TOPICS IN EXPLICIT NUMBER THEORY

ELCHIN HASANALIZADE
Master of Science, Royal Institute of Technology (KTH), 2013

A thesis submitted
in partial fulfilment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

THEORETICAL AND COMPUTATIONAL SCIENCE

Department of Mathematics and Computer Science
University of Lethbridge
LETHBRIDGE, ALBERTA, CANADA

TOPICS IN EXPLICIT NUMBER THEORY

ELCHIN HASANALIZADE

Date of Defence: August 21, 2023

Dr. H. Kadiri	Associate Professor	Ph.D.
Dr. N. Ng	Professor	Ph. D.
Thesis Co -supervisors		
Dr. D. Morris	Professor	Ph.D.
Thesis Examination Committee Member		
Dr. A. Fiori	Associate Professor	Ph.D.
Thesis Examination Committee Member		
Dr. F. Luca	Professor	Ph.D.
External Examiner		
University of Witwatersrand, South Africa		
Dr. J. Sheriff	Assistant Professor	Ph.D.
Chair, Thesis Examination Committee		

Abstract

This doctoral thesis is made up of a collection of papers of the author. All of these papers are already published. The structure of this thesis is as follows. After the introduction, there are four chapters.

In Chapters 2 and 3, we obtain explicit bounds for the number of non-trivial zeros of the Riemann zeta function and Dedekind zeta functions, which improve previous results of Trudgian [120]. This improvement is based on ideas from previous works of Bennett *et al.* [11], Kadiri and Ng [63] and Trudgian [120]. This is a joint work with Quanli Shen and Peng-Jie Wong.

In Chapter 4, we apply explicit results from transcendental number theory due to Bugeaud [20], Laurent [72] and Matveev [79] to completely solve a Diophantine inequality involving the Fibonacci numbers and to study a particular case of the Terai-Shinsho conjecture.

In Chapter 5, using elementary methods and explicit bounds for primes in arithmetic progressions due to Bennett *et al.* [10] we study two Diophantine equations which involve multiplicative functions.

Acknowledgments

First and foremost, I would like to thank my supervisors Dr. Habiba Kadiri and Dr. Nathan Ng for their patient guidance, constant encouragement and support throughout my doctoral studies.

Thank-you to Dr. Florian Luca for being my external examiner. Thank you to Dr. Andrew Fiori and Dr. Dave Morris for being members of my supervisory committee.

I would also like to thank my co-authors Quanli Shen and Peng-Jie Wong for the papers we have written together; it has been enjoyable and rewarding experience to work with you.

The Chapters 2-5 are based on the papers that have been published or submitted for publication. Many thanks to the referees for their useful and constructive comments.

Finally, I would like to thank my family, in particular my wife Jamila and my daughter Fatima, for constant patience and belief in me.

Publications of the author

The following articles were written and published throughout the PhD programme (September 2019- August 2023):

1. Hasanalizade, E. On the distribution of the divisor function at consecutive integers, *Bull. Aust. Math. Soc.*, **103**, (2021), 244-247.

2. Hasanalizade, E. On the equation $\phi(n+k) = 2\phi(n)$, *Integers* **21** (2021), A68, 5 pp.

3. Hasanalizade, E., Shen, Q. and Wong, P.-J. Counting zeros of Dedekind zeta functions, *Math. Comp.*, **91**, (2021), no. 333, 277-293.

4. Hasanalizade, E. On multiplicative functions additive on Goldbach-type sets, *Aequat. Math.*, no. 4, (2022), 827-832.

5. Hasanalizade, E. On a k-additive uniqueness set for multiplicative functions, *Bull. Aust. Math. Soc.*, **106**, (2022), 209-214.

6. Hasanalizade, E., Shen, Q. and Wong, P.-J. Counting zeros of the Riemann zeta function, *J. Number Theory*, **235**, (2022), 219-241.

7. Hasanalizade, E. Multiplicative functions k-additive on generalized pentagonal numbers, *Integers* **22** (2022), A43, 6pp.

8. Hasanalizade, E. On a conjecture of Deaconescu, *Integers* **22** (2022), A99, 6pp.

9. Hasanalizade, E. Sums of Fibonacci numbers close to a power of 2, *Boletín de la Sociedad Matemática Mexicana*, Article number: 19, (2023).

10. Hasanalizade, E. On near-perfect numbers of special forms, *Bull. Aust. Math. Soc.* (2023), 1-7.

11. Hasanalizade, E. On the equation $\sigma^*(n) = 1 + Mn$, *Integers* **23** (2023), A31, 6pp.

12. Hasanalizade, E. A note on the exponential Diophantine equation $(44m^2+1)^x + (5m^2-1)^y = (7m)^z$, (2023), preprint.

Contributions of authors

The Chapters 2,3,4 and 5 are slightly modified versions of the following six papers [50–55].

