EULER'S FUNCTION ON PRODUCTS OF PRIMES IN PROGRESSIONS

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Dedication

This dedication space belongs to Herbie “El Jefe” Campos and Breanna “Thirsty Cali” Jackson. You owe me.

Abstract

In 1962, Rosser and Schoenfeld [21] asked whether there were infinitely many $n \in \mathbb{N}$ for which $n/\varphi(n) > e^{\gamma} \log\log n$. This question was answered in the affirmative in 1983 by Jean-Louis Nicolas [17], who showed that there are infinitely many such n both in the case that the Riemann Hypothesis is true, and in the case that the Riemann Hypothesis is false. Landau's theorem naturally generalizes to the scenario where we restrict our attention to integers whose prime divisors all fall in a fixed arithmetic progression. In this thesis, I will discuss the methods of Nicolas as they relate to the classical result, then turn my attention to answering the relevant analogue of Rosser and Schoenfeld's question for this restricted set of integers. It will be shown that, for certain arithmetic progressions, Nicolas' answer generalizes to a setting where the Generalized Riemann Hypothesis on a particular set of L-functions is concerned.

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Finally, I'd like to thank the Campus Coffee Company, for fueling my addictions and, occasionally, my work.

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$\mathbf{1}$	The trivial character. For any a, $\mathbf{1}(a) = 1$.
C	The Euler-Mascheroni constant.
$\chi(a)$	A Dirichlet character.
$C(q,a)$	The constant term in Mertens' theorem for primes in arithmetic progressions, Theorem 3.5.
$A \overset{\text{def}}{=} B$	means A is *defined to be* B
e	Euler's number.
$\mathcal{F}(\chi)$	$\sum_{\rho \in \mathcal{Z}(\chi')} \frac{1}{\rho(1-\rho)}$.
$\mathcal{F}_q$	$\displaystyle\sum_{\chi} \sum_{\substack{\rho \in \mathcal{Z}(\chi') \\ \Re(\rho)=1/2}} \frac{1}{\rho(1-\rho)}$.
$F_s(x)$	$\int_x^\infty t^s g(t) \mathrm{d}t$.
$f(x;q,a)$	$\displaystyle\frac{(\log(\varphi(q)\theta(x;q,a)))^{\frac{1}{\varphi(q)}}}{C(q,a)} \cdot \prod_{p \leq x} \left(1 - \frac{1}{p}\right)$.
$g(x)$	$\displaystyle\frac{1+\log x}{x^2 \log^2 x}$.
$\Im(s)$	The imaginary part of a complex number s.
$\mathrm{Ind}_q(a)$	The least natural number other than 1 for which that a is an $\mathrm{Ind}_q(a)$-th power modulo q.
$J(x;q,a)$	$\displaystyle\int_x^\infty R(t;q,a) g(t) \mathrm{d}t$.

$K(x;q,a)$	$\int_x^\infty S(t;q,a)g(t)\mathrm{d}t$.
$L(s,\chi)$	The Dirichlet L-function associated with the character χ.
$\mathscr{L}(s;q,a)$	$\sum_\chi \overline{\chi}(a)\frac{L'}{L}(s,\chi)$.
$m_\rho(\chi)$	The multiplicity of ρ as a zero of $L(s,\chi)$.
N_k	The k-th primorial; the product of the first k primes.
$\overline{N}_k$	The product of the first k primes in a fixed arithmetic progression.
$\mathcal{N}_{q,a}(k)$	The Nicolas inequality for the progression $a \pmod q$ at k (Definition 3.9).
$f_1(x) = O(f_2(x))$	means $\lvert f_1(x)\rvert \le M f_2(x)$ for some absolute constant M.
$f_1(x) = o(f_2(x))$	means $\lim_{x\to\infty}\frac{f_1(x)}{f_2(x)} = 0$.
$f_1(x) \sim f_2(x)$	means $\lim_{x\to\infty}\frac{f_1(x)}{f_2(x)} = 1$.
$f_1(x) = \Omega_+(f_2(x))$	means there exists a positive constant c and an increasing real sequence $\{x_i\}$ tending to infinity along which $f_1(x_i) > cf_2(x_i)$.
$f_1(x) = \Omega_-(f_2(x))$	means there exists a positive constant c and an increasing real sequence $\{x_i\}$ tending to infinity along which $f_1(x_i) < -cf_2(x_i)$.
$f_1(x) = \Omega_\pm(f_2(x))$	means $f_1(x) = \Omega_+(f_2(x))$ and $f_1(x) = \Omega_-(f_2(x))$.
$\varphi(n)$	Euler's totient function: $n\prod_{p\mid n}\left(1-\frac{1}{p}\right)$.
p_i	The i-th prime.
$\overline{p}_i$	The i-th prime in a fixed arithmetic progression.
$\psi(x;q,a)$	The second Chebyshev function: $\sum_{\substack{p^k\le x\\ p^k\equiv a \pmod q}} \log p$.

$\Re(s)$	The real part of a complex number s.
$\mathcal{R}_{q,a}$	$\#\{b \in \mathbb{Z}_q^\times \mid b^{\mathrm{Ind}_q(a)} \equiv a \pmod q\}$.
$R(x;q,a)$	$\psi(x;q,a) - \frac{x}{\varphi(q)}$.
$S_{q,a}$	$\{n \in \mathbb{N}\ ;\ p \mid n \implies p \equiv a \pmod q\}$.
$S(x;q,a)$	$\theta(x;q,a) - \frac{x}{\varphi(q)}$.
Θ	The supremum of the real parts of the singularities of $\mathscr{L}(s;q,a)$ in the strip $0 < \Re(s) < 1$.
$\theta(x;q,a)$	The first Chebyshev function: $\sum_{\substack{p\leq x \\ p\equiv a \pmod q}} \log p$.
$\mathcal{Z}(\chi)$	$\{\rho = \beta + i\gamma \in \mathbb{C}\ ;\ L(\rho,\chi) = 0,\ \beta \geq 0 \text{ and } \rho \neq 0\}$.
$\zeta(s)$	The Riemann zeta function.

Chapter 1

Introduction

Number theory is useful,

since one can graduate with it.

– E. Landau

1.1 Analytic Number Theory

This thesis is an expression of the philosophy that analytic number theory is a game of translation. The constituent languages of this game are *arithmetic* (spoken with primes and integers) and *analysis* (spoken with complex-valued functions). Historically, demonstrations of this philosophy appear at least as early as Euler's 1737 proof of the infinitude of primes which, unlike the strictly arithmetic proof of Euclid in his *Elements*, established this fact using the tools of analysis and infinite series. The maturation of number theory introduced more challenging questions and more powerful methods in the vein of Euler's proof. At the forefront of this development were investigations into the density of primes. One such example is Mertens' theorem.

Theorem (Mertens [13]). *Let $x \geq 2$, then*

$$\prod_{p\leq x}\left(1-\frac{1}{p}\right) \sim \frac{e^{-C}}{\log x}$$

as $x \to \infty$, where $e = 2.718\ldots$ is Euler's number and $C = 0.5772\ldots$ is Euler's constant.

The proofs of such theorems use the methods of Euler and other techniques of real analysis in order to study the behavior of functions over primes. Later, complex analysis was ushered into the fray, spurred on by the monumental work of Riemann, *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse*[1]. Key among the tools included in Riemann's paper was the Riemann zeta function $\zeta(s)$, a meromorphic function whose behavior was closely tied to the behavior of primes. The value of complex analysis in the study of the density of primes comes into full display in the independent proofs by Hadamard and de la Vallée-Poussin of the prime number theorem, who appeal to the behavior of $\zeta(s)$ to establish the asymptotic size of the prime counting function.

Theorem ([3, Theorem 6]). *Let $\pi(x)$ be the number of primes less than x. Then,*

$$\pi(x) \sim \frac{x}{\log x},$$

as $x \to \infty$.

Riemann's zeta function is a wellspring of utility in analytic number theory and it is not surprising that one of the most important unsolved problems in 21st century mathematics, the Riemann hypothesis (see Conjecture 2.8), regards the behavior of this function. Naturally, just as the behavior of analytic objects can inform our understanding of arithmetic objects, so can the behavior of arithmetic objects better inform our understanding of analytic objects. A relevant example of this mindset is the 1983 work of Jean-Louis Nicolas [17], who provides an equivalence between the Riemann hypothesis and a statement about the behavior of Euler's totient function

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$$

which counts the natural numbers less than and coprime to n. Specifically, Nicolas proved the following.

[1] *On the Number of Primes Less Than a Given Magnitude*

Theorem ([17, Theorem 2]). *(i) If the Riemann hypothesis is true, then for all primorials $N_k = \prod_{i=1}^{k} p_i$, we have*

$$\frac{N_k}{\varphi(N_k) \log\log N_k} > e^C$$

where p_i is the i-th prime.

(ii) If the Riemann hypothesis is false, then there are infinitely many primorials for which the above inequality holds, and also infinitely many primorials for which the above inequality does not hold.

This theorem is discussed in detail in Chapter 2. Here, observe that e^C is the reciprocal of the constant arising in Mertens' theorem. Nicolas' work, as with other work providing equivalence between the properties of the Riemann zeta function and various arithmetic objects, perfectly captures the connection between the realms of the analytic and the arithmetic.

1.2 Results

In this thesis, the aim will be to generalize the work of Nicolas, replacing the primes with primes in arithmetic progressions. In 1837 Dirichlet showed that there are infinitely many primes in appropriately chosen arithmetic progressions, mirroring the 1737 result of Euler. Subsequent work in this direction unearthed analogues for the results of Chebyshev and Mertens, as well as the prime number theorem, building upon the analytic methods surrounding Riemann's zeta function through the introduction of the analogous Dirichlet L-functions. Unsurprisingly, analogues of the Riemann hypothesis exist in this context as well, for which it is similarly valuable to examine the behavior of primes in arithmetic progressions in order to illuminate the behavior of Dirichlet L-functions.

The primary achievement of this thesis is the procurement of arithmetic-analytic equivalences analogous to those obtained by Nicolas where $\zeta(s)$ is replaced with a set of Dirichlet L-functions and the primorials are replaced with products over the first k primes in an arithmetic progression $qk+a$, where q and a are fixed and coprime. However, as is often the case, the act of generalization introduces new features which must be considered. These features are discussed in detail in Chapter 3. In particular, we require the introduction of a conjecture which is effectively a weakening of the generalized Riemann hypothesis, and in fact is equivalent to the generalized Riemann hypothesis (on a set of L-functions) when $a=1$. Precisely, denoting for a Dirichlet character χ its associated L-function $L(s,\chi)$, the following conjecture will often be considered.

Conjecture (Conjecture 3.17)**.** Let q and a be fixed coprime integers. If there exists a nontrivial zero ρ of $L(s,\chi)$ for some χ modulo q for which $\Re(\rho)$ is neither 0 nor $\frac{1}{2}$, then

$$\sum_{\chi \pmod q} \overline{\chi}(a) m_\rho(\chi) = 0, \tag{1.1}$$

where $m_\rho(\chi)$ is the multiplicity of the zero ρ of $L(s,\chi)$.

More information regarding this conjecture can be found in Section 3.3.

We also must observe that, as we shift to primes in arithmetic progressions, the analogue of Mertens' theorem (Theorem 3.5) will have a different constant than e^{-C}. For fixed, coprime q and a, we can denote this constant by $C(q,a)$. Finally, we must include a condition regarding the zeroes of L-functions along the real line segment $(0,1)$, as captured by the function

$$\mathscr{L}(s;q,a) = \sum_{\chi \pmod q} \overline{\chi}(a) \frac{L'}{L}(s,\chi).$$

In Nicolas' case, such a condition is satisfied for $\zeta(s)$, so no direct reference to it was required. In practice, this condition can be verified by computation. These adjustments provide the main result of Chapter 5.

Theorem (Theorem 5.7)**.** *Let $q \le 400,000$ and a be fixed coprime natural numbers. Write*

$$\overline{N}_k = \prod_{i=1}^{k} \overline{p}_i,$$

where $\overline{p}_i$ is the i-th prime in the arithmetic progression generated by q and a.

If Conjecture 3.17 is false, then there are infinitely many $k \in \mathbb{N}$ for which

$$\frac{\overline{N}_k}{\varphi(\overline{N}_k)(\log(\varphi(q)\log\overline{N}_k))^{\frac{1}{\varphi(q)}}} > \frac{1}{C(q,a)}, \tag{1.2}$$

and also infinitely many $k \in \mathbb{N}$ for which (1.2) *does not hold.*

This theorem corresponds to the second half of Nicolas' theorem [17, Theorem 2]. In Chapter 6, an analogue for the first half is established. With the additional notation that χ' denotes a primitive Dirichlet character which induces the Dirichlet character χ and

$$\mathcal{Z}(\chi) = \{\rho = \beta + i\gamma \in \mathbb{C}\,;\, L(\rho,\chi) = 0,\ \beta \geq 0 \text{ and } \rho \neq 0\},$$

Chapter 6 establishes the following assertion.

Theorem (Theorem 6.4)**.** *Fix q and choose a for which the congruence $x^2 \equiv a \pmod{q}$ has a solution. Suppose the following are true.*

(i) Conjecture 3.17 holds.

(ii)

$$\frac{\mathcal{F}_q}{\sum\limits_{b^2 \equiv a \pmod{q}} 1} < \frac{3}{2},$$

where

$$\mathcal{F}_q = \sum_{\chi} \sum_{\substack{\rho \in \mathcal{Z}(\chi') \\ \Re(\rho) = \frac{1}{2}}} \frac{1}{\rho(1-\rho)}.$$

Then there are at most finitely $k \in \mathbb{N}$ for which

$$\frac{\overline{N}_k}{\varphi(\overline{N}_k)(\log(\varphi(q)\log\overline{N}_k))^{\frac{1}{\varphi(q)}}} < \frac{1}{C(q,a)}$$

is satisfied.

We may combine both of the preceding theorems to establish an equivalence.

Theorem (Theorem 6.5)**.** *Fix q and choose a for which the congruence $x^2 \equiv a \pmod{q}$ has a solution. If*

$$\frac{\mathcal{F}_q}{\sum_{b^2 \equiv a \pmod{q}} 1} < \frac{3}{2}$$

and $\mathscr{L}(s;q,a)$ has no singularities on the segment $(0,1)$, then the following are equivalent

(i) Conjecture 3.17 holds.

(ii) There are at most finitely $k \in \mathbb{N}$ for which

$$\frac{\overline{N}_k}{\varphi(\overline{N}_k)(\log(\varphi(q)\log \overline{N}_k))^{\frac{1}{\varphi(q)}}} \leq \frac{1}{C(q,a)}.$$

Finally, the conditions of Theorem 6.5 are verifiable for specific choices of q and a under the assumption of the Generalized Riemann Hypothesis on the set of L-functions corresponding to Dirichlet characters modulo q (GRH_q). It becomes a matter of computation to determine the following concrete statements.

Theorem. *For $q \in \{2,3,4,5,6,8,10,12\}$, GRH_q is true if and only if there are at most finitely many primorials n in $S_{q,1}$ for which*

$$\frac{n}{\varphi(n)(\log \varphi(q) \log n)^{\frac{1}{\varphi(q)}}} \leq \frac{1}{C(q,1)}.$$

Theorem. *Let $(q,a) \in \{(3,1),(4,1),(5,1),(5,4),(6,1),(8,1),(10,1),(10,9),(12,1)\}$. If GRH_q is true, then for all but finitely many primorials n in $S_{q,a}$, we have*

$$\frac{n}{\varphi(n)(\log \varphi(q) \log n)^{\frac{1}{\varphi(q)}}} > \frac{1}{C(q,a)}.$$

If Conjecture 3.17 is false, then there are infinitely many primorials in $S_{q,a}$ for which the above inequality holds, and infinitely many primorials in $S_{q,a}$ for which the above inequality does not hold.

Chapter 2

Background

...nothing at all takes place

in the universe in which some rule

of maximum or minimum does not appear...

– L. Euler

2.1 Euler's Function

The arithmetic object that our discussion will be framed around is Euler's totient function, this thesis' namesake function, which is defined as follows.

Definition 2.1. Let $n \in \mathbb{N}$. *Euler's totient function* is the multiplicative function

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right) \tag{2.1}$$

which counts the natural numbers less than n that are coprime to n.

Figure 2.1 gives a visualization of $\varphi(n)$ for $n \leq 235$. This and all subsequent point plots were generated using Maple™ [2] [1].

Many properties of Euler's function are natural consequences of this definition. Since no natural number other than 1 is coprime to itself, we know that, for $n > 1$, $\varphi(n) \leq n - 1$. Furthermore, if p is a prime, then by definition $\varphi(p) = p - 1$, so this upper bound is achieved. Consequently, the following proposition holds.

[2] Maple is a trademark of Waterloo Maple, inc.

Proposition 2.2 ([3, Theorem 326])**.** $\limsup_{n\to\infty} \frac{\varphi(n)}{n} = 1$.

The proof follows immediately upon considering the sequence of primes.

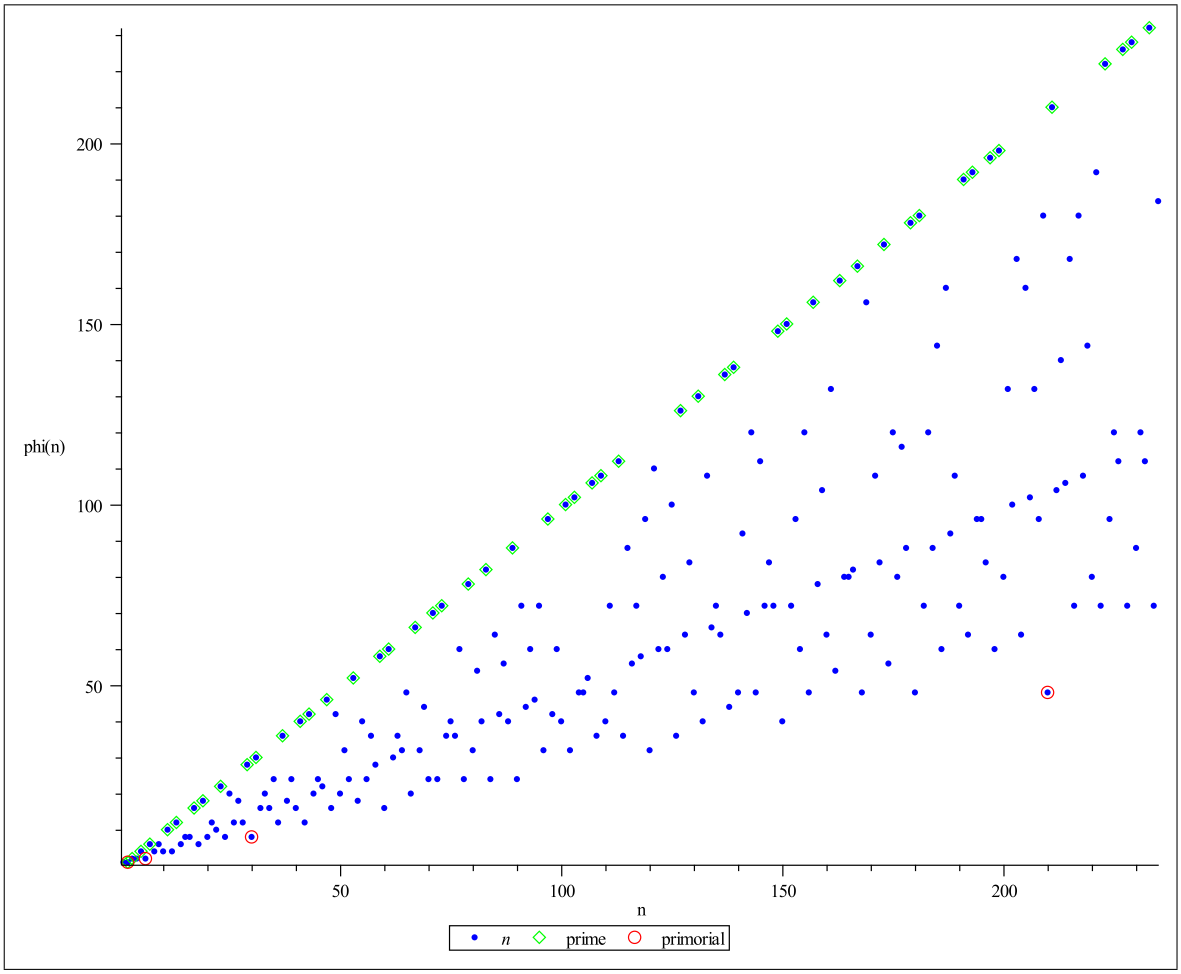

Figure 2.1: Special Values of Euler's Function. Here we have highlighted the maximal behavior along the sequence of primes, as well as the minimal behavior along the sequence $2, 6, 30, 210 \ldots$, a.k.a. the *primorials*.

In the case of minimal values of Euler's function (in the sense that they are small relative to n), the theorem that best captures the behavior of Euler's function in is the 1909 result of Landau.

Theorem 2.3 ([3, Theorem 328])**.** *Let e be Euler's number and C be the Euler-Mascheroni constant. Then,*

$$\limsup_{n\to\infty} \frac{n}{\varphi(n) \log\log n} = e^C.$$

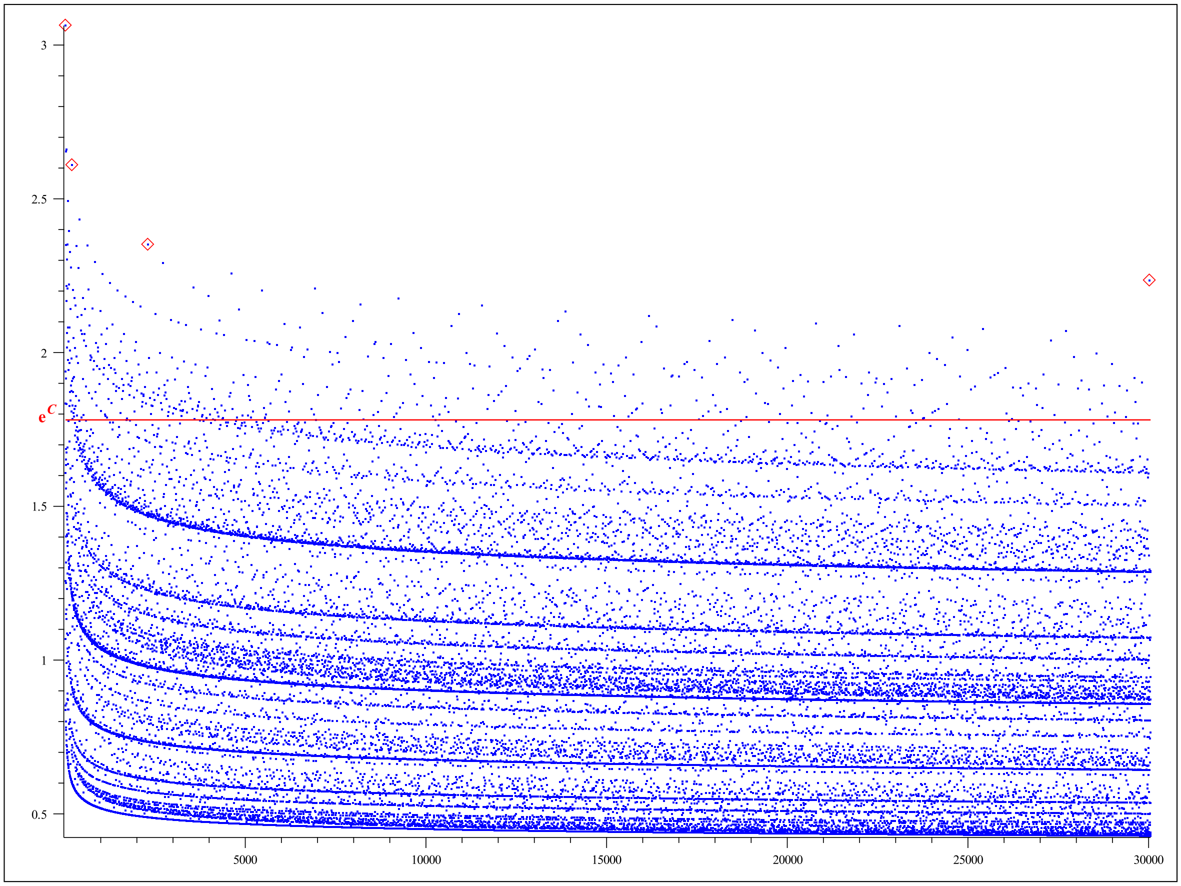

Figure 2.2: Visualization of Theorem 2.3. Points represent $(n, \frac{n}{\varphi(n)\log\log n})$. Points are highlighted where n is a primorial.

That is, for any fixed $\varepsilon > 0$, there are at most finitely many n such that $\frac{n}{\varphi(n)\log\log n} > e^C + \varepsilon$.

This behavior is discussed in a more explicit manner in the celebrated paper of Rosser and Schoenfeld [21], where they prove the following explicit bound.

Theorem 2.4 ([21, Theorem 15])**.** *For $n > 2$,*

$$\frac{n}{\varphi(n)\log\log n} \le e^C + \frac{2.50637}{(\log\log n)^2}.$$

In the same paper, Rosser and Schoenfeld remark that they do not know if there are an infinite number of natural numbers satisfying

$$\frac{n}{\varphi(n)\log\log n} > e^C. \tag{2.2}$$

2.2 Nicolas' Theorem

In his 1983 paper "*Petites valeurs de la fonction d'Euler*"[3] [17], Jean-Louis Nicolas attributes the preceding remark to Rosser and Schoenfeld in the form of the following question.

Question 1 ([17, p. 375])**.** Do there exist infinitely many $n \in \mathbb{N}$ for which $\frac{n}{\varphi(n)} > e^C \log\log n$?

In the same paper, Nicolas resolves this question by proving the following theorem.

Theorem 2.5 ([17, Theorem 1])**.** *There exist infinitely many $n \in \mathbb{N}$ for which*

$$\frac{n}{\varphi(n)} > e^C \log\log n.$$

The method of proof which Nicolas employs here is particularly interesting, but requires the introduction of two concepts, one arithmetic and one analytic. On the arithmetic side, one may observe that the value $\frac{n}{\varphi(n)}$ is, by the definition of $\varphi(n)$, equal to $\prod_{p|n} \frac{p}{p-1}$. From here, we see that the value of the left hand side in (2.2) is the same for any two natural numbers which have the same prime divisors. Moreover, the value of $\frac{p}{p-1}$ is always greater than 1, however this value is strictly decreasing as p increases. Since the right hand side of (2.2) increases as n increases, Nicolas [17, p. 376] observes that the best method of attacking Question 1 will be to choose natural numbers which are squarefree and whose prime divisors are as small as possible. More directly, Nicolas considers the set of *primorials*.

Definition 2.6. The k-th *primorial*, denoted N_k, is the product of the first k primes. That is,

$$N_k = \prod_{i=1}^{k} p_i,$$

[3] "Small values of Euler's function"

where p_i is the i-th prime.

In Nicolas' proof, primorials are the arithmetic object and make up the infinite set of natural numbers referred to in Theorem 2.5. On the analytic side, he considers the *Riemann zeta function*.

Definition 2.7. The *Riemann zeta function* $\zeta(s)$ is the (meromorphic) continuation to the whole complex plane of the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^s},$$

which is defined for $\Re(s) > 1$.

The Riemann zeta function has a pole at $s = 1$ and trivial zeroes at $s = -2, -4, -6 \ldots$. Its *nontrivial* zeroes, denoted $\rho = \beta + i\gamma$, are those found in the *critical strip*, the region for which $0 \leq \sigma \leq 1$. For our purposes it is important to recall the following conjecture regarding the nontrivial zeroes of $\zeta(s)$.

Conjecture 2.8 (The Riemann hypothesis). If $\rho = \beta + i\gamma$ is a nontrivial zero of the Riemann zeta function, then $\beta = 1/2$.

Nicolas establishes Theorem 2.5 in two ways. Without reference to Conjecture 2.8, he obtains his result by appealing to an oscillation result of Landau. Second, he strengthens this result, proving that the claim that there are infinitely many natural numbers (in fact, primorials) which satisfy (2.2) is *independent* of the Riemann Hypothesis. That is, whether or not Conjecture 2.8 holds true, we can determine that there are infinitely many primorials which satisfy (2.2).

Theorem 2.9 ([17, Theorem 2]). *If the Riemann Hypothesis is true, then for all primorials N_k, we have*

$$\frac{N_k}{\varphi(N_k) \log\log N_k} > e^C.$$

If the Riemann Hypothesis is false, then there are infinitely many primorials for which the above inequality holds, and infinitely many primorials for which the above inequality does not hold.

One should note, in particular, that this is stronger than Theorem 2.5. The first statement of Theorem 2.9 automatically implies that there are only finitely many primorials which *do not* satisfy (2.2), and the second statement of Theorem 2.9 is equivalent to the statement "If there are only finitely many primorials which do not satisfy (2.2), then the Riemann Hypothesis is false". In this way, Theorem 2.9 implies the following *equivalence*.

