hep-th/0410209

High frequency quasi-normal modes for black-holes with generic singularities

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(Dated: February 1, 2008)

We compute the high frequency quasi-normal modes (QNM) for scalar perturbations of spherically symmetric single horizon black-holes in $(D+2)$-space-time dimensions with generic curvature singularities and having metrics of the form $ds^2 = \eta x^p(dy^2 - dx^2) + x^q d\Omega_D^2$ near the singularity $x = 0$. The real part of the QN frequencies is shown to be proportional to $\log\left[1 + 2\cos\left(\pi\left[qD - 2\right]/2\right)\right]$ where the constant of proportionality is equal to the Hawking temperature for non-degenerate black-holes and inverse of horizon radius for degenerate black-holes. Apart from agreeing with the QN frequencies that have been computed earlier, our results imply that the horizon area spectrum for the general spherically symmetric black-holes is equispaced. Applying our results, we also find the QNM frequencies for extremal Reissner-Nordström and various stringy black-holes.

PACS numbers: 04.30.-w,04.60.-m,04.70.-s,04.70.Dy

I. INTRODUCTION

Quasi-normal modes (QNM) are classical perturbations with non-vanishing damping propagating in a given gravitational background subject to specific boundary conditions. The frequency and damping of these oscillations depend only on the parameters characterizing the black hole and are completely independent of the particular initial configuration that caused the excitation of such vibrations. Over the last three decades, QNM have been of interest due to their observational significance in the detection of gravitational waves. (For a review, see Ref. [1])

During the last few years there has been renewed interest in QNM for the following two reasons. First, in estimating the thermalization time-scales in connection with the AdS/CFT conjecture [2]. Secondly, and more importantly, it has been conjectured recently that the real part of the QN frequencies for black-hole perturbations (ω_{QNM}) correspond to the minimum energy change of a black-hole undergoing quantum transitions [3, 4].

Earlier, Bekenstein [5, 6] had conjectured using semi-classical arguments that the black-hole area spectrum is equispaced and is of the form

$$A_s = 4\,\log(k)\,\ell_{_{\rm Pl}}^2\,s \qquad s = 1, 2, \cdots, \tag{1}$$

where k is an integer to be determined and $\ell_{_{\rm Pl}}$ is the Planck length. This implies that for a $(D+2)-$dimensional Schwarzschild black-hole, the energy $\omega = \Delta M$ emitted when the black-hole looses one quantum of the area is

$$\omega = \frac{(D-1)\ln k}{4\pi r_h} = (\ln k)\,T_H\,, \tag{2}$$

where r_h is the horizon and T_H is the Hawking temperature.

Hod [3] noticed, based on the numerical result [7], that the QNM spectrum for the Schwarzschild black-hole has a frequency whose real part numerically approached Eq. (2) with $k = 3$ in the limit of infinite imaginary frequency. This was later confirmed analytically by Motl [8] for 4-dimensional Schwarzschild, who showed that the QN frequencies have the following structure:

$$\omega_{_{QNM}} = 2\pi\,i\,T_H\left(n + \frac{1}{2}\right) + T_H\ln(3) + \mathcal{O}(n^{-1/2})\,. \tag{3}$$

Eqs. (1) and (3) were used by Dreyer [4] recently to argue that the Barbero-Immirzi parameter appearing in loop quantum gravity is $\ln(3)/(2\sqrt{2}\pi)$ and that the spin network states in the theory are dominated by $j_m = 1$ representations of the $SU(2)$ group[1]. Together, this also led to the microscopic entropy of the black-hole to be one quarter its horizon area (in Planck units). Turning the argument around, (1) and (3) can also be used to vindicate the point of view that horizon area is equispaced [6, 13].

Subsequently, Motl and Neitzke [14] (see also Ref. [15]) used a more flexible and powerful approach, called the

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[1] Note, however, that the above value of the Barbero-Immirizi parameter does not agree with that predicted by the recent analyses by Domagala and Lewandowski [9], and Meissner [10]. See also Refs. [11, 12]

monodromy method, and were able to compute high-frequency QNM and the asymptotic value of the real part of the QN frequencies for a D-dimensional Schwarzschild. Naturally, attempts were made to compute the QN frequencies for a host of other black-holes. There have been quite a bit of effort to understand the physics underlying the real (see, for instance, Refs. [16–21]) and the imaginary part of the high-frequency QNM (see, for instance, Refs. [22–36]), and there appears to be pieces of evidence in favour of as well as against the predictions referred to above. Thus, it is important to explore the QN frequencies for other black-holes and to probe their full implications.

Even though the monodromy approach does not require the full knowledge of the space-time – except at the singularity, horizons and spatial asymptotic infinity – all the previous analyses have been restricted to space-times whose line-element is known for the whole of the manifold including the recent work by Tamaki and Nomura [37]. In this work, we compute the QN frequencies for $(D+2)-$dimensional spherically symmetric single horizon black-holes with generic singularities and near-horizon properties (which include the ones that have been already explored). Near the horizon, we assume that the spherically symmetric metric takes the form of the Rindler while close to the singularity we use the form of Szekeres-Iyer metric [38–40]. Using the Monodromy approach, we show that (i) the imaginary part of the high frequency QNM are discrete and uniformly spaced and (ii) the real part depends on the horizon radius, space-time dimension and power-law index of S^D near the singularity. We also show that the real part of the high frequency QNM has a logarithmic dependence whose argument need not necessarily be an integer. In order to illustrate this fact, we consider specific black-holes and obtain their QN frequencies.

