MOMENTS AND ZEROS OF L-FUNCTIONS

QUANLI SHEN
Bachelor of Science, Linyi University, 2012
Master of Science, Capital Normal University, 2016

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QUANLI SHEN

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Dr. H. Kadiri Thesis Co-supervisor	Associate Professor	Ph.D.
Dr. N. Ng Thesis Co-supervisor	Professor	Ph.D.
Dr. D. Morris Thesis Examination Committee Member	Professor	Ph.D.
Dr. A. Fiori Internal External Examiner Department of Mathematics and Computer Science Faculty of Arts and Science	Associate Professor	Ph.D.
Dr. C. Turnage-Butterbaugh External Examiner Carleton College Northfield, Minnesota	Assistant Professor	Ph.D.
Dr. J. Sheriff Chair, Thesis Examination Committee	Assistant Professor	Ph.D.

Abstract

We study moments and zeros of L-functions in this thesis.

In Chapter 2, by following closely Soundararajan-Young's method, we prove an asymptotic for the fourth moment of quadratic Dirichlet L-functions under the generalized Riemann hypothesis. Unconditionally, we are able to give a sharp lower bound that agrees with Keating-Snaith's conjecture.

In Chapter 3, we use a recursive method that was pioneered by Heath-Brown and developed by Young to give an asymptotic with an error $O(X^{\frac{1}{2}+\varepsilon})$ for the smoothed first moment of quadratic twists of modular L-functions. The result is analogous to Sono's work on the second moment of quadratic Dirichlet L-functions. It improves previous results of Iwaniec and Soundararajan-Radziwiłł.

In Chapter 4, we obtain an explicit result for the number of zeros, in a box, of Dedekind zeta functions, which improves a result of Trudgian. Our argument is based on previous works of Bennett-Martin-O'Bryant-Rechnitzer, Kadiri-Ng and Trudgian.

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The Chapters 2–4 are based on the papers that have been published or submitted for publication. Many thanks to the referees for their very helpful and constructive comments.

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Chapter 1

Introduction

1.1 L-functions and examples.

Many problems in analytic number theory can be studied via the theory of L-functions. For instance, the prime number theorem and the prime number theorem in arithmetic progressions rely crucially on nonvanishing results of the Riemann zeta function and Dirichlet L-functions, respectively. The Riemann zeta function and Dirichlet L-functions are two specific examples of general L-functions which we now define. The definition is based on Selberg's axiomatic definition [93] (see also Iwaniec-Kowalski [55, Chapter 5]). Generally it is believed that all L-functions arise from automorphic L-functions, but this is far from proven (see Cogdell [15, Chapter 9] for the definition and further details on automorphic L-functions).

We say $L(s, f)$, $s \in \mathbb{C}$ is an L-function if it satisfies the following conditions.

(1) $L(s, f)$ has the Dirichlet series with the Euler product of degree $d \geq 1$,

$$L(s, f) = \sum_{n \geq 1} \lambda_f(n) n^{-s} = \prod_p \left(1 - \alpha_1(p) p^{-s}\right)^{-1} \cdots \left(1 - \alpha_d(p) p^{-s}\right)^{-1},$$

with $\lambda_f(1) = 1$, $\lambda_f(n) \in \mathbb{C}$, $\alpha_i(p) \in \mathbb{C}$. The series and the product must be absolutely convergent in the region $\mathrm{Re}(s) > 1$. We call $\alpha_i(p)$ the local parameters of $L(s, f)$ at p. They satisfy

$$|\alpha_i(p)| < p \text{ for all } p.$$

(2) We have the gamma factor

$$\gamma(s, f) = \pi^{-ds/2} \prod_{j=1}^{d} \Gamma\left(\frac{s + \kappa_j}{2}\right).$$

The numbers $\kappa_j \in \mathbb{C}$ are called the local parameters of $L(s, f)$ at infinity. We assume these numbers are either real or come in conjugate pairs. Moreover, $\mathrm{Re}(\kappa_j) > -1$.

(3) We have the constant $q(f) \geq 1$, called the conductor of $L(s,f)$, such that $\alpha_i(p) \neq 0$ for $p \nmid q(f)$ and $1 \leq i \leq d$.

(4) From (2) and (3), we define the complete L-function

$$\Lambda(s,f) = q(f)^{\frac{s}{2}}\gamma(s,f)L(s,f).$$

It is holomorphic in the half-plane $\mathrm{Re}(s) > 1$ and is analytically extended to an meromorphic function on the entire complex plane $\mathbb{C}$ of order 1 with at most poles at $s = 0$ and $s = 1$. Moreover, it satisfies the functional equation

$$\Lambda(s,f) = \varepsilon(f)\Lambda(1-s,\bar{f}),$$

where $\bar{f}$ is an object associated with f (the dual of f) for which $\lambda_{\bar{f}}(n) = \bar{\lambda}_f(n)$, $\gamma(s,\bar{f}) = \gamma(s,f)$, $q(\bar{f}) = q(f)$. The complex number $\varepsilon(f)$ is called the "root number" of $L(s,f)$. The absolute value of $\varepsilon(f)$ must be 1.

We now give some well-known examples of L-functions.

Riemann zeta function.

The simplest example of an L-function is the Riemann zeta function defined by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} \left(1 - \frac{1}{p^s}\right)^{-1}$$

for $\mathrm{Re}(s) > 1$. One of the best references for $\zeta(s)$ is Titchmarsh [101]. The Riemann zeta function is a degree 1 function with conductor 1. It can be analytically extended to the entire complex plane with only a simple pole at $s = 1$ with residue 1. We define the Λ-function by

$$\Lambda(s) := \pi^{-\frac{s}{2}}\Gamma(\tfrac{s}{2})\zeta(s).$$

It satisfies the functional equation $\Lambda(s) = \Lambda(1-s)$. Due to the existence of the simple pole at $s = 1$, we also often use the so-called ξ-function:

$$\xi(x) := \tfrac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\tfrac{s}{2})\zeta(s).$$

The function $\xi(s)$ is entire and satisfies the functional equation $\xi(s) = \xi(1-s)$.

Dirichlet L-functions.

A Dirichlet L-function is associated to a Dirichlet character. We call a group homomorphism

$$\chi : (\mathbb{Z}/q\mathbb{Z})^{\times} \to \mathbb{C} \setminus \{0\}$$

a Dirichlet character modulo integer q. Here $(\mathbb{Z}/q\mathbb{Z})^{\times}$ is the multiplicative group of integers modulo q. This definition can be extended to all integers by setting $\chi(n) = \chi(\bar{n})$ if $(n,q) = 1$, and $\chi(n) = 0$ if $(n,q) > 1$. The Dirichlet L-function associated to χ modulo q is defined by

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}$$

for $\mathrm{Re}(s) > 1$.

The principal character modulo q is defined to be $\chi(n) = 1$ for all $(n,q) = 1$. If χ is not a principal character, then $L(\frac{1}{2},\chi)$ admits analytic continuation to an entire function on the whole complex plane. In the case that χ is principal, $L(s,\chi) = \zeta(s)\prod_{p|q}(1-p^{-s})$, so $L(s,\chi)$ behaves like the Riemann zeta function.

The definition of χ tells us that χ is a periodic function on $\mathbb{Z}$ of period q. However, if we restrict $(n,q) = 1$, then the period of χ may be smaller than q. It can be proved that this "smaller" period must be a divisor of q. We call χ a primitive character if the period of χ restricted by $(n,q) = 1$ is exactly equal to q. A Dirichlet L-function associated to a primitive Dirichlet character is called primitive Dirichlet L-function, which is of degree 1 and with conductor q. In general, we are interested in studying primitive Dirichlet L-functions. The main reason is that primitive characters are simpler to compute with, and results for primitive characters can usually be extended to imprimitive characters with minor adjustments.

In Chapter 2, we are interested in quadratic primitive characters. The set of Dirichlet characters modulo q forms a group under multiplication. The Dirichlet characters of order 2 in this group are called quadratic characters. It is clear that a Dirichlet character is quadratic if and only if it is real and non-principal. Every primitive quadratic character can be denoted by the Kronecker symbol $(\frac{d}{\cdot})$, where d is a fundamental discriminant. (We also often use χ_d to denote the Kronecker symbol.) A fundamental discriminant d is either $d \equiv 1 \pmod 4$ and square-free, or $d = 4a$ with $a \equiv 2, 3 \pmod 4$ and a being square-free. We can see that a fundamental discriminant is essentially a square-free integer.

Let χ be primitive modulo q. The ξ-function is defined by

$$\xi(s, \chi) := (\tfrac{\pi}{q})^{-\frac{1}{2}(s+\mathfrak{a})}\Gamma[\tfrac{1}{2}(s+\mathfrak{a})]L(s, \chi).$$

Here the number $\mathfrak{a}$, depending on χ, is defined by

$$\mathfrak{a} = \begin{cases} 0 & \text{if } \chi(-1) = 1, \\ 1 & \text{if } \chi(-1) = -1. \end{cases}$$

It is entire and satisfies the functional equation

$$\xi(1-s, \bar{\chi}) = \frac{i^{\mathfrak{a}}\sqrt{q}}{\tau(\chi)}\xi(s, \chi),$$

where $\tau(\chi)$ is the Gaussian sum defined by $\tau(\chi) = \sum_{m=1}^{q} \chi(m) e^{\frac{2\pi m i}{q}}$. One can see more detail in Davenport [22, Chapter 9] and Montgomery-Vaughan [77, Chapter 10].

Modular L-functions.

Modular forms are very important in modern mathematics. For example, the famous Fermat's last theorem was solved by Wiles via proving the modularity theorem which asserts that each elliptic curve L-function over $\mathbb{Q}$ arises from a modular form. Modular forms are also extensively studied in analytic number theory (see Iwaniec-Kowalski [55, Chapters 14–16]).

A modular form of weight κ for the full modular group

$$SL_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

is a complex function f over the upper half-plane $\mathbf{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ that satisfies the following three conditions:

- f is holomorphic on $\mathbf{H}$.
- For any $z \in \mathbf{H}$ and any matrix in $SL_2(\mathbb{Z})$, we have

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^{\kappa} f(z).$$

- f is also holomorphic as $z \to i\infty$. (In other words, the Fourier series of f starts at $n = 0$.)

Hecke produced Dirichlet series by using the coefficients of the Fourier expansion of a modular form at $i\infty$. Let f be a modular form of weight κ for the full modular group $SL_2(\mathbb{Z})$ as defined above. We further assume f is an eigenfunction of all Hecke operators (see [55, Page 370]). The Fourier expansion of f at infinity is

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{\kappa-1}{2}} e(nz),$$

where $\lambda_f(1) = 1$ and $|\lambda_f(n)| \leq \tau(n)$ for $n \geq 1$. Here $e(z) := e^{2\pi i z}$, and $\tau(n)$ is the number of divisors of n. A modular L-function is defined by

$$L(s, f) := \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{1}{p^{2s}}\right)^{-1}$$

for $\mathrm{Re}(s) > 1$, and it has an analytic continuation to the entire complex plane. The completed L-function is defined by

$$\Lambda(s, f) := \left(\frac{1}{2\pi}\right)^s \Gamma(s + \tfrac{\kappa-1}{2}) L(s, f).$$

It satisfies the functional equation

$$\Lambda(s,f) = i^{\kappa}\Lambda(1-s,f).$$

Dedekind zeta functions.

Given a number field K, the Dedekind zeta function $\zeta_K(s)$ of K is defined by

$$\zeta_K(s) = \sum_{\mathfrak{a}\neq 0} \frac{1}{\mathrm{N}(\mathfrak{a})^s} = \prod_{\mathfrak{p}} \left(1 - \frac{1}{\mathrm{N}(\mathfrak{p})^s}\right)^{-1},$$

for $\mathrm{Re}(s) > 1$, where the sum is over non-zero integral ideals of K, and the product is over prime ideals of K. The Dedekind zeta functions are used to study properties of number fields. For example, they can be used to count the number of prime ideals in the ring of integers. If $K = \mathbb{Q}$, then $\zeta_K(s) = \zeta(s)$. A Dedekind zeta function has an analytic continuation to a meromorphic function on $\mathbb{C}$ with only a simple pole at $s = 1$. Let us assume the degree of the field extension $\mathbb{Q} \subset K$ is n_K, and the discriminant of K is d_K. Then $\zeta_K(s)$ is an L-function of degree n_K with conductor d_K. Let $n_K = r_1 + 2r_2$, where r_1 is the number of real embeddings that fix $\mathbb{Q}$ and r_2 is the number of pairs of complex conjugate embeddings. The completed zeta function $\xi_K(s)$ is

$$\xi_K(s) = s(s-1)d_K^{\frac{s}{2}}\gamma_K(s)\zeta_K(s),$$

where

$$\gamma_K(s) = \left(\pi^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right)\right)^{r_2} \left(\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\right)^{r_1+r_2}.$$

It satisfies the functional equation

$$\xi_K(s) = \xi_K(1-s).$$

More information on Dedekind zeta functions can be seen in the book of Neukirch [80, Chapter VII].

In this thesis, we will study Dirichlet L-functions, twisted modular L-functions (which are modular L-functions twisted by Dirichlet characters) and Dedekind zeta functions, respectively, in Chapters 2, 3 and 4. We study these L-functions in two aspects. For the family of Dirichlet L-

functions and twisted modular L-functions, we study their moments; for Dedekind zeta functions, we are interested in the number of non-trivial zeros in a box.

In the rest of this chapter, in Section 1.2, we discuss some background for the field of moments of L-functions, and in Section 1.3, we introduce some history and motivation on counting zeros of L-functions.

1.2 Moments of L-functions.

It is important to understand the distribution of values of L-functions. In particular, where are the zeros of a L-function located, how often does an L-function get large and small, and how often do the values lie in a given interval? In reality, it is hard to study a single function (e.g., Riemann hypothesis, Lindelöf hypothesis). It is often simpler to study a family of L-functions with the hope that statistical results on average give intuitive ideas for a single L-function and may provide partial results. We let $\mathcal{F}$ denote a family (or collection) of L-functions. Let $L^{(i)}(s,f)$ denote the i-th derivative of an L-function $L(s,f)$. The following types of problems are extensively studied in analytic number theory.

1. Continuous moment of a single L-function.

   Estimate
   $$\int_0^T |L^{(i)}(\sigma_0+it,f)|^k dt$$
   where $i \in \mathbb{Z}_{\geq 0}, k \geq 0$ and $\sigma_0 \in \mathbb{R}$.

2. Discrete moment averaged over a family of L-functions at a given point.

   Estimate
   $$\sum_{f\in\mathcal{F}} |L^{(i)}(s_0,f)|^k$$
   where $i \in \mathbb{Z}_{\geq 0}, k \geq 0$ and $s_0 \in \mathbb{C}$.

3. Discrete moment of a fixed L-function averaged over a set of complex numbers.

   Estimate
   $$\sum_{j=1}^{N} |L^{(i)}(s_j,f)|^k$$
   where $i \in \mathbb{Z}_{\geq 0}, k \geq 0$ and $\{s_j\}$ is a complex sequence.

We remark that in some situations we may prefer the problems without the sign of absolute value. In Chapters 2 and 3 we study several problems related to Problem (2). Namely we study moments of quadratic Dirichlet L-functions and moments of quadratic twists of modular L-functions. On the other hand, some examples for Problem (1) and (3) will be given in Subsection 1.2.2: continuous moments of $\zeta(s)$ and discrete moments of $\zeta'(\rho)$.

1.2.1 Conjectures.

The field of moments of L-functions has attracted many mathematicians and many fruitful results have been given in this field. One of many interesting problems in this field is to establish asymptotic formulae for the moments of various families of L-functions. However, only a few moments of L-functions have been asymptotically established. Fortunately, we have nice conjectures for moments of L-functions. In this section, we introduce two methods that are usually used to formulate conjectures: the random matrix theory [62, 63] by Keating and Snaith, and the recipe method [17] by Conrey, Farmer, Keating, Rubinstein and Snaith. We should remark that the method of multiple Dirichlet series (see Diaconu-Goldfeld-Hoffstein [25]), which we do not plan to mention in detail here, is another very powerful tool for making conjectures, as well as giving rigorous proofs.

Symmetries.

There is a surprisingly close connection between the zeros of L-functions and the eigenvalues of characteristic polynomials of matrices in random matrix theory. This was first observed in a conversation between Dyson and Montgomery at a tea party in Princeton, where they found [76] that the pair correlation of the nontrivial zeros of $\zeta(s)$ agrees with the pair correlation of eigenvalues of large random Hermitian matrices. This discovery was strengthened by Odlyzko [83] via a profound numerical study.

Later Katz and Sarnak [60, 61] observed that the low-lying zeros within a family of L-functions follow the statistics of the eigenvalues near 1 of characteristic polynomials of the matrix ensemble associated to this family. The possible matrix groups are: the unitary group $U(N)$, the orthogonal group $O(N)$ and the symplectic group $USp(N)$ (see definitions in Remark 1.1). Katz and Sarnak proposed a classification for families of L-functions, which consists of unitary family, symplectic family, even orthogonal family and odd orthogonal family, according to their associated symmetry types in random matrix theory. The following table gives several examples for each symmetry type.

Table 1.1: Examples for different symmetry types

Symmetry type	Examples
Unitary	$\int_0^T \lvert\zeta(\frac{1}{2}+it)\rvert^{2k}dt$; $\sum_{\chi \text{ is primitive modulo } q} \lvert L(\frac{1}{2},\chi)\rvert^k$; $\sum_{\substack{0<\gamma\leq T \\ \zeta(\frac{1}{2}+i\gamma)=0}} \lvert\zeta'(\frac{1}{2}+i\gamma)\rvert^k$
Symplectic	$\sum_{\substack{d \text{ fund. disc.} \\ 0<d\leq X}} L(\frac{1}{2},\chi_d)^k$
Even orthogonal	$\sum_{f\in H_2(q)} \lvert L(\frac{1}{2},f)\rvert^k$, where $H_2(q)$ is a basis of Hecke new-forms of weight 2 and level q.
Odd orthogonal	$\sum_{\substack{d \text{ fund. disc.} \\ 0<d\leq X}} L(\frac{1}{2},f\otimes\chi_d)^k$, where f is a holomorphic primitive cuspidal eigenform.

Remark 1.1. The unitary group, $U(N)$, is the group of all $N\times N$ complex matrices U that satisfy the condition $UU^\dagger = I_N$, where $U^\dagger$ denotes the complex transpose of U, and I_N is the $N\times N$ identity matrix.

The symplectic group, $USp(N)$, is the group of all $N\times N$ unitary matrices S that satisfy the condition $SJS^t = J$, where

$$J = \begin{bmatrix} 0 & I_N \\ I_N & 0 \end{bmatrix},$$

and S^t means the transpose of S.

The orthogonal group, $O(N)$ is the group of all $N\times N$ real matrices O that satisfy the condition $OO^t = I_N$.

Random matrix theory conjectures.

It can be conjectured (see Conrey and Farmer [16]) that

$$\frac{1}{\mathcal{Q}^*}\sum_{\substack{f\in\mathcal{F} \\ c(f)\leq\mathcal{Q}}} V(L(\tfrac{1}{2},f))^k \sim g_k\frac{a(k)}{\Gamma(1+B(k))}(\log\mathcal{Q}^A)^{B(k)}, \tag{1.1}$$

where $V(x)$ depends on the symmetry type, A is a symmetry-dependent constant, $c(f)$ is the conductor of f and $\mathcal{Q}^*$ is the number of f with $c(f)\leq\mathcal{Q}$. Here g_k and $B(k)$ are only determined by the symmetry type of the family whereas $a(k)$ is an arithmetic factor which depends on the specific family involved. We note (1.1) are discrete moments and continuous moments can be formulated in a similar manner.

The values of a_k can be conjectured by an arithmetic method. The difficulty mainly lies in determining g_k. The precise values of g_k were predicted by Keating and Snaith [62, 63] by using a random matrix model, which, along with values of a_k, completed the conjecture for the leading main term in the asymptotic formula for the moments of each specific family of L-functions. The idea of Keating and Snaith is to suggest the moments of L-functions in a family are comparable to the moments of the characteristic polynomials of random matrices corresponding to this family. For instance, for moments of the Riemann zeta function (the idea is applied to other families as well), Keating and Snaith considered the characteristic polynomial $Z(U,\theta)$ of a matrix U in the group $U(N)$:

$$Z(U,\theta)=\prod_{n=1}^{N}\left(1-e^{i(\theta_n-\theta)}\right),$$

where $e^{i\theta_n}$, $n=1,2,\cdots,N$ are eigenvalues of U. Then they computed the following s-th (s is a complex number) moment of characteristic polynomials of $U(N)$:

$$\begin{aligned}\langle |Z|^s\rangle_{U(N)} :=\frac{1}{(2\pi)^N N!}\int_0^{2\pi}\cdots\int_0^{2\pi}\ d\theta_1\cdots d\theta_N \prod_{1\le j<m\le N}\left|e^{i\theta_j}-e^{i\theta_m}\right|^2\\ \times\prod_{n=1}^{N}\left|\left(1-e^{i(\theta_n-\theta)}\right)\right|^s.\end{aligned}$$

The above complicated integral can be simplified by the Selberg integral. Indeed, one can prove

$$\lim_{N\to\infty}\frac{1}{N^{s^2}}\langle |Z|^s\rangle_{U(N)}=\frac{G^2(1+s)}{G(1+2s)},$$

where $G(s)$ is defined in (1.3). By comparing the above asymptotic formula with the main term for the moments of the Riemann zeta function and letting $N=\log\frac{T}{2\pi}$, Keating-Snaith conjectured precise values for g_k as follows.

Conjecture 1.2 (Keating-Snaith)**.** *For any real $k>0$,*

$$\int_0^T|\zeta(\tfrac{1}{2}+it)|^{2k}dt\sim\frac{g_k a_k}{(k^2)!}T(\log T)^{k^2},$$

where

$$g_k := (k^2)! \frac{G^2(1+k)}{G(1+2k)},$$

and

$$a_k := \prod_p \left(1 - \frac{1}{p}\right)^{k^2} \sum_{m=0}^{\infty} \left(\frac{\Gamma(m+k)}{m!\Gamma(k)}\right)^2 p^{-m}. \tag{1.2}$$

Here G is the Barnes G-function defined by

$$G(z+1) = (2\pi)^{\frac{z}{2}} \exp(-\tfrac{1}{2}(z^2 + \gamma z^2 + z)) \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^n e^{-z+\frac{z^2}{2n}} \tag{1.3}$$

for all $z \in \mathbb{C}$.

Conjectures by the recipe method.

For *integral* moments, Conrey, Farmer, Keating, Rubinstein and Snaith [17] proposed the recipe method which successfully refined the conjecture of Keating and Snaith by obtaining lower-order main terms. The heuristic of their method (called recipe method) is to evaluate the contribution from diagonal terms, and assume certain off-diagonal terms, which are (relatively) complicated, cancel out.

For example, for the $2k$-th moment of the Riemann zeta function, we define the following function

$$Z(s, \alpha) := \zeta(s+\alpha_1) \cdots \zeta(s+\alpha_k)\zeta(1-s-\alpha_{k+1}) \cdots \zeta(1-s-\alpha_{2k}). \tag{1.4}$$

Here α_i are shifts which are very small, and $\alpha = (\alpha_1, \ldots, \alpha_{2k})$. Our goal is to heuristically evaluate

$$\int_{-\infty}^{\infty} Z(\tfrac{1}{2} + it, \alpha) g(t) dt,$$

where $g(t)$ is a suitable weight function. Note that letting $\alpha_i \to 0$ in the above gives us exactly the $2k$-th (weighted) moment of the Riemann zeta function. For each zeta function, we use the approximate functional equation

$$\zeta(s) = \sum_m \frac{1}{m^s} + \chi(s) \sum_n \frac{1}{n^{1-s}} + \text{ remainder}. \tag{1.5}$$

Here

$$\chi(s) := \pi^{s-\frac{1}{2}} \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}.$$

It satisfies

$$\chi(s) = \left(\frac{t}{2\pi}\right)^{\frac{1}{2}-s} e^{it+\frac{\pi}{4}i} \left(1 + O\left(\frac{1}{t}\right)\right).$$

Inserting (1.5) into (1.4) and multiplying out $Z(s, \alpha)$, we obtain 2^{2k} terms. We only keep those terms in which the product of χ-factors is not oscillating rapidly. For instance, an obvious non-oscillating term is the one obtained by always using the "first part" of the approximate functional equation of each zeta factor of $Z(s, \alpha)$ in when expanding $Z(s, \alpha)$. Note this term does not have any χ-factors.

Based on the above philosophy, along with a very delicate combinatorial computation, Conrey, Farmer, Keating, Rubinstein and Snaith made the following conjecture for all integral moments of the Riemann zeta function.

Conjecture 1.3 (Conrey, Farmer, Keating, Rubinstein and Snaith)**.** *For $k \in \mathbb{N}$,*

$$\int_0^T |\zeta(\tfrac{1}{2} + it)|^{2k} dt = T P_{k^2}(\log T) + O(T^{\frac{1}{2}+\varepsilon}),$$

where $P_{k^2}(x)$ is a polynomial of degree k^2 that can be computed explicitly.

1.2.2 Historical results.

Here we provide historical results of three families of L-functions that have been broadly investigated. This section may not cover all the existing results because there were so many results in the past and new results keep coming out constantly.

Moments of the Riemann zeta function.

One of the famous problems in the theory of L-functions is the study of continuous moments of the Riemann zeta function. Let $k > 0$ be real. Let

$$I_k(T) := \int_0^T |\zeta(\tfrac{1}{2} + it)|^{2k} dt$$

denote the $2k$-th moment of the Riemann zeta function. We introduce this moment here in order to give an example for Problem (1) in Section 1.2 while we do not study this problem in our thesis.

Hardy and Littlewood [39] studied $I_k(T)$ by proving

$$I_1(T) \sim T \log T.$$

Ingham [52] refined this result to

$$I_1(T) = TP_1(\log T) + O(T^{\frac{1}{2}+\varepsilon}), \tag{1.6}$$

where $P_1(x)$ is a linear polynomial. The best error for (1.6) up to date is $O(T^{\frac{1515}{4816}+\varepsilon})$ due to Bourgain and Watt [7]. Previously Watt [106] proved the error $O(T^{\frac{131}{416}}(\log T)^{\frac{32587}{8320}})$. Moreover, the fourth moment was obtained by Ingham [52],

$$I_2(T) = \frac{T}{2\pi^2}(\log T)^4 + O(T(\log T)^3).$$

It was improved to

$$I_2(T) = TP_4(\log T) + O(T^{\frac{7}{8}+\varepsilon})$$

by Heath-Brown [45], where $P_4(x)$ is a polynomial of degree 4. The sharpest error term now is $O(T^{\frac{2}{3}+\varepsilon})$ due to Zavorotnyi [109].

In spite of many attempts, computing higher moments of the Riemann zeta function seems beyond current techniques. However, many exciting conjectures have been established. Conrey and Ghosh conjectured [18] that

$$I_3(T) \sim \frac{43a_3}{9!} T(\log T)^9.$$

Conrey and Gonek [19] conjectured that

$$I_4(T) \sim \frac{24024a_4}{16!} T(\log T)^{16}$$

for certain precise constants a_3, a_4 (see values of them in (1.2)). The conjectures of Conrey-Ghosh and Conrey-Gonek are maded by number theoretic techniques. Generally, Keating and Snaith [63] made the conjecture for the leading main term of the $2k$-th moment as shown in Conjecture 1.2, and, Conrey, Farmer, Keating, Rubinstein and Snaith [17] refined this conjecture via obtaining other principal lower-order main terms (see Conjecture 1.3).

We also have many results on the lower and upper bounds of the moments. The lower bound

$$I_k(T) \gg T(\log T)^{k^2}$$

was established by Ramachandra [89] for all positive integers k, by Heath-Brown [46] for all positive rational numbers k, and under RH, by Ramachandra [88] for all positive real numbers k. In the other direction, for real $0 \leq k \leq 2$, assuming RH, Ramachandra [89, 90] and Heath-Brown [46, 47] independently proved that

$$I_k(T) \ll T(\log T)^{k^2}.$$

For all positive real numbers k, assuming RH, Soundararajan [98] proved that

$$I_k(T) \ll T(\log T)^{k^2+\varepsilon}.$$

Building on Soundararajan's work and developing a number of new techniques, under RH, Harper [40] proved that for all positive real numbers k,

$$I_k(T) \ll T(\log T)^{k^2}.$$

Recently, Heap-Radziwiłł-Soundararajan [44] proved unconditionally that for any real number k with $0 \leq k \leq 2$,

$$I_k(T) \ll T(\log T)^{k^2}.$$

Moments of quadratic Dirichlet L-functions.

The family of quadratic Dirichlet L-functions has been extensively studied. Let $k > 0$ and

$$\sideset{}{^\flat}\sum_{|d|\le X} L(\tfrac{1}{2}, \chi_d)^k$$

denote the k-th moment of quadratic Dirichlet L-functions, where $\sideset{}{^\flat}\sum$ means the sum over fundamental discriminants. This is an example for Problem (2) in Section 1.2. This moment is related to the Chowla's conjecture that will be discussed in Subsection 1.2.3.

Jutila proved that

$$\sideset{}{^\flat}\sum_{0<d\le X} L(\tfrac{1}{2}, \chi_d) = \frac{H(1)}{4\zeta(2)} X \left[\log \frac{X}{\pi} + \frac{\Gamma'}{\Gamma}\left(\frac{1}{4}\right) + 4\gamma - 1 + 4\frac{H'}{H}(1)\right] + O(X^{\frac{3}{4}+\varepsilon}),$$

where

$$H(s) := \prod_p \left(1 - \frac{1}{(p+1)p^s}\right),$$

and

$$\sideset{}{^\flat}\sum_{0<d\le X} L(\tfrac{1}{2}, \chi_d)^2 = \frac{c}{\zeta(2)} X(\log X)^3 + O(X(\log X)^{\frac{5}{2}+\varepsilon}),$$

where

$$c = \frac{1}{48}\prod_p \left(1 - \frac{4p^2 - 3p + 1}{p^4 + p^3}\right).$$

Soundararajan established the third moment by proving

$$\sideset{}{^*}\sum_{\substack{0<d\le X \\ (d,2)=1}} L(\tfrac{1}{2}, \chi_{8d})^3 = XQ_6(\log X) + O(X^{\frac{11}{12}+\varepsilon}),$$

where $Q_6(x)$ is an explicit polynomial of degree 6, and $\sideset{}{^*}\sum$ means the sum over square-free integers. Note that the above result is for χ_{8d} instead of χ_d. The reason is to focus on the main methods and techniques. The case of χ_d may require further techniques and computation based on the argument for χ_{8d}.

Keating-Snaith [62] made the following conjecture.

Conjecture 1.4 (Keating-Snaith). *For any positive real number k,*

$$\sideset{}{^*}\sum_{\substack{0<d\leq X\\(d,2)=1}} L\left(\tfrac{1}{2},\chi_{8d}\right)^k \sim \frac{4a_k}{\pi^2}\frac{G(k+1)\sqrt{\Gamma(k+1)}}{\sqrt{G(2k+1)\Gamma(2k+1)}}X(\log X)^{\frac{k(k+1)}{2}},$$

where $G(z)$ is the Barnes G-function, and

$$a_k := 2^{-\frac{k(k+2)}{2}}\prod_{(p,2)=1}\frac{(1-\frac{1}{p})^{\frac{k(k+1)}{2}}}{1+\frac{1}{p}}\left(\frac{(1+\frac{1}{\sqrt{p}})^{-k}+(1-\frac{1}{\sqrt{p}})^{-k}}{2}+\frac{1}{p}\right).$$

Conrey, Farmer, Keating, Rubinstein and Snaith [17] gave a more precise conjecture, including all other principal lower order terms:

Conjecture 1.5 (Conrey, Farmer, Keating, Rubinstein and Snaith). *For any $k\in\mathbb{N}$,*

$$\sideset{}{^*}\sum_{\substack{0<d\leq X\\(d,2)=1}} L(\tfrac{1}{2},\chi_{8d})^k = XQ_{\frac{k(k+1)}{2}}(\log X)+o(X),$$

as $X\to\infty$ where $Q_n(x)$ is an explicit polynomial of degree n.

In this thesis, we proved Conjecture 1.4 for the the case $k=4$ in Chapter 2 (see Theorem 2.1) under the generalized Riemann hypothesis using the method of Soundararajan and Young [99]. For more details, the readers are referred to the introduction part of Chapter 2.

Discrete moments of $\zeta'(\rho)$.

Gonek [36] and Hejhal [49] introduced the following $2k$-th discrete moment of $\zeta'(s)$ given by

$$J_k(T) := \sum_{0<\gamma\leq T}|\zeta'(\rho)|^{2k},$$

where $k\in\mathbb{R}$, and the sum runs over the nontrivial zeros $\rho=\beta+i\gamma$ of $\zeta(s)$. Note the case $k<0$ is also very interesting. These moments provide an example of the type Problem (3) in Section 1.2. One reason the moments $J_k(T)$ are studied is that they are related to the size of Merten's function $M(x)$ defined by

$$M(x) := \sum_{n\leq x}\mu(n).$$

For example, Gonek (unpublished) and Ng [81] proved that

$$M(x) \ll x^{\frac{1}{2}}(\log x)^{\frac{3}{2}},$$

assuming RH and the bound $J_{-1}(T) \ll T$. Furthermore, Ng [81] has shown the same assumptions imply that $M(e^y)e^{-\frac{y}{2}}$ possesses a limiting distribution.

Assuming RH, Gonek [35] showed that

$$J_1(T) \sim \frac{T}{24}\log^4\left(\frac{T}{2\pi}\right).$$

Ng [82] considered the fourth moment and proved that $J_2(T) \asymp T(\log T)^9$. More precisely, he showed the inequality

$$\frac{c_1}{\pi^3}TL^9\left(1+O\left(\frac{\log L}{L}\right)\right) \leq J_2(T) \leq \frac{c_2}{\pi^3}TL^9\left(1+O\left(\frac{\log L}{L}\right)\right),$$

where $L := \log\left(\frac{T}{2\pi}\right)$, $c_1 = 0.0000687\cdots$, and $c_2 = 0.0051561\cdots$.

The asymptotic formulae for the fourth and higher moments are still open problems. Independently, Gonek [36] and Hejhal [49] conjectured that

$$J_k(T) \asymp T(\log T)^{(k+1)^2}.$$

By modelling characteristic polynomials of random matrices, Hughes, Keating and O'Connell [51] refined this conjecture. They predicted that for any $k > -\frac{3}{2}$,

$$J_k(T) \sim c(k)a(k)\frac{T}{2\pi}\log^{(k+1)^2}\left(\frac{T}{2\pi}\right),$$

where

$$\begin{aligned} c(k) &:= \frac{G^2(k+2)}{G(2k+3)}, \\ a(k) &:= \prod_p \left(1-\frac{1}{p}\right)^{k^2}\sum_{m=0}^{\infty}\left(\frac{\Gamma(m+k)}{m!\Gamma(k)}\right)^2 p^{-m}. \end{aligned}$$

Based on the ratios conjecture, Conrey and Snaith [20] proved that

$$J_2(T) = T\tilde{P}_9\left(\log \frac{T}{2\pi}\right) + O(T^{\frac{1}{2}+\varepsilon}),$$

where $\tilde{P}_9(x)$ is a polynomial of degree 9.

Many results on lower and upper bounds of the moments have been obtained. For any $k \in \mathbb{R}$, under RH, Milinovich [73] showed that

$$J_k(T) \ll T(\log T)^{(k+1)^2+\varepsilon}.$$

This result was improved recently by Kirila [65] using Harper's method [40] via showing assuming RH, for $k \geq \frac{1}{2}$,

$$J_k(T) \ll T(\log T)^{(k+1)^2}.$$

On the other hand, under GRH, Milinovich and Ng [75] showed that for any $k \in \mathbb{N}$,

$$J_k(T) \gg T(\log T)^{(k+1)^2}.$$

Gao [32] very recently obtained that under RH, for any real $k > 0$,

$$J_k(T) \gg T(\log T)^{(k+1)^2}.$$

For negative moments, Gonek [36] proved that

$$J_{-1}(T) \gg T$$

assuming RH and the simplicity of the zeros of $\zeta(s)$. The RH condition was removed by Garaev and Sankaranarayanan [33]. Milinovich and Ng [74] obtained a precise inequality

$$J_{-1}(T) \geq \left(\frac{3}{2\pi^3} - \varepsilon\right) T,$$

assuming RH and the simplicity of the zeros of $\zeta(s)$. With the same assumption, Heap, Li and

Zhao [43] proved that for any rational $k \geq 0$,

$$J_{-k}(T) \gg T(\log T)^{(k-1)^2}.$$

1.2.3 Applications.

In this section, we give four applications of bounds and asymptotics for moments of L-functions. There are applications to the size of L-functions (Lindelöf hypothesis), the nonvanishing results of L-functions, the Birch and Swinnerton-Dyer conjecture, and generalized Fermat equations.

Lindelöf hypothesis.

The Lindelöf hypothesis states that for any $\varepsilon > 0$,

$$\zeta(\tfrac{1}{2} + it) \ll_\varepsilon t^\varepsilon.$$

Hardy and Littlewood observed that the Lindelöf hypothesis is equivalent to the following upper bound

$$\int_0^T |\zeta(\tfrac{1}{2} + it)|^{2k} dt \ll_{k,\varepsilon} T^{1+\varepsilon},$$

where $k \in \mathbb{N}$. See details in Titchmarsh [101, Theorem 13.2]. This relation enables us to convert the study of the Lindelöf hypothesis to the research on the moments of the Riemann zeta function. This method has also been widely used in obtaining subconvexity bounds for more general L-functions. For example, see Duke-Friedlander-Iwaniec [27] and Kowalski-Michel-VanderKam [68].

Nonvanishing of Dirichlet L-functions.

It is a hard open problem in analytic number theory to show that $L\left(\frac{1}{2}, \chi\right) \neq 0$ for all primitive characters χ. In 1965, Chowla [14] conjectured this when χ is a quadratic character. Assuming GRH, Katz and Sarnak (unpublished) and Özlük-Snyder [84] independently proved $L\left(\frac{1}{2}, \left(\frac{d}{\cdot}\right)\right) \neq 0$ for at least $\frac{15}{16}$ of the fundamental discriminants $|d| \leq X$. By establishing the mollified first and second moments of quadratic Dirichlet L-functions, Soundararajan [97] showed that there are at least 87.5% of the odd square-free integers $d \geq 0$ such that $L\left(\frac{1}{2}, \left(\frac{8d}{\cdot}\right)\right) \neq 0$.

In the other direction, Balasubramanian and K. Murty [4] proved that a positive proportion (though very small) of Dirichlet L-functions in the family of primitive characters modulo prime modulo q do not vanish at $s = \frac{1}{2}$. Iwaniec and Sarnak [56] refined this by showing $L(\frac{1}{2}, \chi) \neq 0$ for at least 33.33% of the primitive characters modulo integral q. Bui [9] improved this result further to 34%. Khan and Ngo [64] later refined this to 37.49% for prime modulo p.

Birch and Swinnerton-Dyer conjecture.

Let E be a modular elliptic curve with root number 1 over $\mathbb{Q}$. In a celebrated paper [66], Kolyvagin proved that if the Hasse-Weil L-function $L(s, E)$ does not vanish at the central point $s = \frac{1}{2}$, then the group of rational points of E is finite, *provided* that there exists a quadratic character χ_d with $d < 0$ such that $L(s, E \otimes \chi_d)$ has a simple zero at the central point and such that $\chi_d(p) = 1$ for every p that divides the conductor of E. Bump-Friedberg-Hoffstein [12] and Murty-Murty [79] independently proved $L'(\frac{1}{2}, E \otimes \chi_d) \neq 0$ for infinitely many fundamental discriminants d with $d < 0$. (See the definition of $L'(\frac{1}{2}, E)$ in (3.1).) These results successfully verify the assumption in Kolyvagin's theorem. Their methods are to investigate the following types of moments:

$$\sum_{|d| \leq X}^{\flat} c_d L'(\tfrac{1}{2}, E \otimes \chi_d)$$

where c_d are complex numbers. By establishing an asymptotic formula for such sums, they are able to deduce the required nonvanishing result in Kolyvagin's theorem. The readers are referred to Chapter 3 and also Ireland-Rosen [53, Chapter 20] for further details.

Generalized Fermat equations.