Theorem 2.10. *The Riemann Hypothesis is true if and only if there are only finitely many primorials N_k for which*

$$\frac{N_k}{\varphi(N_k)\log\log N_k} \leq e^C.$$

Theorem 2.10 establishes a bridge between the world of the arithmetic, as represented by the behavior of Euler's function evaluated at primorials, and the world of the analytic, as represented by the behavior of the Riemann zeta function. It is the aim of this thesis to use this equivalence as a template. We will follow the work of Nicolas, albeit in a more general setting, with the hope of establishing bridges of the same quality, but with different arithmetic and analytic objects on their ends.

2.3 The Structure of This Thesis

Before we can have any hope of generalizing Theorem 2.10, we will need to convince ourselves that there exists a context where doing so makes sense. This will be the main task of Chapter 3. There we aim to mirror the results of Landau in this chapter, but replacing primes with primes *in arithmetic progressions*. Theorem 2.3, which was the precursor to Question 1, will be found to generalize neatly in this new setting, as evidenced in Theorem 3.8. Using that success as an inspiration, we will implement and discuss the appropriate generalization of primorials in the context of arithmetic progressions. On the analytic side, it is natural that we discuss Dirichlet L-functions at this point as well, including a variety of conjectures related to the Riemann Hypothesis in this setting.

From here, we may turn our attention towards the generalizations themselves. In Chapter 4, we establish a function, $f(x;q,a)$ which captures information about the behavior of $\varphi(n)$ at a generalization of primorials. From there we introduce some important upper and lower bounds on $\log f(x;q,a)$, in anticipation of generalizing Theorem 2.9. Finally, the upper bounds of this chapter, coupled with an oscillation technique will provide us with generalized versions of the statement "There are infinitely many primorials which satisfy (2.2)", without having made reference to Conjecture 2.8.

After having established unconditional results, Chapter 5 will focus on theorems which require us to assume the relevant generalization of the Conjecture 2.8 is false. From this assumption we will be able to prove an analogue of the statement "If the Riemann Hypothesis is false, then there are infinitely many primorials for which (2.2) holds, and also infinitely many primorials for which it does not hold". In contrast, Chapter 6 will focus on assuming that the conjectural information is true, leading to results reminiscent of "If the Riemann Hypothesis is true, then there are finitely many primorials for which (2.2) is false". These two chapters will form a proof that Theorem 2.10 can be generalized in certain cases, as desired.

Chapter 3

Primes in Arithmetic Progressions

When things get too complicated,

it sometimes makes sense to stop and wonder:

have I asked the right question?

– E. Bombieri

The primary goal of this chapter is to provide evidence that it, in fact, makes sense to generalize the results of Nicolas [17]. As discussed in Chapter 2, we particularly wish to replace the primorials and the Riemann zeta function with objects that make sense in our new setting. In order to justify a generalization of Nicolas' theorem, we need to return to Landau's Theorem, that is, Theorem 2.3. There are three crucial elements involved in the proof: the primorials, Mertens' Theorem [3, Theorem 429], and the Prime Number Theorem [3, Theorem 6]. A key observation is that all of these elements have analogues when we move from discussing primes (insofar as they are used to construct primorials) and discuss instead primes in arithmetic progressions.

3.1 Generalizing Landau's Theorem

Definition 3.1. Fix $q, a \in \mathbb{N}$. Take a_0 to be the least natural number for which $a_0 \equiv a \pmod{q}$. Then the sequence

$$a_0,\ a_0+q,\ a_0+2q,\ \ldots,\ a_0+kq,\ \ldots$$

is called the *arithmetic progression* $a \pmod q$. If a prime p can be written in the form $p = a_0 + kq$ for some $k \in \mathbb{N}$, then we say p is a *prime in the arithmetic progression* $a \pmod q$.

By Dirichlet's theorem [14, Corollary 4.10], we know that if q and a are coprime, then there are infinitely many primes in the arithmetic progression $a \pmod q$, and therefore the discussion of this chapter will be meaningful.

In Theorem 2.3, the limit superior of $\frac{n}{\varphi(n)\log\log n}$ was taken over all natural numbers. Here, in order to favor primes which fall into a chosen arithmetic progression we wish to restrict our attention to a smaller set, whose elements avoid prime divisors from outside the chosen progression. Precisely, we have the following definition.

Definition 3.2. For $q, a \in \mathbb{N}$, let

$$S_{q,a} = \{n \in \mathbb{N}\,;\, p \mid n \implies p \equiv a \pmod q\}.$$

Example 3.3.

- Note that $S_{1,1} = \mathbb{N}$.
- Similarly, $S_{2,1}$ is the odd natural numbers.
- For a nontrivial example, consider $S_{5,2} = \{1, 2, 4, 7, 8, 14, 16, 17, 28, 32, \ldots\}$.

Recall that in Chapter 2, the expression $\frac{n}{\varphi(n)}$ was largest relative to the size of n when n was squarefree and had many small prime divisors. If we restrict ourselves to $S_{q,a}$, the only admissible prime divisors of n become those which are in the progression $a \pmod q$. In this setting, we can still consider n squarefree, but now the prime divisors must be both small *and* fall into the progression $a \pmod q$. We can therefore capture the appropriate generalization of the primorials as follows.

Definition 3.4. Let q and a be coprime natural numbers. Then the k-th *primorial in $S_{q,a}$*, denoted $\overline{N}_k$, is the product of the first k primes in the progression $a \pmod{q}$. That is,

$$\overline{N}_k \overset{\text{def}}{=} N_{q,a}(k) = \prod_{i=1}^{k} \overline{p}_i,$$

where $\overline{p}_i$ is the i-th prime in the arithmetic progression $a \pmod{q}$.

Throughout this thesis, q and a will generally be fixed. For notational compactness, we often suppress reference to q and a and use $\overline{N}_k$. Note that for $q = a = 1$, we have $\overline{N}_k = N_k$.

In Figure 3.1, we consider $\varphi(n)$ on the sets $S_{5,a}$ for all a coprime to 5.

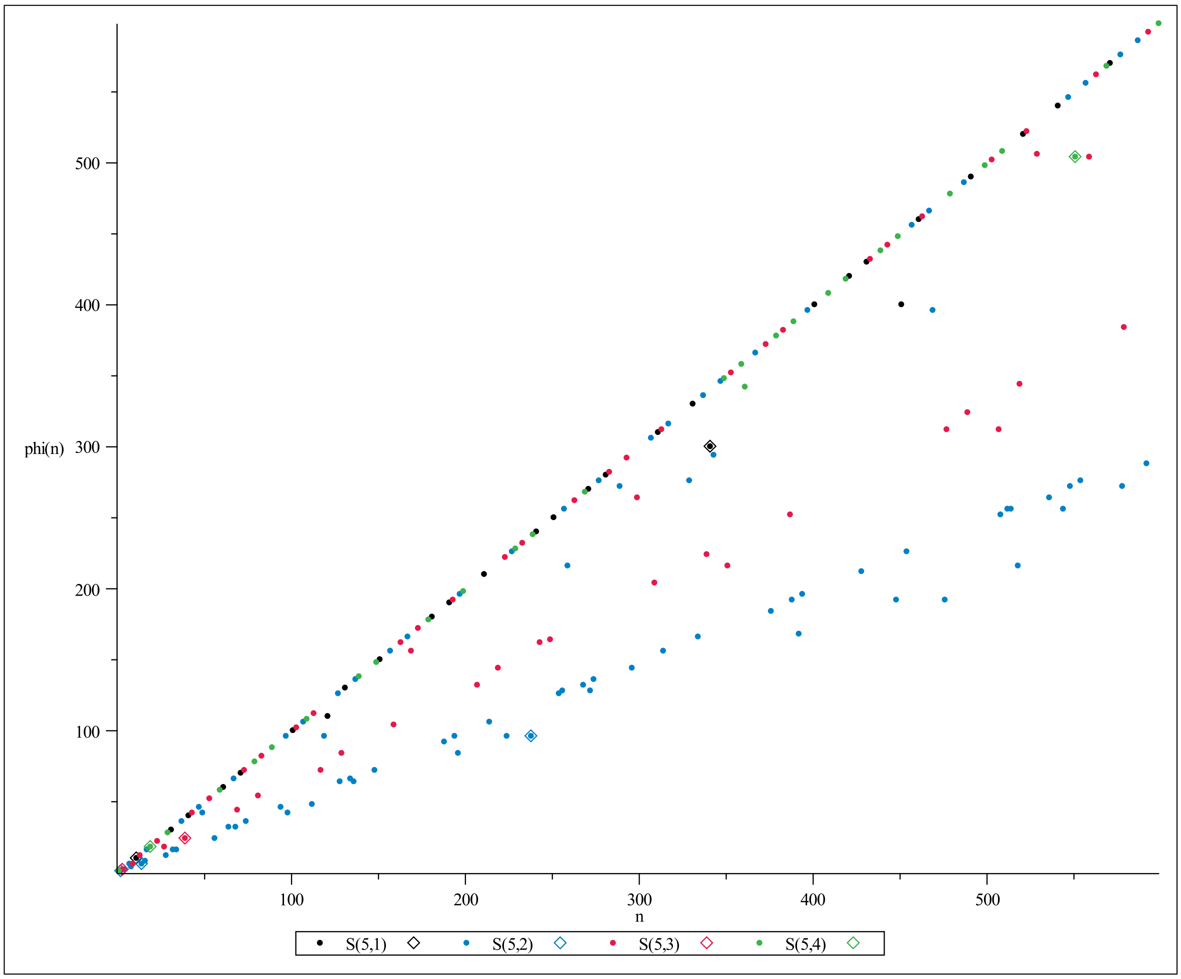

Figure 3.1: Euler's Function on $S_{5,a}$. The highlighted points represent primorials in $S_{5,a}$.

A generalization of Mertens' theorem for primes in arithmetic progressions was provided by Williams [22]. However, here we draw upon the work of Languasco and Zaccagnini ([7], [8], [9]) because they have focused on providing an explicit form for the constant appearing in the generalized Mertens' theorem, $C(q,a)$, and additionally in [8] have done significant work accurately computing these constants .

Theorem 3.5 ([7, p. 46])**.** *Let $x \geq 2$ and $q,a \in \mathbb{N}$ be coprime. Then,*

$$\prod_{\substack{p \leq x \\ p \equiv a \pmod{q}}} \left(1 - \frac{1}{p}\right) \sim \frac{C(q,a)}{(\log x)^{\frac{1}{\varphi(q)}}},$$

as $x \to \infty$, where

$$C(q,a)^{\varphi(q)} = e^{-C} \prod_{p} \left(1 - \frac{1}{p}\right)^{\alpha(p;q,a)}$$

and

$$\alpha(p;q,a) = \begin{cases} \varphi(q) - 1 & \text{if } p \equiv a \pmod{q}, \\ -1 & \text{otherwise.} \end{cases}$$

We may note that, in agreement with the classical Mertens' theorem, $C(1,1)$ is e^{-C} since $\alpha(p;1,1) = 0$ for all primes p.

The final ingredient needed to generalize Landau's theorem is an analogue of the Prime Number Theorem in the context of primes in arithmetic progressions. For technical reasons, it benefits us to weight the counting of primes in the following manner.

Definition 3.6. Let $q,a \in \mathbb{N}$ be coprime. The *first Chebyshev function* will be denoted $\theta(x;q,a)$ and be defined by

$$\theta(x;q,a) = \sum_{\substack{p \leq x \\ p \equiv a \pmod{q}}} \log p.$$

This definition allows us to state the following version of the Prime Number Theorem for Arithmetic Progressions, first proved by Landau.

Theorem 3.7 ([16, Theorem 6.8])**.** *Let $x \geq 2$ and $q, a \in \mathbb{N}$ be coprime. Then,*

$$\theta(x; q, a) \sim \frac{x}{\varphi(q)},$$

as $x \to \infty$.

With these elements in place, we can establish a generalization of Theorem 2.3.

Theorem 3.8. *Let $q, a \in \mathbb{N}$ be coprime. Then*

$$\limsup_{n \in S_{q,a}} \frac{n}{\varphi(n)(\log(\varphi(q)\log n))^{1/\varphi(q)}} = \frac{1}{C(q,a)},$$

where $C(q,a)$ is defined in Theorem 3.5.

Proof. For $n \in S_{q,a}$, let r be the number of prime divisors of n that are larger than $\varphi(q)\log n$. Writing $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, we have $(\varphi(q)\log n)^r < n$ and thus,

$$r < \frac{\log n}{\log(\varphi(q)\log n)}$$

Employing the above bound for r yields

$$\begin{aligned} \frac{n}{\varphi(n)} = \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right)^{-1} &\leq (1 - \frac{1}{\varphi(q)\log n})^{-r} \prod_{\substack{p \leq \varphi(q)\log n \\ p \mid n}} \left(1 - \frac{1}{p}\right)^{-1} \\ &< \left(1 - \frac{1}{\varphi(q)\log n}\right)^{\frac{-\log n}{\log(\varphi(q)\log n)}} \prod_{\substack{p \leq \varphi(q)\log n \\ p \equiv a \pmod q}} \left(1 - \frac{1}{p}\right)^{-1}. \qquad (3.1) \end{aligned}$$

The first factor on the right of (3.1) tends to 1 as $n \to \infty$. Invoking Theorem 3.5 for the latter product, we conclude that

$$(1-\frac{1}{\varphi(q)\log n})^{\frac{-\log n}{\log(\varphi(q)\log n)}} \prod_{\substack{p\leq\varphi(q)\log n\\ p\equiv a \pmod q}}(1-\frac{1}{p})^{-1} \sim \frac{(\log(\varphi(q)\log n))^{\frac{1}{\varphi(n)}}}{C(q,a)}, \tag{3.2}$$

as $n\to\infty$. From (3.1) and (3.2), we deduce

$$\limsup_{n\in S_{q,a}} \frac{n}{\varphi(n)(\log(\varphi(q)\log n))^{1/\varphi(q)}} \leq \frac{1}{C(q,a)}.$$

To establish a sequence which attains this bound, consider $\overline{N}_k$, the k-th primorial in $S_{q,a}$. Then, by Theorem 3.5

$$\frac{\overline{N}_k}{\varphi(\overline{N}_k)} = \prod_{\substack{p\leq\overline{p}_k\\ p\equiv a \pmod q}} \left(1-\frac{1}{p}\right)^{-1} \sim \frac{(\log\overline{p}_k)^{\frac{1}{\varphi(q)}}}{C(q,a)},$$

as $k\to\infty$. Next, we apply Theorem 3.7 to obtain

$$\log(\varphi(q)\log\overline{N}_k) = \log(\varphi(q)\theta(\overline{p}_k;a,q)) \sim \log\overline{p}_k,$$

as $k\to\infty$. Hence, we have

$$\frac{\overline{N}_k}{\varphi(\overline{N}_k)} \sim \frac{\left(\log(\varphi(q)\log\overline{N}_k)\right)^{\frac{1}{\varphi(q)}}}{C(q,a)},$$

as $k\to\infty$. That is,

$$\lim_{k\to\infty} \frac{\overline{N}_k}{\varphi(\overline{N}_k)(\log(\varphi(q)\log\overline{N}_k))^{1/\varphi(q)}} = \frac{1}{C(q,a)}.$$

This concludes the proof. □

It is interesting to note that this version of Theorem 2.3 introduces some new features, particularly the two instances of $\varphi(q)$ in the left-hand expression. The power $\frac{1}{\varphi(q)}$ exists as a direct consequence of the same power appearing in the statement of Theorem 3.5. When

$q = 1$, this is moot, since $\varphi(1) = 1$. Indeed, because $C(1,1) = e^{-C}$, we recover Theorem 2.3 exactly. Figure 3.2 visualizes Theorem 3.8 for $q = 5$. In each fixed set $S_{5,a}$, one can observe behavior congruous with the behavior in Figure 2.2.

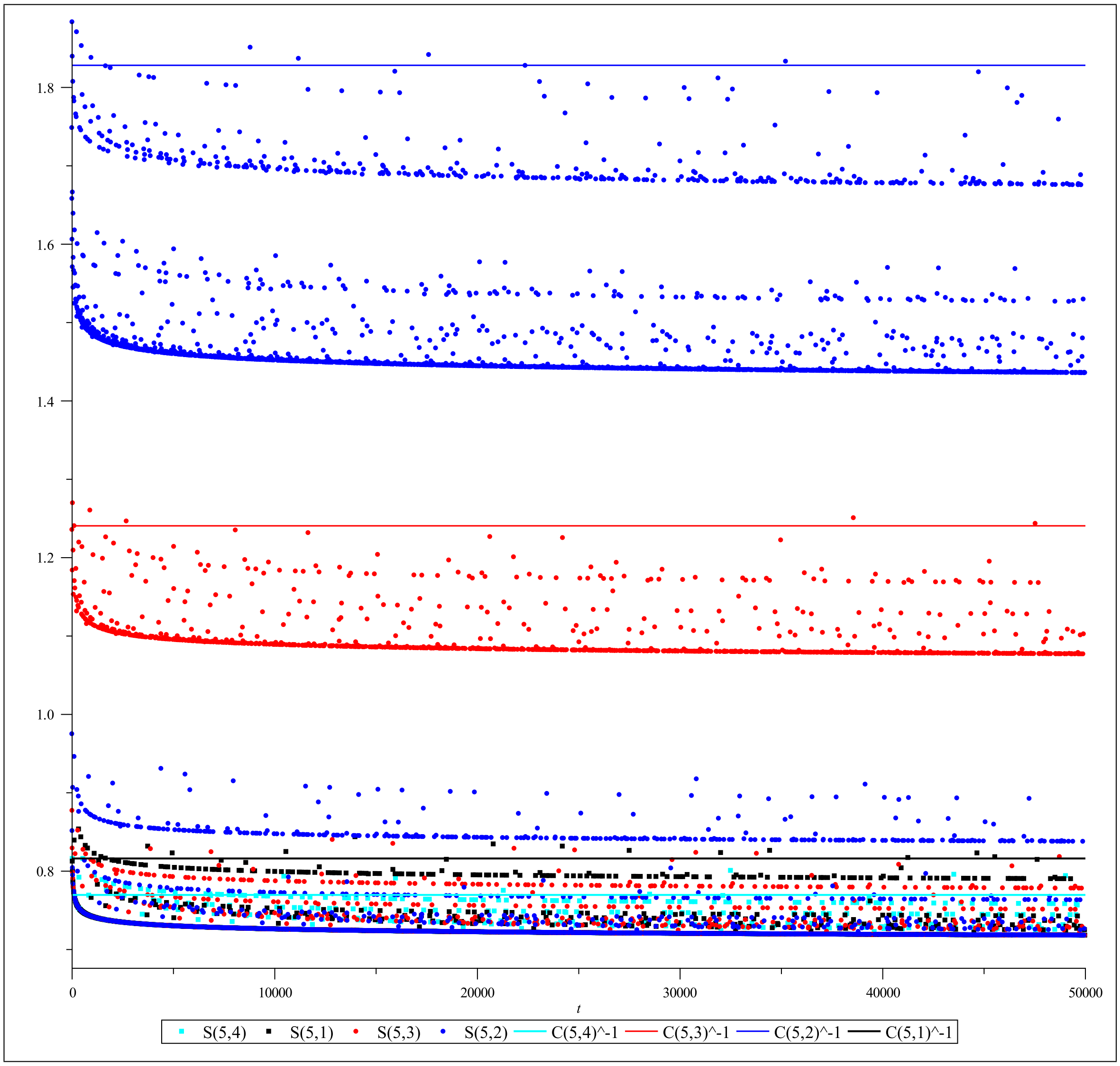

Figure 3.2: Visualization of Theorem 3.8 for $q = 5$ and a coprime to 5.

It seems reasonable at this point to see how far we can extend this generalization, along the line of study begun by Nicolas. A generalization of Question 1 is natural.

Question 2. Let $q, a \in \mathbb{N}$ be coprime and consider the inequality

$$\frac{n}{\varphi(n)(\log \varphi(q) \log n)^{\frac{1}{\varphi(q)}}} > \frac{1}{C(q,a)}. \tag{3.3}$$

Are there infinitely many $n \in S_{q,a}$ for which 3.3 is satisfied?

To facilitate the discussion of this question in contexts where q, a are fixed, we define the following notations.

Definition 3.9. Let q and a be coprime natural numbers. Call the statement of the inequality

$$\frac{\overline{N}_k}{\varphi(\overline{N}_k)(\log \varphi(q) \log \overline{N}_k)^{\frac{1}{\varphi(q)}}} > \frac{1}{C(q,a)} \tag{3.4}$$

the *Nicolas inequality for the progression $a \pmod q$ at k*, denoted $\mathcal{N}_{q,a}(k)$.

If $\mathcal{N}_{q,a}(k)$ is true, then we say the corresponding Nicolas inequality is *satisfied*.

Under this framework, we can refine Question 2 so that we are only focused on primorials in $S_{q,a}$.

Question 3. Let $q, a \in \mathbb{N}$ be coprime. Are there infinitely many k for which $\mathcal{N}_{q,a}(k)$ is true?

3.2 Dirichlet L-functions

The aim of the remainder of this thesis is to establish a criteria for resolving Question 3 using tools and methods from analysis analogous to those used by Nicolas. Where Nicolas establishes his result by appealing to the behavior of the Riemann zeta function under opposing conjectural conditions, we will need to distinguish the appropriate analytic object(s) to replace the Riemann zeta function in the context of primes in arithmetic progressions and provide a corresponding analogue for the Riemann hypothesis (Conjecture 2.8). It is natural to expect that the appropriate replacement will be the Dirichlet L-functions. We begin by covering the definition of Dirichlet characters (see [14, pp. 115-119, 282-285] for an in-depth discussion).

Definition 3.10. Fix q and let $\hat{\chi}$ be a group homomorphism from $(\mathbb{Z}/q\mathbb{Z})^\times$ to $\mathbb{C}^\times$. The extended function $\chi : \mathbb{Z} \to \mathbb{C}^\times$ defined by

$$\chi(a) = \begin{cases} \hat{\chi}(a_0) & \text{if } (q,a) = 1, \\ 0 & \text{otherwise,} \end{cases}$$

where a_0 is the least natural number for which $a_0 \equiv a \pmod q$, is called a *Dirichlet character modulo q*.

One may observe that the set of Dirichlet characters modulo q forms a group under pointwise multiplication and this group is isomorphic to $(\mathbb{Z}/q\mathbb{Z})^\times$ [14, Corollary 4.5]. Hence, there are $\varphi(q)$ characters modulo q. The identity of this group is

$$\chi_0(a) = \begin{cases} 1 & \text{if } (q,a) = 1, \\ 0 & \text{otherwise ,} \end{cases}$$

which is called the *principal* character (modulo q). The only character modulo 1 is the principal character for this modulus which will be denoted $\mathbf{1}(a) = 1$, and referred to as the *trivial* character. Additionally, every character χ has an inverse in this group, which we call the *conjugate of* χ, and denote by $\overline{\chi}$.

The following are properties which hold for all Dirichlet characters and which are relevant to our discussion.

- For any character χ, $\chi(1) = 1$.
- If $(q,a) = 1$, then $\chi(a)$ is a $\varphi(q)$-th root of unity.
- A character modulo q has period q, i.e., $\chi(a+q) = \chi(a)$ for any $a \in \mathbb{Z}$.
- A character χ is completely multiplicative, i.e., for any integers m and n, $\chi(mn) = \chi(m)\chi(n)$.

We can also classify Dirichlet characters in a variety of ways.

Definition 3.11. If χ is a character modulo q such that

$$\chi(a) = \begin{cases} \chi'(a) & \text{if } (q,a) = 1 \\ 0 & \text{otherwise} \end{cases}$$

for some character χ' modulo d, where d is a divisor of q, then we say that χ' *induces* χ (or χ is *induced by* χ').

Definition 3.12. A character χ is said to be *primitive* if the character which induces χ is χ itself. Otherwise, we call χ an *imprimitive* character. A character χ is said to be *real* if $\chi(a) \in \mathbb{R}$ for all $a \in \mathbb{Z}$. Otherwise, we call χ *complex*. A character χ is said to be *odd* if $\chi(-1) = -1$. Otherwise, we call χ *even*.

As will be seen in subsequent chapters, the appropriate replacement for the Riemann zeta function are the Dirichlet *L*-functions corresponding to characters χ modulo q, defined as follows (see [14, pp. 120-121, 333-334] for details).

Definition 3.13. Given a Dirichlet character $\chi \pmod{q}$, we define the corresponding *Dirichlet L-function*, denoted $L(s,\chi)$, to be the (meromorphic) continuation to the whole complex plane of the infinite series

$$\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

defined for $\Re(s) > 1$. Its *nontrivial zeroes* are those zeroes found in the *critical strip*, $\{s \in \mathbb{C} : 0 \le \Re(s) \le 1\}$.

Observe that, by definition, the Riemann zeta function is the function $L(s,\mathbf{1})$ where $\mathbf{1}(n) = 1$ is the trivial character. Since χ is multiplicative, every Dirichlet *L*-function also may be represented by its Euler product

$$L(s,\chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}$$

valid for $\Re(s) > 1$. For characters χ modulo $q > 1$, it is possible that χ is induced by a character χ', where χ' is a primitive character modulo d, d dividing q. In such a situation, the fact that our character is imprimitive leads to a deletion of factors in the Euler product. That is,

$$L(s,\chi) = L(s,\chi') \prod_{\substack{p \mid q \\ p \nmid d}} \left(1 - \frac{\chi'(p)}{p^s}\right).$$

The additional factors in the rightmost product contribute zeroes to the L-function of an imprimitive character. Each factor $1 - \frac{\chi'(p)}{p^s}$ contributes a set of zeroes $\{\rho \mid \rho = \frac{2\pi ik + i\arg(\chi'(p))}{\log p}, k \in \mathbb{Z}\}$ where $\arg(\chi'(p))$ is any real number for which $\chi'(p) = e^{i\arg(\chi'(p))}$. While these zeroes have a different behavior than the zeroes for which $\Re(\rho) \neq 0$, they still play a role in the *explicit formula* (see Theorem 6.1), which is an important tool employed in Chapter 6. For this reason that we include zeroes with $\Re(\rho) = 0$ in the nontrivial zeroes.

Finally, while no direct reference to the functional equations [14, Corollary 10.8] of L-functions is required for this thesis, it is important to note that if $\rho = \beta + i\gamma$ is a zero of $L(s,\chi)$ satisfying $0 < \beta < 1$, then $1 - \overline{\rho}$ is also a zero of $L(s,\chi)$, where $\overline{\rho}$ is the complex conjugate of ρ. Also, $1 - \rho$ and $\overline{\rho}$ are zeroes of $L(s,\overline{\chi})$ [14, p. 333].

3.3 Important Conjectures

It is often the case that we generalize the Riemann Hypothesis across *all* Dirichlet L-functions in the following fashion (see, for example, [14, p. 333]).

Conjecture 3.14 (The generalized Riemann hypothesis). If $\rho = \beta + i\gamma$ is a nontrivial zero of a Dirichlet L-function and $\beta \neq 0$, then $\beta = 1/2$.

Since the ultimate goal of this thesis is to generalize the *equivalence* Nicolas establishes in Theorem 2.10, it becomes important to establish a conjecture which is flexible enough to take the place of the Riemann Hypothesis in answering Question 2. The simplest approach would be to restrict Conjecture 3.14 to only those L-functions whose defining character is a character modulo q.

Conjecture 3.15 (GRH$_q$). For fixed q, let χ be a Dirichlet character modulo q. Then if $\rho = \beta + i\gamma$ is a nontrivial zero of $L(s,\chi)$ and $\beta \neq 0$, then $\beta = 1/2$.

An important function in this thesis is the following Dirichlet series.

Notation 3.16. Let q and a be fixed coprime integers. We define

$$\mathscr{L}(s;q,a) = \sum_{\chi} \overline{\chi}(a) \frac{L'}{L}(s,\chi).$$

The function $\mathscr{L}(s;q,a)$ has potential singularities in the critical strip when s is a zero of a Dirichlet L-function corresponding to a character modulo q. In this setting, we hope to leverage the behavior of $\mathscr{L}(s;q,a)$ at such singularities. For this reason, in place of Conjecture 3.15, we formulate a conjecture which captures information about how the zeroes of L-functions contribute to the singularities of $\mathscr{L}(s;q,a)$.

Conjecture 3.17. Let q and a be fixed coprime integers. If there exists a nontrivial zero ρ of $L(s,\chi)$ for some χ modulo q for which $\Re(\rho)$ is neither 0 nor $\frac{1}{2}$, then

$$\sum_{\chi \ (\mathrm{mod}\ q)} \overline{\chi}(a) m_\rho(\chi) = 0, \tag{3.5}$$

where $m_\rho(\chi)$ is the multiplicity of ρ as a zero of $L(s,\chi)$.

In general, Conjecture 3.17 is weaker than Conjecture 3.15. If we assume that Conjecture 3.15 holds, there will not be any zeroes for which $\Re(\rho)$ is not 0 or $\frac{1}{2}$, and therefore Conjecture 3.17 holds trivially. In the special case where $a = 1$ (for any q), we have that the two conjectures are actually equivalent, since $\overline{\chi}(1) = 1$ for all Dirichlet characters, and $m_\rho(\chi) \geq 0$. That is, for $a = 1$, assuming Conjecture 3.17 implies that no zero ρ which is a counter example to Conjecture 3.15 exists, since this would imply $\sum_\chi m_\rho(\chi) > 0$.

