbb{Z}$ associated to the orbit C_2.

3.2.2 How (non-trivial) Fundamental Groups Arise

In this section, we describe how the non-trivial equivariant fundamental groups arise when passing from $\hat{G} = \mathrm{GL}_n$ to one of its subgroups, $\hat{G}' = \mathrm{SO}$ or Sp. In the previous section, we saw that when $\lambda \in \hat{G}'$, the bilinear pairing J induced a very strong condition on the eigenspaces of λ. Whenever the composition of matrices was skew-symmetric, this

resulted in having parity conditions on the rank-triangle; When the composition of matrices is symmetric, this may result in having a non-trivial equivariant fundamental group.

Proposition 3.30. *The non-trivial equivariant fundamental groups $A_{C'}$ of the H'_λ-orbits of C' in V'_λ may arise for*

1. *$\lambda \in$ SO, when λ consists of integral powers of q;[28] and for*

2. *$\lambda \in$ Sp, when λ consists of half-integral powers of q.[29]*

In either case, the resulting non-trivial equivariant fundamental groups will consist of a product of 2-groups.

The proof of this proposition is non-trivial, as the interaction between the stabilizers on the Vogan varieties becomes increasingly complicated as the number of relations defining the orbits increases. One may leverage parts of Proposition 3.15 in order to prove this, namely Equation (3.2). Since the stabilizer in SO or Sp must be the fixed points of the stabilizer from GL_n under the action of an involution. Based on the classification of possible involutions on groups of the form given in Equation (3.2), the only disconnected group that can arise is the orthogonal group.

While technical to prove, it is clear to see that non-trivial equivariant fundamental groups may arise whenever

$$J_i := v_i v_{i-1} \cdots v^*_{i-1} v^*_i \tag{3.3}$$

is symmetric, since the action we have on it is

$$h_i v_i v_{i-1} \cdots v^*_{i-1} v^*_i h^T_i,$$

and the subgroup $h_i J_i h^T_i = J_i$ stabilizing it will be an "orthogonal group" for a potentially degenerate bilinear form, which is typically disconnected.

---

[28]Due to the $\mathrm{SO}(E_\ell)$ factor of H'_λ acting on $v_{\ell-1} \in \mathrm{hom}(E_\ell, E^*_\ell)$ in V'_λ.
[29]Due to the $\mathrm{Sym}^2(E_\ell)$ map in V'_λ being acted on by $\mathrm{GL}(E_\ell)$ in H'_λ.

This leads to the following corollary:

Corollary 3.31. *The equivariant fundamental group A'_C, can be*

$$A_{C'} = \left(\mathbb{Z}\big/2\mathbb{Z}\right)^{\ell}, \quad \text{for any } \ell \in \mathbb{Z}_{\geq 0},$$

provided the dimension n of the Vogan variety V'_λ is large enough.

We demonstrate this corollary by studying the open orbits for the cases in Proposition 3.30, since there is sufficient combinatorial data encoded in the rank-triangle in order to observe the phenomena of the disconnected equivariant fundamental groups;

Example 3.32. Consider $\lambda \in \hat{G}' = \text{SO}$, for a sufficiently large n (namely $n = \ell^2$) with the corresponding open orbit C' in V'_λ given the overlying dimensions of the eigenspace:

$$1 \quad 2 \quad \cdots \quad \ell-1 \quad \ell \quad \ell-1 \quad \cdots \quad 2 \quad 1.$$

The non-trivial equivariant fundamental group for the open orbit will be

$$A_{C'} = \left(\mathbb{Z}\big/2\mathbb{Z}\right)^{\ell-1}, \quad \text{for } \ell \in \mathbb{Z}_{\geq 0}.$$

Observe, the Vogan variety V'_λ is of the form

$$V'_\lambda = \left\{ \begin{pmatrix} 0 & v_1 & 0 & \cdots & 0 \\ 0 & 0 & v_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & v_1^* \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \,\middle|\, \begin{array}{l} v_1 : E_2 \to E_1 \\ v_2 : E_3 \to E_2 \\ \vdots \\ v_{\ell-1} : E_\ell \to E_{\ell-1} \end{array} \right\} \quad \text{where } v_i^* = -v_i^T.$$

The centralizer H'_λ which acts on this Vogan variety is of the form

$$H'_\lambda = \left\{ \begin{pmatrix} h_1 & 0 & 0 & \cdots & 0 \\ 0 & h_2 & 0 & \cdots & 0 \\ 0 & 0 & h_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & h_1^* \end{pmatrix} \,\middle|\, \begin{array}{l} h_1 \in \text{GL}(E_1) \\ h_2 \in \text{GL}(E_2) \\ \vdots \\ h_\ell \in \text{SO}(E_\ell) \end{array} \right\} \quad \text{where } h_i^* = h_i^{-T}.$$

52

Since we are working on the open orbit, without any loss in generality, we may choose

$$v_i = \begin{pmatrix} \mathrm{Id}_i & 0 \end{pmatrix},$$

where Id_i denotes the rank(i)-identity matrix, with respect to some basis. Let us compute the first few elements of the stabilizer,

v_0:

$$a_{11} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & 0 \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \end{pmatrix} \implies h_1 = \begin{pmatrix} a_{11} & 0 \\ b_{21} & b_{22} \end{pmatrix}.$$

v_1:

$$\begin{pmatrix} a_{11} & 0 \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & 0 & 0 \\ b_{21} & b_{22} & 0 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{pmatrix} \implies h_2 = \begin{pmatrix} a_{11} & 0 & 0 \\ b_{21} & b_{22} & 0 \\ c_{31} & c_{32} & c_{33} \end{pmatrix}.$$

v_2:

$$\begin{pmatrix} a_{11} & 0 & 0 \\ b_{21} & b_{22} & 0 \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} d_{11} & d_{12} & d_{13} & d_{14} \\ d_{21} & d_{22} & d_{23} & d_{24} \\ d_{31} & d_{32} & d_{33} & d_{34} \\ d_{41} & d_{42} & d_{43} & d_{44} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 \\ c_{31} & c_{32} & c_{33} & 0 \end{pmatrix} = \begin{pmatrix} d_{11} & d_{12} & d_{13} & d_{14} \\ d_{21} & d_{22} & d_{23} & d_{24} \\ d_{31} & d_{32} & d_{33} & d_{34} \end{pmatrix} \implies h_3 = \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 \\ c_{31} & c_{32} & c_{33} & 0 \\ d_{41} & d_{42} & d_{43} & d_{44} \end{pmatrix}.$$

In each case of the computation, we find that as we move down the centralizers, they consist of lower triangular matrices, where each ith-row of each block is determined by a block of free variables from the h_i^{th} block in $GL(E_i)$. However, what now distinguishes our case of $\lambda \in GL_n$ to $\lambda \in SO$ is that

1. For the integral powers of q and $\lambda \in SO$, we have $h_\ell \in SO(E_\ell)$ for the middle block.

2. $E_i \cong E_{k-i}^*$ tells us that $v_i^* = -v_i^T$ and $h_i^* = h_i^{-T}$. So we find that

$$h_i = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} \implies h_i^* = \begin{pmatrix} A^* & \star \\ 0 & C^* \end{pmatrix},$$

where we can carry out this process in each block of A and C;

3. Putting these two facts together, we'd have $h_\ell (h_\ell^*)^T = \mathrm{Id}$, which implies that

$$\begin{pmatrix} A & 0 \\ B & C \end{pmatrix} \begin{pmatrix} A^{-1} & \star \\ 0 & C^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \iff \begin{pmatrix} AA^{-1} & A\star \\ BA^{-1} & B\star + CC^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \implies B = 0.$$

If $B = 0$, then $\star = 0$, since $h_\ell^* = h_\ell^{-T}$, i.e., in general, if X is a triangular matrix and $XX^T = \mathrm{Id}$, then X must be a diagonal matrix. This implies that for our middle matrix, we'd have that

$$h_\ell = \mathrm{diag}(a_{11}, b_{22}, c_{33}, \ldots, z_{\ell\ell}).$$

This computation immediately collapses down all of our centralizing elements h_i, from lower triangular to diagonal matrices. Now again imposing the condition that $h_\ell \in \mathrm{SO}(E_\ell)$ yields the following result,

$$h_\ell h_\ell^T = \mathrm{Id} \iff h_\ell^2 = \mathrm{Id}.$$

Thus, we find that

$$\mathrm{Stab}_c(H_\lambda') \cong \left\{ h \in H_\lambda \,\middle|\, \begin{array}{c} a_{11}, b_{22}, c_{33}, \ldots, z_{\ell\ell} = \pm 1 \\ a_{11} b_{22} c_{33} \cdots z_{\ell\ell} = 1 \end{array} \right\}.$$

In conclusion, the off-diagonal needed to vanish in order to stabilize. When we include the imposition that $h_\ell \in \mathrm{SO}(E_\ell)$[30], the diagonal consisted of elements of the form $a_{11}, b_{22}, c_{33}, \ldots, z_{\ell\ell} = \pm 1$, i.e. the non-trivial equivariant fundamental group is

$$A_{C'} = \left(\mathbb{Z} \big/ 2\mathbb{Z} \right)^{\ell - 1}, \quad \text{for any } \ell \in \mathbb{Z}_{\geq 0}.$$

---

[30] When $\lambda \in \mathrm{Sp}$, we would find that $h_\ell \in \mathrm{Sp}(E_\ell)$, we'd have had $XJX^T = J$, which evades this condition.

A similar process will occur when we move from the open orbit to other orbits. One notes that these blocks appearing on the diagonal will be of varying sizes, some of which may be in the orthogonal group.

When working on the open orbit, without any loss in generality, we are able to choose $v_i = \begin{pmatrix} \mathrm{Id}_i & 0 \end{pmatrix}$. Then, the matrices corresponding to the ranks appearing down the middle of the rank-triangle will be of the form $v_i v_i^* = \mathrm{Id}_i$. Since the identity is a symmetric matrix, this will lead to the following stabilization:

$$h_\ell v_\ell h_\ell^T = v_\ell \iff h_\ell h_\ell^T = \mathrm{Id}_i.$$

We can generalize the above example to the following setup by letting all of the variables denote blocks of matrices, then looking at the open orbit $C' \subset V'_\lambda$ with the given overlying eigen-dimensions:

$$\underline{n_1 \quad n_2 \quad \cdots \quad n_{\ell-1} \quad n_\ell \quad n_{\ell-1} \quad \cdots \quad n_2 \quad n_1}$$

we would find the following stabilizer:

$$\mathrm{Stab}_c(H'_\lambda) = \left\{ \begin{pmatrix} h_1 & 0 & 0 & \cdots & 0 \\ 0 & h_2 & 0 & \cdots & 0 \\ 0 & 0 & h_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & h_1^* \end{pmatrix} \middle| \begin{array}{l} h_i = \begin{pmatrix} h_{i-1} & 0 \\ 0 & h'_i \end{pmatrix} \\ h'_i \in \prod_{i=1}^{n_\ell} O(n_i - n_{i-1}) \\ h_\ell \in \mathrm{SO}(E_\ell) \end{array} \right\}, \qquad \text{where } h_i^* = h_i^{-T}.$$

We demonstrate this by a small example which one could inductively use to construct the above fully generalized setup:

Example 3.33. Consider the open orbit $C' \subset V'_\lambda$ with the given overlying dimensions of the eigenspace:

$$\underline{n_1 \quad n_2 \quad n_3 \quad n_2 \quad n_1}$$

We have the following Vogan variety V'_λ and centralizer H'_λ:

$$V'_\lambda = \left\{ \begin{pmatrix} 0 & v_1 & 0 & 0 & 0 \\ 0 & 0 & v_2 & 0 & 0 \\ 0 & 0 & 0 & v_2^* & 0 \\ 0 & 0 & 0 & 0 & v_1^* \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right\} \text{ and } H'_\lambda = \left\{ \begin{pmatrix} h_1 & 0 & 0 & 0 & 0 \\ 0 & h_2 & 0 & 0 & 0 \\ 0 & 0 & h_3 & 0 & 0 \\ 0 & 0 & 0 & h_2^* & 0 \\ 0 & 0 & 0 & 0 & h_1^* \end{pmatrix} \,\middle|\, \begin{matrix} h_1 \in \mathrm{GL}(E_1) \\ h_2 \in \mathrm{GL}(E_2) \\ h_3 \in \mathrm{SO}(E_3) \\ h_i^* = h_i^{-T} \end{matrix} \right\},$$

where $v_1 \in \mathrm{Mat}_{n_1 \times n_2}$, $v_2 \in \mathrm{Mat}_{n_2 \times n_3}$, and $v_i^* = -v_i^T$. Observe the resulting stabilizer, letting

$$v_i = \begin{pmatrix} \mathrm{Id} & 0 \end{pmatrix}, \quad h_1 = \begin{pmatrix} A_{n_1 \times n_1} \end{pmatrix}, \quad h_2 = \begin{pmatrix} B_{n_1 \times n_1} & B_{n_1 \times (n_2-n_1)} \\ B_{(n_2-n_1) \times n_1} & B_{(n_2-n_1) \times (n_2-n_1)} \end{pmatrix},$$

and

$$h_3 = \begin{pmatrix} C_{n_1 \times n_1} & C_{n_1 \times (n_2-n_1)} & C_{n_1 \times (n_3-n_2)} \\ C_{(n_2-n_1) \times n_1} & C_{(n_2-n_1) \times (n_2-n_1)} & C_{(n_2-n_1) \times (n_3-n_2)} \\ C_{(n_3-n_2) \times n_1} & C_{(n_3-n_2) \times (n_2-n_1)} & C_{(n_3-n_2) \times (n_3-n_2)} \end{pmatrix}.$$

We find that

v_0:

$$\begin{pmatrix} A_{n_1 \times n_1} \end{pmatrix} \begin{pmatrix} \mathrm{Id}_{n_1 \times n_1} & 0_{n_1 \times (n_2-n_1)} \end{pmatrix} = \begin{pmatrix} \mathrm{Id}_{n_1 \times n_1} & 0_{n_1 \times (n_2-n_1)} \end{pmatrix} \begin{pmatrix} B_{n_1 \times n_1} & B_{n_1 \times (n_2-n_1)} \\ B_{(n_2-n_1) \times n_1} & B_{(n_2-n_1) \times (n_2-n_1)} \end{pmatrix}$$

$$\begin{pmatrix} A_{n_1 \times n_1} & 0_{n_1 \times (n_2-n_1)} \end{pmatrix} = \begin{pmatrix} B_{n_1 \times n_1} & B_{n_1 \times (n_2-n_1)} \end{pmatrix}$$

$$\implies h_2 = \begin{pmatrix} A_{n_1 \times n_1} & 0_{n_1 \times (n_2-n_1)} \\ B_{(n_2-n_1) \times n_1} & B_{(n_2-n_1) \times (n_2-n_1)} \end{pmatrix} = \begin{pmatrix} A & 0 \\ B' & B \end{pmatrix}.$$

v_1:

$$\begin{pmatrix} A & 0 \\ B' & B \end{pmatrix} \begin{pmatrix} \mathrm{Id} & 0 \end{pmatrix} = \begin{pmatrix} \mathrm{Id} & 0 \end{pmatrix} \begin{pmatrix} C_{n_1 \times n_1} & C_{n_1 \times (n_2-n_1)} & C_{n_1 \times (n_3-n_2)} \\ C_{(n_2-n_1) \times n_1} & C_{(n_2-n_1) \times (n_2-n_1)} & C_{(n_2-n_1) \times (n_3-n_2)} \\ C_{(n_3-n_2) \times n_1} & C_{(n_3-n_2) \times (n_2-n_1)} & C_{(n_3-n_2) \times (n_3-n_2)} \end{pmatrix}$$

$$\begin{pmatrix} A & 0 & 0 \\ B' & B & 0 \end{pmatrix} = \begin{pmatrix} C_{n_1 \times n_1} & C_{n_1 \times (n_2-n_1)} & C_{n_1 \times (n_3-n_2)} \\ C_{(n_2-n_1) \times n_1} & C_{(n_2-n_1) \times (n_2-n_1)} & C_{(n_2-n_1) \times (n_3-n_2)} \end{pmatrix}$$

$$\implies h_3 = \begin{pmatrix} A_{n_1 \times n_1} & 0_{n_1 \times (n_2-n_1)} & 0_{n_1 \times (n_3-n_2)} \\ B_{(n_2-n_1) \times n_1} & B_{(n_2-n_1) \times (n_2-n_1)} & 0_{(n_2-n_1) \times (n_3-n_2)} \\ C_{(n_3-n_2) \times n_1} & C_{(n_3-n_2) \times (n_2-n_1)} & C_{(n_3-n_2) \times (n_3-n_2)} \end{pmatrix} = \begin{pmatrix} A & 0 & 0 \\ B' & B & 0 \\ C' & C'' & C \end{pmatrix}.$$

Lastly, since $h_3 \in \mathrm{SO}(E_3)$, we find that

$$\begin{pmatrix} A & 0 & 0 \\ B' & B & 0 \\ C' & C'' & C \end{pmatrix} \begin{pmatrix} A^T & B'^T & C'^T \\ 0 & B^T & C''^T \\ 0 & 0 & C^T \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} AA^T & AB'^T & AC'^T \\ B'A^T & B'B'^T + BB^T & B'C'^T + BC''^T \\ C'A^T & C'B'^T + C''B^T & C'C'^T + C''C''^T + CC^T \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since the off diagonal vanishes and $A \in O(n_0)$, we must have that B', $C' = 0$; this immediately gives $B \in O(n_1 - n_0)$; this then implies that $C'' = 0$, finally yielding $C \in O(n_2 - n_1)$. As one can see, there is a very natural generalization via induction for the open orbits to higher-dimensional cases.