The rest of the paper is organized as follows. In the next section, we discuss generic properties of the space-time near the horizon and the singularity. In Sec. (III), we obtain the QN frequencies for a general static spherically symmetric black-holes. In Sec. (IV), we apply our general results to specific black-holes, reproducing earlier results and obtaining new ones. Finally, we conclude in Sec. (V) summarising our results and speculating on future directions.

II. SPHERICALLY SYMMETRIC BLACK-HOLE

We start with the general $(D+2)-$dimensional spherically symmetric line element:

$$\begin{aligned} ds^2 &= -f(r)dt^2 + \frac{dr^2}{g(r)} + \rho^2(r)d\Omega_D^2\,, \qquad (4)\\ &= f(r)\left[-dt^2 + dx^2\right] + \rho^2(r)d\Omega_D^2\,, \qquad (5)\end{aligned}$$

where $f(r)$, $g(r)$ and $\rho(r)$ are arbitrary (continuous, differentiable) functions of the radial coordinate r, $d\Omega_D^2$ is the metric on unit S^D and

$$x = \int \frac{dr}{\sqrt{f(r)g(r)}}\ , \qquad (6)$$

denotes the tortoise coordinate. As it can be seen, the line-element (5) factorizes into the product of two spaces $\mathcal{M}^2 \times S^D$, where $\mathcal{M}^2$ is the 2-dimensional space-time with Minkowskian topology.

In order for the line-element (4) to describe a static black-hole, the space-time must contain a singularity (say at $r = 0$) and have horizons. In this work, we will assume that the space-time contains one event-horizon at r_h and that it is asymptotically flat. However, we do not assume any specific form of $f(r)$, $g(r)$ and $\rho(r)$. In the rest of this section, we discuss generic properties of the space-time near the horizon ($r = r_h$) and the singularity ($r = 0$).

A. Horizon structure

For the line-element (4), there exists a time-like Killing vector field $\xi^\mu(x^\nu)$ which can be expressed as $\xi^\mu(x^\nu) = (1, 0, \cdots, 0)$[41]. Substituting $I^\mu = \xi^\mu$ into the definition of the surface gravity, i. e.,

$$\kappa^2 = -\frac{1}{2} I^{\mu,\nu} I_{\mu,\nu} \qquad (7)$$

we obtain [41]

$$\kappa = \frac{1}{2}\left(\sqrt{\frac{g(r)}{f(r)}}\frac{df(r)}{dr}\right)_{r=r_h}\ . \qquad (8)$$

Using the property that the event horizon is a null hypersurface, the location of the horizon is determined by the condition $g^{\mu\nu}\,\partial_\mu N\,\partial_\nu N = 0$. For the line-element (4) N is a function of r characterizing the null hypersurface which gives $g(r_h) = 0$. Thus, the location of the event horizon is given by the roots of the above equation.

Since $\kappa = constant$ and $g(r) = 0$ at the event horizon, using the definition of surface gravity from Eq. (8), we have the condition $f(r_h)/g(r_h) = H(r_h)$ where $H(r_h) \neq 0$. Using the property that $f(r)$ and $g(r)$ are smooth functions, we have the following relation for general black-holes:

$$f(r)/g(r) = H(r) \qquad (9)$$

where $H(r)$ is a smooth function and is non-vanishing at the event-horizon.

In order to obtain the line-element near the horizon, we make the coordinate transformation $(t, r) \rightarrow (t, \gamma)$, which is defined by

$$\gamma \equiv \frac{1}{\kappa}\sqrt{f}, \qquad d\gamma = \frac{1}{2\kappa}\frac{d_r f}{\sqrt{f}}\,dr\,, \qquad (10)$$

where κ is given by (8). Note that the horizon (r_h) is at $\gamma = 0$. The line-element (4) becomes

$$ds^2 = -\kappa^2\gamma^2 dt^2 + 4\,\frac{f}{g}\frac{\kappa^2}{(d_r f)^2}d\gamma^2 + \rho^2(r)d\Omega_D^2\,, \qquad (11)$$

and hence, near the horizon, we have

$$ds^2 \to -\kappa^2\gamma^2 dt^2 + d\gamma^2 + \rho^2(r_0)\, d\Omega_D^2\,. \tag{12}$$

For space-times with single non-degenerate horizon (like Schwarzschild for which $\kappa \neq 0$), $f(r), g(r)$ can be expanded around r_h as

$$f(r) = f'(r_h)\ (r - r_h);\ g(r) = g'(r_h)\ (r - r_h)\,, \tag{13}$$

using (8), we have

$$\kappa = \frac{1}{2}\sqrt{g'(r_h)\, f'(r_h)}\,, \tag{14}$$

and using the relation (6), we have

$$x = c_0\ \ln(r - r_h)\,, \tag{15}$$

where c_0 is a constant and is given in Table (I).

In the case of $(D+2)-$dimensional Schwarzschild, we have

$$f(r) = g(r) = 1 - \left(\frac{r_h}{r}\right)^{D-1}\,, \tag{16}$$

where r_h is related to the black-hole mass (M) by the relation $M = (D\,\Omega_D\, r_h^{D-1})/(16\pi G_{D+2})$, G_{D+2} being the $(D+2)-$dimensional Newton's constant and $\Omega_D = (2\pi^{(D+1)/2}/\Gamma[(D+1)/2]$. Using (8), we have

$$\kappa = \frac{(D-1)}{2\, r_h}\,. \tag{17}$$

For space-times with degenerate horizon, such as the extremal Reissner-Nordström (RN), we have

$$f(r) = g(r) = \frac{(r^{D-1} - r_h^{D-1})^2}{r^{2(D-1)}} \tag{18}$$

where

$$r_h = (8\pi G_{D+2} M)\,/\,(D\ \Omega_D) \tag{19}$$

For these space-times using the relation (6), we have (near the horizon)

$$x \simeq c_0 \log(r - r_h) + \mathcal{O}(r^n) \tag{20}$$

where c_0 is a constant and is given in Table (I). See Table (I) and also Sec. (IV) for properties of the stringy black-holes.