Moments of L-functions and nonvanishing of L-functions have applications to generalized Fermat equations. Ellenberg showed a connection between the generalized Fermat equation

$$x^4 + y^2 = z^p \tag{1.7}$$

and the nonvanishing of certain modular L-functions (see Ellenberg [28, p. 765]). Based on this he showed that (1.7) has no integral solutions with $\gcd(x, y, z) = 1$ for $p \geq 211$. Bennett, Ellenberg and Ng [5] showed that (1.7) has no integral solutions with $\gcd(x, y, z) = 1$ for $p \geq 4$.

The nonvanishing result is obtained by computing the moment of a certain family of modular L-functions.

1.2.4 Original results.

We now mention the original results in this thesis on moments of L-functions. In Chapter 2, we prove an asymptotic formula for the fourth moment of quadratic Dirichlet L-functions under the assumption of the generalized Riemann hypothesis. Unconditionally, we prove a sharp lower bound for this family of L-functions. Our results confirm the conjecture of Keating and Snaith for the case $k = 4$, assuming the generalized Riemann hypothesis. The argument is largely based on Soundararajan and Young's method [99] for the second moment of quadratic twists of modular L-functions and Soundararajan's work [97] on the third moment of quadratic Dirichlet L-functions. More precisely, we use the argument of Soundararajan and Young [99] to shorten the length of the Dirichlet polynomial involved via establishing the shifted version of the fourth moment of quadratic Dirichlet L-functions. While the off-diagonal terms are bounded as an error term in Soundararajan and Young's work, these terms contribute to a main term in our consideration. We use techniques from Soundararajan's article [97] to analyze the off-diagonal terms, which is the main new ingredient of our work. We obtain that

Theorem 1.6 (Chapter 2)**.** *Assume GRH for $L(s, \chi_d)$ for all fundamental discriminants d. For any $\varepsilon > 0$, we have*

$$\sideset{}{^*}\sum_{\substack{0<d\leq X\\(d,2)=1}} L(\tfrac{1}{2}, \chi_{8d})^4 = \frac{a_4}{2^6 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot \pi^2} X(\log X)^{10} + O\left(X(\log X)^{9.75+\varepsilon}\right).$$

Here $\sum^*$ denotes the summation over square-free integers, and a_4 is a constant defined in (2.4)*.*

We also prove a sharp lower bound unconditionally. This was also stated without proof by Rudnick and Soundararajan [92].

Theorem 1.7 (Chapter 2)**.** *Unconditionally, we have*

$$\sideset{}{^*}\sum_{\substack{0<d\leq X\\(d,2)=1}} L(\tfrac{1}{2}, \chi_{8d})^4 \geq \left(\frac{a_4}{2^6 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot \pi^2} + o(1)\right) X(\log X)^{10}.$$

Gao [31] recently established an asymptotic for the fourth moment of quadratic Hecke L-functions in the Gaussian field with using some of techniques for the above theorems.

Another original result in this thesis concerns the family of quadratic twists of modular L-functions. In Chapter 3, We obtain an error term of size $O(X^{\frac{1}{2}+\varepsilon})$ for the smoothed first moment of quadratic twists of modular L-functions. A similar argument allows us to give a comparable result for the smoothed first moment of the first derivative of quadratic twists of modular L-functions. The main idea we use is a recursive method. This method was first used by Heath-Brown [48] to get upper bounds for mean values of sums of real characters, and later by Young [107, 108] to study asymptotics for the first and third moments of quadratic Dirichlet L-functions. Our result is analogous to Sono's work [96] where he considered the second moment of quadratic Dirichlet L-functions. Note that $L(s,\chi_d)^2$ is analogous to $L(s, f\otimes\chi_d)$. These each are degree two L-functions as described in Section 1.1. Many of the same techniques apply, for example, approximate functional equation, Poisson summation formula, etc. We prove that

Theorem 1.8 (Chapter 3)**.** *Let $\kappa \equiv 0 \,(\mathrm{mod}\, 4)$ and $\kappa \neq 0$. Let $\Phi(x) : (0,\infty) \to \mathbb{R}$ be a smooth, compactly supported function. We have*

$$\sideset{}{^*}\sum_{(d,2)=1} L(\tfrac{1}{2}, f\otimes\chi_{8d})\Phi(\tfrac{d}{X}) = \frac{8\tilde{\Phi}(1)}{\pi^2} L(1, \mathrm{sym}^2 f) Z^*(0) X + O(X^{\frac{1}{2}+\varepsilon}),$$

Here Z^* is defined via (3.5) *and the paragraph below Theorem 3.4, and $\tilde{\Phi}$ is the Mellin transform of Φ defined by*

$$\tilde{\Phi}(s) := \int_0^\infty \Phi(x) x^{s-1} dx.$$

We also establish an asymptotic for the first moment of the derivative.

Theorem 1.9 (Chapter 3)**.** *Let $\kappa \equiv 2 \,(\mathrm{mod}\, 4)$. Let $\Phi(x) : (0,\infty) \to \mathbb{R}$ be a smooth, compactly supported function. We have*

$$\sideset{}{^*}\sum_{(d,2)=1} L'(\tfrac{1}{2}, f\otimes\chi_{8d})\Phi(\tfrac{d}{X}) = \frac{8\tilde{\Phi}(1)}{\pi^2} L(1, \mathrm{sym}^2 f) Z^*(0) X \Big[\log X + 2\frac{L'(1,\mathrm{sym}^2 f)}{L(1,\mathrm{sym}^2 f)} + \frac{Z^{*\prime}(0)}{Z^*(0)} + \log\frac{8}{2\pi} + \frac{\Gamma'(\frac{\kappa}{2})}{\Gamma(\frac{\kappa}{2})} + \frac{\tilde{\Phi}'(1)}{\tilde{\Phi}(1)} \Big] + O(X^{\frac{1}{2}+\varepsilon}).$$

We remark that a similar argument can lead to the first moment of higher derivatives of twised modular L-functions. The above two theorems improve the previous results (up to multiplying by a smoothed function): $O(X(\log X)^{1-\rho})$, where ρ is an explicit positive real number, of Murty-Murty [79], $O(X^{\frac{13}{14}+\varepsilon})$ of Iwaniec [54], Stefanicki [100, Theorem 3] and Luo-Ramakrishnan [71, Proposition 3.6], and $O(X^{\frac{7}{8}+\varepsilon})$ of Soundararajan-Radziwiłł [87, Proposition 2]. Recall that such results were used in completing Kolyvagin's work on Birch and Swinnerton-Dyer conjecture as described in Subsection 1.2.3.

1.3 Counting zeros of L-functions.

In this thesis we also prove results about the zeros of L-functions. Let us start with the Riemann zeta function as an example in this section. Zeros of the Riemann zeta function are intimately related to prime numbers via Riemann's explicit formula (see (1.8)). The most important conjecture concerning zeros of the Riemann zeta function is the Riemann hypothesis. It asserts that all non-trivial zeros of $\zeta(s)$ lie on the critical line $\mathrm{Re}(s) = \frac{1}{2}$. Similarly, it is widely believed that all non-trivial zeros of a general L-function also lie on the critical line. Another old problem related to zeros is to give a precise formula for the number of zeros of an L-function in a box.

The following definitions are the number of zeros in a box of two L-functions: the Riemann zeta function $\zeta(s)$ and the Dedekind zeta function $\zeta_K(s)$ associated to a number field K. Set

$$N(T) := \#\{\rho = \beta + i\gamma \in \mathbb{C} \mid \zeta(\rho) = 0,\ 0 < \beta < 1,\ 0 < \gamma \leq T\},$$
$$N_K(T) := \#\{\rho = \beta + i\gamma \in \mathbb{C} \mid \zeta_K(\rho) = 0,\ 0 < \beta < 1,\ |\gamma| \leq T\}.$$

Note $N(T) = \frac{1}{2}N_K(T)$ when $K = \mathbb{Q}$. In Chapter 4, we shall prove a precise explicit asymptotic formula for $N_K(T)$.

1.3.1 Motivation and a brief history.

The Tchebychev ψ-function is

$$\psi(x) = \sum_{n\leq x} \Lambda(n),$$

where $\Lambda(n) = \log p$ if $n = p^k$ for some $k \geq 0$, otherwise, $\Lambda(n) = 0$. The prime number theorem is equivalent to the asymptotic $\psi(x) \sim x$. It was independently proven by Hadamard [38] and de la Vallée Poussin [23, 24]. The basic idea of the proof is to express ψ as a sum over the zeros of $\zeta(s)$. In fact, for $x \notin \mathbb{N}$,

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log(1-x^2), \tag{1.8}$$

where ρ runs through all non-trivial zero of $\zeta(s)$. A version of this formula was stated in Riemann's memoir and later proved by von Mangoldt. In order to prove the prime number theorem one actually requires a version of (1.8) where the sum over zeros is truncated to $\sum_{|\gamma|\le T} \frac{x^{\rho}}{\rho}$ with $T > 0$. Such a formula can be obtained via Perron's formula. Thus estimates for $\psi(x)$ can be deduced from bounds for $\sum_{|\gamma|\le T} \frac{x^{\rho}}{\rho}$. Bounds for this latter sum may be deduced from the zero-free region for $\zeta(s)$ and the size of $N(T)$.

In his memoir, Riemann stated without proof that

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi e} + O(\log T), \tag{1.9}$$

and it was proved by von Mangoldt. The equation (1.9) is crucial for the establishment of the prime number theorem. Naturally, explicit versions for (1.9) are very useful in establishing explicit versions of the prime number theorem (see Faber-Kadiri [29] and Trudgian [104]). Moreover, the explicit version is also applied to sums of zeros of the Riemann zeta function (see Brent-Platt-Trudgian [8]) and the full proof of the ternary Goldbach problem (see Helfgott [50]).

Explicit versions of (1.9) have been established. Namely, there exist positive constants C_1, C_2, C_3 such that for $T \ge T_0 \ge e$,

$$\left|N(T) - \frac{T}{2\pi}\log\left(\frac{T}{2\pi e}\right)\right| \le C_1 \log T + C_2 \log\log T + C_3. \tag{1.10}$$

Below in Table 1.2, we list a table that consists of historical results for the values of C_1, C_2, and C_3.

The first explicit result for $N_K(T)$ was established by Kadiri and Ng [58]. They showed that for $T \ge 1$, one has

$$\left|N_K(T) - \frac{T}{\pi}\log\left(d_K\left(\frac{T}{2\pi e}\right)^{n_K}\right)\right| \le D_1(\log d_K + n_K \log T) + D_2 n_K + D_3, \tag{1.11}$$

Table 1.2: Explicit bounds for $N(T)$ in (1.10)

	C_1	C_2	C_3	T_0
Von Mangoldt [105] (1905)	0.4320	1.9167	13.0788	28.5580
Grossmann [37] (1913)	0.2907	1.7862	7.0120	50
Backlund [3] (1918)	0.1370	0.4430	5.2250	200
Rosser [91] (1941)	0.1370	0.4430	2.4630	2
Trudgian [102] (2014)	0.1120	0.2780	3.3850	e
Hasanalizade, S. and Wong [42]	0.1038	0.2573	9.3675	e

with admissible triple $(D_1, D_2, D_3) = (0.506, 16.950, 7.663)$, where n_K and d_K are the degree and absolute discriminant of K, respectively. They also mentioned that D_1 could be taken as small as $(\pi \log 2)^{-1} \approx 0.459$ at expense of larger D_2 and D_3. This was improved by Trudgian [103] who showed $(D_1, D_2, D_3) = (0.316, 5.872, 3.655)$ is valid, and the constant D_1 in (1.11) can be made as small as 0.247 (with larger D_2 and D_3). Unfortunately, as pointed out by Bennett, Martin, O'Bryant, and Rechnitzer [6], an error appeared in [103]. It will be fixed and also improved in Chapter 4 by using the method of Bennett, Martin, O'Bryant and Rechnitzer [6], and Kadiri and Ng [58], and Trudgian [103].

Our new results on $N_K(T)$ shall be described precisely in the next subsection.

1.3.2 Original results.

We show in Chapter 4 that

Theorem 1.10 (Hasanalizade, S. and Wong, Chapter 4)**.** *Given a number field K of degree n_K and with absolute discriminant d_K, for any $T \geq 1$, we have*

$$\left| N_K(T) - \frac{T}{\pi} \log\left(d_K \Big(\frac{T}{2\pi e} \Big)^{n_K} \right) \right| \leq 0.228(\log d_K + n_K \log T) + 23.108 n_K + 4.520. \qquad (1.12)$$

The above theorem was used in the work of Kadiri-Wong [59] and the Master's thesis of Das [21]. The techniques of proofs of the above theorem includes Jensen's formula from complex analysis, bounds for the Riemann zeta function/Dedekind zeta functions, and Backlund's trick.

Following a similar manner of the proof for Theorem 1.10, in a following paper [42] joint with Hasanalizade and Wong, we study $N(T)$ and prove that

Theorem 1.11 (Hasanalizade, S. and Wong)**.** *For any $T \geq e$, we have*

$$\left| N(T) - \frac{T}{2\pi} \log\left(\frac{T}{2\pi e}\right) \right| \leq 0.1038 \log T + 0.2573 \log\log T + 9.3675. \tag{1.13}$$

See Table 1.2 for a comparison to historical results. Note that the estimate (1.13) of the Riemann zeta function is better than (1.12) of Dedekind zeta functions. (The equation (1.13) needs to be multiplied by a factor 2 when doing the comparison.) The reason is that for the case of the Riemann zeta function, we have better upper bounds for $\zeta(\frac{1}{2}+it)$ and $\zeta(1+it)$ in the literature while only trivial (convexity) bounds are available in the case of Dedekind zeta functions.

1.4 Contributions of Authors

The Chapters $2, 3, 4$ are slightly modified versions of the following three papers [41, 95, 94].

- The fourth moment of quadratic Dirichlet L-functions, *Math. Z.*, 298, 713–745, 2021. arXiv:1907.01107
- The first moment of quadratic twists of modular L-functions, *submitted.* arXiv:2103.12284
- (with Elchin Hasanalizade and Peng-Jie Wong) Counting zeros of Dedekind zeta functions, to appear in *Math. Comp.*, arXiv:2102.04663

Chapter 4 is based on collaborations with Elchin Hasanalizade and Peng-Jie Wong in the article [41]. All authors contributed equally to this project. Specifically, I made significant contributions to Subsection 4.2.1, Lemma 4.11, Proposition 4.17, Section 4.4 and I prepared the Maple computation file. Furthermore, all authors proofread the whole article.

Chapter 2

The fourth moment of quadratic Dirichlet L-functions

2.1 Introduction.

Let $\chi_d = \left(\frac{d}{\cdot}\right)$ be a real primitive Dirichlet character modulo d given by the Kronecker symbol, where d is a fundamental discriminant. The k-th moment of quadratic Dirichlet L-functions is

$$\sum_{0<d\leq X}^{\flat} L(\tfrac{1}{2}, \chi_d)^k, \tag{2.1}$$

where $\sum^{\flat}$ denotes the sum over fundamental discriminants, and k is a positive real number. One great motivation to study (2.1) comes from Chowla's conjecture, which states that $L(\frac{1}{2}, \chi_d) \neq 0$ for all fundamental discriminants d. The current best result toward this conjecture is Soundararajan's celebrated work [97] in 2000, where it was proven that $L(\frac{1}{2}, \chi_{8d}) \neq 0$ for at least 87.5% of the odd square-free integers $d \geq 0$. The key to the proof is the evaluation of mollified first and second moments of quadratic Dirichlet L-functions.

In 2000, using a random matrix model, Keating and Snaith [62] conjectured that for any positive real number k,

$$\sum_{|d|\leq X}^{\flat} L(\tfrac{1}{2}, \chi_d)^k \sim C_k X (\log X)^{\frac{k(k+1)}{2}}, \tag{2.2}$$

where C_k are explicit constants. Various researchers have studied versions of these moments summed over certain subsets of the fundamental discriminants. For instance, in (2.1) we consider positive fundamental discriminants. However, there are no difficulties in also studying negative fundamental discriminants. Some articles even consider characters of the form χ_{8d}, where d are odd positive square-free integers. The main reason researchers study these special cases, rather than consider all fundamental discriminants, is to focus on the methods and techniques. It is possible to establish results for all fundamental discriminants, but this would involve more cases that need to be studied. The conjecture analogous to (2.2) for characters of the form χ_{8d}, which can be established by using Keating and Snaith's method [62], was obtained in Andrade and Keating's paper [2, Conjecture 2]. For any positive real number k, it was conjectured that

$$\sideset{}{^*}\sum_{\substack{0<d\leq X\\(d,2)=1}} L\left(\tfrac{1}{2},\chi_{8d}\right)^k \sim \frac{4a_k}{\pi^2}\frac{G(k+1)\sqrt{\Gamma(k+1)}}{\sqrt{G(2k+1)\Gamma(2k+1)}}X(\log X)^{\frac{k(k+1)}{2}}, \tag{2.3}$$

where $\sum^*$ denotes the sum over square-free integers, $G(z)$ is the Barnes G-function, and

$$a_k := 2^{-\frac{k(k+2)}{2}}\prod_{(p,2)=1}\frac{(1-\frac{1}{p})^{\frac{k(k+1)}{2}}}{1+\frac{1}{p}}\left(\frac{(1+\frac{1}{\sqrt{p}})^{-k}+(1-\frac{1}{\sqrt{p}})^{-k}}{2}+\frac{1}{p}\right). \tag{2.4}$$

In this chapter, we prove the conjecture in (2.3) for $k=4$ assuming the generalized Riemann hypothesis (GRH).

Theorem 2.1. *Assume GRH for $L(s,\chi_d)$ for all fundamental discriminants d. For any $\varepsilon>0$, we have*

$$\sideset{}{^*}\sum_{\substack{0<d\leq X\\(d,2)=1}} L(\tfrac{1}{2},\chi_{8d})^4 = \frac{a_4}{2^6\cdot 3^3\cdot 5^2\cdot 7\cdot \pi^2}X(\log X)^{10} + O\left(X(\log X)^{9.75+\varepsilon}\right).$$

The proof of Theorem 2.1 largely follows Soundararajan and Young's paper [99] in 2010 and Soundararajan's paper [97] in 2000. In [99], Soundararajan and Young proved an asymptotic formula for the second moment of quadratic twists of a modular L-function, obtaining the leading main term. Experts believed that the methods and techniques in [99] could be used to evaluate the fourth moment of quadratic Dirichlet L-functions. Motivated by this expectation, we established Theorem 2.1. In fact, Theorem 2.1 may be viewed as a version of [99, Theorem 1.2] where f is an Eisenstein series. The main difference between this result and [99] is that the off-diagonal terms (see just after (2.23) for a precise definition) contribute to the main term, whereas in [99] they are part of the error term. We use techniques from [97, Sections 5.2, 5.3] to evaluate the off-diagonal terms and this is the main new input. These terms may be written as a certain multiple complex integral. One of the difficulties in evaluating this integral is that the integrand has high order poles, and this makes the calculation more intricate. It should be noted that in 2017 Florea [30] proved an asymptotic formula for the fourth moment of quadratic Dirichlet L-functions in the function field setting, with extra lower main terms.

Similar to [99, Theorem 1.1], we obtain an unconditional lower bound that matches the conjectured asymptotic formula (2.3). This result was stated without proof by Rudnick and Soundararajan [92] in 2006.

Theorem 2.2. *Unconditionally, we have*

$$\sideset{}{^*}\sum_{\substack{0<d\leq X\\(d,2)=1}} L(\tfrac{1}{2},\chi_{8d})^4 \geq \Big(\frac{a_4}{2^6\cdot 3^3\cdot 5^2\cdot 7\cdot \pi^2}+o(1)\Big)X(\log X)^{10}.$$

We now introduce more refined conjectures for the moments of quadratic Dirichlet L-functions and provide a brief history of related results. In 2005, Conrey, Farmer, Keating, Rubinstein and Snaith [17] gave a more precise conjecture, including all other principal lower order terms,

$$\sideset{}{^\flat}\sum_{0<d\leq X} L(\tfrac{1}{2},\chi_d)^k = XP_{\frac{k(k+1)}{2}}(\log X)+E_k(X), \tag{2.5}$$

where k is a positive integer, $P_n(x)$ is an explicit polynomial of degree n, and $E_k(X)=o_k(X)$. For characters of the form χ_{8d}, their conjecture may be written as

$$\sideset{}{^*}\sum_{\substack{0<d\leq X\\(d,2)=1}} L(\tfrac{1}{2},\chi_{8d})^k = XQ_{\frac{k(k+1)}{2}}(\log X)+\hat{E}_k(X), \tag{2.6}$$

where $Q_n(x)$ is another explicit polynomial of degree n, and $\hat{E}_k(X)=o_k(X)$.

In 1981, Jutila [57] established (2.5) for $k=1$ with $E_1(X)=O(X^{\frac{3}{4}+\varepsilon})$. In 1985, Goldfeld and Hoffstein [34] improved this to $E_1(X)=O(X^{\frac{19}{32}+\varepsilon})$ by using multiple Dirichlet series. Their work implies the error $O(X^{\frac{1}{2}+\varepsilon})$ for a smoothed version of the sum in (2.5) when $k=1$. This was later obtained by Young [107] in 2009, using a different technique based on a recursive method and a study of shifted moments. We remark that Alderson and Rubinstein [1] conjectured that

$E_1(X) = O(X^{\frac{1}{4}+\varepsilon})$. In 1981, the second moment was established by Jutila [57],

$$\sideset{}{^\flat}\sum_{|d|\leq X} L(\tfrac{1}{2}, \chi_d)^2 = C_2 X (\log X)^3 + O\left(X(\log X)^{\frac{5}{2}+\varepsilon}\right).$$

In 2000, Soundararajan [97] improved this by obtaining the full main term in (2.6), in the case $k = 2$, with the power savings $\hat{E}_2(X) = O(X^{\frac{5}{6}+\varepsilon})$. In 2020, Sono [96] improved this to $O(X^{\frac{1}{2}+\varepsilon})$ for a smoothed variant of $\hat{E}_2(X)$. In [97] Soundararajan was the first to prove an asymptotic for the third moment, obtaining $\hat{E}_3(X) = O(X^{\frac{11}{12}+\varepsilon})$. In 2003, Diaconu, Goldfeld and Hoffstein [25] improved this to $E_3(X) = O(X^{0.85\cdots+\varepsilon})$ by using multiple Dirichlet series techniques. In 2013, Young [108] further improved this to $O(X^{\frac{3}{4}+\varepsilon})$ for a smoothed version of $\hat{E}_3(X)$ by using similar techniques to [107]. Recently, in 2018, Diaconu and Whitehead [26] improved Young's result by showing that a smoothed version of $\hat{E}_3(X)$ is of size $cX^{\frac{3}{4}} + O(X^{\frac{2}{3}+\varepsilon})$, for some $c \in \mathbb{R}$. This verified a conjecture of Diaconu, Goldfeld and Hoffstein [25] of the presence of a secondary lower order term. Zhang [110] had previously conditionally established a secondary term of size $X^{\frac{3}{4}}$ in 2005.

For the family of quadratic Dirichlet L-functions, moments higher than four have not been asymptotically evaluated. This seems beyond current techniques. However, there are celebrated results on upper and lower bounds of the moments. In 2006, Rudnick and Soundararajan [92] proved the lower bound

$$\sideset{}{^\flat}\sum_{0<d\leq X} L(\tfrac{1}{2}, \chi_d)^k \gg_k X(\log X)^{\frac{k(k+1)}{2}}$$

for all even natural number $k \geq 1$. In 2009, Soundararajan [98] proved under GRH that for all positive real k,

$$\sideset{}{^\flat}\sum_{0<d\leq X} L(\tfrac{1}{2}, \chi_d)^k \ll_{k,\varepsilon} X(\log X)^{\frac{k(k+1)}{2}+\varepsilon}. \tag{2.7}$$

In 2013, Harper [40], assuming GRH, improved this to

$$\sideset{}{^\flat}\sum_{0<d\leq X} L(\tfrac{1}{2}, \chi_d)^k \ll_k X(\log X)^{\frac{k(k+1)}{2}}.$$

The method of this chapter is largely based on the arguments and techniques in [99] and [97]. We use the approximate functional equation for Dirichlet L-functions and then employ the

Poisson summation formula to separate the summation into diagonal terms, off-diagonal terms, and error terms. Both diagonal and off-diagonal terms contribute to the main term. To bound the error terms, by following the argument in [98, 99], under GRH, we established an upper bound for the shifted moments of quadratic Dirichlet L-functions (see Theorem 2.6).

With further effort, one might be able to heuristically obtain all the main terms that are expected from the conjecture of Conrey et al. in (2.6). However, the computation will be complicated. It might be simplified by considering a shifted version of the fourth moment, analogous to the calculation in [107]. Florea considered the function field version of the fourth moment in [30]. In her work she was able to identify all the main terms as given by a conjecture of Andrade-Keating [2, Conjecture 5] (the function field analogue of (2.6)). By using a recursive method, Florea obtained extra lower main terms in this case. It is possible that her techniques may be employed to obtain additional lower main terms in Theorem 2.1, and we hope to revisit this in future work. However, one would need to apply the approximate functional equation for the fourth power of the L-function rather than the second power (2.10). In addition, one would have to eliminate the use of the parameters U_1, U_2 in (2.15). In our work, we use the approximate functional equation for the second power of the L-function as it is necessary to obtain the unconditional lower bound in Theorem 2.2.

The outline of this chapter is as follows. The proof of Theorem 2.1 and 2.2 proceed simultaneously. In Section 2.2, we introduce some tools. In Section 2.3, we set up the evaluation of the fourth moment. We apply the Poisson summation formula to split the fourth moment into diagonal, off-diagonal, and error terms. We evaluate the diagonal terms and off-diagonal terms in Section 2.4 and Section 2.5, respectively. The error terms are bounded in Section 2.6. The proofs of Theorem 2.1 and 2.2 are completed in Section 2.7. Finally, we give the proof of Theorem 2.6 in Section 2.8.

Notation. In this chapter we shall use the convention that $\varepsilon > 0$ denotes an arbitrary small constant which may vary in different situations. For two functions $f(x)$ and $g(x)$, we shall use the notation $f(x) = O(g(x))$, $f(x) \ll g(x)$ to mean there exists a constant C such that $|f(x)| \leq C|g(x)|$ for all sufficiently large x. If we write $f(x) = O_a(g(x))$ or $f(x) \ll_a g(x)$, then we mean that the corresponding constants depend on a. Throughout the chapter, the big O

may depend on ε.

2.2 Basic tools.

In this section, we introduce several tools that shall be used in this chapter.

2.2.1 Approximate functional equation.

For $\xi > 0$, define

$$\omega(\xi) := \frac{1}{2\pi i}\int_{(c)} \pi^s g(s)\xi^{-s}\frac{ds}{s},\ c > 0, \tag{2.8}$$

where

$$g(s) := \pi^{-s}\left(\frac{\Gamma(\frac{s}{2}+\frac{1}{4})}{\Gamma(\frac{1}{4})}\right)^2. \tag{2.9}$$

Here, and henceforth, $\int_{(c)}$ stands for $\int_{c-i\infty}^{c+i\infty}$. It can be shown (see [97, Lemma 2.1]) that $w(\xi)$ is real-valued and smooth on $(0,+\infty)$, bounded as ξ near 0, and decays exponentially as $\xi \to +\infty$. Define

$$A(d) := \sum_{n=1}^{\infty} \frac{\tau(n)\chi_{8d}(n)}{\sqrt{n}}\omega\left(\frac{n\pi}{8d}\right),$$

where $\tau(n)$ is the number of divisors of n. It was proved [97, Lemma 2.2] that for odd, positive, square-free integers d,

$$L(\tfrac{1}{2},\chi_{8d})^2 = 2A(d). \tag{2.10}$$

2.2.2 Poisson summation formula.

The following lemma is [99, Lemma 2.2].

Lemma 2.3. *Let Φ be a smooth function with compact support on the positive real numbers, and suppose that n is an odd integer. Then*

$$\sum_{(d,2)=1}\left(\frac{d}{n}\right)\Phi\left(\frac{d}{Z}\right) = \frac{Z}{2n}\left(\frac{2}{n}\right)\sum_{k\in\mathbb{Z}}(-1)^k G_k(n)\hat{\Phi}\left(\frac{kZ}{2n}\right),$$

where

$$G_k(n) := \left(\frac{1-i}{2} + \left(\frac{-1}{n}\right)\frac{1+i}{2}\right) \sum_{a \,(\mathrm{mod}\, n)} \left(\frac{a}{n}\right) e\left(\frac{ak}{n}\right), \tag{2.11}$$

and

$$\hat{\Phi}(y) := \int_{-\infty}^{\infty} (\cos(2\pi xy) + \sin(2\pi xy))\, \Phi(x) dx$$

is a Fourier-type transform of Φ.

The precise values of the Gauss-type sum $G_k(n)$ have been calculated in [97, Lemma 2.3] as follows.

Lemma 2.4. *If m and n are relatively prime odd integers, then $G_k(mn) = G_k(m)G_k(n)$. Moreover, if p^α is the largest power of p dividing k (setting $\alpha = \infty$ if $k = 0$), then*

$$G_k(p^\beta) = \begin{cases} 0 & \text{if } \beta \leq \alpha \text{ is odd,} \\ \phi(p^\beta) & \text{if } \beta \leq \alpha \text{ is even,} \\ -p^\alpha & \text{if } \beta = \alpha + 1 \text{ is even,} \\ \left(\frac{kp^{-\alpha}}{p}\right) p^\alpha \sqrt{p} & \text{if } \beta = \alpha + 1 \text{ is odd,} \\ 0 & \text{if } \beta \geq \alpha + 2. \end{cases}$$

Here ϕ is the Euler totient function.

2.2.3 Smooth function.

Let Φ be a smooth Schwarz class function that is compactly supported on $[\frac{1}{2}, \frac{5}{2}]$, and $0 \leq \Phi(t) \leq 1$ for all t. For any integer $\nu \geq 0$, define

$$\Phi_{(\nu)} := \max_{0 \leq j \leq \nu} \int_{\frac{1}{2}}^{\frac{5}{2}} |\Phi^{(j)}(t)| dt.$$

For any $s \in \mathbb{C}$, define

$$\check{\Phi}(s) := \int_0^\infty \Phi(t) t^{-s} dt.$$

Note that $\check{\Phi}(s)$ is a holomorphic function of s. Integrating by parts ν times gives us

$$\check{\Phi}(s) = \frac{1}{(s-1)(s-2)\cdots(s-\nu)} \int_0^\infty \Phi^{(\nu)}(t) t^{-s+\nu} dt.$$

Hence, for $\mathrm{Re}(s) < 1$, we see that

$$\check{\Phi}(s) \ll_\nu \frac{3^{|\mathrm{Re}(s)|}}{|s-1|^\nu} \Phi_{(\nu)}. \tag{2.12}$$

2.2.4 Some lemmas.

The following lemma is the sharpest upper bound up to date for the fourth moment of quadratic Dirichlet L-functions, due to Heath-Brown [48, Theorem 2].

Lemma 2.5. *Suppose $\sigma + it$ is a complex number with $\sigma \geq \frac{1}{2}$. Then*

$$\sum_{|d| \leq X}^{\flat} |L(\sigma + it, \chi_d)|^4 \ll X^{1+\varepsilon}(1 + |t|)^{1+\varepsilon}.$$

Assuming GRH, the bound in Lemma 2.5 can be improved by the following theorem.

Theorem 2.6. *Assume GRH for $L(s, \chi_d)$ for all fundamental discriminants d. Let $z_1, z_2 \in \mathbb{C}$ with $0 \leq \mathrm{Re}(z_1), \mathrm{Re}(z_2) \leq \frac{1}{\log X}$, and $|\mathrm{Im}(z_1)|, |\mathrm{Im}(z_2)| \leq X$. Then*

$$\sum_{|d| \leq X}^{\flat} |L(\tfrac{1}{2} + z_1, \chi_d)|^2 |L(\tfrac{1}{2} + z_2, \chi_d)|^2 \ll X(\log X)^{4+\varepsilon} \left(1 + \min\left\{(\log X)^6, \frac{1}{|\mathrm{Im}(z_1) - \mathrm{Im}(z_2)|^6}\right\}\right).$$

The proof of Theorem 2.6 is postponed to Section 2.8. Note that Section 2.8 is self-contained. Theorem 2.6 is similar to [99, Corollary 5.1]. Indeed, the proof of it follows closely the proof of [99, Corollary 5.1] and the argument in [98, Section 4]. Analogous results to Theorem 2.6 were obtained by Chandee [13, Theorem 1.1] for the moments of the Riemann zeta function, and by Munsch [78, Theorem 1.1] for the moments of Dirichlet L-functions modulo q.

We remark that Lemma 2.5 is used to bound the error terms in the proof of Theorem 2.2, while both Lemma 2.5 and Theorem 2.6 are needed to bound the error terms in the proof of Theorem 2.1.

2.3 Setup of the problem.

Let Φ be a smooth function as described in Subsection 2.2.3. We consider the following smoothed version of the fourth moment

$$\sideset{}{^*}\sum_{(d,2)=1} L(\tfrac{1}{2}, \chi_{8d})^4 \Phi\left(\tfrac{d}{X}\right).$$

Using the approximate functional equation (2.10), we have

$$\sideset{}{^*}\sum_{(d,2)=1} L(\tfrac{1}{2}, \chi_{8d})^4 \Phi\left(\tfrac{d}{X}\right) = \sideset{}{^*}\sum_{(d,2)=1} \left(A_{8d}(\tfrac{1}{2}; 8d)\right)^2 \Phi\left(\tfrac{d}{X}\right), \tag{2.13}$$

where

$$A_t(\tfrac{1}{2}; 8d) := 2\sum_{n=1}^{\infty} \frac{\tau(n)\chi_{8d}(n)}{\sqrt{n}} \omega\left(\frac{n\pi}{t}\right). \tag{2.14}$$

Let $X^{\frac{9}{10}} \leq U_1 \leq U_2 \leq X$ be two parameters that will be chosen later. Define

$$S(U_1, U_2) := \sideset{}{^*}\sum_{(d,2)=1} A_{U_1}(\tfrac{1}{2}; 8d) A_{U_2}(\tfrac{1}{2}; 8d) \Phi\left(\tfrac{d}{X}\right). \tag{2.15}$$

We remark that (2.13) is approximately equal to (2.15) by choosing appropriate values for U_1 and U_2. This will be explained in Section 2.7.

Combining (2.14) and (2.15), we obtain that

$$S(U_1, U_2) = 4 \sideset{}{^*}\sum_{(d,2)=1} \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \frac{\tau(n_1)\tau(n_2)\chi_{8d}(n_1 n_2)}{\sqrt{n_1 n_2}} h(d, n_1, n_2), \tag{2.16}$$

where

$$h(x, y, z) := \Phi\left(\frac{x}{X}\right) \omega\left(\frac{y\pi}{U_1}\right) \omega\left(\frac{z\pi}{U_2}\right). \tag{2.17}$$

Using the Möbius inversion to remove the square-free condition in (2.16) gives

$$
\begin{aligned}
&S(U_1, U_2)\\
&= 4 \sum_{(d,2)=1} \sum_{a^2|d} \mu(a) \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \frac{\tau(n_1)\tau(n_2)\chi_{8d}(n_1 n_2)}{\sqrt{n_1 n_2}} h(d, n_1, n_2)\\
&= 4 \sum_{(a,2)=1} \mu(a) \sum_{(d,2)=1} \sum_{(n_1,a)=1} \sum_{(n_2,a)=1} \frac{\tau(n_1)\tau(n_2)\chi_{8d}(n_1 n_2)}{\sqrt{n_1 n_2}} h(a^2 d, n_1, n_2)\\
&= 4 \left(\sum_{\substack{a \le Y \\ (a,2)=1}} + \sum_{\substack{a > Y \\ (a,2)=1}} \right) \mu(a) \sum_{(d,2)=1} \sum_{(n_1,a)=1} \sum_{(n_2,a)=1} \frac{\tau(n_1)\tau(n_2)\chi_{8d}(n_1 n_2)}{\sqrt{n_1 n_2}} h(a^2 d, n_1, n_2)\\
&=: S_1 + S_2. \qquad (2.18)
\end{aligned}
$$

In the above, we let S_1 denote the terms with $a \le Y$, where Y is a parameter that satisfies $Y \le X$. The value of Y will be chosen later. Also, we let S_2 denote the terms with $a > Y$. The terms S_1 contribute to the main term. We will discuss S_1 in Sections 2.4, 2.5, 2.6. The terms S_2 contribute to the error term by the following lemma.

Lemma 2.7. *Unconditionally, we have $S_2 \ll X^{1+\varepsilon} Y^{-1}$. Under GRH, we have $S_2 \ll XY^{-1} \log^{44} X$.*

Proof. Write $d = lb^2$, where l is square-free and b is positive. Grouping terms in S_2 according to $c = ab$, we deduce that

$$
\begin{aligned}
S_2 &= 4 \sum_{(c,2)=1} \sum_{\substack{a>Y \\ a|c}} \mu(a) \sideset{}{^*}\sum_{(l,2)=1} \sum_{(n_1,c)=1} \sum_{(n_2,c)=1} \frac{\tau(n_1)\tau(n_2)\chi_{8l}(n_1 n_2)}{\sqrt{n_1 n_2}} h(c^2 l, n_1, n_2)\\
&= \frac{4}{(2\pi i)^2} \sum_{(c,2)=1} \sum_{\substack{a>Y \\ a|c}} \mu(a) \int_{(\frac12+\varepsilon)} \int_{(\frac12+\varepsilon)} \frac{g(u)g(v)}{uv} U_1^u U_2^v\\
&\quad \times \sideset{}{^*}\sum_{(l,2)=1} \Phi\left(\frac{c^2 l}{X}\right) L_c(\tfrac12 + u, \chi_{8l})^2 L_c(\tfrac12 + v, \chi_{8l})^2 \, du \, dv, \qquad (2.19)
\end{aligned}
$$

where for $\mathrm{Re}(s) > 1$, $L_c(s,\chi)$ is given by the Euler product of $L(s,\chi)$ with omitting all prime factors of c. The last equation follows by the definition of $h(x,y,z)$ in (2.17). Moving the lines of the integral to $\mathrm{Re}(u) = \mathrm{Re}(v) = \frac{1}{\log X}$, the double integral above is bounded by

$$\ll (\log^2 X)\tau^4(c) \int_{(\frac{1}{\log X})} \int_{(\frac{1}{\log X})} |g(u)g(v)| \sideset{}{^*}\sum_{\substack{(l,2)=1 \\ l \le \frac{5X}{2c^2}}} |L(\tfrac{1}{2}+u, \chi_{8l})|^4 \; |du| \; |dv|. \tag{2.20}$$

Here we use the inequalities $2ab \le a^2 + b^2$ and $|L_c(\frac{1}{2}+u, \chi_{8l})| \le \tau(c)|L(\frac{1}{2}+u, \chi_{8l})|$.

By Theorem 2.6, we see that for $|\mathrm{Im}(u)| \le \frac{X}{c^2}$,

$$\sideset{}{^*}\sum_{\substack{(l,2)=1 \\ l \le \frac{5X}{2c^2}}} |L(\tfrac{1}{2}+u, \chi_{8l})|^4 \ll \frac{X}{c^2} \log^{11} X. \tag{2.21}$$

Also, by Lemma 2.5, we get that

$$\sideset{}{^*}\sum_{\substack{(l,2)=1 \\ l \le \frac{5X}{2c^2}}} |L(\tfrac{1}{2}+u, \chi_{8l})|^4 \ll \left(\frac{X}{c^2}\right)^{1+\varepsilon} (1 + |\mathrm{Im}(u)|)^{1+\varepsilon}. \tag{2.22}$$

Substituting both (2.21) and (2.22) in (2.20), we can bound (2.20) by

$$\ll \frac{\tau^4(c)}{c^2} X \log^{13} X.$$

Together with (2.19), this yields

$$S_2 \ll X \log^{13} X \sum_{(c,2)=1} \frac{\tau^4(c)}{c^2} \sum_{\substack{a>Y \\ a|c}} 1 \ll X \log^{13} X \sum_{c>Y} \frac{\tau^5(c)}{c^2} \ll XY^{-1} \log^{44} X.$$

This completes the proof of the conditional part of the lemma. The unconditional part follows similarly by substituting (2.22) in (2.20). □

Now we consider S_1. Using the Poisson summation formula (see Lemma 2.3) for the sum over d in S_1, we obtain that

$$S_1 = 2X \sum_{\substack{a\le Y \\ (a,2)=1}} \frac{\mu(a)}{a^2} \sum_{k\in\mathbb{Z}} (-1)^k \sum_{(n_1,2a)=1} \sum_{(n_2,2a)=1} \frac{\tau(n_1)\tau(n_2)}{\sqrt{n_1n_2}} \frac{G_k(n_1n_2)}{n_1n_2} \times \int_{-\infty}^{\infty} h(xX,n_1,n_2)(\cos+\sin)\left(\frac{2\pi kxX}{2n_1n_2a^2}\right) dx. \quad (2.23)$$

Let $S_1(k=0)$ denote the sum above over $k=0$, which are called diagonal terms. Let $S_1(k\neq 0)$ denote the sum over $k\neq 0$. Write $S_1(k\neq 0) = S_1(k=\square) + S_1(k\neq\square)$, where $S_1(k=\square)$ denotes the terms with square k, and $S_1(k\neq\square)$ denotes the remaining terms. We call $S_1(k=\square)$ off-diagonal terms. We will discuss $S_1(k=0)$, $S_1(k=\square)$, and $S_1(k\neq\square)$ in Section 2.4, 2.5, 2.6, respectively.