We now shift to considering an example in Sp, where λ consists of half-integral powers of q:

Example 3.34. Consider $\lambda \in \hat{G}' = \mathrm{Sp}$, for a sufficiently large n (namely $n = \ell(\ell+1)$) with the corresponding open orbit $C' \subset V'_\lambda$ given the overlying dimensions of the eigenspace:

$$\underline{1 \quad 2 \quad \cdots \quad \ell \quad \ell \quad \cdots \quad 2 \quad 1}$$

The non-trivial equivariant fundamental group for this orbit will be:

$$A_{C'} = \left(\mathbb{Z}/2\mathbb{Z}\right)^\ell, \quad \text{for any } \ell \in \mathbb{Z}_{\geq 0}.$$

Observe that the Vogan variety V'_λ is of the form

$$V'_\lambda = \left\{ \begin{pmatrix} 0 & v_1 & 0 & \cdots & 0 \\ 0 & 0 & v_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & v_1^* \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \middle| \begin{array}{c} v_1 : E_2 \to E_1 \\ v_2 : E_3 \to E_2 \\ \vdots \\ v_{\ell-1} : E_\ell \to E_{\ell-1} \\ v_\ell : E_\ell \to E_\ell^* \end{array} \right\},$$

where $v_i^* = -v_i^T$, for $i = 1, 2, \ldots, \ell - 1$ and $v_i^* = v_i^T$, for $i = \ell$.

The centralizer H'_λ is of the form

$$H'_\lambda = \left\{ \begin{pmatrix} h_1 & 0 & 0 & \cdots & 0 \\ 0 & h_2 & 0 & \cdots & 0 \\ 0 & 0 & h_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & h_1^* \end{pmatrix} \middle| \begin{matrix} h_1 \in \mathrm{GL}(E_1) \\ h_2 \in \mathrm{GL}(E_2) \\ \vdots \\ h_\ell \in \mathrm{GL}(E_\ell) \end{matrix} \right\} \quad \text{where } h_i^* = h_i^{-T}.$$

Since we are working on the open orbit, without any loss in generality, we may choose

$$v_i = \begin{cases} \begin{pmatrix} \mathrm{Id}_i & 0 \end{pmatrix} & i = 1, 2, \ldots, \ell - 1 \\[2ex] \mathrm{Id}_\ell & i = \ell. \end{cases}$$

where Id_i denotes the rank(i)-identity matrix with respect to some basis. We've seen previously, in Example 3.32, that the centralizer will consist of block diagonal matrices; the distinction now is that there are an even number of blocks in the quivers of the Vogan variety and an odd number of centralizing elements. Observe what happens in this middle component:

$$h_\ell = \begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & b_{22} & 0 & \cdots & 0 \\ 0 & 0 & c_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & z_{\ell\ell} \end{pmatrix} \in \mathrm{GL}(E_\ell), \text{ from the conditions on } H'_\lambda.$$

Thus, in order to stabilize the middle computation yields:

$$h_\ell v_\ell (h_\ell^*)^{-1} = v_\ell \iff h_\ell^2 = \mathrm{Id} \implies a_{11}^2, b_{22}^2, c_{33}^2, \ldots, z_{\ell\ell}^2 = 1.$$

In conclusion, since $\lambda \in \mathrm{Sp}$, we find that $h_i \in \mathrm{GL}(E_i)$, for all $i = 0, 1, 2, \ldots, \ell$. In order to stabilize, everything off the diagonal had to go to 0. The imposition that $v_\ell \in \mathrm{Sym}^2(E_\ell)$[31] force the condition on the middle components of the centralizer that $h_\ell \in O(E_\ell)$, which makes the diagonal of the centralizing elements be of the form: $a_{11}, b_{22}, c_{33}, \ldots, z_{\ell\ell} = \pm 1$,

---

[31]When $\lambda \in \mathrm{SO}$, we have $v_\ell \in \mathrm{Alt}^2(E_\ell)$, v_ℓ cannot be the identity matrix, evading this condition.

i.e., the non-trivial equivariant fundamental group is

$$A_{C'} = \left(\mathbb{Z}/2\mathbb{Z} \right)^{\ell}, \quad \text{for any } \ell \in \mathbb{Z}_{\geq 0}.$$

With the following generalization as in the previous example:

Example 3.35. Consider $\lambda \in \hat{G}' = \mathrm{Sp}$, for a sufficiently large n with the corresponding open orbit C' in V'_{λ} given the overlying dimensions of the eigenspace:

$$\begin{array}{ccccccc} n_1 & n_2 & \cdots & n_{\ell} & n_{\ell} & \cdots & n_2 & n_1 \end{array}$$

The non-trivial equivariant fundamental group for this orbit will be:

$$A_{C'} = \left(\mathbb{Z}/2\mathbb{Z} \right)^{\ell}, \quad \text{for any } \ell \in \mathbb{Z}_{\geq 0},$$

since the Vogan variety V'_{λ} is of the form

$$V'_{\lambda} = \left\{ \begin{pmatrix} 0 & v_1 & 0 & \cdots & 0 \\ 0 & 0 & v_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & v_1^* \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \; \middle| \; \begin{array}{c} v_1 \in \mathrm{Mat}_{n_1 \times n_2} \\ v_2 \in \mathrm{Mat}_{n_2 \times n_3} \\ \vdots \\ v_{\ell-1} \in \mathrm{Mat}_{n_{\ell-1} \times n_{\ell}} \\ v_{\ell} \in \mathrm{Mat}_{n_{\ell} \times n_{\ell}} \end{array} \right\},$$

where $v_i^* = -v_i^T$, for $i = 1, 2, \ldots, \ell - 1$ and $v_i^* = v_i^T$, for $i = \ell$.

The centralizer H'_{λ} is of the form

$$H'_{\lambda} = \left\{ \begin{pmatrix} h_1 & 0 & 0 & \cdots & 0 \\ 0 & h_2 & 0 & \cdots & 0 \\ 0 & 0 & h_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & h_1^* \end{pmatrix} \; \middle| \; \begin{array}{c} h_1 \in \mathrm{GL}(E_1) \\ h_2 \in \mathrm{GL}(E_2) \\ \vdots \\ h_{\ell} \in \mathrm{GL}(E_{\ell}) \end{array} \right\}, \quad \text{where } h_i^* = h_i^{-T}.$$

Since we are working on the open orbit, without any loss in generality, we may choose

$$v_i = \begin{cases} \begin{pmatrix} \mathrm{Id}_i & 0 \end{pmatrix} & i = 1, 2, \ldots, \ell - 1 \\ \mathrm{Id}_{\ell} & i = \ell \end{cases}$$

where Id_i denotes the n_i-dimensional identity matrix, with respect to some basis. The conditions on the blocks of H'_λ will be analogous to the conditions in the previous example. Observe,

$$h_\ell = \begin{pmatrix} A_{11} & 0 & 0 & \cdots & 0 \\ B_{21} & B_{22} & 0 & \cdots & 0 \\ C_{31} & C_{32} & C_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Z_{\ell 1} & Z_{\ell 2} & Z_{\ell 3} & \cdots & Z_{\ell\ell} \end{pmatrix} \in \mathrm{GL}(E_\ell), \text{ from the conditions on } H'_\lambda,$$

we find that

$$h_\ell v_\ell (h_\ell^*)^{-1} = v_\ell \iff h_\ell h_\ell^T = \mathrm{Id}$$

$$\implies A_{11} \in O(n_1), \quad B_{22} \in O(n_2 - n_1), \quad C_{33} \in O(n_3 - n_2), \quad \cdots, \quad Z_{\ell\ell} \in O(n_\ell - n_{\ell-1})$$

Where the blocks off the diagonal vanish.

In conclusion, for $\lambda \in \mathrm{Sp}$, we find that $h_i \in \mathrm{GL}(E_i)$, for all $i = 1, 2, \ldots, \ell$. In order to stabilize, everything off the diagonal had to go to 0. The imposition that $v_\ell \in \mathrm{Sym}^2(E_\ell)$ force the condition on the middle components of the centralizer that $h_i \in O(E_i)$, where the diagonal of the centralizing elements be of the form: $h_i \in O(n_i - n_{i-1})$, for $i = 1, 2, \ldots, \ell$. This leads to the following stabilizer

$$\mathrm{Stab}_c(H'_\lambda) = \left\{ \begin{pmatrix} h_1 & 0 & 0 & \cdots & 0 \\ 0 & h_2 & 0 & \cdots & 0 \\ 0 & 0 & h_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & h_1^* \end{pmatrix} \,\middle|\, \begin{array}{c} h_i = \begin{pmatrix} h_{i-1} & 0 \\ 0 & h'_i \end{pmatrix} \\ h'_i \in \prod_{i=1}^{\ell} O(n_i - n_{i-1}) \\ h_i = h_i^{-T} \end{array} \right\},$$

i.e. the non-trivial equivariant fundamental group is

$$A_{C'} = \left(\mathbb{Z} \big/ 2\mathbb{Z} \right)^\ell, \quad \text{for any } \ell \in \mathbb{Z}_{\geq 0}.$$

3.2.3 Equivariant Double Covers

In the previous section, we saw when and how the non-trivial equivariant fundamental groups arose. They arose whenever the middle of the rank-triangle consisted of symmetric matrices. The non-trivial equivariant fundamental groups that arose were 2-groups which came from having orthogonal groups appear within the stabilizers. Since the equivariant fundamental groups will always be 2-groups, we wish to construct equivariant double covers for these groups. To demonstrate the construction, consider the following example:

Example 3.36. Consider Example 3.7, for $\lambda \in$ Sp, then the corresponding Vogan variety V_λ' and centralizer H_λ' are of the form:

$$V_\lambda' = \left\{ \begin{pmatrix} 0 & v_1 \\ 0 & 0 \end{pmatrix} \ \middle|\ v_1 \in \mathrm{Sym}^2(k) \right\} \quad \text{and} \quad H_\lambda = \left\{ \begin{pmatrix} h_1 & 0 \\ 0 & h_1^* \end{pmatrix} \ \middle|\ \begin{matrix} h_1 \in \mathrm{GL}(k) \\ h_1^* = h^{-T} \end{matrix} \right\}.$$

Then the action of H_λ' on V_λ' is given by:

$$h'.v' = h_1 v_1 h_1^T.$$

The orbit closures are given by:

$$\overline{C_r'} = \left\{ v' \in V_\lambda' \ \middle|\ \mathrm{rank}(v') \leq r \right\} \quad \text{such that } 0 \leq r \leq k.$$

We note that the stabilizers for $\overline{C_r'}$ are disconnected, for each r. Observe, without any loss in generality, we may choose as representative

$$v_r = \begin{pmatrix} \mathrm{Id}_r & 0 \\ 0 & 0 \end{pmatrix}.$$

Then, the corresponding stabilizers will be of the form:

$$\mathrm{Stab}_{c_r}(H_\lambda') = \left\{ h \in H_\lambda' \ \middle|\ h_1\, v_r\, h_1^T = v_r \right\}.$$

We'd find that the stabilizer is disconnected for every r. Observe

$$h_1\, v_r\, h_1^T = v_r \iff h_r h_1^T = \mathrm{Id}_r \iff h_r h_r^T = \mathrm{Id}_r \implies h_1 = \begin{pmatrix} h_r & 0 \\ 0 & h_{n-r} \end{pmatrix},$$

where $h_r \in O(r)$ and $h_{n-r} \in \mathrm{GL}(n-r)$. Since $O(r)$ is disconnected,

$$A'_{C_r} = \mathrm{Stab}_{c_r}(H'_\lambda) \Big/ \mathrm{Stab}_{c_r}(H'_\lambda)^\circ = \pm 1.$$

To construct an equivariant double cover, we may use Proposition 2.9, than as an equivariant cover, one has

$$\widehat{C} = H'_\lambda \Big/ \mathrm{Stab}_c(H_\lambda)^\circ.$$

On the open orbit this becomes

$$\widehat{C'_k} = \left\{ (v,d) \in V'_\lambda \times \mathbb{C} \,\big|\, d^2 - \det(v) = 0 \right\},$$

with the following action $h'.v' = (h_1 v_1 h_1^T,\ \det(h_1)d)$.

For the other orbits we must work on coordinate charts in order to take

$$\widehat{C'_r} = \left\{ (v,d) \in V'_\lambda \times \mathbb{C} \,\big|\, d^2 - \det(v_r) = 0 \right\},$$

where v_r is the non-vanishing $(r+1) \times (r+1)$- minor of v, together with the following action $h'.v' = (h_1 v_1 h_1^T,\ \det(h_r)d)$, where h_r is the stabilizing factor of v_r, as our double cover.

Note 3.37. In the above Examples, 3.32 and 3.35, we perform a similar procedure, and provide a natural generalization for the universal equivariant covers on the open orbits, given by the variety

$$\widehat{C'_i} = \left\{ (v,d_i)_{0 < i \leq \ell} \in V'_\lambda \times \mathbb{C} \,\big|\, d_i^2 = \det(J_i) \right\},$$

where $J_i = v_i v_{i-1} \cdots v_{i-1}^* v_i^*$, to define our universal equivariant double cover, since it will cover each one of the sources of the disconnected fundamental groups, i.e., by covering the symmetric matrices arising due to compositions of elements in V_λ, which were the matrices corresponding to the ranks arising down the middle of the rank-triangle.