- (with Quanli Shen and Peng-Jie Wong) Counting zeros of Dedekind zeta functions, *Math. Comp.*, **91**, (2021), no. 333, 277-293
- (with Quanli Shen and Peng-Jie Wong) Counting zeros of the Riemann zeta function, *J. Number Theory*, **235**, (2022), 219-241
- Sums of Fibonacci numbers close to a power of 2, *Boletín de la Sociedad Matemática Mexicana*, Article number: 19, (2023)
- A note on the exponential Diophantine equation $(44m^2+1)^x+(5m^2-1)^y=(7m)^z$, (2023), preprint
- On the equation $\phi(n+k)=2\phi(n)$, *Integers* **21** (2021), A68, 5 pp
- On the equation $\sigma^*(n)=1+Mn$, *Integers* **23** (2023), A31, 6pp

Chapter 2 is based on collaborations with Quanli Shen and Peng-Jie Wong in the article [54]. All authors contributed equally to this project. Specifically, I made significant contributions to Subsection 2.3.1, Lemma 2.8, Proposition 2.10, Subsection 2.3.3, Proposition 2.14, Lemma 2.15, Definition 2.16 and I also performed computations on Maple. Furthermore, all authors proofread the whole article.

Chapter 3 is based on collaborations with Quanli Shen and Peng-Jie Wong in the article [55]. All authors contributed equally to this project. Specifically, I made significant contributions to Subsections 3.4.1-3.4.2, Section 3.5 and I also did Maple computations. Furthermore, all authors proofread the whole article.

Contents

List of Tables

Notations

Below we introduce some basic notations and definitions that we will use throughout this thesis.

- $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ are the set of natural, integer, real, and complex numbers respectively.
- The letter p with or without indices is used to denote prime numbers, q,a,n are integers, x is a real number and $s = \sigma + it$ is a complex number.
- The notation $n \equiv a (\text{mod } q)$ means there exists $k \in \mathbb{Z}$ such that $n = kq + a$.
- $(a,q) = \gcd(a,q)$ is the greatest common divisor of a and q.
- The Euler totient function $\phi(n)$ is defined by $\phi(n) = \#\{m \in \mathbb{N} |\ m \leq n \text{ and } (m,n) = 1\}$.
- The sum of divisors function $\sigma(n)$ is given by

$$\sigma(n) = \sum_{d|n} d.$$

- The unitary sum of divisors function $\sigma^*(n)$ is defined by

$$\sigma^*(n) = \sum_{\substack{d|n \\ (d,\frac{n}{d})=1}} d.$$

.

Suppose f,g are real functions in x.

- $f(x) = O(g(x))$ if there is a constant C such that for all sufficiently large x, $|f(x)| \leq Cg(x)$.

- $f(x) = o(g(x))$ means $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$.
- $f(x) \sim g(x)$ means $\frac{f(x)}{g(x)} \to 1$ as $x \to \infty$.
- The von Mangoldt function $\Lambda(n)$ is an arithmetic function defined by

$$\Lambda(n) = \begin{cases} \log p \text{ if } n = p^k,\ k \geq 0, \\ 0 \text{ otherwise} \end{cases}$$

- Prime counting functions:

$$\pi(x) = \sum_{p\leq x} 1,$$
$$\theta(x) = \sum_{p\leq x} \log p$$
$$\psi(x) = \sum_{n\leq x} \Lambda(n).$$

- More generally, for $(a,q) = 1$ we have

$$\pi(x;q,a) = \sum_{\substack{p\leq x \\ p\equiv a (\text{mod } q)}} 1,$$
$$\theta(x;q,a) = \sum_{\substack{p\leq x \\ p\equiv a (\text{mod } q)}} \log p$$
$$\psi(x;q,a) = \sum_{\substack{p^k\leq x,\ k\geq 1 \\ p\equiv a (\text{mod } q)}} \log p.$$

- Let $q \in \mathbb{N}$, A *Dirichlet character* mod q is a function $\chi : \mathbb{Z} \to \mathbb{C}$ satisfying the following three properties:

  (i) $\chi(n) = 0$ holds if and only if $(n,q) > 1$,

  (ii) $\chi(n+q) = \chi(n)$,

  (iii) $\chi(n_1 n_2) = \chi(n_1)\chi(n_2)$.

- The Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \Re s > 1$$

 and the Dirichlet L-functions

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \quad \Re s > 1,$$

 where $\chi(n)$ are defined above.

- The Gamma function is given by

$$\Gamma(s) = \int_0^{\infty} x^{s-1} e^{-x} dx, \ \ \Re s > 0.$$

The following are related to the number fields.

- $\mathbb{K}$ is a number field with ring of integers $O_{\mathbb{K}}$.
- r_1 and $2r_2$ are the number of real and complex embeddings of $\mathbb{K}$.
- $n_{\mathbb{K}} = r_1 + 2r_2$ is the degree of $\mathbb{K}$.
- $d_{\mathbb{K}}$ is the discriminant of $\mathbb{K}$.
- $\mathfrak{a} \subset O_{\mathbb{K}}$ is an integral ideal.
- $\mathfrak{p} \subset O_{\mathbb{K}}$ is a prime ideal.
- $N_{\mathbb{K}}(\mathfrak{a})$ is the norm of $\mathfrak{a}$ in $\mathbb{K}$.
- The Dedekind zeta function is defined by

$$\zeta_{\mathbb{K}}(s) = \sum_{\mathfrak{a} \neq 0} \frac{1}{N_{\mathbb{K}}(\mathfrak{a})^s}$$

The following are related to the zeros of $\zeta(s)$ and $\zeta_{\mathbb{K}}(s)$:

- For $\beta, \gamma \in \mathbb{R}$, $\rho = \beta + i\gamma$ denotes a nontrivial zero of either $\zeta(s)$ or $\zeta_{\mathbb{K}}(s)$.
- Let $T > 0$.

$$N(T) = \#\{\rho = \beta + i\gamma \in \mathbb{C} |\ \zeta(\rho) = 0,\ 0 < \beta < 1, 0 < \gamma \leq T\}.$$

- Similarly

$$N_K(T) = \#\{\rho = \beta + i\gamma \in \mathbb{C} |\ \zeta_K(\rho) = 0,\ 0 < \beta < 1, |\gamma| \leq T\}.$$

Chapter 1

Introduction

1.1 Prime Number Theorem (PNT)

Prime numbers have always been a fascinating subject for many generations of mathematicians - from ancient times of Euclid all the way up to modern times. They stand apart from other sequences of numbers that have a regular pattern. I could not have put it better myself but quote from Don Zagier's inaugural lecture *The first 50 million prime numbers* [124]:

> There are two facts about the distribution of prime numbers of which I hope to convince you so overwhelmingly that they will be permanently engraved in your hearts. The first is that, despite their simple definition and role as the building blocks of the natural numbers, the prime numbers belong to the most arbitrary and ornery objects studied by mathematicians: they grow like weeds among the natural numbers, seeming to obey no other law than that of chance, and nobody can predict where the next one will sprout. The second fact is even more astonishing, for it states just the opposite: that the prime numbers exhibit stunning regularity, that there are laws governing their behaviour, and that they obey these laws with almost military precision.

But wherever there is a regular structure among natural numbers there are mathematicians looking for the explanation. After examining tables of large prime numbers, Gauss was led to conjecture that the prime-counting function

$$\pi(x) = \#\{p \leq x |\ p \text{ is a prime}\}$$

is asymptotic to $\frac{x}{\log x}$ as $x \to \infty$. F.W. Bessel in his letter to W. Olbers [37] mentioned for the first time that the number $\pi(x)$ is well approximated by the *logarithmic integral* defined by

$$li(x) = \lim_{\varepsilon \to 0} \left(\int_0^{1-\varepsilon} + \int_{1+\varepsilon}^{x} \right) \frac{dt}{\log t} = \int_2^x \frac{dt}{\log t} - 1.04\ldots$$

In 1808 A.M. Legendre conjectured that a good approximation to $\pi(x)$ is given by

$$\frac{x}{\log x - 1.08366}.$$

The first result about the asymptotic behaviour of $\pi(x)$ was obtained by P.L. Chebyshev, who showed that if the limit

$$\lim_{x \to \infty} \frac{\pi(x) \log x}{x}$$

exists, then it must be 1. In 1852 he derived rather close lower and upper bounds for $\pi(x)$. Namely, if x is sufficiently large, then

$$0.9212 \frac{x}{\log x} \leq \pi(x) \leq 1.1056 \frac{x}{\log x}.$$

He also introduced what are now known as the Chebyshev functions:

$$\theta(x) = \sum_{p \leq x} \log p, \text{ and } \psi(x) = \sum_{p^k \leq x,\ k \geq 1} \log p.$$

In his seminal memoire *Über die Anzahl der Primzahlen unter einer gegebener Größe* [95] published in 1859 Gauss's student Riemann considered for the first time at complex arguments the function

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p (1 - p^{-s})^{-1}$$

now called the *Riemann zeta function*. He showed that $\pi(x)$ is intimately connected to the analytic properties of this function. Moreover, he proved that $\zeta(s)$ extends to a meromorphic function on $\mathbb{C}$ which has only a simple pole at $s = 1$ and satisfies the functional equation

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s).$$

Riemann zeta function $\zeta(s)$ vanishes at $s = -2, -4, -6, \ldots$ (these are called the *trivial zeros*. All other zeros are located in the so-called *critical strip* $0 \leq \Re(s) \leq 1$. Riemann conjectured that all non-trivial zeros of $\zeta(s)$ lie on the vertical line $\Re(s) = \frac{1}{2}$ called the *critical line*. This is known as the Riemann Hypothesis. But what do the zeros of the Riemann zeta function have to do with the prime numbers? The connection between the zeros $\rho = \beta + i\gamma$ of the Riemann zeta function with the prime numbers is given by the remarkable formula

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \log 2\pi - \frac{1}{2}\log(1 - x^{-2})$$

valid for all $x \geq 2$ not a prime power. This beautiful formula shows that zeros of the Riemann zeta function, an analytic object are intimately related to the distribution of prime numbers, a “discrete” arithmetic object. If the Riemann Hypothesis is true a variant of the the above formula implies that

$$\psi(x) = x + O(x^{1/2}(\log x)^2).$$

In 1896 Hadamard and de la Vallée Poussin independently proved the conjecture of Gauss, which is now the celebrated *Prime Number Theorem*. It can be stated in an equivalent form as

$$\psi(x) \sim x \text{ and } \theta(x) \sim x \text{ as } x \to \infty.$$

The crucial part of the proof was the fact that $\zeta(s)$ has no zeros on the line $\sigma = 1$.