2.4 Evaluation of $S_1(k=0)$.

In this section, we shall extract one main term of S_1 from $S_1(k=0)$. The argument here is similar to [99, Section 3.2].

It follows from the definition of $G_k(n)$ in (2.11) that $G_0(n)=\phi(n)$ if $n=\square$, and $G_0(n)=0$ otherwise. By this fact and (2.23), we see that

$$\begin{aligned} S_1(k=0) &= 2X \sum_{\substack{a\le Y \\ (a,2)=1}} \frac{\mu(a)}{a^2} \sum_{\substack{(n_1n_2,2a)=1 \\ n_1n_2=\square}} \frac{\tau(n_1)\tau(n_2)}{\sqrt{n_1n_2}} \frac{\phi(n_1n_2)}{n_1n_2} \int_{-\infty}^{\infty} h(xX,n_1,n_2)dx \\ &= 2X \sum_{\substack{(n_1n_2,2)=1 \\ n_1n_2=\square}} \frac{\tau(n_1)\tau(n_2)}{\sqrt{n_1n_2}} \frac{\phi(n_1n_2)}{n_1n_2} \sum_{\substack{a\le Y \\ (a,2n_1n_2)=1}} \frac{\mu(a)}{a^2} \int_{-\infty}^{\infty} h(xX,n_1,n_2)dx. \end{aligned} \quad (2.24)$$

Observe that

$$\sum_{\substack{a\le Y \\ (a,2n_1n_2)=1}} \frac{\mu(a)}{a^2} = \frac{8}{\pi^2} \prod_{p|n_1n_2} \left(1-\frac{1}{p^2}\right)^{-1} + O\left(Y^{-1}\right).$$

Inserting this into (2.24), combined with

$$\frac{\phi(n_1n_2)}{n_1n_2} \prod_{p|n_1n_2} \left(1-\frac{1}{p^2}\right)^{-1} = \prod_{p|n_1n_2} \frac{p}{p+1},$$

we obtain that

$$S_1(k=0)=\frac{16X}{\pi^2}\sum_{\substack{(n_1n_2,2)=1\\ n_1n_2=\square}}\frac{\tau(n_1)\tau(n_2)}{\sqrt{n_1n_2}}\prod_{p|n_1n_2}\frac{p}{p+1}\int_{-\infty}^{\infty}h(xX,n_1,n_2)dx$$
$$+O\left(\frac{X}{Y}\sum_{\substack{(n_1n_2,2)=1\\ n_1n_2=\square}}\frac{\tau(n_1)\tau(n_2)}{\sqrt{n_1n_2}}\int_{-\infty}^{\infty}|h(xX,n_1,n_2)|dx\right). \quad (2.25)$$

Now we simplify the error term above. Recall that $w(\xi)$ is bounded as ξ near 0 and decreases exponentially as $\xi\to+\infty$. It follows that

$$\sum_{\substack{(n_1n_2,2)=1\\ n_1n_2=\square}}\frac{\tau(n_1)\tau(n_2)}{\sqrt{n_1n_2}}\int_{-\infty}^{\infty}|h(xX,n_1,n_2)|dx$$
$$\ll\sum_{\substack{(n_1n_2,2)=1\\ n_1n_2=\square}}\frac{\tau(n_1)\tau(n_2)}{\sqrt{n_1n_2}}\left(1+\frac{n_1}{U_1}\right)^{-100}\left(1+\frac{n_2}{U_2}\right)^{-100}$$
$$\ll\log^{11}X. \quad (2.26)$$

The last inequality follows by separating the sum into two parts corresponding to whether $n_1,n_2\le U_1U_2$. Combining (2.25) and (2.26), we have

$$S_1(k=0)=\frac{16X}{\pi^2}\sum_{\substack{(n_1n_2,2)=1\\ n_1n_2=\square}}\frac{\tau(n_1)\tau(n_2)}{\sqrt{n_1n_2}}\prod_{p|n_1n_2}\frac{p}{p+1}\int_{-\infty}^{\infty}h(xX,n_1,n_2)dx+O\left(XY^{-1}\log^{11}X\right).$$

Recall $h(x,y,z)$ from (2.17) and $\omega(\xi)$ from (2.8). We have

$$S_1(k=0)$$
$$=\frac{16X}{\pi^2}\int_{-\infty}^{\infty}\Phi\left(x\right)dx\frac{1}{(2\pi i)^2}\int_{(1)}\int_{(1)}\frac{g(u)g(v)}{uv}U_1^uU_2^v\sum_{\substack{(n_1n_2,2)=1\\ n_1n_2=\square}}\frac{\tau(n_1)\tau(n_2)}{n_1^{\frac12+u}n_2^{\frac12+v}}\prod_{p|n_1n_2}\frac{p}{p+1}\,du\,dv$$
$$+O\left(XY^{-1}\log^{11}X\right). \quad (2.27)$$

Lemma 2.8. *For* $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > \frac{1}{2}$, *we have*

$$\sum_{\substack{(n_1n_2,2)=1\\ n_1n_2=\square}} \frac{\tau(n_1)\tau(n_2)}{n_1^{\alpha}n_2^{\beta}} \prod_{p|n_1n_2} \frac{p}{p+1} = \zeta(2\alpha)^3\zeta(2\beta)^3\zeta(\alpha+\beta)^4 Z_1(\alpha,\beta), \tag{2.28}$$

where $Z_1(\alpha,\beta)$ *is defined by*

$$Z_1(\alpha,\beta) := \prod_p Z_{1,p}(\alpha,\beta).$$

Here

$$Z_{1,2}(\alpha,\beta) := \left(1-\frac{1}{4^{\alpha}}\right)^3\left(1-\frac{1}{4^{\beta}}\right)^3\left(1-\frac{1}{2^{\alpha+\beta}}\right)^4,$$

and for $p \nmid 2$,

$$\begin{aligned} Z_{1,p}(\alpha,\beta) := \left(1-\frac{1}{p^{2\alpha}}\right)\left(1-\frac{1}{p^{2\beta}}\right)\left(1-\frac{1}{p^{\alpha+\beta}}\right)^4 \Bigg[1 &+ \frac{4}{p^{\alpha+\beta}} + \frac{1}{p^{2\alpha}} + \frac{1}{p^{2\beta}} + \frac{1}{p^{2\alpha+2\beta}} - \frac{1}{p+1} \\ \times\left(\frac{3}{p^{2\alpha}} + \frac{3}{p^{2\beta}} + \frac{4}{p^{\alpha+\beta}} - \frac{1}{p^{4\alpha}} - \frac{1}{p^{4\beta}} - \frac{3}{p^{2\alpha+2\beta}} + \frac{2}{p^{2\alpha+4\beta}} + \frac{2}{p^{4\alpha+2\beta}} - \frac{1}{p^{4\alpha+4\beta}}\right)\Bigg]. \end{aligned}$$

Furthermore, $Z_1(\alpha,\beta)$ *is analytic and uniformly bounded in the region* $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) \geq \frac{1}{4}+\varepsilon$.

Proof. We have

$$\sum_{\substack{(n_1n_2,2)=1\\ n_1n_2=\square}} \frac{\tau(n_1)\tau(n_2)}{n_1^{\alpha}n_2^{\beta}} \prod_{p|n_1n_2} \frac{p}{p+1} = \prod_{(p,2)=1}\left(1+\frac{p}{p+1}\left(\sum_{r=1}^{\infty}\sum_{n_1n_2=p^{2r}} \frac{\tau(n_1)\tau(n_2)}{n_1^{\alpha}n_2^{\beta}}\right)\right).$$

Note that

$$\sum_{r=1}^{\infty}\sum_{n_1n_2=p^{2r}} \frac{\tau(n_1)\tau(n_2)}{n_1^{\alpha}n_2^{\beta}} = \frac{(1+\frac{1}{p^{2\alpha}})(1+\frac{1}{p^{2\beta}})}{(1-\frac{1}{p^{2\alpha}})^2(1-\frac{1}{p^{2\beta}})^2} + \frac{4}{p^{\alpha+\beta}}\frac{1}{(1-\frac{1}{p^{2\alpha}})^2(1-\frac{1}{p^{2\beta}})^2} - 1.$$

Thus,

$$\sum_{\substack{(n_1n_2,2)=1\\ n_1n_2=\square}} \frac{\tau(n_1)\tau(n_2)}{n_1^{\alpha}n_2^{\beta}} \prod_{p|n_1n_2} \frac{p}{p+1} = \prod_{(p,2)=1} \frac{1}{(1-\frac{1}{p^{2\alpha}})^2(1-\frac{1}{p^{2\beta}})^2} \Bigg[1+\frac{4}{p^{\alpha+\beta}}+\frac{1}{p^{2\alpha}}+\frac{1}{p^{2\beta}}+\frac{1}{p^{2\alpha+2\beta}}$$
$$-\frac{1}{p+1}\left(\frac{3}{p^{2\alpha}}+\frac{3}{p^{2\beta}}+\frac{4}{p^{\alpha+\beta}}-\frac{1}{p^{4\alpha}}-\frac{1}{p^{4\beta}}-\frac{3}{p^{2\alpha+2\beta}}+\frac{2}{p^{2\alpha+4\beta}}+\frac{2}{p^{4\alpha+2\beta}}-\frac{1}{p^{4\alpha+4\beta}}\right)\Bigg].$$

Then (2.28) follows by comparing Euler factors on both sides. The remaining part of the lemma follows directly from the definition of $Z_1(\alpha,\beta)$. □

It follows from (2.27) and Lemma 2.8 that

$$S_1(k=0) = \frac{16X}{\pi^2}\int_{-\infty}^{\infty} \Phi\left(x\right) dx \frac{1}{(2\pi i)^2}\int_{(1)}\int_{(1)} \frac{g(u)g(v)}{uv} U_1^u U_2^v \zeta(1+2u)^3\zeta(1+2v)^3\zeta(1+u+v)^4$$
$$\times Z_1(\tfrac{1}{2}+u,\tfrac{1}{2}+v)\ du\ dv + O\left(XY^{-1}\log^{11}X\right). \quad (2.29)$$

The double integral in (2.29) can be written as

$$\frac{1}{(2\pi i)^2}\int_{(1)}\int_{(1)} \frac{U_1^u U_2^v}{uv(2u)^3(2v)^3(u+v)^4}\mathcal{E}(u,v)\ du\ dv,$$

where

$$\mathcal{E}(u,v) := g(u)g(v)\zeta(1+2u)^3(2u)^3\zeta(1+2v)^3(2v)^3\zeta(1+u+v)^4(u+v)^4 Z_1(\tfrac{1}{2}+u,\tfrac{1}{2}+v).$$

Clearly, $\mathcal{E}$ is analytic for $\mathrm{Re}(u),\mathrm{Re}(v) \geq -\frac{1}{4}+\varepsilon$.

Now move the lines of the integral above to $\mathrm{Re}(u)=\mathrm{Re}(v)=\frac{1}{10}$ without encountering any poles. Next move the line of the integral over v to $\mathrm{Re}(v)=-\frac{1}{5}$. We may encounter two poles of order at most 4 at both $v=0$ and $v=-u$. Thus,

$$\frac{1}{(2\pi i)^2}\int_{(\frac{1}{10})}\int_{(\frac{1}{10})} \frac{U_1^u U_2^v}{uv(2u)^3(2v)^3(u+v)^4}\mathcal{E}(u,v)\ du\ dv$$
$$= \frac{1}{2\pi i}\int_{(\frac{1}{10})} \left(\operatorname*{Res}_{v=0}+\operatorname*{Res}_{v=-u}\right)\left[\frac{U_1^u U_2^v}{uv(2u)^3(2v)^3(u+v)^4}\mathcal{E}(u,v)\right] du + O\left(U_1^{\frac{1}{10}}U_2^{-\frac{1}{5}}\right). \quad (2.30)$$

The integral of the residue at $v=-u$ in (2.30) will contribute to an error term. In fact, we have

$$\begin{aligned}
&\operatorname*{Res}_{v=-u}\left[\frac{U_1^uU_2^v}{uv(2u)^3(2v)^3(u+v)^4}\mathcal{E}(u,v)\right]\\
&=\frac{1}{3!}\frac{\partial^3}{\partial v^3}\bigg|_{v=-u}\left[\frac{U_1^uU_2^v}{uv(2u)^3(2v)^3}\mathcal{E}(u,v)\right]\\
&=\frac{U_1^uU_2^{-u}}{384u^{11}}\Big[\mathcal{E}(u,-u)\left(u^3\log^3U_2+12u^2\log^2U_2+60u\log U_2+120\right)\\
&\quad+\mathcal{E}^{(0,1)}(u,-u)\left(3u^3\log^2U_2+24u^2\ln U_2+60u\right)\\
&\quad+\mathcal{E}^{(0,2)}(u,-u)\left(3u^3\log U_2+12u^2\right)+\mathcal{E}^{(0,3)}(u,-u)u^3\Big],
\end{aligned}$$

where $\mathcal{E}^{(i,j)}(u,v) := \frac{\partial^{i+j}\mathcal{E}}{\partial u^i\partial v^j}(u,v)$. It follows that

$$\frac{1}{2\pi i}\int_{(\frac{1}{10})}\operatorname*{Res}_{v=-u}\left[\frac{U_1^uU_2^v}{uv(2u)^3(2v)^3(u+v)^4}\mathcal{E}(u,v)\right]du \ll U_1^{\frac{1}{10}}U_2^{-\frac{1}{10}}\log^3X. \tag{2.31}$$

It remains to consider the integral of the residue at $v=0$ in (2.30). Note that

$$\begin{aligned}
I_1(u) &:= \operatorname*{Res}_{v=0}\left[\frac{U_1^uU_2^v}{uv(2u)^3(2v)^3(u+v)^4}\mathcal{E}(u,v)\right]\\
&=\frac{U_1^u}{384u^{11}}\Big[\mathcal{E}(u,0)(u^3\log^3U_2-12u^2\log^2U_2+60u\log U_2-120)\\
&\quad+\mathcal{E}^{(0,1)}(u,0)(3u^3\log^2U_2-24u^2\log U_2+60u)\\
&\quad+\mathcal{E}^{(0,2)}(u,0)(3u^3\log U_2-12u^2)+\mathcal{E}^{(0,3)}(u,0)u^3\Big].
\end{aligned}$$

Moving the line of the integral below from $\mathrm{Re}(u)=\frac{1}{10}$ to $\mathrm{Re}(u)=-\frac{1}{10}$ with encountering a pole at $u=0$, we see that

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{(\frac{1}{10})}I_1(u)du\\
&=\operatorname*{Res}_{u=0}I_1(u)+O(U_1^{-\frac{1}{10}}\log^3X)\\
&=\frac{\mathcal{E}(0,0)}{11612160}\left(-\log^{10}U_1+5\log^9U_1\log U_2-9\log^8U_1\log^2U_2+6\log^7U_1\log^3U_2\right)\\
&\quad+O\left(\log^9X\right)+O\left(U_1^{-\frac{1}{10}}\log^3X\right).
\end{aligned} \tag{2.32}$$

Combining (2.29), (2.30), (2.31) and (2.32), we obtain that

$$
\begin{aligned}
&S_1(k=0)\\
&= \frac{16X}{\pi^2}\tilde{\Phi}(1)\cdot\frac{\mathcal{E}(0,0)}{11612160}\left(-\log^{10}U_1+5\log^9 U_1\log U_2-9\log^8 U_1\log^2 U_2+6\log^7 U_1\log^3 U_2\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + O\left(X\log^9 X+XY^{-1}\log^{11}X\right). \qquad (2.33)
\end{aligned}
$$

where $\tilde{\Phi}(s)$ is defined in (2.38).

Now we compute $\mathcal{E}(0,0)$ above. Clearly, $\mathcal{E}(0,0)=Z_1(\frac12,\frac12)$. By the definition of $Z_1(u,v)$ in Lemma 2.8, it follows that

$$
\begin{aligned}
Z_1(\tfrac12,\tfrac12) &= \frac{1}{2^{10}}\prod_{(p,2)=1}\left(1-\frac1p\right)^6\left[1+\frac6p+\frac{1}{p^2}-\frac{1}{p+1}\left(\frac{10}{p}-\frac{5}{p^2}+\frac{4}{p^3}-\frac{1}{p^4}\right)\right]\\
&= \frac{1}{2^{10}}\prod_{(p,2)=1}\frac{(1-\frac1p)^6}{1+\frac1p}\left(1+\frac7p-\frac{3}{p^2}+\frac{6}{p^3}-\frac{4}{p^4}+\frac{1}{p^5}\right). \qquad (2.34)
\end{aligned}
$$

On the other hand, recalling the definition of a_4 from (2.4), we have

$$
\begin{aligned}
a_4 &= \frac{1}{2^{12}}\prod_{(p,2)=1}\frac{(1-\frac1p)^{10}}{1+\frac1p}\left(\frac{(1+\frac{1}{\sqrt p})^{-4}+(1-\frac{1}{\sqrt p})^{-4}}{2}+\frac1p\right)\\
&= \frac{1}{2^{12}}\prod_{(p,2)=1}\frac{(1-\frac1p)^{10}}{1+\frac1p}\frac{1}{(1+\frac{1}{\sqrt p})^4(1-\frac{1}{\sqrt p})^4}\left[\frac12\left(1-\frac{1}{\sqrt p}\right)^4+\frac12\left(1+\frac{1}{\sqrt p}\right)^4+\frac1p\left(1-\frac1p\right)^4\right]\\
&= \frac{1}{2^{12}}\prod_{(p,2)=1}\frac{(1-\frac1p)^6}{1+\frac1p}\left(1+\frac7p-\frac{3}{p^2}+\frac{6}{p^3}-\frac{4}{p^4}+\frac{1}{p^5}\right). \qquad (2.35)
\end{aligned}
$$

Comparing (2.34) with (2.35), we conclude $Z_1(\frac12,\frac12)=4a_4$, which implies $\mathcal{E}(0,0)=4a_4$. Together with (2.33), it follows that

Lemma 2.9. *We have*

$$
\begin{aligned}
&S_1(k=0)\\
&= \frac{a_4\tilde{\Phi}(1)X}{2^6\cdot 3^4\cdot 5\cdot 7\cdot\pi^2}\left(-\log^{10}U_1+5\log^9 U_1\log U_2-9\log^8 U_1\log^2 U_2+6\log^7 U_1\log^3 U_2\right)\\
&\quad + O\left(X\log^9 X+XY^{-1}\log^{11}X\right).
\end{aligned}
$$

2.5 Evaluation of $S_1(k = \square)$.

In this section, we compute another part of the main term of S_1 which arises from $S_1(k = \square)$. Many of the techniques used here are from Sections 5.2, 5.3 of [97].

Recall from (2.23) that

$$S_1(k \neq 0) = 2X \sum_{\substack{a \leq Y \\ (a,2)=1}} \frac{\mu(a)}{a^2} \sum_{k \neq 0} (-1)^k \sum_{(n_1,2a)=1} \sum_{(n_2,2a)=1} \frac{\tau(n_1)\tau(n_2)}{\sqrt{n_1 n_2}} \frac{G_k(n_1 n_2)}{n_1 n_2} \times \int_{-\infty}^{\infty} h(xX, n_1, n_2)(\cos + \sin)\left(\frac{2\pi k x X}{2 n_1 n_2 a^2}\right) dx. \quad (2.36)$$

To proceed, we need the following lemma.

Lemma 2.10. *Let $f(x)$ be a smooth function on $\mathbb{R}_{>0}$. Suppose f decays rapidly as $x \to \infty$, and $f^{(n)}(x)$ converges as $x \to 0^+$ for every $n \in \mathbb{Z}_{\geq 0}$. Then we have*

$$\int_0^{\infty} f(x) \cos(2\pi x y) dx = \frac{1}{2\pi i} \int_{(\frac{1}{2})} \tilde{f}(1-s)\Gamma(s) \cos\left(\frac{\operatorname{sgn}(y)\pi s}{2}\right) (2\pi |y|)^{-s} ds, \quad (2.37)$$

where $\tilde{f}$ is the Mellin transform of f defined by

$$\tilde{f}(s) := \int_0^{\infty} f(x) x^{s-1} dx. \quad (2.38)$$

In addition, the equation (2.37) *is also valid when* cos *is replaced by* sin.

Proof. See [99, Section 3.3]. $\square$

Taking $f(x) = h(xX, n_1, n_2)$ in Lemma 2.10, we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} h(xX, n_1, n_2)(\cos + \sin)\left(\frac{2\pi k x X}{2 n_1 n_2 a^2}\right) dx \\
&= \frac{X^{-1}}{2\pi i} \int_{(\frac{1}{2})} \tilde{h}(1-s; n_1, n_2)\Gamma(s)(\cos + \operatorname{sgn}(k)\sin)\left(\frac{\pi s}{2}\right)\left(\frac{n_1 n_2 a^2}{\pi |k|}\right)^s ds,
\end{aligned}$$

where

$$\tilde{h}(1-s; n_1, n_2) = \int_0^{\infty} h(x, n_1, n_2) x^{-s} dx.$$

Recall from (2.17) the definition of $h(x, y, z)$. The above contour integral is

$$\frac{1}{(2\pi i)^3}\int_{(\varepsilon)}\int_{(\varepsilon)}\int_{(\varepsilon)}\Gamma(s)\left(\frac{a^2}{|k|}\right)^s \mathcal{J}(s,k)g(u)g(v)\frac{1}{n_1^{u-s}n_2^{v-s}}\frac{U_1^uU_2^vX^{-s}}{uv}\,ds\,du\,dv,$$

where

$$\mathcal{J}(s,k) = \tilde{\Phi}(1-s)(\cos + \operatorname{sgn}(k)\sin)\left(\frac{\pi s}{2}\right)\pi^{-s}.$$

Move the lines of the triple integral to $\mathrm{Re}(s) = \frac{1}{2}+\varepsilon$, $\mathrm{Re}(u) = \mathrm{Re}(v) = \frac{1}{2}+2\varepsilon$, and change the variables $u' = u - s$, $v' = v - s$. We obtain that

$$\begin{aligned}&\int_{-\infty}^{\infty} h(xX, n_1, n_2)(\cos + \sin)\left(\frac{2\pi kxX}{2n_1n_2a^2}\right)dx\\
&= \frac{1}{(2\pi i)^3}\int_{(\varepsilon)}\int_{(\varepsilon)}\int_{(\frac{1}{2}+\varepsilon)}\Gamma(s)\left(\frac{a^2}{|k|}\right)^s \mathcal{J}(s,k)g(u+s)g(v+s)\frac{1}{n_1^un_2^v}\frac{U_1^{u+s}U_2^{v+s}X^{-s}}{(u+s)(v+s)}\,ds\,du\,dv.\end{aligned}$$

Substituting this in (2.36), we get that

$$\begin{aligned}S_1(k\neq 0) = 2X\sum_{\substack{a\le Y\\(a,2)=1}}\frac{\mu(a)}{a^2}\sum_{k\neq 0}(-1)^k\frac{1}{(2\pi i)^3}\int_{(\varepsilon)}\int_{(\varepsilon)}\int_{(\frac{1}{2}+\varepsilon)}\Gamma(s)\left(\frac{a^2}{|k|}\right)^s\mathcal{J}(s,k)g(u+s)g(v+s)\\
\times\frac{U_1^{u+s}U_2^{v+s}X^{-s}}{(u+s)(v+s)}\sum_{(n_1,2a)=1}\sum_{(n_2,2a)=1}\frac{\tau(n_1)\tau(n_2)}{n_1^{\frac{1}{2}+u}n_2^{\frac{1}{2}+v}}\frac{G_k(n_1n_2)}{n_1n_2}\,ds\,du\,dv.\end{aligned}\tag{2.39}$$

Lemma 2.11. *Write $4k = k_1k_2^2$, where k_1 is a fundamental discriminant (possibly $k_1 = 1$), and k_2 is a positive integer. In the region $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > \frac{1}{2}$, we have*

$$\sum_{(n_1,2a)=1}\sum_{(n_2,2a)=1}\frac{\tau(n_1)\tau(n_2)}{n_1^{\alpha}n_2^{\beta}}\frac{G_k(n_1n_2)}{n_1n_2} = L(\tfrac{1}{2}+\alpha,\chi_{k_1})^2L(\tfrac{1}{2}+\beta,\chi_{k_1})^2Z_2(\alpha,\beta,a,k).\tag{2.40}$$

Here $Z_2(\alpha,\beta,a,k)$ is defined as follows:

$$Z_2(\alpha,\beta,a,k) := \prod_p Z_{2,p}(\alpha,\beta,a,k),$$

where

$$Z_{2,p}(\alpha,\beta,a,k) := \left(1-\frac{\chi_{k_1}(p)}{p^{\frac{1}{2}+\alpha}}\right)^2\left(1-\frac{\chi_{k_1}(p)}{p^{\frac{1}{2}+\beta}}\right)^2 \qquad \textit{if } p|2a,$$

and

$$Z_{2,p}(\alpha,\beta,a,k) := \left(1 - \frac{\chi_{k_1}(p)}{p^{\frac{1}{2}+\alpha}}\right)^2 \left(1 - \frac{\chi_{k_1}(p)}{p^{\frac{1}{2}+\beta}}\right)^2 \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \frac{\tau(p^{n_1})\tau(p^{n_2})}{p^{n_1\alpha+n_2\beta}} \frac{G_k(p^{n_1+n_2})}{p^{n_1+n_2}} \quad \text{if } p \nmid 2a.$$

In addition, $Z_2(\alpha,\beta,a,k)$ *is analytic in the region* $\mathrm{Re}(\alpha),\mathrm{Re}(\beta) > 0$*, and we have*

$$Z_2(\alpha,\beta,a,k) \ll \tau^4(a)\tau^8(|k|)\log^{10} X \tag{2.41}$$

in the region $\mathrm{Re}(\alpha),\mathrm{Re}(\beta) \geq \frac{1}{\log X}$*, where the implied constant is absolute.*

Proof. The formula (2.40) follows from the joint multiplicativity of $G_k(n_1n_2)$ with variables n_1 and n_2. In fact,

$$\sum_{(n_1,2a)=1}\sum_{(n_2,2a)=1} \frac{\tau(n_1)\tau(n_2)}{n_1^{\alpha}n_2^{\beta}} \frac{G_k(n_1n_2)}{n_1n_2} = \prod_{(p,2a)=1}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \frac{\tau(p^{n_1})\tau(p^{n_2})}{p^{n_1\alpha+n_2\beta}} \frac{G_k(p^{n_1+n_2})}{p^{n_1+n_2}}.$$

Then we obtain (2.40) by comparing Euler factors on both sides.

For $p \nmid 2ak$, by Lemma 2.4, we know that

$$Z_{2,p}(\alpha,\beta,a,k) = \left(1 - \frac{\chi_{k_1}(p)}{p^{\frac{1}{2}+\alpha}}\right)^2 \left(1 - \frac{\chi_{k_1}(p)}{p^{\frac{1}{2}+\beta}}\right)^2 \left(1 + \frac{2\chi_{k_1}(p)}{p^{\frac{1}{2}+\alpha}} + \frac{2\chi_{k_1}(p)}{p^{\frac{1}{2}+\beta}}\right). \tag{2.42}$$

This shows that $Z_2(\alpha,\beta,a,k)$ is analytic in the region $\mathrm{Re}(\alpha),\mathrm{Re}(\beta) > 0$.

It remains to prove the upper bound of $Z_2(\alpha,\beta,a,k)$. It follows from (2.42) that for $p \nmid 2ak$,

$$\prod_{(p,2ak)=1} Z_{2,p}(\alpha,\beta,a,k) = \prod_{(p,2ak)=1}\left(1 - \frac{3}{p^{1+2\alpha}} - \frac{3}{p^{1+2\beta}} - \frac{4}{p^{1+\alpha+\beta}} + O\left(\frac{1}{p^{3/2}}\right)\right) \ll \log^{10} X.$$

For $p|2a$, we get that

$$\prod_{p|2a} Z_{2,p}(\alpha,\beta,a,k) \leq \prod_{p|2a}\left(1 + \frac{1}{\sqrt{p}}\right)^4 \ll \tau^4(a).$$

For $p \nmid 2a, p|k$, using the trivial bound $G_k(p^n) \leq p^n$, we obtain that

$$\prod_{p|k, p\nmid 2a} Z_{2,p}(\alpha,\beta,a,k) \leq \prod_{p|k,p\nmid 2a}\left(1 + \frac{1}{\sqrt{p}}\right)^4 \sum_{0\leq n_1+n_2\leq \mathrm{ord}_p(k)+1} (n_1+1)(n_2+1) \ll \tau^8(|k|).$$

By the above three bounds, we have obtained (2.41). $\square$

By (2.39) and Lemma 2.11, it follows that

$$\begin{aligned} &S_1(k \neq 0) \\ &= \frac{2X}{(2\pi i)^3} \sum_{\substack{a \le Y \\ (a,2)=1}} \frac{\mu(a)}{a^2} \sum_{k \neq 0} (-1)^k \int_{(\varepsilon)} \int_{(\varepsilon)} \int_{(\frac{1}{2}+\varepsilon)} \Gamma(s) \mathcal{J}(s,k) a^{2s} g(u+s) g(v+s) \frac{U_1^{u+s} U_2^{v+s} X^{-s}}{(u+s)(v+s)} \\ &\quad \times \frac{1}{|k|^s} L(1+u, \chi_{k_1})^2 L(1+v, \chi_{k_1})^2 Z_2(\tfrac{1}{2}+u, \tfrac{1}{2}+v, a, k) \, ds \, du \, dv. \end{aligned} \tag{2.43}$$

Note that when moving the lines of integration of the variables u, v to the left, then we may encounter poles only when $k = \square$ (then $k_1 = 1$). Thus, we break the sum in (2.43) into two parts depending on whether $k = \square$.

Write

$$\begin{aligned} S_1(k = \square) := \frac{2X}{(2\pi i)^3} \sum_{\substack{a \le Y \\ (a,2)=1}} \frac{\mu(a)}{a^2} \sum_{\substack{k \neq 0 \\ k = \square}} (-1)^k \int_{(\varepsilon)} \int_{(\varepsilon)} \int_{(\frac{1}{2}+\varepsilon)} \Gamma(s) \mathcal{J}(s,k) a^{2s} g(u+s) g(v+s) \\ \times \frac{U_1^{u+s} U_2^{v+s} X^{-s}}{(u+s)(v+s)} \frac{1}{|k|^s} \zeta(1+u)^2 \zeta(1+v)^2 Z_2(\tfrac{1}{2}+u, \tfrac{1}{2}+v, a, k) \, ds \, du \, dv, \end{aligned}$$

and

$$\begin{aligned} S_1(k \neq \square) := \frac{2X}{(2\pi i)^3} \sum_{\substack{a \le Y \\ (a,2)=1}} \frac{\mu(a)}{a^2} \sum_{k \neq 0, \square} (-1)^k \int_{(\varepsilon)} \int_{(\varepsilon)} \int_{(\frac{1}{2}+\varepsilon)} \Gamma(s) \mathcal{J}(s,k) a^{2s} g(u+s) g(v+s) \\ \times \frac{U_1^{u+s} U_2^{v+s} X^{-s}}{(u+s)(v+s)} \frac{1}{|k|^s} L(1+u, \chi_{k_1})^2 L(1+v, \chi_{k_1})^2 Z_2(\tfrac{1}{2}+u, \tfrac{1}{2}+v, a, k) \, ds \, du \, dv. \end{aligned} \tag{2.44}$$

We will give an upper bound for $S_1(k \neq \square)$ in the next section. In the rest of this section, we focus on $S_1(k = \square)$ and obtain a main term. By the change of variables (replace k by k^2), we get that

$$\begin{aligned} S_1(k = \square) = \frac{2X}{(2\pi i)^3} \sum_{\substack{a \le Y \\ (a,2)=1}} \frac{\mu(a)}{a^2} \sum_{k=1}^{\infty} (-1)^k \int_{(\varepsilon)} \int_{(\varepsilon)} \int_{(\frac{1}{2}+\varepsilon)} \Gamma(s) \mathcal{J}(s,1) a^{2s} g(u+s) g(v+s) \\ \times \frac{U_1^{u+s} U_2^{v+s} X^{-s}}{(u+s)(v+s)} \frac{1}{k^{2s}} \zeta(1+u)^2 \zeta(1+v)^2 Z_2(\tfrac{1}{2}+u, \tfrac{1}{2}+v, a, k^2) \, ds \, du \, dv. \end{aligned} \tag{2.45}$$

Lemma 2.12. *In the region* $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0$, $\mathrm{Re}(\gamma) > \frac{1}{2}$,

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{2\gamma}} Z_2(\alpha, \beta, a, k^2) = (2^{1-2\gamma} - 1)\zeta(2\gamma) Z_3(\alpha, \beta, \gamma, a). \tag{2.46}$$

Here $Z_3(\alpha, \beta, \gamma, a)$ *is defined by*

$$Z_3(\alpha, \beta, \gamma, a) := \zeta(2\alpha + 2\gamma)^2 \zeta(2\beta + 2\gamma)^2 \prod_p Z_{3,p}(\alpha, \beta, \gamma, a),$$

where for $p|2a$,

$$Z_{3,p}(\alpha, \beta, \gamma, a) := \left(1 - \frac{1}{p^{\frac{1}{2}+\alpha}}\right)^2 \left(1 - \frac{1}{p^{\frac{1}{2}+\beta}}\right)^2 \left(1 - \frac{1}{p^{2\alpha+2\gamma}}\right)^2 \left(1 - \frac{1}{p^{2\beta+2\gamma}}\right)^2, \tag{2.47}$$

and for $p \nmid 2a$,

$$\begin{aligned} Z_{3,p}(\alpha, \beta, \gamma, a) := &\left(1 - \frac{1}{p^{\frac{1}{2}+\alpha}}\right)^2 \left(1 - \frac{1}{p^{\frac{1}{2}+\beta}}\right)^2 \left[\left(1 - \frac{1}{p}\right) \left(1 + \frac{1}{p^{2\alpha+2\gamma}}\right) \left(1 + \frac{1}{p^{2\beta+2\gamma}}\right) \right. \\ &+ \frac{1}{p} \left(1 - \frac{1}{p^{2\alpha+2\gamma}}\right)^2 \left(1 - \frac{1}{p^{2\beta+2\gamma}}\right)^2 + \left(1 - \frac{1}{p}\right) \frac{4}{p^{\alpha+\beta+2\gamma}} \\ &\left. + 2\left(1 - \frac{1}{p^{2\gamma}}\right) \left(\frac{1}{p^{\frac{1}{2}+\alpha}} + \frac{1}{p^{\frac{1}{2}+\beta}} + \frac{1}{p^{\frac{1}{2}+2\alpha+\beta+2\gamma}} + \frac{1}{p^{\frac{1}{2}+\alpha+2\beta+2\gamma}} \right) \right]. \end{aligned} \tag{2.48}$$

Moreover,

(1) $Z_3(\alpha, \beta, \gamma, a)$ *is analytic and uniformly bounded in the region* $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) \geq \frac{1}{2} + \varepsilon$, $\mathrm{Re}(\gamma) \geq 2\varepsilon$.

(2) $Z_3(\alpha, \beta, \gamma, a)$ *is analytic and* $Z_3(\alpha, \beta, \gamma, a) \ll \log^{14} X$ *in the region* $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) \geq \frac{1}{2} + \frac{1}{\log X}$, $\mathrm{Re}(\gamma) \geq \frac{2}{\log X}$. *The implied constant is absolute.*

Proof. We first compute the left-hand side of (2.46) without $(-1)^k$. Note that

$$\sum_{k=1}^{\infty} \frac{1}{k^{2\gamma}} Z_2(\alpha, \beta, a, k^2) = \sum_{k=1}^{\infty} \frac{1}{k^{2\gamma}} \prod_p Z_{2,p}(\alpha, \beta, a, k^2) = \prod_p \sum_{b=0}^{\infty} \frac{Z_{2,p}(\alpha, \beta, a, p^{2b})}{p^{2b\gamma}}. \tag{2.49}$$

We remark here that $Z_{2,p}(\alpha, \beta, a, 1)$ may not be 1. If $p|2a$, we have

$$\sum_{b=0}^{\infty} \frac{Z_{2,p}(\alpha,\beta,a,p^{2b})}{p^{2b\gamma}} = \frac{1}{1-\frac{1}{p^{2\gamma}}} \left(1-\frac{1}{p^{\frac{1}{2}+\alpha}}\right)^2 \left(1-\frac{1}{p^{\frac{1}{2}+\beta}}\right)^2. \tag{2.50}$$

If $p \nmid 2a$, we have

$$\begin{aligned}\sum_{b=0}^{\infty} \frac{Z_{2,p}(\alpha,\beta,a,p^{2b})}{p^{2b\gamma}} &= \left(1-\frac{1}{p^{\frac{1}{2}+\alpha}}\right)^2 \left(1-\frac{1}{p^{\frac{1}{2}+\beta}}\right)^2 \\ &\times \sum_{b=0}^{\infty} \frac{1}{p^{2b\gamma}} \left(\sum_{\substack{n_1,n_2\geq 0\\ n_1+n_2=2b+1}} \frac{\tau(p^{n_1})\tau(p^{n_2})}{p^{n_1\alpha+n_2\beta}} \frac{p^{2b}\sqrt{p}}{p^{n_1+n_2}} + \sum_{\substack{n_1,n_2\geq 0\\ n_1+n_2\leq 2b\\ n_1+n_2 \text{ even}}} \frac{\tau(p^{n_1})\tau(p^{n_2})}{p^{n_1\alpha+n_2\beta}} \frac{\phi(p^{n_1+n_2})}{p^{n_1+n_2}} \right).\end{aligned} \tag{2.51}$$

Note that

$$\begin{aligned}&\sum_{b=0}^{\infty} \frac{1}{p^{2b\gamma}} \sum_{\substack{n_1,n_2\geq 0\\ n_1+n_2=2b+1}} \frac{\tau(p^{n_1})\tau(p^{n_2})}{p^{n_1\alpha+n_2\beta}} \frac{p^{2b}\sqrt{p}}{p^{n_1+n_2}} \\ &= \frac{1}{p^{-\gamma+\frac{1}{2}}} \frac{1}{(1-\frac{1}{p^{2\alpha+2\gamma}})^2(1-\frac{1}{p^{2\beta+2\gamma}})^2} \left[\frac{2}{p^{\alpha+\gamma}}\left(1+\frac{1}{p^{2\beta+2\gamma}}\right) + \frac{2}{p^{\beta+\gamma}}\left(1+\frac{1}{p^{2\alpha+2\gamma}}\right)\right],\end{aligned}$$

and

$$\begin{aligned}&\sum_{b=0}^{\infty} \frac{1}{p^{2b\gamma}} \sum_{\substack{n_1,n_2\geq 0\\ n_1+n_2\leq 2b\\ n_1+n_2 \text{ even}}} \frac{\tau(p^{n_1})\tau(p^{n_2})}{p^{n_1\alpha+n_2\beta}} \frac{\phi(p^{n_1+n_2})}{p^{n_1+n_2}} = \frac{1}{1-\frac{1}{p^{2\gamma}}} \\ &\times \left[\frac{1}{p} + \left(1-\frac{1}{p}\right) \frac{1}{(1-\frac{1}{p^{2\alpha+2\gamma}})^2(1-\frac{1}{p^{2\beta+2\gamma}})^2} \left(\left(1+\frac{1}{p^{2\alpha+2\gamma}}\right)\left(1+\frac{1}{p^{2\beta+2\gamma}}\right) + \frac{4}{p^{\alpha+\beta+2\gamma}}\right)\right].\end{aligned}$$

Inserting them into (2.51), combined with (2.49), (2.50), we obtain that

$$\sum_{k=1}^{\infty} \frac{1}{k^{2\gamma}} Z_2(\alpha,\beta,a,k^2) = \zeta(2\gamma) Z_3(\alpha,\beta,\gamma,a). \tag{2.52}$$

Now we prove (2.46). It is clear that $G_{4k}(n) = G_k(n)$ for any odd n, so $Z_2(\alpha, \beta, a, 4k^2) =$

$Z_2(\alpha, \beta, a, k^2)$. Thus,

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{2\gamma}} Z_2(\alpha, \beta, a, k^2) &= \frac{1}{4^\gamma} \sum_{k=1}^{\infty} \frac{1}{k^{2\gamma}} Z_2(\alpha, \beta, a, 4k^2) - \sum_{k \text{ odd}} \frac{1}{k^{2\gamma}} Z_2(\alpha, \beta, a, k^2) \\ &= (2^{1-2\gamma} - 1) \sum_{k=1}^{\infty} \frac{1}{k^{2\gamma}} Z_2(\alpha, \beta, a, k^2).\end{aligned}$$

Together with (2.52), this yields (2.46).