B. Generic power-law singularities

The analysis of Motl-Neitzke [14] depends crucially on the behavior of the metric near the singularity. Thus, in order to assess the generality of the result, one needs to understand the generality of the space-time singularities. A decade ago, Szekeres-Iyer [38, 39] (see also, Ref. [40]) investigated a large class of four-dimensional spherically symmetric space-times with power-law singularities. These space-times practically encompass all known spherically symmetric solutions of the Einstein equations such as Schwarzschild(-de Sitter), Reissner-Norström and other type of metrics with null singularities. In Sec. (IV), we will show that certain stringy black-holes and higher dimensional Gibbons-Maeda type black-holes [42–44] also fall into this class.

Szekeres-Iyer had shown that near the singularity, the spherically symmetric metric (4) takes the following form(s):

$$\begin{aligned} ds^2 &\overset{r\to 0}{\simeq} \eta r^{2p/q} dy^2 - \frac{4\eta}{q^2} r^{2(p-q+2)/q} dr^2 + r^2 d\Omega_D^2\,, && (21)\\ &= \eta x^p (dy^2 - dx^2) + x^q d\Omega_D^2\,, && (22) \end{aligned}$$

where $y = \beta t,\ \beta > 0,\ \eta = 1, 0, -1$ correspond to space-like, null and time-like singularities respectively and p, q are constants and capture the dominant behavior near the singularity. Note that the notation of t and r is adapted to the case of $\eta = -1$ where the singularity is time-like and t is time. However, we will continue to use this notation even for space-like singularities where t is actually space-like. The line-element (22) clearly shows that near the singularity the product spaces have different singularity structure. The curvature invariants – Ricci and Kretschmann scalars – for the line-element (21) go as

$$\begin{aligned} R &= \frac{a x^{-p} + b x^{2-q}}{x^2}\,, && (23)\\ R_{abcd}R^{abcd} &= \frac{c x^{-2p} + d x^{-2q+4} + e x^{-p-q+2}}{x^4}\,, && (24) \end{aligned}$$

where $a(p,q), \ldots, e(p,q) = \mathcal{O}(1)$. The form of the invariants show that the Szekers-Iyer line-element indeed describe the spherically symmetric space-time near the singularity.

Comparing Eqs. (4, 21), we have

$$f(r) = -\eta\beta^2 r^{2p/q};\ \frac{1}{g(r)} = -\frac{4\eta}{q^2} r^{2(p-q+2)/q};\ \rho(r) = r\ . \tag{25}$$

Substituting the above expressions in Eq. (6), we have

$$x = \frac{r^{2/q}}{\beta}\,. \tag{26}$$

Note that near the curvature singularity, the tortoise coordinate depends on q but not on p. This will be crucial in obtaining the real part of the high frequency QNM. In Table (I), we have given the values of p, q for various black-holes.

For example, in the case of $(D+2)$-dimensional Schwarzschild:

$$p = \frac{1-D}{D}\ ; \quad q = \frac{2}{D}\ ; \quad x = \frac{r^D}{\beta}\ , \tag{27}$$

whereas for $(D+2)$-dimensional (non-)extremal Reissner-Nordström:

$$p = -2\left(\frac{D-1}{2D-1}\right) \; ; \quad q = \frac{2}{2D-1} \; ; \quad x = \frac{r^{2D-1}}{\beta} \, . \tag{28}$$

III. QUASI-NORMAL MODES FOR STATIC BLACK-HOLES

In this section, we obtain the high (imaginary) frequency QNM, corresponding to the scalar gravitational perturbations, for the general spherically symmetric $(D+2)-$dimensional black-holes discussed in the previous section.

A. Scalar Perturbations

The perturbations of a $(D+2)$-dimensional static black-holes (4) can result in three kinds – scalar, vector and tensor – of gravitational perturbations (see for example, Ref. [45]). The higher dimensional scalar gravitational effective potential, which is of our interest in this work, correspond to the well-known four-dimensional Regge-Wheeler potential. The evolution equation for the scalar gravitational perturbations follows directly from the massless, minimally coupled scalar field propagating in the line element (4), i. e.,

$$\begin{aligned} \Box \, \Phi &\equiv \frac{1}{\sqrt{-g}} \, \partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu \Phi\right) = 0 \, , \\ \frac{d^2 R(r)}{dx^2} &+ \left[\omega^2 - V(r)\right] R(r) = 0 \, , \end{aligned} \tag{29}$$

where $\Phi(x^\mu) = \rho(r)^{-D/2} \, R(r) \, \exp(i\omega t) \, Y_{l m_1 \dots m_{D-1}}$ and $V(r)$ is the higher dimensional analog of the Regge-Wheeler potential and is given by

$$\begin{aligned} V(r) &= \frac{l(l+D-1)}{r^2} \, f(r) + \left(\frac{D}{2}\right) \rho(r)^{-\frac{D}{2}} \sqrt{f(r) \, g(r)} \\ &\times \frac{d}{dr} \left(\rho(r)^{\frac{D-2}{2}} \, \frac{d\rho(r)}{dr} \sqrt{f(r) \, g(r)}\right) \, . \end{aligned} \tag{30}$$