Another condition we must consider when applying the analytic tools of Chapters 4 and 5 is whether there exist any zeroes on the real line segment $(0,1)$. It is a well-known fact (see [17, p. 385] for a proof) that this condition holds for $\zeta(s)$, but it may not hold in general.

Such zeroes have the potential to hamper the utility of so-called *oscillation* theorems which we plan to employ. Fortunately, there are a significant number of cases where this concern may be immediately eliminated, primarily thanks to the work of Platt [19]. In tandem, the following theorems allow us to conclude, for given ranges of q, that there are no real zeroes on the segment $(0,1)$ of $L(s,\chi)$, where χ is a primitive character of modulus q.

Theorem 3.18 ([19, Theorem 10.1]). *Let χ be a primitive Dirichlet character of modulus q where $q \leq 400,000$. If q is even, then any nontrivial, nonzero zero $\rho = \beta + i\gamma$ of $L(s,\chi)$ satisfying $|\gamma| < \max(\frac{10^8}{q}, \frac{7.5\cdot 10^7}{q} + 200)$ also satisfies $\beta = \frac{1}{2}$. If q is odd, then any nontrivial, nonzero zero $\rho = \beta + i\gamma$ of $L(s,\chi)$ satisfying $|\gamma| < \max(\frac{10^8}{q}, \frac{3.75\cdot 10^7}{q} + 200)$ also satisfies $\beta = \frac{1}{2}$.*

For our purposes, all we take this to imply is that, for the relevant $L(s,\chi)$, the only possible zero on the segment $(0,1)$ is $\rho = \frac{1}{2}$. As it happens, this is not a problem either.

Theorem 3.19 ([19, Theorem 10.2]). *Let χ be a primitive Dirichlet character of modulus q where $q \leq 2,000,000$. Then*

$$L(\frac{1}{2},\chi) \neq 0.$$

Hence for L-functions of primitive Dirichlet characters whose modulus q is less than $400,000$, there are no zeroes on the segment $(0,1)$. However, the prevailing belief seems to be that $L(1/2,\chi) \neq 0$ for any primitive Dirichlet character χ, based on a conjecture of Chowla [2, pp. xv+119] regarding the central value of *real* primitive characters. Finally, the methods of Chapter 4 require that $\mathscr{L}(s;q,a)$ has a singularity at ρ where $0 < \Re(\rho) < 1$. Such a singularity can be shown to exist for a fixed q, since ρ may be taken to be any zero on the $1/2$-line of $\prod_\chi L(s,\chi)^{\overline{\chi}(a)}$.

By now, evidence is mounting in favor of a generalization of Theorem 2.10. In part, the arithmetic objects and theorems which motivated Nicolas' approach to Question 1 all generalized neatly. While the generalization was less direct, Conjecture 2.8 likewise was elevated to a statement which accommodates the goals of this thesis. What remains in the following sections is to employ the techniques of Nicolas, modified for the new setting, in order to bring all of these pieces together.

Chapter 4

Unconditional Results

It is the story that matters

not just the ending.

– P. Lockhart

This chapter is dedicated to establishing results and techniques which are "conjecture-free" in the sense that they do not rely on Conjecture 3.17. Particularly, for certain q, we can answer Question 3 in the affirmative; there *are* infinitely many primorials in $S_{q,a}$ for which (3.4) holds. In other words, this chapter will verify the existence of generalizations of Theorem 2.5. In subsequent chapters we will use and refine these results to establish the constituent pieces necessary to generalize Theorem 2.9 and, for certain cases of q and a, Theorem 2.10.

4.1 The Function $f(x;q,a)$

To start, is beneficial to translate our problem from the arithmetic language of Question 3 to a setting where analytic techniques and theorems apply. In [17], Nicolas observed that we can encode information regarding the Nicolas inequality at k, $\mathcal{N}_{q,a}(k)$, using a real-valued function $f(x)$ [17, pp. 376-77]. In what follows, we mimic this construction to encode information regarding (3.4).

Let $\overline{p}$ represent any prime in the progression $a \pmod q$. Observe that, for $x \in [\overline{p}_k, \overline{p}_{k+1})$,

$$\prod_{\overline{p} \leq x} \left(1 - \frac{1}{p}\right) = \frac{\varphi(\overline{N}_k)}{\overline{N}_k}$$

and

$$\log(\varphi(q)\theta(x;q,a)) = \log(\varphi(q) \sum_{\overline{p} \leq x} \log \overline{p}) = \log(\varphi(q) \log \overline{N}_k).$$

With these observations in mind, define

$$f(x;q,a) = \frac{(\log(\varphi(q)\theta(x;q,a)))^{\frac{1}{\varphi(q)}}}{C(q,a)} \cdot \prod_{\overline{p} \leq x} \left(1 - \frac{1}{p}\right).$$

Hence, for any $x \in [\overline{p}_k, \overline{p}_{k+1})$,

$$f(x;q,a) = \frac{\left(\log(\varphi(q) \log \overline{N}_k)\right)^{\frac{1}{\varphi(q)}}}{C(q,a)} \cdot \frac{\varphi(\overline{N}_k)}{\overline{N}_k}.$$

It is therefore apparent that $\mathcal{N}_{q,a}(k)$ holds if and only if $f(x;q,a) < 1$ for any $x \in [\overline{p}_k, \overline{p}_{k+1})$. We note, as Nicolas did in [17, p. 377], that this is also equivalent to establishing

$$\log f(x;q,a) = \frac{\log\log(\varphi(q)\theta(x;q,a))}{\varphi(q)} + \sum_{\overline{p} \leq x} \log\left(1 - \frac{1}{p}\right) - \log C(q,a) < 0, \qquad (4.1)$$

for $x \in [\overline{p}_k, \overline{p}_{k+1})$. In the next section, we provide a comparison of $\log f(x;q,a)$ for some values of q, and make some observations that will become relevant in Chapter 6. We continue by determining useful upper and lower bounds for $\log f(x;q,a)$. The upper bounds, in particular, will allow us to establish that there are infinitely k for which $\mathcal{N}_{q,a}(k)$ holds.

4.2 Plots of $\log f(x;q,a)$

In this section we supply plots of $\log f(\overline{p}_k;q,a)$ for several values of q. Since $\log f(x;q,a)$ is fixed between primes in the progression $a \pmod q$, the horizontal axis in each case is k, the index of $\overline{p}_k$, rather than x. An observation, for which more evidence will be provided in Chapter 6, is that the behavior of $\log f(x;q,a)$ depends, in part, in whether or not a is a quadratic residue of q.

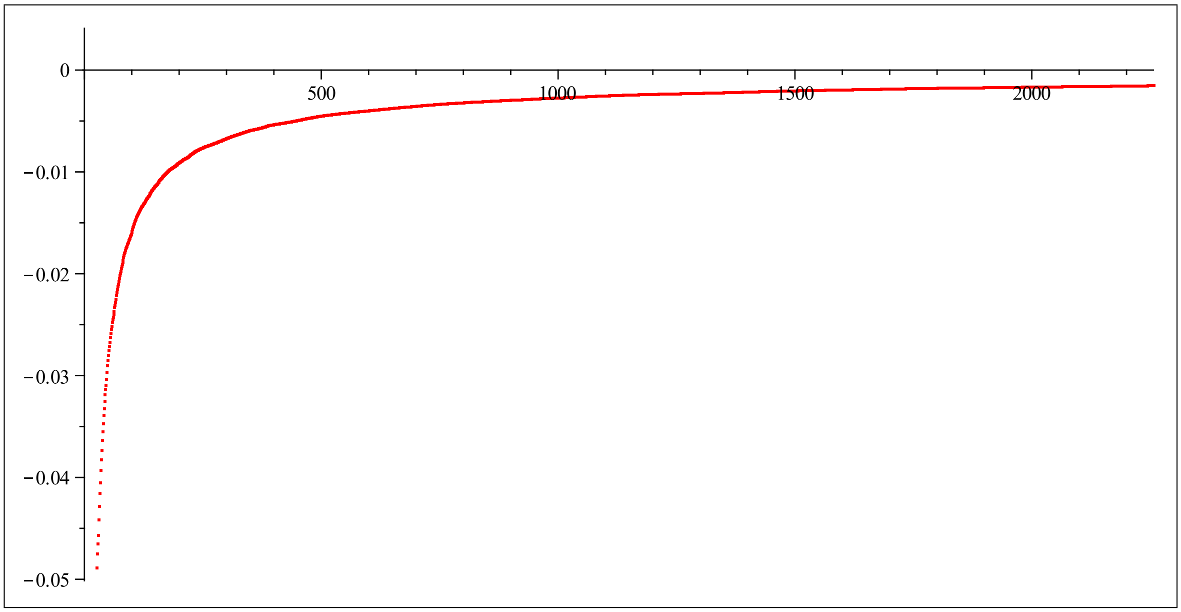

Figure 4.1: Plot of $\log f(p_k;1,1)$. As observed by Nicolas [17], this function is negative for small values of k. Assuming RH, it will remain negative for all values of k.

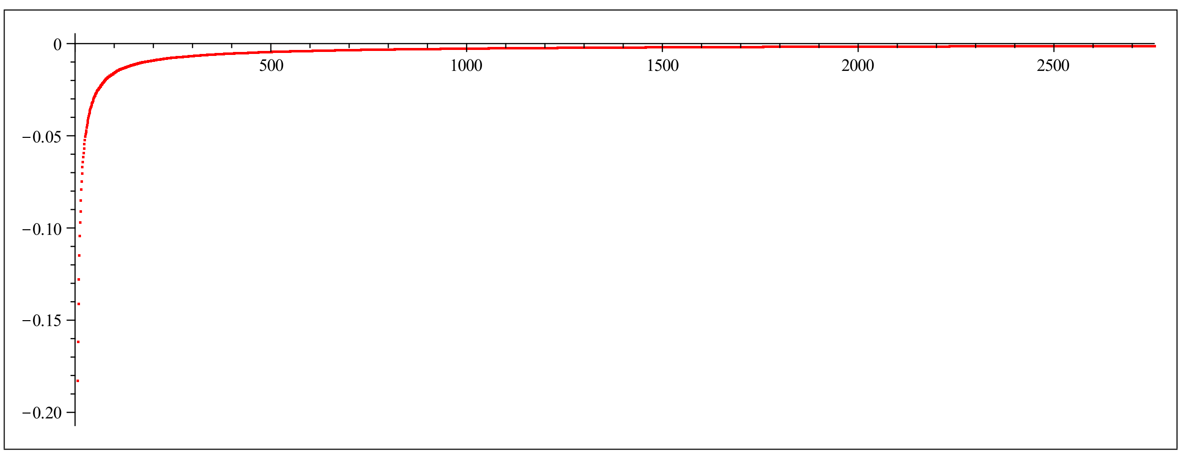

Figure 4.2: Plot of $\log f(\overline{p}_k;2,1)$.

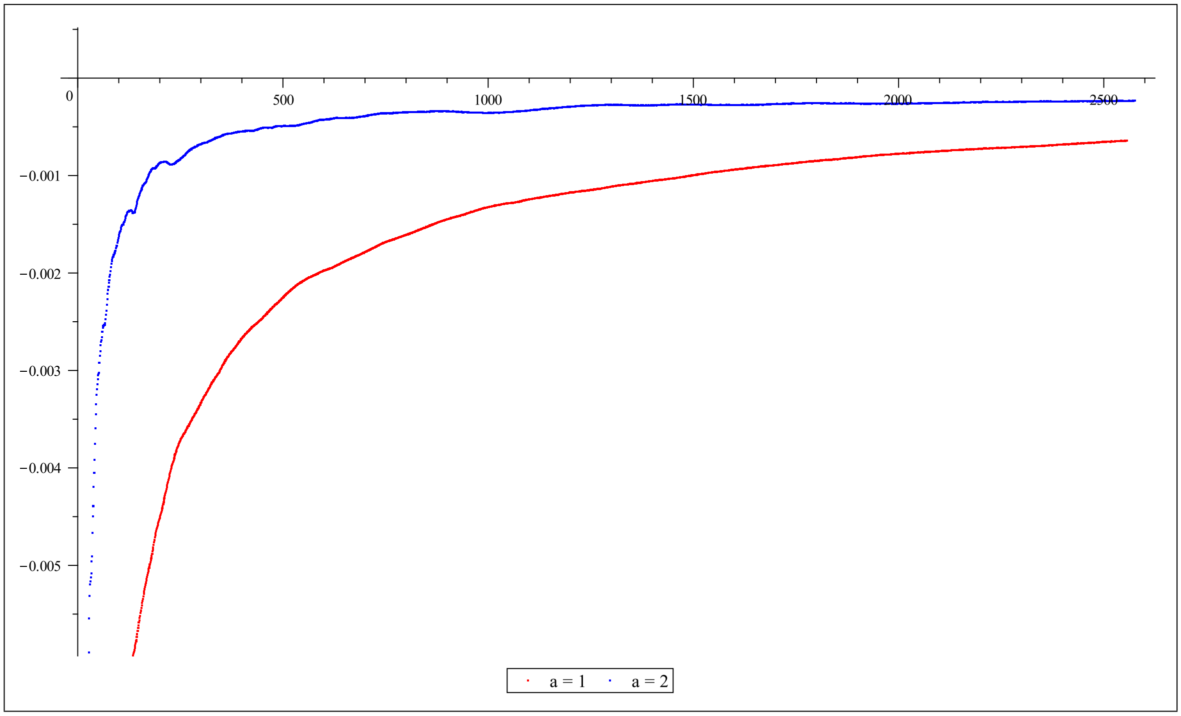

Figure 4.3: Plot of $\log f(\overline{p}_k; 3, a)$.

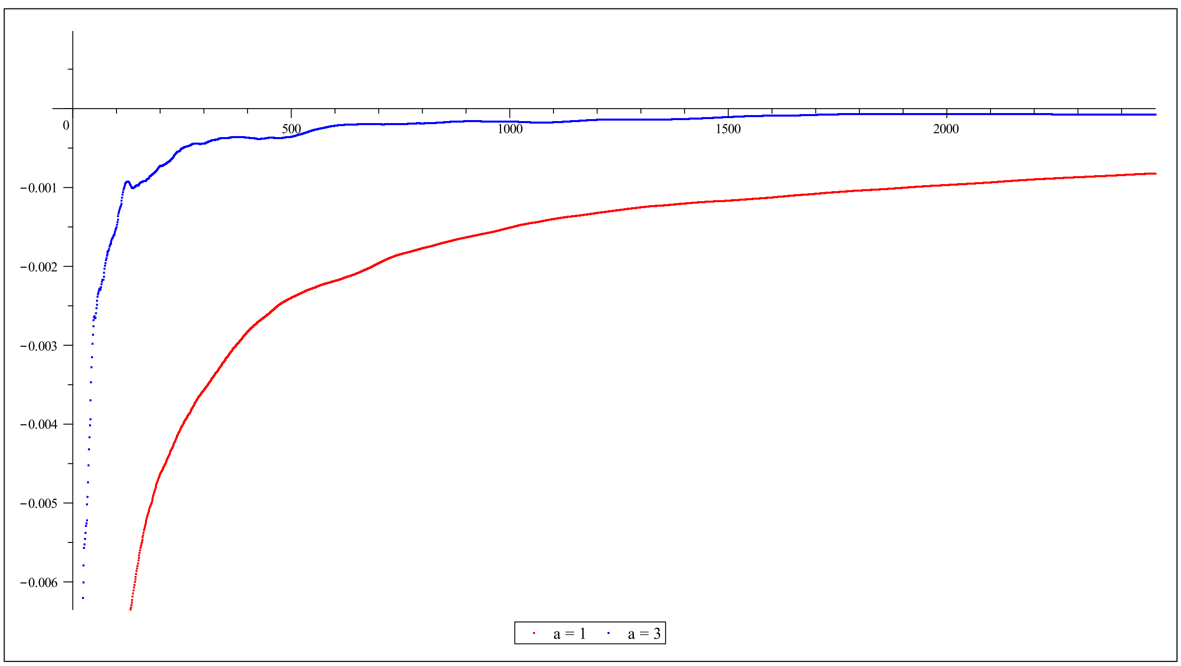

Figure 4.4: Plot of $\log f(\overline{p}_k; 4, a)$.

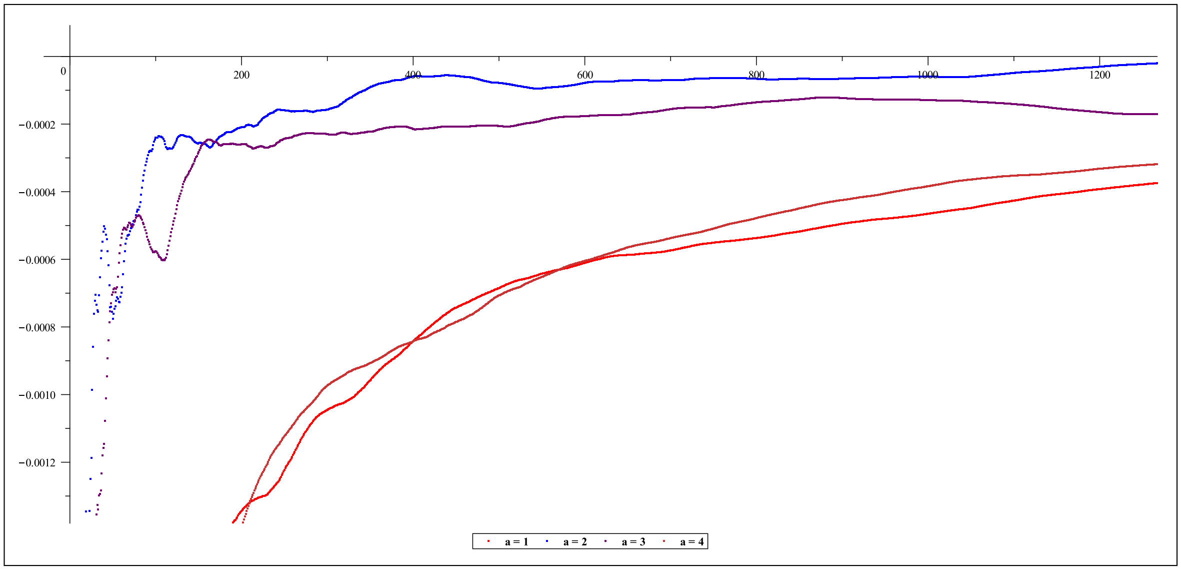

Figure 4.5: Plot of $\log f(\overline{p}_k; 5, a)$.

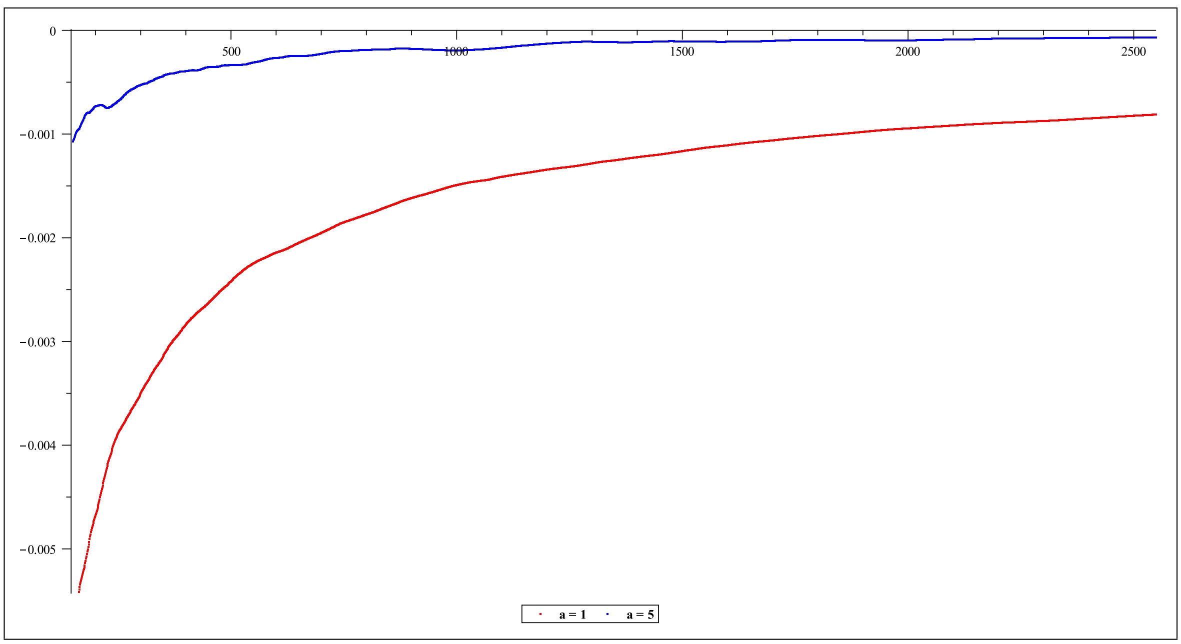

Figure 4.6: Plot of $\log f(\overline{p}_k; 6, a)$.

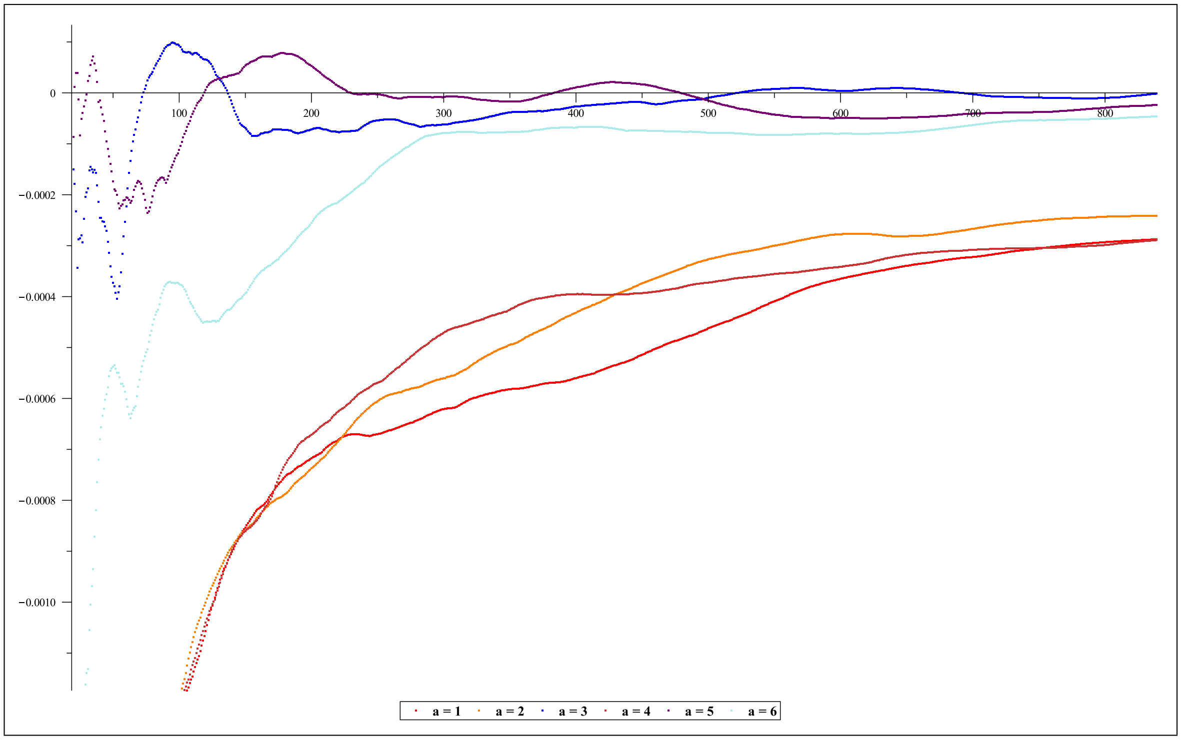

Figure 4.7: Plot of $\log f(\overline{p}_k;7,a)$. Here we observe that $\log f(x;7,a)$ is not always negative. However this new behavior is only observed in cases where a is not a square $\pmod q$.

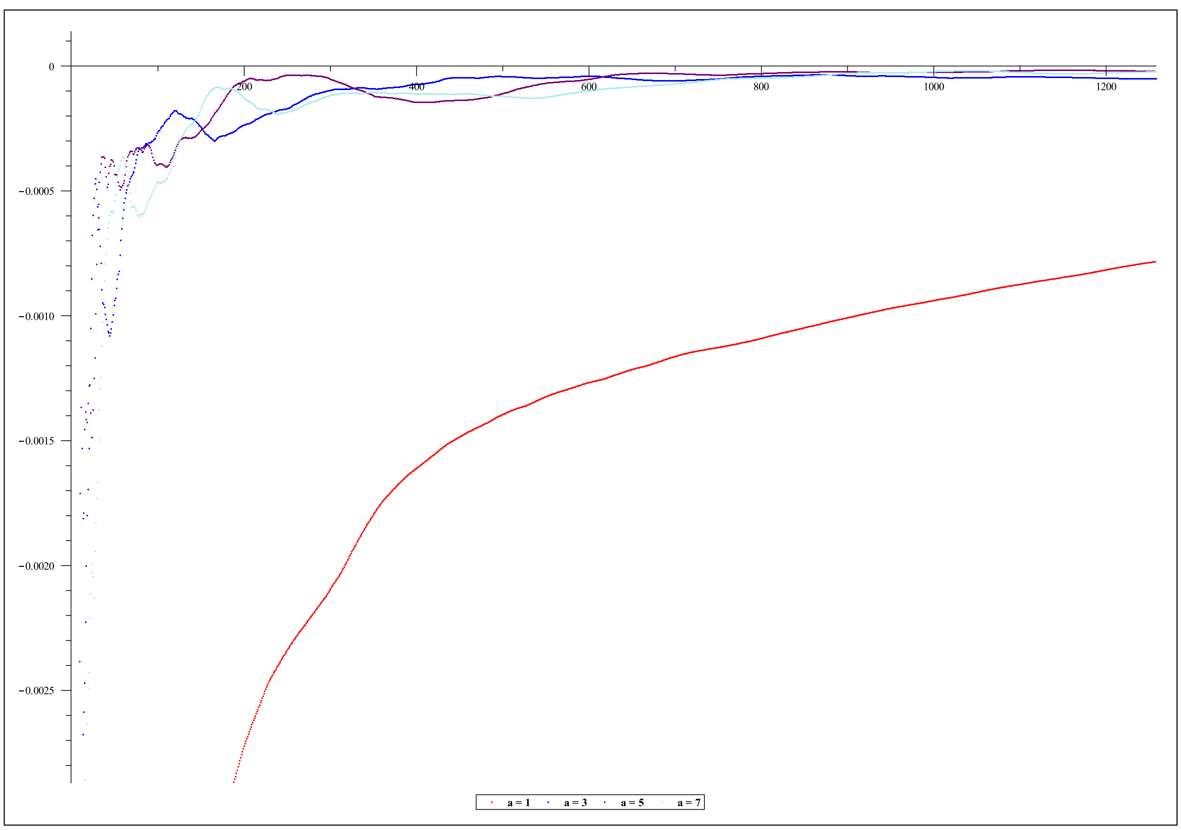

Figure 4.8: Plot of $\log f(\overline{p}_k;8,a)$.

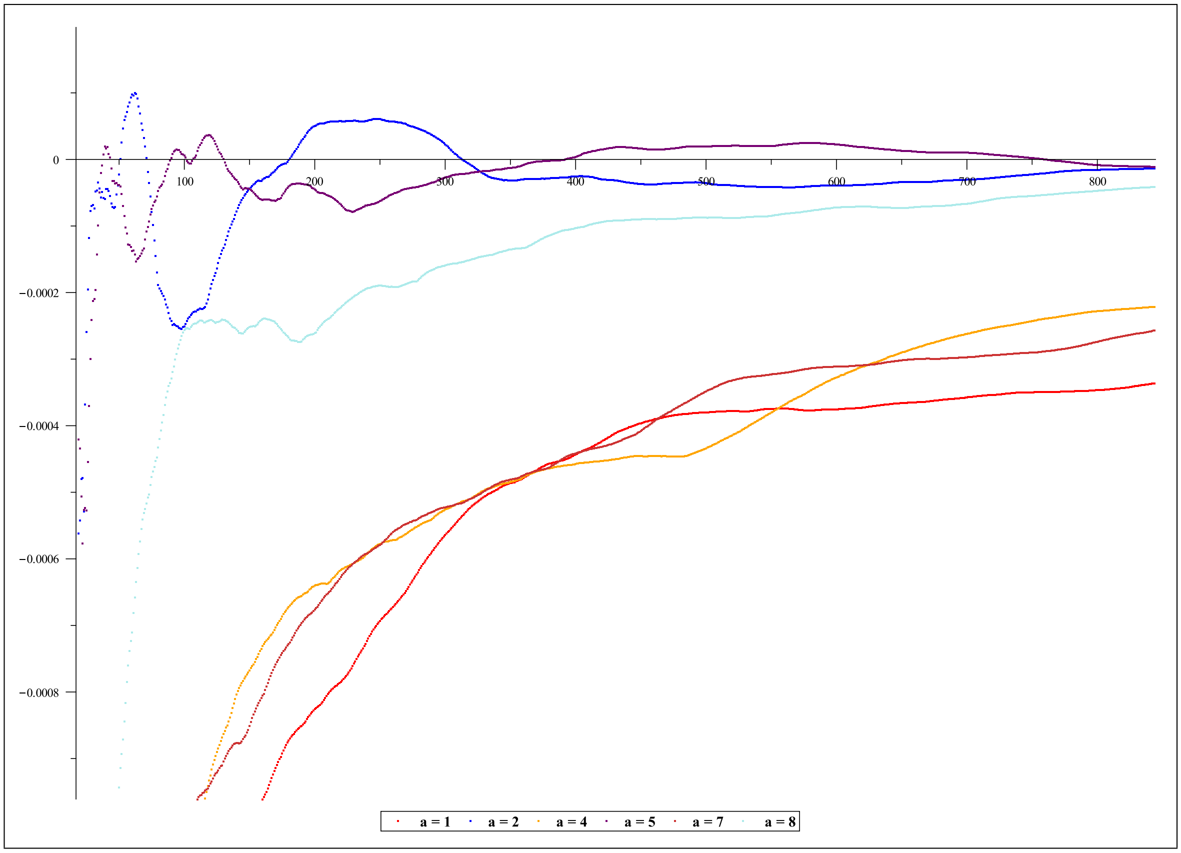

Figure 4.9: Plot of $\log f(\overline{p}_k; 9, a)$.