Chapter 4

Resolutions of Singularities

In this section, we will be studying the geometry of the H_λ orbit closures $\overline{C} \subseteq V_\lambda$. The geometry of such orbit closures was first studied by Abaeisis, Del Fra and Kraft in [1] over 40 years ago. They proved that for equioriented quivers of type A over fields of characteristic 0, the orbit closures are normal and have rational singularities. Their results have since been generalized to fields of arbitrary characteristic, and for arbitrary quivers of type A_n and D_n.

4.1 The Singularities Found on the H_λ-orbit closures of V_λ

Before discussing resolutions of singularities, we shall first recall the definitions of singularities and illustrate how to identify them in our context. For further reading, one recommends Robin Hartshorne's Algebraic Geometry [12] or Igor Shafarevich's Basic Algebraic Geometry [17] both of which cover the topic in-depth; in the former, we reference many useful definitions, theorems, and propositions for our proofs.

Consider a variety V defined by the function f. In the classical calculus context, a point $p \in V$ is said to be singular if and only if $f(p) = 0$ and $f'(p) = 0$. Graphically, these cases may manifest as cusps, self-intersections, or isolated points. For the classical algebraic context, we generalize this notion by formally defining the tangent space, which we can apply in a broader context.

Definition 4.1 (Tangent Space). Suppose that X is a variety and x is a point in X. We denote by $\Theta_x{}^{32}$ the tangent space of X at x.

---

[32] This is sometimes denoted by $T_x X$, $D_x X$, or $\mathcal{J}_x(X)$ in the literature.

Then we may define singular and nonsingular points with respect to the dimensions of the variety and its tangent space.

Definition 4.2 (Singular and nonsingular points on an irreducible variety). Suppose that X is an irreducible variety, and define $s := \min_{x \in X}(\dim \Theta_x)$. Then for any point $x \in X$,

$$x \text{ is a } \textit{singular point} \iff \dim \Theta_x > s$$

$$x \text{ is a } \textit{nonsingular point} \iff \dim \Theta_x = s.$$

In fact, $\min_{x \in X}(\dim \Theta_x)$ is just the dimension of the variety, that is, $s = \dim X$.

We can restate the above criterion for singular and non-singular points on an irreducible variety in terms of the following proposition, known in the literature as the Jacobian criterion.

Proposition 4.3 (Jacobian criterion for smoothness). *[12, Chapter 5, p.31] Suppose that $X \subseteq \mathbb{A}^n$, is an affine variety generated by $(f_1, \ldots, f_t) = I(X) \subset k[x_1, \ldots, x_n]$, we define the Jacobian matrix of X at a point $x \in X$ as the matrix given by*

$$\mathcal{I}_x(f_1, \ldots, f_t) = \left(\frac{\partial f_i}{\partial x_j} \right)\Big|_x.$$

X is said to be non-singular at $x \in X$ if $\operatorname{rank} \mathcal{I}_x(f_1, \ldots, f_t) = n - d$ [33]*, where d is the dimension of X, and is singular otherwise.*

In our context, we apply the Jacobian criterion by computing the Jacobian on the generating functions for the H_λ-orbit C in V_λ, to identify where on the boundary of $\overline{C}$ is singular.

Example 4.4. We note that two of our orbits will always be smooth.

i. The largest orbit or open orbit $C = \{v \in V_\lambda \mid r_{ij} = \min(\dim(E_\ell))_{i \le \ell \le j}, \forall\, 1 \le i \le j < k\}$, since $\overline{C} = V_\lambda \cong \mathbb{A}^n$ which is smooth;

ii. and the smallest orbit or closed orbit $C_0 = \{v \in V_\lambda \mid r_{ii} = \operatorname{rank}(v_i) = 0, \forall\, 1 \le i \le k\}$, since $\overline{C_0} = 0$ is a point, which is always smooth in $\mathbb{A}^n$.

---

[33]$n - d$ is the codimension of X in $\mathbb{A}^n$, and will occur when the matrix is "maximal" among all points in the variety.

Example 4.5. In Example 3.14, the orbits that are always smooth are C_4 and C_0. Thus, we begin with C_3, which is generated by the ideal $f = xy$, and verify the Jacobian condition on this ideal to determine where the singularities lie. Here we see that

$$\mathcal{J}(f) = \begin{bmatrix} y_1 & y_2 & y_3 & x_1 & x_2 & x_3 \end{bmatrix}$$

Evaluating it on each orbit, C_i, for $C_i \leq C_3$, we pick the following base points;

$$c_3 = \left(\begin{pmatrix} 1 & i & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \right), \quad c_2 = \left(\begin{pmatrix} 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right), \quad \text{and} \quad c_1 = \left(\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right).$$

To see that:

$$\operatorname{rank} \mathcal{J}_{c_3}(f) = \operatorname{rank} \begin{bmatrix} 1 & i & 0 & 1 & i & 0 \end{bmatrix} = 1,$$

$$\operatorname{rank} \mathcal{J}_{c_2}(f) = \operatorname{rank} \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} = 1,$$

$$\operatorname{rank} \mathcal{J}_{c_1}(f) = \operatorname{rank} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = 1, \text{ and}$$

$$\operatorname{rank} \mathcal{J}_{c_0}(f) = \operatorname{rank} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = 0.$$

Indeed, here we see that $\overline{C_3}$ is singular on the boundary at C_0. We wish to construct a minimal (in the sense of Definition 2.15) resolution of singularities for $\overline{C_3}$, this process will be demonstrated in the following Sections 4.2 and 4.3.

We note that the class of varieties studied in this thesis are generalized determinantal (rank) varieties, that is, varieties defined by rank equations, i.e., by determinant equations, which are highly singular. We illustrate this by referring to Example 3.7, which is the classical example of a determinantal variety.

Example 4.6. In Example 3.7, we recall that the orbits were described by

$$C_r = \{ v \in V_\lambda \mid \operatorname{rank} v = r \},$$

whose orbit closure $\overline{C_r} = \bigcup_{i \leq r} C_i$, was generated by the $(r+1) \times (r+1)$-minors of $v \in V_\lambda$.

The determinant of this variety will result in a degree $r+1$ polynomial in $\mathbb{C}[v_{ij}]$, which are in general highly singular. In fact, the variety $\overline{C_r}$ is singular on the entire boundary, for every orbit C_i, for $0 \leq i < r$, since the Jacobian of the $(r+1) \times (r+1)$-minors will result in a matrix of all of the $r \times r$-minors. These clearly vanish on C_{r-1} (by definition of the orbit), and will hence vanish for every other sub variety with rank$(v) < r$.

4.2 Resolving Singularities via Blowing Up at a Point

In this section, we review the classical method of resolution of singularities via blowing up at a point, describe how we can apply this in the classical setting for more manageable varieties, and then describe a more specialized method that applies the same principles but is designed for resolving rank varieties. Firstly, what is a resolution of singularities?

Definition 4.7 (Resolution of Singularities)**.** For any variety X, if there exists a smooth and irreducible variety $\widetilde{X}$ and a proper birational morphism $\pi : \widetilde{X} \to X$, then π is called a *resolution of singularities* of X.

Loosely speaking, a map $\pi : Y \to X$ between topological spaces is said to be proper provided that $\pi^{-1}(K)$ is compact in Y, for all compact subsets $K \subseteq X$. In particular, its fibres are compact [16, p. 227]. More formally speaking,

Definition 4.8 (A Proper Morphism)**.** A morphism $\pi : Y \to X$ is said to be proper if it is separated, universally closed, and is of finite type. This notion is meant to capture the following universal property: that for any base change from $Y' \to Y$ the morphism $X \times_Y Y' \to Y'$ is closed.

That is to say proper morphisms are closed under composition, base change (extension), and taking cartesian products. They are very closely related to projective morphisms; in fact, Hartshorne [12, Theorem 4.9, p.103] tells us that any projective morphism over a field is proper.

Definition 4.9 (A Birational Morphism). A morphism $\pi : \widetilde{X} \to X$ is said to be birational if there exists a dense open algebraic subvariety $Y \subsetneq X$, such that $Y \cong \pi^{-1}(Y)$.

We wish to construct minimal resolutions of singularities, minimal in the sense that the resolutions $\pi : \widetilde{X} \to X$ will ideally be either small or semi-small (see Definition 2.15) for the use of other theorems regarding the p-adic Kazhdan-Lusztig hypothesis.

The existence of a resolution of singularities was proved by Heisuke Hironaka for the case of char $k = 0$ in his paper "Resolution of Singularities of an Algebraic Variety over a Field of Characteristic Zero" [13]. It is also known in many cases for char $k = p > 0$ for some curves and surfaces, but is still an open conjecture in general.

In 1964, Heisuke Hironaka proved that every quasi-projective variety admitted a resolution of singularities (every variety is birationally equivalent to a smooth projective variety) through iterative blow-ups. The method is called "blowing up" since the resulting variety looks like a grenade that extracts a line from each base point being blown up, and each line has the same direction as a line that passes through the base point. The idea is to record the incidence direction when passing through a point, while leaving the variety away from this point unchanged. This section follows a survey on blow-ups by See-Hak Seong [15].

Definition 4.10 (The Blowing Up of $\mathbb{A}^n$ at the Origin). The blowup of $\mathbb{A}^n$ at the origin is a set $B(\mathbb{A}^n) \subset \mathbb{A}^n \times \mathbb{P}^{n-1}$ such that

$$B(\mathbb{A}^n) = \{(x, \ell) \in \mathbb{A}^n \times \mathbb{P}^{n-1} \mid (x) \in \ell\}.$$

Note 4.11. Here, the origin is called the centre of the blowing up. In practice, we perform a linear shift in coordinates away from the origin to any arbitrary point $p \in \mathbb{A}^n$, this will be considered the new centre of the blowing up.

Definition 4.12 (A Point on the Projective Line). In general if $(x) = (x_1, x_2, \ldots, x_n) \in \mathbb{A}^n$ and $\ell = [X] = [X_1 : X_2 : \cdots : X_n] \in \mathbb{P}^{n-1}$:

$$(x) \in [X] \iff x_i X_j = x_j X_i, \quad \forall\, 0 < i, j \leq n.$$

67

Moreover, the variety $B(\mathbb{A}^n)$ is equipped with its natural projection:

$$\pi : B(\mathbb{A}^n) \to \mathbb{A}^n$$

$$(x, \ell) \mapsto x$$

is called the *one-point* blowup of $\mathbb{A}^n$.

We recall the notion of a "vector bundle". It is a topological construction that makes precise the notion of a family of vector spaces that is parametrized by another space.

Definition 4.13. A *vector bundle* over a variety X is a variety E together with a map $\varpi : E \to X$, such that $\varpi^{-1}(x)$ has the structure of a vector space for every $x \in X$ and that the following local triviality condition is satisfied:

For every point $x \in X$, there is an open neighbourhood $U \subset X$ of x, and a homeomorphism

$$\varphi : \varpi^{-1}(U) \to U \times \mathbb{C}^n$$

taking $\varpi^{-1}(\{x\})$ to $\{x\} \times \mathbb{C}^n$ by a vector space isomorphism for each $x \in U$.

φ is called a local trivialization of the vector bundle. The space X is called the base space; the space E is called the total space; and the vector spaces $\varpi^{-1}(\{x\})$ are called the fibres over x.

The simplest example of this is when the family of vector spaces is constant, i.e., we can consider a fixed vector space V such that $V(x) = V$, for every $x \in X$. We define the *trivial vector bundle* as $X \times V$ with the direct product projection for some vector space V.

We can think of $B(\mathbb{A}^n) \subset \mathbb{A}^n \times \mathbb{P}^{n-1}$ as sub-bundle of a trivial bundle by considering the projection onto $\mathbb{P}^{n-1}$, where $\mathbb{P}^{n-1}$ is considered the base space

$$\varpi : B(\mathbb{A}^n) \to \mathbb{P}^{n-1}$$

$$(x, \ell) \mapsto \ell.$$

$B(\mathbb{A}^n)$ is called the tautological line bundle over $\mathbb{P}^{n-1}$.

Let us demonstrate the process of blowing up via example and apply this method to Example 3.14, the variety

$$C_3 = \left\{ (x,y) \in \mathbb{A}^3 \times \mathbb{A}^3 \cong \mathbb{A}^6 \,\middle|\, \begin{array}{l} x \neq 0 \\ y \neq 0 \\ x \cdot y = 0 \end{array} \right\},$$

is singular on the boundary of C_0 at the origin (as seen in the previous section), the method of blowups suggests that we blowup the line through the origin by attaching a projective line at the origin together with placing the source of the singularity on this line to record how we approach and leave the singularity. To do so, we will be considering a new space $\widetilde{C_3} \subset B(\mathbb{A}^6)$ which projects down onto $\overline{C_3}$, and will be defined by placing the point (x,y) on the line $[X:Y] \in \mathbb{P}^5$.

If $\pi : B(C_3) \to \overline{C_3}$, then the resolution $\widetilde{C_3} = \overline{\pi^{-1}(C_3 - C_0)}^{34}$ will then be given by,

$$\widetilde{C_3} = \left\{ \begin{pmatrix} x_1,x_2,x_3,y_1,y_2,y_3, \\ [X_1:X_2:X_3:Y_1:Y_2:Y_3] \end{pmatrix} \in \mathbb{A}^6 \times \mathbb{P}^5 \,\middle|\, \begin{array}{c} x_1y_1 + x_2y_2 + x_3y_3 = 0 \\ (x,y) \in [X_1:X_2:X_3:Y_1:Y_2:Y_3] \\ X_1Y_1 + X_2Y_2 + X_3Y_3 = 0 \end{array} \right\},$$

for $[X_1:X_2:X_3:Y_1:Y_2:Y_3] \in \mathbb{P}^5$, together with the projection map $\pi : \widetilde{C} \to \overline{C}$ defined above. We note that the third equation, which defines the relation, is derived by combining the first two equations.

This resolution, though well understood, is neither unique nor an efficient method of resolving singularities. Indeed, observe that there are 2 irregular points, namely, $(y) = 0$ or $(x) = 0$. We note that while neither point is singular, they represent the boundaries C_1 and C_2 of the singular variety $\overline{C_3}$, and the source of the singularity is the boundary of C_0. Thus, we can consider blowing up each line as they pass through the origin; this method admits an alternative resolution[35], as

$$\widetilde{C_3} = \left\{ (x,y,[X],[Y]) \in \mathbb{A}^3 \times \mathbb{A}^3 \times \mathbb{P}^2 \times \mathbb{P}^2 \,\middle|\, \begin{array}{c} x_1y_1 + x_2y_2 + x_3y_3 = 0 \\ (x) \in [X], \ (y) \in [Y] \\ X_1Y_1 + X_2Y_2 + X_3Y_3 = 0 \end{array} \right\}.$$

---

[34] Where $\overline{\cdot}$ is taken to be the Zariski closure.
[35] This resolution is smaller over the origin, while larger over C_1 and C_2.

We note that the projection $\pi : \widetilde{C_3} \to \overline{C_3}$ in either case is not even semi-small. Ideally, we wish to construct a minimal resolution, minimal in that the dimensions or sizes of the fibres $\pi^{-1}(x)$ are as small as possible.

In this example, we note that for the points $x = 0$ or $y = 0$, their fibres are given by $\pi^{-1}((0, y)) = \mathbb{P}^2$ or $\pi^{-1}((x, 0)) = \mathbb{P}^2$, respectively, and while neither point is singular, their preimage is infinite. However, the method we will ultimately use to resolve this singularity will result in a minimal 1-step resolution by attaching a $\mathbb{P}^2$ at the origin, where the preimage $\pi^{-1}(x)$ is finite away from the origin and where $\pi^{-1}(0) = \mathbb{P}^2$. The method that results in this resolution will be described in full detail in Section 4.3.2.