The first property of $Z_3(\alpha, \beta, \gamma, a)$ comes directly from its definition. Now we prove the second property. We know that for $\operatorname{Re}(\alpha), \operatorname{Re}(\beta) \geq \frac{1}{2} + \frac{1}{\log X}$, $\operatorname{Re}(\gamma) \geq \frac{2}{\log X}$,

$$\begin{aligned}&Z_3(\alpha, \beta, \gamma, a) \\ &\ll (\log^4 X) \prod_{p|2a} \left(1 + \frac{1}{p^{1+\frac{1}{\log X}}}\right)^8 \prod_{p\nmid 2a} \left(1 + \frac{6}{p^{1+\frac{6}{\log X}}} + \frac{4}{p^{1+\frac{5}{\log X}}} + O\left(\frac{1}{p^2}\right)\right) \ll \log^{14} X,\end{aligned}$$

as desired. $\square$

It follows from (2.45) and Lemma 2.12 that

$$\begin{aligned}S_1(k = \square) = \frac{2X}{(2\pi i)^3} \sum_{\substack{a \leq Y \\ (a,2)=1}} \frac{\mu(a)}{a^2} \int_{(\varepsilon)} \int_{(\varepsilon)} \int_{(\frac{1}{2}+\varepsilon)} \Gamma(s) \mathcal{J}(s, 1) a^{2s} g(u + s) g(v + s) \frac{U_1^{u+s} U_2^{v+s} X^{-s}}{(u + s)(v + s)} \\ \times \zeta(1 + u)^2 \zeta(1 + v)^2 (2^{1-2s} - 1) \zeta(2s) Z_3(\tfrac{1}{2} + u, \tfrac{1}{2} + v, s, a) \, ds \, du \, dv.\end{aligned}$$

Note that $Z_3(\frac{1}{2} + u, \frac{1}{2} + v, s)$ is analytic in the region $\operatorname{Re}(u), \operatorname{Re}(v) \geq \varepsilon$, $\operatorname{Re}(s) \geq 2\varepsilon$ by (1) of Lemma 2.12, so we move the lines of the integral above to $\operatorname{Re}(u) = \operatorname{Re}(v) = 1$, $\operatorname{Re}(s) = \frac{1}{10}$ without encountering any poles. (The only possible pole lies in $\zeta(2s)$ at $s = \frac{1}{2}$, but is cancelled by the simple zero arising from $2^{1-2s} - 1$.) Hence,

$$\begin{aligned}S_1(k = \square) = \frac{2X}{(2\pi i)^3} \sum_{\substack{a \leq Y \\ (a,2)=1}} \frac{\mu(a)}{a^2} \int_{(1)} \int_{(1)} \int_{(\frac{1}{10})} \Gamma(s) \mathcal{J}(s, 1) a^{2s} g(u + s) g(v + s) \frac{U_1^{u+s} U_2^{v+s} X^{-s}}{(u + s)(v + s)} \\ \times \zeta(1 + u)^2 \zeta(1 + v)^2 (2^{1-2s} - 1) \zeta(2s) Z_3(\tfrac{1}{2} + u, \tfrac{1}{2} + v, s, a) \, ds \, du \, dv. \quad (2.53)\end{aligned}$$

Note that we may extend the sum over a to infinity with an error term

$$\frac{2X}{(2\pi i)^3} \sum_{\substack{a>Y\\(a,2)=1}} \frac{\mu(a)}{a^2} \int_{(1)} \int_{(1)} \int_{(\frac{1}{10})} \Gamma(s)\mathcal{J}(s,1)a^{2s} g(u+s)g(v+s) \frac{U_1^{u+s}U_2^{v+s}X^{-s}}{(u+s)(v+s)}$$
$$\times \zeta(1+u)^2\zeta(1+v)^2(2^{1-2s}-1)\zeta(2s)Z_3(\tfrac{1}{2}+u, \tfrac{1}{2}+v, s, a)\; ds\; du\; dv.$$

Move the lines of the integral above to $\mathrm{Re}(u) = \mathrm{Re}(v) = \frac{1}{\log X}$, $\mathrm{Re}(s) = \frac{2}{\log X}$ without encountering any poles. Then by (2) of Lemma 2.12, this is bounded by

$$\ll X \log^{20} X \sum_{\substack{a>Y\\(a,2)=1}} \frac{1}{a^{2-\frac{4}{\log X}}}$$
$$\times \int_{(\frac{1}{\log X})} \int_{(\frac{1}{\log X})} \int_{(\frac{2}{\log X})} (1+|2s|)|\mathcal{J}(s,1)| \left|\Gamma(s)\Gamma(\tfrac{u+s}{2}+\tfrac{1}{4})^2\Gamma(\tfrac{v+s}{2}+\tfrac{1}{4})^2\right| \; |ds|\; |du|\; |dv|$$
$$\ll X(\log^{20} X)Y^{-1} \int_{(\frac{2}{\log X})} (1+|2s|)|\Gamma(s)||\tilde{\Phi}(1-s)| \left|(\cos+\sin)(\tfrac{\pi s}{2})\right| |ds|$$
$$\ll XY^{-1}(\log^{21} X)\Phi_{(5)}.$$

The last inequality is due to (2.12) and the fact $|\Gamma(s)(\cos+\sin)(\frac{\pi s}{2})| \ll |s|^{\mathrm{Re}(s)-\frac{1}{2}}$. Together with (2.53), it implies that

$$S_1(k=\Box) = \frac{2X}{(2\pi i)^3} \sum_{(a,2)=1} \frac{\mu(a)}{a^2} \int_{(1)} \int_{(1)} \int_{(\frac{1}{10})} \Gamma(s)\mathcal{J}(s,1)a^{2s} g(u+s)g(v+s) \frac{U_1^{u+s}U_2^{v+s}X^{-s}}{(u+s)(v+s)}$$
$$\times \zeta(1+u)^2\zeta(1+v)^2(2^{1-2s}-1)\zeta(2s)Z_3(\tfrac{1}{2}+u, \tfrac{1}{2}+v, s, a)\; ds\; du\; dv$$
$$+ O\left(XY^{-1}(\log^{21} X)\Phi_{(5)}\right). \quad (2.54)$$

Let $K_1(\alpha, \beta, \gamma; p), K_2(\alpha, \beta, \gamma; p)$ denote the expressions of (2.47) and (2.48), respectively. We have the following lemma.

Lemma 2.13. *In the region* $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > \frac{1}{2}$, $0 < \mathrm{Re}(\gamma) < \frac{1}{2}$,

$$\sum_{(a,2)=1} \frac{\mu(a)}{a^{2-2\gamma}} Z_3(\alpha, \beta, \gamma, a) = \frac{\zeta(2\alpha+2\gamma)^3\zeta(2\beta+2\gamma)^3\zeta(\alpha+\beta+2\gamma)^4}{\zeta(\frac{1}{2}+\alpha+2\gamma)^2\zeta(\frac{1}{2}+\beta+2\gamma)^2} Z_4(\alpha, \beta, \gamma), \quad (2.55)$$

where

$$Z_4(\alpha,\beta,\gamma) := K_1(\alpha,\beta,\gamma;2)$$
$$\times \prod_p \frac{(1-\frac{1}{p^{2\alpha+2\gamma}})(1-\frac{1}{p^{2\beta+2\gamma}})(1-\frac{1}{p^{\alpha+\beta+2\gamma}})^4}{(1-\frac{1}{p^{\frac12+\alpha+2\gamma}})^2(1-\frac{1}{p^{\frac12+\beta+2\gamma}})^2} \prod_{(p,2)=1} \left(K_2(\alpha,\beta,\gamma;p) - \frac{1}{p^{2-2\gamma}} K_1(\alpha,\beta,\gamma;p) \right).$$

Moreover, $Z_4(\alpha,\beta,\gamma)$ is analytic and uniformly bounded in the region $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) \ge \frac{3}{8}$, $-\frac{1}{16} \le \mathrm{Re}(\gamma) \le \frac{1}{8}$.

Proof. We have

$$\sum_{(a,2)=1} \frac{\mu(a)}{a^{2-2\gamma}} Z_3(\alpha,\beta,\gamma,a)$$
$$= \zeta(2\alpha+2\gamma)^2\zeta(2\beta+2\gamma)^2 \sum_{(a,2)=1} \frac{\mu(a)}{a^{2-2\gamma}} \prod_{p|2a} K_1(\alpha,\beta,\gamma;p) \prod_{p\nmid 2a} K_2(\alpha,\beta,\gamma;p)$$
$$= \zeta(2\alpha+2\gamma)^2\zeta(2\beta+2\gamma)^2 K_1(\alpha,\beta,\gamma;2) \prod_{(p,2)=1} \left(K_2(\alpha,\beta,\gamma;p) - \frac{1}{p^{2-2\gamma}} K_1(\alpha,\beta,\gamma;p) \right).$$

This implies the equation (2.55). The later part of the lemma can be proved directly by the definition of $Z_4(\alpha,\beta,\gamma)$. $\square$

It follows from (2.54) and Lemma 2.13 that

$$S_1(k=\square) = \frac{2X}{(2\pi i)^3} \int_{(1)} \int_{(1)} \int_{(\frac{1}{10})} \mathcal{J}(s,1)(2^{1-2s}-1)\zeta(2s) g(u+s) g(v+s) \frac{U_1^{u+s} U_2^{v+s} X^{-s}}{(u+s)(v+s)}$$
$$\times \zeta^2(1+u)\zeta^2(1+v) \frac{\Gamma(s)\zeta(1+2u+2s)^3\zeta(1+2v+2s)^3\zeta(1+u+v+2s)^4}{\zeta(1+u+2s)^2\zeta(1+v+2s)^2}$$
$$\times Z_4(\tfrac12+u, \tfrac12+v, s)\, ds\, du\, dv + O\left(XY^{-1}(\log^{21} X)\Phi_{(5)}\right), \tag{2.56}$$

where $Z_4(\frac12+u,\frac12+v,s)$ is analytic and uniformly bounded in the region $\mathrm{Re}(u), \mathrm{Re}(v) \ge -\frac{1}{8}$, $-\frac{1}{16} \le \mathrm{Re}(s) \le \frac{1}{8}$.

Move the lines of the triple integral above to $\mathrm{Re}(u) = \mathrm{Re}(v) = \mathrm{Re}(s) = \frac{1}{100}$ without encountering any poles. Then move the line of the integral over v to $\mathrm{Re}(v) = -\frac{1}{50} + \frac{1}{\log X}$. There is a pole of order at most 2 at $v = 0$, and a pole of order at most 4 at $v = -s$, so the triple integral in (2.56) is

$$\frac{1}{(2\pi i)^2}\int_{(\frac{1}{100})}\int_{(\frac{1}{100})} I_2(u,s)+I_3(u,s)\ du\ ds + O\left(U_1^{\frac{1}{50}}U_2^{-\frac{1}{100}}X^{-\frac{1}{100}}(\log^2 X)\Phi_{(5)}\right), \tag{2.57}$$

where $I_2(u,s), I_3(u,s)$ are the residues of the integrand in (2.56) at $v=0$ and $v=-s$, respectively.

The double integral of $I_3(u,s)$ in (2.57) is bounded. To see this, note that

$$I_3(u,v) = \mathcal{J}(s,1)(2^{1-2s}-1)\zeta(2s)g(u+s)\frac{U_1^{u+s}X^{-s}}{u+s}\frac{\zeta(1+u)^2\Gamma(s)\zeta(1+2u+2s)^3}{\zeta(1+u+2s)^2}\frac{1}{3!}\frac{d^3}{dv^3}\bigg|_{v=-s}$$
$$\left(\frac{g(v+s)U_2^{v+s}\zeta(1+v)^2\zeta(1+2v+2s)^3(v+s)^3\zeta(1+u+v+2s)^4 Z_4(\frac{1}{2}+u,\frac{1}{2}+v,s)}{\zeta(1+v+2s)^2}\right).$$

Moving the line of the following integral in terms of u from $\mathrm{Re}(u)=\frac{1}{100}$ to $\mathrm{Re}(u)=\frac{1}{\log X}$ gives

$$\frac{1}{(2\pi i)^2}\int_{(\frac{1}{100})}\int_{(\frac{1}{100})} I_3(u,s)\ du\ ds \ll U_1^{\frac{1}{100}}X^{-\frac{1}{100}}(\log^5 X)\Phi_{(5)}. \tag{2.58}$$

Now we handle the double integral of $I_2(u,s)$ in (2.57). Write the integrand in (2.56) in the form of

$$\frac{U_1^{u+s}U_2^{v+s}X^{-s}}{(u+s)(v+s)}\frac{1}{u^2v^2}\frac{(u+2s)^2(v+2s)^2}{s(2u+2s)^3(2v+2s)^3(u+v+2s)^4}\mathcal{F}(u,v,s).$$

Clearly, $\mathcal{F}(u,v,s)$ is analytic in the region $\mathrm{Re}(u+2s),\mathrm{Re}(v+2s)>0$, $\mathrm{Re}(u),\mathrm{Re}(v)\ge -\frac{1}{8}$ and $-\frac{1}{16}\le \mathrm{Re}(s)\le\frac{1}{8}$. We have

$$I_2(u,s) = \frac{U_1^{u+s}U_2^{s}X^{-s}}{16(u+2s)^3(u+s)^4s^4u^2}$$
$$\times\left[\mathcal{F}(u,0,s)(us\log U_2+2s^2\log U_2-10s-3u)+\mathcal{F}^{(0,1,0)}(u,0,s)(us+2s^2)\right].$$

Move the line of the double integral below from $\mathrm{Re}(u)=\frac{1}{100}$ to $\mathrm{Re}(u)=-\frac{1}{100}+\frac{1}{\log X}$. There is one possible pole at $u=0$. Hence,

$$\frac{1}{(2\pi i)^2}\int_{(\frac{1}{100})}\int_{(\frac{1}{100})} I_2(u,s)\ du\ ds = \frac{1}{2\pi i}\int_{(\frac{1}{100})}\operatorname*{Res}_{u=0}(I_2(u,s))\,ds + O\left(U_2^{\frac{1}{100}}X^{-\frac{1}{100}}\log^8 X\right). \tag{2.59}$$

Note that

$$\operatorname*{Res}_{u=0} I_2(u,s) = \frac{U_1^s U_2^s X^{-s}}{64 s^{11}} \Big(\mathcal{F}(0,0,s)(s^2 \log U_1 \log U_2 - 5s \log U_1 - 5s \log U_2 + 26) \\ + \mathcal{F}^{(1,0,0)}(0,0,s)(s^2 \log U_2 - 5s) + \mathcal{F}^{(0,1,0)}(0,0,s)(s^2 \log U_1 - 5s) + \mathcal{F}^{(1,1,0)}(0,0,s)s^2 \Big).$$

We see that the expression in the brackets above is analytic for $-\frac{1}{16} \leq \operatorname{Re}(s) \leq \frac{1}{8}$. Then we move the line of the integral below to $\operatorname{Re}(s) = -\frac{1}{100}$ with only a possible pole at $s = 0$, and get that

$$\frac{1}{2\pi i} \int_{(\frac{1}{100})} \operatorname*{Res}_{u=0} (I_2(u,s))\, ds = \frac{\mathcal{F}(0,0,0)}{64} \sum_{\substack{j_1+j_2+j_3+j_4=10 \\ j_1,j_2,j_3,j_4 \geq 0}} \frac{(-1)^{j_3} B(j_4)}{j_1! j_2! j_3! j_4!} (\log^{j_1} U_1)(\log^{j_2} U_2)(\log^{j_3} X) \\ + O\left(U_1^{-\frac{1}{100}} U_2^{-\frac{1}{100}} X^{\frac{1}{100}} \log^2 X + \log^9 X \right), \quad (2.60)$$

where

$$B(j) = \begin{cases} 26 & \text{if } j = 0, \\ -5(\log U_1 + \log U_2) & \text{if } j = 1, \\ 2 \log U_1 \log U_2 & \text{if } j = 2, \\ 0 & \text{if } j \geq 3. \end{cases} \quad (2.61)$$

Next we compute $\mathcal{F}(0,0,0)$ above. Note that

$$\mathcal{F}(0,0,0) = \mathcal{J}(0,1) g(0)^2 Z_4(\tfrac{1}{2}, \tfrac{1}{2}, 0) = -\frac{1}{2} \tilde{\Phi}(1) Z_4(\tfrac{1}{2}, \tfrac{1}{2}, 0).$$

Recalling the definition of $Z_4(\alpha, \beta, \gamma)$ from Lemma 2.13, we have

$$\begin{aligned}
&Z_4(\tfrac{1}{2}, \tfrac{1}{2}, 0) \\
&= K_1(\tfrac{1}{2}, \tfrac{1}{2}, 0; 2) \prod_p \left(1 - \frac{1}{p}\right)^2 \prod_{(p,2)=1} \left(K_2(\tfrac{1}{2}, \tfrac{1}{2}, 0; p) - \frac{1}{p^2} K_1(\tfrac{1}{2}, \tfrac{1}{2}, 0; p)\right) \\
&= \frac{1}{4} K_1(\tfrac{1}{2}, \tfrac{1}{2}, 0; 2) \prod_{(p,2)=1} \left(1 - \frac{1}{p}\right)^7 \left(1 + \frac{7}{p} - \frac{3}{p^2} + \frac{6}{p^3} - \frac{4}{p^4} + \frac{1}{p^5}\right) \\
&= \frac{1}{4\zeta_2(2)} K_1(\tfrac{1}{2}, \tfrac{1}{2}, 0; 2) \prod_{(p,2)=1} \frac{(1 - \frac{1}{p})^6}{1 + \frac{1}{p}} \left(1 + \frac{7}{p} - \frac{3}{p^2} + \frac{6}{p^3} - \frac{4}{p^4} + \frac{1}{p^5}\right) \\
&= \frac{32 a_4}{\pi^2}.
\end{aligned}$$

The last equality is due to (2.35). Thus,

$$\mathcal{F}(0,0,0) = -\frac{16\tilde{\Phi}(1) a_4}{\pi^2}.$$

Combining (2.56) with (2.57), (2.58), (2.59), (2.60), and the identity above, it follows that

Lemma 2.14. *We have*

$$\begin{aligned}
S_1(k = \square) = &-\frac{a_4 \tilde{\Phi}(1) X}{2\pi^2} \sum_{j_1+j_2+j_3+j_4=10} \frac{(-1)^{j_3} B(j_4)}{j_1! j_2! j_3! j_4!} (\log^{j_1} U_1)(\log^{j_2} U_2)(\log^{j_3} X) \\
&+ X \cdot O\Big(\log^9 X + U_1^{\frac{1}{100}} X^{-\frac{1}{100}} (\log^5 X) \Phi_{(5)} \\
&\qquad + U_1^{\frac{1}{50}} U_2^{-\frac{1}{100}} X^{-\frac{1}{100}} (\log^2 X) \Phi_{(5)} + Y^{-1} (\log^{21} X) \Phi_{(5)} \Big).
\end{aligned}$$

2.6 Upper bounds for $S_1(k \neq \square)$.

In this section, we shall prove the following upper bounds for $S_1(k \neq \square)$. The techniques applied here are from [97, Section 5.4] and the last part of [99, Section 3].

Lemma 2.15. *Unconditionally, we have*

$$S_1(k \neq \square) \ll U_1^{\frac{1}{2}} U_2^{\frac{1}{2}} Y X^{\varepsilon} \Phi_{(5)}.$$

Under GRH, we have

$$S_1(k \neq \square) \ll U_1^{\frac{1}{2}} U_2^{\frac{1}{2}} Y (\log X)^{2^{17}} \Phi_{(5)}.$$

Proof. It follows from (2.44) that

$$\begin{aligned} &S_1(k \neq \square) \\ &\ll X \sum_{\substack{a \leq Y \\ (a,2)=1}} \frac{1}{a^2} \sideset{}{^\flat}\sum_{k_1 \neq 0,1} \sum_{k_2=1}^{\infty} \bigg| \int_{(\varepsilon)} \int_{(\varepsilon)} \int_{(\frac{1}{2}+\varepsilon)} \Gamma(s)\mathcal{J}(s,k_1)a^{2s}g(u+s)g(v+s)\frac{U_1^{u+s}U_2^{v+s}X^{-s}}{(u+s)(v+s)} \\ &\qquad \times \frac{4^s}{|k_1k_2^2|^s} L(1+u,\chi_{k_1})^2 L(1+v,\chi_{k_1})^2 Z_2(\tfrac{1}{2}+u,\tfrac{1}{2}+v,a,k_1k_2^2)\ ds\ du\ dv \bigg|. \quad (2.62) \end{aligned}$$

Separate the sum over k_1 to the sum over $|k_1| \leq T := U_1U_2Y^2X^{-1}$, and that over $|k_1| > T$. Clearly, $X^{\frac{4}{5}} \leq T \leq X^3$ since $X^{\frac{9}{10}} \leq U_1 \leq U_2 \leq X$ and $1 \leq Y \leq X$. For the first category, we move the the lines of the integral to $\mathrm{Re}(u) = \mathrm{Re}(v) = -\frac{1}{2} + \frac{1}{4\log X}$, $\mathrm{Re}(s) = \frac{3}{4}$, while for the second category, we move the lines to $\mathrm{Re}(u) = \mathrm{Re}(v) = -\frac{1}{2} + \frac{1}{4\log X}$, $\mathrm{Re}(s) = \frac{5}{4}$.

By (2.41), the terms in the first category are bounded by

$$\begin{aligned} &\ll X^{\frac{1}{4}}U_1^{\frac{1}{4}}U_2^{\frac{1}{4}} \log^{10} X \sum_{\substack{a \leq Y \\ (a,2)=1}} \frac{\tau^4(a)}{\sqrt{a}} \int_{(-\frac{1}{2}+\frac{1}{\log X})} \int_{(-\frac{1}{2}+\frac{1}{\log X})} \int_{(\frac{3}{4})} |\mathcal{J}(s,k_1)\Gamma(s)g(u+s)g(v+s)| \\ &\qquad \times \sideset{}{^\flat}\sum_{|k_1| \leq T} \frac{\tau^8(|k_1|)}{|k_1|^{\frac{3}{4}}} |L(1+u,\chi_{k_1})|^4\ |ds|\ |du|\ |dv|. \quad (2.63) \end{aligned}$$

Note that

$$\sideset{}{^\flat}\sum_{|k_1| \leq T} \frac{\tau^8(|k_1|)}{|k_1|^{\frac{3}{4}}} |L(1+u,\chi_{k_1})|^4 \ll \sum_{1 \leq 2^l \leq T} \sideset{}{^\flat}\sum_{2^l < |k_1| \leq 2^{l+1}} \frac{\tau^8(|k_1|)}{|k_1|^{\frac{3}{4}}} |L(1+u,\chi_{k_1})|^4. \quad (2.64)$$

By (2.64) and Lemma 2.5, it follows that

$$\sideset{}{^\flat}\sum_{|k_1| \leq T} \frac{\tau^8(|k_1|)}{|k_1|^{\frac{3}{4}}} |L(1+u,\chi_{k_1})|^4 \ll T^{\frac{1}{4}+\varepsilon}(1+|\mathrm{Im}(u)|)^{1+\varepsilon}. \quad (2.65)$$

This bound can be improved under GRH. In fact, we split the left-hand side of (2.65) into

$$\begin{aligned} &\sideset{}{^\flat}\sum_{|k_1| \leq T} \frac{\tau^8(|k_1|)}{|k_1|^{\frac{3}{4}}} |L(1+u,\chi_{k_1})|^4 \\ &= \sideset{}{^\flat}\sum_{|k_1| \leq X^{\frac{1}{5}}} \frac{\tau^8(|k_1|)}{|k_1|^{\frac{3}{4}}} |L(1+u,\chi_{k_1})|^4 + \sideset{}{^\flat}\sum_{X^{\frac{1}{5}} < |k_1| \leq T} \frac{\tau^8(|k_1|)}{|k_1|^{\frac{3}{4}}} |L(1+u,\chi_{k_1})|^4. \end{aligned}$$

By Theorem 2.6, we have for $|\mathrm{Im}(u)| \leq X^{\frac{1}{5}}$,

$$\sideset{}{^\flat}\sum_{|k_1|\leq X^{\frac{1}{5}}} \frac{\tau^8(|k_1|)}{|k_1|^{\frac{3}{4}}} |L(1+u,\chi_{k_1})|^4 \ll X^{\frac{1}{5}} \log^{11} X.$$

Later in (2.84) of Section 2.8, under GRH, it will be proved that for $-\frac{1}{2} \leq \mathrm{Re}(u) \leq -\frac{1}{2} + \frac{1}{\log X}$ and $|\mathrm{Im}(u)| \leq X$,

$$\sideset{}{^\flat}\sum_{|k_1|\leq X} |L(1+u,\chi_{k_1})|^8 \ll X(\log X)^{37}.$$

Using dyadic blocks and Cauchy-Schwarz inequality, combined with the above bound, we can deduce that for $|\mathrm{Im}(u)| \leq X^{\frac{1}{5}}$,

$$\sideset{}{^\flat}\sum_{X^{\frac{1}{5}}<|k_1|\leq T} \frac{\tau^8(|k_1|)}{|k_1|^{\frac{3}{4}}} |L(1+u,\chi_{k_1})|^4 \ll T^{\frac{1}{4}} \log^{2^{16}} X.$$

Thus for $|\mathrm{Im}(u)| \leq X^{\frac{1}{5}}$,

$$\sideset{}{^\flat}\sum_{|k_1|\leq T} \frac{\tau^8(|k_1|)}{|k_1|^{\frac{3}{4}}} |L(1+u,\chi_{k_1})|^4 \ll T^{\frac{1}{4}} \log^{2^{16}} X. \tag{2.66}$$

Recall the definition of T. Substituting both (2.65) and (2.66) in (2.63), we have proved the contribution of the terms in the first category is $\ll U_1^{\frac{1}{2}} U_2^{\frac{1}{2}} Y(\log X)^{2^{17}} \Phi_{(5)}$. Similarly, we can deduce that the contribution of the terms in the second category is also $\ll U_1^{\frac{1}{2}} U_2^{\frac{1}{2}} Y(\log X)^{2^{17}} \Phi_{(5)}$.

The conditional part of the lemma is proved now. The unconditional part can be proved similarly by substituting (2.65) in (2.63). □

2.7 Proof of main theorems.

In this section, we complete the proof of Theorem 2.1 and Theorem 2.2. The argument is similar to [99, Section 5].

2.7.1 Proof of Theorem 2.1

Recall the definition of $S(U_1, U_2)$ from (2.15). Write $U = \frac{X}{(\log X)^{2^{50}}}$. Take $U_1 = U_2 = U$ and $Y = X^{\frac{1}{2}} U_1^{-\frac{1}{4}} U_2^{-\frac{1}{4}}$.

Using these values, we can simplify Lemmas 2.7, 2.9, 2.14 and 2.15. In the following, we give the detail of the simplification for Lemma 2.14. The summation in Lemma 2.14 is

$$\sum_{j_1+j_2+j_3+j_4=10} \frac{(-1)^{j_3} B(j_4)}{j_1! j_2! j_3! j_4!} (\log^{j_1} U)(\log^{j_2} U)(\log^{j_3} X). \tag{2.67}$$

We consider the case $j_4 = 0$, and other cases can be done similarly. Assume $j_4 = 0$ in (2.67). Then by (2.61), we have

$$\begin{aligned}
&\sum_{j_1+j_2+j_3=10} \frac{(-1)^{j_3} B(0)}{j_1! j_2! j_3!} (\log^{j_1} U)(\log^{j_2} U)(\log^{j_3} X) \\
&= 26 \sum_{j_1+j_2+j_3=10} \frac{(-1)^{j_3}}{j_1! j_2! j_3!} (\log^{j_1} U)(\log^{j_2} U)(\log^{j_3} X) \\
&= 26(\log^{10} X) \sum_{j_1+j_2+j_3=10} \frac{(-1)^{j_3}}{j_1! j_2! j_3!} + O\left(\log^{9+\varepsilon} X\right) \\
&= \frac{26}{10!} \log^{10} X + O\left(\log^{9+\varepsilon} X\right).
\end{aligned}$$

The second last equality is due to $\log^j U = \log^j X + O(\log^{j-1+\varepsilon} X)$ for $j \geq 0$. The last equality is obtained by

$$\sum_{j_1+j_2+j_3=10} \frac{(-1)^{j_3}}{j_1! j_2! j_3!} = \frac{1}{10!} \left. \frac{d^{10}}{dx^{10}} \right|_{x=0} (e^x e^x e^{-x}) = \frac{1}{10!}.$$

Similarly, we can compute other cases in (2.67). Combining all cases we can show (2.67) is

$$\left(\frac{26}{10!} - \frac{10}{9!} + \frac{1}{8!}\right) \log^{10} X + O\left(\log^{9+\varepsilon} X\right) = \frac{1}{2^4 \cdot 3^4 \cdot 5^2 \cdot 7} \log^{10} X + O\left(\log^{9+\varepsilon} X\right).$$

Using this fact, Lemma 2.14 can be simplified to

$$S_1(k = \square) = -\frac{a_4 \tilde{\Phi}(1)}{2^5 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot \pi^2} X \log^{10} X + O\left(X \log^{9+\varepsilon} X + X(\log^{-20} X)\Phi_{(5)}\right).$$

Now by (2.15), (2.18), combined with Lemmas 2.7, 2.9, 2.14 and 2.15, we can obtain that

$$\begin{aligned}
S(U_1, U_2) &= \sum_{(d,2)=1}^{*} \left|A_U(\tfrac{1}{2}; 8d)\right|^2 \Phi\left(\tfrac{d}{X}\right) \\
&= \frac{a_4 \tilde{\Phi}(1)}{2^6 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot \pi^2} X \log^{10} X + O\left(X \log^{9+\varepsilon} X + X(\log^{-20} X)\Phi_{(5)}\right).
\end{aligned} \tag{2.68}$$

Define $B_U(\frac{1}{2};8d) = L(\frac{1}{2},\chi_{8d})^2 - A_U(\frac{1}{2};8d)$. We claim that

$$\sideset{}{^*}\sum_{(d,2)=1} |B_U(\tfrac{1}{2};8d)|^2 \Phi\left(\tfrac{d}{X}\right) \ll X \log^{9.5+\varepsilon} X. \tag{2.69}$$

In fact, we have

$$B_U(\tfrac{1}{2};8d) = \frac{1}{\pi i}\int_{(c)} g(s) L(\tfrac{1}{2}+s,\chi_{8d})^2 \frac{(8d)^s - U^s}{s} ds.$$

Since $\frac{(8d)^s-U^s}{s}$ is entire, we move the line of the integral to $\mathrm{Re}(s)=0$. By the bound $|\frac{(8d)^{it}-U^{it}}{it}| \ll \log(\frac{8d}{U})$, $t\in\mathbb{R}$, we get that

$$B_U(\tfrac{1}{2};8d) \ll \log\left(\frac{8d}{U}\right)\int_{-\infty}^{\infty} |g(it)||L(\tfrac{1}{2}+it,\chi_{8d})|^2 dt.$$

This implies that the left-hand side of (2.69) is

$$\ll \left(\log\frac{X}{U}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |g(it_1)||g(it_2)| \sideset{}{^*}\sum_{(d,2)=1} |L(\tfrac{1}{2}+it_1,\chi_{8d})|^2 |L(\tfrac{1}{2}+it_2,\chi_{8d})|^2 \Phi(\tfrac{d}{X}) dt_1 dt_2. \tag{2.70}$$

Split the integral according to whether $|t_1|,|t_2|\le X$. If $|t_1|,|t_2|\le X$, then use Theorem 2.6. Otherwise, use Lemma 2.5. This will establish (2.69).

Note that

$$\begin{aligned}\sideset{}{^*}\sum_{(d,2)=1} L(\tfrac{1}{2},\chi_{8d})^4 \Phi\left(\tfrac{d}{X}\right) &= \sideset{}{^*}\sum_{(d,2)=1} (A_U(\tfrac{1}{2};8d) + B_U(\tfrac{1}{2};8d))^2 \Phi\left(\tfrac{d}{X}\right) \\ &= \sideset{}{^*}\sum_{(d,2)=1} A_U(\tfrac{1}{2};8d)^2 \Phi\left(\tfrac{d}{X}\right) + \sideset{}{^*}\sum_{(d,2)=1} B_U(\tfrac{1}{2};8d)^2 \Phi\left(\tfrac{d}{X}\right) \\ &\quad + 2\sideset{}{^*}\sum_{(d,2)=1} A_U(\tfrac{1}{2};8d) B_U(\tfrac{1}{2};8d) \Phi\left(\tfrac{d}{X}\right).\end{aligned}$$

Using the Cauchy-Schwarz inequality on the third term, combined with (2.68) and (2.69), we obtain that

$$\sum_{(d,2)=1}^{*} L(\tfrac{1}{2},\chi_{8d})^4\Phi(\tfrac{d}{X}) = \frac{a_4\tilde{\Phi}(1)}{2^6\cdot 3^3\cdot 5^2\cdot 7\cdot \pi^2}X\log^{10}X + O\left(X\log^{9.75+\varepsilon}X + X(\log^{-5}X)\Phi_{(5)}\right). \tag{2.71}$$

In the following we remove the function $\Phi(\frac{d}{X})$ in the above summation. Choose Φ such that $\Phi(t)=1$ for all $t\in(1+Z^{-1},2-Z^{-1})$, $\Phi(t)=0$ for all $t\notin(1,2)$, and $\Phi^{(\nu)}(t)\ll_\nu Z^\nu$ for all $\nu\geq 0$. This implies that $\Phi_{(\nu)}\ll_\nu Z^\nu$, and that $\tilde{\Phi}(1)=\check{\Phi}(0)=1+O(Z^{-1})$. Then by (2.71), we get that

$$\begin{aligned}&\sum_{(d,2)=1}^{*} L(\tfrac{1}{2},\chi_{8d})^4\Phi(\tfrac{d}{X})\\ &= \frac{a_4}{2^6\cdot 3^3\cdot 5^2\cdot 7\cdot \pi^2}X\log^{10}X + O\left(X(\log^{10}X)Z^{-1} + X\log^{9.75+\varepsilon}X + X(\log^{-5}X)Z^5\right).\end{aligned}$$

Take $Z=\log X$. We have

$$\sum_{\substack{X<d\leq 2X\\(d,2)=1}}^{*} L(\tfrac{1}{2},\chi_{8d})^4 \geq \sum_{(d,2)=1}^{*} L(\tfrac{1}{2},\chi_{8d})^4\Phi(\tfrac{d}{X}) = \frac{a_4}{2^6\cdot 3^3\cdot 5^2\cdot 7\cdot \pi^2}X\log^{10}X + O\left(X\log^{9.75+\varepsilon}X\right). \tag{2.72}$$

Similarly, we can choose $\Phi(t)$ in (2.71) such that $\Phi(t)=1$ for all $t\in[1,2]$, $\Phi(t)=0$ for all $t\notin(1-Z^{-1},2+Z^{-1})$, and $\Phi^{(\nu)}(t)\ll_\nu Z^\nu$ for all $\nu\geq 0$. Taking $Z=\log X$, we can deduce that

$$\sum_{\substack{X<d\leq 2X\\(d,2)=1}}^{*} L(\tfrac{1}{2},\chi_{8d})^4 \leq \sum_{(d,2)=1}^{*} L(\tfrac{1}{2},\chi_{8d})^4\Phi(\tfrac{d}{X}) = \frac{a_4}{2^6\cdot 3^3\cdot 5^2\cdot 7\cdot \pi^2}X\log^{10}X + O\left(X\log^{9.75+\varepsilon}X\right). \tag{2.73}$$

Combining (2.72) and (2.73), we obtain that

$$\sum_{\substack{X<d\leq 2X\\(d,2)=1}}^{*} L(\tfrac{1}{2},\chi_{8d})^4 = \frac{a_4}{2^6\cdot 3^3\cdot 5^2\cdot 7\cdot \pi^2}X\log^{10}X + O\left(X\log^{9.75+\varepsilon}X\right).$$

Applying the above with $X=\frac{x}{2}$, $X=\frac{x}{4}$, $\ldots$, we have proved Theorem 2.1.

2.7.2 Proof of Theorem 2.2.

Write $U = X^{1-4\varepsilon}$. By the Cauchy-Schwarz inequality, we obtain that

$$\sum_{(d,2)=1}^{*} L(\tfrac{1}{2}, \chi_{8d})^4 \Phi\left(\tfrac{d}{X}\right) \geq \frac{\left(\sum^{*}_{(d,2)=1} A_U(\frac{1}{2}, 8d) L(\frac{1}{2}, \chi_{8d})^2 \Phi\left(\frac{d}{X}\right)\right)^2}{\sum^{*}_{(d,2)=1} \left(A_U(\frac{1}{2}, 8d)\right)^2 \Phi\left(\frac{d}{X}\right)}. \tag{2.74}$$

Let A^2 and B denote the numerator and denominator of the right-hand side in (2.74), respectively.

We first handle B. By (2.15) and (2.18), combined with Lemmas 2.7, 2.9, 2.14 and 2.15, taking $Y = X^{\frac{1}{2}} U_1^{-\frac{1}{4}} U_2^{-\frac{1}{4}}$ and $U_1 = U_2 = U$, we get that

$$B = S(U_1, U_2) = \frac{a_4\left(1 - \frac{80}{3}\varepsilon + O(\varepsilon^2)\right)}{2^6 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot \pi^2} \tilde{\Phi}(1) X \log^{10} X + O\left(X \log^9 X + X \Phi_{(5)}\right),$$

where the implied constant in $O(\varepsilon^2)$ is absolute.

For A, we have

$$A = 4 \sum_{(d,2)=1}^{*} \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \frac{\tau(n_1)\tau(n_2)\chi_{8d}(n_1 n_2)}{\sqrt{n_1 n_2}} h_1(d, n_1, n_2),$$

where

$$h_1(x, y, z) := \Phi\left(\frac{x}{X}\right) \omega\left(\frac{y\pi}{U}\right) \omega\left(\frac{z\pi}{8x}\right).$$

Note that the difference between A and B lies in the difference between $h(x, y, z)$ and $h_1(x, y, z)$. By slightly modifying the argument for computing B, taking $Y = X^{\frac{1}{2}} U^{-\frac{1}{4}} X^{-\frac{1}{4}}$, we can deduce that

$$A = \frac{a_4\left(1 - \frac{40}{3}\varepsilon + O(\varepsilon^2)\right)}{2^6 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot \pi^2} \tilde{\Phi}(1) X \log^{10} X + O\left(X \log^9 X + X \Phi_{(5)}\right),$$

where the implied constant in $O(\varepsilon^2)$ is absolute.