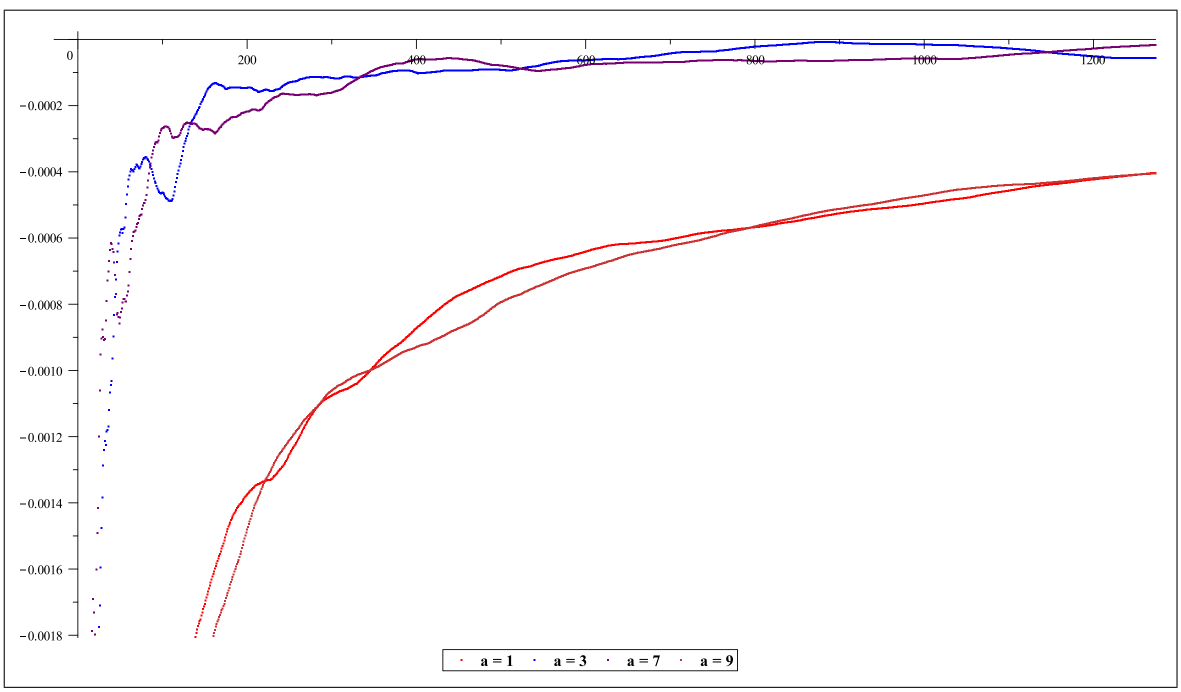

Figure 4.10: Plot of $\log f(\overline{p}_k; 10, a)$.

4.3 Bounds for $\log f(x;q,a)$

To establish an initial bound for $\log f(x;q,a)$, it is beneficial to develop the following notations. First, let

$$g(x) = -\frac{d^2}{dx^2}(\log\log x) = \frac{1+\log x}{x^2\log^2 x}.$$

Second, consider, for a given arithmetic progression, the *error term in the prime number theorem*, which will be denoted

$$S(x;q,a) = \theta(x;q,a) - \frac{x}{\varphi(q)}.$$

These two notations together allow us to define

$$K(x;q,a) = \int_x^\infty S(t;q,a)g(t)\,dt.$$

The next proposition is analogous to [17, Proposition 1].

Proposition 4.1. *Let q and a be coprime natural numbers. For all x for which $\theta(x;q,a) > \frac{4x}{5\varphi(q)}$, we have*

$$K(x;q,a) - \frac{(\varphi(q)S(x;q,a))^2(\log(\frac{4x}{5})+1)}{2\varphi(q)(\frac{4x}{5})^2\log^2(\frac{4x}{5})} \le \log f(x;q,a) \le K(x;q,a) + \frac{1}{2(x-1)}.$$

Proof. Throughout, let us presume an arithmetic progression $a \pmod q$, denoting the k-th prime in the progression by $\overline{p}_k$. Consider the mean value theorem applied to $\log\log t$ on the interval $[m,n]$, with $m = \min(x, \varphi(q)\theta(x;q,a))$ and $n = \max(x, \varphi(q)\theta(x;q,a))$. Then, there exists c_1 between x and $\varphi(q)\theta(x;q,a)$ for which

$$\log\log(\varphi(q)\theta(x;q,a)) = \log\log x + \frac{\varphi(q)S(x;q,a)}{c_1\log c_1} \tag{4.2}$$

Likewise, applying Taylor's theorem to $\log\log t$, centered at x and evaluated at $\varphi(q)\theta(x;q,a)$, there exists c_2 between x and $\varphi(q)\theta(x;q,a)$ for which

$$\log\log(\varphi(q)\theta(x;q,a)) = \log\log x + \frac{\varphi(q)S(x;q,a)}{x\log x} - \frac{(\varphi(q)S(x;q,a))^2(\log c_2+1)}{2c_2^2\log^2 c_2}. \quad (4.3)$$

In (4.2), observe that for both $\varphi(q)\theta(x;q,a) < c_1 < x$ and $x \le c_1 \le \varphi(q)\theta(x;q,a)$, we have

$$\frac{\varphi(q)S(x;q,a)}{c_1\log c_1} \le \frac{\varphi(q)S(x;q,a)}{x\log x}$$

and therefore

$$\log\log(\varphi(q)\theta(x;q,a)) \le \log\log x + \frac{\varphi(q)S(x;q,a)}{x\log x}. \quad (4.4)$$

In (4.3), Theorem 3.7, tells us that for large enough x, we will have $\varphi(q)\theta(x;q,a) > \frac{4x}{5}$, and therefore $c_2 > \frac{4x}{5}$ for such x. Observing that

$$-\frac{\log c_2+1}{2c_2^2\log^2 c_2}$$

is an increasing function of c_2, we have

$$\log\log(\varphi(q)\theta(x;q,a)) \ge \log\log x + \frac{\varphi(q)S(x;q,a)}{x\log x} - \frac{(\varphi(q)S(x;q,a))^2(\log\frac{4x}{5}+1)}{2(\frac{4x}{5})^2\log^2\frac{4x}{5}}. \quad (4.5)$$

By partial summation

$$\begin{aligned}\sum_{\substack{p\le x\\ p\equiv a \pmod q}} \frac{1}{p} &= \sum_{\substack{p\le x\\ p\equiv a \pmod q}} \frac{1}{p\log p}\log p\\ &= \frac{\theta(x;q,a)}{x\log x} + \int_{\overline{p}_1}^{x} \theta(t;q,a)g(t)\,\mathrm{d}t.\end{aligned}$$

With the substitution $\theta(t;q,a) = S(t;q,a) + \frac{t}{\varphi(q)}$, we obtain

$$\begin{aligned}\sum_{\substack{p\le x\\ p\equiv a \pmod q}} \frac{1}{p} &= \frac{S(x;q,a)}{x\log x} + \frac{1}{\varphi(q)\log x} + \int_{\overline{p}_1}^{x} S(t;q,a)g(t)\,\mathrm{d}t + \frac{1}{\varphi(q)}\int_{\overline{p}_1}^{x}\frac{1}{t}\frac{\log t+1}{\log^2 t}\,\mathrm{d}t\\ &= \frac{S(x;q,a)}{x\log x} + \frac{\log\log x}{\varphi(q)} - \int_{x}^{\infty} S(t;q,a)g(t)\,\mathrm{d}t\\ &\quad + \int_{\overline{p}_1}^{\infty} S(t;q,a)g(t)\,\mathrm{d}t + \frac{1}{\varphi(q)\log\overline{p}_1} - \frac{\log\log\overline{p}_1}{\varphi(q)}. \end{aligned} \tag{4.6}$$

Hence, we may write (4.6) as

$$\sum_{\substack{p\le x\\ p\equiv a \pmod q}} \frac{1}{p} = \frac{S(x;q,a)}{x\log x} + \frac{\log\log x}{\varphi(q)} - K(x;q,a) + M(q,a), \tag{4.7}$$

where

$$M(q,a) = \int_{\overline{p}_1}^{\infty} S(t;q,a)g(t)\,\mathrm{d}t + \frac{1}{\varphi(q)\log\overline{p}_1} - \frac{\log\log\overline{p}_1}{\varphi(q)}$$

is the constant term. By comparison with Mertens' second theorem for arithmetic progressions ([9, (1-1)]), we see that this constant has the useful expression [9, (1-3)].

$$M(q,a) = \sum_{p\equiv a \pmod q}\left\{\log\left(1-\frac{1}{p}\right)+\frac{1}{p}\right\} - \log C(q,a). \tag{4.8}$$

Now it can be readily verified, by (4.8), that

$$\begin{aligned}&\varphi(q)\log f(x;q,a)\\ &= \log\log(\varphi(q)\theta(x;q,a)) + \varphi(q)\sum_{\substack{p\le x\\ p\equiv a \pmod q}}\log\left(1-\frac{1}{p}\right) - \varphi(q)\log C(q,a)\\ &= \overline{U}(x) + \underline{u}(x),\end{aligned} \tag{4.9}$$

where

$$\overline{U}(x) = \log\log(\varphi(q)\theta(x;q,a)) - \sum_{\substack{p\le x\\ p\equiv a \pmod q}}\frac{\varphi(q)}{p} + \varphi(q)M(q,a) \tag{4.10}$$

and

$$\begin{aligned}
\underline{u}(x) &= \varphi(q) \sum_{\substack{p\le x \\ p\equiv a \pmod q}} \log\left(1-\frac{1}{p}\right) + \sum_{\substack{p\le x \\ p\equiv a \pmod q}} \frac{\varphi(q)}{p} - \varphi(q)\log C(q,a) - \varphi(q)M(q,a) \\
&= \varphi(q) \sum_{\substack{p> x \\ p\equiv a \pmod q}} -\left\{\log\left(1-\frac{1}{p}\right)+\frac{1}{p}\right\}.
\end{aligned} \tag{4.11}$$

By expanding the latter expression for $\underline{u}$ in (4.11) and crudely bounding it by a geometric series over all natural numbers, we see that

$$0 < \underline{u}(x) \le \frac{\varphi(q)}{2(x-1)}. \tag{4.12}$$

Now, we have for $x \ge \overline{p}_1$ that

$$\varphi(q)\log f(x;q,a) = \log\log(\varphi(q)\theta(x;q,a)) - \sum_{\substack{p\le x \\ p\equiv a \pmod q}} \frac{1}{p} + \varphi(q)M(q,a) + \underline{u}(x)$$

by replacing $\overline{U}(x)$ from (4.10) in (4.9). Substituting equation (4.7) for the series in the above equation yields

$$\begin{aligned}
\varphi(q)\log f(x;q,a) &= \log\log(\varphi(q)\theta(x;q,a)) - \frac{\varphi(q)S(x;q,a)}{x\log(x)} - \log\log x \\
&\quad + \varphi(q)K(x;q,a) - \varphi(q)M(q,a) + \varphi(q)M(q,a) + \underline{u}(x). \\
&= \left(\log\log(\varphi(q)\theta(x;q,a)) - \frac{\varphi(q)S(x;q,a)}{x\log(x)} - \log\log x\right) \\
&\quad + \varphi(q)K(x;q,a) + \underline{u}(x).
\end{aligned} \tag{4.13}$$

The bracketed expression in (4.13) is bounded above by 0, as established in (4.4). Hence, by (4.12)

$$\varphi(q)\log f(x;q,a) \le \varphi(q)K(x;q,a) + \underline{u}(x) \le \varphi(q)K(x;q,a) + \frac{\varphi(q)}{2(x-1)}. \tag{4.14}$$

On the other hand, for x satisfying $\varphi(q)\theta(x;q,a) > \frac{4x}{5}$, the bracketed expression is bounded below by

$$-\frac{(\varphi(q)S(x;q,a))^2(\log\frac{4x}{5}+1)}{2(\frac{4x}{5})^2\log^2\frac{4x}{5}}$$

as established in (4.5). Therefore,

$$\begin{aligned}\varphi(q)\log f(x;q,a) &\geq \varphi(q)K(x;q,a)+\underline{u}(x)-\frac{(\varphi(q)S(x;q,a))^2(\log\frac{4x}{5}+1)}{2(\frac{4x}{5})^2\log^2\frac{4x}{5}}\\ &\geq \varphi(q)K(x;q,a)-\frac{(\varphi(q)S(x;q,a))^2(\log\frac{4x}{5}+1)}{2(\frac{4x}{5})^2\log^2\frac{4x}{5}}.\end{aligned} \tag{4.15}$$

Dividing both sides of (4.14) and (4.15) by $\varphi(q)$ gives the stated result. □

Throughout this chapter and Chapter 6, only the upper bound for $\log f(x;q,a)$ will be employed, whereas the lower bound will find its use in Chapter 5.

We would at this point recall the following definition.

Definition 4.2. Let $q,a \in \mathbb{N}$ be coprime. The *second Chebyshev function* will be denoted $\psi(x;q,a)$ and be defined by

$$\psi(x;q,a) = \sum_{\substack{p^k\leq x\\ p^k\equiv a \pmod q}} \log p.$$

That is, where $\theta(x;q,a)$ takes a weighted sum over primes, $\psi(x;q,a)$ takes a weighted sum over prime *powers*. It follows from these definitions that $\theta(x;q,a) \leq \psi(x;q,a)$. Consequently, we wish to consider the ψ-analogues of $S(x;q,a)$ and $K(x;q,a)$. We can denote

$$R(x;q,a) = \psi(x;q,a) - \frac{x}{\varphi(q)}$$

and

$$J(x;q,a) = \int_x^\infty R(t;q,a)g(t)\,\mathrm{d}t.$$

Observe that $K(x;q,a) \leq J(x;q,a)$. This observation is prudent, in part, because we have available to us the *explicit formula for* $\psi(x;q,a)$ [12, Lemma 3.1, p. 271], which will allow us to use information regarding the zeroes of Dirichlet L-functions to better estimate the upper bounds on $\log f(x;q,a)$, especially when we assume Conjecture 3.15 in Chapter 6.

Consider the following notations, from which a stronger estimate than $K(x;q,a) \leq J(x;q,a)$ will follow.

Definition 4.3. Let $q,a \in \mathbb{N}$ be coprime. We denote by $\mathrm{Ind}_q(a)$ the *index of* $a \pmod q$, the least natural number $m > 1$ for which a is an m-th power modulo q. Furthermore, we denote by $\mathcal{R}_{q,a}$ the number of $\mathrm{Ind}_q(a)$-th "roots" of a, by which we mean

$$\mathcal{R}_{q,a} = \#\{b \in \mathbb{Z}_q^\times \mid b^{\mathrm{Ind}_q(a)} \equiv a \pmod q\}.$$

Since q and a are coprime, we note that $a^{\varphi(q)+1} \equiv a \pmod q$ by the Fermat-Euler theorem. Therefore, $2 \leq \mathrm{Ind}_q(a) \leq \varphi(q)+1$, and $\mathrm{Ind}_q(a)$ is well-defined.

It will be useful in some circumstances to have a closed form for $\mathcal{R}_{q,a}$.

Proposition 4.4. *Let $q,a \in \mathbb{N}$ be coprime. Write $q = 2^\alpha p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$, where p_i are the distinct odd prime divisors of q. Let $m = \mathrm{Ind}_q(a)$. We have*

$$\mathcal{R}_{q,a} = \begin{cases} \prod_{i=1}^r \left(m, \varphi(p_i^{\alpha_i})\right) & \text{if } \alpha \leq 1, \\ (m,2)\left(m,2^{\alpha-2}\right)\prod_{i=1}^r \left(m, \varphi(p_i^{\alpha_i})\right) & \text{otherwise.} \end{cases}$$

Proof. Let $q = 2^\alpha p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$ and $m = \mathrm{Ind}_q(a)$.Then, $x^m \equiv a \pmod q$ has a solution. In fact, by [10, Theorem 3.21], we know that its solutions are in 1-1 correspondence with the solutions of the system

$$\begin{cases} x^m \equiv a \pmod{2^\alpha} \\ x^m \equiv a \pmod{p_1^{\alpha_1}} \\ \vdots \\ x^m \equiv a \pmod{p_r^{\alpha_r}} \end{cases}$$

For each odd prime p_i, [10, Theorem 4.13] establishes that there are $\left(m,\varphi(p_i^{\alpha_i})\right)$ solutions to each congruence $x^m \equiv a \pmod{p_i^{\alpha_i}}$. On the other hand, the congruence $x^m \equiv a \pmod{2^\alpha}$ has 1 solution if $\alpha = 1$ and $(m,2)$ solutions if $\alpha = 2$, again by [10, Theorem 4.13]. If $\alpha \geq 3$, then $x^m \equiv a \pmod{2^\alpha}$ has $(m,2)\cdot(m,2^{\alpha-2})$ solutions via [10, Theorem 4.14]. The formula for $\mathcal{R}_{q,a}$ follows by taking the product of the number of solutions as we range over congruences corresponding to prime divisors of q.

□

In particular, the cases where $\mathrm{Ind}_q(a) = 2$ are relevant in Chapter 6.

Corollary 4.5. *Fix $q \in \mathbb{N}$ and let a be coprime to q such that $\mathrm{Ind}_q(a) = 2$. Then*

$$\mathcal{R}_{q,a} = \begin{cases} 2^{\omega(q)-1} & \text{if } 2\|q, \\ 2^{\omega(q)} & \text{if } 2\nmid q \text{ or } 4\|q, \\ 2^{\omega(q)+1} & \text{if } 8|q, \end{cases}$$

where $\omega(q)$ is the number of distinct prime divisors of q.

Proof. This follows immediately from Proposition 4.4 upon observing that, for any odd prime, $\left(2,\varphi(p_i^{\alpha_i})\right) = 2$ and $(2,2)\cdot\left(2,2^{\alpha-2}\right)$ is 2 when $4\|q$ and is 4 when $8|q$. □

Example 4.6. One may see that $\mathrm{Ind}_3(1) = 2$ since $1^2 \equiv 2^2 \equiv 1 \pmod 3$, but $\mathrm{Ind}_3(2) = 3$, since $2^3 \equiv 2 \pmod 3$ and no squares are congruent to $2 \pmod 3$.

This definition allows us to capture an important distinction between $\theta(x;q,a)$ and $\psi(x;q,a)$. Namely, that the congruence class of a prime is not necessarily the same as the congruence class of the powers of that prime. We wish to remove as much of the discrepancy between $\theta(x;q,a)$ and $\psi(x;q,a)$ in order to more tightly bound $K(x;q,a)$ above. To this end we have the following proposition.

Proposition 4.7. *Let $q, a \in \mathbb{N}$ be coprime. Then, for $0 < \varepsilon < 1/\varphi(q)$, there exists x_0 such that for all $x > x_0$,*

$$\theta(x;q,a) \leq \psi(x;q,a) - cx^{\frac{1}{\mathrm{Ind}_q(a)}}$$

where

$$c = \mathcal{R}_{q,a}\left(\frac{1}{\varphi(q)} - \varepsilon\right).$$

Proof. Consider:

$$\psi(x;q,a) - \theta(x;q,a) = \sum_{k=2}^{\infty} \sum_{\substack{p^k \leq x \\ p^k \equiv a \pmod q}} \log p.$$

We wish to break up the inner sum into sums over fixed residue classes modulo q for p. That is,

$$\psi(x;q,a) - \theta(x;q,a) = \sum_{k=2}^{\infty} \sum_{b \in \mathbb{Z}_q^\times} \sum_{\substack{p^k \leq x \\ p^k \equiv a \pmod q \\ p \equiv b \pmod q}} \log p, \tag{4.16}$$

where the innermost sums may be empty. However, when $k = \mathrm{Ind}_q(a)$, there must be at least one $b \in \mathbb{Z}_q^\times$ for which $b^{\mathrm{Ind}_q(a)} \equiv a \pmod q$. Let $Q = \{b \in \mathbb{Z}_q^\times \mid b^{\mathrm{Ind}_q(a)} \equiv a \pmod q\}$. Since all of the terms on the right of (4.16) are non-negative, we may ignore all of them except for those corresponding to $k = \mathrm{Ind}_q(a)$ and $b \in Q$ to obtain

$$\psi(x;q,a) - \theta(x;q,a) \geq \sum_{b \in Q} \sum_{\substack{p^{\mathrm{Ind}_q(a)} \leq x \\ p^{\mathrm{Ind}_q(a)} \equiv a \pmod q \\ p \equiv b \pmod q}} \log p = \sum_{b \in Q} \sum_{\substack{p^{\mathrm{Ind}_q(a)} \leq x \\ p \equiv b \pmod q}} \log p.$$

Isolating $\theta(x;q,a)$ in the above inequality, we determine

$$\theta(x;q,a) \leq \psi(x;q,a) - \sum_{b \in Q} \theta(x^{\frac{1}{\mathrm{Ind}_q(a)}};q,b).$$

Now, we know by Theorem 3.7 that upon fixing $0 < \varepsilon < \frac{1}{\varphi(q)}$, there exists, for each $b \in Q$, an x_b so that for all $x > x_b$,

$$\left(\frac{1}{\varphi(q)} - \varepsilon\right) x^{\frac{1}{\mathrm{Ind}_q(a)}} < \theta(x^{\frac{1}{\mathrm{Ind}_q(a)}}; q, b) < \left(\frac{1}{\varphi(q)} + \varepsilon\right) x^{\frac{1}{\mathrm{Ind}_q(a)}}.$$

Upon observing that, by definition, $|Q| = \mathcal{R}_{q,a}$ and taking $c = \mathcal{R}_{q,a}\left(\frac{1}{\varphi(q)} - \varepsilon\right)$, we have, for $x > x_0 = \max x_b$,

$$\theta(x; q, a) \leq \psi(x; q, a) - c x^{\frac{1}{\mathrm{Ind}_q(a)}}$$

as desired. □

Explicit evaluations of x_0 and c corresponding to q and a for a given ε can be found, for example, in Ramaré and Rumely [20].

We may consider integrating over x^s in a fashion akin to $J(x;q,a)$ and $K(x;q,a)$ to obtain

$$F_s(x) = \int_x^\infty t^s g(t)\,\mathrm{d}t.$$

Proposition 4.7 and Proposition 4.1 provide the following upper bound for $\log f(x;q,a)$.

Corollary 4.8. *Let $q, a \in \mathbb{N}$ be coprime and $m = \mathrm{Ind}_q(a)$. Then, for $0 < \varepsilon < 1/\varphi(q)$, there exist x_0, c as given in Proposition 4.7 for which, when $x > x_0$,*

$$\log f(x;q,a) \leq J(x;q,a) - cF_{\frac{1}{m}}(x) + \frac{1}{2(x-1)},$$

where, for a complex number s,

$$F_s(x) = \int_x^\infty t^s g(t)\,\mathrm{d}t.$$

Nicolas discusses the function $F_s(x)$ for $\Re(s) = 1/2$ in [17], and later for $\Re(s) < 1$ in [18]. Since our interest is when $\Re(s) = 1/m$ for $m \in \mathbb{N}$, we can adapt [17, Lemma 1] and

[18, Lemma 2.2] to estimate $F_{\frac{1}{m}+it}(x)$. The first assertion here is Lemma 2.2 of [18].

Lemma 4.9. *Let s be a complex number such that $\Re(s) < 1$. Then, for $x > 1$,*

$$F_s(x) = -\frac{x^{s-1}}{(s-1)\log x} + r_s(x),$$

where

$$r_s(x) = -\frac{s}{1-s}\left(\frac{x^{s-1}}{(1-s)\log^2 x} + \int_x^\infty \frac{2t^{s-2}}{(s-1)\log^3 x}\,\mathrm{d}t\right).$$

Moreover, if $s = \frac{1}{m} + it$, for an integer $m > 1$, we have

$$|r_s(x)| \le \left|\frac{s}{(1-s)^2}\right|\left(\frac{x^{\frac{1-m}{m}}}{\log^2 x}\right)\left(1 + \frac{2}{|s-1|\log x}\right).$$

Proof. For the first statement, we reproduce the proof given in Lemma 2.2 of [18]. An application of integration by parts yields

$$F_s(x) = -\frac{x^{s-1}}{(s-1)\log x} - \frac{x^{s-1}}{(s-1)\log^2 x} + \int_x^\infty \frac{t^{s-2}}{s-1}\cdot\left(\frac{1}{\log^2 t} + \frac{2}{\log^3 t}\right)\mathrm{d}t.$$

Now write

$$F_s(x) = -\frac{x^{s-1}}{(s-1)\log x} + r_s(x),$$

where

$$r_s(x) = -\frac{x^{s-1}}{(s-1)\log^2 x} + \int_x^\infty \frac{t^{s-2}}{s-1}\cdot\left(\frac{1}{\log^2 t} + \frac{2}{\log^3 t}\right)\mathrm{d}t.$$

Observing that

$$\int_x^\infty \frac{t^{s-2}}{(s-1)\log^2 t}\,\mathrm{d}t = -\frac{x^{s-1}}{(s-1)^2\log^2 x} + \int_x^\infty \frac{2t^{s-2}}{(s-1)\log^3 t}\,\mathrm{d}t$$

then

$$r_s(x) = -\frac{s}{(1-s)}\left(\frac{x^{s-1}}{(1-s)\log^2 x} + \int_x^\infty \frac{2t^{s-2}}{\log^3 t}\,\mathrm{d}t\right).$$

Now, letting $s = \frac{1}{m} + it$, taking the absolute value of $r_s(x)$ yields

$$\begin{aligned}
|r_s(x)| &= \left|\frac{s}{(1-s)}\right| \left|\frac{x^{s-1}}{(1-s)\log^2 x} + \int_x^{\infty} \frac{2t^{s-2}}{(s-1)\log^3 t}\,\mathrm{d}t\right| \\
&\leq \left|\frac{s}{(1-s)}\right| \left(\frac{x^{\frac{1-m}{m}}}{|1-s|\log^2 x} + \frac{2}{|s-1|\log^3 x}\left|\frac{x^{s-1}}{s-1}\right|\right) \\
&= \left|\frac{s}{(1-s)^2}\right| \left(\frac{x^{\frac{1-m}{m}}}{\log^2 x}\right)\left(1 + \frac{2}{|s-1|\log x}\right).
\end{aligned}$$

□

For $s = \frac{1}{m}$, we immediately obtain the following result.

Corollary 4.10. *Let $m > 1$. Then for $x > 1$,*

$$F_{\frac{1}{m}}(x) = \frac{m}{(m-1)x^{\frac{m-1}{m}}\log x} + r_{\frac{1}{m}}(x)$$

where

$$|r_{\frac{1}{m}}(x)| \leq \left(\frac{m}{(m-1)^2 x^{\frac{m-1}{m}}\log^2 x}\right)\left(1 + \frac{2m}{(m-1)\log x}\right).$$

Moreover,

$$F_{\frac{1}{m}}(x) \geq \frac{m+1}{m x^{\frac{m-1}{m}}\log x},$$

whenever $x > e^{\frac{m^2}{m-1}}$.

Proof. The first claim follows immediately from Lemma 4.9 with $s = \frac{1}{m}$. The second claim follows from the first, as we have

$$\begin{aligned}
F_{\frac{1}{m}}(x) &\geq \frac{m}{(m-1)x^{\frac{m-1}{m}}\log x} - \frac{m}{(m-1)^2 x^{\frac{m-1}{m}}\log^2 x}\left(1 + \frac{2m}{(m-1)\log x}\right) \\
&= \frac{m}{(m-1)x^{\frac{m-1}{m}}\log x}\left(1 - \frac{1}{(m-1)\log x}\right)\left(1 + \frac{2m}{(m-1)\log x}\right) \\
&\geq \frac{m}{(m-1)x^{\frac{m-1}{m}}\log x}\left(1 - \frac{1}{(m-1)\log x}\right) \cdot 1.
\end{aligned}$$

Then, since $\frac{m}{m-1}(1-\frac{1}{(m-1)\log x})$ is increasing as a function of x and takes the value $\frac{m+1}{m}$ when $x = e^{\frac{m^2}{m-1}}$, we are done. □

The preceding discussion allows us to make the following important observation.