However, in general, a one-point blowup may not resolve the singularity[36]. It may need to be iterated several times for each singularity. Even when these iterative blowups are performed, each iteration may result in a new singularity, hopefully of a lower dimension. When the method of a one-point blowup fails, we may extend the construction to the blowing up of a line or a hyperplane rather than a point in order to achieve the desired result.

We will now shift our focus to a method of resolution of singularities tailored to our context.

4.3   Generalized Method for Resolving Singularities in Orbit Closures

S. Abaesis, A. Del Fra, and H. Kraft proved over 40 years ago in their paper [1] that the orbit closures of equioriented quivers of type A were normal, Cohen-Macaulay, with rational singularities. They provide a method of resolving singularities along with proof. This method of resolving singularities has since been well understood and generalized to a wider array of problems.

We will be further expanding upon their work and extending the construction to our context using what is now known by some in the literature as the Kempf-Lascoux-Weyman geometric technique, this method is demonstrated in [5] and detailed in [21] is a more

---

[36]Although the method described above will resolve any homogeneous polynomial in 1-step

elegant, generalized way of characterizing resolutions of generalized determinantal/rank
varieties in terms of what are called generalized flag varieties.

Weyman studied these varieties in the context of computing cohomologies, and [5] gave
an example of this construction applied to our context, we focus on the latter application and
detail it in the following section. The geometric technique is a method that is analogous
to the blowups defined in the previous section. Though the method for choosing which
projective space(s) and the relation(s) that we place on them will differ, it will provide a
desingularization for the H_λ-orbit closures $\overline{C} \subseteq V_\lambda$. This will first be done for $\hat{G} = \mathrm{GL}_n$,
which we are hopeful can be extended to the context of $\hat{G}' = \mathrm{SO}$ and Sp in the context
where the equivariant fundamental group is trivial, and then for when the equivariant fun-
damental group is non-trivial. The background for much of the construction is detailed in
"Cohomology of Vector Bundles and Syzygies" by Jerzy Weyman [21].

The general idea is to construct a desingularization $\widetilde{C}$ of the H_λ-orbit closure $\overline{C}$ of V_λ
such that $\widetilde{C}$ is the total space of a vector bundle over some projective variety $\mathcal{F}$, which is a
sub-bundle of the trivial bundle $V_\lambda \times \mathcal{F}$ over $\mathcal{F}$.

4.3.1 Generalized Flag varieties

In this section, we recall the basic properties of the various projective spaces that we
will be considering, and how we will use them in our construction of the resolutions for our
H_λ-orbit closures in V_λ.

Let E be a vector space of dimension n over $\mathbb{C}$. We define the *Grassmannian* $\mathrm{Gr}(r, E)$
as the set of all r-dimensional subspaces of E, for an integer, $0 < r < n$, explicitly

$$\mathrm{Gr}(r, E) = \{F^{(r)} \subseteq E \mid \dim(F^{(r)}) = r\}.$$

When $r = 1$ the Grassmannian $\mathrm{Gr}(1, E)$ is simply $\mathbb{P}(E)$ and hence is a compact smooth
projective variety. In fact, the Grassmannian is a compact smooth projective variety for all

$0 < r < n$ of dimension $r(n-r)$. This can be shown via the Plücker embedding:

$$\phi : \mathrm{Gr}(r,E) \to \mathbb{P}\left(\overset{r}{\bigwedge} E\right)$$

$$U \mapsto \mathbb{C}(u_1 \wedge u_2 \wedge \cdots \wedge u_r),$$

where $\{u_1, u_2, \ldots, u_r\}$ is a basis for U. We note that GL_n acts transitively on $\mathrm{Gr}(r,E)$ via

its natural action on E, and that $\mathrm{Gr}(r,E)$ is the unique orbit of this action. GL_n also acts

on $\mathbb{P}(\wedge^r E)$ via its linear action on $\wedge^r E$ and this inclusion is equivariant with respect to this

action. Thus, $\mathrm{Gr}(r,E)$ is a homogeneous GL_n-space; as a consequence, it is projective and

non-singular of dimension $r(n-r)$ (see [21, Chapter 3]).

We may denote the elements in $\mathrm{Gr}(r,E)$ by $F^{(r)}$, since the Grassmannian is a collection

of all r-dimensional subspaces.

Note 4.14. To introduce affine charts for E we will first fix a basis for E, so write $E = \mathbb{C}^n$.

We can then consider the standard affine chart $U \subset \mathrm{Gr}(r,E)$ given by

$$U = \left\{ \begin{bmatrix} Id_r \\ A \end{bmatrix} \;\middle|\; A \in \mathrm{Mat}_{(n-r) \times r} \right\},$$

where the inclusion $U \to \mathrm{Gr}(r,E)$ is given by $F^{(r)} = \mathrm{Im}\left(\begin{bmatrix} Id_r \\ A \end{bmatrix}\right)$, so that the matrix A,

determines an element of $\mathrm{Gr}(r,E)$.

In fact, if we consider the collection of charts of this form, where we introduce a per-

mutation matrix $\sigma_\alpha \in \mathrm{GL}_n$, we may obtain an open covering using the sets:

$$U_\alpha = \left\{ \sigma_\alpha \left(\begin{bmatrix} Id_r \\ A \end{bmatrix} \right) \;\middle|\; A \in \mathrm{Mat}_{(n-r) \times r} \right\}, \tag{4.1}$$

for every permutation σ_α of the rows. This collection of charts naturally covers $\mathrm{Gr}(r,E)$.

These charts represent equivalence up to ECO's (elementary column operations), since the

rank of a matrix is invariant under ECO's. This construction is analogous to the set of lines

through the origin $\mathbb{P}^1$, where the standard chart is given by $[1 : x]$, and each element $x \in \mathbb{A}^1$ defines a new line through the origin. This chart also has extensive overlap with the second chart $[y : 1]$ in $\mathbb{P}^1$, they overlap everywhere except $x, y = 0$ and as a transition map between the charts we have $x \mapsto 1/x$.

There is a canonical bijection between $\mathrm{Gr}(r, E)$ and its dual $\mathrm{Gr}(n - r, E^*)$:

$$\mathrm{Gr}(r, E) \cong \mathrm{Gr}(n - r, E^*)$$

$$\begin{bmatrix} Id_r \\ A \end{bmatrix} \mapsto \begin{bmatrix} -A & Id_{n-r} \end{bmatrix},$$

where we use the dual basis for E^* and represent elements of E^* with row vectors instead of column vectors. We may write $F^{(r)\perp} = \mathrm{Im}\left(\begin{bmatrix} -A & Id_{n-r} \end{bmatrix} \right)$ such that $F^{(r)\perp} F^{(r)} = \{0\}$.

We extend the above construction of Grassmannians to generalized flag varieties. A *flag* in an n-dimensional vector space E over $\mathbb{C}$ is an increasing[37] sequence of subspaces

$$\{0\} \subseteq F^{(r_1)} \subseteq F^{(r_2)} \subseteq \cdots \subseteq F^{(r_k)} = E.$$

Where $\dim(F^{(r_j)}) = r_j$ and $0 \leq r_1 \leq r_2 \leq \cdots \leq r_k = n$. The sequence of integers $\underline{r} = (r_1, r_2, \ldots, r_k)$ is called the signature of the flag, and the flag is said to be full if $\underline{r} = (1, 2, \ldots, n)$, and is said to be partial otherwise. Now, we define the flag variety as

$$\mathrm{Flag}(\underline{r}, E) = \left\{ \left(F^{(\underline{r})} \right) \in \prod_{r_j \in \underline{r}} \mathrm{Gr}(r_j, E) \;\middle|\; F^{(r_1)} \subseteq \cdots \subseteq F^{(r_k)} \right\},$$

where $(F^{(\underline{r})}) = (F^{(r_1)}, F^{(r_2)}, \ldots, F^{(r_k)})$. The flag variety, $\mathrm{Flag}(\underline{r}, E)$ is also a compact smooth[38] projective variety, and this can also be shown via the Plücker embedding, since we have the

---

[37]Conventionally, one defines generalized flag varieties using proper containment. Here we allow non-proper containment for notational convenience. Geometrically, it is equivalent to simply deleting repeated subspaces. Additionally, we include the dimension of the eigenspace in the signature for this construction so that our flags may be represented as square invertible matrices. Though this is not a requirement for flags in general.

[38]Since GL_n still acts transitively on the flags (of a given signature), the flag variety is still homogeneous and hence smooth.

obvious embedding

$$\pi : \text{Flag}(\underline{r}, E) \to \text{Gr}(r_1, E) \times \text{Gr}(r_2, E) \times \cdots \times \text{Gr}(r_k, E).$$

The action of $GL(E)$ on a flag variety is transitive, and is defined by letting $h \in GL(E)$ send this chain to

$$\{0\} \subseteq h(F^{(r_1)}) \subseteq h(F^{(r_2)}) \subseteq \cdots \subseteq h(F^{(r_k)}) = h(E). \tag{4.2}$$

Similarly to the Grassmannian setting, upon fixing a basis for E, we may represent coordinate charts for the flag using matrices. This allows us to realize a coordinate chart description for the flags as an invertible matrices in $GL(E)$. The standard chart is

$$F^{(\underline{r})} \longleftrightarrow \begin{bmatrix} Id_{r_1} & & & & \\ \begin{bmatrix} | \\ A_1 \\ | \end{bmatrix} & Id_{r_2-r_1} & & & \\ & \begin{bmatrix} | \\ A_2 \\ | \end{bmatrix} & \ddots & & \\ & & & Id_{r_{k-1}-r_{k-2}} & \\ & & & \begin{bmatrix} A_{k-1} \end{bmatrix} & Id_{n-r_{k-1}} \end{bmatrix}. \tag{4.3}$$

If we let $(e_j)_{0<j\leq k}$ denote the column vectors of the above matrix, we have the natural map $\pi_j : \text{Flag}(\underline{r}, E) \to \text{Gr}(r_j, E)$, via projection onto $\text{span}(e_1, e_2, \ldots, e_{r_j})$, is a map between the varieties. As with Grassmannians, the other open charts are given by applying the permutation matrix $\sigma_\alpha \in GL(E)$ to the sets of matrices in the form of the above standard chart.

Finally, we define *Generalized Flag Varieties*, by considering products of the above flag varieties

$$\mathcal{F} = \prod_{i=1}^{k} \text{Flag}\left(\underline{r_i}, E_i\right) \quad \text{for integers } 0 < i \leq k,$$

where each E_i is a vector space of dimension n_i over $\mathbb{C}$. This construction is analogous to the above setup, and as such, the generalized flag varieties are naturally compact smooth projective varieties.

Remark 4.15. The action of $\prod_{i=1}^{k} GL(E_i)$ on the generalized flag variety $\mathcal{F}$ is transitive.[39]

---

[39]This follows from Equation (4.2), and it is this fact that makes our construction of resolutions equivariant!

Generalized flag varieties are precisely the varieties we want to consider in our setup. Since a generalized flag variety $\mathcal{F}$, consists of linear subspaces of λ's eigenspaces, the elements of the quiver representation $v = (v_1, \ldots, v_{k-1})$ will naturally map between these subspaces. We will construct a resolution of singularities by attaching an appropriate generalized flag variety using an orbit's rank-triangle. This construction is meant to be analogous to the general procedure given in Section 4.2 regarding blowups.

4.3.2 Resolutions of Singularities for the H_λ-Orbit Closures in V_λ

In this section, we will be giving the full proof and construction for the resolution of singularities for the H_λ-orbit closures $\overline{C} \subset V_\lambda$. The proof will be given by proving several lemmata in order to simplify the presentation of the proof. We begin by explaining the intuition given at the end of the previous section.

Recall. Two points, $x, y \in V_\lambda$ are in the same H_λ-orbit C if and only if they share the same rank-triangle:

$$
C : \quad
\begin{array}{ccccccc}
n_1 & & n_2 & & \cdots & & n_k \\
\hline
& r_{11} & & r_{22} & \cdots & r_{k-1\ k-1} & \\
& & r_{12} & & \vdots & & \\
& & & \ddots & & & \\
& & & r_{1\ k-1} & & &
\end{array}
$$

by Proposition 3.8, and that by Proposition 3.11, we have that $r_{i(j+1)} \leq r_{ij}$, for all i, j.

Using this, we have a natural ordering on the ranks appearing in the rank-triangle. In particular, for each i we can construct a flag for the eigenspace E_i, whose signature is given by $(r_{ij})_{ij}$ where $k - 1 \leq j \leq i - 1$, for each $1 \leq i \leq k$. Then the generalized flag variety is of the form

$$
\mathcal{F} = \prod_{i=1}^{k} \mathrm{Flag}\big((r_{i(k-1)}, \ldots, r_{i(i-1)}), E_i\big).
$$

Thus, we give the following definition as our resolution of singularities.

Definition 4.16 (Resolution of Singularities). Let $\overline{C}$ be an H_λ-orbit closure in V_λ. Then we define the subvariety $\widetilde{C} \subset V_\lambda \times \mathcal{F}$, given by

$$\widetilde{C} := \left\{ (v, F) \in V_\lambda \times \mathcal{F} \mid v_i(F_{(i+1)j}) \subseteq F_{ij}, \ \forall \, i, j \right\},$$

where $F_{ij} := F_i^{(r_{ij})} \in \mathcal{F} = \prod_{i=1}^k \mathrm{Flag}((r_{i(k-1)}, \ldots, r_{i(i-1)}), E_i)$ is the generalized flag variety of partial flags over $E(\lambda)$, and the signatures (r_{ij}) are given by the ranks appearing in the rank-triangle for C.

We note that $\widetilde{C}$ has an H_λ-action.

Lemma 4.17 ($\widetilde{C}$ has an H_λ-Action). *Let C denote an H_λ-orbit in V_λ, and let $\widetilde{C}$ be defined as above in Definition 4.16. Then, $\widetilde{C}$ is equipped with a natural action by H_λ.*

Proof. H_λ naturally acts on $\widetilde{C}$, since for $(v, F) \in \widetilde{C}$:

- H_λ acts on $v \in C$ via $h.v = (h_i \, v_i \, h_{i+1}^{-1})_i$; and

- H_λ acts on $F \in \prod_{i=1}^k \mathrm{Flag}((r_{ij})_j, E_i)$ via $h.F = (h_i \, F_{ij})_i$.

Then if $v_i F_{(i+1)j} \subseteq F_{ij}$ for all i, j, we have that

$$h_i v_i F_{(i+1)j} \subseteq h_i F_{ij}$$

$$\implies h_i v_i (h_{i+1}^{-1} h_{i+1}) F_{(i+1)j} \subseteq h_i F_{ij}$$

$$\implies (h_i v_i h_{i+1}^{-1})(h_{i+1} F_{(i+1)j}) \subseteq h_i F_{ij}$$

$$\implies v_i' F_{(i+1)j}' \subseteq F_{ij}'.$$

I.e., for $(v, F) \in \widetilde{C}$, we have that $(v', F') = h(v, F) \in \widetilde{C}$. $\qquad\square$

The construction of $\widetilde{C}$ is partially motivated by the following lemma.