Later, in 1899 de la Vallée Poussin obtained more precise result. He showed that the non-existence of zeta zeros in the region $\sigma > 1 - c/\log(|t| + 2)$ for some constant $c > 0$

yields the prime number theorem with the error term $\psi(x) = x + O(xe^{-c\sqrt{\log x}})$. In 1958 Korobov [68] and Vinogradov [122] proved that there exists a constant $c_1 > 0$ such that the error term is of size $O\left(x\exp\left(-c_1\frac{(\log x)^{3/5}}{(\log\log x)^{1/5}}\right)\right)$. This is currently asymptotically the strongest error term.

Rosser and Schoenfeld [99, 100] using analytic methods and heavy computations determined explicit bounds for the Chebyshev functions. In particular, they showed that the maximum of the function $\frac{\psi(x)}{x}$ is $1.03883\ldots$ at $x = 113$ [99, Theorem 12]. Moreover,

$$0.980x \leq \theta(x) \leq 1.019x, \text{ for all } x \geq e^{20}$$
$$|\theta(x) - x|, |\psi(x) - x| < 0.0242269\frac{x}{\log x} \text{ for all } x \geq 10^8.$$

as proved in [98, Equation 12] and [100, Theorem 7] respectively.

One also has

$$\frac{x}{\log x} < \pi(x) < 1.25506\frac{x}{\log x}$$

for all $x > 17$ [99, Corollary 1].

Over the years many researchers have worked at improving the work of Rosser and Schoenfeld. For further details of subsequent improvements, the reader may consult: Axler [3], Büthe [24], Dusart [35], Faber and Kadiri [38],Trudgian [121] and Fiori, Kadiri and Swidinsky [42, 43].

Platt and Trudgian [89] showed that $\theta(x) \leq (1 + 7.5 \cdot 10^{-7})x$ for all $x \geq 0$. Broadbent et al. [19] improved this bound to

$$\theta(x) \leq (1 + 1.93378 \cdot 10^{-8})x \text{ for all } x \geq 0.$$

Fiori et al. [42] proved that for all $x > 2$

$$|\psi(x) - x| < 9.22022x(\log x)^{3/2} \exp\left(-0.8476836\sqrt{\log x}\right)$$

and that, for all $x \geq e^{300}$,

$$|\psi(x) - x| < 4.9678 \cdot 10^{-15} x.$$

Using an explicit Vinogradov-Korobov zero-free region for $\zeta(s)$, Johnston and Yang [61] established that for all $x \geq 23$

$$|\psi(x) - x| \leq 0.026x(\log x)^{1.801} \exp\Big(-0.1853 \log^{3/5} x (\log\log x)^{-1/5}\Big).$$

1.1.1 Applications of the explicit bounds for primes

Explicit Chebyshev type bounds have numerous applications in number theory. Below we provide some interesting examples.

1. A *distinct covering system of congruences* is a finite collection of congruences

   $$a_i \bmod m_i, \ 1 < m_1 < m_2 < \ldots < m_k$$

   such that every integer satisfies at least one of them. Erdős asked if there exist distinct covering systems for which the least modulus is arbitrarily large. Hough [59] answered this question negatively. Using numerical calculations together with explicit estimate for $\theta(x)$, he proved that the least modulus of a distinct covering system is at most 10^{16}.

2. Robin's criterion states that the Riemann Hypothesis is true if and only if the inequality $\sigma(n) < e^{\gamma} n \log\log n$ is satisfied for $n \geq 5041$, where $\sigma(n)$ is the sum of all positive divisors of n and $\gamma = 0.5772\ldots$ denotes the Euler-Mascheroni constant. Us-

ing a bound of Rosser and Schoenfeld, Choie et al. [28] showed that if n does not satisfy Robin's criterion then there exists $a > 1$ such that $a^5|n$.

3. A set of m natural numbers $\{a_1, a_2, \ldots, a_m\}$ is called a *Diophantine m-tuple* with property $D(n)$ if $a_i a_j + n$ is a perfect square for all $1 \leq i < j \leq m$. Define $M_n :=$ $\sup\{|S| : S \text{ has property } D(n)\}$, where $|S|$ denotes the number of elements in S. Using Rosser and Schoenfeld lower bound for $\pi(x)$, Dujella [33] obtained that

$$M_n \leq 32 \text{ for } |n| \leq 400,$$
$$M_n < 267.81 \log|n|(\log\log|n|)^2 \text{ for } |n| > 400.$$

4. Using Rosser and Schoenfeld bound for $\theta(x)$, Moree [82] proved that the Diophantine equation of Erdős-Moser type

$$1^k + 2^k + \cdots + (x-1)^k = ax^k$$

has no integer solutions (a, x, k) with

$$k > 1, \; x < \max(10^{10^6}, a \cdot 10^{22}).$$

1.2 PNT in arithmetic progressions

Euler and Legendre asked if there are infinitely many primes in arithmetic progressions with coprime first element and difference. The first attempt to answer this question was made by Legendre but his argument was fallacious. The first correct proof of the infinitude of primes in any progression $a + qn$ with $(a, q) = 1$ was given by Dirichlet in 1837. For this purpose, he introduced a new class of functions which are now called *Dirichlet's L-*

functions. A group homomorphism

$$\chi : \ (\mathbb{Z}/q\mathbb{Z})^{\times} \to \mathbb{C}\backslash\{0\}$$

is called a *Dirichlet character* modulo integer q, where $(\mathbb{Z}/q\mathbb{Z})^{\times}$ is the multiplicative group of reduced residue classes (mod q). We extend χ to an arithmetic function $\chi : \ \mathbb{Z} \to \mathbb{C}$ by setting $\chi(n) = \chi(n \bmod q)$ if $(n,q) = 1$ and $\chi(n) = 0$ if $(n,q) > 1$.