The analysis of Motl and Neitzke requires the extension of Eq. (29) beyond the physical region $r_h < r < \infty$. In order to perform the analysis we need to know the nature of the singularity of the differential equation (29) at $r = 0, r_h,$ and ∞. Assuming that the space-time is asymptotically flat at radial infinity and using the relation (13) near the horizon, it is easy to shown that $r \to \infty$ and $r = r_h$ are irregular and regular singular points of the differential equation (29) respectively. In the case of power-law singularities as discussed in the previous section, in order for $r = 0$ to be a regular singular point of the differential equation (29), it can be easily shown that p, q *must* satisfy the following conditions:

$$q > 0 \;\; \text{and} \;\; p - q + 2 > 0 \tag{31}$$

The above conditions will be useful in reducing the generalized Regge-Wheeler potential (30) near the singularity similar to the near-origin form of the potential derived by Motl and Neitzke [14]. Using the above conditions and x continued to the whole complex plane (say z), Eq. (29) is an ordinary differential equation with regular singular points at $r = 0, \; r_h$ and an irregular singular point at $r = \infty$. Thus, by the general theory of differential equations [46], any solution of (29) in the physical region extends to a solution on the $r-$plane. However, this solution may be multi-valued around the singular points at $r = 0, \; r_h$ [14].

B. Computation of quasi-normal modes

QNM are solutions to the differential equation (29) whose frequency is allowed to be complex. In the case of asymptotically flat space-times, which is of our interest, the modes are required to have purely outgoing boundary conditions both at the horizon and in the asymptotic region, i. e.,

$$R(x) \sim e^{\pm i\omega x} \quad \text{as} \quad x \to \mp\infty \, . \tag{32}$$

In order for the black-hole to be stable, the modes should decay in time, hence $\Im(\omega) > 0$. In the monodromy approach [14], unlike the earlier approaches, the authors analytically continued x (in the complex plane z), instead of ω, and introduced the boundary conditions as the product $\omega \, z \to \pm\infty$, instead of $x \to \pm\infty$. In this approach, we need to know the solution of Eq. (29) near $r = 0, \; r_h$ and compare their monodromies.

For the general spherically symmetric space-time – with the power-law singularity at the origin and generic horizon structure – the generalized Regge-Wheeler potential (30) near $r = 0$ and $r = r_h$ is

$$V(r) \overset{r \to r_h}{\simeq} \left[\frac{l(l+D-1)}{\rho^2(r_h)} \, f'(r_h) + \frac{D}{2} \, f'(r_h) \, g'(r_h) \, \frac{\rho'(r_h)}{\rho(r_h)}\right] (r - r_h) + \mathcal{O}[(r - r_h)^2] \, , \tag{33}$$

$$\overset{r \to 0}{\simeq} \left(\frac{q\beta}{2}\right)^2 \, \frac{D}{2} \, \left(\frac{D}{2} - \frac{2}{q}\right) \, r^{-4/q} - \eta \beta^2 \, l \, (l + D - 1) \, r^{2(p-q)/q} \, . \tag{34}$$

Following points are worth noting regarding the above result:
(i) Using the conditions (31) in the second equation above, it is clear that near the origin the first term in the RHS dominates the second term. Hence, it would suffice to consider first term in the rest of the analysis. Rewriting the potential near the origin in-terms of z, we have

$$V[r[z]] \stackrel{r\to 0}{=} \frac{D\,q}{8}\left(\frac{qD}{2}-2\right)\frac{1}{z^2}\,. \qquad (35)$$

(ii) It is also interesting to note that the generalized Regge-Wheeler potential near the origin depends *only* on q, D. From the line-element (22), this implies that the potential depends on the singularity structure of the S^D space and not on $\mathcal{M}^2$.
(iii) Remarkably, the generalized Regge-Wheeler potential near the origin is similar to the form of the near-origin potential derived by Motl and Neitzke [14] except for the coefficients[2]. In view of these observations, the monodromy calculation [14] should carry through relatively unchanged for the general spherically symmetric metrics with power-law singularities. In the rest of the section, we describe this calculation for the general spherically symmetric metrics and obtain the high (imaginary) frequency QNM. [We follow the notation of Motl and Neitzke closely to provide easy comparison.]

Substituting the generalized Regge-Wheeler potential (35) in the differential equation (29), we get

$$R(z) = A_+ c_+ \sqrt{\omega z} J_{+\nu}(\omega z) + A_- c_- \sqrt{\omega z} J_{-\nu}(\omega z)\,, \quad (36)$$

where $\nu = (D\,q-2)/4$, while the products c_+A_+ and c_-A_- represent constant coefficients. Following [14], we will choose the "normalization factors" (denoted by $c_\pm$) so that

$$c_\pm \sqrt{\omega z} J_{\pm\nu}(\omega z) \sim 2\cos\left(\omega z - \alpha_\pm\right) \quad \text{as} \quad \omega z \to \infty\,, \qquad (37)$$

with

$$\alpha_\pm \equiv \frac{\pi}{4}\left[1 \pm 2\nu\right]\,. \qquad (38)$$

From Eqs. (36) and (37), as well as the boundary condition (32) in-terms of $\omega\, z$ we get the following constraint

$$A_+ e^{-i\alpha_+} + A_- e^{-i\alpha_-} = 0 \qquad (39)$$

and obtain the asymptotic form for $R(z)$ as

$$R(z) \sim \left[A_+ e^{+i\alpha_+} + A_- e^{+i\alpha_-}\right] e^{-i\omega z} \quad \text{as} \quad \omega z \to \infty\,. \qquad (40)$$