Lemma 4.18 (The Morphism $s : C \to \pi'^{-1}(C)$ is Well-Defined.). *Let C denote an H_λ-orbit in V_λ, and let $\widetilde{C}$ be defined as above in Definition 4.16. Then, there exists a well-defined map*

$$s : C \to \pi'^{-1}(C) \quad \text{given by} \quad c \mapsto (c, \mathrm{Im}(v_i \cdots v_j)_{ij}). \tag{4.4}$$

Proof. Let $c = (v_1, \ldots, v_{k-1})$ be an arbitrary element of C, then $s(c) = (c, \text{Im}(v_i \cdots v_j)_{ij})$. In particular, $\text{Im}(v_i \cdots v_j)$ is an r_{ij}-dimensional subspace of E_i, i.e., $\text{Im}(v_i \cdots v_j) \in \text{Gr}(r_{ij}, E_i)$. Then,

1. the collection of images forms a flag, since $\text{Im}(v_i \cdots v_{j+1}) \subseteq \text{Im}(v_i \cdots v_j)$ for all i, j;

2. the elements v_i of v map between the flags, since $v_i(\text{Im}(v_{i+1} \cdots v_j)) = \text{Im}(v_i \cdots v_j)$.

Thus, the map $s : C \to \pi'^{-1}(C)$ is well-defined. $\qquad \square$

Consider the following example, which demonstrates how we can construct generalized flag varieties from the rank-triangle of a given H_λ-orbit C.

Example 4.19. Recall that in Example 3.14, we had 3 eigenspaces, E_1, E_2, E_3 with dimensions n_1, n_2, n_3, which we relabel as r_{10}, r_{21}, r_{32} (see Proposition 3.11) respectively:
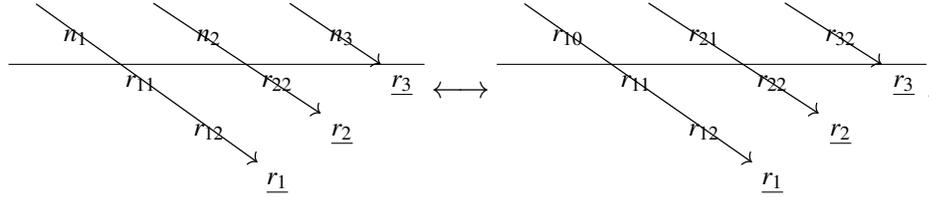
Then we may write

$$\underline{r_1} = (r_{12}, r_{11}, r_{10}),$$

$$\underline{r_2} = (r_{22}, r_{21}),$$

$$\underline{r_3} = (r_{32}),$$

for the signatures of the flags for each eigenspace $\text{Flag}(\underline{r_i}, E_i)$. Then, if we consider an element $c = (v_1, v_2) \in C \subset V_\lambda$, we may construct the following illustrative diagram, given in Figure 4.1 of a generalized flag variety, together with the maps v_i that map between the flags.

Figure 4.1 displays a collection of linear maps $v_i(F_{(i+1)j}) \subseteq F_{ij}$, for all i, j, where $F_i^{(r_{ij})} = F_{ij} = \text{Im}(v_i \cdots v_j)_{ij}$.
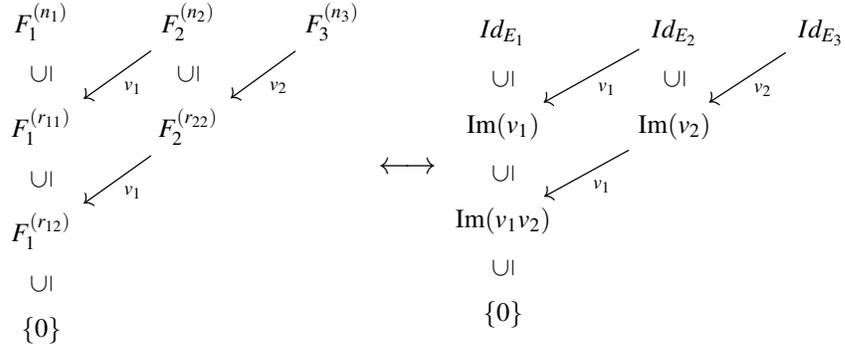
77

$$
\begin{array}{ccc}
F_1^{(n_1)} \qquad F_2^{(n_2)} \qquad F_3^{(n_3)} & \qquad & Id_{E_1} \qquad Id_{E_2} \qquad Id_{E_3} \\
\cup| \nearrow v_1 \quad \cup| \nearrow v_2 & & \cup| \nearrow v_1 \quad \cup| \nearrow v_2 \\
F_1^{(r_{11})} \qquad F_2^{(r_{22})} & & Im(v_1) \qquad Im(v_2) \\
\cup| \nearrow v_1 & \longleftrightarrow & \cup| \nearrow v_1 \\
F_1^{(r_{12})} & & Im(v_1 v_2) \\
\cup| & & \cup| \\
\{0\} & & \{0\}
\end{array}
$$

Figure 4.1: Generalized flag variety for Example 4.29.

We now state the key result of this construction.

Theorem 4.20. *Let $\widetilde{C}$ be defined as in Definition 4.16. If we denote by $\pi : V_\lambda \times \mathcal{F} \to V_\lambda$ the projection onto the first coordinate and by π' the restriction of π to $\widetilde{C}$, then*

1. *We have the following diagram*

$$
\begin{array}{ccc}
\widetilde{C} & \subset & V_\lambda \times \mathcal{F} \\
\downarrow \pi' & & \downarrow \pi \\
\overline{C} & \subset & V_\lambda.
\end{array}
$$

   In particular $\pi(\widetilde{C}) = \overline{C}$ and π' is equivariant.

2. *π' is birational, in particular $\pi'^{-1}(C) \cong C$.*

3. *π' is a proper morphism.*

4. *$\widetilde{C}$ is a connected, smooth, and irreducible algebraic variety.*

In particular, together these facts imply that $\pi' : \widetilde{C} \to \overline{C}$ defines a resolution of singularities.

Proof. We show in Lemma 4.21 that $\widetilde{C}$ together with the projection ϖ onto the second coordinate is a vector bundle over $\mathcal{F}$, a smooth connected variety, and as a consequence $\widetilde{C}$ is also a connected, smooth, and irreducible algebraic variety. In Lemma 4.22, we prove

that the induced projection π' onto the first coordinate is a proper map. Then in Lemma 4.24, we prove that $\text{Im}(\pi') = \overline{C}$, and in Lemma 4.23, we prove that $\pi'^{-1}(C) \simeq C$. Since C is open in $\overline{C}$, it is an open subset on which π' admits an inverse, we have that π' is a birational morphism. Lastly, since we know that the morphism $\pi' : \widetilde{C} \to V_\lambda$ is proper and proper morphisms are stable under base change, we may first map $\widetilde{C}$ to the image of π', which is simply $\overline{C}$ by Lemma 4.24, and π' will still be proper when we change its codomain. Hence, $\pi' : \widetilde{C} \to \overline{C}$ defines a resolution of singularities over $\overline{C}$. $\qquad\square$

Lemma 4.21 ($\widetilde{C}$ is a Smooth Algebraic Variety). *The projection $\varpi' : \widetilde{C} \to \mathcal{F}$, onto the generalized flag variety $\mathcal{F} = \prod_{i=1}^{k} \text{Flag}((r_{i(k-1)}, \ldots, r_{i(i-1)}), E_i)$ gives $\widetilde{C}$ the structure of a subbundle of the trivial vector bundle $\varpi : V_\lambda \times \mathcal{F} \to \mathcal{F}$. In particular, $\widetilde{C}$ is a smooth algebraic variety.*

This proof is similar to the one from [21, Proposition 3.3.1] which concerns the tautological vector bundle over $\text{Gr}(r, E)$. While it is important to understand the tautological vector bundle, this is not the variety that we have constructed, and as such, we provide an analogous proof for our varieties $\widetilde{C}$.

Proof. Let C denote an H_λ-orbit in V_λ, and let $\widetilde{C}$ be defined as above in Definition 4.16. The map $\varpi : V_\lambda \times \mathcal{F} \to \mathcal{F}$ defines on $V_\lambda \times \mathcal{F}$ the structure of the trivial vector bundle of rank N over $\mathcal{F}$ together with the map ϖ, since $V_\lambda \cong \mathbb{A}^N$, for $N = \sum_{i=1}^{k-1} n_{i+1} n_i$, where $n_i = \dim(E_i)$. It is clear that $\widetilde{C}$ is a subset of $V_\lambda \times \mathcal{F}$, then if we restrict to its image over $\widetilde{C}$, we claim that the induced map $\varpi' := \varpi \mid_{\widetilde{C}}$ defines on $\widetilde{C}$ the structure of a subbundle over $\mathcal{F}$.

Indeed, if we let F be an arbitrary element of the generalized flag variety $\mathcal{F}$, then its pre-image $\varpi'^{-1}(\{F\}) = W \times \{F\}$, for some linear subspace W of V_λ, since the conditions $v_i(F_{(i+1)j}) \subseteq F_{ij}$ on $v_i \in \hom(E_{i+1}, E_i)$ gives a linear subspace, for each j (some of which may be repeated). Since W is a linear subspace of V_λ, it is isomorphic to $\mathbb{A}^{N'}$ for some $N' \leq N \in \mathbb{N}$. In fact, the isomorphism class of W is independent of the particular choice of $F \in \mathcal{F}$.

Since H_λ acts transitively on $\mathcal{F}$, for any $F, G \in \mathcal{F}$, there exists an invertible element $h \in H_\lambda$ for which $hF = G$. Thus, $h : V_\lambda \to V_\lambda$ gives an isomorphism from $\varpi'^{-1}(\{F\}) = W_F \times \{F\}$ to $\varpi'^{-1}(\{G\}) = W_G \times \{G\}$, since ϖ' is equivariant, h takes $\varpi'^{-1}(\{F\})$ to $\varpi'^{-1}(\{hF\}) = \varpi'^{-1}(\{G\})$. That is to say the corresponding vector space W is uniquely determined (up to isomorphism).

We now define the local trivialization of the vector bundle. That is, for any $F \in \mathcal{F}$, there is an open neighbourhood $U \subset \mathcal{F}$ of F such that $\varpi'^{-1}(U) \cong W \times U$. It suffices to work with the collection of standard charts U_α (See Equation (4.1)); we will fix a base point in the chart U_α to be $F_\alpha = \sigma_\alpha$. We recall that σ_α is the collection of permutations that give the chart U_α. We now, define $\varpi'^{-1}(\{F_\alpha\}) =: W$. Then, there exists an isomorphism

$$h_\alpha : W \times U_\alpha \to \varpi'^{-1}(U_\alpha),$$

given by the map

$$W \times U_\alpha \to \varpi'^{-1}(U_\alpha)$$

$$(v, \, F) \mapsto (\varphi_\alpha(F)v, F),$$

where $\varphi_\alpha : U_\alpha \hookrightarrow H_\lambda$ is the inclusion of the chart into H_λ, we recall by Equation (4.3), the open charts U_α will be a product of square $n_i \times n_i$ matrices for each i.

Then for any other point $F \in \mathcal{F}$, we have that $\varphi_\alpha(F)F_\alpha = F$, which implies that

$$\varphi_\alpha(F) : W \to \varpi'^{-1}(F).$$

This, together with the obvious inverse h_α^{-1} given by $(v, F) \mapsto (\varphi^{-1}(F)v, F)$, gives us an isomorphism $\varpi'^{-1}(U_\alpha) \cong W \times U_\alpha$. We remark that this map is compatible with the projection, which completes the proof. $\qquad\square$

Since vector bundles (and hence subbundles) are smooth, we have that $\widetilde{C}$ is a smooth variety. Next, to prove that the variety $\widetilde{C}$ we've constructed is indeed a resolution of singularities, we now focus on proving the properties of the induced map π' which gives $\widetilde{C}$ the structure of a resolution of singularities.

Lemma 4.22 (π' is Proper)**.** *Let C denote an H_λ-orbit in V_λ, and let $\widetilde{C}$ be defined as above
in Definition 4.16. Then, the induced map $\pi' : \widetilde{C} \to V_\lambda$ is proper.*

Proof. We begin by noting that the inclusion ι of $\widetilde{C}$ into $V_\lambda \times \mathcal{F}$ is a closed embedding,
which is proper by [12, Corollary 4.8, (a)]. The natural projection π of $V_\lambda \times \mathcal{F}$ onto V_λ
is proper since π is a projective morphism (see [12, page 103]) which is proper by [12,
Theorem 4.9] and lastly, the composition of proper maps is proper by [12, Corollary 4.8(b)].
That is to say, the induced map $\pi' : \widetilde{C} \to V_\lambda$ is proper since the following diagram commutes:

$$\begin{array}{ccc} \widetilde{C} & \xrightarrow{\iota} & V_\lambda \times \mathcal{F} \\ & {\scriptstyle \pi'} \searrow & \downarrow {\scriptstyle \pi} \\ & & V_\lambda. \end{array} \qquad \square$$

Lemma 4.23 (π' is Birational)**.** *Let C denote an H_λ-orbit in V_λ, and let $\widetilde{C}$ be defined as
above in Definition 4.16. Then, the map*

$$s : C \to \pi'^{-1}(C) \quad \text{given by} \quad c \mapsto (c, \mathrm{Im}(v_i \cdots v_j)_{ij}), \qquad (4.5)$$

given in Lemma 4.18 defines an isomorphism such that $\pi' \circ s = \mathrm{id}$.

Proof. By Lemma 4.18, this map is well-defined. It is obvious that $\pi' \circ s = \mathrm{id}_C$. So, to
prove that s is an isomorphism, we only need to prove that $s \circ \pi' = \mathrm{id}_{\pi'^{-1}(C)}$.

Suppose that (c, F) is an arbitrary element of $\pi'^{-1}(C)$. Then we have that

$$(s \circ \pi')(c, F) = s(\pi'(c, F)) = s(c) = (c, \mathrm{Im}(v_i \cdots v_j)_{ij}).$$

Now,

- since $F_{(j+1)j)} := E_{j+1}$, we obtain $\mathrm{Im}(v_i \cdots v_j) = v_i \cdots v_j(F_{(j+1)j)}) \subseteq F_{ij}$; and

- since $c \in C$, we have that $\dim(\mathrm{Im}(v_i \cdots v_j)) = r_{ij} = \dim(F_{ij})$.

In particular, we have that $\mathrm{Im}(v_i \cdots v_j)$ is a linear subspace of F_{ij} with equal dimension.
Thus, we must have that $\mathrm{Im}(v_i \cdots v_j) = F_{ij}$, for all $0 < i \le j < k$. I.e., $s \circ \pi' = \mathrm{id}_{\pi'^{-1}(C)}$. $\square$

Lemma 4.24 ($\pi(\widetilde{C}) = \overline{C}$). *Let C denote an H_λ-orbit in V_λ, and let $\widetilde{C}$ be defined as above in Definition 4.16. Then, we have that $\mathrm{Im}(\pi') = \overline{C}$.*

Proof. ($\subseteq$) Let $(v, F) \in \widetilde{C}$, then by the definition of $\widetilde{C}$, as in the proof of Lemma 4.23, we have $\mathrm{Im}(v_i \cdots v_j) \subseteq F_{ij}$, for all i, j. This then implies that

$$\mathrm{rank}(v_i \cdots v_j) \leq \dim(F_{ij}) = r_{ij}, \ \forall \ i, j.$$

Since these are precisely the defining equations for the orbit closure, we immediately have that $\mathrm{Im}(\pi') \subseteq \overline{C}$.

($\supseteq$) Let s be the map defined in Equation (4.4). Then by Lemma 4.23, since $(\pi' \circ s)(C) = C$ we immediately have that $C \subseteq \mathrm{Im}(\pi')$. Now, since π' is proper, $\mathrm{Im}(\pi')$ is closed, so we may obtain $\overline{C} \subseteq \mathrm{Im}(\pi')$ by taking the closure.