Choose Φ such that $\Phi(t) = 1$ for all $t \in (1 + Z^{-1}, 2 - Z^{-1})$, $\Phi(t) = 0$ for all $t \notin (1, 2)$, and $\Phi^{(\nu)}(t) \ll_\nu Z^\nu$ for all $\nu \geq 0$. Take $Z = \log X$. Combining (2.74) with the estimates for A and

B, we have

$$\sum_{\substack{(d,2)=1\\ X<d\le 2X}}^{*} L(\tfrac{1}{2}, \chi_{8d})^4 \ge \left(1+O(\varepsilon^2)\right) \frac{a_4}{2^6 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot \pi^2} X \log^{10} X.$$

Having summed this with $X = \frac{x}{2}$, $X = \frac{x}{4}$, ..., we obtain Theorem 2.2.

2.8 Proof of Theorem 2.6.

In this section, we shall prove Theorem 2.6. The proof here closely follows [99, Section 6]. The argument in this section does not depend on Section 2.2 – Section 2.7.

Let $x \in \mathbb{R}$ with $x \ge 10$, and $z \in \mathbb{C}$. Define

$$\mathcal{L}(z,x) := \begin{cases} \log\log x & |z| \le (\log x)^{-1}, \\ -\log|z| & (\log x)^{-1} < |z| \le 1, \\ 0 & |z| > 1. \end{cases}$$

Let $z_1, z_2 \in \mathbb{C}$. We define

$$\mathcal{M}(z_1,z_2,x) := \frac{1}{2}\left(\mathcal{L}(z_1,x) + \mathcal{L}(z_2,x)\right),$$

and

$$\mathcal{V}(z_1,z_2,x) := \frac{1}{2}(\mathcal{L}(2z_1,x) + \mathcal{L}(2z_2,x) + \mathcal{L}(2\mathrm{Re}(z_1),x) + \mathcal{L}(2\mathrm{Re}(z_2),x) + 2\mathcal{L}(z_1+z_2,x) + 2\mathcal{L}(z_1+\overline{z_2},x)).$$

Remark 2.16. We see that the definition of $\mathcal{M}(z_1,z_2,x)$ is different from that in [99, Section 6] by a factor -1, while $\mathcal{V}(z_1,z_2,x)$ is the same. The difference is due to the different symmetry types of families of L-functions (see Katz-Sarnak [60]). The family of quadratic Dirichlet L-functions is symplectic, whereas the family of quadratic twists of a modular L-function in [99] is orthogonal. For further explanation, we refer readers to [99, p. 1111] and [98, p. 991].

Proposition 2.17. *Assume GRH for $L(s,\chi_d)$ for all fundamental discriminants d. Let X be large. Let $z_1, z_2 \in \mathbb{C}$ with $0 \le \mathrm{Re}(z_1), \mathrm{Re}(z_2) \le \frac{1}{\log X}$, and $|\mathrm{Im}(z_1)|, |\mathrm{Im}(z_2)| \le X$. Let*

$\mathcal{N}(V; z_1, z_2, X)$ denote the number of fundamental discriminants $|d| \leq X$ such that

$$\log |L(\tfrac{1}{2} + z_1, \chi_d) L(\tfrac{1}{2} + z_2, \chi_d)| \geq V + \mathcal{M}(z_1, z_2, X).$$

Then for $10\sqrt{\log\log X} \leq V \leq \mathcal{V}(z_1, z_2, X)$, we have

$$\mathcal{N}(V; z_1, z_2, X) \ll X \exp\left(-\frac{V^2}{2\mathcal{V}(z_1, z_2, X)}\left(1 - \frac{25}{\log\log\log X}\right)\right);$$

for $\mathcal{V}(z_1, z_2, X) < V \leq \frac{1}{16}\mathcal{V}(z_1, z_2, X) \log\log\log X$, we have

$$\mathcal{N}(V; z_1, z_2, X) \ll X \exp\left(-\frac{V^2}{2\mathcal{V}(z_1, z_2, X)}\left(1 - \frac{15V}{\mathcal{V}(z_1, z_2, X)\log\log\log X}\right)^2\right);$$

finally, for $\frac{1}{16}\mathcal{V}(z_1, z_2, X) \log\log\log X < V$, we have

$$\mathcal{N}(V; z_1, z_2, X) \ll X \exp\left(-\frac{1}{1025} V \log V\right).$$

Proof. It is helpful to keep in mind that $\log\log X + O(1) \leq \mathcal{V}(z_1, z_2, x) \leq 4 \log\log X$. By slightly modifying the proof of the main proposition in [98], we obtain that for any $2 \leq x \leq X$,

$$\begin{aligned} &\log |L(\tfrac{1}{2} + z_i, \chi_d)| \\ &\leq \operatorname{Re}\left(\sum_{2 \leq n \leq x} \frac{\Lambda(n)\chi_d(n)}{n^{\frac{1}{2} + \frac{\lambda_0}{\log x} + z_i} \log n} \frac{\log(\frac{x}{n})}{\log x}\right) + (1 + \lambda_0)\frac{\log X}{\log x} + O\left(\frac{1}{\log x}\right), \ i = 1, 2, \end{aligned}$$

where $\lambda_0 = 0.56\ldots$ is the unique real number satisfying $e^{-\lambda_0} = \lambda_0$. It follows that

$$\begin{aligned} &\log |L(\tfrac{1}{2} + z_1, \chi_d)||L(\tfrac{1}{2} + z_2, \chi_d)| \\ &\leq \operatorname{Re}\left(\sum_{\substack{p^l \leq x \\ l \geq 1}} \frac{\chi_d(p^l)}{l p^{l(\frac{1}{2} + \frac{\lambda_0}{\log x})}} (p^{-lz_1} + p^{-lz_2}) \frac{\log(\frac{x}{p^l})}{\log x}\right) + 2(1 + \lambda_0)\frac{\log X}{\log x} + O\left(\frac{1}{\log x}\right). \end{aligned} \tag{2.75}$$

The terms with $l \geq 3$ in the the above sum contribute $O(1)$. Using the fact $\sum_{p|d} \frac{1}{p} \ll \log\log\log d$, we get that

$$\begin{aligned}
&\mathrm{Re}\left(\sum_{p^2 \leq x} \frac{\chi_d(p^2)}{2p^{1+\frac{2\lambda_0}{\log x}}} (p^{-2z_1} + p^{-2z_2}) \frac{\log(\frac{x}{p^2})}{\log x} \right) \\
&= \mathrm{Re}\left(\sum_{p \leq \sqrt{x}} \frac{1}{2p^{1+\frac{2\lambda_0}{\log x}}} (p^{-2z_1} + p^{-2z_2}) \frac{\log(\frac{x}{p^2})}{\log x} \right) + O(\log\log\log X). \qquad (2.76)
\end{aligned}$$

By RH, we can deduce that

$$\sum_{p \leq y} (p^{-2z_1} + p^{-2z_2}) \log p = \frac{y^{1-2z_1}}{1-2z_1} + \frac{y^{1-2z_2}}{1-2z_2} + O\left(\sqrt{y}(\log Xy)^2\right). \qquad (2.77)$$

The above sum also has a trivial bound $\ll y$. Combining (2.76) with these two bounds, by partial summation, we have

$$\sum_{p \leq \sqrt{x}} \frac{1}{2p^{1+\frac{2\lambda_0}{\log x}}} (p^{-2z_1} + p^{-2z_2}) \frac{\log(\frac{x}{p^2})}{\log x} = \mathcal{M}(z_1, z_2, x) + O(\log\log\log X).$$

Inserting above estimates into (2.75), by $\mathcal{M}(z_1, z_2, x) \leq \mathcal{M}(z_1, z_2, X)$, we obtain that

$$\begin{aligned}
&\log |L(\tfrac{1}{2} + z_1, \chi_d)||L(\tfrac{1}{2} + z_2, \chi_d)| \\
&\leq \mathrm{Re}\left(\sum_{2 < p \leq x} \frac{\chi_d(p)}{p^{\frac{1}{2}+\frac{\lambda_0}{\log x}}} (p^{-z_1} + p^{-z_2}) \frac{\log(\frac{x}{p})}{\log x} \right) + \mathcal{M}(z_1, z_2, X) + \frac{4\log X}{\log x} + O(\log\log\log X). \qquad (2.78)
\end{aligned}$$

For brevity, put $\mathcal{V} := \mathcal{V}(z_1, z_2, X)$. Set

$$A := \begin{cases} \frac{1}{2}\log\log\log X & 10\sqrt{\log\log X} \leq V \leq \mathcal{V}, \\ \frac{\mathcal{V}}{2V}\log\log\log X & \mathcal{V} < V \leq \frac{1}{16}\mathcal{V}\log\log\log X, \\ 8 & V > \frac{1}{16}\mathcal{V}\log\log\log X. \end{cases}$$

By taking $x = \log X$ in (2.78) and bounding the sum over p in (2.78) trivially, we know that $\mathcal{N}(V; z_1, z_2, X) = 0$ for $V > \frac{5\log X}{\log\log X}$. Thus, we can assume $V \leq \frac{5\log X}{\log\log X}$.

From now on, we set $x = X^{A/V}$ and $z = x^{1/\log\log X}$. Let S_1 be the sum in (2.78) truncated to $p \le z$, and S_2 be the sum over $z < p \le x$. It follows from (2.78) that

$$\log|L(\tfrac{1}{2}+z_1,\chi_d)||L(\tfrac{1}{2}+z_2,\chi_d)| \le S_1 + S_2 + \mathcal{M}(z_1,z_2,X) + \frac{5V}{A}.$$

Note that if d satisfies $\log|L(\frac{1}{2}+z_1,\chi_d)||L(\frac{1}{2}+z_2,\chi_d)| \ge V + \mathcal{M}(z_1,z_2,X)$, then either

$$S_2 \ge \tfrac{V}{A}, \text{ or } S_1 \ge V_1 := V(1-\tfrac{6}{A}).$$

Write

$$\begin{aligned}
\text{meas}(X;S_1) &:= \#\{|d| \le X \ : \ d \text{ is a fundamental discriminant, } S_1 \ge V_1\},\\
\text{meas}(X;S_2) &:= \#\{|d| \le X \ : \ d \text{ is a fundamental discriminant, } S_2 \ge \tfrac{V}{A}\}.
\end{aligned}$$

For any $m \le \frac{V}{2A} - 1$, by [99, Lemma 6.3], we have

$$\sideset{}{^\flat}\sum_{|d|\le X} |S_2|^{2m} \ll X\frac{(2m)!}{m!2^m}\left(\sum_{z<p\le x}\frac{4}{p}\right)^m \ll X(3m\log\log\log X)^m.$$

By choosing $m = \lfloor \frac{V}{2A} \rfloor - 1$, we get that

$$\text{meas}(X;S_2) \ll X\exp\left(-\frac{V}{4A}\log V\right). \tag{2.79}$$

We next estimate $\text{meas}(X;S_1)$. For any $m \le \frac{\frac{1}{2}\log X - \log\log X}{\log z}$, by [99, Lemma 6.3], we obtain that

$$\sideset{}{^\flat}\sum_{|d|\le X} |S_1|^{2m} \ll X\frac{(2m)!}{m!2^m}\left(\sum_{p\le z}\frac{|a(p)|^2}{p}\right)^m, \tag{2.80}$$

where

$$a(p) = \frac{\text{Re}(p^{-z_1}+p^{-z_2})\log(\frac{x}{p})}{p^{\frac{\lambda_0}{\log x}}\log x}.$$

By using (2.77) and the partial summation, we can show that

$$\sum_{p\leq z}\frac{|a(p)|^2}{p}\leq\frac{1}{4}\sum_{p\leq\sqrt{X}}\frac{1}{p}(p^{-z_1}+p^{-\overline{z_1}}+p^{-z_2}+p^{-\overline{z_2}})^2=\mathcal{V}(z_1,z_2,X)+O(\log\log\log X).$$

Together with (2.80), this yields

$$\operatorname{meas}(X;S_1)\ll XV_1^{-2m}\frac{(2m)!}{m!2^m}(\mathcal{V}+O(\log\log\log X))^m\ll X\left(\frac{2m}{e}\cdot\frac{\mathcal{V}+O(\log\log\log X)}{V_1^2}\right)^m.$$

Taking $m=\lfloor\frac{V_1^2}{2\mathcal{V}}\rfloor$ when $V\leq\frac{(\log\log X)^2}{\log\log\log X}$, and taking $m=\lfloor 10V\rfloor$ otherwise, we obtain that

$$\operatorname{meas}(X;S_1)\ll X\exp\left(-\frac{V_1^2}{2\mathcal{V}}\left(1+O\left(\frac{\log\log\log X}{\log\log X}\right)\right)\right)+X\exp\left(-V\log V\right). \tag{2.81}$$

Using the estimates (2.79) and (2.81), we can establish Proposition 2.17. This completes the proof. □

For convenience, in the following we show a rough form of Proposition 2.17. Let $k\in\mathbb{R}_{>0}$ be fixed. For $10\sqrt{\log\log X}\leq V\leq 4k\mathcal{V}(z_1,z_2,X)$, we have

$$\mathcal{N}(V;z_1,z_2,X)\ll X(\log X)^{o(1)}\exp\left(-\frac{V^2}{2\mathcal{V}(z_1,z_2,X)}\right), \tag{2.82}$$

and for $V>4k\mathcal{V}(z_1,z_2,X)$, we have

$$\mathcal{N}(V;z_1,z_2,X)\ll X(\log X)^{o(1)}\exp(-4kV). \tag{2.83}$$

Observe that

$$\begin{aligned}\sideset{}{^\flat}\sum_{|d|\leq X}|L(\tfrac{1}{2}+z_1,\chi_d)L(\tfrac{1}{2}+z_2,\chi_d)|^k&=-\int_{-\infty}^{\infty}\exp(kV+k\mathcal{M}(z_1,z_2,X))d\mathcal{N}(V;z_1,z_2,X)\\&=k\int_{-\infty}^{\infty}\exp(kV+k\mathcal{M}(z_1,z_2,X))\mathcal{N}(V;z_1,z_2,X)dV.\end{aligned}$$

Inserting the rough bounds (2.82) and (2.83) into the integral above, we can deduce that

Theorem 2.18. *Assume GRH for $L(s,\chi_d)$ for all fundamental discriminants d. Let X be large. Let $z_1,z_2\in\mathbb{C}$ with $0\leq\operatorname{Re}(z_1),\operatorname{Re}(z_2)\leq\frac{1}{\log X}$, and $|\operatorname{Im}(z_1)|,|\operatorname{Im}(z_2)|\leq X$. Then for any positive* *real number k and any $\varepsilon > 0$, we have*

$$\sum_{|d|\leq X}^{\flat} |L(\tfrac{1}{2}+z_1,\chi_d)L(\tfrac{1}{2}+z_2,\chi_d)|^k \ll_{k,\varepsilon} X(\log X)^{\varepsilon} \exp\left(k\mathcal{M}(z_1,z_2,X)+\frac{k^2}{2}\mathcal{V}(z_1,z_2,X)\right).$$

In the rest of this section, we complete the proof of Theorem 2.6.

Proof of Theorem 2.6. By Theorem 2.18 and the fact that $\mathcal{L}(z,x) \leq \log\log x$ for $z \in \mathbb{C}, x \geq 10$, we can trivially get that

$$\sum_{|d|\leq X}^{\flat} |L(\tfrac{1}{2}+z_1,\chi_d)|^k|L(\tfrac{1}{2}+z_2,\chi_d)|^k \ll_{k,\varepsilon} X(\log X)^{2k^2+k+\varepsilon}. \tag{2.84}$$

Now we assume $|\mathrm{Im}(z_1)-\mathrm{Im}(z_2)| \geq \frac{1}{\log X}$. Write $t_1 = \mathrm{Im}(z_1)$ and $t_2 = \mathrm{Im}(z_2)$.

If $t_1t_2 \geq 0$, then $|t_1-t_2| \leq |t_1+t_2| \leq \max(2|t_1|, 2|t_2|)$, say $|t_1+t_2| \leq 2|t_1|$. Note that $\mathcal{L}(y,X)$ is a decreasing function for $y \geq 0$. Thus, we have

$$\begin{aligned}\mathcal{L}(z_1,X),\ \mathcal{L}(2z_1,X),\ \mathcal{L}(z_1+z_2,X),\ \mathcal{L}(z_1+\overline{z_2},X) &\leq \mathcal{L}(|t_1-t_2|,X)+O(1)\\ &\leq \max(0,-\log|t_1-t_2|)+O(1).\end{aligned}$$

This together with

$$\mathcal{L}(z_2,X),\ \mathcal{L}(2z_2,X),\ \mathcal{L}(2\mathrm{Re}(z_1),X),\ \mathcal{L}(2\mathrm{Re}(z_2),X) \leq \log\log X$$

implies

$$2\mathcal{M}(z_1,z_2,X)+2\mathcal{V}(z_1,z_2,X) \leq 4\log\log X+\max\{0,-6\log|t_1-t_2|\}+O(1). \tag{2.85}$$

On the other hand, if $t_1t_2 < 0$, then $|t_1-t_2| = |t_1|+|t_2| \leq \max\{|2t_1|, |2t_2|\}$, say $|t_1-t_2| \leq |2t_2|$. It implies that $|t_1| \leq |t_2|$ and that $\mathcal{L}(2t_2,X) \leq \mathcal{L}(|t_1-t_2|,X)$. Note $|t_1-t_2| = 2|t_1|+|t_1+t_2|$, so $|t_1-t_2| \leq \max\{4|t_1|, 2|t_1+t_2|\}$. In fact, if $|t_1-t_2| > 4|t_1|$, then $2|t_1|+|t_1+t_2| > 4|t_1|$, which implies $|t_1| \leq \frac{1}{2}|t_1+t_2|$. It means $|t_1-t_2| = 2|t_1|+|t_1+t_2| \leq 2|t_1+t_2|$. Without loss of generality, we can say $|t_1-t_2| \leq 4|t_1|$. It follows that $\mathcal{L}(z_1,X), \mathcal{L}(2z_1,X) \leq \mathcal{L}(|t_1-t_2|,X)+O(1)$. Now we have

$$\mathcal{L}(z_1, X),\ \mathcal{L}(2z_1, X),\ \mathcal{L}(z_2, X),\ \mathcal{L}(2z_2, X),\ \mathcal{L}(z_1 + \overline{z_2}, X) \leq \mathcal{L}(|t_1 - t_2|, X) + O(1)$$
$$\leq \max(0, -\log|t_1 - t_2|) + O(1).$$

This combined with

$$\mathcal{L}(2\mathrm{Re}(z_1), X),\ \mathcal{L}(2\mathrm{Re}(z_2), X),\ \mathcal{L}(z_1 + z_2, X) \leq \log\log X$$

also implies (2.85).

By inserting (2.85) into Theorem 2.18, we can show for $|\mathrm{Im}(z_1) - \mathrm{Im}(z_2)| \geq \frac{1}{\log X}$,

$$\sum_{|d|\leq X}^{\flat} |L(\tfrac{1}{2} + z_1, \chi_d)|^2 |L(\tfrac{1}{2} + z_2, \chi_d)|^2 \ll X(\log X)^{4+\varepsilon}\left(1 + \frac{1}{|t_1 - t_2|^6}\right). \tag{2.86}$$

By combining (2.86) and (2.84) with $k = 2$, we have proved Theorem 2.6. $\square$

Chapter 3

The first moment of quadratic twists of modular L-functions

3.1 Introduction.

The study of moments of L-functions is of much interest to researchers in number theory due to its fruitful applications. One example is that Bump-Friedberg-Hoffstein [12] and Murty-Murty [79] independently proved $L'(\frac{1}{2}, E \otimes \chi_d) \neq 0$ for infinitely many fundamental discriminants d with $d < 0$, where E is a modular elliptic curve with root number 1 over $\mathbb{Q}$ and $\chi_d(\cdot) := \left(\frac{d}{\cdot}\right)$ denotes the Kronecker symbol. The method of their work is to investigate moments of the derivative of quadratic twists of modular L-functions. Their result verifies the assumption of Kolyvagin's theorem [66] on the Birch-Swinnerton-Dyer conjecture, where it was proven that if the Hasse-Weil L-function $L(s, E)$ does not vanish at the center point $s = \frac{1}{2}$, then the group of rational points of E is finite, *provided* that there exists a quadratic character χ_d with $d < 0$ such that $L(s, E \otimes \chi_d)$ has a simple zero at the central point and such that $\chi_d(p) = 1$ for every p that divides the conductor of E.

In particular, Murty-Murty [79] proved the asymptotic formula for the first moment of the derivative of quadratic twists of modular L-functions with an error term $O(X(\log X)^{1-\rho})$, where ρ is an explicit positive real number. It was later improved by Iwaniec [54] to a power savings $O(X^{\frac{13}{14}+\varepsilon})$ for a smoothed version. In [11] Bump-Friedberg-Hoffstein claimed the error term $O(X^{\frac{3}{5}+\varepsilon})$ without proof. Note that in [54, 79] they considered quadratic twists of elliptic curve L-functions, but it is no doubt that the methods there would extend to all modular newforms. This chapter obtains an error term of the size $O(X^{\frac{1}{2}+\varepsilon})$. The improvement is due to the recursive method developed by Young in his works on the moments of quadratic Dirichlet L-functions [107, 108]. The argument here also allows us to obtain an error term of the same size $O(X^{\frac{1}{2}+\varepsilon})$ for the first moment of quadratic twists of modular L-functions, which improves the error term $O(X^{\frac{13}{14}+\varepsilon})$ of Stefanicki [100, Theorem 3] and Luo-Ramakrishnan [71, Proposition 3.6] and $O(X^{\frac{7}{8}+\varepsilon})$ of Soundararajan-Radziwiłł [87, Proposition 2]. Also, with slightly more effort, we can obtain similar results for the first moment of higher derivatives of twisted modular L-functions.

To precisely state our result, we shall introduce some notation. Let f be a modular form of weight κ for the full modular group $SL_2(\mathbb{Z})$. (Our argument may extend to congruent subgroups.) We assume f is an eigenfunction of all Hecke operators. The Fourier expansion of f at infinity is

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{\kappa-1}{2}} e(nz),$$

where $\lambda_f(1) = 1$ and $|\lambda_f(n)| \leq \tau(n)$ for $n \geq 1$. Here $e(z) := e^{2\pi i z}$, and $\tau(n)$ is the number of divisors of n. The twisted modular L-function is defined by

$$L(s, f \otimes \chi_d) := \sum_{n=1}^{\infty} \frac{\lambda_f(n)\chi_d(n)}{n^s} = \prod_{p \nmid d} \left(1 - \frac{\lambda_f(p)\chi_d(p)}{p^s} + \frac{1}{p^{2s}}\right)^{-1} \tag{3.1}$$

for $\mathrm{Re}(s) > 1$, and it extends to the entire complex plane. The completed L-function is defined by

$$\Lambda(s, f \otimes \chi_d) := \left(\frac{|d|}{2\pi}\right)^s \Gamma(s + \tfrac{\kappa-1}{2}) L(s, f \otimes \chi_d).$$

It satisfies the functional equation

$$\Lambda(s, f \otimes \chi_d) = i^{\kappa} \epsilon(d) \Lambda(1 - s, f \otimes \chi_d), \tag{3.2}$$

where $\epsilon(d) = 1$ if d is positive, and $\epsilon(d) = -1$ if d is negative. In this chapter we consider the case $d > 0$, so $\epsilon = 1$. The case $d < 0$ can be done similarly. We prove the following assertions.

Theorem 3.1. *Let $\kappa \equiv 0 \pmod 4$. Let $\Phi(x) : (0, \infty) \to \mathbb{R}$ be a smooth, compactly supported function. We have*

$$\sideset{}{^*}\sum_{(d,2)=1} L(\tfrac{1}{2}, f \otimes \chi_{8d}) \Phi(\tfrac{d}{X}) = \frac{8\tilde{\Phi}(1)}{\pi^2} L(1, \mathrm{sym}^2 f) Z^*(0) X + O(X^{\frac{1}{2}+\varepsilon}),$$

Here $\sum^*$ *denotes the summation over square-free integers,* Z^* *is defined via* (3.5), (3.6), *and* $\tilde{\Phi}$ *is the Mellin transform of* Φ *defined by*

$$\tilde{\Phi}(s) := \int_0^\infty \Phi(x)x^{s-1}dx.$$

Theorem 3.2. *Let* $\kappa \equiv 2 \,(\bmod\, 4)$. *Let* $\Phi(x) : (0, \infty) \to \mathbb{R}$ *be a smooth, compactly supported function. We have*

$$\sum_{(d,2)=1}^{*} L'(\tfrac{1}{2}, f \otimes \chi_{8d})\Phi(\tfrac{d}{X}) = \frac{8\tilde{\Phi}(1)}{\pi^2} L(1, \mathrm{sym}^2 f)Z^*(0)X\Big[\log X + 2\frac{L'(1, \mathrm{sym}^2 f)}{L(1, \mathrm{sym}^2 f)} + \frac{Z^{*\prime}(0)}{Z^*(0)}$$
$$+ \log\frac{8}{2\pi} + \frac{\Gamma'(\frac{\kappa}{2})}{\Gamma(\frac{\kappa}{2})} + \frac{\tilde{\Phi}'(1)}{\tilde{\Phi}(1)}\Big] + O(X^{\frac{1}{2}+\varepsilon}).$$

In the above, the symmetric square L-function is defined by

$$L(s, \mathrm{sym}^2 f) := \zeta(2s)\sum_{n=1}^{\infty}\frac{\lambda_f(n^2)}{n^s} = \prod_p \left(1 - \frac{\alpha_f(p)^2}{p^s}\right)^{-1}\left(1 - \frac{\alpha_f(p)\beta_f(p)}{p^s}\right)^{-1}\left(1 - \frac{\beta_f(p)^2}{p^s}\right)^{-1},$$

where $\mathrm{Re}(s) > 1$, $\alpha_f(p) + \beta_f(p) = \lambda_f(p)$ and $\alpha_f(p)\beta_f(p) = 1$. We see that the main term in Theorem 3.2 coincides with [85, Theorem 2.3]. Note that the form of the moment and the definition of Z^* in [85, Theorem 2.3] are slightly different from ours.

It is worth mentioning that recently Bui–Florea–Keating–Roditty-Gershon [10] obtained the error term of the same size $O(X^{\frac{1}{2}+\varepsilon})$ for the function field analogue. The second moment, expected to be much more difficult, was computed asymptotically by Soundararajan and Young [99] under the generalized Riemann hypothesis. Their method was also used by Petrow [85] for studying moments of derivatives of twisted modular L-functions. The computation of asymptotic formulas for higher moments is believed beyond current techniques, whereas we do have beautiful conjectures due to Keating-Snaith [62] and Conrey-Farmer-Keating-Rubinstein-Snaith [17].

The moments of quadratic twists of modular L-functions are comparable to the moments of quadratic Dirichlet L-functions. The iterative method, initially used by Heath-Brown [48] to study mean values of real characters, was applied by Young [107] in obtaining the error term $O(X^{\frac{1}{2}+\varepsilon})$ in the asymptotic formula for the first moment of quadratic Dirichlet L-functions. The error term $O(X^{\frac{1}{2}+\varepsilon})$ was also essentially implicit in Goldfeld-Hoffstein's work [34]. In addition, by using the recursive method, the third moment of quadratic Dirichlet L-functions was improved to $O(X^{\frac{3}{4}+\varepsilon})$ by Young [108], and recently the second moment was improved to $O(X^{\frac{1}{2}+\varepsilon})$ by Sono [96]. The moment in Theorem 3.1 is analogous to the second moment of quadratic Dirichlet L-functions, so it should not be a coincidence that Sono's work [96] and Theorem 3.1 have the same error term $O(X^{\frac{1}{2}+\varepsilon})$. The conjectured error term for the second moment of quadratic Dirichlet L-functions is $O(X^{\frac{1}{2}+\varepsilon})$, so it may be hard to improve Theorems 3.1 and 3.2.

The proof for Theorems 3.1 and 3.2 is similar to [96, 107, 108]. To adapt to the recursive method, we consider the shifted first moment twisted by a quadratic character as follows:

$$M(\alpha,\ell) := \sideset{}{^*}\sum_{(d,2)=1} \chi_{8d}(\ell) L(\tfrac{1}{2}+\alpha, f\otimes\chi_{8d})\Phi(\tfrac{d}{X}), \tag{3.3}$$

where ℓ is a positive, odd integer. Write $\ell = \ell_1\ell_2^2$, where ℓ_1 is square-free. We may make the following conjecture.

Conjecture 3.3. *Let $h \geq \frac{1}{2}$. Let $\Phi(x) : (0,\infty) \to \mathbb{C}$ be a smooth, compactly supported function. Assume $|\mathrm{Re}(\alpha)| \ll \frac{1}{\log X}$ and $|\mathrm{Im}(\alpha)| \ll (\log X)^2$. Then for any $\varepsilon > 0$, we have*

$$\begin{aligned} M(\alpha,\ell) = {} & \frac{4X\tilde{\Phi}(1)}{\pi^2 \ell_1^{\frac{1}{2}+\alpha}} L(1+2\alpha, \mathrm{sym}^2 f) Z(\tfrac{1}{2}+\alpha, \ell) \\ & + i^\kappa \frac{4\gamma_\alpha X^{1-2\alpha}\tilde{\Phi}(1-2\alpha)}{\pi^2 \ell_1^{\frac{1}{2}-\alpha}} L(1-2\alpha, \mathrm{sym}^2 f) Z(\tfrac{1}{2}-\alpha, \ell) + O(\ell^{\frac{1}{2}+\varepsilon} X^{h+\varepsilon}). \end{aligned} \tag{3.4}$$

Here the big O is depending on ε, h and Φ, and we define, for $\mathrm{Re}(\gamma) > 0$*,*

$$Z(\tfrac{1}{2}+\gamma, \ell) := L(1+2\gamma, \mathrm{sym}^2 f)^{-1} \prod_{(p,2)=1} Z_p(\tfrac{1}{2}+\gamma, \ell), \tag{3.5}$$

where $Z_p(\frac{1}{2}+\gamma,\ell)$ is defined by

$$
\begin{aligned}
&Z_p(\tfrac{1}{2}+\gamma,\ell)\\
&:= \begin{cases}
p^{\frac{1}{2}+\gamma}\left(\frac{p}{p+1}\right)\left[\frac{1}{2}(1-\frac{\lambda_f(p)}{p^{\frac{1}{2}+\gamma}}+\frac{1}{p^{1+2\gamma}})^{-1}-\frac{1}{2}(1+\frac{\lambda_f(p)}{p^{\frac{1}{2}+\gamma}}+\frac{1}{p^{1+2\gamma}})^{-1}\right] & \text{if } p|\ell_1,\\
\frac{p}{p+1}\left[\frac{1}{2}(1-\frac{\lambda_f(p)}{p^{\frac{1}{2}+\gamma}}+\frac{1}{p^{1+2\gamma}})^{-1}+\frac{1}{2}(1+\frac{\lambda_f(p)}{p^{\frac{1}{2}+\gamma}}+\frac{1}{p^{1+2\gamma}})^{-1}\right] & \text{if } p\nmid \ell_1, p|\ell_2,\\
1+\frac{p}{p+1}\left[\frac{1}{2}(1-\frac{\lambda_f(p)}{p^{\frac{1}{2}+\gamma}}+\frac{1}{p^{1+2\gamma}})^{-1}+\frac{1}{2}(1+\frac{\lambda_f(p)}{p^{\frac{1}{2}+\gamma}}+\frac{1}{p^{1+2\gamma}})^{-1}-1\right] & \text{if } (p,2\ell)=1.
\end{cases}
\end{aligned}
$$

The function $Z(\frac{1}{2}+\gamma,\ell)$ has an analytic continuation to the region $\mathrm{Re}(\gamma) > -\frac{1}{4}$, *by Lemma 3.10.*

The main term in (3.4) can be conjectured by heuristically following this chapter's argument or using the recipe method in [17]. To obtain Theorems 3.1 and 3.2, it suffices to prove the following theorem.

Theorem 3.4. *If Conjecture 3.3 is true for some $h \geq \frac{1}{2}$ (for any ε and any Φ that satisfy the conditions described in Conjecture 3.3), then it is true for $\frac{4h-1}{4h}$ replacing h (for any ε and any Φ that satisfy the conditions described in Conjecture 3.3).*

Proof of Theorems 3.1 and 3.2. We see Conjecture 3.3 is true for $h = 1$ by Lemma 3.9 in the next section. By Theorem 3.4 we can reduce it to $h = 1, \frac{3}{4}, \frac{2}{3}, \cdots$, which tends to $h = \frac{1}{2}$. Set $\ell = 1$ and write

$$Z^*(\alpha) := Z(\tfrac{1}{2}+\alpha, 1). \tag{3.6}$$

Then Theorem 3.1 follows by letting $\alpha \to 0$ in (3.4). We can differentiate both sides of (3.4) in terms of α. Note that the error term in (3.4) is holomorphic on the disc centred at $(0,0)$ with radius $\ll \frac{1}{\log X}$. Hence the size of the derivative of the error term is still $O(X^{\frac{1}{2}+\varepsilon})$ by Cauchy's integral formula. This gives Theorem 3.2 by letting $\alpha \to 0$. Note that we can compute asymptotic formulas for the first moment of higher derivatives of twisted L-functions in a similar way. □

The rest of the chapter will focus on proving Theorem 3.4. The idea is as follows. We first apply the approximate functional equation in the twisted L-function in (3.3). Then the Möbius inversion is used to remove the square-free condition where the new parameter a is introduced. We split the summation over a into two pieces. For large a, the Poisson summation formula is employed to separate the summation into diagonal terms and non-diagonal terms (see their definitions below (3.14)). On the other hand, for small a, we convert the summation back to that with the square-free condition, where we will use the induction hypothesis (3.4). We obtain partial main terms and error terms there. These partial main terms can be perfectly combined with the diagonal terms after some simplification, finally leading to the main term in (3.4). We remark that there are nice cancellations between specific terms in the moment of quadratic Dirichlet L-functions (see [107, 108, 96]), which seem to not appear in our case.

3.2 Preliminary lemmas.

Lemma 3.5. *Let $G(s)$ be an even, entire function with $G(0) = 1$, bounded in any fixed strip $|\mathrm{Re}(s)| \leq A$, and decaying rapidly as $|\mathrm{Im}(s)| \to \infty$. Let*

$$\omega_\alpha(\xi) := \frac{1}{2\pi i} \int_{(1)} \frac{G(s)}{s} g_\alpha(s) \xi^{-s} ds,$$

where

$$g_\alpha(s) := (2\pi)^{-s} \frac{\Gamma(\frac{\kappa}{2} + \alpha + s)}{\Gamma(\frac{\kappa}{2} + \alpha)},$$

and $\int_{(c)}$ denotes the contour integral $\int_{c-i\infty}^{c+i\infty}$. Set

$$X_{\alpha,d} := \left(\frac{|d|}{2\pi}\right)^{-2\alpha} \frac{\Gamma(\frac{\kappa}{2} - \alpha)}{\Gamma(\frac{\kappa}{2} + \alpha)}.$$

Then we have

$$L(\tfrac{1}{2} + \alpha, f \otimes \chi_d) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)\chi_d(n)}{n^{\frac{1}{2}+\alpha}} \omega_\alpha\left(\frac{n}{|d|}\right) + i^\kappa \epsilon(d) X_{\alpha,d} \sum_{n=1}^{\infty} \frac{\lambda_f(n)\chi_d(n)}{n^{\frac{1}{2}-\alpha}} \omega_{-\alpha}\left(\frac{n}{|d|}\right).$$

Proof. Set

$$\begin{aligned}
I &:= \frac{1}{2\pi i}\int_{(1)} \left(\frac{|d|}{2\pi}\right)^s \frac{\Gamma(\frac{\kappa}{2}+\alpha+s)}{\Gamma(\frac{\kappa}{2}+\alpha)} L(\tfrac{1}{2}+\alpha+s, f\otimes\chi_d)\frac{G(s)}{s}ds \\
&= \frac{1}{2\pi i}\int_{(1)} \frac{\Lambda(\frac{1}{2}+\alpha+s)}{\Gamma(\frac{\kappa}{2}+\alpha)}\frac{G(s)}{s}\left(\frac{d}{2\pi}\right)^{-\frac{1}{2}-\alpha} ds.
\end{aligned} \tag{3.7}$$

Move the line of integration to $\mathrm{Re}(s) = -1$. The residue theorem gives

$$L(\tfrac{1}{2}+\alpha, f\otimes\chi_d) = I - I', \tag{3.8}$$

where

$$I' := \frac{1}{2\pi i}\int_{(-1)} \frac{\Lambda(\frac{1}{2}+\alpha+s)}{\Gamma(\frac{\kappa}{2}+\alpha)}\frac{G(s)}{s}\left(\frac{d}{2\pi}\right)^{-\frac{1}{2}-\alpha} ds.$$

By changing the variable $s \to -s$ and the functional equation (3.2),

$$\begin{aligned}
I' &= -i^\kappa\epsilon(d)\frac{1}{2\pi i}\int_{(1)} \frac{\Lambda(\frac{1}{2}-\alpha+s)}{\Gamma(\frac{\kappa}{2}+\alpha)}\frac{G(s)}{s}\left(\frac{|d|}{2\pi}\right)^{-\frac{1}{2}-\alpha} ds \\
&= -i^\kappa\epsilon(d)X_{\alpha,d}\frac{1}{2\pi i}\int_{(1)} \left(\frac{|d|}{2\pi}\right)^s \frac{\Gamma(\frac{\kappa}{2}-\alpha+s)}{\Gamma(\frac{\kappa}{2}-\alpha)} L(\tfrac{1}{2}-\alpha+s, f\otimes\chi_d)\frac{G(s)}{s}ds.
\end{aligned} \tag{3.9}$$

Insert (3.7) and (3.9) back into (3.8) and write $L(\frac{1}{2}\pm\alpha+s, f\otimes\chi_d)$ as their Dirichlet series. This completes the proof. ∎

Remark 3.6. Write

$$\mathcal{Z}(\alpha,s) := \zeta(2+4\alpha+4s)(1+4\alpha+4s)(1-4\alpha-4s).$$

We can take

$$G(s) = e^{s^2}\frac{\mathcal{Z}(\alpha,s)\mathcal{Z}(\alpha,-s)\cdot\mathcal{Z}(-\alpha,s)\mathcal{Z}(-\alpha,-s)}{\mathcal{Z}(\alpha,0)^2\mathcal{Z}(-\alpha,0)^2}.$$

The purpose of adding some zeta factors into $G(s)$ is that they cancel out certain terms in $Z(\frac{1}{2}\pm\alpha\pm s)$. See Lemma 3.10 and (3.48) for an example.

The following lemma is a generalized version of Poisson summation formula established by

Soundararajan [97, Lemma 2.6] (also see [99, Lemma 2.2]).

Lemma 3.7. *Let Φ be a smooth function with compact support on the positive real numbers, and suppose that n is an odd integer. Then*

$$\sum_{(d,2)=1} \left(\frac{d}{n}\right) \Phi\left(\frac{d}{Z}\right) = \frac{Z}{2n}\left(\frac{2}{n}\right)\sum_{k\in\mathbb{Z}}(-1)^k G_k(n)\hat{\Phi}\left(\frac{kZ}{2n}\right),$$

where

$$G_k(n) := \left(\frac{1-i}{2} + \left(\frac{-1}{n}\right)\frac{1+i}{2}\right)\sum_{a\,(\mathrm{mod}\,n)}\left(\frac{a}{n}\right)e\left(\frac{ak}{n}\right),$$

and

$$\hat{\Phi}(y) := \int_{-\infty}^{\infty}\left(\cos(2\pi xy) + \sin(2\pi xy)\right)\Phi(x)dx$$

is a Fourier-type transform of Φ.

The Gauss-type sum $G_k(n)$ above can be explicitly computed in the following lemma (see [97, Lemma 2.3]).

Lemma 3.8. *If m and n are relatively prime odd integers, then $G_k(mn) = G_k(m)G_k(n)$. Moreover, if p^α is the largest power of p dividing k (setting $\alpha = \infty$ if $k = 0$), then*

$$G_k(p^\beta) = \begin{cases} 0 & \text{if } \beta \le \alpha \text{ is odd},\\ \phi(p^\beta) & \text{if } \beta \le \alpha \text{ is even},\\ -p^\alpha & \text{if } \beta = \alpha+1 \text{ is even},\\ \left(\frac{kp^{-\alpha}}{p}\right)p^\alpha\sqrt{p} & \text{if } \beta = \alpha+1 \text{ is odd},\\ 0 & \text{if } \beta \ge \alpha+2. \end{cases}$$

Here ϕ is the Euler totient function.