We follow the contour from the the negative imaginary axis of z ($\Im(\omega z) \to \infty$) to the positive imaginary axis ($\Im(\omega z) \to -\infty$). Now from Eq.(26) and the relation $z = 0$, it follows that the the rotation of the contour on the r plane is by an angle $3\pi q/2$, i.e. $r \to r\exp(i3\pi q/2)$. This translates to a rotation of $3\pi q/2 \times 2/q = 3\pi$ in the z plane. That is: $z \to \exp(3i\pi) z$. Further, using $J_\nu(ze^{im\pi}) = e^{im\nu\pi} J_\nu(z)$, we have (as $\omega z \to -\infty$)

$$\begin{aligned} R(z) \;\sim\; & \left(A_+ e^{5i\alpha_+} + A_- e^{5i\alpha_-}\right) e^{-i\omega z} \\ + \; & \left(A_+ e^{7i\alpha_+} + A_- e^{7i\alpha_-}\right) e^{+i\omega z}\,. \end{aligned} \qquad (41)$$

Then, from Eqs. (40,41), following [14], we obtain the monodromy about the specified contour around the singularity to be

$$\frac{A_+ e^{5i\alpha_+} + A_- e^{5i\alpha_-}}{A_+ e^{i\alpha_+} + A_- e^{i\alpha_-}}\,. \qquad (42)$$

Eliminating the constants $A_\pm$ using the constraint (39), we get

$$-\frac{e^{6i\alpha_+} - e^{6i\alpha_-}}{e^{2i\alpha_+} - e^{2i\alpha_-}} = \frac{\sin(3\pi\nu)}{\sin(\pi\nu)} = 1 + 2\cos\left(\frac{\pi}{2}[Dq-2]\right)\,. \qquad (43)$$

We can evaluate the monodromy of $R(z)$ by choosing a contour which passes through the horizon. The mode function near the horizon (as $z \to -\infty$) is given by

$$R(z) \sim e^{+i\omega z} \sim \exp\left[i\, c_0\, \omega \ln(x - x_h)\right]\,. \qquad (44)$$

Thus, the monodromy by choosing the contour near the horizon is $\exp(4\pi\omega c_0)$. Equating the two monodromies, we obtain

$$\omega_{{}_{QNM}} = \frac{i}{2c_0}\left[n + \frac{1}{2}\right] \pm \frac{1}{4\pi c_0}\log\left[1 + 2\cos\left(\frac{\pi}{2}[Dq-2]\right)\right]\,, \qquad (45)$$

where n is an integer. This is the main result of our paper, regarding which we would like to stress a few points. First, the above result is valid for a general spherically symmetric space-times which is asymptotically flat and has a single horizon at $r = r_h$. Second, it is clear, from the above expression, that the imaginary part of the high frequency QNM are discrete and are equally spaced. Third, the real part of the high frequency QNM has a logarithmic dependence and has a prefactor which depends on r_h (See Table I). Lastly and more importantly, it is clear from the above expression that the argument of the logarithm is not always an integer. It is a non-negative integer *only* if

$$a)\;\; \left(\frac{Dq-2}{2}\right)\pi = 2\,j\pi \quad \text{where} \quad j \in \mathcal{Z} \qquad (46)$$

$$b)\;\; \left(\frac{Dq-2}{2}\right)\pi = \frac{m\,\pi}{3} \quad \text{where} \quad m \in \mathcal{Z}\,. \qquad (47)$$

For the case (a), the real part of high frequency QNM is proportional to $\log(3)$ for all j. For the case (b), the real part of high frequency QNM is (i) proportional to $\log(2)$ for $m = \pm1, \pm5, \pm7, \cdots$, and (ii) blows up for $m = \pm2, \pm4, \pm8, \cdots$. It is also interesting to note that the real part of the QN frequencies *vanish* for all half odd integers in Eq. (46).

[2] The above behaviour of the Regge-Wheeler potential was also observed in Ref. [47]. Other consequences of this universal behaviour will be explored in Ref. [48].

IV. APPLICATION TO SPECIFIC BLACK-HOLES

In the previous section, we have obtained the high frequency QNM for a spherically symmetric black hole with a generic singularity. As we have shown, the real part of the high frequency QNM is not necessarily proportional to ln(3) as in the case of $(D+2)$-dimensional Schwarzschild. In order to illustrate this fact, we take specific examples and obtain their QNM.

A. $(D+2)-$dimensional Schwarzschild

As noted earlier, for these black-holes

$$4\pi c_0 = \frac{1}{T_H} \quad \text{and} \quad q = \frac{2}{D} \ . \tag{48}$$

Thus from (45),

$$\omega_{_{QNM}} = 2\pi i T_H \left[n + \frac{1}{2}\right] \pm T_H \log 3 \ , \tag{49}$$

as found by previous authors [14].

B. $(D+2)-$dimensional extremal Reissner-Nordström

From Table I, we see:

$$c_0 = \frac{D r_h}{(D-1)^2} \quad \text{and} \quad q = \frac{2}{2D-1} \ . \tag{50}$$

Thus

$$\begin{aligned} \omega_{_{QNM}} &= \frac{i(D-1)^2}{2Dr_h}\left[n+\frac{1}{2}\right] \\ &\pm \frac{(D-1)^2}{4\pi D r_h} \log\left[1 + 2\cos\left(\frac{\pi(D-1)}{2D-1}\right)\right] \end{aligned} \tag{51}$$

Thus, for example in four dimensions ($D=2$), we get:

$$\omega_{_{QNM}} = \frac{i}{4r_h}\left[n+\frac{1}{2}\right] \pm \frac{1}{8\pi r_h}\log 2 \ . \tag{52}$$

Note that the logarithmic nature of the real part of the QN frequency persists although its coefficient is no longer the Hawking temperature. The real part is in agreement with the argument of Motl and Neitzke [14], although their analysis is for non-extremal Reissner-Nördstrom. Note however, that the above result appears to disagree with that stated in Ref. [49] and [50].