That is to say we have that $\overline{C} \subseteq \mathrm{Im}(\pi') \subseteq \overline{C}$, thus $\mathrm{Im}(\pi') = \overline{C}$. $\square$

4.3.3 Example Resolutions of Singularities in $\hat{G} = \mathrm{GL}_n$

In this section, we demonstrate the method of resolutions described in Section 4.3.2. We will be giving many examples to explicitly demonstrate the construction for the resolutions of singularities of H_λ-orbit closures $\overline{C} \subseteq V_\lambda$, for $\lambda \in \hat{G} = \mathrm{GL}_n$.

In order to illustrate Lemma 4.24 and demonstrate that our algorithm does indeed produce smooth vector space condition, we begin with the following motivating example:

Example 4.25. Consider $\lambda = \mathrm{diag}(q^{\frac{1}{2}}, q^{\frac{1}{2}}, q^{\frac{1}{2}}, q^{-\frac{1}{2}}, q^{-\frac{1}{2}}, q^{-\frac{1}{2}})$. Then, writing E_1, E_2 for the respective $E_{q^{1/2}}, E_{q^{-1/2}}$ eigenspaces, we may write the Vogan variety V_λ and centralizer H_λ as:

$$V_\lambda = \left\{ \begin{pmatrix} 0 & v_1 \\ 0 & 0 \end{pmatrix} \ \middle| \ v_1 : E_2 \to E_1 \right\} \quad \text{and} \quad H_\lambda = \left\{ \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} \ \middle| \ \begin{matrix} h_1 \in \mathrm{GL}(E_1) \\ h_2 \in \mathrm{GL}(E_2) \end{matrix} \right\}$$

The variety V_λ is the prototypical example of a determinantal variety. We note that the orbits are given by $C_r = \{v \in V_\lambda \mid \mathrm{rank}(v_1) = r\}$ are understood by the following partial ordering:

$$C_0 < C_1 < C_2 < C_3.$$

The open and closed orbits C_3 and C_0 resp. are smooth, while the orbits C_1 and C_2 are singular. We may resolve both of these orbits via the resolutions:

$$\widetilde{C_3} = \left\{ v \in V_\lambda \times \mathrm{Gr}(3,3) \mid v_1(E_2) \subseteq F_1^{(3)} \right\},$$

$$\widetilde{C_2} = \left\{ v \in V_\lambda \times \mathrm{Gr}(2,3) \mid v_1(E_2) \subseteq F_1^{(2)} \right\},$$

$$\widetilde{C_1} = \left\{ v \in V_\lambda \times \mathrm{Gr}(1,3) \mid v_1(E_2) \subseteq F_1^{(1)} \right\}, \text{ and}$$

$$\widetilde{C_0} = \left\{ v \in V_\lambda \times \mathrm{Gr}(0,3) \mid v_1(E_2) \subseteq F_1^{(0)} \right\}.$$

Where $F_1^{(r)}$ corresponds to the r-dimensional subspace of E_1. Then, we demonstrate how the equations $v_1(E_2) \subseteq F_1^{(r)}$ determine linear subspace conditions:

i. We note that $\widetilde{C_3} \simeq \overline{C_3}$. Indeed, if we were to resolve the open (smooth) orbit, C_3, as a linear subspace condition we would find $v_1(E_2) \subseteq F_i^{(3)}$. Which is automatic, since $v_1 \in \mathrm{Mat}_{3 \times 3}$ and taking the rank of this equation yields: $\mathrm{rank}(v_1) \leq 3$, which is precisely the equation defining the closure of $\overline{C_3}$.

ii. Next, we consider $v_1(E_2) \subseteq F_i^{(2)} \implies \det(v_i) = 0$, since $v_1 \in \mathrm{Mat}_{3 \times 3}$ to map into a less than 3-dimensional subspace, v_1 must not be invertible.

iii. Similarly, $v_1(E_2) \subseteq F_i^{(1)} \implies \det(A_{ij}(v_1)) = 0$, for all $1 \leq i, j \leq 3$, where $A_{ij}(v_1)$ denotes 2×2-minors of v_1.

iv. As with our first point, we have that $\widetilde{C_0} \simeq C_0$. Since, if we were to compute a resolution for the closed (smooth) orbit, C_0, as a linear subspace condition we would find $v_1(E_2) \subseteq F_1^{(0)} \implies v_1 \equiv 0$. Which is automatic, since if $\mathrm{Im}(v_1) = \{0\}$, v_1 must be the zero map, thus we would recover C_0.

The condition on the vanishing $(r+1) \times (r+1)$-minors is precisely the condition given in Equation (3.1) for our generalized determinantal varieties. This idea yields the linear subspace conditions for when $v_{ij} := v_i \cdots v_j$ becomes a product of quiver representations.

Next, we can consider the generalization of this example which will resolve each one
of the singular orbits for the determinantal variety example.

Example 4.26 (k, k-example, i.e., Example 3.7). The Vogan variety V_λ and centralizer H_λ
were of the form:

$$V_\lambda = \left\{ \begin{pmatrix} 0 & v_1 \\ 0 & 0 \end{pmatrix} \,\middle|\, v_1 : E_2 \to E_1 \right\} \quad \text{and} \quad H_\lambda = \left\{ \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} \,\middle|\, h_1, h_2 \in \mathrm{GL}(k) \right\}.$$

Where E_1, E_2 are k-dimensional eigenspaces. We saw that each orbit C_r was singular for all
$0 < r < k$, then using the above construction, we resolve the singularities as:

$$
\begin{array}{ccccc}
 & & F_1^{(k)} \quad F_2^{(k)} & & E_2 \\
 & & \cup\text{I} \quad \swarrow{\scriptstyle v_1} & & \swarrow{\scriptstyle v_1} \\
\dfrac{k \quad\quad k}{r} & \longleftrightarrow & F_1^{(r)} & \longleftrightarrow & F_1^{(r)} \\
 & & \cup\text{I} & & \cup \\
 & & \{0\} & & \{0\}
\end{array}
$$

Then the resolution is given by the addition of a single Grassmannian, namely $F_1^{(r)} \in$
$\mathrm{Gr}(r, E_1)$,

$$\widetilde{C_r} = \left\{ (v, F) \in V_\lambda \times \mathrm{Gr}(r, E_1) \,\middle|\, v_1(E_2) \subseteq F_1^{(r)} \right\}.$$

Then the subspace condition is given by:

$$v_1(E_2) \subseteq F_1^{(r)} \implies \det(A_m(v_1)) = 0$$

Where $A_m(v_1)$ denotes the collection of all $(r+1) \times (r+1)$-minors of v_1, indexed by the
set M. In general this will be a collection of degree $r+1$ polynomials in $\mathbb{C}[x_{ij}]$. The variety
$\widetilde{C_r}$ is smooth by Lemma 4.21, which states that it is a vector bundle over $\mathrm{Gr}(r, E_1)$.

We may naturally apply this algorithm to larger examples, consider applying our method
to the C_ψ-singularity computed in [5]:

Example 4.27. Consider the infinitesimal parameter

$$\lambda := 2\lambda_0 \oplus 4\lambda_1 \oplus 4\lambda_2 \oplus 4\lambda_3 \oplus 2\lambda_4.$$

The quiver variety $v = (v_1, v_2, v_3, v_4, v_1) \in V_\lambda$ such that $v_i \in \hom(E_{i+1}, E_i)$, in this setup is will be given as:

$$\underset{\bullet}{\overset{E_1}{\bullet}} \xleftarrow{\quad v_1 \quad} \underset{\bullet}{\overset{E_2}{\bullet}} \xleftarrow{\quad v_2 \quad} \underset{\bullet}{\overset{E_2}{\bullet}} \xleftarrow{\quad v_3 \quad} \underset{\bullet}{\overset{E_4}{\bullet}} \xleftarrow{\quad v_4 \quad} \underset{\bullet}{\overset{E_5}{\bullet}}$$

Consider the H_λ-orbit, C_ψ, given by the following rank-triangle:

$$C_\psi : \quad \begin{array}{ccccccccc} 2 & & 4 & & 4 & & 4 & & 2 \\ & 2 & & 3 & & 3 & & 2 & \\ & & 1 & & 2 & & 1 & & \\ & & & 1 & & 1 & & & \\ & & & & 0 & & & & \end{array} \quad .$$

This orbit is singular (see [5]), and so we use the integer values given in the rank-triangle to construct the following diagram, which is meant to represent the generalized flag variety together with the conditions on the linear maps between them
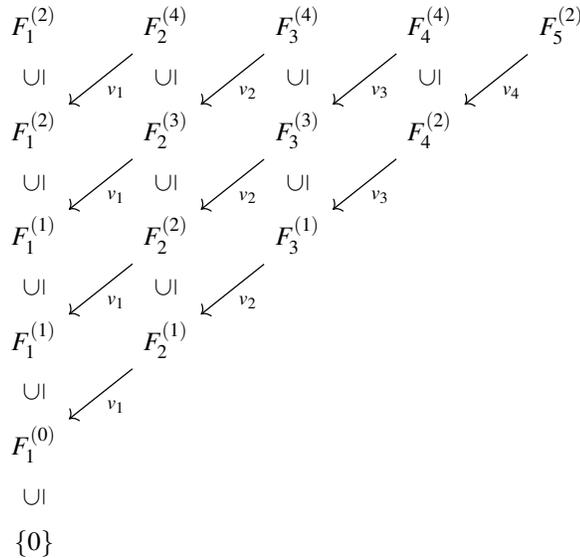
Where the sequences of arrows give a collection of linear maps defined by $v_i(F_{i+1}^{(r_{(i+1)j})}) \subseteq$

$F_i^{(r_{ij})}$. One may in principal reduce the number of subspaces shown in each flag to clean up the presentation of the generalized flag variety $\mathcal{F}$. Since the geometry of this setup doesn't "see" repeated subspaces, we may simplify the previous diagram
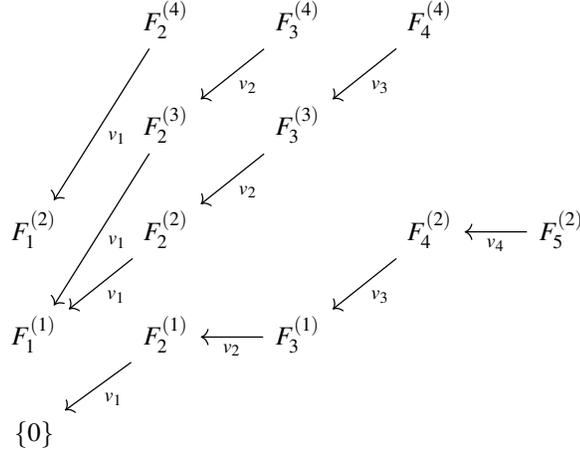
with the implied subspace inclusions. This new collection, together with the linear subspace conditions, will yield a resolution of singularities together with the projection π down to $\overline{C}$. One writes

$$\widetilde{C} = \left\{ (c,F) \in V_\lambda \times \prod_{i=1}^{k} \mathrm{Flag}((r_{i(k-1)},\ldots,r_{i(i-1)})E_i) \;\middle|\; v_i(F_{i+1}^{(r_{(i+1)j})}) \subseteq F_i^{r_{ij}} \right\}.$$

Next, we revisit the principle example, Example 3.14, given in Section 4.2. This example became the foundation of the idea, Example 4.19, which demonstrated the general construction of our resolutions:

Example 4.28 (Resolution of Example 3.14). The Vogan variety V_λ and centralizer H_λ were of the form:

$$V_\lambda = \left\{ \begin{pmatrix} 0 & v_1 & 0 \\ 0 & 0 & v_2 \\ 0 & 0 & 0 \end{pmatrix} \;\middle|\; \begin{matrix} v_1 : E_2 \to E_1 \\ v_2 : E_3 \to E_2 \end{matrix} \right\} \quad \text{and} \quad H_\lambda = \left\{ \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix} \;\middle|\; \begin{matrix} h_1 \in GL(E_1) \\ h_2 \in GL(E_2) \\ h_3 \in GL(E_3) \end{matrix} \right\}.$$

Where E_1, E_2, E_3 are the eigenspaces $E_{q^1}, E_{q^0}, E_{q^{-1}}$ respectively, where we denote by:

$$v_1 = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

Then, singular orbit C_3 was given by the following rank-triangle, to which we will now apply the construction given in Example 4.19:

$$
\begin{array}{ccc}
1 & 3 & 1 \\
\hline
& 1 & 1 \\
& 0 &
\end{array}
\quad \longleftrightarrow \quad
\begin{array}{c}
F_1^{(1)} \qquad F_2^{(3)} \qquad F_3^{(1)} \\
\cup | \quad\swarrow_{v_1} \quad \cup | \quad\swarrow_{v_2} \\
F_1^{(1)} \qquad F_2^{(1)} \\
\cup | \quad\swarrow_{v_1} \\
F_1^{(0)} \\
\cup | \\
\{0\}
\end{array}
\quad \longleftrightarrow \quad
\begin{array}{c}
E_2 \\
\cup \quad\swarrow_{v_1} \\
E_1 \qquad F_2^{(1)} \xleftarrow{v_2} E_3. \\
\cup \quad\swarrow_{v_1} \\
\{0\}
\end{array}
$$

This resolution only required adding in one linear subspace, namely $F_1^{(1)} \in \mathrm{Gr}(1, E_1) = \mathbb{P}^2$, so we may let $F_1^{(1)} = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{P}^2 \right\}$, together with the mapping conditions

- $v_2(E_3) \subseteq F_2^{(1)}$ is given by the following subspace conditions:

$$
\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \begin{bmatrix} a \\ b \\ c \end{bmatrix} \iff
\begin{array}{c}
by_1 - ay_2 = 0 \\
cy_1 - ay_3 = 0 \\
cy_2 - by_3 = 0
\end{array}
$$

- $v_1(F_2^{(1)}) \subseteq \{0\}$ is given by the following condition:

$$
\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0 \iff ax_1 + bx_2 + cx_3 = 0
$$

Together, these four equations resolve the singularity of our variety. Then, we have a refinement of the blowup shown in Section 4.2 for Example 3.14, given by:

$$
\widetilde{C}_3 = \left\{ (v, F) \in V_\lambda \times \mathbb{P}^2 \;\middle|\; \begin{array}{l} v_1(F_2^{(1)}) \subseteq \{0\} \\ v_2(E_3) \subseteq F_2^{(1)} \end{array} \right\}.
$$

Indeed, this variety is smooth, observe

$$v_1(F_2^{(1)}) \subseteq \{0\} \implies f = ax_1 + bx_2 + cx_3$$

$$v_2(E_3) \subseteq F_2^{(1)} \implies g_1 = by_1 - ay_2, \; g_2 = cy_1 - ay_3, \text{ and } g_3 = cy_2 - by_3.$$

Then, we compute the Jacobian as follows

$$
\mathcal{J}(f, g_1, g_2, g_3) = \begin{array}{c|ccccccccc}
 & \partial x_1 & \partial x_2 & \partial x_3 & \partial y_1 & \partial y_2 & \partial y_3 & \partial a & \partial b & \partial c \\
\hline
\partial f & a & b & c & 0 & 0 & 0 & x_1 & x_2 & x_3 \\
\partial g_1 & 0 & 0 & 0 & b & -a & 0 & -y_2 & y_1 & 0 \\
\partial g_2 & 0 & 0 & 0 & c & 0 & -a & -y_3 & 0 & y_1 \\
\partial g_3 & 0 & 0 & 0 & 0 & c & -b & 0 & -y_3 & y_2
\end{array}.
$$

Now restricting this Jacobian to the previously singular orbit, C_0, we find that

$$
\mathcal{J}_{c_0}(f, g_1, g_2, g_3) = \begin{bmatrix}
a & b & c & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & b & -a & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c & 0 & -a & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c & -b & 0 & 0 & 0
\end{bmatrix}.
$$

Which since a, b, c cannot simultaneously be 0, we have that $\text{rank}(\mathcal{J}_{c_0}) = 3$, which is the codimension of $\widetilde{C_3}$ in $V_\lambda \times \mathbb{P}^2$. Since

- π is birational, we have that $\dim(\widetilde{C_3}) = \dim(C_3) = 5$;[40] and

- $\dim(V_\lambda \times \mathbb{P}^2) = \dim(V_\lambda) + \dim(\mathbb{P}^2) = 6 + 2 = 8$.