The Dirichlet L-function associated to χ modulo q is defined by

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}$$

for $\Re s > 1$.

Let $\phi(n)$ denote the Euler totient function and let

$$\pi(x;q,a) = \sum_{\substack{p \le x \\ p \equiv a (\text{mod } q)}} 1,$$

$$\theta(x;q,a) = \sum_{\substack{p \le x \\ p \equiv a (\text{mod } q)}} \log p,$$

$$\psi(x;q,a) = \sum_{\substack{p^k \le x, \ k \ge 1 \\ p \equiv a (\text{mod } q)}} \log p.$$

Dirichlet's prime number theorem for arithmetic progressions asserts that, for any fixed q, we have

$$\pi(x;a,q) \sim \frac{x}{\phi(q)\log x} \text{ as } x \to \infty.$$

This is equivalent to

$$\theta(x;a,q) \sim \frac{x}{\phi(q)} \text{ and } \psi(x;a,q) \sim \frac{x}{\phi(q)} \text{ as } x \to \infty.$$

As with the PNT, we cite some explicit results for the primes in arithmetic progressions. Let

$$E(x;a,q) = \left| \frac{\psi(x;a,q) - \frac{x}{\phi(q)}}{\frac{x}{\phi(q)}} \right|.$$

McCurley [81] proved that $E(x;a,q) \leq 11.50$ for $x \geq e^{(\log q)^2}$ and $q \geq 10^{13}$. Ramaré and Rumely [94] showed that $E(x;a,q) \leq 0.002$ for $x \geq 10^{100}$ and $q = 13$ and in Kadiri and Lumley [76] improved this bound showing that $E(x;a,q) \leq 4.992 \cdot 10^{-6}$ for $x \geq 10^{100}$ and $q = 13$. These results are actually proven for a finite family of "small" moduli (including $q = 13$) for which the Generalized Riemann Hypothesis (GRH) was partially verified. Bennett et al. [10] proved the following general result.

Theorem 1.1. *Let $q \geq 3$ be an integer and let a be an integer that is coprime to q. There exist explicit positive constants $c_\psi(q)$ and $x_\psi(q)$ such that*

$$\left| \psi(x;a,q) - \frac{x}{\phi(q)} \right| \leq c_\psi(q) \frac{x}{\log x} \textit{ for all } x \geq x_\psi(q)$$

Moreover, $c_\psi(q)$ and $x_\psi(q)$ satisfy $c_\psi(q) \leq c_0(q)$ and $x_\psi(q) \leq x_0(q)$ where

$$c_0(q) = \begin{cases} \frac{1}{840}, & \textit{if } 3 \leq q \leq 10^4, \\ \frac{1}{160} & \textit{if } q > 10^4, \end{cases}$$

and

$$x_0(q) = \begin{cases} 8 \cdot 10^9, & \textit{if } 3 \leq q \leq 10^5, \\ \exp\left(0.03\sqrt{q}\log^3 q\right), & \textit{if } q > 10^5. \end{cases}$$

They also derived explicit bounds for $\pi(x;q,a)$ and $p_n(q,a)$ for certain moduli q, where $p_n(q,a)$ is the nth smallest prime that is congruent to a modulo q.

Theorem 1.2. *([10, Corollary 1.6]) Let $1 \leq q \leq 1200$ be an integer, and let a be an integer that is coprime to q.*

- *For all $x \geq 50q^2$, we have*

$$\frac{x}{\phi(q)\log x} < \pi(x;q,a) < \frac{x}{\phi(q)\log x}\left(1+\frac{5}{2\log x}\right).$$

- *For all positive integers n such that $p_n(q,a) \geq 22q^2$, we have*

$$n\phi(q)\log(n\phi(q)) < p_n(q,a) < n\phi(q)\Big(\log(n\phi(q))+\frac{4}{3}\log\log(n\phi(q))\Big).$$

1.2.1 Applications of the explicit bounds for primes in arithmetic progressions

Here is a non-exhaustive list of some interesting applications of the above explicit bounds for primes in arithmetic progressions:

1. Suppose $3 < q < 10^5$. Francis and Lee [45] proved that every integer greater than two can be written as the sum of a prime and a square-free number co-prime to q.

2. Dalton and Trifonov [31] proved that every positive integer n can be represented in the form $n = s + p$ where s is a squarefree number and p is a prime with $p \leq \sqrt{n}$ except when $n \in \{1,2,3,6,11,30,155,247\}$.