C. 2-dimensional stringy black-holes

Let us consider the generic 2-dimensional black-hole solution in string theory, which encompasses the solutions found in Refs. [51–53], with or without matter fields [3] :

$$ds^2 = -\left(1 - \frac{M}{\lambda}\, e^{2\lambda r}\right) dt^2 + \left(1 - \frac{M}{\lambda}\, e^{2\lambda r}\right)^{-1} dr^2 \ , \tag{53}$$

where λ is a constant and M can be interpreted as the mass of the black-hole. It has a horizon at:

$$r = \frac{1}{2\lambda}\ln\left(\frac{\lambda}{M}\right) \ , \tag{54}$$

whose Hawking temperature is

$$T_H = \frac{\lambda}{2\pi} \ . \tag{55}$$

(Note that it is independent of M.) $D=0$ for these black-holes renders q irrelevant for QNM. QN frequencies can thus be read-off from (45):

$$\omega_{_{QNM}} = 2\pi i T_H(n+m+1) \text{ or } 2\pi i T_H(n-m), \ m \in \mathcal{Z} \ . \tag{56}$$

It is clear from the above expression that the real part of the high QN frequencies for generic 2-dimensional (stringy) black-holes vanish. In Ref. [21], the authors have obtained high-frequency QNM for a generic 2-dimensional dilaton gravity. Our result is in agreement with the results their case for $h(\phi)=1$ which corresponds to pure 2-dimensional gravity.

D. 4-dimensional Stringy Black-Holes

The line-element of the 4-dimensional generalization of a charged black-hole solution [42, 43] is given by (4) where

$$\begin{aligned} f(r) &= g(r) = \left(1 - \frac{r_h}{r}\right)\left(1 - \frac{r_0}{r}\right)^{(1-\alpha^2)/(1+\alpha^2)} ; \\ \rho(r) &= r\left(1 - \frac{r_0}{r}\right)^{\alpha^2/(1+\alpha^2)} . \end{aligned} \tag{57}$$

r_h, r_0 are related to the physical mass and charge by the relation

$$M = \frac{r_h}{2} + \left[\frac{1-\alpha^2}{1+\alpha^2}\right]\frac{r_0}{2}; \qquad Q = \left[\frac{r_h\, r_0}{1+\alpha^2}\right]^{1/2} \ . \tag{58}$$

Note that $\alpha=0$ reduces it to the familiar 4-dimensional Reissner-Nordström solution, whereas for $\alpha=1$ the line-element takes the form of the charged stringy black-hole solution of [43]. The above solution has a regular horizon at r_h. For any non-zero value of α, the inner-horizon (r_0)

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[3] The transformations $\eta = -\tanh^{-1}\left[1 - \frac{M}{\lambda}e^{2\lambda r}\right]^{-1}/\lambda$ and $x_\pm = \exp(-\lambda(r' \pm t))$ (where $\exp(2\lambda r') = (-\lambda^2\exp(2\lambda r))/(1 - \mu\exp(2\lambda r)/\lambda)$ convert the metric (53) into the forms assumed in [51] and [53] respectively.

is a space-like singularity. Thus, we will focus on the situation where $\alpha \neq 0$ which corresponds to a black-hole with a singular horizon.

It can be easily shown that the above black-hole has a non-zero Hawking temperature given by:

$$T_H = \frac{1}{4\pi r_h}\left(1 - \frac{r_0}{r_h}\right)^{(1-\alpha^2)/(1+\alpha^2)} . \tag{59}$$

Near the singularity ($r \to r_0$), we have [$\epsilon \equiv r - r_0$]:

$$\begin{aligned} f(r) &= \left(1 - \frac{r_h}{r_0}\right)\left(\frac{\epsilon}{r_0}\right)^{(1-\alpha^2)/(1+\alpha^2)} , \\ \rho(r) &= \left(r_0\ \epsilon^{\alpha^2}\right)^{1/(1+\alpha^2)} , \\ x &= (r_0\ \epsilon)^{2\alpha^2/(1+\alpha^2)} . \end{aligned} \tag{60}$$

Thus, the 4-dimensional stringy black-hole line-element near the singularity becomes

$$ds^2 \stackrel{r\to r_0}{\simeq} h\, x^{(1-\alpha^2)/(2\alpha^2)}\left[dt^2 - dx^2\right] + x\, d\Omega^2 , \tag{61}$$

where $h = h(r_0, r_h, \alpha) = \mathcal{O}(1)$. Comparing the above line-element with (22), it follows that:

$$p = \frac{1-\alpha^2}{2\alpha^2}; \quad q = 1 . \tag{62}$$

Thus, from (45), we get:

$$\omega_{_{QNM}} = 2\pi i T_H \left[n + \frac{1}{2}\right] \pm T_H \log(3) . \tag{63}$$

The above result is in agreement with Ref. [37].