We need the following upper bound for the first moment of twisted modular L-functions. It is analogous to [48, Theorem 2] of Heath-Brown.

Lemma 3.9. *For $\sigma \ge \frac{1}{2}$, we have*

$$\sideset{}{^\flat}\sum_{|d|\le X} |L(\sigma+it, f\otimes\chi_d)| \ll_\varepsilon X^{1+\varepsilon}(1+|t|)^{\frac{1}{2}+\varepsilon},$$

where $\sum^\flat$ *denotes the summation over fundamental discriminants.*

Proof. It follows from [99, Corollary 2.5] and the Cauchy-Schwarz inequality. $\square$

Lemma 3.10. *(1) The function* $Z(\frac{1}{2}+\gamma,\ell)$ *defined in* (3.5) *is analytic and absolutely convergent in the region* $\mathrm{Re}(\gamma) > -\frac{1}{4}$.

(2) Let $\mathrm{Re}(\gamma) > 0$*. Then for any integer* $N \geq 0$*, we have*

$$L(1+2\gamma, \mathrm{sym}^2 f)Z(\tfrac{1}{2}+\gamma,\ell) = L(1+2\gamma, \mathrm{sym}^2 f)\frac{\zeta(2^{N+1}+2^{N+2}\gamma)}{\zeta(2+4\gamma)}Z^N(\tfrac{1}{2}+\gamma,\ell).$$

Here $Z^N(\frac{1}{2}+\gamma,\ell)$ *is analytic and is bounded by* ℓ^ε *in the region* $\mathrm{Re}(\gamma) > \max(-\frac{1}{2}+\varepsilon, -\frac{1}{2}+\frac{1}{2^{N+2}}+\frac{\varepsilon}{2^{N+2}})$*. Note* $Z^0 = Z$*.*

Proof. Recall the definition of $L(1+2\gamma, \mathrm{sym}^2 f)Z(\frac{1}{2}+\gamma,\ell)$ in (3.5). We see that in the region $\mathrm{Re}(\gamma) > 0$, for $(p, 2\ell) = 1$,

$$Z_p(\tfrac{1}{2}+\gamma,\ell) = \left(1 - \frac{\lambda_f(p)}{p^{\frac{1}{2}+\gamma}} + \frac{1}{p^{1+2\gamma}}\right)^{-1}\left(1 + \frac{\lambda_f(p)}{p^{\frac{1}{2}+\gamma}} + \frac{1}{p^{1+2\gamma}}\right)^{-1}\left(1 + \frac{1}{p^{1+2\gamma}} + P(\gamma)\right), \quad (3.10)$$

where

$$P(\gamma) = -\frac{1}{p+1}\left[1 + \frac{1}{p^{1+2\gamma}} - \left(1 - \frac{\lambda_f(p)}{p^{\frac{1}{2}+\gamma}} + \frac{1}{p^{1+2\gamma}}\right)\left(1 + \frac{\lambda_f(p)}{p^{\frac{1}{2}+\gamma}} + \frac{1}{p^{1+2\gamma}}\right)\right]$$

We see that $P(\gamma) = O(\frac{1}{p^{1+\varepsilon}})$ when $\mathrm{Re}(\gamma) > -\frac{1}{2}+\varepsilon$. In (3.10), we factor out $1+\frac{1}{p^{1+2\gamma}}$ by

$$\begin{aligned} 1 + \frac{1}{p^{1+2\gamma}} + P(\gamma) &= \frac{1}{1-\frac{1}{p^{1+2\gamma}}}\left(1 - \frac{1}{p^{1+2\gamma}}\right)\left(1 + \frac{1}{p^{1+2\gamma}} + P(\gamma)\right) \\ &= \frac{1}{1-\frac{1}{p^{1+2\gamma}}}\left(1 - \frac{1}{p^{2+2^2\gamma}} + P(\gamma) - \frac{1}{p^{1+2\gamma}}P(\gamma)\right) \end{aligned} \quad (3.11)$$

Note that $P(\gamma) - \frac{1}{p^{1+2\gamma}}P(\gamma) = O(\frac{1}{p^{1+\varepsilon}})$ when $\mathrm{Re}(\gamma) > -\frac{1}{2}+\varepsilon$. It is clear that the expression $1 - \frac{1}{p^{2+2^2\gamma}} + P(\gamma) - \frac{1}{p^{1+2\gamma}}P(\gamma)$ is exactly the Euler factor of $Z(\frac{1}{2}+\gamma,\ell)$ in the case of $(p, 2\ell) = 1$. This proves that $Z(\frac{1}{2}+\gamma,\ell)$ is analytic and absolutely convergent in the region $\mathrm{Re}(\gamma) > -\frac{1}{4}$.

Factoring out $1-\frac{1}{p^{2+2^2\gamma}}$ in (3.11), we obtain

$$\begin{aligned}
&1-\frac{1}{p^{2+2^2\gamma}}+P(\gamma)-\frac{1}{p^{1+2\gamma}}P(\gamma)\\
&=\frac{1}{1+\frac{1}{p^{2+2^2\gamma}}}\left(1+\frac{1}{p^{2+2^2\gamma}}\right)\left(1-\frac{1}{p^{2+2^2\gamma}}+P(\gamma)-\frac{1}{p^{1+2\gamma}}P(\gamma)\right)\\
&=\frac{1}{1+\frac{1}{p^{2+2^2\gamma}}}\left(1-\frac{1}{p^{2^2+2^3\gamma}}+P(\gamma)-\frac{1}{p^{1+2\gamma}}P(\gamma)+\frac{1}{p^{2+2\gamma}}P(\gamma)-\frac{1}{p^{3+6\gamma}}P(\gamma)\right)
\end{aligned}$$

Note that the terms with $P(\gamma)$ are $O(\frac{1}{p^{1+\varepsilon}})$ when $\mathrm{Re}(\gamma)>-\frac{1}{2}+\varepsilon$.

Repeating the above process continuously we can get

$$\begin{aligned}
&\frac{1}{1+\frac{1}{p^{2+2^2\gamma}}}\left(1-\frac{1}{p^{2^2+2^3\gamma}}+P(\gamma)-\frac{1}{p^{1+2\gamma}}P(\gamma)+\frac{1}{p^{2+2\gamma}}P(\gamma)-\frac{1}{p^{3+6\gamma}}P(\gamma)\right)\\
&=\prod_{m=1}^{N}\frac{1}{1+\frac{1}{p^{2^m+2^{m+1}\gamma}}}\left(1-\frac{1}{p^{2^{N+1}+2^{N+2}\gamma}}+Q(\gamma)\right)\\
&=\frac{1-\frac{1}{p^{2+4\gamma}}}{1-\frac{1}{p^{2^{N+1}+2^{N+2}\gamma}}}\left(1-\frac{1}{p^{2^{N+1}+2^{N+2}\gamma}}+Q(\gamma)\right),
\end{aligned}$$

where $Q(\gamma)$ is a certain expression satisfying $Q(\gamma)=O_{N,\varepsilon}(\frac{1}{p^{1+\varepsilon}})$ when $\mathrm{Re}(\gamma)>-\frac{1}{2}+\varepsilon$.

Note the expression $1-\frac{1}{p^{2^{N+1}+2^{N+2}\gamma}}+Q(\gamma)$ is the Euler factor of $Z^N(\frac{1}{2}+\gamma,\ell)$ when $(p,2\ell)=1$. For $\mathrm{Re}(\gamma)>\max(-\frac{1}{2}+\varepsilon,-\frac{1}{2}+\frac{1}{2^{N+2}}+\frac{\varepsilon}{2^{N+2}})$, we have

$$\prod_{(p,2\ell)=1}\left(1-\frac{1}{p^{2^{N+1}+2^{N+2}\gamma}}+Q(\gamma)\right)\ll 1.$$

In addition, it is easy to derive the Euler factors of $Z^N(\frac{1}{2}+\gamma,\ell)$ corresponding to $p|2\ell$ and to prove that they contribute $\ll \ell^{\varepsilon}$, as desired. □

3.3 Setup of the problem.

By (3.3) and Lemma 3.5, we get

$$M(\alpha,\ell)=M^{+}(\alpha,\ell)+M^{-}(\alpha,\ell),$$

where

$$M^+(\alpha,\ell) := \sum_{(d,2)=1}^{*} \Phi\left(\frac{d}{X}\right)\sum_{n=1}^{\infty}\frac{\lambda_f(n)\chi_{8d}(\ell n)}{n^{\frac{1}{2}+\alpha}}\omega_\alpha\left(\frac{n}{8d}\right),$$
$$M^-(\alpha,\ell) := i^\kappa \sum_{(d,2)=1}^{*} \Phi\left(\frac{d}{X}\right) X_{\alpha,8d}\sum_{n=1}^{\infty}\frac{\lambda_f(n)\chi_{8d}(\ell n)}{n^{\frac{1}{2}-\alpha}}\omega_{-\alpha}\left(\frac{n}{8d}\right).$$

Remark 3.11. Define $\Phi_z(x) := x^z\Phi(x)$. We then can write

$$M^-(\alpha,\ell) = i^\kappa\gamma_\alpha X^{-2\alpha}\sum_{(d,2)=1}^{*} \Phi_{-2\alpha}\left(\frac{d}{X}\right)\sum_{n=1}^{\infty}\frac{\lambda_f(n)\chi_{8d}(\ell n)}{n^{\frac{1}{2}-\alpha}}\omega_{-\alpha}\left(\frac{n}{8d}\right),$$

where

$$\gamma_\alpha := \left(\frac{8}{2\pi}\right)^{-2\alpha}\frac{\Gamma(\frac{\kappa}{2}-\alpha)}{\Gamma(\frac{\kappa}{2}+\alpha)}. \tag{3.12}$$

Notice that $M^-(\alpha,\ell)$ is equal to $i^\kappa\gamma_\alpha X^{-2\alpha}M^+(-\alpha,\ell)$ with $\Phi_{-2\alpha}$ in place of Φ. Thus we just need to evaluate $M^+(\alpha,\ell)$, and the results for $M^-(\alpha,\ell)$ can be obtained immediately.

The square-free condition in $M^+(\alpha,\ell)$ can be removed by using Möbius inversion. This gives

$$\begin{aligned} M^+(\alpha,\ell) &= \sum_{(d,2)=1}\sum_{a^2|d}\mu(a)\Phi\left(\frac{d}{X}\right)\sum_{n=1}^{\infty}\frac{\lambda_f(n)\chi_{8d}(\ell n)}{n^{\frac{1}{2}+\alpha}}\omega_\alpha\left(\frac{n}{8d}\right)\\ &= \sum_{(a,2\ell)=1}\mu(a)\sum_{(d,2)=1}\sum_{(n,a)=1}\frac{\lambda_f(n)\chi_{8d}(\ell n)}{n^{\frac{1}{2}+\alpha}}\omega_\alpha\left(\frac{n}{8a^2d}\right)\Phi\left(\frac{a^2d}{X}\right)\\ &=: M_N^+(\alpha,\ell)+M_R^+(\alpha,\ell), \end{aligned} \tag{3.13}$$

where $M_N^+(\alpha,\ell)$ and $M_R^+(\alpha,\ell)$ denote the sums over $a \le Y$ and $a > Y$, respectively. Here $Y(\le X)$ is a parameter chosen later.

We use the Poisson summation formula to split $M_N^+(\alpha,\ell)$. Using Lemma 3.7 on the summation over d in $M_N^+(\alpha,\ell)$, we derive

$$\begin{aligned} M_N^+(\alpha,\ell) = \frac{X}{2}\sum_{\substack{(a,2\ell)=1\\ a\le Y}}\frac{\mu(a)}{a^2}\sum_{(n,2a)=1}\frac{\lambda_f(n)}{n^{\frac{1}{2}+\alpha}}\sum_{k\in\mathbb{Z}}(-1)^k\frac{G_k(\ell n)}{\ell n}\\ \times\int_{-\infty}^{\infty}(\cos+\sin)\left(\frac{2\pi kxX}{2n\ell a^2}\right)\omega_\alpha\left(\frac{n}{8xX}\right)\Phi(x)dx. \end{aligned} \tag{3.14}$$

Let $M_N^+(\alpha,\ell,k=0)$ denote the term $k=0$ above, and let $M_N^+(\alpha,\ell,k\neq 0)$ denote the remaining terms. We call $M_N^+(\alpha,\ell,k=0)$ diagonal terms and $M_N^+(\alpha,\ell,k\neq 0)$ off-diagonal terms.

On the other hand, we convert $M_R^+(\alpha,\ell)$ in (3.13) back to the summation over square-free integers, and then appeal to the induction hypothesis (3.4). To see this, recall that

$$M_R^+(\alpha,\ell)=\sum_{\substack{(a,2\ell)=1\\ a>Y}}\mu(a)\sum_{(d,2)=1}\sum_{(n,a)=1}\frac{\lambda_f(n)\chi_{8d}(\ell n)}{n^{\frac{1}{2}+\alpha}}\omega_\alpha\left(\frac{n}{8a^2d}\right)\Phi\left(\frac{a^2d}{X}\right).$$

Write $d=eb^2$, where e is square-free and b is positive. Group terms according to $c=ab$. It follows that

$$\begin{aligned}
&M_R^+(\alpha,\ell)\\
&=\sum_{(c,2\ell)=1}\sum_{\substack{a>Y\\ a|c}}\mu(a)\sideset{}{^*}\sum_{(e,2)=1}\sum_{(n,2c)=1}\frac{\lambda_f(n)\chi_{8e}(\ell n)}{n^{\frac{1}{2}+\alpha}}\omega_\alpha\left(\frac{n}{8c^2e}\right)\Phi\left(\frac{c^2e}{X}\right)\\
&=\sum_{(c,2\ell)=1}\sum_{\substack{a>Y\\ a|c}}\mu(a)\frac{1}{2\pi i}\int_{(1)}\sideset{}{^*}\sum_{(e,2)=1}\chi_{8e}(\ell)\Phi_s\left(\frac{e}{X'}\right)L_c(\tfrac{1}{2}+\alpha+s,f\otimes\chi_{8e})X^s8^sg_\alpha(s)\frac{G(s)}{s}ds,
\end{aligned}\tag{3.15}$$

where $X':=\frac{X}{c^2}$. Here $L_c(s,f\otimes\chi_{8e})$, $\mathrm{Re}(s)>1$ is given by the Euler product of $L(s,f\otimes\chi_{8e})$ with omitting all prime factors of c. In the first equation, the condition $(c,\ell)=1$ is due to $\chi_{8ed^2}(\ell)=0$ if $(d,\ell)\neq 1$. We use the following lemma to change $L_c(\frac{1}{2}+\alpha+s,f\otimes\chi_{8e})$ back to the form of $L(\frac{1}{2}+\alpha+s,f\otimes\chi_{8e})$. It is similar to Lemma 9 of Kowalski and Michel [67].

Lemma 3.12. *Let d be a fundamental discriminant. Then*

$$\prod_{p|c}\left(1-\frac{\lambda_f(p)\chi_d(p)}{p^s}+\frac{\chi_d(p^2)}{p^{2s}}\right)=\sum_{m|c}\sum_{n|c}\mu(m)\mu(mn)^2\lambda_f(m)\chi_d(m)\chi_d(n^2)\frac{1}{m^s}\frac{1}{n^{2s}}.\tag{3.16}$$

Proof. Note that the summand on the right-hand side of (3.16) is jointly multiplicative. Thus

$$\begin{aligned}
&\sum_{m|c}\sum_{n|c}\mu(m)\mu(mn)^2\lambda_f(m)\chi_d(m)\chi_d(n^2)\frac{1}{m^s}\frac{1}{n^{2s}} \\
&=\prod_{p|c}\sum_{0\le r_1,r_2\le \mathrm{order}_p(c)}\mu(p^{r_1})\mu(p^{r_1+r_2})^2\lambda_f(p^{r_1})\chi_d(p^{r_1})\chi_d(p^{2r_2})\frac{1}{p^{r_1 s}}\frac{1}{p^{2r_2 s}} \\
&=\prod_{p|c}\left(1-\frac{\lambda_f(p)\chi_d(p)}{p^s}+\frac{\chi_d(p^2)}{p^{2s}}\right),
\end{aligned}$$

as desired. $\square$

It follows from (3.15) and Lemma 3.12 that

$$\begin{aligned}
M_R^+(\alpha,\ell) = &\sum_{(c,2\ell)=1}\sum_{\substack{a>Y\\ a|c}}\mu(a)\sum_{r_1|c}\frac{\mu(r_1)\lambda_f(r_1)}{r_1^{\frac{1}{2}+\alpha}}\sum_{r_2|c}\frac{\mu(r_1r_2)^2}{r_2^{1+2\alpha}}\frac{1}{2\pi i}\int_{(\frac{1}{\log X})} \\
&\times\sideset{}{^*}\sum_{(e,2)=1}\chi_{8e}(\ell_1 r_1\ell_2^2 r_2^2)\Phi_s\left(\frac{e}{X'}\right)L(\tfrac{1}{2}+\alpha+s,f\otimes\chi_{8e})\frac{1}{r_1^s r_2^{2s}}X^s 8^s g_\alpha(s)\frac{G(s)}{s}ds.
\end{aligned}$$

We can truncate the above integral for $|\mathrm{Im}(s)| \ll (\log X)^2$ with an error $O(1)$ by the rapid decay of $|G(s)|$ as $|\mathrm{Im}(s)| \to \infty$. For $|\mathrm{Im}(s)| \ll (\log X)^2$, we are allowed to employ the inductive hypothesis (3.4). Hence we have

$$M_R^+(\alpha,\ell) = M_{R,1}^+(\alpha,\ell) + M_{R,2}^+(\alpha,\ell) + M_{R,3}^+(\alpha,\ell) + O(1),$$

where

$$M^+_{R,1}(\alpha,\ell) := \frac{1}{\ell_1^{\frac{1}{2}+\alpha}} \sum_{(c,2\ell)=1} \sum_{\substack{a>Y\\ a|c}} \mu(a) \sum_{r_1|c} \frac{\mu(r_1)\lambda_f(r_1)}{r_1^{1+2\alpha}} \sum_{r_2|c} \frac{\mu(r_1r_2)^2}{r_2^{1+2\alpha}} \frac{1}{2\pi i} \int_{(\frac{1}{\log X})} \frac{4X^{1+s}\tilde{\Phi}(1+s)}{\pi^2 c^2}$$
$$\times L(1+2\alpha+2s, \mathrm{sym}^2 f) Z(\tfrac{1}{2}+\alpha+s, \ell r_1 r_2^2) \frac{8^s}{\ell_1^s r_1^{2s} r_2^{2s}} g_\alpha(s) \frac{G(s)}{s} ds. \tag{3.17}$$

$$M^+_{R,2}(\alpha,\ell) := \frac{i^\kappa}{\ell_1^{\frac{1}{2}-\alpha}} \sum_{(c,2\ell)=1} \sum_{\substack{a>Y\\ a|c}} \mu(a) \sum_{r_1|c} \frac{\mu(r_1)\lambda_f(r_1)}{r_1} \sum_{r_2|c} \frac{\mu(r_1r_2)^2}{r_2^{1+2\alpha}} \frac{1}{2\pi i} \int_{(\frac{1}{\log X})} \frac{4X^{1-2\alpha-s}\gamma_{\alpha+s}}{\pi^2 c^{2-4\alpha-4s}}$$
$$\times \tilde{\Phi}(1-2\alpha-s) L(1-2\alpha-2s, \mathrm{sym}^2 f) Z(\tfrac{1}{2}-\alpha-s, \ell r_1 r_2^2) \frac{\ell_1^s 8^s}{r_2^{2s}} g_\alpha(s) \frac{G(s)}{s} ds. \tag{3.18}$$

$$M^+_{R,3}(\alpha,\ell) := \sum_{(c,2\ell)=1} \sum_{\substack{a>Y\\ a|c}} \mu(a) \sum_{r_1|c} \frac{\mu(r_1)\lambda_f(r_1)}{r_1^{\frac{1}{2}+\alpha}} \sum_{r_2|c} \frac{\mu(r_1r_2)^2}{r_2^{1+2\alpha}} \frac{1}{2\pi i} \int_{\substack{s=\frac{1}{\log X}+it\\ |t|\ll(\log X)^2}}$$
$$\times O\left((\ell r_1 r_2^2)^{\frac{1}{2}+\frac{\varepsilon}{100}} X'^{h+\varepsilon} \right) \frac{1}{r_1^s r_2^{2s}} X^s 8^s g_\alpha(s) \frac{G(s)}{s} ds. \tag{3.19}$$

Note that in (3.17) and (3.18) we have extended the range of integrals from $|\mathrm{Im}(s)| \ll (\log X)^2$ to the vertical line $\mathrm{Re}(s) = \frac{1}{\log X}$ with an error $O(1)$.

Now we have separated $M^+(\alpha,\ell)$ into several parts. In summary, we have obtained

$$M(\alpha,\ell) = M^+(\alpha,\ell) + M^-(\alpha,\ell), \tag{3.20}$$

and

$$\begin{aligned} M^+(\alpha,\ell) &= M^+_N(\alpha,\ell) + M^+_R(\alpha,\ell), \\ M^+_N(\alpha,\ell) &= M^+_N(\alpha,\ell,k=0) + M^+_N(\alpha,\ell,k\neq 0), \\ M^+_R(\alpha,\ell) &= M^+_{R,1}(\alpha,\ell) + M^+_{R,2}(\alpha,\ell) + M^+_{R,3}(\alpha,\ell) + O(1). \end{aligned} \tag{3.21}$$

We can also split $M^-(\alpha,\ell)$ similarly using Remark 3.11. We will evaluate $M^+_N(\alpha,\ell,k=0)$, $M^+_N(\alpha,\ell,k\neq 0)$, respectively, in Sections 3.4, 3.5. The analysis for $M^+_{R,1}(\alpha,\ell)$, $M^+_{R,2}(\alpha,\ell)$ and $M^+_{R,3}(\alpha,\ell)$ will be done in Section 3.6. We complete the proof of Theorem 3.4 in Section 3.7.

3.4 Evaluation of $M_N^+(\alpha, \ell, k = 0)$.

Recall $M_N^+(\alpha, \ell, k = 0)$ in (3.14). By the definition of $G_k(n)$ in Lemma 3.7, we know $G_0(n) = \phi(n)$ if $n = \square$, and $G_0(n) = 0$ otherwise. Here $n = \square$ means n is a square number. Hence

$$\begin{aligned} M_N^+(\alpha, \ell, k = 0) &= \frac{X}{2} \sum_{\substack{(a,2\ell)=1 \\ a \le Y}} \frac{\mu(a)}{a^2} \sum_{\substack{(n,2a)=1 \\ \ell n = \square}} \frac{\lambda_f(n)}{n^{\frac{1}{2}+\alpha}} \frac{\phi(\ell n)}{\ell n} \int_{-\infty}^{\infty} \omega_\alpha\left(\frac{n}{8xX}\right) \Phi(x) dx \\ &= \frac{X}{2} \sum_{\substack{(a,2\ell)=1 \\ a \le Y}} \frac{\mu(a)}{a^2} \frac{1}{2\pi i} \int_{(1)} \tilde{\Phi}(s+1) Z_1(\tfrac{1}{2} + \alpha + s, a, \ell) 8^s X^s g_\alpha(s) \frac{G(s)}{s} ds, \quad (3.22) \end{aligned}$$

where

$$Z_1(\tfrac{1}{2} + \gamma, a, \ell) := \sum_{\substack{(n,2a)=1 \\ \ell n = \square}} \frac{\lambda_f(n)}{n^{\frac{1}{2}+\gamma}} \frac{\phi(\ell n)}{\ell n}.$$

For simplicity we use $E_1(\gamma; p), E_2(\gamma; p), E_3(\gamma; p)$ to denote the three Euler factors in (3.6), respectively. For $\mathrm{Re}(\gamma) > 0$, write

$$\mathcal{A}(\gamma, a, \ell) := \prod_{p \mid \ell_1} E_1(\gamma; p) \prod_{\substack{p \nmid \ell_1 \\ p \mid \ell_2}} E_2(\gamma; p) \prod_{(p,2a\ell)=1} \left(E_3(\gamma; p) + \frac{1}{p^2 - 1} \right). \quad (3.23)$$

Lemma 3.13. *For* $\mathrm{Re}(\gamma) > 0$*, we have*

$$Z_1(\tfrac{1}{2} + \gamma, a, \ell) = \frac{1}{\ell_1^{\frac{1}{2}+\gamma} \zeta_{2a}(2)} \mathcal{A}(\gamma, a, \ell).$$

Proof. For each prime p, let b_1, b_2 be integers such that $p^{b_1} || \ell_1$ and $p^{b_2} || \ell_2$. We change the variable $n \to \ell_1 n^2$. We can do this because $\ell n = \square$ implies $\ell_1 n = \square$. It gives

$$\begin{aligned} Z_1(\tfrac{1}{2} + \gamma, a, \ell) &= \frac{1}{\ell_1^{\frac{1}{2}+\gamma}} \sum_{(n,2a)=1} \frac{\lambda_f(\ell_1 n^2)}{n^{1+2\gamma}} \prod_{p \mid \ell_1 \ell_2 n} \left(1 - \frac{1}{p}\right) \\ &= \frac{1}{\ell_1^{\frac{1}{2}+\gamma}} \prod_{(p,2a)=1} \sum_{r=0}^{\infty} \frac{\lambda_f(p^{b_1+2r})}{p^{(1+2\gamma)r}} \prod_{q \mid p^{b_1+b_2+r}} \left(1 - \frac{1}{q}\right). \end{aligned}$$

In the following, we consider three cases for the sum over r above.

If $(p,2a\ell)=1$, then $b_1=b_2=0$. Thus

$$\begin{aligned}&\sum_{r=0}^{\infty}\frac{\lambda_f(p^{2r})}{p^{(1+2\gamma)r}}\prod_{q|p^r}\left(1-\frac{1}{q}\right)=1+\left(1-\frac{1}{p}\right)\sum_{r=1}^{\infty}\frac{\lambda_f(p^{2r})}{p^{(\frac{1}{2}+\gamma)2r}}\\&=1+\left(1-\frac{1}{p}\right)\left[\frac{1}{2}\left(1-\frac{\lambda_f(p)}{p^{\frac{1}{2}+\gamma}}+\frac{1}{p^{1+2\gamma}}\right)^{-1}+\frac{1}{2}\left(1+\frac{\lambda_f(p)}{p^{\frac{1}{2}+\gamma}}+\frac{1}{p^{1+2\gamma}}\right)^{-1}-1\right].\end{aligned}\tag{3.24}$$

If $(p,2a)=1, p|\ell_1$, then $b_1=1$ since ℓ_1 is square-free. Hence

$$\begin{aligned}&\sum_{r=0}^{\infty}\frac{\lambda_f(p^{1+2r})}{p^{(1+2\gamma)r}}\left(1-\frac{1}{p}\right)=p^{\frac{1}{2}+\gamma}\left(1-\frac{1}{p}\right)\sum_{r=0}^{\infty}\frac{\lambda_f(p^{1+2r})}{p^{(\frac{1}{2}+\gamma)(1+2r)}}\\&=p^{\frac{1}{2}+\gamma}\left(1-\frac{1}{p}\right)\left[\frac{1}{2}\left(1-\frac{\lambda_f(p)}{p^{\frac{1}{2}+\gamma}}+\frac{1}{p^{1+2\gamma}}\right)^{-1}-\frac{1}{2}\left(1+\frac{\lambda_f(p)}{p^{\frac{1}{2}+\gamma}}+\frac{1}{p^{1+2\gamma}}\right)^{-1}\right].\end{aligned}\tag{3.25}$$

If $(p,2a)=1, p\nmid\ell_1, p|\ell_2$, then $b_1=0, b_2\geq 1$. This gives

$$\sum_{r=0}^{\infty}\frac{\lambda_f(p^{2r})}{p^{(1+2\gamma)r}}\left(1-\frac{1}{p}\right)=\left(1-\frac{1}{p}\right)\left[\frac{1}{2}\left(1-\frac{\lambda_f(p)}{p^{\frac{1}{2}+\gamma}}+\frac{1}{p^{1+2\gamma}}\right)^{-1}+\frac{1}{2}\left(1+\frac{\lambda_f(p)}{p^{\frac{1}{2}+\gamma}}+\frac{1}{p^{1+2\gamma}}\right)^{-1}\right].\tag{3.26}$$

We then complete the proof by taking out the factor $1-\frac{1}{p^2}$ from (3.24), (3.25) and (3.26). □

It follows from (3.22) and Lemma 3.13 that

Lemma 3.14. *We have*

$$\begin{aligned}&M_N^+(\alpha,\ell,k=0)\\&=\frac{4X}{\pi^2\ell_1^{\frac{1}{2}+\alpha}}\sum_{\substack{(a,2\ell)=1\\ a\leq Y}}\frac{\mu(a)}{a^2}\prod_{p|a}\frac{1}{1-\frac{1}{p^2}}\frac{1}{2\pi i}\int_{(1)}\tilde{\Phi}(s+1)\mathcal{A}(s+\alpha,a,\ell)\frac{1}{\ell_1^s}8^sX^sg_\alpha(s)\frac{G(s)}{s}ds.\end{aligned}$$

3.5 Upper bound for $M_N^+(\alpha, \ell, k \neq 0)$.

We shall prove an upper bound for $M_N^+(\alpha, \ell, k \neq 0)$ in this section. Recall in (3.14) that

$$M_N^+(\alpha, \ell, k \neq 0) = \frac{X}{2} \sum_{\substack{(a,2\ell)=1 \\ a \leq Y}} \frac{\mu(a)}{a^2} \sum_{(n,2a)=1} \frac{\lambda_f(n)}{n^{\frac{1}{2}+\alpha}} \sum_{k \neq 0} (-1)^k \frac{G_k(\ell n)}{\ell n}$$
$$\times \int_{-\infty}^{\infty} (\cos + \sin) \left(\frac{2\pi k x X}{2n\ell a^2} \right) \omega_\alpha \left(\frac{n}{8xX} \right) \Phi(x) dx. \quad (3.27)$$

Lemma 3.15. *Let $f(x)$ be a smooth function on $\mathbb{R}_{>0}$. Suppose f decays rapidly as $x \to \infty$, and $f^{(n)}(x)$ converges as $x \to 0^+$ for every $n \in \mathbb{Z}_{\geq 0}$. Then we have*

$$\int_0^\infty f(x) \cos(2\pi x y) dx = \frac{1}{2\pi i} \int_{(\frac{1}{2})} \tilde{f}(1-u) \Gamma(u) \cos \left(\frac{\operatorname{sgn}(y) \pi u}{2} \right) (2\pi |y|)^{-u} du, \quad (3.28)$$

In addition, the equation (3.28) *is also valid when* cos *is replaced by* sin.

Proof. See [99, Section 3.3]. ☐

By Lemma 3.15, the integral in (3.27) is

$$\frac{1}{2\pi i} \int_{(\frac{1}{2})} X^{-u} \Gamma(u) (\cos + \operatorname{sgn}(k) \sin) \left(\frac{\pi u}{2} \right) \left(\frac{\ell n a^2}{\pi |k|} \right)^u \int_0^\infty \Phi(x) x^{-u} \omega_\alpha \left(\frac{n}{8xX} \right) dx du$$
$$= \frac{1}{(2\pi i)^2} \int_{(\frac{1}{2})} \int_{(1)}$$
$$\times \tilde{\Phi}(1+s-u) X^{-u+s} \Gamma(u) (\cos + \operatorname{sgn}(k) \sin) \left(\frac{\pi u}{2} \right) \left(\frac{\ell a^2}{\pi |k|} \right)^u 8^s \frac{1}{n^{s-u}} g_\alpha(s) \frac{G(s)}{s} ds du.$$

Move the contour of the above integral to $\operatorname{Re}(u) = \frac{1}{2} + \varepsilon, \operatorname{Re}(s) = \frac{1}{2} + 2\varepsilon$, and change the variable $s' = s - u$. This implies

$$\frac{1}{(2\pi i)^2} \int_{(\frac{1}{2}+\varepsilon)} \int_{(\varepsilon)} \tilde{\Phi}(1+s) X^s \Gamma(u) (\cos + \operatorname{sgn}(k) \sin) \left(\frac{\pi u}{2} \right) \left(\frac{\ell a^2}{\pi |k|} \right)^u$$
$$\times 8^{s+u} \frac{1}{n^s} g_\alpha(s+u) \frac{G(s+u)}{s+u} ds du.$$

Together with (3.27), it follows that

$$\begin{aligned}
&M_N^+(\alpha, \ell, k \neq 0) \\
&= \frac{X}{2\ell} \sum_{\substack{(a,2\ell)=1 \\ a \leq Y}} \frac{\mu(a)}{a^2} \sum_{k \neq 0} (-1)^k \frac{1}{(2\pi i)^2} \int_{(\frac{1}{2}+\varepsilon)} \int_{(\varepsilon)} \tilde{\Phi}(1+s) X^s \Gamma(u)(\cos + \operatorname{sgn}(k) \sin)\left(\frac{\pi u}{2}\right)\left(\frac{\ell a^2}{\pi |k|}\right)^u \\
&\times \; 8^{s+u} g_\alpha(s+u) \frac{G(s+u)}{s+u} Z_2(\tfrac{1}{2}+\alpha+s, a, k, \ell) ds du,
\end{aligned} \tag{3.29}$$

where

$$Z_2(\gamma, a, k, \ell) := \sum_{(n,2a)=1} \frac{\lambda_f(n)}{n^\gamma} \frac{G_k(\ell n)}{n}.$$

Lemma 3.16. *Write $4k = k_1 k_2^2$, where k_1 is a fundamental discriminant (possibly $k_1 = 1$) and k_2 is positive. Then for $\mathrm{Re}(\gamma) > \frac{1}{2}$, we have*

$$Z_2(\gamma, a, k, \ell) = L(\tfrac{1}{2}+\gamma, f \otimes \chi_{k_1}) Z_3(\gamma, a, k, \ell). \tag{3.30}$$

Here

$$Z_3(\gamma, a, k, \ell) := \prod_p Z_{3,p}(\gamma, a, k, \ell),$$

where

$$\begin{aligned}
Z_{3,p}(\gamma, a, k, \ell) &:= 1 - \frac{\lambda_f(p)\chi_{k_1}(p)}{p^{\frac{1}{2}+\gamma}} + \frac{\chi_{k_1}(p)^2}{p^{1+2\gamma}} \quad \textit{if } p|2a, \quad \textit{and} \\
Z_{3,p}(\gamma, a, k, \ell) &:= \left(1 - \frac{\lambda_f(p)\chi_{k_1}(p)}{p^{\frac{1}{2}+\gamma}} + \frac{\chi_{k_1}(p)^2}{p^{1+2\gamma}}\right) \sum_{r=0}^\infty \frac{\lambda_f(p^r)}{p^{r\gamma}} \frac{G_k(p^{r+\mathrm{ord}_p(\ell)})}{p^r} \quad \textit{if } p \nmid 2a.
\end{aligned}$$

Moreover, $Z_3(\gamma, a, k, \ell)$ is analytic in the region $\mathrm{Re}(\gamma) > 0$ and is uniformly bounded by $a^\varepsilon |k|^\varepsilon \ell^{\frac{1}{2}+\varepsilon}(\ell, k_2^2)^{\frac{1}{2}}$ in the region $\mathrm{Re}(\gamma) > \frac{\varepsilon}{2}$.

Proof. The proof is similar to [97, Lemma 5.3]. Note that $G_k(n)$ is multiplicative. Hence

$$Z_2(\gamma, a, k, \ell) = \prod_{(p,2a)=1} \sum_{r=0} \frac{\lambda_f(p^r)}{p^{r\gamma}} \frac{G_k(p^{r+\mathrm{ord}_p(\ell)})}{p^r}.$$

Then the identity (3.30) follows directly from a comparison of both sides.

When $p \nmid 2ak\ell$, by the definition of $Z_{3,p}(\gamma, a, k, \ell)$ and Lemma 3.8, we know

$$\begin{aligned} Z_{3,p}(\gamma, a, k, \ell) &= \left(1 - \frac{\lambda_f(p)\chi_{k_1}(p)}{p^{\frac{1}{2}+\gamma}} + \frac{\chi_{k_1}(p)^2}{p^{1+2\gamma}}\right)\left(1 + \frac{\lambda_f(p)\chi_{k_1}(p)}{p^{\frac{1}{2}+\gamma}}\right) \\ &= 1 + \frac{\chi_{k_1}(p)^2}{p^{1+2\gamma}} - \frac{\lambda_f(p)^2\chi_{k_1}(p)^2}{p^{1+2\gamma}} + \frac{\lambda_f(p)\chi_{k_1}(p)^3}{p^{\frac{3}{2}+3\gamma}}. \end{aligned} \tag{3.31}$$

Hence $Z_3(\gamma, a, k, \ell)$ is analytic in the region $\mathrm{Re}(\gamma) > 0$.

It remains to prove the upper bound of $Z_3(\gamma, a, k, \ell)$. For $p \nmid 2ak\ell$, by (3.31) and the fact $|\lambda_f(n)| \leq \tau(n)$, we get

$$\prod_{(p,2ak)=1} Z_{3,p}(\gamma, a, k, \ell) \ll 1. \tag{3.32}$$

For $p|2a$, we have

$$\prod_{p|2a} Z_{3,p}(\gamma, a, k, \ell) \ll a^{\varepsilon}. \tag{3.33}$$

For $p \nmid 2a, p|k\ell$, we let $p^{b_1}||k, p^{b_2}||\ell$. We can assume $b_2 \leq b_1+1$ since $G_k(p^{r+b_2}) = 0$ otherwise (by Lemma 3.8). We claim $Z_{3,p}(\gamma, a, k, \ell) \ll (1+b_1+b_2)^2 p^{\min(b_2, \lfloor\frac{b_1}{2}\rfloor + \frac{b_2}{2})}$. In fact, the trivial bound $G_k(p^n) \leq p^n$ gives $Z_{3,p}(\gamma, a, k, \ell) \ll (1+b_1+b_2)^2 p^{b_2}$, which proves the case $b_2 \leq \lfloor\frac{b_1}{2}\rfloor + \frac{b_2}{2}$. The remaining cases include: b_1 even and $b_2 = b_1+1$, or b_1 odd and $b_2 = b_1$, or b_1 odd and $b_2 = b_1+1$. For b_1 even and $b_2 = b_1+1$, by Lemma 3.8, we know $Z_{3,p}(\gamma, a, k, \ell) \ll p^{b_1}\sqrt{p} = p^{\lfloor\frac{b_1}{2}\rfloor + \frac{b_2}{2}}$. The other two cases can be done similarly. This combined with (3.32) and (3.33) gives the upper bound for $Z_3(\gamma, a, k, \ell)$. $\square$

By (3.29) and Lemma 3.16, we have

$$\begin{aligned} &M_N^+(\alpha, \ell, k \neq 0) \\ &= \frac{X}{2\ell}\sum_{\substack{(a,2\ell)=1 \\ a\leq Y}} \frac{\mu(a)}{a^2}\sum_{k\neq 0}(-1)^k \frac{1}{(2\pi i)^2}\int_{(\frac{1}{2}+\varepsilon)}\int_{(\varepsilon)} \tilde{\Phi}(1+s)X^s\Gamma(u)(\cos + \mathrm{sgn}(k)\sin)\left(\frac{\pi u}{2}\right)\left(\frac{\ell a^2}{\pi|k|}\right)^u \\ &\quad \times 8^{s+u} g_\alpha(s+u)\frac{G(s+u)}{s+u} L(1+\alpha+s, f\otimes\chi_{k_1})Z_3(\tfrac{1}{2}+\alpha+s, a, k, \ell)dsdu. \end{aligned}$$

Move the lines of the integral to $\mathrm{Re}(s) = -\frac{1}{2} - \alpha + \varepsilon$, $\mathrm{Re}(u) = 1 + \varepsilon$ without encountering any poles. Together with Lemma 3.9 and Lemma 3.16, it follows that

Lemma 3.17. *We have*

$$M_N^+(\alpha, \ell, k \neq 0) \ll \ell^{\frac{1}{2}+\varepsilon} X^{\frac{1}{2}+\varepsilon} Y.$$

3.6 Evaluation of $M_R^+(\alpha, \ell)$.

In this section we shall simplify $M_{R,1}^+(\alpha,\ell)$, and derive upper bounds for $M_{R,2}^+(\alpha,\ell)$, $M_{R,3}^+(\alpha,\ell)$ by proving the follow lemma.