3. Consider the following generalization of a Diophantine m-tuple defined in the section 1.1.1(3). Fix a natural number $k \geq 2$. Given a non-zero integer n, a set of natural numbers $S = \{a_1,a_2,\dots,a_m\}$ is called a *Diophantine m-tuple* with property $D_k(n)$ if $a_ia_j + n$ is a k-th power for all $1 \leq i < j \leq m$. Define $M_k := \sup\{|S| :$ S has property $D_k(n)\}$. Bhattarcharjee et al. [13] showed that if $k = o(\log\log n)$ then $M_k(n) \leq 3\phi(k)\log n + O\Big(\frac{(\phi(k))^2\log n}{\log\log n}\Big)$.

4. Let $a,b,c,m \in \mathbb{N}$ be fixed with $(b,c) = 1$ and $m \geq 2$. Saunders [101] showed that there are only finitely many solutions to the Diophantine equation $\phi(ax^m) = \frac{b\cdot n!}{c}$ and

these solutions satisfy $n \le \max\{61, 3a, 3b, 3c\}$.

5. Let $\mathcal{S} = \{n : \phi(n) = \phi(n+1)\} = \{1, 3, 15, \dots\}$. It is not known if $\mathcal{S}$ is an infinite set but the reciprocal sum of the members of $\mathcal{S}$ is finite. Kinlaw et al. [66] proved that

$$\sum_{n \in \mathcal{S}} \frac{1}{n} < 7.8358.$$

In this thesis we also apply Bennett et al. explicit bounds in Chapter 5 to some Diophantine problems.

1.3 Primes in the Chebotarev Density Theorem

Since the establishment of the classical PNT, mathematicians developed analogues of PNT in other contexts where one can define the notion of ”prime element”. Chebotarev's density theorem is a grand generalization of the prime number theorem and the prime number theorem in arithmetic progressions. Before we state it we require some notations and definitions from algebraic number theory.

Let L/K be a Galois extension of number fields with Galois group G. Since O_L is also a Dedekind domain for any prime ideal $\mathfrak{p}$ of O_K one has

$$\mathfrak{p} O_L = \mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_\ell^{e_\ell}, \tag{1.1}$$

where $\mathfrak{P}_j$'s are distinct prime ideals in O_L and e_j's are positive integers. Note that since L/K is Galois, $e_i = e_j$ for all i, j. We say that $\mathfrak{P}_j$ are above $\mathfrak{p}$. If $e_j = 1$ for every j, then $\mathfrak{p}$ is called unramified. Otherwise $\mathfrak{p}$ is ramified.

Since the Galois group G acts on the prime factors of $\mathfrak{p} O_L$ it is natural to consider the stabiliser subgroup of $\mathfrak{P}|\mathfrak{p}$, i.e. $D_{\mathfrak{P}} = \{\sigma \in G |\ \sigma(\mathfrak{P}) = \mathfrak{P}\}$ which is called the decomposition group at $\mathfrak{P}$. One also has the inertia group at $\mathfrak{P}$

$$I_{\mathfrak{P}} = \{\sigma \in G \mid \sigma(x) \equiv x (\text{mod } \mathfrak{P}) \text{ for all } x \in O_L\}, \tag{1.2}$$

which is a normal subgroup of $D_{\mathfrak{P}}$. Note that the inertia group $I_{\mathfrak{P}}$ is trivial when $\mathfrak{p}$ is unramified in L. Since every non-zero prime ideal is maximal in a Dedekind domain both $O_L/\mathfrak{P}$ and $O_K/\mathfrak{p}$ are fields and in fact finite. Moreover, it can be shown that the extension $(O_L/\mathfrak{P})/(O_K/\mathfrak{p})$ is Galois and $\mathrm{Gal}((O_L/\mathfrak{P})/(O_K/\mathfrak{p}))$ is a cyclic group with a generator $x \to x^{N\mathfrak{p}}$. Therefore, one can choose an element $\sigma_{\mathfrak{P}} \in D_{\mathfrak{P}}$ whose image in $\mathrm{Gal}((O_L/\mathfrak{P})/(O_K/\mathfrak{p}))$ is the generator described above. Such an element is called a Frobenius automorphism at $\mathfrak{P}$.

Let K be a number field and L/K be a finite Galois extension of number fields. For any unramified prime $\mathfrak{p}$ of O_K and any prime $\mathfrak{P}$ of L lying over $\mathfrak{p}$, the *Artin symbol* $\left[\frac{L/K}{\mathfrak{P}}\right]$ is the unique element $\sigma \in \mathrm{Gal}(L/K)$ such that

$$\sigma(\alpha) = \alpha^{N_{K/\mathbb{Q}}(\mathfrak{p})} (\text{mod } \mathfrak{P}) \text{ for all } \alpha \in L. \tag{1.3}$$

The fact that $\mathfrak{p}$ is unramified ensures that such map is well-defined and unique up to the choice of the prime $\mathfrak{P}$ lying over $\mathfrak{p}$: the values of $\left[\frac{L/K}{\mathfrak{P}}\right]$ lying over $\mathfrak{p}$ are all conjugate in $\mathrm{Gal}(L/K)$. We denote by $\left[\frac{L/K}{\mathfrak{p}}\right]$ the conjugacy class of $\left[\frac{L/K}{\mathfrak{P}}\right]$ in $G = \mathrm{Gal}(L/K)$ for any $\mathfrak{P}$ lying over $\mathfrak{p}$ and we will refer to this as the *Artin symbol* of $\mathfrak{p}$.