E. 5-dimensional Stringy black-holes

The line-element for RR charged 5-dimensional stringy black-hole [44] formed by wrapping $D1$ and $D5$ branes on $T^4 \times S^1$ is given by (4) where

$$\begin{aligned} f(r) &= F^{-2/3}\left(1 - \frac{r_h^2}{r^2}\right); \quad \rho(r) = F^{1/6}\, r; \\ g(r) &= F^{-1/3}\left(1 - \frac{r_h^2}{r^2}\right), \end{aligned} \tag{64}$$

and

$$F = \left[1 + \frac{r_h^2 \sinh^2\alpha}{r^2}\right]\left[1 + \frac{r_h^2 \sinh^2\gamma}{r^2}\right]\left[1 + \frac{r_h^2 \sinh^2\sigma}{r^2}\right] .$$

The black-hole carries three $U(1)$ charges, which are proportional to the number of $D1-$branes, $D5-$branes and open string momentum along the compact dimension common between these branes, are related to the black-hole parameters as:

$$\begin{aligned} Q_1 &= \frac{V r_h^2}{2g}\sinh 2\alpha\ ;\ Q_5 = \frac{r_h^2}{2g}\sinh 2\gamma; \\ n &= \frac{R^2 V r_h^2}{2g^2}\sinh 2\sigma , \end{aligned} \tag{65}$$

where $(2\pi)^4 V$ and $2\pi R$ are the volume and radius of the T^4 and S^1 respectively and g is the string coupling. The above solution has a regular event horizon at r_h. When all the three charges are nonzero, the surface $r = 0$ is a smooth inner horizon. When at least one of the charges is zero (say $\sigma = 0$), the event horizon remains, however the surface $r = 0$ becomes singular. Thus, we will focus on the situation where $\sigma = 0$ which corresponds to a black-hole with a singularity at $r = 0$. The surface gravity for this resultant black-hole is

$$\kappa = (r_h \cosh\alpha \cosh\gamma)^{-1} . \tag{66}$$

Near the singularity, we have

$$\begin{aligned} f(r) = -k\ r^{2/3}; &\quad g(r) = -l\ r^{2/3}; \\ \rho(r) = m\ r^{1/3}; &\quad x = \left(m\sqrt{kl}\right)\ r . \end{aligned} \tag{67}$$

Thus, the 5-dimensional stringy black-hole line-element near the singularity becomes

$$ds^2 \stackrel{r\to 0}{\simeq} s\, x^{2/3}\left[dt^2 - dx^2\right] + x^{2/3}\, d\Omega^2 \tag{68}$$

where $k, l, m, s = k, l, m, s\ (r_h, \alpha, \gamma) = \mathcal{O}(1)$. Comparing with (22), it follows that:

$$p = \frac{2}{3}; \quad q = \frac{2}{3} . \tag{69}$$

Thus, from (45), we get:

$$\omega_{_{QNM}} = 2\pi i T_H \left[n + \frac{1}{2}\right] \pm T_H \log(3) . \tag{70}$$

We would like to clarify the following point regarding our result: As we had mentioned earlier, the analysis is *strictly* valid for single-horizon black-hole space-times. In obtaining the high-frequency QNMs for the 5-dimensional stringy black-holes, we have assumed that one of the charges to the zero ($\sigma = 0$). Note, however, for these stringy black-holes with a regular inner horizon, it has been shown in Ref. [54] that the frequencies determined by the monodromies (of the two horizons) coincide with the QNM of the near horizon BTZ metric and that these are the ones that are relevant for quantum gravity (the so called 'non-quasinormal modes'). It will be interesting to closely examine the relation between the two results.

V. DISCUSSION

In this paper, we have computed the high frequency QNM for scalar perturbations of spherically symmetric single horizon and asymptotically flat black-holes in $(D+2)-$dimensional space-times. We have computed these modes using the monodromy approach [14]. We have shown that the real part of these modes depends on the horizon radius (r_h), dimension (D) and the power-law index (q) of S^D near the singularity.

TABLE I: The table gives the list of physical quantities for different black-holes. k is the surface gravity of the black-hole; c_0 is the constant which appears in the near-horizon relation between x and r; p and q are the power-law index for $\mathcal{M}^2$ and S^D respectively, and $(D\,q-2)/2$ is the quantity which determines the real part of QN frequency ($\Re[\omega_{QNM}]$). The properties of the various black-holes (BH) are discussed in Secs. (II, IV).

Space-Time	Near horizon properties		Near singularity properties			$\Re[\omega_{\rm QNM}]$
	κ	c_0	p	q	$(Dq-2)/2$	
Non-degenerate Horizons	$\frac{\sqrt{f'(r_h)g'(r_h)}}{2}$	$\frac{1}{2\kappa}=\frac{1}{4\pi T_H}$	$p > q-2$	$q > 0$	Any real value	$\frac{1}{4\pi c_0}\log\left[1+2\cos\left(\frac{\pi(Dq-2)}{2}\right)\right]$
(D + 2)-dimens. Schwarzchild	$\frac{(D-1)}{2r_h}$	$\frac{r_h}{D-1}$	$\frac{1-D}{D}$	$\frac{2}{D}$	0	$T_H\log 3$
4-dimensional stringy BH	$\frac{1}{2r_h}\left[1-\frac{r_0}{r_h}\right]^{\frac{1-\alpha^2}{1+\alpha^2}}$	$\frac{1}{2\kappa}$	$\frac{1-\alpha^2}{2\alpha^2}$	1	0	$T_H\log 3$
5-dimensional stringy BH	$\frac{r_h^{-1}}{\cosh\alpha\cosh\gamma}$	$\frac{1}{2\kappa}$	$-\frac{2}{3}$	$\frac{2}{3}$	0	$T_H\log 3$
2-dimensional stringy BH	1	$\frac{1}{2}$	1	0	-1	0
(D + 2)-dimens. Degenerate RN	0	$\frac{D}{(D-1)^2}r_h$	$-2\left(\frac{D-1}{2D-1}\right)$	$\frac{2}{2D-1}$	$\left(\frac{1-D}{2D-1}\right)$	$\frac{1}{8\pi r_h}\log 2$

We have also shown that the real part of the high frequency modes has a logarithmic dependence, although the argument of the logarithm is not necessarily an integer. However, when we applied our result to specific examples and obtained their QNM, the argument turned out to be an integer. In particular, for $(D+2)$-dimensional Schwarzschild and $4,5$-dimensional stringy black-holes, we found that the real part of the QN frequencies is proportional to $\ln(3)$. However for 4-dimensional Reissner-Nordström it is proportional to $\ln(2)$ and vanishes in the case of 2-dimensional stringy black-holes. It would be interesting to compute the QNM for black holes with multiple horizons as well as those which are not spherically symmetric or asymptotically flat and also to compare our results with others that are related to a generic class of black holes [21].