Theorem 4.11. *Fix q and a coprime, and let $m = \mathrm{Ind}_q(a)$. Choose ε satisfying $0 < \varepsilon < \frac{m+1}{(m+2)\varphi(q)}$, for which we have c, x_0 arising from Corollary 4.8. There exists x_1 so that for all $x > x_1$, we have*

$$\frac{\frac{m+1}{m}c - \frac{\varepsilon}{m}}{x^{\frac{m-1}{m}}\log x} > \frac{1}{2(x-1)}.$$

Then, for $x > \max\{e^{\frac{m^2}{m-1}}, x_0, x_1\}$, we have

$$\log f(x;q,a) < J(x;q,a).$$

Proof. For the relevant x, Corollary 4.8 provides

$$\log f(x;q,a) \leq J(x;q,a) - cF_{\frac{1}{m}}(x) + \frac{1}{2(x-1)}.$$

Note that x_1 exists since $x^{\frac{m-1}{m}}\log x = O(x^{1-\alpha})$ for an $\alpha > 0$ and

$$\frac{m+1}{m}c - \frac{\varepsilon}{m} > 0.$$

Furthermore, since $x > e^{\frac{m^2}{m-1}}$, we may apply Corollary 4.10 yielding

$$-cF_{\frac{1}{m}}(x) + \frac{1}{2(x-1)} < -\frac{\frac{m+1}{m}c}{x^{\frac{m-1}{m}}\log x} + \frac{\frac{m+1}{m}c - \frac{\varepsilon}{m}}{x^{\frac{m-1}{m}}\log x} = -\frac{\varepsilon}{mx^{\frac{m-1}{m}}\log x},$$

which is negative since $\varepsilon > 0$. This implies

$$J(x;q,a) - cF_{\frac{1}{m}}(x) + \frac{1}{2(x-1)} < J(x;q,a),$$

for x as large as specified. □

4.4 The Oscillation of $J(x;q,a)$

Having established Theorem 4.11, a natural approach towards answering the analogous question of Rosser and Schoenfeld (Question 3) is apparent. Fix an arithmetic progression $a \pmod q$. Given x such that $\overline{p}_k \le x < \overline{p}_{k+1}$, if we have $J(x;q,a) < 0$, we obtain by way of Theorem 4.11 that, for large x, $\log f(x;q,a) < J(x;q,a)$, and therefore $\mathcal{N}_{q,a}(k)$ holds. Thus, to establish that there are infinitely many elements of $S_{q,a}$ satisfying (3.4), all that is needed is to show that $J(x;q,a)$ does not stay strictly greater than 0 on $[x',\infty)$ for any x'. Such a claim can be established with the help of an oscillation theorem, courtesy of Landau.

Theorem 4.12 (**Landau's Oscillation Theorem**). *Let $h:[1,\infty)\to\mathbb{R}$ be a function which is bounded and Riemann-integrable on intervals of the form $[1,T]$, $1<T<\infty$. Consider the integral*

$$H(s) = \int_1^\infty \frac{h(x)}{x^s}\,dx.$$

Suppose that the line $\Re(s) = \sigma_0$ is the line of convergence for H, and the function $h(x)$ is of constant sign on an interval of the form $[x',\infty)$. Then the real point $s=\sigma_0$ on the line of convergence must be a singularity of $H(s)$.

Proof. See [5, Theorem H]. □

With this tool in hand, we are in position to prove that $J(x;q,a)$ changes sign infinitely often. Given the discussion which opened this section, we prove the following result.

Theorem 4.13. *Let q and a be two coprime integers. There are infinitely many primorials $\overline{N}_k$ in $S_{q,a}$ for which*

$$\frac{\overline{N}_k}{\varphi(\overline{N}_k)(\log(\varphi(q)\log\overline{N}_k))^{\frac{1}{\varphi(q)}}} > \frac{1}{C(q,a)},$$

where $C(q,a)$ is the constant in Theorem 3.5, provided that

(i) The function

$$\mathscr{L}(s;q,a) = \sum_{\chi \pmod q} \overline{\chi}(a)\frac{L'}{L}(s,\chi)$$

has no singularities on the line segment $(0,1)$.

(ii) $\mathscr{L}(s;q,a)$ *has a singularity in the strip* $0<\Re(s)<1$.

Proof. Let q and a be two coprime integers. We will establish that $J(x;q,a)$ changes sign infinitely often by an appeal to Theorem 4.12. Then Theorem 4.11 confirms the claim of this theorem.

Write

$$h(x;q,a)=\begin{cases} J(x;q,a) & \text{if } x\geq \overline{p}_1 \\ 0 & \text{if } 1\leq x<\overline{p}_1 \end{cases}.$$

First, we aim to prove that

$$H(s)=\int_1^\infty \frac{h(x;q,a)}{x^s}\,\mathrm{d}t=\int_{\overline{p}_1}^\infty \frac{J(x;q,a)}{x^s}\,\mathrm{d}t=\int_{\overline{p}_1}^\infty \frac{1}{x^s}\int_x^\infty \frac{R(t;q,a)}{t^2}\frac{(\log t+1)}{\log^2 t}\,\mathrm{d}t\mathrm{d}x, \quad (4.17)$$

defined for $\Re(s)>1$, extends to a function without singularity for $0<\Re(s)\leq 1$.

The variant of Theorem 3.7 which is applicable for $\psi(x;q,a)$ [14, pp. 379-381] establishes that

$$|R(t;q,a)|=\left|\psi(t;q,a)-\frac{t}{\varphi(q)}\right|=O\left(\frac{t}{\log t}\right)$$

and hence the inner integral in (4.17) is absolutely convergent. Therefore, we can change the order of integration to obtain

$$\begin{aligned} H(s)&=\int_{\overline{p}_1}^\infty \frac{R(t;q,a)}{t^2}\frac{(\log t+1)}{\log^2 t}\left(\int_{\overline{p}_1}^t \frac{1}{x^s}\mathrm{d}x\right)\mathrm{d}t \\ &=\int_{\overline{p}_1}^\infty \frac{R(t;q,a)}{t^2}\frac{(\log t+1)}{\log^2 t}\left(\frac{(\overline{p}_1)^{1-s}-t^{1-s}}{s-1}\right)\mathrm{d}t \\ &=\frac{1}{s-1}\left((\overline{p}_1)^{1-s}J(\overline{p}_1;q,a)-\int_{\overline{p}_1}^\infty \frac{R(t;q,a)}{t^{s+1}}\frac{(\log t+1)}{\log^2 t}\mathrm{d}t\right). \end{aligned} \quad (4.18)$$

It will become important later in the proof to have observed that, when $s=1$, the last bracketed expression is $J(\overline{p}_1;q,a)-J(\overline{p}_1;q,a)=0$.

Now we turn our attention to rewriting

$$\int_{\overline{p}_1}^{\infty} \frac{R(t;q,a)}{t^{s+1}} \frac{(\log t + 1)}{\log^2 t} \,\mathrm{d}t$$

in a form more suitable to our purpose. It follows from Theorem 12 of [21] that

$$\psi(x;q,a) \le \psi(x) = O(x).$$

The above, coupled with the definition

$$\psi(x;q,a) = \sum_{\substack{n \le x \\ n \equiv a \ (\mathrm{mod}\ q)}} \Lambda(n),$$

where $\Lambda(n)$ is the *von Mangoldt function*, provides (by way of Exercise 2.1.5 of [15])

$$\sum_{n \equiv a \ (\mathrm{mod}\ q)} \frac{\Lambda(n)}{n^s} = s \int_{\overline{p}_1}^{\infty} \frac{\psi(t;q,a)}{t^{s+1}} \,\mathrm{d}t,$$

where the equality holds for $\Re(s) > 1$. The identity

$$\sum_{n \equiv a \ (\mathrm{mod}\ q)} \frac{\Lambda(n)}{n^s} = -\frac{1}{\varphi(q)} \sum_{\chi \ (\mathrm{mod}\ q)} \overline{\chi}(a) \frac{L'}{L}(s,\chi)$$

also holds for $\Re(s) > 1$ [14, (4.28)], and therefore we have established in this region that

$$-\frac{1}{s\varphi(q)} \sum_{\chi \ (\mathrm{mod}\ q)} \overline{\chi}(a) \frac{L'}{L}(s,\chi) = \int_{\overline{p}_1}^{\infty} \frac{\psi(t;q,a)}{t^{s+1}} \,\mathrm{d}t. \tag{4.19}$$

Since $R(t;q,a) = \psi(t;q,a) - t/\varphi(q)$, we also have

$$\int_{\overline{p}_1}^{\infty} \frac{R(t;q,a)}{t^{s+1}} \,\mathrm{d}t = \int_{\overline{p}_1}^{\infty} \frac{\psi(t;q,a)}{t^{s+1}} \,\mathrm{d}t - \frac{1}{\varphi(q)} \int_{1}^{\infty} \frac{1}{t^s} \,\mathrm{d}t + \frac{1}{\varphi(q)} \int_{1}^{\overline{p}_1} \frac{1}{t^s} \,\mathrm{d}t.$$

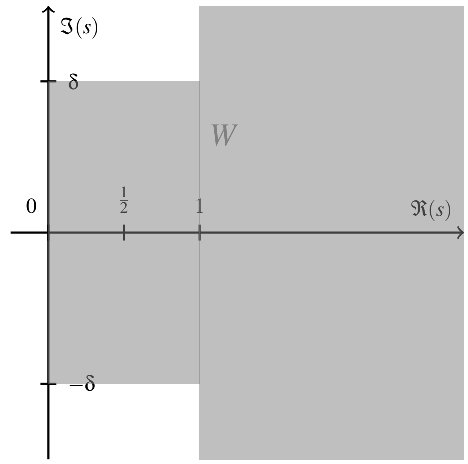

Figure 4.11: The region W.

We therefore have, for $\Re(s) > 1$

$$\begin{aligned}\int_{\overline{p}_1}^{\infty} \frac{R(t;q,a)}{t^{s+1}}\,\mathrm{d}t &= \frac{1}{\varphi(q)}\left(-\frac{1}{s}\sum_{\chi \pmod q}\overline{\chi}(a)\frac{L'}{L}(s,\chi) - \frac{1}{s-1} + \int_1^{\overline{p}_1}\tfrac{1}{t^s}\,\mathrm{d}t\right)\\ &= \frac{1}{\varphi(q)}\left(-\frac{1}{s}\mathscr{L}(s;q,a) - \frac{1}{s-1} + \int_1^{\overline{p}_1}\tfrac{1}{t^s}\,\mathrm{d}t\right). \end{aligned} \tag{4.20}$$

Now suppose that $\mathscr{L}(s;q,a)$ has no singularity on the critical real line. In such a scenario, we can construct a simply connected region W (Figure 4.11) where the right hand side of (4.20) is singularity-free by taking

$$W = \{\, s \mid \Re(s) > 1 \,\} \cup \{\, s \mid 0 < \Re(s) \le 1 \text{ and } |\Im(s)| < \delta \,\}$$

where δ is specifically chosen to be less than the minimum absolute height of the first singularity of $\mathscr{L}(s;q,a)$.

One may observe that the right hand side of (4.20) is holomorphic in the neighborhood of $s = 1$ and in W, since the only pole arising from $\mathscr{L}(s;q,a)$ is a simple one with residue -1 corresponding to the term $\overline{\chi}_0(a)\frac{L'}{L}(s,\chi_0)$. In the Laurent expansion of $\mathscr{L}(s;q,a)$ around $s = 1$, the term $\frac{1}{s-1}$ arising from $\frac{L'}{L}(s,\chi_0)$ is therefore eliminated by the $-\frac{1}{s-1}$ term in (4.20).

That is, in a neighborhood of $s = 1$,

$$\begin{aligned}
&\frac{1}{\varphi(q)}\left(-\frac{1}{s}\mathscr{L}(s;q,a) - \frac{1}{s-1} + \int_1^{\overline{p}_1} \tfrac{1}{t^s}\,\mathrm{d}t\right)\\
&= \frac{1}{\varphi(q)(s-1)} - \frac{1}{\varphi(q)(s-1)} + \frac{\log \overline{p}_1}{\varphi(q)} - \frac{1}{\varphi(q)} \sum_{\chi\neq\chi_0} \overline{\chi}(a)\frac{L'}{L}(1,\chi) + O_{q,a}(|s-1|)\\
&= \frac{\log \overline{p}_1}{\varphi(q)} - \frac{1}{\varphi(q)} \sum_{\chi\neq\chi_0} \overline{\chi}(a)\frac{L'}{L}(1,\chi) + O_{q,a}(|s-1|),
\end{aligned}$$

which is analytic. Hence, the right hand side of (4.20) admits on W an antiderivative, call it $H_1(s)$, and a second antiderivative $H_2(s)$ in W. By definition, we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(H_1(s)\right) = \frac{1}{\varphi(q)}\left(-\frac{1}{s}\sum_{\chi \pmod q} \overline{\chi}(a)\frac{L'}{L}(s,\chi) - \frac{1}{s-1} + \int_1^{\overline{p}_1} \tfrac{1}{t^s}\,\mathrm{d}t\right).$$

On the other hand,

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(-\int_{\overline{p}_1}^{\infty} \frac{R(t;q,a)}{t^{s+1}\log t}\,\mathrm{d}t\right) = \int_{\overline{p}_1}^{\infty} \frac{R(t;q,a)}{t^{s+1}}\,\mathrm{d}t.$$

Thus, for $\Re(s) > 1$, (4.20) establishes

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(H_1(s)\right) = \frac{\mathrm{d}}{\mathrm{d}s}\left(-\int_{\overline{p}_1}^{\infty} \frac{R(t;q,a)}{t^{s+1}\log t}\,\mathrm{d}t\right).$$

Now, let $\gamma : [0,1] \to \mathbb{C}$ be a smooth curve with $\gamma(0) = s_0$, a fixed point in the half-plane with $\Re(s_0) > 1$ and $\gamma(1) = s$, where $\Re(s) > 1$. Then we have

$$\int_\gamma \frac{\mathrm{d}}{\mathrm{d}s}\left(H_1(s)\right)\mathrm{d}s = \int_\gamma \frac{\mathrm{d}}{\mathrm{d}s}\left(-\int_{\overline{p}_1}^{\infty} \frac{R(t;q,a)}{t^{s+1}\log t}\,\mathrm{d}t\right)\mathrm{d}s,$$

from which we obtain

$$H_1(s) + \lambda_1 = -\int_{\overline{p}_1}^{\infty} \frac{R(t;q,a)}{t^{s+1}\log t}\,\mathrm{d}t + \lambda_2.$$

Rearranging yields

$$\int_{\overline{p}_1}^{\infty} \frac{R(t;q,a)}{t^{s+1}\log t}\,\mathrm{d}t = -H_1(s) + (\lambda_2 - \lambda_1). \tag{4.21}$$

Recalling that the antiderivative of H_1 is H_2, we integrate along γ once more to obtain

$$\int_{\overline{p}_1}^{\infty} \frac{R(t;q,a)}{t^{s+1}\log^2 t}\,\mathrm{d}t = H_2(s) + (\lambda_2 - \lambda_1)s + \mu. \tag{4.22}$$

By (4.18), we have

$$H(s) = \frac{1}{s-1}\left((\overline{p}_1)^{1-s} J(\overline{p}_1;q,a) - \int_{\overline{p}_1}^{\infty} \frac{R(t;q,a)}{t^{s+1}\log t}\,\mathrm{d}t - \int_{\overline{p}_1}^{\infty} \frac{R(t;q,a)}{t^{s+1}\log^2 t}\,\mathrm{d}t \right).$$

We substitute the integrals in the above equation using their respective identities, (4.21) and (4.22). With the notation

$$E(s) \stackrel{\text{def}}{=} (\overline{p}_1)^{1-s} J(\overline{p}_1;q,a) + -(\lambda_2 - \lambda_1) - \mu - (\lambda_2 - \lambda_1)s,$$

this substitution yields, for $\Re(s) > 1$,

$$\begin{aligned} H(s) &= \frac{1}{s-1}\left((\overline{p}_1)^{1-s} J(\overline{p}_1;q,a) + H_1(s) - H_2(s) - (\lambda_2 - \lambda_1) - \mu - (\lambda_2 - \lambda_1)s \right) \\ &= \frac{1}{s-1}\left(H_1(s) - H_2(s) + E(s) \right), \end{aligned} \tag{4.23}$$

where $E(s)$ is entire. The following can be observed from (4.18) and (4.23). As mentioned, the bracketed expression (in either equation) is 0 when $s = 1$, and thus it cancels the pole in $H(s)$ at $s = 1$. Therefore, the right hand side of Equation (4.23) is holomorphic on W. Consequently, we have an analytic continuation for $H(s)$ into W. Crucially, we have extended $H(s)$ to the real line in the critical strip. Hence, if we suppose $J(x;q,a)$ is of constant sign for some interval $[x', \infty)$, then Theorem 4.12 establishes that the line of convergence must satisfy $\Re(s) \le 0$, since no point with $s = \sigma > 0$ is a singularity. That is, $H(s)$ must extend to a function which is holomorphic in the half-plane $\Re(s) > 0$.

Reconsidering (4.23), we see that the holomorphy of $H(s)$ implies that $H_1(s)-H_2(s)$ is holomorphic on $\Re(s)>0$, and therefore $\frac{\mathrm{d}^2}{\mathrm{d}s^2}(H_1(s)-H_2(s))$ is holomorphic in this region as well. We have assumed $\mathscr{L}(s;q,a)$ has a singularity at $s=\rho$, where $0<\Re(\rho)<1$ and $|\Im(\rho)|>0$. Such a singularity must be simple, since the zeroes of $L(s,\chi)$ contribute simple poles with residue $m_\rho(\chi)$ in the logarithmic derivative. Therefore, in an appropriate deleted neighborhood of ρ,

$$\mathscr{L}(s;q,a)=\frac{\sum_\chi\overline{\chi}(a)m_\rho(\chi)}{s-\rho}+c_0+c_1(s-\rho)+c_2(s-\rho)^2\ldots,$$

where $\sum_\chi\overline{\chi}(a)m_\rho(\chi)\neq 0$. In the same neighborhood, we therefore have

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}s^2}H_1(s)&=\frac{1}{\varphi(q)}\left(\frac{-1}{s}\mathscr{L}(s;q,a)-\frac{1}{s-1}+\int_1^{\overline{p}_1}\tfrac{1}{t^s}\,\mathrm{d}t\right)\\&=\frac{1}{\varphi(q)}\left(\frac{-1}{\rho}\frac{\sum_\chi\overline{\chi}(a)m_\rho(\chi)}{s-\rho}-\frac{1}{\rho-1}+\int_1^{\overline{p}_1}\tfrac{1}{t^\rho}\,\mathrm{d}t+O_{q,a}(|s-\rho|)\right)\end{aligned}$$

Consequently,

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}s^2}H_2(s)&=\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\varphi(q)}\left(\frac{-1}{\rho}\frac{\sum_\chi\overline{\chi}(a)m_\rho(\chi)}{s-\rho}-\frac{1}{\rho-1}+\int_1^{\overline{p}_1}\tfrac{1}{t^\rho}\,\mathrm{d}t+O_{q,a}(|s-\rho|)\right)\right)\\&=\frac{1}{\varphi(q)}\left(\frac{\sum_\chi\overline{\chi}(a)m_\rho(\chi)}{\rho(s-\rho)^2}+c'+O_{q,a}(|s-\rho|)\right),\end{aligned}$$

where c' is a constant. This implies that $\frac{\mathrm{d}^2}{\mathrm{d}s^2}H_2(s)$ has a pole of order 2 at ρ. However, we have claimed $\frac{\mathrm{d}^2}{\mathrm{d}s^2}(H_1(s)-H_2(s))$ is holomorphic for $\Re(s)>0$. This is a contradiction, and so $J(x;q,a)$ must *not* be of constant sign on some interval $[x',\infty)$. By Theorem 4.11, this means that $\log f(x;q,a)<0$ in infinitely many intervals $\overline{p}_k\leq x<\overline{p}_{k+1}$. Therefore, by way of the reasoning at the beginning of Section 4.4 and (4.1), we must have infinitely many primorials $\overline{N}_k$ in $S_{q,a}$ for which

$$\frac{\overline{N}_k}{\varphi(\overline{N}_k)(\log(\varphi(q)\log\overline{N}_k))^{\frac{1}{\varphi(q)}}}>\frac{1}{C(q,a)},$$

as desired. □

Theorem 4.13 answers Question 3, proviso two caveats. If we can show that there are no singularities for $\mathscr{L}(s;q,a)$ on the segment $(0,1)$ but that there is a singularity ρ for which $0 < \Re(\rho) < 1$, then there *are* infinitely many primorials in $S_{q,a}$ satisfying their respective Nicolas inequality (3.4). By way of Theorem 3.18 and Theorem 3.19, we know that this condition holds for $q \leq 400,000$, since any singularity of $\mathscr{L}(s;q,a)$ in this region must arise from a zero of a Dirichlet L-function whose character is $\chi \pmod q$. Thus, the following assertion holds.

Theorem 4.14. *Let $q \leq 400,000$ and a be coprime natural numbers. If $\mathscr{L}(s;q,a)$ has a singularity ρ for which $0 < \Re(\rho) < 1$, then there are infinitely many primorials $\overline{N}_k$ in $S_{q,a}$ for which*

$$\frac{\overline{N}_k}{\varphi(\overline{N}_k)(\log(\varphi(q)\log\overline{N}_k))^{\frac{1}{\varphi(q)}}} > \frac{1}{C(q,a)},$$

where $C(q,a)$ is the constant in Theorem 3.5.

Practically, the condition of Theorem 4.14 may be disjointed by picking the first zero of any $L(s,\chi)$, χ a Dirichlet character modulo q on the half line and verifying that it indeed contributes a singularity in $\mathscr{L}(s;q,a)$. What remains now is to answer the alternative question: "Are there infinitely many primorials in $S_{q,a}$ which *do not* satisfy (3.3)?". In the case that Conjecture 3.17 is false, we rely on the lower bounds of Theorem 4.1 to find an infinitude of exceptions in $S_{q,a}$ to (3.3), as we will see in the next chapter. In the case that Conjecture 3.17 is true, we may verify computationally that (3.3) is, in certain circumstances, satisfied for all but finitely many primorials in $S_{q,a}$, as in Chapter 6.

Chapter 5

Lower Bounds for $\log f(x;q,a)$

A minimum put to good use

is enough for anything.

– J. Verne

5.1 Omega Theorems

In the previous chapter, we established that there are infinitely many primorials in $S_{q,a}$ for which

$$\frac{n}{\varphi(n)(\log(\varphi(q)\log n))^{\frac{1}{\varphi(q)}}} > \frac{1}{C(q,a)},$$

conditional upon

$$\mathscr{L}(s;q,a) = \sum_{\chi \pmod q} \overline{\chi}(a)\frac{L'}{L}(s,\chi)$$

having no singularities on the segment $(0,1)$ and having a singularity at s for which $0 < \Re(s) < 1$. However, we did not make reference to Conjecture 3.17. In the case that Conjecture 3.17 is false, we may apply the methods of Chapter 4, albeit with more care than before, to determine that there are also infinitely many primorials in $S_{q,a}$ for which

$$\frac{n}{\varphi(n)(\log(\varphi(q)\log n)^{\frac{1}{\varphi(q)}}} \leq \frac{1}{C(q,a)}.$$

However, one should recall that the approach in Chapter 4 was to show that $\log f(x;q,a)$ was negative, and therefore only the upper bounds of Proposition 4.1 were of critical concern.

In this chapter, we instead concern ourselves with *lower* bounds on $\log f(x;q,a)$, to aid us in proving that $\log f(x;q,a)$ is positive infinitely often. In these endeavors, we introduce the following notation.

Definition 5.1. Let q and a be two coprime integers. We will write Θ to denote the supremum of the real parts of the singularities of $\mathscr{L}(s;q,a)$ in the strip $0 < \Re(s) < 1$.

Notice that $\mathscr{L}(s;q,a) = \sum_{\chi \pmod q} \overline{\chi}(a)\frac{L'}{L}(s,\chi)$ has a simple pole of residue -1 at $s = 1$ arising from the term $\frac{L'}{L}(s,\chi_0)$, but this pole is ignored in the definition of Θ. Supposing Conjecture 3.17 is false, there must be a character χ modulo q for which $L(s,\chi)$ has a zero ρ of real part neither 0 nor $1/2$ for which

$$\sum_{\chi} \overline{\chi}(a) m_\rho(\chi) \neq 0,$$

where $m_\rho(\chi)$ is the multiplicity of the zero ρ of $L(s,\chi)$.

First, consider the case that $\Re(\rho) > \frac{1}{2}$. Then $\sum_\chi \overline{\chi}(a)\frac{L'}{L}(s,\chi)$ admits a simple pole of residue $\sum_\chi \overline{\chi}(a)m_\rho(\chi)$ at $s = \rho$. Since $\Re(\rho) > \frac{1}{2}$, this implies $\Theta \geq \Re(\rho) > \frac{1}{2}$. In the case that $0 < \Re(\rho) < \frac{1}{2}$, recall that $1-\rho$ is a zero of $L(s,\overline{\chi})$, and $\Re(1-\rho) > \frac{1}{2}$. Therefore, $m_\rho(\chi) = m_{1-\rho}(\overline{\chi})$ for every character χ modulo q. Therefore,

$$\sum_{\chi} \overline{\chi}(a) m_\rho(\chi) = \sum_{\chi} \overline{\chi}(a) m_{1-\rho}(\overline{\chi}) \neq 0.$$

In this instance, $\sum_{\chi \pmod q} \overline{\chi}(a)\frac{L'}{L}(s,\chi)$ again admits a simple pole of residue $\sum_\chi \overline{\chi}(a)m_\rho(\chi)$ at $s = 1-\rho$, and therefore $\Theta \geq \Re(1-\rho) > \frac{1}{2}$. Hence, if we suppose Conjecture 3.17 is false, we have that $\Theta > \frac{1}{2}$. We will use this fact to obtain what are known as Ω theorems regarding the behavior of $\log f(x;q,a)$.

Definition 5.2. Let $f_1(x)$ be a real-valued function, $f_2(x)$ be a positive real-valued function, and $c > 0$ be a constant. We say that

$$f_1(x) = \Omega_+(f_2(x))$$

if there exists an increasing real sequence $\{x_i\}$ tending to infinity along which $f_1(x_i) > cf_2(x_i)$. Likewise we may say

$$f_1(x) = \Omega_-(f_2(x))$$

if $f_1(x_i) < -cf_2(x_i)$. One may write

$$f_1(x) = \Omega_\pm(f_2(x))$$

whenever $f_1(x) = \Omega_+(f_2(x))$ and $f_1(x) = \Omega_-(f_2(x))$.

To establish an Ω *theorem* in this context, then, is to establish that one of the above relations holds for a function of interest.

5.2 Omega Theorems for $J(x;q,a)$ and $K(x;q,a)$

We adapt the techniques of Theorem 4.13 to establish an Ω theorem for $J(x;q,a)$, under the assumption that there are singularities of $\mathscr{L}(s;q,a)$ off the critical line.

Theorem 5.3. *If* $\Theta > \frac{1}{2}$ *and* $\mathscr{L}(s;q,a)$ *has no singularities on the line* $(0,1)$*, then for* b *satisfying* $1-\Theta < b < \frac{1}{2}$*, we have*

$$J(x;q,a) = \int_x^\infty R(t;q,a)g(t)\,\mathrm{d}t = \Omega_\pm(x^{-b}),$$

and

$$K(x;q,a) = \int_x^\infty S(t;q,a)g(t)\,\mathrm{d}t = \Omega_\pm(x^{-b}).$$

Proof. Let b satisfy $1-\Theta < b < \frac{1}{2}$. Then there exists a singularity ρ of $\mathscr{L}(s;q,a)$ not at 1

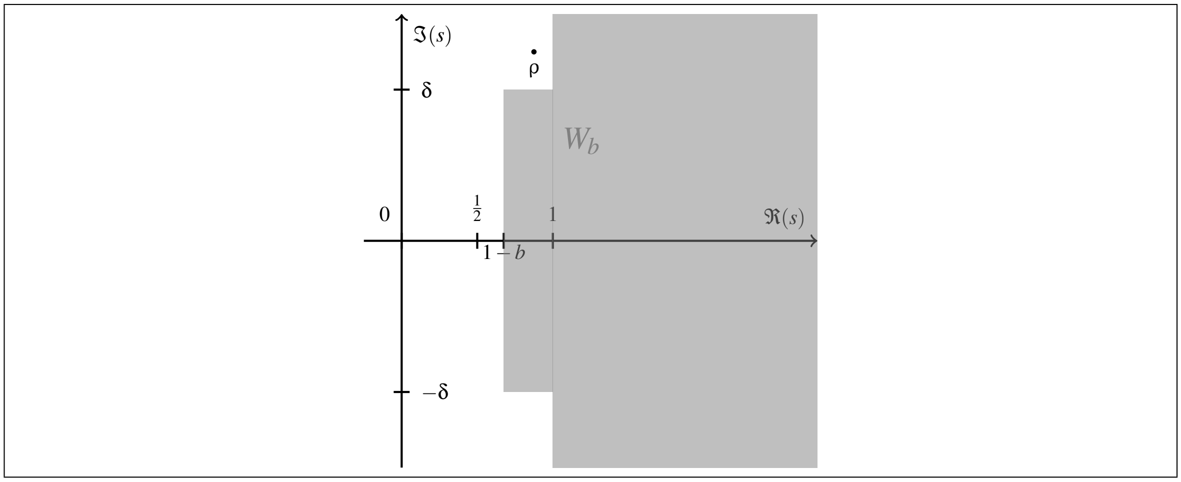

Figure 5.1: The region W_b and the singularity ρ.

for which $\Re(\rho) = \beta > 1 - b$. If we consider the integral

$$G(s) = \int_{\overline{p}_1}^{\infty} \frac{J(x;q,a) - x^{-b}}{x^s}\,\mathrm{d}x,$$

then, for $\Re(s) > 1$, the methods of Theorem 4.13, in particular (4.23), establish

$$\begin{aligned} G(s) &= H(s) - \frac{1}{s-1+b} + \int_1^{\overline{p}_1} \frac{x^{-b}}{x^s}\,\mathrm{d}x \\ &= \frac{1}{s-1}\left(H_1(s) - H_2(s) + E(s)\right) - \frac{1}{s-1+b} + \int_1^{\overline{p}_1} \frac{x^{-b}}{x^s}\,\mathrm{d}x \end{aligned} \tag{5.1}$$

where $H_1(s)$ and $H_2(s)$ are the first and second antiderivatives of

$$\frac{1}{\varphi(q)}\left(-\frac{1}{s}\mathscr{L}(s;q,a) - \frac{1}{s-1} + \int_1^{\overline{p}_1} \tfrac{1}{t^s}\,\mathrm{d}t\right)$$

in the region W. Furthermore, and $E(s)$ and $\int_1^{\overline{p}_1} \frac{x^{-b}}{x^s}\,\mathrm{d}x$ are analytic on

$$W_b = \{\, s \mid \Re(s) > 1 - b \,\} \cap W.$$

The right-hand side of (5.1) therefore extends to a holomorphic function in W_b.