Then by Proposition 4.3, the variety $\widetilde{C_3}$ is smooth on $\pi^{-1}(C_0)$.

We have the following very natural generalization for when the middle eigenspace (E_2) is an arbitrary size.

---

[40]Since the variety C_3 is defined by having 6 variables satisfying 1 equation.

Example 4.29 $(1, k, 1$-Example$)$. In the previous example, suppose that $\dim(E_2) = k$. Then, given the above setup, we have the following construction for the resolution of singularities:

$$
\begin{array}{c}
\dfrac{\begin{array}{ccc} 1 & k & 1 \\ \hline 1 & 1 \end{array}}{0}
\end{array}
\longleftrightarrow
\begin{array}{c}
F_1^{(1)} \qquad F_2^{(k)} \qquad F_3^{(1)} \\
\cup| \; \swarrow_{v_1} \;\; \cup| \; \swarrow_{v_2} \\
F_1^{(1)} \qquad F_2^{(1)} \\
\cup| \; \swarrow_{v_1} \\
F_1^{(0)} \\
\cup| \\
\{0\}
\end{array}
\longleftrightarrow
\begin{array}{c}
\qquad\qquad E_2 \\
\qquad \swarrow_{v_1} \;\; \cup \\
E_1 \qquad F_2^{(1)} \xleftarrow[v_2]{} E_3. \\
\cup \; \swarrow_{v_1} \\
\{0\}
\end{array}
$$

Again, formally speaking, the only subspace that we've needed to add is $F_1^{(1)} \in \mathrm{Gr}(1, k) = \mathbb{P}^{k-1}$, together with the mapping conditions:

- $v_2(E_3) \subseteq F_2^{(1)}$ is given by the following subspace conditions

$$
\begin{pmatrix} y_1 \\ \vdots \\ y_k \end{pmatrix} \in \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix} \iff a_i y_j = a_j y_j, \; \forall\, 1 \leq i, j \leq k.
$$

- $v_1(F_2^{(1)}) \subseteq \{0\}$ is given by the following condition

$$
\begin{pmatrix} x_1 & \cdots & x_k \end{pmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix} = 0 \iff a_1 x_1 + \cdots + a_k x_k = 0.
$$

Again as a resolution we find

$$
\widetilde{C}_3 = \left\{ (v, F) \in V_\lambda \times \mathbb{P}^{k-1} \;\middle|\; \begin{array}{l} v_1(E_2) \subseteq E_1 \\ v_1(F_2^{(1)}) \subseteq \{0\} \\ v_2(E_3) \subseteq F_2^{(1)} \end{array} \right\}.
$$

This is an example of an infinite family which share "the same" resolution.

4.3.4   Resolutions of Singularities arising in $\hat{G}' = $ SO or Sp

In this section, we describe how the transition from $\hat{G} = \mathrm{GL}_n$ to one of the subgroups
$\hat{G}' = $ SO or Sp affects the construction detailed in Section 4.3.2 for the resolution of singu-
larities of H_λ-orbit closures arising in GL_n. In Section 3.2, Proposition 3.19, we saw that
when passing to $\lambda \in \hat{G}' = $ SO or Sp, there were heavy restrictions imposed on the infinites-
imal parameters λ. Specifically, λ needed to be symmetric, and when we impose this re-
striction, we find that $\lceil k/2 \rceil$ of λ's eigenspaces are identified (with possibly one eigenspace
being identified with itself). Therefore, we must redefine the construction of the resolution
of singularities for $\lambda \in \hat{G} = \mathrm{GL}_n$, as stated in Definition 4.16, in order to create resolutions
for H'_λ-orbits occurring in SO and Sp.

As a result, the proof of Lemma 4.21 will no longer be true, and our construction will
no longer ensure that the $\widetilde{C}$ will be a vector bundle for H'_λ-orbits arising in SO or Sp (the
step of the proof which fails is specifically when defining the local trivialization). The
construction will generally not be enough because, when we move to H'_λ in SO or Sp, we
find that H'_λ does not always act transitively on a pair of flags in E_i, $E_i^* \simeq E_{k-i}$, at the same
time (sometimes it will). For example,

Example 4.30. Consider the following H'_λ-orbits arising in $\hat{G}' = $ Sp, defined by the follow-
ing rank-triangles:

$$
C : \begin{array}{ccccccc} 2 & & 3 & & 3 & & 2 \\ & 1 & & 0 & & 1 & \\ & & 0 & & 0 & & \\ & & & 0 & & & \end{array}
\quad \text{and} \quad
C' : \begin{array}{ccccccc} 2 & & 3 & & 3 & & 2 \\ & 1 & & 1 & & 1 & \\ & & 0 & & 0 & & \\ & & & 0 & & & \end{array} .
$$

While $C < C'$ and both $\overline{C}$ and $\overline{C}'$ are singular on the boundary at C_0, our algorithm would
provide a resolution for C but not for C'. Observe, the resolutions would be given by:

$$
\widetilde{C} = \left\{ (c, F) \in V'_\lambda \times \mathcal{F} \mid v_i(F_{(i+1)j}) \subseteq F_{ij} \right\},
$$
$$
\widetilde{C'} = \left\{ (c', F') \in V'_\lambda \times \mathcal{F}' \mid v_i(F_{(i+1)j}) \subseteq F_{ij} \right\},
$$

where $v_i(F_{(i+1)j}) \subseteq F_{ij}$ denotes the appropriate relations, $\mathcal{F} = \mathrm{Gr}(1, E_1) \times \mathrm{Gr}(1, E_3)$, and $\mathcal{F}' = \mathrm{Gr}(1, E_1) \times \mathrm{Gr}(1, E_2) \times \mathrm{Gr}(1, E_3)$. One can observe that $\widetilde{C}$ will be smooth since H'_λ still acts transitively on $\mathcal{F}$ for this orbit, while H'_λ does not act transitively on $\mathcal{F}'$ for $\widetilde{C}'$. Observe,

$$\mathrm{Gr}(1, E_2) \times \mathrm{Gr}(1, E_3) \cong \mathrm{Gr}(1, E_2) \times \mathrm{Gr}(1, E_2^*) \text{ by the identification } E_i^* \cong E_{k-i}$$

$$\cong \mathrm{Gr}(1, E_2) \times \mathrm{Gr}(2, E_2) \text{ by duality.}$$

There are two H'_λ-orbits in this generalized flag variety, namely

$$(F_2^{(1)}, F_2^{(2)}) \quad \text{where} \quad F_2^{(1)} \subset F_2^{(2)} \text{ or } F_2^{(1)} \not\subset F_2^{(2)}.$$

Thus, our algorithm produces a resolution of singularities for C, but not for C'. We note the following ad hoc construction yields a resolution for $\overline{C'}$

$$\widetilde{C}' = \left\{ (c, F_1^{(1)}, F_2^{(1)}) \in V_\lambda \times \mathrm{Gr}(1, E_1) \times \mathrm{Gr}(1, E_2) \,\left|\, \begin{array}{l} v_1(E_2) \subset F_1^{(1)} \\ v_2(E_3) \subset F_2^{(1)} \\ v_1(F_2^{(1)}) = \{0\} \end{array} \right. \right\}.$$

Thus, one needs to be careful when defining the generalized flag variety $\mathcal{F}'$ in $\hat{G}' = \mathrm{SO}$ or Sp.

Example 4.31. In some very simple examples, such as Example 3.7, the resolutions of singularities we describe still works, since H'_λ will still act transitively on the flag variety we construct for each of the following orbits:

$$C_r : \quad \frac{k \quad k}{r}, \quad \text{for all } 1 \le r \le k.$$

The resolution is given by

$$\widetilde{C}_r = \left\{ (v, F) \in V_\lambda \times \mathrm{Gr}(r, E_1) \,\left|\, v_1(E_1^*) \subseteq F_1^{(r)} \right. \right\}.$$

We note that this resolution is identical to the one in the GL_n, and we remark for the same reasons as in GL_n that it is smooth.

However, a new issue will arise as seen in Section 3.2.3, since each one of these orbits will carry a non-trivial equivariant fundamental group. As an equivariant cover, for the open orbit C_k, we considered the following variety:

$$\widehat{C}_k = \left\{ (v,d) \in V_\lambda \times \mathbb{C} \,\big|\, d^2 = \det(v_1) \right\}.$$

The H_λ-orbit C_k was previously smooth since $\overline{C_k} = \mathbb{A}^{k^2}$, while this cover is certainly not smooth, the defining equation $d^2 - \det(v_1) = 0$ is in general highly singular.

In Section 2.2, we explained that the local Langlands correspondence was able to be reinterpreted as a bijection between the category of representations $\mathrm{Rep}_\lambda(G/F)$ and the category of perverse sheaves $\mathrm{Per}_{H_\lambda}(V_\lambda)$. Working on the geometric side, the simple objects in the category of perverse sheaves were parametrized by pairs (C, ρ), where C is an H_λ-orbit in V_λ and ρ is an irreducible representation of the equivariant fundamental group A_C. In Section 3.1.3, we saw that the equivariant fundamental group for the H_λ-orbits C arising for infinitesimal parameters $\lambda \in \hat{G} = \mathrm{GL}_n$ were always trivial (see Proposition 3.15). While, in Section 3.2.2, we saw that when we pass from GL_n to one of the subgroups SO or Sp, the equivariant groups may be non-trivial.

Thus, we need to perform additional resolutions on the equivariant covers of C, since each cover is associated to some $\mathcal{L}_i$, we need to perform a resolution, one for each $IC(C, \mathcal{L}_i)$. i.e., a distinct resolution for each cover of $\mathcal{L}_i$ will be required for $\lambda \in \hat{G}' = \mathrm{SO}$ or Sp, in order to apply the theorems in Section 2.2.

Only sporadic examples have been worked out for constructing smooth equivariant birational double covers.

Example 4.32. [6, 15.2.3 (c)] Consider $\lambda = \text{diag}(q^{\frac{1}{2}}, q^{\frac{1}{2}}, q^{-\frac{1}{2}}, q^{-\frac{1}{2}}) \in \text{Sp}$. The Vogan variety and centralizer is given by:

$$V_\lambda' = \left\{ \begin{pmatrix} 0 & v_1 \\ 0 & 0 \end{pmatrix} \;\middle|\; X \in \text{Sym}^2(E_{q^{1/2}}) \right\} \quad \text{and} \quad H_\lambda' = \left\{ \begin{pmatrix} h_1 & 0 \\ 0 & h_1^* \end{pmatrix} \;\middle|\; h_1^* = h_1^{-T} \right\}.$$

Then, we may remark that the open orbit, i.e., the orbit corresponding to the rank-triangle:

$$C_2': \quad \begin{array}{ccc} 2 & & 2 \\ & 2 & \end{array}$$

carries a non-trivial equivariant fundamental group, then as a smooth equivariant birational double cover, we may consider the variety:

$$\widehat{C_2'} = \left\{ \left(\begin{pmatrix} x & y \\ y & z \end{pmatrix}, \begin{bmatrix} a \\ b \end{bmatrix} \right) \in V_\lambda' \times \mathbb{P}^1 \;\middle|\; \begin{bmatrix} a & b \end{bmatrix} \begin{pmatrix} x & y \\ y & z \end{pmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0 \right\}.$$

This is again a resolution using flag varieties, exploiting the fact that $E_1 \cong E_2^*$. We note that we can interpret the above condition as:

$$v_1 \left(\begin{bmatrix} a \\ b \end{bmatrix} \right) \subset \begin{bmatrix} a \\ b \end{bmatrix}^\perp.$$

Unfortunately, a complete/coherent construction for the resolutions of singularities arising in the H_λ'-orbit closures in $\hat{G}' = \text{SO}$ or Sp, falls beyond the scope of this thesis. While we provided a detailed exposition towards this end, we were unable to complete the construction or proof for these resolutions.

4.4 Explicit Computation of the Multiplicity Matrix

In this section, we apply the methods described in Section 2.3, to give a demonstration of computing the geometric multiplicity matrix. In particular, we will be explaining the computation of a multiplicity matrix, namely m^λ_{geo} on the geometric side of the local Langlands correspondence for a family of infinitesimal parameters $\lambda \in \hat{G} = \mathrm{GL}_n$, as shown in Section 4.3.3.

Parts of the following example has been facilitated through conversations with the Automorphic Representations group at the University of Calgary.

Example 4.33. Consider Example 4.29, for $\lambda = \mathrm{diag}(q, \underbrace{1, \ldots, 1}_{k-\text{times}}, q^{-1}) \in \hat{G} = \mathrm{GL}_n$.

Then the Vogan variety V_λ and centralizer H_λ are given by:

$$V_\lambda = \left\{ \begin{pmatrix} 0 & v_1 & 0 \\ 0 & 0 & v_2 \\ 0 & 0 & 0 \end{pmatrix} \middle| \begin{matrix} v_1 \in \mathrm{Mat}_{1 \times k} \\ v_2 \in \mathrm{Mat}_{k \times 1} \end{matrix} \right\} \quad \text{and} \quad H_\lambda = \left\{ \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix} \middle| \begin{matrix} h_1, h_3 \neq 0 \\ h_2 \in \mathrm{GL}(k) \end{matrix} \right\}.$$

The orbits are given by the rank-triangles described in Section 4.2 as:
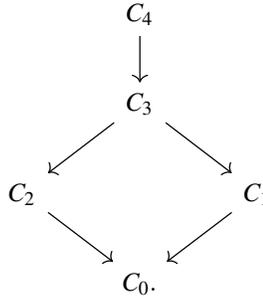
Since the equivariant fundamental groups in GL_n are trivial, the equivariant local system associated to each orbit is simply the trivial representation $\mathcal{L}_{\mathbb{1}_{C_i}} := \mathbb{1}_{C_i}$, and as such, the simple objects in $\mathrm{Per}^{\text{simple}}_{H_\lambda}(V_\lambda)$ belong to

$$\mathrm{Per}^{\text{simple}}_{H_\lambda}(V_\lambda) = \{ IC(C_0, \mathbb{1}_{C_0}), IC(C_1, \mathbb{1}_{C_1}), IC(C_2, \mathbb{1}_{C_2}), IC(C_3, \mathbb{1}_{C_3}), IC(C_4, \mathbb{1}_{C_4}) \}.$$

Now, in order to construct the geometric multiplicity matrix, we consider the restriction of the above simple objects to each orbit; this may be more succinctly understood via Table 4.1.