In the light of the above results, let us re-examine the argument presented in [4]. For black-holes with non-zero Hawking temperature, it follows from (45) that the real part of QN frequencies is of the form:

$$\Re(\omega_{QNM}) = T_H \log\left[1+2\cos\left(\frac{\pi(Dq-2)}{2}\right)\right] \quad . \tag{71}$$

Identifying the above with the minimum change of black-hole mass (due to Hawking radiation or accretion), we get, using the first law of black-hole thermodynamics and the area law for entropy:

$$\Delta A = 4\ell_{\rm Pl}^2 \log\left[1+2\cos\left(\frac{\pi(Dq-2)}{2}\right)\right] \quad , \tag{72}$$

where $A = \Omega_D r_h^D$ is the horizon area of the black-hole and $\ell_{\rm Pl}$ is the $(D+2)-$dimensional Planck length in this case. Equating with the minimum area quantum in loop quantum gravity, namely $\Delta A = 8\pi\ell_{\rm Pl}^2\gamma\sqrt{j_m(j_m+1)}$ and subsequently setting $D=2$, we get the following prediction for the Immirzi parameter:

$$\gamma = \frac{\log\left[1+2\cos\left(\pi(q-1)\right)\right]}{2\pi\sqrt{j_m(j_m+1)}} \quad , \tag{73}$$

where j_m denotes the representation of $SU(2)$ for the spin-network states in loop quantum gravity (for supersymmetric spin networks the area spectrum is derived in Ref. [55]). Assuming that the number of points where the spin-network states puncture the horizon is given by:

$$N = A/\Delta A \tag{74}$$

one gets the microscopic entropy of the black-hole, as the logarithm of the dimension of the boundary Hilbert space, to be

$$\begin{aligned} S &= N\log(1+2j_m) \\ &= \frac{A}{4\ell_{\rm Pl}^2}\frac{\ln\left(1+2j_m\right)}{\log\left[1+2\cos\left(\pi(q-1)\right)\right]} \end{aligned} \tag{75}$$

Thus, for the above relation to agree with the Bekenstein-Hawking entropy, one must have:

$$j_m = \cos\left(\pi(q-1)\right) \quad . \tag{76}$$

Even though, for $q=1$, (76) agrees with the prediction of Ref. [4], one would like to have a better understanding of the above result. More importantly, the above result is valid *only* for non-extremal black-holes. Since $\kappa = 0$ for the extremal black-holes, the real part of the QN frequencies (71) cannot be related to the temperature implying that the relation for the change in the horizon area (72) is no more valid. We hope to address this issue elsewhere.

Finally, following [13] we find that the adiabatic invariant in our case is:

$$\begin{aligned} I &= \int \frac{dE}{\omega_{QNM}} \\ &= \left(\log\left[1 + 2\cos\left(\frac{\pi(Dq-2)}{2}\right)\right]\right)^{-1} \int \frac{dE}{T_H} \\ &= \left(\log\left[1 + 2\cos\left(\frac{\pi(Dq-2)}{2}\right)\right]\right)^{-1} S \ , \end{aligned} \qquad (77)$$

which again confirms Bekenstein's conjecture that horizon area (and hence black-hole entropy) is an adiabatic invariant. The crucial ingredient in the above is:

$$\Re(\omega_{QNM}) \propto T_H \ . \qquad (78)$$

This, along with the Bohr-Sommerfeld quantization rule $I = n$ implies the equispaced nature of the horizon spectrum:

$$A \propto n \, \ell_{_{\rm Pl}}^D \ , \qquad (79)$$

with the following proportionality constant:

$$4 \times \left(\log\left[1 + 2\cos\left(\frac{\pi(Dq-2)}{2}\right)\right]\right)^{-1} \qquad (80)$$

Such equispaced spectrum has been verified earlier in several other approaches, albeit with different proportionality constants and often with a 'zero-point' or 'ground state' energy, interpreted as a Planck-sized remnant left over when the black-hole evaporates (see for e.g. [56–58] and references therein). It is also interesting to note from (52) and the relation $r_h \sim M^{1/(D-1)}$ that the relation $I \propto A$ continues to hold for extremal Reissner-Nördstrom.

We end with a couple of caveats: (i) The area spectrum in Loop Quantum Gravity is not equispaced in general [59, 60]. However, it is equispaced in the large area limit, as well as if one uses a different representation [61]. (ii) Strictly speaking, the QNM are only associated with transitions of short durations. Thus, other transitions can modify the equispaced area spectrum obtained here. We hope to further examine these and related issues elsewhere.

Acknowledgments

This work was supported by the Natural Sciences and Engineering Research Council of Canada. We would like to thank G. Kunstatter, A. J. M. Medved, J. M. Natario and R. Schiappa for useful comments. SS would like to thank the Department of Physics, University of Lethbridge, Canada for hospitality where this work was completed. SS would also like to thank Martin O'Loughlin and F. Hussain for discussions.

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