By Theorem 4.12, if we assume $J(x;q,a) - x^{-b}$ maintains a constant sign on intervals of the form $[x', \infty)$, then the abscissa of convergence, σ_0, of $G(s)$ must satisfy $\sigma_0 \leq 1 - b \leq \beta$. This is impossible since at ρ the second derivative of $H_1(s) - H_2(s)$ will have a pole of order 2, as in Theorem 4.13, contradicting the holomorphy of the integral $G(s)$ in the half-plane $\Re(s) > 1 - b$. We have a contradiction and therefore $J(x;q,a) - x^{-b} > 0$ on some sequence tending to infinity. Hence,

$$J(x;q,a) = \Omega_+(x^{-b}).$$

Taking the integral for $J(x;q,a) + x^{-b}$ and repeating the above proof establishes that $J(x;q,a) + x^{-b} < 0$ on another infinite sequence, i.e.,

$$J(x;q,a) = \Omega_-(x^{-b}).$$

For all $x > 0$, [21, Equation (3.36)] provides

$$0 \leq \psi(x;q,a) - \theta(x;q,a) \leq \psi(x) - \theta(x) \leq 1.4260x^{1/2}. \tag{5.2}$$

Recall that

$$R(x;q,a) = \psi(x;q,a) - \frac{x}{\varphi(q)}$$

is greater than or equal to

$$S(x;q,a) = \theta(x;q,a) - \frac{x}{\varphi(q)}.$$

We conclude from (5.2) that

$$R(x;q,a) - \frac{3}{2}x^{\frac{1}{2}} \leq S(x;q,a) \leq R(x;q,a). \tag{5.3}$$

If we weight the above inequality by $g(x)$ and integrate from x to ∞, we obtain

$$J(x;q,a) - \frac{3}{2}F_{\frac{1}{2}}(x) \leq K(x;q,a) \leq J(x;q,a), \tag{5.4}$$

where

$$F_s(x) = \int_x^\infty t^s g(t)\,\mathrm{d}t.$$

The inequality on the right of (5.4) automatically yields

$$K(x;q,a) = \Omega_-(x^{-b}),$$

since $J(x;q,a) = \Omega_-(x^{-b})$. On the other hand, Corollary 4.10 established that

$$F_{\frac{1}{2}}(x) = \frac{2}{x^{\frac{1}{2}}\log x} + r_{\frac{1}{2}}(x),$$

where

$$|r_{\frac{1}{2}}(x)| \leq \left(\frac{2}{x^{\frac{1}{2}}\log^2 x}\right)\left(1+\frac{4}{\log x}\right).$$

Therefore, along the sequence which gives us $J(x;q,a) = \Omega_+(x^{-b})$, we have

$$\begin{aligned} J(x;q,a) - \frac{3}{2}F_{\frac{1}{2}}(x) &\geq \frac{1}{x^b} - \frac{3}{x^{\frac{1}{2}}\log x} - \frac{3}{x^{\frac{1}{2}}\log^2 x}\left(1+\frac{4}{\log x}\right) \\ &= \frac{1}{x^b}\left(1 - \frac{3}{x^{\frac{1}{2}-b}\log x} - \frac{3}{x^{\frac{1}{2}-b}\log^2 x}\left(1+\frac{4}{\log x}\right)\right). \qquad (5.5) \end{aligned}$$

Since $b < \frac{1}{2}$, the bracketed expression approaches 1 as x goes to infinity. Hence, by (5.4), for $x > e^6$ and along the sequence corresponding to $J(x;q,a) = \Omega_+(x^{-b})$,

$$K(x;q,a) \geq J(x;q,a) - \frac{3}{2}F_{\frac{1}{2}}(x) > 0.36x^{-b}.$$

Therefore,

$$K(x;q,a) = \Omega_+(x^{-b}).$$

□

5.3 Omega Theorems for $\log f(x;q,a)$

By Corollary 4.11, $\log f(x;q,a) \le J(x;q,a)$ for large enough x and by Theorem 5.3, if $\Theta > \frac{1}{2}$ then $J(x;q,a) < -cx^{-b}$ for infinitely many x. Hence, when $\Theta > 1/2$, we obtain

$$\log f(x;q,a) = \Omega_-(x^{-b}). \tag{5.6}$$

It remains to establish that $\log f(x;q,a) = \Omega_+(x^{-b})$. A preliminary step in this endeavor is a modification of a result of Ingham [5, Theorem 32].

Theorem 5.4. *Let q and a be coprime integers. If $0 < \nu < \Theta$ is fixed, then*

$$R(x;q,a) = \psi(x;q,a) - \frac{x}{\varphi(q)} = \Omega_\pm(x^\nu),$$

provided

$$\mathscr{L}(s;q,a) = \sum_{\chi \pmod q} \overline{\chi}(a) \frac{L'}{L}(s,\chi)$$

has no singularities on the segment $(0,1)$.

Proof. Let $0 < \nu < \Theta$. For $\Re(s) > 1$, recall (4.19), which established

$$\int_{\overline{p}_1}^\infty \frac{\psi(t;q,a)}{t^{s+1}}\,\mathrm{d}t = -\frac{1}{s\varphi(q)} \sum_{\chi \pmod q} \overline{\chi}(a) \frac{L'}{L}(s,\chi). \tag{5.7}$$

Furthermore,

$$\frac{1}{\varphi(q)} \int_1^\infty \frac{t^\nu}{t^{s+1}}\,\mathrm{d}t = \frac{1}{\varphi(q)(s-\nu)} \tag{5.8}$$

and

$$\frac{1}{\varphi(q)} \int_1^\infty \frac{t}{t^{s+1}}\,\mathrm{d}t = \frac{1}{\varphi(q)(s-1)}. \tag{5.9}$$

From (5.7), (5.8), and (5.9), for $\Re(s) > 1$, we have

$$\begin{aligned}\int_1^\infty \frac{\frac{\psi(x;q,a)-\frac{x}{\varphi(q)}-\frac{x^\nu}{\varphi(q)}}{x}}{x^s}\,\mathrm{d}x &= \frac{1}{\varphi(q)}\left(-\frac{1}{s}\sum_{\chi \pmod q}\overline{\chi}(a)\frac{L'}{L}(s,\chi)-\frac{1}{s-1}-\frac{1}{s-\nu}\right) \\ &= -\frac{\mathscr{L}(s;q,a)}{s\varphi(q)}-\frac{1}{\varphi(q)(s-1)}-\frac{1}{\varphi(q)(s-\nu)}.\end{aligned} \tag{5.10}$$

Let σ_0 be the abscissa of convergence of the integral represented above. If the half-plane $\Re(s) > \sigma_0$ contains any singularities of $\mathscr{L}(s;q,a)$, then we have failed to correctly identify σ_0. Hence, we must have $\sigma_0 \geq \Theta$. Moreover, the right hand side of (5.10) will have no singularities on the real axis for $s > \nu$ by our assumption. Observe, then, that $s = \sigma_0 \geq \Theta > \nu$ is *not* a singularity of the integral under consideration and therefore Theorem 4.12 maintains that

$$\frac{\psi(x;q,a)-\frac{x}{\varphi(q)}-\frac{x^\nu}{\varphi(q)}}{x}$$

cannot be of constant sign on any interval of the form $[x',\infty)$. That is, there is a sequence tending to infinity for which $\psi(x;q,a)-\frac{x}{\varphi(q)} > \frac{x^\nu}{\varphi(q)}$, and so for any $0 < \nu < \Theta$, we have $R(x;q,a) = \Omega_+(x^\nu)$. Repeating this argument for the integral from x to ∞ of

$$\frac{\psi(x;q,a)-\frac{x}{\varphi(q)}+\frac{x^\nu}{\varphi(q)}}{x}$$

establishes $R(x;q,a) = \Omega_-(x^\nu)$ □

Corollary 5.5. *Let q and a be coprime integers. If $\frac{1}{2} < \nu < \Theta$ is fixed, then*

$$S(x;q,a) = \theta(x;q,a) - \frac{x}{\varphi(q)} = \Omega_\pm(x^\nu),$$

provided

$$\mathscr{L}(s;q,a) = \sum_{\chi \pmod q}\overline{\chi}(a)\frac{L'}{L}(s,\chi)$$

has no singularities on the segment $(0,1)$.

Proof. Let $\frac{1}{2} < \nu < \Theta$. Recall (5.3), which established

$$R(x;q,a) - \frac{3}{2}x^{\frac{1}{2}} \leq S(x;q,a) \leq R(x;q,a).$$

By Theorem 5.4, $R(x;q,a) = \Omega_-(x^\nu)$, and therefore we have $S(x;q,a) = \Omega_-(x^\nu)$. On the other hand, for the sequence which provides us $R(x;q,a) = \Omega_+(x^\nu)$, we have

$$S(x;q,a) \geq cx^\nu - \frac{3}{2}x^{\frac{1}{2}} = cx^\nu\left(1 - \frac{3}{2c}x^{\frac{1}{2}-\nu}\right),$$

where $c > 0$. Since $\nu > \frac{1}{2}$, the bracketed expression tends to 1, and for large enough x may be bounded below by a positive constant. Hence, $S(x;q,a) = \Omega_+(x^\nu)$. □

We are now in a position to prove that $\log f(x;q,a) = \Omega_+(x^{-b})$.

Theorem 5.6. *Suppose* $\Theta > \frac{1}{2}$ *and* $\mathscr{L}(s;q,a)$ *has no singularities on the segment* $(0,1)$. *Then for*

$1 - \Theta < b < \frac{1}{2}$,

$$\log f(x;q,a) = \Omega_\pm(x^{-b}).$$

Proof. The discussion at the beginning of this section provided us with $\log f(x;q,a) = \Omega_-(x^{-b})$. For the other case, we wish to establish an infinite sequence for which

$$\log f(x;q,a) \geq cx^{-b}$$

for some positive constant c. To achieve this goal, we will consider the lower bound in Proposition 4.1, which showed that for large enough x,

$$K(x;q,a) - \frac{(\varphi(q)S(x;q,a))^2(\log(\frac{4x}{5})+1)}{2\varphi(q)(\frac{4x}{5})^2\log^2(\frac{4x}{5})} \leq \log f(x;q,a). \tag{5.11}$$

We can determine the behavior of the above lower bound by considering the local extrema of the function $y(x;q,a) = K(x;q,a) - x^{-b}$. At primes in the arithmetic progression

$a \pmod q$, $y(x;q,a)$ has discontinuities arising from $K(x;q,a)$ but it is otherwise differentiable. Therefore, we have

$$y'(x;q,a) = -\frac{S(x;q,a)}{x^2}\left(\frac{\log x + 1}{\log^2 x}\right) + \frac{b}{x^{b+1}},$$

when x is not a prime in the arithmetic progression $a \pmod q$.

Therefore, we may look for extrema of $y(x;q,a)$ by examining the sign changes in $y'(x;q,a)$. It is possible that $y'(x;q,a)$ changes sign around a prime in the arithmetic progression $a \pmod q$. Between primes, we may examine the behavior of $y'(x;q,a)$ by inspecting

$$\left(\frac{x}{\varphi(q)} - T\right)(\log x + 1) + bx^{1-b}\log^2 x = 0, \tag{5.12}$$

where T is fixed, since on any interval $[\overline{p}_k, \overline{p}_{k+1})$, $\theta(x;q,a)$ is fixed. Observe that the left-hand of (5.12) side has at best one root in $[1,\infty)$ and in fact this root lies in $[1, \varphi(q)T]$. This is true because the expression in (5.12) represents a continuous function that can only be negative while $(\frac{x}{\varphi(q)} - T)$ is negative. Hence, on the fixed interval $[\overline{p}_k, \overline{p}_{k+1})$, the left-hand side of (5.12) is $x^2\log^2(x)y'(x)$, revealing that $y'(x)$ has *at most* one root between consecutive primes in the given arithmetic progression. Hence, the sign changes of y' are "well-spaced" in the sense that they do not accumulate to a limit point. Additionally, on the sequences corresponding to Corollary 5.5, y' must eventually be both negative and positive (corresponding to the Ω_+ and the Ω_- results, respectively). Therefore, denoting the sequence of sign changes of y' by $\{x_i\}$, we conclude that $\{x_i\}$ is an increasing sequence which tends to infinity.

Now, if we were to suppose that $y(x) = K(x;q,a) - x^{-b}$ is negative or zero for all local extrema occurring on an interval of the form $[x',\infty)$, then we naturally must have that $K(x;q,a) - x^{-b} \leq 0$ for all $x > x'$, but we know by Theorem 5.3 that $K(x;q,a) = \Omega_+(x^{-b'})$

where $1-\Theta < b' = \frac{1+b-\Theta}{2} < b$. This implies that there are arbitrarily large x for which

$$cx^{-b'} \le K(x;q,a) \le x^{-b},$$

but this is false since $-b' < b$. We have a contradiction, and hence $y(x) = K(x;q,a) - x^{-b}$ must be positive on an infinite subsequence of $\{x_i\}$, say $\{x_{i_j}\} = \{t_i\}$. That is, we have

$$K(t_i;q,a) > t_i^{-b}. \tag{5.13}$$

Observe that for a point x_i at which y' changes sign, x_i will also be a sign change of the expression

$$-\frac{y'(x;q,a)x^2\log^2 x}{\log x+1} = S(x;q,a) - \frac{bx^{1-b}\log^2 x}{\log x+1}.$$

If x_i is not a prime congruent to a modulo q, then $x_i \in (\overline{p}_k, \overline{p}_{k+1})$ for some $k \in \mathbb{N}$. In this instance,

$$S(x_i;q,a) - \frac{bx_i^{1-b}\log^2 x_i}{\log x_i+1} = \theta(x_i;q,a) - \frac{x_i}{\varphi(q)} - \frac{bx_i^{1-b}\log^2 x_i}{\log x_i+1} = 0,$$

since the function represented here is continuous around x_i. On the other hand, if $x_i = \overline{p}$, then

$$\theta(\overline{p}-1;q,a) - \frac{\overline{p}}{\varphi(q)} - \frac{b\overline{p}^{1-b}\log^2\overline{p}}{\log\overline{p}+1} \tag{5.14}$$

and

$$\theta(\overline{p};q,a) - \frac{\overline{p}}{\varphi(q)} - \frac{b\overline{p}^{1-b}\log^2\overline{p}}{\log\overline{p}+1}. \tag{5.15}$$

The difference between (5.15) and (5.14) is $\log\overline{p}$ and we know a sign change occurs at $\overline{p}$. More precisely, (5.14) must be negative, while (5.15) is positive. Accordingly,

$$S(x_i;q,a) - \frac{bx_i^{1-b}\log^2 x_i}{\log x_i+1} \in [0, \log x_i).$$

In either case,

$$0 \le S(x_i;q,a) - \frac{bx_i^{1-b}\log^2 x_i}{\log x_i + 1} < \log x_i,$$

which implies

$$S(x_i;q,a) < \frac{\log x_i + \log^2 x_i + bx_i^{1-b}\log^2 x_i}{\log x_i}$$

That is,

$$S(x_i;q,a) = O(x_i^{1-b}\log x_i)$$

Hence,

$$\frac{\varphi(q)S^2(x_i;q,a)(\log(\frac{4x_i}{5})+1)}{2(\frac{4x_i}{5})^2\log(\frac{4x_i}{5})} = O\left(\frac{(\log x_i)^2(\log(\frac{4x_i}{5})+1)}{x^{2b}\log^2(\frac{4x_i}{5})}\right). \tag{5.16}$$

Bringing (5.11), (5.13), and (5.16) together on the sequence $\{t_i\}$ for large enough x, we have

$$\begin{aligned}\log f(x;q,a) &\ge K(t_i;q,a) - \frac{\varphi(q)S^2(t_i;q,a)(\log(\frac{4t_i}{5})+1)}{2(\frac{4t_i}{5})^2\log(\frac{4t_i}{5})} > t_i^{-b} + O\left(\frac{(\log t_i)^2(\log(\frac{4t_i}{5})+1)}{t^{2b}\log^2(\frac{4t_i}{5})}\right)\\ &= t_i^{-b}\left(1 + O\left(\frac{(\log t_i)^2(\log(\frac{4t_i}{5})+1)}{t^{2b}\log^2(\frac{4t_i}{5})}\right)\right) = t_i^{-b}\left(1+O\left(\frac{\log t_i}{t_i^b}\right)\right)\end{aligned} \tag{5.17}$$

Therefore, $\log f(x;q,a) = \Omega_+(x^{-b})$.

□

Recalling the discussions in Section 4.1 and at the beginning of this chapter, we may restate the above theorem as follows. Supposing Conjecture 3.17 is false, we have that $\Theta > \frac{1}{2}$. In this case, if $\mathscr{L}(s;q,a)$ has no singularities on the segment $(0,1)$, then $\log f(x;q,a)$ is both greater than and less than 0 infinitely often. More plainly, under these conditions there are infinitely many primorials in $S_{q,a}$ for which

$$\frac{n}{\varphi(n)(\log(\varphi(q)\log n))^{\frac{1}{\varphi(q)}}} > \frac{1}{C(q,a)}$$

holds and *also* infinitely many primorials in $S_{q,a}$ for which

$$\frac{n}{\varphi(n)(\log(\varphi(q)\log n))^{\frac{1}{\varphi(q)}}} \leq \frac{1}{C(q,a)}.$$

As with the end of Chapter 4, we have by Theorems 3.18 and 3.19 that there are no singularities of $\mathscr{L}(s;q,a)$ on the segment $(0,1)$ when $q \leq 400,000$. We may therefore conclude the following.

Theorem 5.7. *Let $q \leq 400,000$ and a be coprime natural numbers. If Conjecture 3.17 is false, then there are infinitely many primorials $\overline{N}_k$ in $S_{q,a}$ for which*

$$\frac{\overline{N}_k}{\varphi(\overline{N}_k)(\log(\varphi(q)\log \overline{N}_k))^{\frac{1}{\varphi(q)}}} > \frac{1}{C(q,a)},$$

holds and also infinitely many primorials $\overline{N}_k$ in $S_{q,a}$ for which this inequality does not hold.

Chapter 6

Upper Bounds for $\log f(x;q,a)$

Thus the hair-splitters can render us a double service,

first by teaching us to do as they do if necessary,

but more especially, by enabling us as often as possible

not to do as they do,

and yet make no sacrifice of exactness.

– H. Poincaré

In Chapter 4, we saw that, with no reference to Conjecture 3.17, there are infinitely many $k \in \mathbb{N}$ for which $\mathcal{N}_{q,a}(k)$ is true. Chapter 5, in contrast, showed that if Conjecture 3.17 is false and $\mathscr{L}(s;q,a)$ had no singularities on the segment $(0,1)$, then there were infinitely many $k \in \mathbb{N}$ for which $\mathcal{N}_{q,a}(k)$ is false. This generalizes part of Theorem 2(b) of Nicolas [17], in which it was shown that the assumption of the falsehood of the Riemann hypothesis led to two infinite sets of primorials; those which satisfied (2.2) and those which did not satisfy (2.2). In this chapter, we establish a generalization of Theorem 2(a). More precisely, the goal is to show that if we assume Conjecture 3.17 is true, then

$$\frac{n}{\varphi(n)\log(\varphi(q)\log(n))^{\frac{1}{\varphi(q)}}} > \frac{1}{C(q,a)}$$

for all but finitely many primorials n in $S_{q,a}$.

The method of attack will be as follows. First, we recall Corollary 4.8, which allows us to establish the upper bound

$$\log f(x;q,a) \leq J(x;q,a) - cF_{\frac{1}{m}}(x) + \frac{1}{2(x-1)} \tag{6.1}$$

for large enough x and a positive constant c. Based upon estimates of Theorem 4.11, $-cF_{\frac{1}{m}}(x) + \frac{1}{2(x-1)}$ is negative for large enough x. It remains only to establish that, under the assumption that Conjecture 3.17 is true, $J(x;q,a)$ is small enough when x is large so that the upper bound in (6.1) becomes negative.

6.1 $J(x;q,a)$ as an Expression of Zeroes of L-functions

Towards an estimate of $J(x;q,a)$ under Conjecture 3.17, we would like to determine a new expression for $J(x;q,a)$ in terms of the zeroes of L-functions corresponding to Dirichlet characters modulo q. In order for this to occur, we need to recall the explicit formula.

Theorem 6.1 ([12, Lemma 3.1])**.** *For a Dirichlet character χ modulo q, write*

$$\mathcal{Z}(\chi) = \{\rho = \beta + i\gamma \in \mathbb{C}\ ;\ L(\rho,\chi) = 0,\ \beta \geq 0 \text{ and } \rho \neq 0\},$$

Let α be 1 if χ is odd and 0 otherwise, $b(\chi)$, $c(\chi)$ be the constant terms in the Laurent expansion of $\frac{L'}{L}(s,\chi)$ about 0 and -1, respectively, and $m_0(\chi)$ be the multiplicity of the zero of $L(s,\chi)$ at 0. Then, for $x > 1$, we have

$$\begin{aligned}\int_1^x \psi(t;q,a)\,dt = &\frac{x^2}{2\varphi(q)} - \frac{1}{\varphi(q)}\sum_{\chi}\overline{\chi}(a)\sum_{\rho\in\mathcal{Z}(\chi)}\frac{x^{\rho+1}}{\rho(\rho+1)} \\ &- \frac{1}{\varphi(q)}\sum_{\chi}\overline{\chi}(a)\sum_{n=1}^{\infty}\frac{x^{-2n+1-\alpha}}{2n(2n-1+2\alpha)} \\ &+ x\left(\frac{1}{\varphi(q)}\sum_{\chi}\overline{\chi}(a)(m_0(\chi)-b(\chi))\right)\end{aligned}$$

$$-x\log x\left(\frac{1}{\varphi(q)}\sum_{\chi}\overline{\chi}(a)m_0(\chi)\right)+\log x\left(\frac{1}{\varphi(q)}\sum_{\chi\text{ odd}}\overline{\chi}(a)\right)$$
$$+\left(\frac{1}{\varphi(q)}\sum_{\chi\text{ even}}\overline{\chi}(a)\frac{L'}{L}(-1,\chi)+\frac{1}{\varphi(q)}\sum_{\chi\text{ odd}}\overline{\chi}(a)(c(\chi)+1)\right).$$

The explicit formula allows us to establish a new expression for $J(x;q,a)$.

Lemma 6.2. *Let q and a be coprime integers. Suppose that Conjecture 3.17 is true. Then, for $x>1$, we have*

$$J(x;q,a)=-\frac{1}{\varphi(q)}\sum_{\chi}\overline{\chi}(a)\sum_{\rho\in\mathcal{Z}(\chi)}\frac{F_\rho(x)}{\rho}+\hat{J}(x;q,a),$$

where

$$F_\rho(x)=\int_x^\infty t^\rho g(t)\,dt,$$
$$\hat{J}(x;q,a)=\int_x^\infty g(t)\hat{g}'(t;q,a)\,dt$$

with

$$g(t)=\frac{\log x+1}{x^2\log^2 x},$$

and

$$\hat{g}'(t;q,a)=-\frac{1}{\varphi(q)}\sum_{\chi}\overline{\chi}(a)\sum_{n=1}^{\infty}\left(\frac{-t^{-(2n+\alpha)}}{2n+\alpha}\right)+\frac{1}{\varphi(q)}\sum_{\chi}\overline{\chi}(a)(m_0(\chi)-b(\chi))$$
$$-\frac{1}{\varphi(q)}\sum_{\chi}\overline{\chi}(a)m_0(\chi)(\log t+1)+\frac{1}{\varphi(q)}\sum_{\chi\text{ odd}}\overline{\chi}(a)t^{-1}.$$

Proof. Since we are assuming Conjecture 3.17, recall that for any zero ρ in $\cup_\chi\mathcal{Z}(\chi)$ whose real part is neither 0 nor $\frac{1}{2}$, we have that

$$\sum_{\chi\ (\mathrm{mod}\ q)}\overline{\chi}(a)m_\rho(\chi)=0. \tag{6.2}$$

This assumption allows us to prove that

$$\sum_{\chi}\overline{\chi}(a)\sum_{\rho\in\mathcal{Z}(\chi)}g'(t)\frac{t^{\rho+1}}{\rho(\rho+1)}$$

is integrable on (x,∞) for any $x>1$. In anticipation of an application of the Dominated Convergence Theorem, for $n\geq 1$, consider the sequence of functions

$$f_n(t)=\sum_{\chi}\overline{\chi}(a)\sum_{\substack{\rho\in\mathcal{Z}(\chi)\\ |\Im(\rho)|\leq n}}g'(t)\frac{t^{\rho+1}}{\rho(\rho+1)}.$$

We see that, for all $t\in(x,\infty)$,

$$\lim_{n\to\infty}f_n(t)=\sum_{\chi}\overline{\chi}(a)\sum_{\rho\in\mathcal{Z}(\chi)}g'(t)\frac{t^{\rho+1}}{\rho(\rho+1)}.$$

Moreover,

$$|f_n(t)|=\left|g'(t)\sum_{\chi}\overline{\chi}(a)\sum_{\substack{\rho\in\mathcal{Z}(\chi)\\ |\Im(\rho)|\leq n}}\frac{t^{\rho+1}}{\rho(\rho+1)}\right|$$
$$\leq|g'(t)|\cdot\left|\sum_{\chi}\overline{\chi}(a)\left(\sum_{\substack{\rho\in\mathcal{Z}(\chi)\\ \Re(\rho)=1/2\\ |\Im(\rho)|\leq n}}\frac{t^{\rho+1}}{\rho(\rho+1)}+\sum_{\substack{\rho\in\mathcal{Z}(\chi)\\ \Re(\rho)=0\\ |\Im(\rho)|\leq n}}\frac{t^{\rho+1}}{\rho(\rho+1)}\right)+\sum_{\chi}\overline{\chi}(a)\sum_{\substack{\rho\in\mathcal{Z}(\chi)\\ \Re(\rho)\neq 0,1/2\\ |\Im(\rho)|\leq n}}\frac{t^{\rho+1}}{\rho(\rho+1)}\right|,$$

where, since zeroes are counted with multiplicity, the final sum becomes

$$\sum_{\substack{\rho\in\cup_{\chi}\mathcal{Z}(\chi)\\ \Re(\rho)\neq 0,1/2\\ |\Im(\rho)\leq n}}\frac{\sum_{\chi}\overline{\chi}(a)m_{\rho}(\chi)}{\rho(\rho+1)}t^{\rho+1}=0,$$

by way of (6.2). We now have

$$|f_n(t)| \leq |g'(t)| \left(t^{\frac{3}{2}} \sum_{\chi} \sum_{\substack{\rho \in \mathcal{Z}(\chi) \\ \Re(\rho)=1/2 \\ |\Im(\rho)| \leq n}} \frac{1}{|\rho(\rho+1)|} + t \sum_{\chi} \sum_{\substack{\rho \in \mathcal{Z}(\chi) \\ \Re(\rho)=0 \\ |\Im(\rho)| \leq n}} \frac{1}{|\rho(\rho+1)|} \right).$$