Informally, the celebrated Chebotarev density theorem says that for any finite Galois extension of number fields L/K and any conjugacy class $\mathcal{C} \subset \mathrm{Gal}(L/K)$ the number of primes $\mathfrak{p}$ of O_K such that $\left[\frac{L/K}{\mathfrak{p}}\right] = \mathcal{C}$ is proportional to the size of $\mathcal{C}$.

For a conjugacy class $\mathcal{C}$ of G and $x > 1$ define

$$\pi_{\mathcal{C}}(x) := |\{\mathfrak{p} \mid \mathfrak{p} \text{ a prime ideal of } K \text{ unramified in } L, \left[\frac{L/K}{\mathfrak{p}}\right] = \mathcal{C} \text{ and } N_{K/\mathbb{Q}}(\mathfrak{p}) \leq x\}| \tag{1.4}$$

In modern language, the Chebotarev's density theorem can be restated as

$$\pi_C(x) \sim \frac{|C|}{|G|}\mathrm{Li}(x)\ (x \to \infty), \tag{1.5}$$

where Li(x) $= \int_2^x \frac{dt}{\log t}$.

Let ρ be a complex finite dimensional representation of $G = \mathrm{Gal}(L/K)$. The Artin L-function attached to ρ is defined by

$$L(s,\rho,L/K) = \prod_{\mathfrak{p}} L_{\mathfrak{p}}(s,\rho,L/K) \tag{1.6}$$

where the local L-function at $\mathfrak{p}$ is defined by

$$L_{\mathfrak{p}}(s,\rho,L/K) = \det(1-\rho|^{V^{I_{\mathfrak{P}}}}(\sigma_{\mathfrak{P}})N\mathfrak{p}^{-s})^{-1} \tag{1.7}$$

for $\Re(s) > 1$. Here the product runs over prime ideals in O_K, $\mathfrak{P}$ a prime ideal above $\mathfrak{p}$, V is the underlying complex vector space on which ρ acts and $V^{I_{\mathfrak{P}}}$ is the subspace of V on which $I_{\mathfrak{P}}$ acts trivially.

While the Chebotarev density theorem is a statement about how prime ideals behave under Galois number fields, it is also a generalization of Dirichlet's theorem on primes in arithmetic progressions. Indeed, apply Chebotarev theorem to $K = \mathbb{Q}$ and $L = \mathbb{Q}(\zeta_q)$, where ζ_q is a primitive qth root of unity. Then $G \simeq (\mathbb{Z}/q\mathbb{Z})^\times$ where the isomorphism is given explicitly by identifying an element $a \in (\mathbb{Z}/q\mathbb{Z})^\times$ with the unique automorphism $\sigma_a \in G$ such that $\sigma_a(\zeta_q) = \zeta_q^a$. Now let $p \nmid q$ be a prime number. Then

$$\left(\frac{\mathbb{Q}(\zeta_q)/\mathbb{Q}}{p}\right) = \sigma_p \tag{1.8}$$

and therefore for any $a \in (\mathbb{Z}/q\mathbb{Z})^\times$

$$\left(\frac{\mathbb{Q}(\zeta_q)/\mathbb{Q}}{p}\right) = \sigma_a \Leftrightarrow \zeta_q^p = \zeta_q^a \Leftrightarrow p \equiv a (\mathrm{mod}\ q) \tag{1.9}$$

By Chebotarev density theorem, we infer that the Dirichlet and natural densities of primes $p \equiv a (\mathrm{mod}\ q)$ is $\frac{1}{|(\mathbb{Z}/q\mathbb{Z})^{\times}|} = \frac{1}{\phi(q)}$.

As with the prime number theorem for arithmetic progressions, proving Chebotarev density theorem requires knowledge about the distribution of zeros of special L-functions attached to L/K. In particular, one must analyze the *Dedekind zeta-function* of L defined by

$$\zeta_L(s) = \sum_{\mathfrak{a}} \frac{1}{N_{L/\mathbb{Q}}(\mathfrak{a})^s} = \prod_{\mathfrak{p}} \Big(1 - \frac{1}{N_{L/\mathbb{Q}}(\mathfrak{p})^s}\Big)^{-1} \text{ for } \Re(s) > 1 \tag{1.10}$$

where the sum is taken over non-zero integral ideals and the product runs through all prime ideals of O_L. Note that for $K = \mathbb{Q}$ and the trivial one-dimensional representation ρ^0 we have $L(s, \rho^0, L/\mathbb{Q}) = \zeta_L(s)$, where $\rho^0(\sigma) = 1$ for all $\sigma \in G$. The Dedekind zeta-function $\zeta_L(s)$ has an analytic continuation to a meromorphic function in the whole complex plane with a simple pole at $s = 1$. Its non-trivial zeros lie in the critical strip $0 < \Re(s) < 1$ and its trivial zeros are at certain non-positive integers. The Dedekind zeta-function suffers from an exceptional zero: that is we cannot eliminate the existence of a real non-trivial zero. Conjecturally such zeros do not exist and the Generalized Riemann Hypothesis (GRH) states that all of the non-trivial zeros lie on the critical line $\Re(s) = \frac{1}{2}$.