Lemma 3.18. *We have*

$$\begin{aligned} &M_{R,1}^+(\alpha,\ell) \\ &= \frac{4X}{\pi^2 \ell_1^{\frac{1}{2}+\alpha}} \sum_{\substack{a>Y \\ (a,2\ell)=1}} \frac{\mu(a)}{a^2} \prod_{p|a} \frac{1}{1-\frac{1}{p^2}} \frac{1}{2\pi i} \int_{(1)} \tilde{\Phi}(1+s) \mathcal{A}(\alpha+s, a, \ell) \frac{1}{\ell_1^s} X^s 8^s g_\alpha(s) \frac{G(s)}{s} ds. &(3.34) \\ &M_{R,2}^+(\alpha,\ell) \ll \ell^\varepsilon X^{\frac{1}{2}+\varepsilon} Y. &(3.35) \\ &M_{R,3}^+(\alpha,\ell) \ll \ell^{\frac{1}{2}+\varepsilon} \frac{X^{h+\varepsilon}}{Y^{2h-1}}. &(3.36) \end{aligned}$$

We give a proof for the above lemma in the rest of the section. Recall $M_{R,1}^+(\alpha,\ell)$ in (3.17). By interchanging summations and integrals, we know

$$\begin{aligned} M_{R,1}^+(\alpha,\ell) = \frac{4X}{\pi^2 \ell_1^{\frac{1}{2}+\alpha}} \sum_{\substack{a>Y \\ (a,2\ell)=1}} \mu(a) \frac{1}{2\pi i} \int_{(1)} \tilde{\Phi}(1+s) \sum_{(r_1,2\ell)=1} \frac{\mu(r_1)\lambda_f(r_1)}{r_1^{1+2\alpha+2s}} \sum_{(r_2,2\ell)=1} \frac{\mu(r_1 r_2)^2}{r_2^{1+2\alpha+2s}} \\ \times L(1+2\alpha+2s, \mathrm{sym}^2 f) Z(\tfrac{1}{2}+\alpha+s, \ell r_1 r_2^2) \sum_{\substack{(c,2\ell)=1 \\ a,r_1,r_2|c}} \frac{1}{c^2} \frac{1}{\ell_1^s} X^s 8^s g_\alpha(s) \frac{G(s)}{s} ds. \end{aligned} \tag{3.37}$$

Lemma 3.19. *For* $\mathrm{Re}(\gamma)>0$,

$$\sum_{(r_1,2\ell)=1}\frac{\mu(r_1)\lambda_f(r_1)}{r_1^{1+2\gamma}}\sum_{(r_2,2\ell)=1}\frac{\mu(r_1r_2)^2}{r_2^{1+2\gamma}}\sum_{\substack{(c,2\ell)=1\\ a,r_1,r_2|c}}\frac{1}{c^2}L(1+2\gamma,\mathrm{sym}^2 f)Z(\tfrac{1}{2}+\gamma,\ell r_1r_2^2)$$
$$=\frac{1}{a^2}\prod_{p|a}\frac{1}{1-\frac{1}{p^2}}\mathcal{A}(\gamma,a,\ell). \tag{3.38}$$

Proof. The left-hand side of (3.38) is

$$\sum_{(r_1,2\ell)=1}\frac{\mu(r_1)\lambda_f(r_1)}{r_1^{1+2\gamma}}\sum_{(r_2,2\ell)=1}\frac{\mu(r_1r_2)^2}{r_2^{1+2\gamma}}\sum_{\substack{(c,2\ell)=1\\ a,r_1,r_2|c}}\frac{1}{c^2}\prod_{p|\ell_1r_1}E_1(\gamma;p)\prod_{\substack{p\nmid\ell_1r_1\\ p|\ell_2r_2}}E_2(\gamma;p)\prod_{(p,2\ell r_1r_2)=1}E_3(\gamma;p). \tag{3.39}$$

Note

$$\sum_{\substack{(c,2\ell)=1\\ a,r_1,r_2|c}}\frac{1}{c^2}=\sum_{\substack{[r_1,r_2,a]|c\\ (c,2\ell)=1}}\frac{1}{c^2}=\frac{1}{[r_1,r_2,a]^2}\sum_{(c,2\ell)=1}\frac{1}{c^2}=\frac{1}{a^2}\frac{([r_1,r_2],a)^2}{[r_1,r_2]^2}\zeta_{2\ell}(2). \tag{3.40}$$

We also see that

$$\prod_{\substack{p\nmid\ell_1r_1\\ p|\ell_2r_2}}E_2(\gamma;p)=\prod_{p|\ell_2r_2}E_2(\gamma;p)\prod_{p|(r_1,r_2)}E_2(\gamma;p)^{-1}\prod_{p|(\ell_1,\ell_2)}E_2(\gamma;p)^{-1}. \tag{3.41}$$

Inserting (3.40) and (3.41) into (3.39), the expression (3.39) now is

$$\frac{\zeta_{2\ell}(2)}{a^2}\prod_{p|\ell_1}E_1(\gamma;p)\prod_{\substack{p|\ell_2\\ p\nmid\ell_1}}E_2(\gamma;p)\prod_{(p,2\ell)=1}E_3(\gamma;p)\sum_{(r_1,2\ell)=1}\sum_{(r_2,2\ell)=1}H(r_1,r_2), \tag{3.42}$$

where

$$H(r_1,r_2)$$
$$:=\frac{\mu(r_1)\lambda_f(r_1)}{r_1^{1+2\gamma}}\frac{\mu(r_1r_2)^2}{r_2^{1+2\gamma}}\frac{([r_1,r_2],a)^2}{[r_1,r_2]^2}\prod_{p|r_1}E_1(\gamma;p)\prod_{p|r_2}E_2(\gamma;p)\prod_{p|(r_1,r_2)}E_2(\gamma;p)^{-1}\prod_{p|r_1r_2}E_3(\gamma;p)^{-1}.$$

Clearly $H(r_1,r_2)$ is joint multiplicative. Then

$$\begin{aligned}&\sum_{(r_1,2\ell)=1}\sum_{(r_2,2\ell)=1} H(r_1,r_2)\\ &=\prod_{(p,2\ell)=1}\left(1-\frac{\lambda_f(p)}{p^{1+2\gamma}}\frac{(p,a)^2}{p^2}E_1(\gamma;p)E_3(\gamma;p)^{-1}+\frac{1}{p^{1+2\gamma}}\frac{(p,a)^2}{p^2}E_2(\gamma;p)E_3(\gamma;p)^{-1}\right).\end{aligned}$$

It follows that

$$\begin{aligned}&\prod_{(p,2\ell)=1} E_3(\gamma;p)\cdot\sum_{(r_1,2\ell)=1}\sum_{(r_2,2\ell)=1} H(r_1,r_2)\\ &=\prod_{(p,2a\ell)=1}\left(E_3(\gamma;p)-\frac{\lambda_f(p)}{p^{3+2\gamma}}E_1(\gamma;p)+\frac{1}{p^{3+2\gamma}}E_2(\gamma;p)\right)\prod_{p|a}1\\ &=\prod_{(p,2a\ell)=1}\left(1-\frac{1}{p^2}\right)\left(E_3(\gamma;p)+\frac{1}{p^2-1}\right).\end{aligned}\tag{3.43}$$

Substituting (3.43) in (3.42) completes the proof. □

We can complete the proof for (3.34) by using (3.37) and Lemma 3.19.

Next recall $M_{R,2}^+(\alpha,\ell)$ in (3.18), which is of the form

$$M_{R,2}^+(\alpha,\ell)=\frac{i^\kappa}{\ell_1^{\frac12-\alpha}}\sum_{(c,2\ell)=1}\sum_{\substack{a>Y\\ a|c}}\mu(a)T(c,\alpha,\ell).$$

We extend the sum over $a>Y$ to that over all positive integers. Then

$$M_{R,2}^+(\alpha,\ell)=\frac{i^\kappa}{\ell_1^{\frac12-\alpha}}\sum_{(c,2\ell)=1}\sum_{a|c}\mu(a)T(c,\alpha,\ell)-\frac{i^\kappa}{\ell_1^{\frac12-\alpha}}\sum_{(c,2\ell)=1}\sum_{\substack{a\le Y\\ a|c}}\mu(a)T(c,\alpha,\ell).\tag{3.44}$$

We know $\sum_{a|c}\mu(a)=1$ when $c=1$, and is zero otherwise. Thus

$$\begin{aligned}&\frac{i^\kappa}{\ell_1^{\frac12-\alpha}}\sum_{(c,2\ell)=1}\sum_{a|c}\mu(a)T(c,\alpha,\ell)\\ &=\frac{i^\kappa}{\ell_1^{\frac12-\alpha}}\frac{1}{2\pi i}\int_{(\frac{1}{\log X})}\frac{4X^{1-2\alpha-s}\gamma_{\alpha+s}\tilde\Phi(1-2\alpha-s)}{\pi^2}L(1-2\alpha-2s,\mathrm{sym}^2 f)\\ &\quad\times Z(\tfrac12-\alpha-s,\ell)\ell_1^s8^sg_\alpha(s)\frac{G(s)}{s}ds.\end{aligned}$$

Move the line of the above integral from $\mathrm{Re}(s)=\frac{1}{\log X}$ to $\mathrm{Re}(s)=\frac12-\varepsilon$. We encounter no poles due to Lemma 3.10 and Remark 3.6. It follows that

$$\frac{i^\kappa}{\ell_1^{\frac12-\alpha}}\sum_{(c,2\ell)=1}\sum_{a|c}\mu(a)T(c,\alpha,\ell)\ll \ell^\varepsilon X^{\frac12+\varepsilon}. \tag{3.45}$$

For the second term of (3.44), we move the contour of the integral in $T(c,\alpha,\ell)$ to $\mathrm{Re}(s)=\frac{1}{10}$ without encountering any poles. We have

$$\begin{aligned}
&\frac{i^\kappa}{\ell_1^{\frac12-\alpha}}\sum_{(c,2\ell)=1}\sum_{\substack{a\le Y\\ a|c}}\mu(a)T(c,\alpha,\ell)\\
&=\frac{i^\kappa}{\ell_1^{\frac12-\alpha}}\sum_{\substack{a\le Y\\(a,2\ell)=1}}\mu(a)\sum_{(r_1,2\ell)=1}\frac{\mu(r_1)\lambda_f(r_1)}{r_1}\sum_{(r_2,2\ell)=1}\frac{\mu(r_1r_2)^2}{r_2^{1+2\alpha}}\frac{1}{2\pi i}\int_{(\frac{1}{10})}\\
&\times\frac{4X^{1-2\alpha-s}\gamma_{\alpha+s}\tilde\Phi(1-2\alpha-s)}{\pi^2}\\
&\times\sum_{\substack{(c,2\ell)=1\\ a,r_1,r_2|c}}\frac{1}{c^{2-4\alpha-4s}}L(1-2\alpha-2s,\mathrm{sym}^2 f)Z(\tfrac12-\alpha-s,\ell r_1r_2^2)\frac{\ell_1^s8^s}{r_2^{2s}}g_\alpha(s)\frac{G(s)}{s}ds.
\end{aligned}$$

Treat $\sum_{\substack{(c,2\ell)=1\\ a,r_1,r_2|c}}\frac{1}{c^{2-4\alpha-4s}}$ as in (3.40). The above is

$$\begin{aligned}
&\frac{i^\kappa}{\ell_1^{\frac12-\alpha}}\sum_{\substack{a\le Y\\(a,2\ell)=1}}\mu(a)\sum_{(r_1,2\ell)=1}\frac{\mu(r_1)\lambda_f(r_1)}{r_1}\sum_{(r_2,2\ell)=1}\frac{\mu(r_1r_2)^2}{r_2^{1+2\alpha}}\frac{1}{2\pi i}\int_{(\frac{1}{10})}\frac{4X^{1-2\alpha-s}\gamma_{\alpha+s}\tilde\Phi(1-2\alpha-s)}{\pi^2}\\
&\times\frac{1}{a^{2-4\alpha-4s}}\frac{([r_1,r_2],a)^{2-4\alpha-4s}}{[r_1,r_2]^{2-4\alpha-4s}}\zeta_{2\ell}(2-4\alpha-4s)L(1-2\alpha-2s,\mathrm{sym}^2 f)\\
&\times Z(\tfrac12-\alpha-s,\ell r_1r_2^2)\frac{\ell_1^s}{r_2^{2s}}8^sg_\alpha(s)\frac{G(s)}{s}ds.
\end{aligned}$$

Move the contour of the integral above to $\mathrm{Re}(s)=\frac12-\varepsilon$ without encountering any poles by Lemma 3.10 and Remark 3.6. In particular, the pole of $\zeta(2-4\alpha-4s)$ is canceled by the factor $1-4\alpha-4s$ in $G(s)$. By the fact

$$\left|\frac{([r_1,r_2],a)^{2-4\alpha-4s}}{a^{2-4\alpha-4s}}\frac{1}{[r_1,r_2]^{2-4\alpha-4s}}\right|\le\frac{1}{r_1^\varepsilon},$$

we obtain

$$\frac{i^\kappa}{\ell_1^{\frac{1}{2}-\alpha}} \sum_{(c,2\ell)=1} \sum_{\substack{a\le Y\\ a|c}} \mu(a) T(c,\alpha,\ell) \ll \ell^\varepsilon X^{\frac{1}{2}+\varepsilon} Y. \tag{3.46}$$

Combining (3.44), (3.45) and (3.46) gives (3.35).

Finally, recall $M^+_{R,3}(\alpha,\ell)$ in (3.19). Note $h \ge \frac{1}{2}$. Then

$$M^+_{R,3}(\alpha,\ell) \ll \ell^{\frac{1}{2}+\frac{\varepsilon}{10}} \sum_{(c,2\ell)=1} \left(\frac{X}{c^2}\right)^{h+\varepsilon} \sum_{\substack{a>Y\\ a|c}} \sum_{r_1|c} \sum_{r_2|c} (r_1 r_2^2)^{\frac{\varepsilon}{10}} \ll \ell^{\frac{1}{2}+\varepsilon} \frac{X^{h+\varepsilon}}{Y^{2h-1}},$$

which gives (3.36).

3.7 Proof of Theorem 3.4.

By Lemmas 3.14, 3.18,

$$\begin{aligned} &M^+_{R,1}(\alpha,\ell) + M^+_N(\alpha,\ell,k=0) \\ &= \frac{4X}{\pi^2 \ell_1^{\frac{1}{2}+\alpha}} \sum_{(a,2\ell)=1} \frac{\mu(a)}{a^2} \prod_{p|a} \frac{1}{1-\frac{1}{p^2}} \frac{1}{2\pi i} \int_{(1)} \tilde{\Phi}(s+1) \mathcal{A}(s+\alpha,a,\ell) \frac{1}{\ell_1^s} 8^s X^s g_\alpha(s) \frac{G(s)}{s} ds, \end{aligned} \tag{3.47}$$

where $\mathcal{A}(s+\alpha,a,\ell)$ is defined in (3.23). It can be deduced that

$$\begin{aligned} \sum_{(a,2\ell)=1} \frac{\mu(a)}{a^2} \prod_{p|a} \frac{1}{1-\frac{1}{p^2}} \mathcal{A}(s+\alpha,a,\ell) &= \prod_{p|\ell_1} E_1(\alpha+s;p) \prod_{\substack{p\nmid \ell 1\\ p|\ell_2}} E_2(\alpha+s;p) \prod_{(p,2\ell)=1} E_3(\alpha+s;p) \\ &= L(1+2\alpha+2s, \operatorname{sym}^2 f) Z(\tfrac{1}{2}+\alpha+s,\ell). \end{aligned}$$

This combined with (3.47) gives

$$\begin{aligned} &M^+_{R,1}(\alpha,\ell) + M^+_N(\alpha,\ell,k=0) \\ &= \frac{4X}{\pi^2 \ell_1^{\frac{1}{2}+\alpha}} \frac{1}{2\pi i} \int_{(1)} \tilde{\Phi}(s+1) L(1+2\alpha+2s, \operatorname{sym}^2 f) Z(\tfrac{1}{2}+\alpha+s,\ell) \frac{1}{\ell_1^s} 8^s X^s g_\alpha(s) \frac{G(s)}{s} ds, \end{aligned} \tag{3.48}$$

Move the integration to the line $\mathrm{Re}(s) = -\frac{1}{2} + \varepsilon$ with encountering one simple pole at $s = 0$ by Lemma 3.10 and Remark 3.6. This gives

$$M_{R,1}^{+}(\alpha, \ell) + M_{N}^{+}(\alpha, \ell, k = 0) = \frac{4X}{\pi^2 \ell_1^{\frac{1}{2}+\alpha}} \tilde{\Phi}(1) L(1 + 2\alpha, \mathrm{sym}^2 f) Z(\tfrac{1}{2} + \alpha, \ell) + O(X^{\frac{1}{2}+\varepsilon} \ell^{\varepsilon}). \tag{3.49}$$

By Remark 3.11, we know

$$\begin{aligned} &M_{R,1}^{-}(\alpha, \ell) + M_{N}^{-}(\alpha, \ell, k = 0) \\ &= i^{\kappa} \frac{4\gamma_\alpha X^{1-2\alpha}}{\pi^2 \ell_1^{\frac{1}{2}-\alpha}} \tilde{\Phi}(1 - 2\alpha) L(1 - 2\alpha, \mathrm{sym}^2 f) Z(\tfrac{1}{2} - \alpha, \ell) + O(X^{\frac{1}{2}+\varepsilon} \ell^{\varepsilon}). \end{aligned} \tag{3.50}$$

Similarly we can derive same upper bounds for $M_N^{-}(\alpha, \ell, k \neq 0)$, $M_{R,2}^{-}(\alpha, \ell)$ and $M_{R,3}^{-}(\alpha, \ell)$ as those for $M_N^{+}(\alpha, \ell, k \neq 0)$, $M_{R,2}^{+}(\alpha, \ell)$ and $M_{R,3}^{+}(\alpha, \ell)$ in Lemmas 3.17, 3.18. Therefore it follows from (3.20), (3.21), (3.49), (3.50), and Lemmas 3.17, 3.18 that

$$\begin{aligned} M(\alpha, \ell) = &\frac{4X}{\pi^2 \ell_1^{\frac{1}{2}+\alpha}} \tilde{\Phi}(1) L(1 + 2\alpha, \mathrm{sym}^2 f) Z(\tfrac{1}{2} + \alpha, \ell) \\ &+ i^{\kappa} \frac{4\gamma_\alpha X^{1-2\alpha}}{\pi^2 \ell_1^{\frac{1}{2}-\alpha}} \tilde{\Phi}(1 - 2\alpha) L(1 - 2\alpha, \mathrm{sym}^2 f) Z(\tfrac{1}{2} - \alpha, \ell) \\ &+ O(\ell^{\frac{1}{2}+\varepsilon} X^{\frac{1}{2}+\varepsilon} Y) + O\left(\ell^{\frac{1}{2}+\varepsilon} \frac{X^{h+\varepsilon}}{Y^{2h-1}}\right). \end{aligned}$$

Taking $Y = X^{\frac{2h-1}{4h}}$ completes the proof of Theorem 3.4.

Chapter 4

Counting zeros of Dedekind zeta functions

4.1 Introduction.

Given a number field K, the Dedekind zeta function $\zeta_K(s)$ of K is defined by

$$\zeta_K(s) = \sum_{\mathfrak{a} \neq 0} \frac{1}{\mathrm{N}(\mathfrak{a})^s},$$

for $\mathrm{Re}(s) > 1$, where the sum is over non-zero integral ideals of K. It is known that $\zeta_K(s)$ has an analytic continuation to a meromorphic function on $\mathbb{C}$ with only a simple pole at $s = 1$, and its zeros $\rho = \beta + i\gamma$ encode deep arithmetic information of K. For instance, the generalised Riemann hypothesis, asserting that if $\zeta_K(\rho) = 0$ and $\beta \in (0, 1)$, then $\beta = \frac{1}{2}$, leads to the strongest form of the prime ideal theorem. A related prominent question is to count the zeros of $\zeta_K(s)$ in the critical strip $0 < \mathrm{Re}(s) < 1$. For $T \geq 0$, we set

$$N_K(T) = \#\{\rho \in \mathbb{C} \mid \zeta_K(\rho) = 0,\ 0 < \beta < 1,\ |\gamma| \leq T\},$$

counted with multiplicity if there are any multiple zeros. The estimate of $N_K(T)$ is crucial for proving effective versions of the Chebotarev density theorem as well as bounding the least prime in the Chebotarev density theorem (see [70, 69]). Moreover, to make these results explicit, it is natural to further require a determination of the implied constants for the estimate of $N_K(T)$.

Adapting the arguments of Backlund [3], McCurley [72], and Rosser [91], in [58], Kadiri and Ng showed that for $T \geq 1$, one has

$$\left| N_K(T) - \frac{T}{\pi} \log\left(d_K \left(\frac{T}{2\pi e} \right)^{n_K} \right) \right| \leq D_1(\log d_K + n_K \log T) + D_2 n_K + D_3, \tag{4.1}$$

with admissible $(D_1, D_2, D_3) = (0.506, 16.950, 7.663)$, where n_K and d_K are the degree and absolute discriminant of K, respectively; also, D_1 can be taken as small as $(\pi \log 2)^{-1} \approx 0.459$ at expense of larger D_2 and D_3. This was improved by Trudgian [103] (not only for Dedekind zeta functions but also for Dirichlet L-functions). In particular, as asserted in [103], the estimate (4.1) is valid with $(D_1, D_2, D_3) = (0.316, 5.872, 3.655)$, and the constant D_1 in (4.1) could be made as small as 0.247 (with larger D_2 and D_3). Unfortunately, as pointed out by Bennett, Martin, O'Bryant, and Rechnitzer [6], there is an error in [103] that appears as the ranges of various parameters used in the argument of [103] were not verified properly. In [6], Bennett *et al.* fixed this problem for Dirichlet L-functions.

The objective of this chapter is to prove the following theorem.

Theorem 4.1. *Given a number field K of degree n_K and with absolute discriminant d_K and r_1 real places, for any $T \geq 1$, we have*

$$\Big|N_K(T) - \frac{T}{\pi}\log\Big(d_K\Big(\frac{T}{2\pi e}\Big)^{n_K}\Big) + \frac{r_1}{4}\Big| \leq 0.22737\log\Big(\frac{d_K(T+2)^{n_K}}{(2\pi)^{n_K}}\Big) + 23.02528 n_K + 4.51954. \tag{4.2}$$

In addition, writing the right of (4.2) as $C_1 \log\big(\frac{d_K(T+2)^{n_K}}{(2\pi)^{n_K}}\big) + C_2 n_K + C_3$, we have further admissible triples (C_1, C_2, C_3) recorded in Table 4.2 in Section 4.4. Moreover, recalling that for $T \geq T_0$, $\log(T+2) - \log T \leq \log(1 + \frac{2}{T_0})$, from the above theorem and the triangle inequality, we derive the following improved bound for $N_K(T)$.

Corollary 4.2. *Given a number field K of degree n_K and with absolute discriminant d_K, for any $T \geq 1$, we have*

$$\Big|N_K(T) - \frac{T}{\pi}\log\Big(d_K\Big(\frac{T}{2\pi e}\Big)^{n_K}\Big)\Big| \leq 0.228(\log d_K + n_K \log T) + 23.108 n_K + 4.520. \tag{4.3}$$

Furthermore, by Table 4.2, writing the right of (4.3) as $D_1(\log d_K + n_K \log T) + D_2 n_K + D_3$, we have the following Table 4.1 of admissible (D_1, D_2, D_3) that not only repair but also improve all triples given in [103, Table 2]. (Note that, for all number fields K, our D_2 and D_3 yield a smaller vlaue of $D_2 n_K + D_3$ than the one given by Trudgian [103].)

The proof of Theorem 4.1 follows closely the arguments of Bennett, Martin, O'Bryant, and Rechnitzer [6], Kadiri and Ng [58], and Trudgian [103], which are an adaption of the methods of Backlund [3], McCurley [72], and Rosser [91]. We also take advantage of the refined estimates for

Table 4.1: Admissible (D_1, D_2, D_3) in Corollary 4.2 and in [103]

Trudgian [103]					Our improvement				
	$T \geq 1$		$T \geq 10$			$T \geq 1$		$T \geq 10$	
D_1	D_2	D_3	D_2	D_3	D_1	D_2	D_3	D_2	D_3
0.247	8.851	3.024	8.726	2.081	0.245	6.735	4.213	6.449	3.124
0.265	7.521	3.178	7.396	2.101	0.264	5.276	4.082	4.968	3.051
0.282	6.776	3.335	6.651	2.123	0.281	4.478	4.010	4.149	3.012
0.299	6.262	3.494	6.138	2.146	0.296	3.971	3.969	3.622	2.990

gamma factors obtained in [6]. Moreover, following the strategy of Bennett *et al.* [6], we extend Rademacher's convexity bound for $\zeta_K(s)$ (cf. Propositions 4.13 and 4.14) that, together with "Backlund's trick" (see Section 4.3.2), plays a central role in improving the leading constants C_1 and D_1. Furthermore, we track all the parameters and related inequalities in a similar manner of Bennett *et al.* [6] to fix the aforementioned error appearing in [103]. Last but not least, we note that we obtain our results by a direct numerical computation (with help from Maple) and that it may be possible to use the "interval analysis" as in [6] to prove an estimate similar to [6, Theorem 1.1]. Nonetheless, since Corollary 4.2 is already as strong as [6, Corollary 1.2], and it is sufficient for most applications, we shall not devote ourselves to do such an interval analysis here.

4.2 The main term and the gamma factor.

4.2.1 The main term.

Let K be a number field of degree n_K and with absolute discriminant d_K. We let r_1 and r_2 be the numbers of real and complex places, respectively, of K and note that $n_K = r_1 + 2r_2$. We define the completed zeta function $\xi_K(s)$ as

$$\xi_K(s) = s(s-1)d_K^{s/2}\gamma_K(s)\zeta_K(s), \tag{4.4}$$

where

$$\gamma_K(s) = \left(\pi^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right)\right)^{r_2}\left(\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\right)^{r_1+r_2}.$$

We recall that $\xi_K(s)$ extends to an entire function of order 1 and satisfies the functional equation

$$\xi_K(s) = \xi_K(1-s). \tag{4.5}$$

As in the introduction, we set

$$N_K(T) = \#\{\rho \in \mathbb{C} \mid \zeta_K(\rho) = 0,\ 0 < \beta < 1,\ |\gamma| \leq T\}.$$

To estimate $N_K(T)$, we shall apply the argument principle as follows. For any fixed $\sigma_1 > 1$, we consider the rectangle $\mathcal{R}$ with vertices $\sigma_1 - iT,\ \sigma_1 + iT,\ 1 - \sigma_1 + iT$, and $1 - \sigma_1 - iT$ (that is away from zeros of $\xi_K(s)$).[1] As $\xi_K(s)$ is entire, it follows from the argument principle that

$$N_K(T) = \frac{1}{2\pi}\Delta_{\mathcal{R}} \arg \xi_K(s).$$

Let $\mathcal{C}$ be the part of the contour of $\mathcal{R}$ in $\mathrm{Re}(s) \geq \frac{1}{2}$ and $\mathcal{C}_0$ be the part of the contour of $\mathcal{R}$ in $\mathrm{Re}(s) \geq \frac{1}{2}$ and $\mathrm{Im}(s) \geq 0$. Since $\overline{\xi_K(s)} = \xi_K(\bar{s})$, the functional equation (4.5) then yields

$$\Delta_{\mathcal{R}} \arg \xi_K(s) = 2\Delta_{\mathcal{C}} \arg \xi_K(s) = 4\Delta_{\mathcal{C}_0} \arg \xi_K(s),$$

which implies that

$$N_K(T) = \frac{2}{\pi}\Delta_{\mathcal{C}_0} \arg \xi_K(s). \tag{4.6}$$

Writing $B = d_K/\pi^{n_K}$, by (4.4), we have

$$\begin{aligned}\Delta_{\mathcal{C}_0} \arg \xi_K(s) &= \Delta_{\mathcal{C}_0} \arg s + \Delta_{\mathcal{C}_0} \arg B^{s/2} \\ &\quad + (r_1 + r_2)\Delta_{\mathcal{C}_0} \arg \Gamma\left(\frac{s}{2}\right) + r_2\Delta_{\mathcal{C}_0} \arg \Gamma\left(\frac{s+1}{2}\right) + \Delta_{\mathcal{C}_0} \arg\left((s-1)\zeta_K(s)\right).\end{aligned} \tag{4.7}$$

[1] Throughout our argument, we will always assume T is away from zeros of $\xi_K(s)$. As shall be seen in Section 4.4, with this assumption, we will prove (4.27) for T away from zeros of $\xi_K(s)$. Nonetheless, if T is the exact height of a zero, we know that $N_K(T) = N_K(T+\varepsilon)$ for all sufficiently small $\varepsilon > 0$ (in other words, $T+\varepsilon$ is away from zeros). Then, by the triangle inequality, applying (4.27) with $T+\varepsilon$, we see that

$$\begin{aligned}&\Big|N_K(T) - \frac{T}{\pi}\log\Big(d_K\Big(\frac{T}{2\pi e}\Big)^{n_K}\Big) + \frac{r_1}{4}\Big| \\ &\leq \Big|N_K(T+\varepsilon) - \frac{T+\varepsilon}{\pi}\log\Big(d_K\Big(\frac{T+\varepsilon}{2\pi e}\Big)^{n_K}\Big) + \frac{r_1}{4}\Big| + \Big|\frac{T+\varepsilon}{\pi}\log\Big(d_K\Big(\frac{T+\varepsilon}{2\pi e}\Big)^{n_K}\Big) - \frac{T}{\pi}\log\Big(d_K\Big(\frac{T}{2\pi e}\Big)^{n_K}\Big)\Big| \\ &\leq C_1\log\Big(\frac{d_K(T+\varepsilon+2)^{n_K}}{(2\pi)^{n_K}}\Big) + C_2 n_K + C_3 + \Big|\frac{T+\varepsilon}{\pi}\log\Big(d_K\Big(\frac{T+\varepsilon}{2\pi e}\Big)^{n_K}\Big) - \frac{T}{\pi}\log\Big(d_K\Big(\frac{T}{2\pi e}\Big)^{n_K}\Big)\Big|.\end{aligned}$$

Now, taking $\varepsilon \to 0^+$, we conclude that (4.27) is also valid when T is the exact height of a zero.

It is clear that

$$\begin{aligned}
&\Delta_{\mathcal{C}_0} \arg s = \arctan(2T), \\
&\Delta_{\mathcal{C}_0} \arg B^{s/2} = \frac{T}{2}\log B = \frac{T}{2}\log\Big(\frac{d_K}{\pi^{n_K}}\Big), \\
&\Delta_{\mathcal{C}_0} \arg\Gamma(s) = \Delta_{\mathcal{C}_0}(\operatorname{Im}\log\Gamma(s)) = \operatorname{Im}\log\Gamma\Big(\frac{1}{2}+iT\Big).
\end{aligned} \tag{4.8}$$

To control the gamma factor, we shall appeal to the improved numerical bound established in [6, Sec. 3]. For $a \in \{0,1\}$, we set

$$g_a(T) = \frac{2}{\pi}\operatorname{Im}\log\Gamma\Big(\frac{1}{4}+\frac{a}{2}+i\frac{T}{2}\Big) - \frac{T}{\pi}\log\Big(\frac{T}{2e}\Big) - \frac{2a-1}{4}.$$

It follows from [6, Proposition 3.2] that for $a \in \{0,1\}$ and $T \geq 5/7$,

$$|g_a(T)| \leq \frac{2-a}{50T}.$$

Hence, setting

$$g_K(T) = (r_1+r_2)g_0(T) + r_2 g_1(T), \tag{4.9}$$

we then obtain

$$|g_K(T)| \leq \frac{2n_K}{50T} - \frac{r_2}{50T}. \tag{4.10}$$

Now, gathering (4.6), (4.7), (4.8), and (4.9), we obtain

$$N_K(T) = \frac{2}{\pi}\arctan(2T) + g_K(T) + \frac{T}{\pi}\log\Big(d_K\Big(\frac{T}{2\pi e}\Big)^{n_K}\Big) - \frac{r_1}{4} + \frac{2}{\pi}\Delta_{\mathcal{C}_0}\arg((s-1)\zeta_K(s)). \tag{4.11}$$

Let $\mathcal{C}_1$ denote the vertical line from σ_1 to σ_1+iT and $\mathcal{C}_2$ denote the horizontal line from σ_1+iT to $\frac{1}{2}+iT$. We require the following two estimates.

Lemma 4.3. *For $s=\sigma+it$ with $\sigma>1$, one has*

$$\frac{\zeta_K(2\sigma)}{\zeta_K(\sigma)} \leq |\zeta_K(s)| \leq \zeta(\sigma)^{n_K},$$

where, as later, $\zeta(s)$ denotes the Riemann zeta function.

Lemma 4.4. *For $\sigma_1 > 1$,*

$$|\Delta_{\mathcal{C}_1} \arg(s-1)\zeta_K(s)| \leq \frac{\pi}{2} + n_K \log \zeta(\sigma_1).$$

Proof. Note that

$$\Delta_{\mathcal{C}_1} \arg(s-1)\zeta_K(s) = \Delta_{\mathcal{C}_1} \arg(s-1) + \Delta_{\mathcal{C}_1} \arg \zeta_K(s) = \arctan\Big(\frac{T}{\sigma_1 - 1}\Big) + \Delta_{\mathcal{C}_1} \arg \zeta_K(s).$$

Now, the lemma follows from the estimate

$$|\Delta_{\mathcal{C}_1} \arg \zeta_K(s)| = |\arg \zeta_K(\sigma_1 + iT)| \leq |\log \zeta_K(\sigma_1 + iT)| \leq \log \zeta_K(\sigma_1) \leq n_K \log \zeta(\sigma_1),$$

where the last inequality is due to Lemma 4.3. □

Thus, by Lemma 4.4 and (4.11), we arrive at

$$\Big|N_K(T) - \frac{T}{\pi} \log\Big(d_K \Big(\frac{T}{2\pi e}\Big)^{n_K}\Big) + \frac{r_1}{4}\Big| \leq 2 + g_K(T) + \frac{2n_K}{\pi} \log \zeta(\sigma_1) + \frac{2}{\pi} |\Delta_{\mathcal{C}_2} \arg((s-1)\zeta_K(s))|. \tag{4.12}$$

4.2.2 Bounding the gamma factor.

For $a \in \{0, 1\}$, $0 \leq d < 9/2$ and $T \geq 5/7$, we set

$$\mathcal{E}_a(T, d) = \Big|\operatorname{Im} \log \Gamma\Big(\frac{\sigma + a + iT}{2}\Big)\Big|_{\sigma=\frac{1}{2}}^{\frac{1}{2}+d} + \operatorname{Im} \log \Gamma\Big(\frac{\sigma + a + iT}{2}\Big)\Big|_{\sigma=\frac{1}{2}}^{\frac{1}{2}-d}\Big|,$$

and we define

$$\mathcal{E}_K(T, d) = (r_1 + r_2)\mathcal{E}_0(T, d) + r_2 \mathcal{E}_1(T, d). \tag{4.13}$$

Following [6, p. 7], we let

$$\begin{aligned}
E_a(T,d) = &\frac{2T/3}{(2a+2d+17)^2+4T^2} + \frac{2T/3}{(2a-2d+17)^2+4T^2} - \frac{4T/3}{(2a+17)^2+4T^2} \\
&+ \frac{T}{2}\log\Big(1+\frac{(2a+17)^2}{4T^2}\Big) - \frac{T}{4}\log\Big(1+\frac{(2a+2d+17)^2}{4T^2}\Big) \\
&- \frac{T}{4}\log\Big(1+\frac{(2a-2d+17)^2}{4T^2}\Big) \\
&+ \frac{(8+6\pi)/45}{((2a+2d+17)^2+4T^2)^{3/2}} + \frac{(8+6\pi)/45}{((2a-2d+17)^2+4T^2)^{3/2}} + \frac{2(8+6\pi)/45}{((2a+17)^2+4T^2)^{3/2}} \\
&+ \sum_{k=0}^{3}\Big(2\arctan\frac{2a+1+4k}{2T} - \arctan\frac{2a+2d+1+4k}{2T} - \arctan\frac{2a-2d+1+4k}{2T}\Big) \\
&+ \frac{2a+2d+15}{4}\arctan\frac{2a+2d+17}{2T} + \frac{2a-2d+15}{4}\arctan\frac{2a-2d+17}{2T} \\
&- \frac{2a+15}{2}\arctan\frac{2a+17}{2T}.
\end{aligned}$$

We shall further set

$$E_K(T,d) = (r_1+r_2)E_0(T,d) + r_2E_1(T,d). \tag{4.14}$$

As shown in [6, p. 6], $\mathcal{E}_a(T,d) \le E_a(T,d)$ for $0 \le d < 9/2$ and $T \ge 5/7$, and thus

$$\mathcal{E}_K(T,d) \le E_K(T,d) \tag{4.15}$$

for $0 \le d < 9/2$ and $T \ge 5/7$. In addition, from [6, Lemma 3.4] and our definition of $E_K(T,d)$, we have the following lemma.

Lemma 4.5. *For $0 \le \delta_1 \le d < 9/2$ and $T \ge 5/7$,*

$$0 < E_K(T,\delta_1) \le E_K(T,d).$$

Furthermore, for $d \in [\frac{1}{4},\frac{5}{8}]$ and $T \ge 5/7$,

$$\frac{E_K(T,d)}{\pi} \le (r_1+r_2)\frac{640d-112}{1536(3T-1)} + r_2\frac{(640+216)d-112-39}{1536(3T+3-1)} + \frac{n_K}{2^{10}}.$$

4.3 Backlund's trick and the Jensen integral.

4.3.1 Introducing the auxiliary function f_N.

For the sake of convenience, we shall set $\mathcal{Z}(w) = (w-1)\zeta_K(w)$. In order to analyse the variation of the argument of $\mathcal{Z}(w)$ on $\mathcal{C}_2$, we shall introduce an auxiliary function

$$f_N(s) = \frac{1}{2}\Big(\mathcal{Z}(s+iT)^N + \mathcal{Z}(s-iT)^N\Big)$$

for $N \in \mathbb{N}$. For $\sigma \in \mathbb{R}$, it is clear that

$$f_N(\sigma) = \frac{1}{2}\Big(\mathcal{Z}(\sigma+iT)^N + \mathcal{Z}(\sigma-iT)^N\Big) = \frac{1}{2}\Big(\mathcal{Z}(\sigma+iT)^N + \overline{\mathcal{Z}(\sigma+iT)^N}\Big) = \mathfrak{Re}(\mathcal{Z}(\sigma+iT)^N).$$

We need the following definition that measures the variation of the argument of $\mathcal{Z}(w)^N$ on $\mathcal{C}_2$.

Definition 4.6. Let b_N denote the non-negative integer, depending on N, such that

$$b_N \leq \frac{1}{\pi}\Big|\Delta_{\mathcal{C}_2} \arg \mathcal{Z}(w)^N\Big| < b_N + 1.$$

From this definition and the fact that $\arg \mathcal{Z}(w)^N = N \arg \mathcal{Z}(w)$, we immediately obtain

$$\frac{b_N}{N} \leq \frac{1}{\pi}\Big|\Delta_{\mathcal{C}_2} \arg \mathcal{Z}(w)\Big| < \frac{b_N+1}{N}. \tag{4.16}$$

In addition, we have the following lemma concerning the zeros of $f_N(\sigma)$.

Lemma 4.7. *In the notation of Definition 4.6, the function $f_N(\sigma)$ has at least b_N zeros in $[\frac{1}{2}, \sigma_1]$.*

Proof. By Definition 4.6, there are at least b_N different values of σ such that $\frac{1}{2} + \frac{1}{\pi} \arg \mathcal{Z}(\sigma + iT)^N \in \mathbb{Z}$. Thus, for such values of σ, $\mathcal{Z}(\sigma+iT)^N$ is purely imaginary, which means that

$$f_N(\sigma) = \mathfrak{Re}(\mathcal{Z}(\sigma+iT)^N) = 0$$

for at least b_N different values σ. $\square$

We shall also require the following lemma regarding the limiting behaviour of f_N.