Removing the restriction on $\Im(\rho)$ yields

$$|f_n(t)| \leq |g'(t)| \left(t^{\frac{3}{2}} \sum_{\chi} \sum_{\substack{\rho \in \mathcal{Z}(\chi) \\ \Re(\rho)=1/2}} \frac{1}{|\rho(\rho+1)|} + t \sum_{\chi} \sum_{\substack{\rho \in \mathcal{Z}(\chi) \\ \Re(\rho)=0}} \frac{1}{|\rho(\rho+1)|} \right),$$

where this upper bound is integrable on (x, ∞) for $x > 1$, since

$$g'(t) = -\frac{1}{t^3} \left(\frac{2}{\log t} + \frac{3}{\log^2 t} + \frac{2}{\log^3 t} \right).$$

Hence, the dominated convergence theorem allows us to note that

$$\sum_{\chi} \overline{\chi}(a) \sum_{\rho \in \mathcal{Z}(\chi)} g'(t) \frac{t^{\rho+1}}{\rho(\rho+1)}$$

is integrable, i.e.,

$$\int_x^{\infty} \sum_{\chi} \overline{\chi}(a) \sum_{\rho \in \mathcal{Z}(\chi)} g'(t) \frac{t^{\rho+1}}{\rho(\rho+1)} \, \mathrm{d}t < \infty.$$

Dividing the previous integral by $\varphi(q)$, it follows that

$$\int_x^{\infty} \frac{1}{\varphi(q)} \sum_{\chi} \overline{\chi}(a) \sum_{\rho \in \mathcal{Z}(\chi)} g'(t) \frac{t^{\rho+1}}{\rho(\rho+1)} \, \mathrm{d}t = \frac{1}{\varphi(q)} \sum_{\chi} \overline{\chi}(a) \Big(\sum_{\rho \in \mathcal{Z}(\chi)} \int_x^{\infty} g'(t) \frac{t^{\rho+1}}{\rho(\rho+1)} \, \mathrm{d}t \Big). \tag{6.3}$$

In Theorem 6.1 let $\hat{g}(x;q,a)$ represent the expression

$$\begin{aligned}\hat{g}(x;q,a) = &-\frac{1}{\varphi(q)}\sum_{\chi}\overline{\chi}(a)\sum_{n=1}^{\infty}\frac{x^{-2n+1-\alpha}}{2n(2n-1+2\alpha)}\\ &+x\left(\frac{1}{\varphi(q)}\sum_{\chi}\overline{\chi}(a)(m_0(\chi)-b(\chi))\right)\\ &-x\log x\left(\frac{1}{\varphi(q)}\sum_{\chi}\overline{\chi}(a)m_0(\chi)\right)+\log x\left(\frac{1}{\varphi(q)}\sum_{\chi\text{ odd}}\overline{\chi}(a)\right)\\ &+\left(\frac{1}{\varphi(q)}\sum_{\chi\text{ even}}\overline{\chi}(a)\frac{L'}{L}(-1,\chi)+\frac{1}{\varphi(q)}\sum_{\chi\text{ odd}}\overline{\chi}(a)(c(\chi)+1)\right).\end{aligned} \tag{6.4}$$

Then by isolating the sums over zeroes in Theorem 6.1, we have

$$\frac{1}{\varphi(q)}\sum_{\chi}\overline{\chi}(a)\sum_{\rho\in Z(\chi)}\frac{x^{\rho+1}}{\rho(\rho+1)} = \frac{x^2}{2\varphi(q)} - \int_1^x \psi(t;q,a)\,\mathrm{d}t + \hat{g}(x;q,a). \tag{6.5}$$

Now consider the left-hand of (6.3). By (6.5), we have

$$\int_x^{\infty} g'(t)\Big(\frac{1}{\varphi(q)}\sum_{\chi}\overline{\chi}(a)\sum_{\rho\in Z(\chi)}\frac{t^{\rho+1}}{\rho(\rho+1)}\Big)\,\mathrm{d}t = \int_x^{\infty} g'(t)\Big(\frac{t^2}{2\varphi(q)} - \int_1^t \psi(r;q,a)\,\mathrm{d}r + \hat{g}(t;q,a)\Big)\,\mathrm{d}t.$$

Integrating by parts with $v = g(t)$ and $u = \frac{t^2}{2\varphi(q)} - \int_1^t \psi(r;q,a)\,\mathrm{d}r + \hat{g}(t;q,a)$ yields

$$\begin{aligned}&\int_x^{\infty} g'(t)\Big(\frac{1}{\varphi(q)}\sum_{\chi}\overline{\chi}(a)\sum_{\rho\in Z(\chi)}\frac{t^{\rho+1}}{\rho(\rho+1)}\Big)\,\mathrm{d}t\\ &= g(t)\left(\frac{t^2}{2\varphi(q)} - \int_1^t \psi(r;q,a)\,\mathrm{d}r + \hat{g}(t;q,a)\right)\Big|_x^{\infty}\\ &- \int_x^{\infty} g(t)\mathrm{d}\left(\frac{t^2}{2\varphi(q)} - \int_1^t \psi(r;q,a)\,\mathrm{d}r + \hat{g}(t;q,a)\right).\end{aligned}$$

Upon evaluating the endpoints in the first term, while simultaneously extracting the integral over $g(t)\hat{g}'(t;q,a)$ from the second term, we have

$$\begin{aligned}
&\int_x^\infty g'(t)\Big(\frac{1}{\varphi(q)}\sum_\chi \overline{\chi}(a)\sum_{\rho\in\mathcal{Z}(\chi)}\frac{t^{\rho+1}}{\rho(\rho+1)}\Big)\,\mathrm{d}t \\
&=\lim_{t\to\infty}\left[g(t)\left(\frac{t^2}{2\varphi(q)}-\int_1^t\psi(r;q,a)\,\mathrm{d}r+\hat{g}(t;q,a)\right)\right] \\
&+g(x)\left(\int_1^x\psi(t;q,a)\,\mathrm{d}t-\hat{g}(x;q,a)-\frac{x^2}{2\varphi(q)}\right) \\
&+\int_x^\infty g(t)\left(\psi(t;q,a)-\frac{t}{\varphi(q)}\right)\mathrm{d}t-\int_x^\infty g(t)\hat{g}'(t;q,a)\,\mathrm{d}t. \qquad (6.6)
\end{aligned}$$

The bracketed expression

$$\left(\frac{t^2}{2\varphi(q)}-\int_1^t\psi(r;q,a)\,\mathrm{d}r+\hat{g}(t;q,a)\right) \qquad (6.7)$$

in the limit above is simply $\frac{1}{\varphi(q)}\sum_\chi\overline{\chi}(a)\sum_{\rho\in\mathcal{Z}(\chi)}\frac{t^{\rho+1}}{\rho(\rho+1)}$ by (6.5) and every ρ in $\cup_\chi\mathcal{Z}(\chi)$ has real part less than 1. Hence, (6.7) is $o\left(t^2\right)$. Recalling that

$$g(t)=\frac{\log t+1}{t^2\log^2 t}=\frac{1}{t^2}\cdot\frac{\log t+1}{\log^2 t},$$

we observe that in (6.6) the limit vanishes. As we know that

$$J(x;q,a)=\int_x^\infty R(t;q,a)g(t)\,\mathrm{d}t=\int_x^\infty g(t)\left(\psi(t;q,a)-\frac{t}{\varphi(q)}\right)\mathrm{d}t,$$

we have determined

$$\begin{aligned}
&\int_x^\infty\frac{1}{\varphi(q)}\sum_\chi\overline{\chi}(a)\sum_{\rho\in\mathcal{Z}(\chi)}g'(t)\frac{t^{\rho+1}}{\rho(\rho+1)}\,\mathrm{d}t \\
=&g(x)\left(\int_1^x\psi(t;q,a)\,\mathrm{d}t-\hat{g}(x;q,a)-\frac{x^2}{2\varphi(q)}\right)+J(x;q,a)-\hat{J}(x;q,a), \qquad (6.8)
\end{aligned}$$

where $\hat{J}(x;q,a)=\int_x^\infty g(t)\hat{g}'(x;q,a)\mathrm{d}t$.

Now we consider the right-hand side of (6.3). Here, we integrate $\int_x^\infty g'(t)\frac{t^{\rho+1}}{\rho(\rho+1)}\,\mathrm{d}t$ by parts with $v = g(t)$ and $u = \frac{t^{\rho+1}}{\rho(\rho+1)}$ to obtain

$$\begin{aligned}
&\frac{1}{\varphi(q)}\sum_{\chi}\overline{\chi}(a)\Big(\sum_{\rho\in\mathcal{Z}(\chi)}\int_x^{\infty} g'(t)\frac{t^{\rho+1}}{\rho(\rho+1)}\,\mathrm{d}t\Big)\\
&=\frac{1}{\varphi(q)}\sum_{\chi}\overline{\chi}(a)\left(\sum_{\rho\in\mathcal{Z}(\chi)}\left(-\frac{g(x)x^{\rho+1}}{\rho(\rho+1)}-\int_x^{\infty}\frac{g(t)t^{\rho}}{\rho}\,\mathrm{d}t\right)\right).
\end{aligned}$$

Distributing both sums, we see that

$$\begin{aligned}
&\frac{1}{\varphi(q)}\sum_{\chi}\overline{\chi}(a)\left(\sum_{\rho\in\mathcal{Z}(\chi)}-\frac{g(x)x^{\rho+1}}{\rho(\rho+1)}-\sum_{\rho\in\mathcal{Z}(\chi)}\int_x^{\infty}\frac{g(t)t^{\rho}}{\rho}\,\mathrm{d}t\right)\\
&=-g(x)\left(\frac{1}{\varphi(q)}\sum_{\chi}\overline{\chi}(a)\sum_{\rho\in\mathcal{Z}(\chi)}\frac{x^{\rho+1}}{\rho(\rho+1)}\right)-\frac{1}{\varphi(q)}\sum_{\chi}\overline{\chi}(a)\sum_{\rho\in\mathcal{Z}(\chi)}\frac{1}{\rho}\int_x^{\infty}g(t)t^{\rho}\,\mathrm{d}t.
\end{aligned}$$

Using (6.5) and recalling that $F_\rho(x) = \int_x^{\infty} g(t)t^{\rho}\,\mathrm{d}t$, we can write

$$\begin{aligned}
&\frac{1}{\varphi(q)}\sum_{\chi}\overline{\chi}(a)\Big(\sum_{\rho\in\mathcal{Z}(\chi)}\int_x^{\infty} g'(t)\frac{t^{\rho+1}}{\rho(\rho+1)}\,\mathrm{d}t\Big)\\
&=g(x)\left(\int_1^x \psi(t;q,a)\,\mathrm{d}t-\hat{g}(x;q,a)-\frac{x^2}{2\varphi(q)}\right)-\frac{1}{\varphi(q)}\sum_{\chi}\overline{\chi}(a)\sum_{\rho\in\mathcal{Z}(\chi)}\frac{F_\rho(x)}{\rho}. \qquad (6.9)
\end{aligned}$$

Taking (6.3), (6.8), and (6.9) in tandem (and isolating $J(x;q,a)$), we arrive at the identity

$$J(x;q,a) = -\frac{1}{\varphi(q)}\sum_{\chi}\overline{\chi}(a)\sum_{\rho\in\mathcal{Z}(\chi)}\frac{F_\rho(x)}{\rho}+\hat{J}(x;q,a). \qquad (6.10)$$

□

6.2 Estimates for $J(x;q,a)$

The identity (6.10) for $J(x;q,a)$ established in Lemma 6.2 allows us to estimate the size of $J(x;q,a)$, conditional upon Conjecture 3.17. Consider, in the aforementioned identity, the expression

$$-\frac{1}{\varphi(q)}\sum_{\chi}\overline{\chi}(a)\sum_{\rho\in\mathcal{Z}(\chi)}\frac{F_\rho(x)}{\rho}. \tag{6.11}$$

We know that $\mathcal{Z}(\chi)$ will contain zeroes ρ satisfying $\Re(\rho)=0$ if and only if χ is an imprimitive character, since these will be the zeroes arising from the deleted factors in the Euler product of $L(s,\chi)$. Separating these zeroes from the others in (6.11), we obtain

$$\begin{aligned}&-\frac{1}{\varphi(q)}\sum_{\chi}\overline{\chi}(a)\sum_{\rho\in\mathcal{Z}(\chi)}\frac{F_\rho(x)}{\rho}\\ &=\frac{1}{\varphi(q)}\left[\sum_{\chi}\overline{\chi}(a)\sum_{\rho\in\mathcal{Z}(\chi')}\frac{-F_\rho(x)}{\rho}+\sum_{\chi\text{ imp.}}\overline{\chi}(a)\sum_{\substack{\rho\in\mathcal{Z}(\chi)\\ \Re(\rho)=0}}\frac{-F_\rho(x)}{\rho}\right],\end{aligned} \tag{6.12}$$

where χ' is the character which induces χ. Conjecture 3.17 implies for zeroes with real part neither 0 nor $1/2$ in the above sums,

$$\sum_{\chi}\overline{\chi}(a)m_\rho(\chi)=0,$$

where $m_\rho(\chi)$ is the multiplicity of ρ as a zero of $L(s,\chi)$. Therefore

$$\sum_{\chi}\overline{\chi}(a)\sum_{\rho\in\mathcal{Z}(\chi')}\frac{-F_\rho(x)}{\rho}=\sum_{\chi}\overline{\chi}(a)\sum_{\substack{\rho\in\mathcal{Z}(\chi')\\ \Re(\rho)=1/2}}\frac{-F_\rho(x)}{\rho}. \tag{6.13}$$

Our goal moving forward will be to individually bound the two sums over χ in (6.11) and bound $\hat{J}(x;q,a)$ in order to provide an overall bound for $J(x;q,a)$ via (6.10).

By Lemma 4.9, we can write

$$-\frac{F_\rho(x)}{\rho}=\frac{x^{\rho-1}}{\rho(\rho-1)\log x}-\frac{r_\rho(x)}{\rho}, \tag{6.14}$$

where, if $\Re(\rho)=1/2$ and $x>e^2$,

$$\begin{aligned}\left|\frac{r_\rho(x)}{\rho}\right| &\le \left|\frac{\rho}{\rho(1-\rho)^2}\right|\left(\frac{1}{\sqrt{x}\log^2 x}\right)\left(1+\frac{2}{|\rho-1|\log x}\right)\\ &\le \frac{3}{|\rho(\rho-1)|\sqrt{x}\log^2 x}.\end{aligned} \tag{6.15}$$

Here we applied the representation $\rho=|\rho|e^{i\theta}$ to obtain

$$\left|\frac{\rho}{1-\rho}\right| = \left|\frac{|\rho|e^{i\theta}}{|\rho|e^{-i\theta}}\right| = |e^{2i\theta}| = 1$$

and $|\rho-1|=\sqrt{1/4+\Im(\rho)^2}\ge 1/2$, implying, for $x>e^2$

$$\left(1+\frac{2}{|\rho-1|\log x}\right)\le 3.$$

Applying (6.14) to the first sum over χ on the right of (6.12), we have

$$\begin{aligned}&\sum_\chi \overline{\chi}(a)\sum_{\substack{\rho\in\mathcal{Z}(\chi')\\ \Re(\rho)=1/2}}\frac{-F_\rho(x)}{\rho}\\ &= \frac{-1}{\sqrt{x}\log x}\sum_\chi\overline{\chi}(a)\sum_{\substack{\rho\in\mathcal{Z}(\chi')\\ \Re(\rho)=1/2}}\frac{x^{i\Im(\rho)}}{\rho(1-\rho)} - \sum_\chi\overline{\chi}(a)\sum_{\substack{\rho\in\mathcal{Z}(\chi')\\ \Re(\rho)=1/2}}\frac{r_\rho(x)}{\rho}.\end{aligned} \tag{6.16}$$

Now, we have

$$\left|\frac{-1}{\sqrt{x}\log x}\sum_\chi\overline{\chi}(a)\sum_{\substack{\rho\in\mathcal{Z}(\chi')\\ \Re(\rho)=1/2}}\frac{x^{i\Im(\rho)}}{\rho(1-\rho)}\right| \le \frac{1}{\sqrt{x}\log x}\sum_\chi\sum_{\substack{\rho\in\mathcal{Z}(\chi')\\ \Re(\rho)=1/2}}\frac{1}{|\rho(1-\rho)|} = \frac{\mathcal{F}_q}{\sqrt{x}\log x}, \tag{6.17}$$

where $\mathcal{F}_q$ is

$$\mathcal{F}_q = \sum_\chi\sum_{\substack{\rho\in\mathcal{Z}(\chi')\\ \Re(\rho)=1/2}}\frac{1}{\rho(1-\rho)}.$$

Note that $\mathcal{F}_q$ is positive because $\Re(\rho)=1/2$ implies $\rho(1-\rho)>0$. Furthermore, for $x>e^2$, (6.15) implies

$$\left|-\sum_{\chi}\overline{\chi}(a)\sum_{\substack{\rho\in\mathcal{Z}(\chi')\\ \Re(\rho)=1/2}}\frac{r_\rho(x)}{\rho}\right|\le\sum_{\chi}\sum_{\substack{\rho\in\mathcal{Z}(\chi')\\ \Re(\rho)=1/2}}\frac{3}{\rho(1-\rho)\sqrt{x}\log^2 x}=\frac{3}{\sqrt{x}\log^2 x}\mathcal{F}_q. \tag{6.18}$$

In summary, for $x>e^2$ by (6.14), (6.17), and (6.18) we have

$$\begin{aligned}\left|\sum_{\chi}\overline{\chi}(a)\sum_{\substack{\rho\in\mathcal{Z}(\chi')\\ \Re(\rho)=1/2}}\frac{-F_\rho(x)}{\rho}\right| &\le\left|\frac{1}{\sqrt{x}\log x}\sum_{\chi}\overline{\chi}(a)\sum_{\substack{\rho\in\mathcal{Z}(\chi')\\ \Re(\rho)=1/2}}\frac{x^{i\Im(\rho)}}{\rho(1-\rho)}\right|+\left|\sum_{\chi}\overline{\chi}(a)\sum_{\substack{\rho\in\mathcal{Z}(\chi')\\ \Re(\rho)=1/2}}\frac{r_\rho(x)}{\rho}\right|\\ &\le\frac{\mathcal{F}_q}{\sqrt{x}\log x}+\frac{3\mathcal{F}_q}{\sqrt{x}\log^2 x}=\frac{\left(1+\frac{3}{\log x}\right)\mathcal{F}_q}{\sqrt{x}\log x}.\end{aligned} \tag{6.19}$$

We now turn our attention to the sum over imprimitive χ in (6.12). Mirroring the method by which (6.15) was obtained, we may apply Lemma 4.9 for zeroes with $\Re(\rho)=0$ and $x>e^2$, yielding

$$-\frac{F_\rho(x)}{\rho}=\frac{x^{\rho-1}}{\rho(\rho-1)\log x}-\frac{r_\rho(x)}{\rho}, \tag{6.20}$$

with

$$\begin{aligned}\left|\frac{r_\rho(x)}{\rho}\right| &\le\left|\frac{\rho}{\rho(1-\rho)^2}\right|\left(\frac{x^{-1}}{|(1-\rho)|\log^2 x}+\frac{2x^{-1}}{|\rho-1|^2\log^3 x}\right)\\ &\le\left|\frac{\rho}{\rho(1-\rho)^2}\right|\left(\frac{1}{|1-\rho|x\log^2 x}\right)\left(1+\frac{2}{|\rho-1|\log x}\right)\\ &\le\frac{2}{|\rho(1-\rho)|x\log^2 x}.\end{aligned} \tag{6.21}$$

Here we applied

$$\left|\frac{\rho}{1-\rho}\right|=\frac{|\Im(\rho)|}{\sqrt{\Im(\rho)^2+1}}\le 1$$

and $|\rho-1|=\sqrt{1+\Im(\rho)^2}>1$. Applying (6.20) to the sum over imprimitive characters in

(6.12), we obtain

$$\begin{aligned}
&\sum_{\chi \text{ imp.}} \overline{\chi}(a) \sum_{\substack{\rho\in\mathcal{Z}(\chi)\\ \Re(\rho)=0}} \frac{-F_\rho(x)}{\rho} \\
&= \frac{-1}{x\log x} \sum_{\chi \text{ imp.}} \overline{\chi}(a) \sum_{\substack{\rho\in\mathcal{Z}(\chi)\\ \Re(\rho)=0}} \frac{x^{i\Im(\rho)}}{\rho(1-\rho)} - \sum_{\chi \text{ imp.}} \overline{\chi}(a) \sum_{\substack{\rho\in\mathcal{Z}(\chi)\\ \Re(\rho)=0}} \frac{r_\rho(x)}{\rho}. \qquad (6.22)
\end{aligned}$$

Taking absolute values in (6.22) we obtain

$$\begin{aligned}
&\left|\sum_{\chi \text{ imp.}} \overline{\chi}(a) \sum_{\substack{\rho\in\mathcal{Z}(\chi)\\ \Re(\rho)=0}} \frac{-F_\rho(x)}{\rho}\right| \\
&\leq \left|\frac{-1}{x\log x} \sum_{\chi \text{ imp.}} \overline{\chi}(a) \sum_{\substack{\rho\in\mathcal{Z}(\chi)\\ \Re(\rho)=0}} \frac{x^{i\Im(\rho)}}{\rho(1-\rho)}\right| + \left|\sum_{\chi \text{ imp.}} \overline{\chi}(a) \sum_{\substack{\rho\in\mathcal{Z}(\chi)\\ \Re(\rho)=0}} \frac{r_\rho(x)}{\rho}\right|. \qquad (6.23)
\end{aligned}$$

Therefore,

$$\left|\sum_{\chi \text{ imp.}} \overline{\chi}(a) \sum_{\substack{\rho\in\mathcal{Z}(\chi)\\ \Re(\rho)=0}} \frac{-F_\rho(x)}{\rho}\right| w \leq \frac{1}{x\log x} \sum_{\chi \text{ imp.}} \sum_{\substack{\rho\in\mathcal{Z}(\chi)\\ \Re(\rho)=0}} \frac{1}{|\rho(1-\rho)|} + \sum_{\chi \text{ imp.}} \sum_{\substack{\rho\in\mathcal{Z}(\chi)\\ \Re(\rho)=0}} \left|\frac{r_\rho(x)}{\rho}\right| \qquad (6.24)$$

Applying (6.21) in the last sum of (6.23), we establish for $x > e^2$

$$\begin{aligned}
&\left|\sum_{\chi \text{ imp.}} \overline{\chi}(a) \sum_{\substack{\rho\in\mathcal{Z}(\chi)\\ \Re(\rho)=0}} \frac{-F_\rho(x)}{\rho}\right| \\
&\leq \frac{1}{x\log x} \sum_{\chi \text{ imp.}} \sum_{\substack{\rho\in\mathcal{Z}(\chi)\\ \Re(\rho)=0}} \frac{1}{|\rho(1-\rho)|} + \sum_{\chi \text{ imp.}} \sum_{\substack{\rho\in\mathcal{Z}(\chi)\\ \Re(\rho)=0}} \frac{2}{|\rho(1-\rho)|x\log^2 x} \\
&\leq \frac{\mathcal{G}_q}{x\log x} + \frac{2\mathcal{G}_q}{x\log^2 x} = \frac{\left(1+\frac{2}{\log x}\right)\mathcal{G}_q}{x\log x}, \qquad (6.25)
\end{aligned}$$

where we let

$$\mathcal{G}_q = \sum_{\chi} \sum_{\substack{\rho \in \mathcal{Z}(\chi) \\ \Re(\rho)=0}} \frac{1}{|\rho(1-\rho)|}.$$

Recalling (6.12) and bringing (6.19) and (6.25) together, we deduce the inequality

$$\left| -\frac{1}{\varphi(q)} \sum_{\chi} \overline{\chi}(a) \sum_{\rho \in \mathcal{Z}(\chi)} \frac{F_\rho(x)}{\rho} \right| \leq \frac{1}{\varphi(q)} \left[\frac{\left(1+\frac{3}{\log x}\right)\mathcal{F}_q}{\sqrt{x}\log x} + \frac{\left(1+\frac{2}{\log x}\right)\mathcal{G}_q}{x\log x} \right]. \tag{6.26}$$

Referring to (6.10), in order to bound $J(x;q,a)$, we now need to consider $\hat{J}(x;q,a)$. Recall from Lemma 6.2 that

$$\hat{J}(x;q,a) = \int_x^\infty g(t)\hat{g}'(t;q,a)\,\mathrm{d}t = \int_x^\infty \hat{g}'(t;q,a)\,\mathrm{d}\left(\frac{-1}{t\log t}\right), \tag{6.27}$$

with

$$g(t) = \frac{\log x + 1}{x^2 \log^2 x} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{-1}{t\log t}\right)$$

and $\hat{g}'(t;q,a)$ is given by

$$\begin{aligned}\hat{g}'(t;q,a) = \frac{1}{\varphi(q)} &\left[\sum_{\chi \text{ odd}} \overline{\chi}(a) \sum_{n=1}^{\infty} \frac{t^{-2n-1}}{2n+1} + \sum_{\chi \text{ even}} \overline{\chi}(a) \sum_{n=1}^{\infty} \frac{t^{-2n}}{2n} + \sum_{\chi \text{ odd}} \overline{\chi}(a) t^{-1} \right] \\ &- \frac{1}{\varphi(q)} \left[\sum_{\chi} \overline{\chi}(a) b(\chi) + \sum_{\chi} \overline{\chi}(a) m_0(\chi) \log t \right]. \end{aligned} \tag{6.28}$$

In order to bound $\hat{J}(x;q,a)$, we aim to bound $\hat{g}'(t;q,a)$ in absolute value. Starting with the sums over characters with specific parity, if we write

$$\nu(k;q,a) = \begin{cases} \sum_{\chi \text{ odd}} \overline{\chi}(a) & \text{if } k \text{ is odd,} \\ \sum_{\chi \text{ even}} \overline{\chi}(a) & \text{if } k \text{ is even,} \end{cases}$$

then we may observe that the sums over odd characters in (6.28) index all of the terms corresponding to odd k in

$$\sum_{k=1}^{\infty} \frac{\nu(k;q,a)t^{-k}}{k}.$$

Likewise, the sum over even characters indexes all of the terms corresponding to even k, so that we obtain

$$\sum_{\chi \text{ odd}} \overline{\chi}(a) \sum_{n=1}^{\infty} \frac{t^{-2n-1}}{2n+1} + \sum_{\chi \text{ even}} \overline{\chi}(a) \sum_{n=1}^{\infty} \frac{t^{-2n}}{2n} + \sum_{\chi \text{ odd}} \overline{\chi}(a)t^{-1} = \sum_{k=1}^{\infty} \frac{\nu(k;q,a)t^{-k}}{k}. \tag{6.29}$$

Since there are $\frac{\varphi(q)}{2}$ even and odd characters respectively, $|\nu(x;q,a)| \leq \frac{\varphi(q)}{2}$ and therefore, for $t > 1$,

$$\left| \sum_{k=1}^{\infty} \frac{\nu(k;q,a)t^{-k}}{k} \right| \leq \frac{\varphi(q)}{2} \sum_{k=1}^{\infty} \frac{t^{-k}}{k} = -\frac{\varphi(q)}{2} \log\left(1 - \frac{1}{t}\right).$$

Observe that $-\log(1-\frac{1}{t})$ is always positive and decreasing on $(1,\infty)$, and therefore for $t > e^4$,

$$\begin{aligned} &\frac{1}{\varphi(q)} \left[\sum_{\chi \text{ odd}} \overline{\chi}(a) \sum_{n=1}^{\infty} \frac{t^{-2n-1}}{2n+1} + \sum_{\chi \text{ even}} \overline{\chi}(a) \sum_{n=1}^{\infty} \frac{t^{-2n}}{2n} + \sum_{\chi \text{ odd}} \overline{\chi}(a)t^{-1} \right] \\ &= -\frac{1}{2} \log\left(1 - \frac{1}{t}\right) \leq -\frac{1}{2} \log\left(1 - \frac{1}{e^4}\right) \leq 0.01. \end{aligned} \tag{6.30}$$

Now consider from (6.28) the expression $-\sum_{\chi} \overline{\chi}(a)b(\chi)$. Taking absolute values, we have

$$\left| \sum_{\chi} \overline{\chi}(a)b(\chi) \right| \leq \sum_{\chi} |b(\chi)| \stackrel{\text{def}}{=} \mathcal{B}_q, \tag{6.31}$$

where $\mathcal{B}_q$ is a constant independent of t. Finally we consider from (6.28) the expression $-\sum_{\chi} \overline{\chi}(a)m_0(\chi)\log t$. In absolute value, we have

$$\left| \sum_{\chi} \overline{\chi}(a)m_0(\chi)\log t \right| \leq \log t \sum_{\chi} m_0(\chi) = \mathcal{M}_q \log t, \tag{6.32}$$

where $\mathcal{M}_q \overset{\text{def}}{=} \sum_\chi m_0(\chi)$ is also independent of t. Returning to (6.28), we can take the absolute value and apply (6.30), (6.31), and (6.32) to determine

$$\begin{aligned}|\hat{g}'(t;q,a)| &\le \frac{1}{\varphi(q)}\left|\sum_{\chi \text{ odd}} \overline{\chi}(a)\sum_{n=1}^{\infty}\frac{t^{-2n-1}}{2n+1} + \sum_{\chi \text{ even}} \overline{\chi}(a)\sum_{n=1}^{\infty}\frac{t^{-2n}}{2n} + \sum_{\chi \text{ odd}} \overline{\chi}(a)t^{-1}\right| \\ &\quad + \frac{1}{\varphi(q)}\left|\sum_{\chi}\overline{\chi}(a)b(\chi)\right| + \frac{1}{\varphi(q)}\left|\sum_{\chi}\overline{\chi}(a)m_0(\chi)\log t\right| \\ &\le 0.01 + \frac{\mathcal{B}_q}{\varphi(q)} + \frac{\mathcal{M}_q \log t}{\varphi(q)}, \end{aligned} \tag{6.33}$$

for $t > e^4$. With this bound in place, we may now turn our attention to estimating $\hat{J}'(x;q,a)$. Suppose $x > e^4$. Then we have, by (6.33),

$$\begin{aligned}\left|\hat{J}(x;q,a)\right| &= \left|\int_x^\infty \hat{g}'(t;q,a)\,\mathrm{d}\left(\frac{-1}{t\log t}\right)\right| \le \int_x^\infty \left|\hat{g}'(t;q,a)\right|\,\mathrm{d}\left(\frac{-1}{t\log t}\right) \\ &\le \int_x^\infty \left(0.01 + \frac{\mathcal{B}_q}{\varphi(q)}\right)\mathrm{d}\left(\frac{-1}{t\log t}\right) + \int_x^\infty \frac{\mathcal{M}_q \log t}{\varphi(q)}\,\mathrm{d}\left(\frac{-1}{t\log t}\right). \end{aligned} \tag{6.34}$$

The first integral in (6.34) is equal to

$$\int_x^\infty \left(0.01 + \frac{\mathcal{B}_q}{\varphi(q)}\right)\mathrm{d}\left(\frac{-1}{t\log t}\right) = \frac{0.01 + \frac{\mathcal{B}_q}{\varphi(q)}}{x\log x}. \tag{6.35}$$

For the second integral in (6.34), integration by parts with $u = \log t$ and $v = \frac{-1}{t\log t}$ provides

$$\begin{aligned}\int_x^\infty \frac{\mathcal{M}_q \log t}{\varphi(q)}\,\mathrm{d}\left(\frac{-1}{t\log t}\right) &= \frac{\mathcal{M}_q}{\varphi(q)}\left[\frac{1}{x} + \int_x^\infty \frac{1}{t^2\log t}\,\mathrm{d}t\right] \\ &\le \frac{\mathcal{M}_q}{\varphi(q)}\left[\frac{1}{x} + \frac{1}{\log x}\int_x^\infty \frac{1}{t^2}\,\mathrm{d}t\right] \\ &= \frac{\mathcal{M}_q}{\varphi(q)x} + \frac{\mathcal{M}_q}{\varphi(q)x\log x}. \end{aligned} \tag{6.36}$$

Combining (6.34) with (6.35) and (6.36), we obtain

$$\left|\hat{J}(x;q,a)\right| \le \frac{0.01\varphi(q) + \mathcal{B}_q + \mathcal{M}_q}{\varphi(q)x\log x} + \frac{\mathcal{M}_q}{\varphi(q)x}. \tag{6.37}$$

The discussion of this section may now be summarized as follows.