Table 4.1: Intersection Cohomology Sheaf Computation Table for the Multiplicity Matrix Associated to an Infinitesimal Parameter λ.

m_{geo}^{λ}	$\mid_{C_0}$	$\mid_{C_1}$	$\mid_{C_2}$	$\mid_{C_3}$	$\mid_{C_4}$
$IC(C_0, \mathbb{1}_{C_0})$	$\mathbb{1}_{C_0}[0]$	0	0	0	0
$IC(C_1, \mathbb{1}_{C_1})$	$\mathbb{1}_{C_0}[k]$	$\mathbb{1}_{C_1}[k]$	0	0	0
$IC(C_2, \mathbb{1}_{C_2})$	$\mathbb{1}_{C_0}[k]$	0	$\mathbb{1}_{C_2}[k]$	0	0
$IC(C_3, \mathbb{1}_{C_3})$	$\displaystyle\bigoplus_{i=0}^{k-1} \mathbb{1}_{C_0}[2k-1-2i]$	$\mathbb{1}_{C_1}[2k-1]$	$\mathbb{1}_{C_2}[2k-1]$	$\mathbb{1}_{C_3}[2k-1]$	0
$IC(C_4, \mathbb{1}_{C_4})$	$\mathbb{1}_{C_0}[2k]$	$\mathbb{1}_{C_1}[2k]$	$\mathbb{1}_{C_2}[2k]$	$\mathbb{1}_{C_3}[2k]$	$\mathbb{1}_{C_4}[2k].$

Many of the entries of Table 4.1 are given by a simple application of Proposition 2.14.

1. By part (1), we have that $IC(C, \mathcal{L}_C)\mid_C = \mathcal{L}_C[\dim C]$. Thus,

$$IC(C_i, \mathbb{1}_{C_i})\mid_{C_i} = \mathbb{1}_{C_i}[\dim C_i], \text{ for each } i;$$

2. By part (2), many of the entries vanish, since $IC(C, \mathcal{L})\mid_D = 0$ whenever $D \not\subseteq \overline{C}$; lastly

3. By part (3), if $\overline{C_i}$ is smooth (which it will be for all $i \neq 3$), we have that $IC(C_i, \mathbb{1}_{C_i})\mid_{C_j} = \mathbb{1}_{C_j}[\dim C_i]$ if $C_j \subseteq \overline{C_i}$ and 0 otherwise.

Since $\overline{C_3}$ is not smooth, we must compute a resolution of singularities in order apply the theorems of Section 2.3, which will allow us to solve for $IC(C_3, \mathbb{1}_{C_3})\mid_{C_i}$, for $i = 0, 1, 2$ in Table 4.1.

Thus, by Example 4.29, we compute the following resolution of singularities:

$$\widetilde{C_3} = \left\{ (v, F) \in V_\lambda \times \text{Gr}(1, k) \; \middle| \; \begin{array}{l} v_1(E_3) \subseteq F_2^{(1)} \\ v_2(F_2^{(1)}) = \{0\} \end{array} \right\}.$$

In this resolution, we note that $\text{Gr}(1, k) \equiv \mathbb{P}^{k-1}$, and that $\pi^{-1}(\{c_0\}) = \mathbb{P}^{k-1}$.

Recall. Definition 2.15, which allows us to compute the size of our resolution. A resolution $\pi : \widetilde{C} \to C$ is said to be small provided that:

$$\dim(C') + 2\dim(\pi^{-1}(c')) < \dim(\widetilde{C}), \text{ for all } C' < C.$$

Since π is birational, we have that $\dim(\widetilde{C_3}) = \dim(\overline{C_3}) = 2k - 1$. Here, we also have that $C_0, C_1, C_2, C_3 \leq C_3$. Thus we compute:

	C_3	C_2	C_1	C_0
$\dim(C_i)$	$2k - 1$	k	k	0
$\dim(\pi^{-1}(c_i))$	0	0	0	$k - 1.$

Since $\dim(C') + 2\dim(\pi^{-1}(c')) < \dim(\widetilde{C})$, for all $C' < C$, we have that this resolution is small, hence

$$\pi_* \mathbb{1}_{\widetilde{C_3}}[\dim(C_3)] = IC(C_3, \mathbb{1}_{C_3}).$$

To compute $\pi_* \mathbb{1}_{\widetilde{C_3}}[\dim(C_3)]|_{c_i}$ we shall use proper base change, given by Equation 2.5. That is we have

$$\pi_* \mathbb{1}_{\widetilde{C_3}}[\dim(C_3)]|_{c_i} = \bigoplus_j H^j(\pi^{-1}(c_i))[\dim(C_3) - j].$$

Thus, to evaluate $IC(C_3, \mathbb{1}_{C_3})|_{c_i}$, we compute the cohomology of the various fibres.

Here, since $\pi^{-1}(c_i) = \{\text{pt}\}$, for $i = 1, 2, 3$, we have that

$$H^j(\{\text{pt}\}) = \begin{cases} \mathbb{C} & j = 0 \\ 0 & \text{otherwise.} \end{cases}$$

i.e., one has that $IC(C_3, \mathbb{1}_{C_3})|_{C_i} = \mathbb{1}_{C_i}[2k - 1]$, for $i = 1, 2, 3$.

Next, in the case of c_0, the cohomology is

$$H^j(\pi^{-1}(c_0)) = H^j(\mathbb{P}^{k-1}) = \begin{cases} \mathbb{C} & 0 \le j \le 2(k-1), \ j \text{ even} \\ \\ 0 & \text{otherwise.} \end{cases}$$

i.e., one has that $IC(C_3, \mathbb{1}_{C_3})|_{C_0} = \bigoplus\limits_{j=0}^{k-1} \mathbb{1}_{C_0}[2k-1-2j]$.

Now, for our final step, we write out the actual geometric multiplicity matrix, m_{geo}, taking careful account of the shifts from Table 4.1 and the shift from Equation (2.3):

$$m_{\text{geo}} = \begin{pmatrix} (-1)^{(0)+(0)} & 0 & 0 & 0 & 0 \\ (-1)^{(k)+(0)} & (-1)^{(k)+(k)} & 0 & 0 & 0 \\ (-1)^{(k)+(0)} & 0 & (-1)^{(k)+(k)} & 0 & 0 \\ \sum\limits_{i=0}^{k-1}(-1)^{(2k-1-2i)+(0)} & (-1)^{(2k-1)+(k)} & (-1)^{(2k-1)+(k)} & (-1)^{(2k-1)+(2k-1)} & 0 \\ (-1)^{(2k)+(0)} & (-1)^{(2k)+(k)} & (-1)^{(2k)+(k)} & (-1)^{(2k+(2k-1))} & (-1)^{(2k)+(2k)} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ (-1)^{(k)} & 1 & 0 & 0 & 0 \\ (-1)^{(k)} & 0 & 1 & 0 & 0 \\ -k & (-1)^{k-1} & (-1)^{k-1} & 1 & 0 \\ 1 & (-1)^k & (-1)^k & -1 & 1 \end{pmatrix}.$$

We can see that when we re-normalize by the factor given in Equation (2.4), the geometric multiplicity matrix m^*_{geo} will consist entirely of positive entries. One could then compare m^*_{geo} with the representation multiplicity matrix m_{rep} on the representation theoretic side, towards the confirmation of the p-adic Kazhdan-Lusztig Hypothesis detailed in Section 2.2, which states that $m^T_{\text{rep}} = m^*_{\text{geo}}$. Given that we are working exclusively on the geometric side of the correspondence, we will be stopping here.

The University of Calgary's Automorphic Representations Group, led by Clifton Cunningham, is currently working on various aspects of the p-adic Kazhdan-Lusztig Hypothesis on both the representation theoretic side and the geometric side using our algorithm (or rather, the algorithm that we have proven works for any resolution of an orbit arising in $\hat{G} = \mathrm{GL}_n$). In this thesis, we have made explicit many of the aspects that they had been

employing implicitly in their construction. We anticipate that by providing them with the framework laid out in this thesis, we will have provided foothold towards breaking ground in examples arising in SO and Sp, by making explicit the structure of Langlands parameters arising in these groups.

Chapter 5

Future Work

Much of the literature surrounding the Langlands program, and particularly the *p*-adic local Langlands program, has been written by experts for experts. In Chapter 2, our thesis has aimed to provide an exposition on the topic at an approachable level for any mathematics enthusiast interested in getting to know the topic, with plenty of references for anyone wishing to fill in the gaps. The local Langlands conjecture is currently a very active area of research, and we will have provided foothold towards breaking ground in understanding deeper categorical aspects of the local Langlands correspondence.

In Section 3.1, we gave exposition detailing the geometry of infinitesimal parameters arising in GL_n, and in Section 3.2, we extended the construction to the subgroups SO or Sp. This involved an in-depth study on the decomposition of the types of Vogan varieties V_λ and centralizer groups H_λ that could arise in either setting. We classified the types of H_λ-orbits C in V_λ that could arise, using Propositions 3.8, which classified the orbits; 3.11, which gave the basic rank conditions on the orbits arising in GL_n; and lastly 3.22, which described the symmetric rank conditions, these were additional restrictions on the orbits that could arise for $\lambda \in \hat{G}' = \mathrm{SO}$ and Sp. In studying these varieties, various notable aspects of the geometry arose when passing to $\hat{G}' = \mathrm{SO}$ or Sp. These restrictions not previously seen in GL_n, but which appear in SO and Sp have imposed further restrictions on the decomposition of these new Vogan varieties V_λ' and centralizer groups H_λ', stemming from the identification of λ's eigenspaces.

When passing from $\hat{G} = \mathrm{GL}_n$ to one of the subgroups $\hat{G}' = \mathrm{SO}$ or Sp, one would like to

be able to construct equivariant resolutions of singularities for the H'_λ-orbit closures which arise in this setting. As seen in Section 4.3.4, our construction described in Section 4.3.2 fails, and thus we need to redefine this construction and tailor it to this setting; the details of how this is done remain future work. One is likely to be able to use the ranks given in the rank-triangle for these orbits to define an alternative generalized flag variety $\mathcal{F}'$, which would be used to construct an equivariant resolution of singularities, together with reasonable mapping conditions that respect the identification between eigenspaces. It is unclear at this stage what the size of these resolutions would be.

Additionally, when passing to $\lambda \in \hat{G}' = \mathrm{SO}$ or Sp, we saw that the equivariant fundamental groups were non-trivial, while in $\hat{G} = \mathrm{GL}_n$ the equivariant fundamental groups were always trivial, by Proposition 3.15. Since the equivariant fundamental groups for the H'_λ-orbits were in general non-trivial in two of the four cases detailed in Proposition 3.21. In Proposition 3.30, we saw that the non-trivial equivariant fundamental groups were always 2-groups; as such, we must construct equivariant double covers, one for each non-trivial local system $\mathcal{L}_i$ on the orbit. One must then construct equivariant resolutions for these covers since they will introduce singularities on orbits that were previously smooth. In order to apply the theorems discussed in Section 2.3, one requires smooth equivariant birational double covers $(\widetilde{C}, \pi)$ for each H'_λ-orbit C. We provide the general construction of the equivariant double covers on the open orbit and discuss how, in general, it is done for the other orbits. For future work, one hopes to give the full construction of the smooth equivariant birational double covers $(\widehat{C}, \pi)$, for each orbit.

This work would function as a useful mechanism with which to explore examples arising in the p-adic Kazhdan-Lusztig Conjecture. One would hopefully be able to study various geometric aspects of the construction, such as size conditions (small or semi-small) on families of resolutions arising in GL_n, for non-symmetric infinitesimal parameters, or symmetric infinitesimal parameters (infinitesimal parameters $\lambda \in \hat{G} = \mathrm{GL}_n$ which can be passed to the subgroups SO or Sp).

In summary, moving forward, one would like to

1. Give a generalized method for resolving singularities for the H'_λ-orbit closures arising in $\hat{G}' = \mathrm{SO}$ and Sp;

2. Give the construction of the equivariant double covers for the orbits arising in $\hat{G}' = \mathrm{SO}$ and Sp;

3. Subsequently, provide a method of resolving the new singularities arising from the equivariant double covers for the H'_λ-orbits arising in $\hat{G}' = \mathrm{SO}$ and Sp, and hence give the general construction for smooth equivariant birational double covers for these orbits; and

4. Lastly, study the various geometric properties of these resolutions, such as providing the sufficient and necessary conditions for when these resolutions are small or semi-small. Additionally, one may look for refinements of the constructions, that could reduce the size of the resolutions.

Much work remains to be done in this field, which has provided a plethora of seemingly never-ending problems towards the union of many seemingly distinct areas of mathematics.

Bibliography

[1] Silvana Abeasis, Alberto Del Fra, and Hanspeter Kraft. The geometry of representations ofa m. *Mathematische Annalen*, 256(3):401–418, 1981.

[2] Emil Artin and John Torrence Tate. *Class field theory*, volume 366. American Mathematical Soc., 1968.

[3] Alexander Beilinson, Jacob Bernstein, and Pierre Deligne. Analyse et topologie sur les espaces singuliers (i). *Astérisque*, 1(100), 1982.

[4] Michel Brion. Representations of quivers. Lecture Notes, https://cel.hal.science/cel-00440026, 2008.

[5] Clifton Cunningham, Andrew Fiori, and Nicole Kitt. Appearance of the kashiwara-saito singularity in the representation theory of p-adic GL_{16}. *arXiv*, 2021. https://arxiv.org/abs/2103.04538.

[6] Clifton Cunningham, Andrew Fiori, Ahmed Moussaoui, James Mracek, and Bin Xu. Arthur packets for p-adic groups by way of microlocal vanishing cycles of perverse sheaves, with examples. *American Mathematical Society*, Mar 2022. https://www.ams.org/books/memo/1353/.

[7] Clifton Cunningham, Andrew Fiori, and Qing Zhang. Arthur packets for G_2 and perverse sheaves on cubics. *Advances in Mathematics*, 395:108074, 2022. https://www.sciencedirect.com/science/article/pii/S0001870821005132.

[8] Clifton Cunningham and Mishty Ray. Proof of Vogan's conjecture on Arthur packets: simple parameters of p-adic general linear groups. *arXiv*, 2022. https://arxiv.org/abs/2206.01027.

[9] Mark de Cataldo and Luca Migliorini. The decomposition theorem, perverse sheaves and the topology of algebraic maps. *Bulletin of the American Mathematical Society*, 46(4):535–633, 2009.

[10] Harm Derksen and Jerzy Weyman. *An introduction to quiver representations*, volume 184. American Mathematical Soc., 2017.

[11] William Fulton and Joe Harris. *Representation theory: a first course*, volume 129. Springer Science & Business Media, 2013.

[12] Robin Hartshorne. *Algebraic geometry*, volume 52. Springer Science & Business Media, 2013.

[13] Heisuke Hironaka. Resolution of singularities of an algebraic variety over a field of characteristic zero: II. *Annals of Mathematics*, pages 205–326, 1964.

[14] Connor David Riddlesden. Combinatorial approach to abv-packets for gln. Master's thesis, University of Lethbridge. Faculty of Arts and Science, Lethbridge, AB, aug 2022. https://opus.uleth.ca/handle/10133/6377.

[15] See-Hak Seong. Introduction to resolution of singularities : Blow up. *See-Hak Seong - University of Illinois Chicago*, Apr 2017. Lecture Notes, http://homepages.math.uic.edu/ seehak/.

[16] Igor Rostislavovich Shafarevich. *Basic algebraic geometry. 2: Schemes and complex manifolds*. Springer, 1996.

[17] Igor Rostislavovich Shafarevich and Miles Reid. *Basic algebraic geometry*, volume 2. Springer, 1994.

[18] Norman Earl Steenrod. Homology with local coefficients. *Annals of Mathematics*, pages 610–627, 1943.

[19] David Alexander Vogan. The local Langlands conjecture. *Representation theory of groups and algebras Contemporary Mathematics*, page 305–379, 1993.

[20] Geoffrey Mark Vooys. *Equivariant Functors and Sheaves*. PhD thesis, University of Calgary, 2021. https://prism.ucalgary.ca/handle/1880/113726.

[21] Jerzy Weyman. *Cohomology of Vector Bundles and Syzygies*. Cambridge Tracts in Mathematics. Cambridge University Press, 2003.