Lemma 4.8. *For any $c > 1$, there is an infinite sequence of natural numbers $(N_m)_{m=1}^{\infty}$ such that $f_{N_m}(c) \neq 0$. Moreover, we have*

$$\limsup_{m\to\infty}\Big(-\frac{1}{N_m}\log|f_{N_m}(c)|\Big) \leq \log\Big(\frac{1}{\sqrt{(c-1)^2+T^2}}\frac{\zeta_K(c)}{\zeta_K(2c)}\Big).$$

Proof. Write $\mathcal{Z}(c+iT) = Re^{i\phi}$ for some $R, \phi \in \mathbb{R}$. It is clear that $\mathcal{Z}(c-iT) = Re^{-i\phi}$. Also, as $\mathcal{Z}(c+iT) \neq 0$ for any $c > 1$, we know that $R > 0$. Thus, we have

$$\frac{f_N(c)}{\mathcal{Z}(c+iT)^N} = \frac{1}{2}\Big(1+\frac{\mathcal{Z}(c-iT)^N}{\mathcal{Z}(c+iT)^N}\Big) = \frac{1}{2}(1+e^{-2N\phi i})$$

for any $N \in \mathbb{N}$.

Now, applying Dirichlet's approximation theorem, for any ϕ, there is an infinite sequence of natural numbers $(N_m)_{m=1}^{\infty}$ such that as $m \to \infty$, $-2N_m\phi \to 0$ modulo 2π and $N_m \to \infty$. Thus, $\frac{f_{N_m}(c)}{\mathcal{Z}(c+iT)^{N_m}} \to 1$ as $m \to \infty$, and hence

$$\lim_{m\to\infty}\Big(-\frac{1}{N_m}(\log|f_{N_m}(c)|-N_m\log|\mathcal{Z}(c+iT)|)\Big) = \Big(\lim_{m\to\infty}\frac{-1}{N_m}\Big)\Big(\lim_{m\to\infty}\log\Big|\frac{f_{N_m}(c)}{\mathcal{Z}(c+iT)^{N_m}}\Big|\Big) = 0.$$

Moreover, by the left inequality of Lemma 4.3, we have

$$|\mathcal{Z}(c+iT)| \geq \sqrt{(c-1)^2+T^2}\frac{\zeta_K(2c)}{\zeta_K(c)},$$

which, combined with the above identity, gives

$$\begin{aligned} 0 &\geq \limsup_{m\to\infty}\Big(-\frac{1}{N_m}\log|f_{N_m}(c)| + \log\Big(\sqrt{(c-1)^2+T^2}\frac{\zeta_K(2c)}{\zeta_K(c)}\Big)\Big) \\ &= \limsup_{m\to\infty}\Big(-\frac{1}{N_m}\log|f_{N_m}(c)|\Big) + \log\Big(\sqrt{(c-1)^2+T^2}\frac{\zeta_K(2c)}{\zeta_K(c)}\Big). \end{aligned}$$

Herein, we complete the proof. □

Let $D(c,r)$ be the open disk centred at c with radius r. Let $(N_m)_{m=1}^{\infty}$ be given as in Lemma 4.8. For any $N \in (N_m)_{m=1}^{\infty}$, we set

$$S_N(c,r) = \frac{1}{N}\sum_{z\in\mathcal{S}_N(D(c,r))}\log\frac{r}{|z-c|},$$

where $\mathcal{S}_N(D(c,r))$ denotes the set of zeros of $f_N(s)$ in $D(c,r)$. As in [6, Theorem 5.1], we have the following version of Jensen's formula.

Theorem 4.9 (Jensen's formula)**.** *For $c \in \mathbb{C}$ and $r > 0$, if $f_N(c) \neq 0$, then*

$$S_N(c,r) = -\frac{1}{N}\log|f_N(c)| + \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{N}\log|f_N(c+re^{i\theta})|d\theta.$$

Applying Jensen's formula and Lemma 4.8, we obtain the following upper bound for $S_N(c,r)$.

Proposition 4.10. *Let c, r, and σ_1 be real numbers such that*

$$c - r < \frac{1}{2} < 1 < c < \sigma_1 < c + r.$$

Let $F_{c,r} : [-\pi,\pi] \to \mathbb{R}$ be an even function such that $F_{c,r}(\theta) \geq \frac{1}{N_m}\log|f_{N_m}(c+re^{i\theta})|$. Then we have

$$\limsup_{m\to\infty} S_{N_m}(c,r) \leq \log\left(\frac{1}{\sqrt{(c-1)^2+T^2}}\frac{\zeta_K(c)}{\zeta_K(2c)}\right) + \frac{1}{\pi}\int_0^{\pi}F_{c,r}(\theta)d\theta.$$

4.3.2 Backlund's trick.

We start with the following technical estimate.

Lemma 4.11. *Let $0 \leq d < 1/2$ and $T \geq 5/7$. Then we have*

$$\begin{aligned}\left|\arg\Big((\sigma-1+iT)\zeta_K(\sigma+iT)\Big)^N\Big|_{\sigma=\frac{1}{2}}^{\frac{1}{2}+d}\right| \leq \left|\arg\Big((\sigma-1+iT)\zeta_K(\sigma+iT)\Big)^N\Big|_{\sigma=\frac{1}{2}}^{\frac{1}{2}-d}\right|\\ + N\mathcal{E}_K(T,d) + N\frac{\pi}{2},\end{aligned}$$

where $\mathcal{E}_K(T,d)$ is defined as in (4.13).

Proof. By the functional equation (4.5) and the fact that $\xi_K(s) = \overline{\xi_K(\bar{s})}$, we have

$$\arg\xi_K(\sigma+iT)\Big|_{\sigma=\frac{1}{2}}^{\frac{1}{2}+d} = -\arg\xi_K(\sigma+iT)\Big|_{\sigma=\frac{1}{2}}^{\frac{1}{2}-d}. \tag{4.17}$$

Since

$$\arg(\sigma+iT) + \arg B^{(\sigma+iT)/2} = \arctan\frac{T}{\sigma} + \frac{T}{2}\log B,$$

by (4.4), we have

$$\begin{aligned}\arg \xi_K(\sigma+iT) = \arctan\frac{T}{\sigma} + \frac{T}{2}\log B + (r_1+r_2)\mathrm{Im}\log\Gamma\Big(\frac{\sigma+iT}{2}\Big) + r_2\mathrm{Im}\log\Gamma\Big(\frac{\sigma+iT+1}{2}\Big)\\ + \arg\Big((\sigma+iT-1)\zeta_K(\sigma+iT)\Big).\end{aligned} \tag{4.18}$$

As we know that for $\pm xy < 1$,

$$\arctan x \pm \arctan y = \arctan\frac{x\pm y}{1\mp xy},$$

for $0 \le d < 1/2$, we have

$$\begin{aligned}&\Big|\arctan\frac{T}{\frac12+d} - \arctan\frac{T}{\frac12} + \arctan\frac{T}{\frac12-d} - \arctan\frac{T}{\frac12}\Big|\\ &= \Big|\arctan\frac{\frac{T}{\frac12+d}-\frac{T}{\frac12}}{1+\frac{T}{\frac12+d}\frac{T}{\frac12}} + \arctan\frac{\frac{T}{\frac12-d}-\frac{T}{\frac12}}{1+\frac{T}{\frac12-d}\frac{T}{\frac12}}\Big|\\ &\le \frac{\pi}{2}.\end{aligned} \tag{4.19}$$

Now, applying the triangle inequality, by (4.17), (4.18), and (4.19), we obtain

$$\Big|\arg\Big((\sigma-1+iT)\zeta_K(\sigma+iT)\Big)\Big|_{\sigma=\frac12}^{\frac12+d}\Big| \le \Big|\arg\Big((\sigma-1+iT)\zeta_K(\sigma+iT)\Big)\Big|_{\sigma=\frac12}^{\frac12-d}\Big| + \mathcal{E}_K(T,d) + \frac{\pi}{2}.$$

Recalling that

$$\arg\Big((\sigma-1+iT)\zeta_K(\sigma+iT)\Big)^N\Big|_{\sigma=\frac12}^{\frac12\pm d} = N\arg\Big((\sigma-1+iT)\zeta_K(\sigma+iT)\Big)\Big|_{\sigma=\frac12}^{\frac12\pm d},$$

we conclude the proof. □

As argued in [6] and [103], we require the following version of "Backlund's trick".

Proposition 4.12 (Backlund's trick)**.** *Let c and r be real numbers. Set*

$$\sigma_1 = c + \frac{(c-1/2)^2}{r} \quad \textit{and} \quad \delta = 2c - \sigma_1 - \frac12.$$

If $1 < c < r$ *and* $0 < \delta < \frac{1}{2}$, *then*

$$\left| \arg\Big((\sigma + iT - 1)\zeta_K(\sigma + iT)\Big)\Big|_{\sigma=\sigma_1}^{1/2} \right| \leq \frac{\pi S_N(c,r)}{2\log(r/(c-1/2))} + \frac{E_K(T,\delta)}{2} + \frac{\pi}{N} + \frac{\pi}{2N} + \frac{\pi}{4}.$$

Proof. By the conditions on c and r and the definitions of σ_1 and δ, we know that

$$c - r < \frac{1}{2} - \delta \leq \frac{1}{2} \leq \frac{1}{2} + \delta = 2c - \sigma_1 \leq c \leq \sigma_1 < c + r.$$

As $\log \frac{r}{|z-c|} > 0$ for $z \in D(c,r)$, we see that

$$S_N(c,r) = \frac{1}{N} \sum_{z \in \mathcal{S}_N(D(c,r))} \log \frac{r}{|z-c|} \geq \frac{1}{N} \sum_{z \in \mathcal{S}_N((c-r,\sigma_1])} \log \frac{r}{|z-c|}.$$

Recall that by Lemma 4.7, there are at least b_N values of σ satisfying $\sigma \in [1/2, \sigma_1]$ and $f_N(\sigma) = 0$, where b_N is defined as in Definition 4.6. For $1 \leq k \leq b_N$, we then set δ_k as the smallest non-negative real number such that

$$f_N(1/2 + \delta_k) = 0 \quad \text{and} \quad k - 1 \leq \frac{1}{\pi}\left| \arg\Big((\sigma + iT - 1)\zeta_K(\sigma + iT)\Big)^N \Big|_{\sigma=1/2}^{1/2+\delta_k} \right|. \tag{4.20}$$

Writing $z_k = \frac{1}{2} + \delta_k$, we let x_1 denote the number of z_k with $z_k \in [1/2, 1/2 + \delta) = [1/2, 2c - \sigma_1)$ and let x_2 denote the number of z_k with $z_k \in [2c - \sigma_1, \sigma_1]$. We note that $x_2 = b_N - x_1$ and that

$$0 \leq \delta_1 < \delta_2 < \cdots < \delta_{x_1} < \delta \leq \delta_{x_1+1} < \cdots < \delta_{b_N} \leq \sigma_1 - 1/2.$$

From (4.15), (4.20), and Lemma 4.11, it follows that

$$k - 1 \leq \frac{1}{\pi}\left| \arg\Big((\sigma - 1 + iT)\zeta_K(\sigma + iT)\Big)^N \Big|_{\sigma=\frac{1}{2}}^{\frac{1}{2}-\delta_k} \right| + \frac{1}{\pi} N E_K(T,\delta_k) + \frac{N}{2}$$

whenever $1 \leq k \leq x_1$ (which implies that $\delta_k < \delta < \frac{1}{2}$).

For each $j \geq 1$, if there exists a k (chosen to be minimal) such that

$$k - 1 - \frac{1}{\pi} N E_K(T,\delta_k) - \frac{N}{2} \geq j,$$

then f_N has at least j zeros in $[1/2-\delta_k, 1/2)$ since

$$\frac{1}{\pi}\Big|\arg\Big((\sigma+iT-1)\zeta_K(\sigma+iT)\Big)^N\Big|_{\sigma=1/2}^{1/2-\delta_k}\Big| \geq k-1-\frac{1}{\pi}NE_K(T,\delta_k)-\frac{N}{2}\geq j.$$

For such an instance, we define δ_{-k} as the smallest values of these zeros (to avoid possible repetition), and we shall say that the zero $z_k = 1/2+\delta_k$ has a pair $z_{-k} = 1/2-\delta_{-k}$. We note that $\delta_{-k}\leq \delta_k$ by the construction.

By the same argument as in [6, p. 11], we have

$$S_N(c,r)\geq \frac{2b_N - \frac{NE_K(T,\delta)+\frac{N\pi}{2}+\pi}{\pi}}{N}\log\Big(\frac{r}{c-1/2}\Big),$$

and thus

$$\frac{b_N}{N}\leq \frac{S_N(c,r)}{2\log(r/(c-1/2))}+\frac{E_K(T,\delta)}{2\pi}+\frac{1}{4}+\frac{1}{2N},$$

which combined with (4.16) completes the proof. □

4.3.3 Constructing and bounding $F_{c,r}$.

We first recall the convexity bound for $\zeta_K(s)$ established by Rademacher [86, Theorem 4].

Proposition 4.13. *Let $\eta\in(0,\frac{1}{2}]$ and $s=\sigma+it$. If $-\eta\leq\sigma\leq 1+\eta$, then one has*

$$|\zeta_K(s)|\leq 3\Big|\frac{1+s}{1-s}\Big|\Big(d_K\Big(\frac{|1+s|}{2\pi}\Big)^{n_K}\Big)^{\frac{1+\eta-\sigma}{2}}\zeta(1+\eta)^{n_K}.$$

Also, for $\sigma\in[-\frac{1}{2},0)$, one has

$$|\zeta_K(s)|\leq 3\Big|\frac{1+s}{1-s}\Big|\Big(d_K\Big(\frac{|1+s|}{2\pi}\Big)^{n_K}\Big)^{\frac{1}{2}-\sigma}\zeta(1-\sigma)^{n_K}. \tag{4.21}$$

We note that the second inequality follows from the first bound by taking $\eta=-\sigma$. Moreover, Rademacher's argument [86] can be used to extend (4.21) for $\sigma<0$ as follows (cf. [6, Theorem 5.7]). For $x\in\mathbb{R}$, let $[x]$ be the integer closest to x; when there are two integers equally close to x, we shall choose the one closer to 0.

Proposition 4.14. *Let $s = \sigma + it$ with $\sigma < 0$. Then we have*

$$|\zeta_K(s)| \le \Big(\frac{d_K}{(2\pi)^{n_K}}\Big)^{\frac{1}{2}-\sigma} |1+s-[\sigma]|^{n_K(\frac{1}{2}+[\sigma]-\sigma)} \prod_{j=1}^{-[\sigma]} |s+j-1|^{n_K} \zeta(1-\sigma)^{n_K}.$$

Proof. From the functional equation (4.5) we have

$$\begin{aligned}|\zeta_K(s)| &\le d_K^{1/2-\sigma}\Big|\frac{\gamma_K(1-s)}{\gamma_K(s)}\Big||\zeta_K(1-s)| \\ &= d_K^{1/2-\sigma}\pi^{(\sigma-\frac{1}{2})n_K}\Big|\frac{\Gamma(\frac{1}{2}+\frac{1-s}{2})}{\Gamma(\frac{1}{2}+\frac{s}{2})}\Big|^{r_2}\Big|\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}\Big|^{r_1+r_2}|\zeta_K(1-s)|.\end{aligned}$$

As $\sigma < 0$, by Lemma 4.3, we have $|\zeta_K(1-s)| \le \zeta(1-\sigma)^{n_K}$. It remains to estimate the ratios of gamma functions. It was obtained in the proof of [6, Theorem 5.7] that for $a, b \in \{0, 1\}$ and $k \in \mathbb{Z}$,

$$\frac{\Gamma(\frac{a}{2}+\frac{1-s}{2})}{\Gamma(\frac{a}{2}+\frac{s}{2})} = \frac{\Gamma(\frac{b}{2}+\frac{1-(s+k)}{2})}{\Gamma(\frac{b}{2}+\frac{s+k}{2})}2^{-k}\Big(\prod_{j=1}^{k}(s+j-1)\Big)\frac{\sin(\frac{\pi}{2}(s+k+1-b))}{\sin(\frac{\pi}{2}(s+1-a))}.$$

Setting $a = 0$ and $a = 1$ and taking $b \equiv k \pmod 2$ and $b \equiv k+1 \pmod 2$, respectively, we can make sine factors ± 1. Thus, upon choosing $k = -[\sigma]$ and applying [86, Lemmata 1 and 2] to $\frac{\Gamma(\frac{b}{2}+\frac{1-(s+k)}{2})}{\Gamma(\frac{b}{2}+\frac{s+k}{2})}$, we conclude that

$$\Big|\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}\Big|^{r_1+r_2} \le \Big(\frac{1}{2}|1+s-[\sigma]|\Big)^{(\frac{1}{2}+[\sigma]-\sigma)(r_1+r_2)} 2^{[\sigma](r_1+r_2)}\Big(\prod_{j=1}^{-[\sigma]}|s+j-1|\Big)^{r_1+r_2}$$

and

$$\Big|\frac{\Gamma(\frac{1}{2}+\frac{1-s}{2})}{\Gamma(\frac{1}{2}+\frac{s}{2})}\Big|^{r_2} \le \Big(\frac{1}{2}|1+s-[\sigma]|\Big)^{(\frac{1}{2}+[\sigma]-\sigma)r_2} 2^{[\sigma]r_2}\Big(\prod_{j=1}^{-[\sigma]}|s+j-1|\Big)^{r_2}.$$

Collecting above estimates and recalling the fact that $n_K = r_1 + 2r_2$, we obtain the desired result. □

Lemma 4.15. *Let $\eta \in (0, \frac{1}{2}]$, $s = \sigma + it$, and $T > 0$. If $\sigma \ge 1 + \eta$, then we have*

$$\frac{1}{N}\log|f_N(s)| \le \frac{1}{2}\log((\sigma-1)^2 + (|t|+T)^2) + n_K\log\zeta(\sigma).$$

If $-\eta \leq \sigma \leq 1+\eta$, then we have

$$\begin{aligned}\frac{1}{N}\log|f_N(s)| \leq \log 3 + \frac{n_K(1+\eta-\sigma)+2}{4}\log((\sigma+1)^2+(|t|+T)^2) \\ + \frac{1+\eta-\sigma}{2}\log\Big(\frac{d_K}{(2\pi)^{n_K}}\Big) + n_K\log\zeta(1+\eta).\end{aligned}$$

If $\sigma \leq -\eta$, then we have

$$\begin{aligned}\frac{1}{N}\log|f_N(s)| &\leq n_K\log\zeta(1-\sigma) + \frac{1}{2}\log((\sigma-1)^2+(|t|+T)^2) \\ &\quad + \frac{1-2\sigma}{2}\log\Big(\frac{d_K}{(2\pi)^{n_K}}\Big) + \frac{(1-2\sigma+2[\sigma])n_K}{4}\log((1+\sigma-[\sigma])^2+(|t|+T)^2) \\ &\quad + \frac{n_K}{2}\sum_{j=1}^{-[\sigma]}\log((\sigma+j-1)^2+(|t|+T)^2).\end{aligned}$$

Proof. Since $\sigma \geq 1+\eta > 1$, by Lemma 4.3, we derive

$$\begin{aligned}|f_N(s)| &\leq \frac{1}{2}\Big(|s+iT-1|^N|\zeta_K(s+iT)|^N + |s-iT-1|^N|\zeta_K(s-iT)|^N\Big) \\ &\leq \Big((\sigma-1)^2+(|t|+T)^2\Big)^{\frac{N}{2}}\zeta(\sigma)^{n_K N}.\end{aligned}$$

Now, the first estimate follows from taking logarithms and dividing both sides by N.

Secondly, if $-\eta \leq \sigma \leq 1+\eta$, then by Proposition 4.13, we see that $|f_N(s)|$ is at most

$$\begin{aligned}&\frac{1}{2}\Big(3^N|s+iT+1|^N + 3^N|s-iT+1|^N\Big)\Big(d_K\Big(\frac{\sqrt{(\sigma+1)^2+(|t|+T)^2}}{2\pi}\Big)^{n_K}\Big)^{\frac{(1+\eta-\sigma)N}{2}}\zeta(1+\eta)^{n_K N} \\ &\leq 3^N\Big((\sigma+1)^2+(|t|+T)^2\Big)^{\frac{N}{2}}\Big(d_K\Big(\frac{\sqrt{(\sigma+1)^2+(|t|+T)^2}}{2\pi}\Big)^{n_K}\Big)^{\frac{(1+\eta-\sigma)N}{2}}\zeta(1+\eta)^{n_K N}.\end{aligned}$$

Again, taking logarithms yields the second bound.

Lastly, for $\sigma \leq -\eta$, it follows from Proposition 4.14 that

$$\begin{aligned}|f_N(s)| \leq \Big((\sigma-1)^2+(|t|+T)^2\Big)^{\frac{N}{2}}\Big(\frac{d_K}{(2\pi)^{n_K}}\Big)^{N(\frac{1}{2}-\sigma)}|(1+\sigma-[\sigma])^2+(|t|+T)^2|^{\frac{(1-2\sigma+2[\sigma])Nn_K}{4}} \\ \times\Big(\prod_{j=1}^{-[\sigma]}((\sigma+j-1)^2+(|t|+T)^2)\Big)^{\frac{n_K N}{2}}\zeta(1-\sigma)^{n_K N}.\end{aligned}$$

We then conclude the proof by taking logarithms. □

Following [6], to proceed further, we introduce some notation and auxiliary functions. We first set

$$L_j(\theta) = \log \frac{(j+c+r\cos\theta)^2 + (|r\sin\theta| + T)^2}{(T+2)^2}.$$

and note that $L_j(\theta)$ is an even function of θ. Moreover, if $\theta \in [0,\pi]$ and $T \geq 5/7$, by the inequality $\log x \leq x-1$, one has $L_j(\theta) \leq \frac{L_j^\star(\theta)}{T+2}$, where

$$L_j^\star(\theta) = 2r\sin\theta - 4 + \frac{7}{19}((j+c+r\cos\theta)^2 + (r\sin\theta - 2)^2).$$

In light of the choice of $F_{c,r}(\theta)$ (for Dirichlet L-functions) in [6, Definition 5.10], we shall use the following $F_{c,r}(\theta)$ for $\zeta_K(s)$.

Definition 4.16. For $\theta \in [-\pi,\pi]$, we let $\sigma = c + r\cos\theta$, with $c - r > -\frac{1}{2}$, and $t = r\sin\theta$. For $\sigma \geq 1+\eta$, we define

$$F_{c,r}(\theta) = n_K \log\zeta(\sigma) + \frac{1}{2}L_{-1}(\theta) + \log(T+2).$$

For $-\eta \leq \sigma \leq 1+\eta$, we define

$$\begin{aligned} F_{c,r}(\theta) = {} & n_K \log\zeta(1+\eta) + \frac{n_K(1+\eta-\sigma)+2}{4}L_1(\theta) + \frac{n_K(1+\eta-\sigma)+2}{2}\log(T+2) \\ & + \frac{1+\eta-\sigma}{2}\Big(\log\frac{d_K}{(2\pi)^{n_K}}\Big) + \log 3. \end{aligned}$$

For $\sigma < -\eta$, we define

$$\begin{aligned} F_{c,r}(\theta) = {} & n_K \log\zeta(1-\sigma) + \frac{1}{2}L_{-1}(\theta) + \log(T+2) + \frac{1-2\sigma}{2}\log\Big(\frac{d_K(T+2)^{n_K}}{(2\pi)^{n_K}}\Big) \\ & + \frac{(1-2\sigma+2[\sigma])n_K}{4}L_{1-[\sigma]}(\theta) + \frac{n_K}{2}\sum_{j=1}^{-[\sigma]} L_{j-1}(\theta). \end{aligned}$$

We note that $F_{c,r}(\theta)$ is an even function of θ satisfying $F_{c,r}(\theta) \geq \frac{1}{N}\log|f_N(c+re^{i\theta})|$. In order to bound $F_{c,r}(\theta)$, following [6], for $c \in \mathbb{R}$ and $r > 0$, we define

$$\theta_y = \begin{cases} 0 & \text{if } c + r \le y; \\ \arccos \frac{y-c}{r} & \text{if } c - r \le y \le c + r; \\ \pi & \text{if } y \le c - r. \end{cases}$$

For the sake of convenience, we define

$$\kappa_1 = \int_{\theta_{1+\eta}}^{\theta_{-\eta}} \frac{1+\eta-\sigma}{2} d\theta + \int_{\theta_{-\eta}}^{\pi} \frac{1-2\sigma}{2} d\theta,$$

For $J_1, J_2 \in \mathbb{N}$, we shall set

$$\kappa_2(J_1) = \frac{\pi}{4J_1} \Big(\log \zeta(c+r) + 2 \sum_{j=1}^{J_1-1} \log \zeta \Big(c + r \cos \frac{\pi j}{2J_1} \Big) \Big),$$

and

$$\kappa_3(J_2) = \frac{\pi - \theta_{1-c}}{2J_2} \Big(\log \zeta(1-c+r) + 2 \sum_{j=1}^{J_2-1} \log \zeta \Big(1 - c - r \cos \Big(\frac{\pi j}{J_2} + \Big(1 - \frac{j}{J_2} \Big) \theta_{1-c} \Big) \Big) \Big).$$

In addition, we define

$$\kappa_4 = \frac{1}{4} \int_{\theta_{1+\eta}}^{\theta_{-\eta}} (1+\eta-\sigma) L_1^\star(\theta) d\theta,$$
$$\kappa_5 = \frac{1}{4} \int_{\theta_{-\eta}}^{\theta_{-1/2}} (1-2\sigma) L_1^\star(\theta) d\theta.$$

Similar to [6, Proposition 5.13], we have the following proposition regarding the upper bound of $\int_0^\pi F_{c,r}(\theta) d\theta$.

Proposition 4.17. *Let c, r, and η be positive real numbers satisfying*

$$-\frac{1}{2} < c - r < -\eta < 1 + \eta < c \tag{4.22}$$

and $0 < \eta \le \frac{1}{2}$. *Then for* $T \ge \frac{5}{7}$, *we have*

$$\begin{aligned}\int_0^\pi F_{c,r}(\theta)d\theta \le{}& n_K \int_0^{\theta_{1+\eta}} \log\zeta(\sigma)d\theta + \frac{1}{2(T+2)}\int_0^{\theta_{1+\eta}} L^\star_{-1}(\theta)d\theta + \theta_{1+\eta}\log(T+2) \\ &+ n_K(\log\zeta(1+\eta))(\theta_{-\eta}-\theta_{1+\eta}) + \Big(\log\frac{d_K(T+2)^{n_K}}{(2\pi)^{n_K}}\Big)\kappa_1 \\ &+ \frac{n_K}{T+2}\kappa_4 + \frac{1}{2(T+2)}\int_{\theta_{1+\eta}}^{\theta_{-\eta}} L^\star_1(\theta)d\theta + (\theta_{-\eta}-\theta_{1+\eta})\log(3(T+2)) \\ &+ n_K\int_{\theta_{-\eta}}^{\pi}\log\zeta(1-\sigma)d\theta + \frac{1}{2(T+2)}\int_{\theta_{-\eta}}^{\pi} L^\star_{-1}(\theta)d\theta + (\pi-\theta_{-\eta})\log(T+2) \\ &+ \frac{n_K}{T+2}\kappa_5.\end{aligned}$$

Proof. We first write

$$\int_0^\pi F_{c,r}(\theta)d\theta = \int_0^{\theta_{1+\eta}} F_{c,r}(\theta)d\theta + \int_{\theta_{1+\eta}}^{\theta_{-\eta}} F_{c,r}(\theta)d\theta + \int_{\theta_{-\eta}}^{\pi} F_{c,r}(\theta)d\theta.$$

By the definition of $F_{c,r}(\theta)$, we have

$$\begin{aligned}\int_0^{\theta_{1+\eta}} F_{c,r}(\theta)d\theta &= n_K\int_0^{\theta_{1+\eta}}\log\zeta(\sigma)d\theta + \frac{1}{2}\int_0^{\theta_{1+\eta}} L_{-1}(\theta)d\theta + \int_0^{\theta_{1+\eta}}\log(T+2)d\theta \\ &\le n_K\int_0^{\theta_{1+\eta}}\log\zeta(\sigma)d\theta + \frac{1}{2(T+2)}\int_0^{\theta_{1+\eta}} L^\star_{-1}(\theta)d\theta + \theta_{1+\eta}\log(T+2).\end{aligned}$$

Secondly, we compute

$$\begin{aligned}\int_{\theta_{1+\eta}}^{\theta_{-\eta}} F_{c,r}(\theta)d\theta ={}& n_K\int_{\theta_{1+\eta}}^{\theta_{-\eta}}\log\zeta(1+\eta)d\theta + \Big(\log\frac{d_K}{(2\pi)^{n_K}}\Big)\int_{\theta_{1+\eta}}^{\theta_{-\eta}}\frac{1+\eta-\sigma}{2}d\theta + \log 3\int_{\theta_{1+\eta}}^{\theta_{-\eta}} 1d\theta \\ &+ \int_{\theta_{1+\eta}}^{\theta_{-\eta}}\frac{n_K(1+\eta-\sigma)+2}{4}L_1(\theta)d\theta + \log(T+2)\int_{\theta_{1+\eta}}^{\theta_{-\eta}}\frac{n_K(1+\eta-\sigma)+2}{2}d\theta.\end{aligned} \tag{4.23}$$

The first three integrals on the right of (4.23) are

$$n_K(\log\zeta(1+\eta))(\theta_{-\eta}-\theta_{1+\eta}) + \Big(\log\frac{d_K}{(2\pi)^{n_K}}\Big)\int_{\theta_{1+\eta}}^{\theta_{-\eta}}\frac{1+\eta-\sigma}{2}d\theta + (\log 3)(\theta_{-\eta}-\theta_{1+\eta}).$$

As $1+\eta-\sigma \ge 0$ for $\theta\in[\theta_{1+\eta},\theta_{-\eta}]$, it follows that the last two integrals on the right of (4.23) are

$$
\begin{aligned}
&\frac{n_K}{4}\int_{\theta_{1+\eta}}^{\theta_{-\eta}}(1+\eta-\sigma)L_1(\theta)d\theta+\frac{1}{2}\int_{\theta_{1+\eta}}^{\theta_{-\eta}}L_1(\theta)d\theta+n_K\log(T+2)\int_{\theta_{1+\eta}}^{\theta_{-\eta}}\frac{1+\eta-\sigma}{2}d\theta\\
&+(\theta_{-\eta}-\theta_{1+\eta})\log(T+2)\\
&\leq\frac{n_K}{(T+2)}\kappa_4+\frac{1}{2(T+2)}\int_{\theta_{1+\eta}}^{\theta_{-\eta}}L_1^\star(\theta)d\theta+n_K\log(T+2)\int_{\theta_{1+\eta}}^{\theta_{-\eta}}\frac{1+\eta-\sigma}{2}d\theta\\
&+(\theta_{-\eta}-\theta_{1+\eta})\log(T+2).
\end{aligned}
$$

Lastly, we have

$$
\begin{aligned}
\int_{\theta_{-\eta}}^{\pi}F_{c,r}(\theta)d\theta&=n_K\int_{\theta_{-\eta}}^{\pi}\log\zeta(1-\sigma)d\theta+\frac{1}{2}\int_{\theta_{-\eta}}^{\pi}L_{-1}(\theta)d\theta+\int_{\theta_{-\eta}}^{\pi}\log(T+2)d\theta\\
&+\Big(\log\frac{d_K(T+2)^{n_K}}{(2\pi)^{n_K}}\Big)\int_{\theta_{-\eta}}^{\pi}\frac{1-2\sigma}{2}d\theta+n_K\int_{\theta_{-\eta}}^{\theta_{-\frac{1}{2}}}\frac{1-2\sigma}{4}L_1(\theta)d\theta\\
&+n_K\sum_{j=1}^{\infty}\int_{\theta_{-j+\frac{1}{2}}}^{\theta_{-j-\frac{1}{2}}}\Big(\frac{1-2\sigma-2j}{4}L_{j+1}(\theta)+\frac{1}{2}\sum_{k=1}^{j}L_{k-1}(\theta)\Big)d\theta.
\end{aligned}\tag{4.24}
$$

The first four integrals on the right of (4.24) are

$$
\begin{aligned}
&\leq n_K\int_{\theta_{-\eta}}^{\pi}\log\zeta(1-\sigma)d\theta+\frac{1}{2(T+2)}\int_{\theta_{-\eta}}^{\pi}L_{-1}^\star(\theta)d\theta+\log(T+2)(\pi-\theta_{-\eta})\\
&+\Big(\log\frac{d_K(T+2)^{n_K}}{(2\pi)^{n_K}}\Big)\int_{\theta_{-\eta}}^{\pi}\frac{1-2\sigma}{2}d\theta.
\end{aligned}
$$

Note that as $-\frac{1}{2}<c-r$, we have $\theta_{-j+\frac{1}{2}}=\theta_{-j-\frac{1}{2}}=\pi$ for $j\geq 1$. Thus, the remaining integral and sum on the the right of (4.24) is

$$
n_K\int_{\theta_{-\eta}}^{\theta_{-\frac{1}{2}}}\frac{1-2\sigma}{4}L_1(\theta)d\theta\leq\frac{n_K}{T+2}\int_{\theta_{-\eta}}^{\theta_{-\frac{1}{2}}}\frac{1-2\sigma}{4}L_1^\star(\theta)d\theta=\frac{n_K}{T+2}\kappa_5.
$$

Putting all the estimates together, we complete the proof. □

To control "zeta integrals" in the above proposition, we shall borrow two estimates from [6, Lemmata 5.14 and 5.15] as follows.

Lemma 4.18. *Let c, r and η be positive real numbers, satisfying* (4.22), *and J_1 and J_2 be positive* *integers. If* $\theta_{1+\eta} \le 2.1$*, then for* $\sigma = c + r\cos\theta$*, one has*

$$\int_0^{\theta_{1+\eta}} \log\zeta(\sigma)d\theta \le \frac{\log\zeta(1+\eta)+\log\zeta(c)}{2}\Big(\theta_{1+\eta}-\frac{\pi}{2}\Big)+\frac{\pi}{4J_1}\log\zeta(c)+\kappa_2(J_1).$$

In addition, assuming further $r > 2c-1$*, one has*

$$\int_{\theta_{-\eta}}^{\pi} \log\zeta(1-\sigma)d\theta \le \frac{\log\zeta(1+\eta)+\log\zeta(c)}{2}(\theta_{1-c}-\theta_{-\eta})+\frac{\pi-\theta_{1-c}}{2J_2}\log\zeta(c)+\kappa_3(J_2).$$

4.4 Completing the proof

. Gathering (4.12) and Propositions 4.10 and 4.12, for

$$\begin{aligned} -\frac{1}{2} < c-r < 1-c < -\eta < 0 < \frac{1}{4} \le \delta = 2c-\sigma_1-\frac{1}{2} \\ < \frac{1}{2} < 1 < 1+\eta < c < \sigma_1 = c+\frac{(c-1/2)^2}{r} < c+r, \end{aligned} \tag{4.25}$$

satisfying $\theta_{1+\eta} \le 2.1$, we have

$$\begin{aligned} &\Big|N_K(T)-\frac{T}{\pi}\log\Big(d_K\Big(\frac{T}{2\pi e}\Big)^{n_K}\Big)+\frac{r_1}{4}\Big| \\ &\le \frac{5}{2}+g_K(T)+\frac{2n_K}{\pi}\log\zeta(\sigma_1)+\frac{\log\Big(\frac{1}{\sqrt{(c-1)^2+T^2}}\frac{\zeta_K(c)}{\zeta_K(2c)}\Big)}{\log\frac{r}{c-\frac{1}{2}}}+\frac{1}{\pi\log\frac{r}{c-\frac{1}{2}}}\int_0^{\pi}F_{c,r}(\theta)d\theta+\frac{E_K(T,\delta)}{\pi}, \end{aligned} \tag{4.26}$$

where $g_K(T)$ and $E_K(T,\delta)$ are defined as in (4.9) and (4.14), respectively, and

$$\log\frac{\zeta_K(c)}{\zeta_K(2c)} = \int_c^{2c} -\frac{\zeta_K'}{\zeta_K}(\sigma)d\sigma \le n_K\int_c^{2c} -\frac{\zeta'}{\zeta}(\sigma)d\sigma \le n_K\log\frac{\zeta(c)}{\zeta(2c)}.$$

Finally, using (4.10), Lemma 4.5, Proposition 4.17, and Lemma 4.18 to bound (4.26) and recalling that $r_1+2r_2=n_K$, for any $T_0 \ge \frac{5}{7}$, we obtain

$$\Big|N_K(T)-\frac{T}{\pi}\log\Big(d_K\Big(\frac{T}{2\pi e}\Big)^{n_K}\Big)+\frac{r_1}{4}\Big| \le C_1\log\Big(\frac{d_K(T+2)^{n_K}}{(2\pi)^{n_K}}\Big)+C_2n_K+C_3 \tag{4.27}$$

whenever $T \geq T_0$, where

$$C_1 = \kappa_1 \Big(\pi \log \frac{r}{c-\frac{1}{2}} \Big)^{-1},$$

$$\begin{aligned} C_2 = &\frac{1}{25T_0} + \frac{2}{\pi}\log\zeta(\sigma_1) + \frac{640\delta - 112}{1536(3T_0-1)} + \max\Big\{0, \frac{856\delta-151}{1536(3T_0+2)} - \frac{640\delta-112}{1536(3T_0-1)}\Big\} + \frac{1}{2^{10}} \\ &+ \Big(\pi\log\frac{r}{c-\frac{1}{2}}\Big)^{-1}\Big(\frac{\log\zeta(1+\eta)+\log\zeta(c)}{2}\Big(\theta_{1+\eta}-\frac{\pi}{2}\Big) + \frac{\pi}{4J_1}\log\zeta(c) + \kappa_2(J_1)\Big) \\ &+ \Big(\pi\log\frac{r}{c-\frac{1}{2}}\Big)^{-1}\Big(\frac{\log\zeta(1+\eta)+\log\zeta(c)}{2}(\theta_{1-c}-\theta_{-\eta}) + \frac{\pi-\theta_{1-c}}{2J_2}\log\zeta(c) + \kappa_3(J_2)\Big) \\ &+ \Big(\pi\log\frac{r}{c-\frac{1}{2}}\Big)^{-1}\Big((\log\zeta(1+\eta))(\theta_{-\eta}-\theta_{1+\eta}) + \max\Big\{0, \frac{\kappa_4+\kappa_5}{T_0+2}\Big\} + \pi\log\frac{\zeta(c)}{\zeta(2c)}\Big), \end{aligned}$$

$$\begin{aligned} C_3 = &\frac{5}{2} + \Big(\pi\log\frac{r}{c-\frac{1}{2}}\Big)^{-1}\Big(\pi\log\Big(1+\frac{2}{T_0}\Big) + (\theta_{-\eta}-\theta_{1+\eta})\log 3\Big) \\ &+ \max\Big\{0, \Big(\pi\log\frac{r}{c-\frac{1}{2}}\Big)^{-1}\Big(\frac{1}{2(T_0+2)}\Big(\int_0^{\theta_{1+\eta}} L^\star_{-1}(\theta)d\theta + \int_{\theta_{1+\eta}}^{\theta_{-\eta}} L^\star_1(\theta)d\theta + \int_{\theta_{-\eta}}^{\pi} L^\star_{-1}(\theta)d\theta\Big)\Big)\Big\}. \end{aligned}$$

For $T_0 = 1$ and $T_0 = 10$, choosing $J_1 = 64$ and $J_2 = 39$, via a Maple numerical computation, we have the following table of admissible (C_1, C_2, C_3).

Table 4.2: Choices of parameters (c, r, η) and resulting admissible (C_1, C_2, C_3)

				$T \geq 1$		$T \geq 10$	
c	r	η	C_1	C_2	C_3	C_2	C_3
1.000011314	1.064340602	$4.2826451 \cdot 10^{-6}$	0.22737	23.02528	4.51954	22.97204	3.30668
1.042877508	1.259860485	0.01737451737	0.24493	6.66558	4.21201	6.60397	3.12362
1.079779637	1.410370323	0.03441682600	0.26304	5.22032	4.08149	5.15251	3.05074
1.114294066	1.538391756	0.05247813411	0.28032	4.43521	4.00936	4.36214	3.01124
1.145720440	1.645584376	0.07107039918	0.29590	3.93889	3.96852	3.86136	2.98903

One may find functioning Maple code at `https://arxiv.org/abs/2102.04663`

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