Proposition 6.3. *Let q and a be coprime natural numbers. If Conjecture 3.17 holds, then, for $x > e^4$, we have*

$$J(x;q,a) \leq \frac{1}{\varphi(q)}\left[\frac{\left(1+\frac{3}{\log x}\right)\mathcal{F}_q}{\sqrt{x}\log x} + \frac{\left(1+\frac{2}{\log x}\right)\mathcal{G}_q + 0.01\varphi(q) + \mathcal{B}_q + \mathcal{M}_q}{x\log x} + \frac{\mathcal{M}_q}{x}\right]. \tag{6.38}$$

Proof. The proof follows upon taking the absolute value of (6.10) and then applying (6.26) and (6.37) to their respective terms. □

6.3 Negativity of $\log f(x;q,a)$

Now that we have established upper bounds for $J(x;q,a)$, recall Proposition 4.7 and Corollary 4.8, from which we determine, for fixed $\varepsilon \in (0, \frac{1}{\varphi(q)})$ and x large, that

$$\log f(x;q,a) \leq J(x;q,a) - cF_{\frac{1}{m}}(x) + \frac{1}{2(x-1)}, \tag{6.39}$$

where $m = \mathrm{Ind}_q(a)$ (see Definition 4.3) and c is the positive constant given by

$$c = \mathcal{R}_{q,a}\left(\frac{1}{\varphi(q)} - \varepsilon\right),$$

where, writing $q = 2^{\alpha} p_1^{\alpha_1} \dots p_r^{\alpha_r}$,

$$\mathcal{R}_{q,a} = \begin{cases} \prod_{i=1}^{r}\left(m, \varphi(p_i^{\alpha_i})\right) & \text{if } \alpha \leq 1, \\ (m,2)\left(m,2^{\alpha-2}\right)\prod_{i=1}^{r}\left(m,\varphi(p_i^{\alpha_i})\right) & \text{otherwise.} \end{cases}$$

Applying Proposition 6.3 in (6.39), we have, for $0 < \varepsilon < \frac{1}{\varphi(q)}$ fixed,

$$\log f(x;q,a) \leq \frac{1}{\varphi(q)} \left[\frac{\left(1+\frac{3}{\log x}\right)\mathcal{F}_q}{\sqrt{x}\log x} + \frac{\left(1+\frac{2}{\log x}\right)\mathcal{G}_q + 0.01\varphi(q) + \mathcal{B}_q + \mathcal{M}_q}{x\log x} + \frac{\mathcal{M}_q}{x} \right] - cF_{\frac{1}{m}}(x) + \frac{1}{2(x-1)}. \tag{6.40}$$

Remembering that our goal here is to show that $\log f(x;q,a)$ is negative for large enough x, we observe that the only source of negativity in (6.40) is $-cF_{\frac{1}{m}}(x)$, and that the highest order term in the upper bound is

$$\frac{\mathcal{F}_q}{\varphi(q)\sqrt{x}\log x}.$$

Recall that Corollary 4.10 establishes

$$F_{\frac{1}{m}}(x) \geq \frac{m+1}{mx^{\frac{m-1}{m}}\log x}. \tag{6.41}$$

If $m = 2$,

$$\frac{m+1}{mx^{\frac{m-1}{m}}\log x} = \frac{3}{2\sqrt{x}\log x} \tag{6.42}$$

and thus our negative term is of the same order as the main positive term in (6.40). However, if $m > 2$,

$$\frac{m+1}{mx^{\frac{m-1}{m}}\log x} = o\left(\frac{1}{\sqrt{x}\log x}\right)$$

since $\sqrt{x} < x^{\frac{m-1}{m}}$, and therefore the negative term in (6.40) will fail to negate the highest order term as x tends to infinity. When this happens, the methods of this chapter fail to prove the negativity of $\log f(x;q,a)$. Therefore, in order to proceed we must choose q and a for which $m = 2$. (Note that $a = 1$ will satisfy this condition for every q, since $1^2 \equiv 1$

(mod q)). In this setting, we are concerned with the difference

$$\frac{\mathcal{F}_q}{\varphi(q)\sqrt{x}\log x} - cF_{\frac{1}{2}}(x).$$

By employing (6.41) we have

$$\frac{\mathcal{F}_q}{\varphi(q)\sqrt{x}\log x} - cF_{\frac{1}{2}}(x) \leq \frac{\mathcal{F}_q}{\varphi(q)\sqrt{x}\log x} - \frac{3}{2}\cdot\frac{\left(\frac{1}{\varphi(q)} - \varepsilon\right)\mathcal{R}_{q,a}}{\sqrt{x}\log x}. \tag{6.43}$$

Since ε may be taken to be arbitrarily small by choosing x large, the upper bound of (6.43) can be made negative provided

$$\mathcal{F}_q < \frac{3}{2}\mathcal{R}_{q,a}. \tag{6.44}$$

In summary, when $\mathrm{Ind}_q(a) = 2$, we have by (6.40) and (6.43) that

$$\log f(x;q,a) = \frac{\mathcal{F}_q}{\varphi(q)\sqrt{x}\log x} - \frac{3}{2}\cdot\frac{\left(\frac{1}{\varphi(q)} - \varepsilon\right)\mathcal{R}_{q,a}}{\sqrt{x}\log x} + O\left(\frac{1}{x}\right),$$

implying that for x large enough, if (6.44) holds, $\log f(x;q,a)$ will be negative. Thus, the following has been proven.

Theorem 6.4. *Fix q and choose a for which $Ind_q(a) = 2$. If Conjecture 3.17 holds, then there are at most finitely many primorials n in $S_{q,a}$ for which*

$$\frac{n}{\varphi(n)(\log(\varphi(q)\log n))^{\frac{1}{\varphi(q)}}} \leq \frac{1}{C(q,a)}$$

is satisfied, as long as

$$\frac{\mathcal{F}_q}{\mathcal{R}_{q,a}} \leq \frac{3}{2}.$$

We may combine Theorem 6.4 with Theorem 5.7 to recover an equivalence analogous to Theorem 2.10.

Theorem 6.5. *Fix q and choose a for which* $\mathrm{Ind}_q(a) = 2$ *(i.e., a is a square modulo q). If*

$$\frac{\mathcal{F}_q}{\mathcal{R}_{q,a}} < \frac{3}{2},$$

and $\mathscr{L}(s;q,a)$ *has no singularities on the segment* $(0,1)$*, then the following are equivalent:*

(i) Conjecture 3.17 is true.

(ii) There are at most finitely many primorials in $S_{q,a}$ *for which*

$$\frac{n}{\varphi(n)(\log(\varphi(q)\log n))^{\frac{1}{\varphi(q)}}} \leq \frac{1}{C(q,a)}.$$

6.4 Examples with Small Modulus

It is prudent to verify that the conditions of Theorem 6.5 hold for a variety of choices of coprime q and a. Regarding the singularities of $\mathscr{L}(s;q,a)$, recall that in Section 3.3 two results of Platt [19], Theorems 3.18 and 3.19, allow us to conclude that for L-functions of primitive Dirichlet characters whose modulus q is less than $400,000$, there are no zeroes on the segment $(0,1)$. Since the singularities of

$$\mathscr{L}(s;q,a) = \sum_{\chi \pmod q} \overline{\chi}(a)\frac{L'}{L}(s,\chi)$$

arise directly from the zeroes of $L(s,\chi)$ where χ is a character modulo q, we may conclude that for $q < 400,000$, $\mathscr{L}(s;q,a)$ has no singularities on the segment $(0,1)$.

Therefore, for $q < 400,000$ and a coprime to q such that $\mathrm{Ind}_q(a) = 2$, we only need to verify

$$\frac{\mathcal{F}_q}{\mathcal{R}_{q,a}} < \frac{3}{2}.$$

Recalling that

$$\mathcal{F}_q = \sum_{\chi} \sum_{\substack{\rho \in \mathcal{Z}(\chi') \\ \Re(\rho)=1/2}} \frac{1}{\rho(1-\rho)},$$

we can observe that, under GRH_q,

$$\mathcal{F}_q = \sum_{\chi} \mathcal{F}(\chi), \tag{6.45}$$

where

$$\mathcal{F}(\chi) = \sum_{\rho \in \mathcal{Z}(\chi')} \frac{1}{\rho(1-\rho)}.$$

In particular, when $a = 1$, Conjecture 3.17 is equivalent to GRH_q, and therefore we may employ the results of Corollary 10.18 of [14] in order to determine that

$$\sum_{\rho \in \mathcal{Z}(\chi')} \frac{1}{\rho(1-\rho)} = \log \frac{q}{\pi} + 2\Re\left(\frac{L'}{L}(1, \overline{\chi'})\right) - C - (1-\alpha) 2 \log 2. \tag{6.46}$$

Recall that χ' is the primitive character which induces χ, C is the Euler-Mascheroni constant, and α is 1 if χ' is odd and 0 otherwise. Throughout this section, the matter of computing $\mathcal{F}(\chi)$ via (6.46) has been left to the Python package MPMATH [6], in particular for the computation of the logarithmic derivative

$$\frac{L'}{L}(1, \overline{\chi'}).$$

We begin with a simple example.

Example 6.6. Let $q = 1$. Corollary 10.14 of [14] establishes

$$\mathcal{F}_1 = \sum_{\rho} \frac{1}{\rho(1-\rho)} = 2 + C - \log \pi - 2 \log 2 \approx 0.04619 < 1.5, \tag{6.47}$$

where the sum is over the non-trivial zeroes of $\zeta(s)$. Observe in this context that Conjecture 3.17 is simply Conjecture 2.8, the Riemann Hypothesis. Moreover, the primorials in $S_{1,1}$ are simply the primorials. Since $\varphi(1) = 1$, this example applied to Theorem 6.5 recovers Theorem 2.10.

Example 6.7. Let $q = 2$ and $a = 1$. The only character modulo 2 is imprimitive and induced by **1**. Therefore,

$$\mathcal{F}_2 = \mathcal{F}_1 = 2 + C - \log\pi - 2\log 2 \approx 0.04619.$$

Since $a = 1$, Conjecture 3.17 is equivalent to GRH_2 and the above computation verifies that these are equivalent to there being at most finitely many primorials n in $S_{2,1}$ for which

$$\frac{n}{\varphi(n)\log\log n} < \frac{e^C}{2}.$$

For larger q, we keep (6.47) in mind, since the principal character modulo q will always be induced by the trivial character, and therefore

$$\sum_{\rho\in\mathcal{Z}(\chi_0')} \frac{1}{\rho(1-\rho)} = 2 + C - \log\pi - 2\log 2 \approx 0.04619$$

for any q. When $a = 1$, Conjecture 3.17 is equivalent to GRH_q and therefore may appeal to (6.46) to determine $\mathcal{F}_q$ for several q.

Example 6.8. The values of $\mathcal{F}_q$ for several values of q, computed under GRH_q, are summarized in Table 6.1. In every case, 1 is a square modulo q and by Corollary 4.5

$$\mathcal{R}_{q,1} = \begin{cases} 2^{\omega(q)-1} & \text{if } 2\|q, \\ 2^{\omega(q)} & \text{if } 2\nmid q \text{ or } 4\|q, \\ 2^{\omega(q)+1} & \text{if } 8|q, \end{cases} \geq 2.$$

Therefore, by Theorem 6.5, for $q \in \{3,4,5,6,8,10,12\}$, GRH_q is true if and only if there are at most finitely many primorials n in $S_{q,1}$ for which

$$\frac{n}{\varphi(n)(\log(\varphi(q)\log n))^{\frac{1}{\varphi(q)}}} < \frac{1}{C(q,1)}.$$

Some values associated with Table 6.1 via (6.46) may be found in Tables 6.2 through 6.7 at the end of this section. For consistency, the numbering of the Dirichlet characters

Table 6.1: $\mathcal{F}_q$ for some small values of q

q	$\mathcal{F}_q$	q	$\mathcal{F}_q$
3	0.15942	8	0.75326
4	0.20176	10	0.60919
5	0.60919	12	0.64516
6	0.15942		

follows that of [11].

Example 6.8 is predicated on the fact that Conjecture 3.17 *is* GRH_q when $a = 1$. If a is not 1, the direct assumption of GRH_q implies that (6.45) holds, so that the computations for $\mathcal{F}_q$ hold for a other than 1, although we lose the equivalence found in Theorem 6.5.

Example 6.9. Assume GRH_q. For $q \in \{2,3,4,5,6,8,10,12\}$, observe in Table 6.1 that $\mathcal{F}_q < 1.5$. Pick a for which $\mathrm{Ind}_q(a) = 2$. Then, by Theorem 6.4, there are at most finitely many primorials in $S_{q,a}$ for which

$$\frac{n}{\varphi(n)(\log\varphi(q)\log n)^{\frac{1}{\varphi(q)}}} < \frac{1}{C(q,a)}.$$

One may ask if we can use the results of Theorem 6.4 and Theorem 6.5 to obtain similar results for arbitrarily large q. We provide some evidence that this will not be the case. Consider the following result, due to Ihara-Murty-Shimura.

Theorem 6.10 ([4, Theorem 3])**.** *Under GRH,*

$$\max_{\chi\in X_q}\left|\frac{L'}{L}(1,\chi)\right| \le (2+o(1))\log\log q,$$

where X_q is the set of non-principal, primitive Dirichlet characters modulo q.

If p is a prime, then $|X_p| = p-2$. From the above theorem and (6.46), we therefore conclude that

$$\mathcal{F}_p = \sum_{\chi} \sum_{\rho \in \mathcal{Z}(\chi')} \frac{1}{\rho(1-\rho)} = \varphi(p) \log \frac{p}{\pi} + O(\varphi(p) \log\log p)$$

as $p \to \infty$. On the other hand, for a coprime to p for which $\mathrm{Ind}_q(a) = 2$, Corollary 4.5 established that $\mathcal{R}_{p,a} = 2$. Therefore, under GRH, the ratio

$$\frac{\mathcal{F}_p}{\mathcal{R}_{p,a}}$$

has order of magnitude $\frac{\varphi(p)}{2} \log p$ and will eventually exceed $\frac{3}{2}$ for large enough p. It appears that a similar behavior holds for composite q, though more care is required in handling the contributions of imprimitive characters.

6.5 Tables of Computations

Here we include values related to (6.46) towards the computation of $\mathcal{F}_q$. The numbering of the characters follows [11]. Note that $\chi_q(1,\cdot)$ is always induced by **1** and therefore (6.47) applies for those characters.

Table 6.2: Values relevant to the computation of $\mathcal{F}_3$.

χ	α	$\Re(\frac{L'}{L}(1,\overline{\chi}))$	$\mathcal{F}(\chi)$
$\chi_3(1,\cdot)$	0	———	0.0461914
$\chi_3(2,\cdot)$	1	0.3682816	0.1132300
$\mathcal{F}_3$	0.1594214		

Table 6.3: Values relevant to the computation of $\mathcal{F}_4$.

χ	α	$\Re(\frac{L'}{L}(1,\overline{\chi}))$	$\mathcal{F}(\chi)$
$\chi_4(1,\cdot)$	0	———	0.0461914
$\chi_4(3,\cdot)$	1	0.2456096	0.1555680
$\mathcal{F}_4$	0.2017594		

Table 6.4: Values relevant to the computation of $\mathcal{F}_5$.

χ	α	$\Re(\frac{L'}{L}(1,\overline{\chi}))$	$\mathcal{F}(\chi)$
$\chi_5(1,\cdot)$	0	———	0.0461914
$\chi_5(2,\cdot)$	1	0.1578645	0.2032214
$\chi_5(3,\cdot)$	1	0.1578645	0.2032214
$\chi_5(4,\cdot)$	0	0.8276795	0.1565570
$\mathcal{F}_5$	0.6091912		

Table 6.5: Values relevant to the computation of $\mathcal{F}_6$.

χ	α	$\Re(\frac{L'}{L}(1,\overline{\chi}))$	$\mathcal{F}(\chi)$
$\chi_6(1,\cdot)$	0	———	0.0461914
$\chi_6(5,\cdot)$	1	0.3682816	0.1132300
$\mathcal{F}_6$	0.1594214		

Table 6.6: Values relevant to the computation of $\mathcal{F}_8$.

χ	α	$\Re(\frac{L'}{L}(1,\overline{\chi}))$	$\mathcal{F}(\chi)$
$\chi_8(1,\cdot)$	0	———	0.0461914
$\chi_8(3,\cdot)$	1	-0.0207114	0.3160731
$\chi_8(5,\cdot)$	0	0.6321150	0.2354316
$\chi_8(7,\cdot)$	1	0.2456096	0.1555680
$\mathcal{F}_8$	0.7532641		

Table 6.7: Values relevant to the computation of $\mathcal{F}_{10}$.

χ	α	$\Re(\frac{L'}{L}(1,\overline{\chi}))$	$\mathcal{F}(\chi)$
$\chi_{10}(1,\cdot)$	0	———	0.0461914
$\chi_{10}(3,\cdot)$	1	0.1578645	0.2032214
$\chi_{10}(7,\cdot)$	1	0.1578645	0.2032214
$\chi_{10}(9,\cdot)$	0	0.8276795	0.1565570
$\mathcal{F}_{10}$	0.6091912		

Table 6.8: Values relevant to the computation of $\mathcal{F}_{12}$.

χ	α	$\Re(\frac{L'}{L}(1,\overline{\chi}))$	$\mathcal{F}(\chi)$
$\chi_{12}(1,\cdot)$	0	———	0.0461914
$\chi_{12}(5,\cdot)$	1	0.3682816	0.1132300
$\chi_{12}(7,\cdot)$	1	0.2456096	0.1555680
$\chi_{12}(11,\cdot)$	0	0.4767499	0.3301666
$\mathcal{F}_{12}$	0.6451560		

Chapter 7

Conclusion

Just because some of us can read and write and do a little math,

that doesn't mean we deserve to conquer the Universe.

– Kurt Vonnegut

Recall that we established at the inception of this thesis that our goal was to generalize the results found in [17], as recounted in Chapter 2. The first step in this direction was to generalize Theorem 2.3. This occurred in Chapter 3.

Theorem (Theorem 3.8)**.** *Let $q,a \in \mathbb{N}$ be coprime. Then*

$$\limsup_{n \in S_{q,a}} \frac{n}{\varphi(n)(\log(\varphi(q)\log n))^{1/\varphi(q)}} = \frac{1}{C(q,a)},$$

where $C(q,a)$ is defined in Theorem 3.5.

This led us to ask questions akin to Question 1.

Question (Question 2)**.** Let $q,a \in \mathbb{N}$ be coprime and consider the inequality

$$\frac{n}{\varphi(n)(\log\varphi(q)\log n)^{\frac{1}{\varphi(q)}}} > \frac{1}{C(q,a)}. \tag{7.1}$$

Are there infinitely many $n \in S_{q,a}$ for which (7.1) is satisfied?

In answering this question, we wished to generalize the following equivalence derived from the work of Nicolas [17].

Theorem (Theorem 2.10)**.** *The Riemann Hypothesis is true if and only if there are only finitely many primorials N_k for which*

$$\frac{N_k}{\varphi(N_k)\log\log N_k} \leq e^C.$$

The degree to which we were able to achieve these goals depended, in part, on the choice of a. We distinguish three cases: $a = 1$, a is a square modulo q but not 1, and a is not a square modulo q.

7.1 Primes Congruent to 1 Modulo q

An answer to Question 2 was first determined in Chapter 4.

Theorem (Theorem 4.14)**.** *Let $q \leq 400{,}000$ and a be coprime natural numbers. If $\mathscr{L}(s;q,a)$ has a singularity ρ for which $0 < \Re(\rho) < 1$, then there are infinitely many primorials n in $S_{q,a}$ for which*

$$\frac{n}{\varphi(n)(\log(\varphi(q)\log n))^{\frac{1}{\varphi(q)}}} > \frac{1}{C(q,a)},$$

Here, we see that, when $a = 1$, we may do away with the condition that $\mathscr{L}(s;q,a)$ has a singularity for which $0 < \Re(\rho) < 1$, since

$$\mathscr{L}(s;q,1) = \sum_{\chi \pmod q} \frac{L'}{L}(s,\chi)$$

must have such a singularity arising from the first zero of $L(s,\chi)$ for any character χ modulo q (or simply the first zero of $\zeta(s)$).

However, a conceit of this thesis was that we wished extend this answer to bridge the worlds of the analytic and the arithmetic in a way analogous to Theorem 2.10. One step toward this goal was the following result of Chapter 5.

Theorem (Theorem 5.7)**.** *Let $q \leq 400{,}000$ and a be coprime natural numbers. If Conjec-*

ture 3.17 is false, then there are infinitely many primorials n in $S_{q,a}$ for which

$$\frac{n}{\varphi(n)(\log(\varphi(q)\log n))^{\frac{1}{\varphi(q)}}} > \frac{1}{C(q,a)}$$

holds and also infinitely many primorials n in $S_{q,a}$ for which this inequality does not hold.

Chapter 6 verified the other half of the equivalence required for an analogue of Theorem 2.10.

Theorem (Theorem 6.4)**.** *Fix q and choose a for which $Ind_q(a) = 2$. If Conjecture 3.17 holds, then there are at most finitely many primorials n in $S_{q,a}$ for which*

$$\frac{n}{\varphi(n)(\log(\varphi(q)\log n))^{\frac{1}{\varphi(q)}}} \leq \frac{1}{C(q,a)}$$

is satisfied, as long as

$$\frac{\mathcal{F}_q}{\mathcal{R}_{q,a}} \leq \frac{3}{2}.$$

For $a = 1$, we have observed that GRH_q is equivalent to Conjecture 3.17, and therefore may stand in for Conjecture 3.17 in Theorem 5.7 and Theorem 6.4. The examples of Chapter 6, as supplemented by Theorem 6.5 verify that the goal of generalizing Theorem 2.10 has been achieved for small q, where here "small" means that

$$\frac{\mathcal{F}_q}{\mathcal{R}_{q,1}} < \frac{3}{2}.$$

Exactly, Example 6.8 establishes the following.

Theorem 7.1. *For $q \in \{2,3,4,5,6,8,10,12\}$, GRH_q is true if and only if there are at most finitely many primorials n in $S_{q,1}$ for which*

$$\frac{n}{\varphi(n)(\log\varphi(q)\log n)^{\frac{1}{\varphi(q)}}} \leq \frac{1}{C(q,1)}.$$

As observed in Chapter 6, we expect the ratio

$$\frac{\mathcal{F}_q}{\mathcal{R}_{q,a}}$$

to eventually exceed 3/2 when q is large. However, the computations conducted in Section 6.4 have not been extensive enough to expose this behavior. One may be interested in performing more computational work in order to extend the ranges (for q) of Theorem 7.1. There are two approaches we may consider here. Firstly, we may simply extend our computations of $\mathcal{F}_q$ to larger q. Secondly, we may be interested in sharpening the estimates made in Chapters 4 and 6, to see if we may improve the condition of Theorem 6.4 so that it holds for a wider range of q. However, as discussed at the end of Section 6.4, it seemed that only finitely many q are "small". Another branch of future work in the case $a = 1$ would be to consider the behavior of (7.1) for primorials in $S_{q,1}$, where q is large, to see if we might still expect results similar to Theorem 7.1 to hold in this case.

7.2 Primes Congruent to Squares Other Than 1

The mismatch between GRH_q and Conjecture 3.17 limited the equivalences of Theorem 7.1 to the case where $a = 1$. When a is a square other than 1 modulo q, one may observe in Theorem 4.14 that we cannot be certain that $\mathscr{L}(s;q,a)$ has a singularity ρ for which $0 < \Re(\rho) < 1$. It seems very unlikely that this is false, since that would mean for any zero ρ of some $L(s,\chi)$, we have

$$\sum_{\chi} \overline{\chi}(a) m_{\rho}(\chi) = 0.$$

Theorem 5.7, conversely, never required the assumption regarding singularities of $\mathscr{L}(s;q,a)$. Therefore the only distinction between $a = 1$ and other squares here is that we assume the falsehood of Conjecture 3.17, instead of the falsehood of GRH_q. When a is a square modulo q, we may still obtain versions of Theorem 2.10 by way of Theorems 5.7 and 6.4. However, in this context

$$\mathcal{F}_q = \sum_{\chi} \sum_{\substack{\rho \in \mathcal{Z}(\chi') \\ \Re(\rho)=1/2}} \frac{1}{\rho(1-\rho)}$$

cannot be computed via (6.46). For future work, we could consider instead estimating $\mathcal{F}_q$ using precomputed lists of zeroes up to some height. Alternatively, we may assume GRH_q and compute $\mathcal{F}_q$ as we did for $a = 1$. Because GRH_q implies Conjecture 3.17, we should still expect that, when q is small, (7.1) is satisfied for all but finitely many primorials n in $S_{q,a}$. Again, we may examine the behavior of (7.1) on primorials in $S_{q,a}$ for q large to see if we might expect (7.1) to be similarly satisfied when q is large.

7.3 Primes Congruent to Non-Squares Modulo q

Whether or not a is a square did not factor into the results of Chapters 4 and 5. Therefore Theorems 4.14 and 5.7 are maintained in the same manner as they were for squares other than 1. Nevertheless, the methods of Chapter 6 did not reveal anything regarding the behavior of (7.1) for non-squares a modulo q. In particular, we did not have evidence predicated upon GRH_q that there are only finitely many n in $S_{q,a}$ for which (7.1) is not satisfied. Experimentation may reveal that the behavior of primorials arising from primes congruent to non-squares differs from that of primorials arising from squares.

7.4 Other Considerations

For certain q and a, the results of Chapter 6 established that, $\log f(x;q,a) < 0$ for all x "large enough". However, this work is not explicit in the sense that we did not provide a c_0 for which $\log f(x;q,a) < 0$ for all $x > c_0$. We may consider repeating the arguments of this thesis with explicit constants in order to provide such c_0.

This thesis also has given little attention to generalizing Theorem 3(b) of [17], which establishes that if $f(x) = e^C \log\theta(x) \prod_{p\le x}(1-1/p)$, then

$$\liminf \log f(x)\sqrt{x}\log x = \log\pi + 2\log 2 - 4 - C$$

and

$$\limsup \log f(x)\sqrt{x}\log x \le C - \log\pi - 2\log 2.$$

We may explore similar limits for the generalized function $\log f(x;q,a)$.

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