Let $S(c,d)$ denote the region given by all $s = \beta + i\gamma$ with $\beta \geq \sigma_c = 1 - c/\log d_K$ and $\gamma \leq t_d = d/\log d_K$. In 1974, Stark [106] proved that a Dedekind zeta function has at most one simple and real zero in the region $S(1/4, 1/4)$. Kadiri [62, Theorem 1.1] showed that the same conclusion holds true in the region $S(1/12.74, 1)$ for d_L large enough. Recently, Lee [73] improved this to the region $S(1/12.44, 1)$.

Early proofs of the Chebotarev density theorem gave no information about the error terms. In 1977 Lagarias and Odlyzko made progress towards resolving this problem by proving the following conditional "effective" version of the Chebotarev density theorem.

Theorem 1.3. *([71, Theorem 1.1]) Let L/K be a Galois extension of number fields with Galois group G, $d_L = |disc(L/\mathbb{Q})|$ and put $n_L = [L : \mathbb{Q}]$. Assume the GRH holds for the Dedekind zeta function $\zeta_L(s)$. Then, there is an effectively computable positive absolute* *constant c such that for every $x > 2$,*

$$\left|\pi_C(x) - \frac{|C|}{|G|}Li(x)\right| \leq c\left(\frac{|C|}{|G|}x^{1/2}\log\left(d_L x^{n_L}\right)\right) \tag{1.11}$$

Lagarias and Odlyzko also proved an unconditional version of their ”effective” Chebotarev density theorem.

Theorem 1.4. *([71, Theorem 1.3]) Let $n_L > 1$. Then there exist effectively computable absolute constants c_1 and c_2 such that if $x \geq exp(10n_L(\log d_L)^2)$ then*

$$\left|\pi_C(x) - \frac{|C|}{|G|}Li(x)\right| \leq \frac{|C|}{|G|}Li(x^{\beta_0}) + c_1 x exp\left(-c_2 n_L^{-1/2}(\log x)^{1/2}\right), \tag{1.12}$$

where β_0 is a possible real and simple zero in the region $S(1/4, 1/4)$.

The natural question in connection with the Chebotarev density theorem is when does a prime ideal $\mathfrak{p}$ of least norm occur? This interesting problem has been studied by many mathematicians. Lagarias and Odlyzko [71] state the following theorem.

Theorem 1.5. *Let L/K be a normal extension of number fields. Assume GRH for $\zeta_L(s)$. The least prime ideal $\mathfrak{p}$ with $\left[\frac{L/K}{\mathfrak{p}}\right] = C$ satisfies*

$$N_{K/\mathbb{Q}}(\mathfrak{p}) \ll (\log d_L)^2. \tag{1.13}$$

- Lagarias and Odlyzko only proved the result $N_{K/\mathbb{Q}}(\mathfrak{p}) \ll (\log d_L)^2(\log\log d_L)^4$ in [71].

- Bach and Sorenson [4] gave an explicit version of this by proving that if the GRH holds, then

$$N_{K/\mathbb{Q}}(\mathfrak{p}) \leq (4\log d_L + 2.5n_L + 5)^2. \tag{1.14}$$

- Fiori [41] showed that there exists an infinite family of number fields L, Galois over $\mathbb{Q}$, for which the smallest prime p of $\mathbb{Q}$ which splits completely in L has size at least $(\log d_L)^{2+o(1)}$.

Nontrivial, unconditional results are much harder to prove. The first nontrivial unconditional bound is due to Lagaris, Montgomery and Odlyzko [70]. They proved the following theorem.

Theorem 1.6. *([70, Theorem1.1]) There is an absolute effectively computable constant B such that for every finite extension K of $\mathbb{Q}$, every finite Galois extension L of K and every conjugacy class $\mathcal{C} \subset G$ there is an unramified prime $\mathfrak{p}$ of K with $\left[\frac{L/K}{\mathfrak{p}}\right] = \mathcal{C}$ which satisfies*

$$N_{K/\mathbb{Q}}(\mathfrak{p}) \leq 2d_L^B. \tag{1.15}$$

1.3.1 Explicit versions of the least prime ideal in the Chebotarev density theorem

Zaman made (1.15) explicit proving that

Theorem 1.7. *([125, Theorem 1.1]) Let K be a number field and suppose L/K is a finite Galois extension. For every conjugacy class C of $Gal(L/K)$, there exists a prime ideal $\mathfrak{p}$ of K unramified in L, for which its Artin symbol $\left[\frac{L/K}{\mathfrak{p}}\right] = C$, for which its norm $N_{K/\mathbb{Q}}(\mathfrak{p})$ is a rational prime, and which satisfies*

$$N_{K/\mathbb{Q}}(\mathfrak{p}) \ll d_L^{40} \tag{1.16}$$

where $d_L = |disc(L/Q)|$. The implied constant is effective and absolute

Further advances and improvements were made by Kadiri, Ng and Wong [64] who reduced the exponent $B = 40$ to 16 for d_L sufficiently large.

Theorem 1.8. *([64, Theorem 1.1]) Let L/K be a Galois extension of number fields with $G = Gal(L/K)$ and let $\mathcal{C} \subset G$ be a conjugacy class. There exists an unramified prime ideal $\mathfrak{p}$ of O_K, of degree one, such that $\sigma_{\mathfrak{p}} = \mathcal{C}$ and $N_{K/\mathbb{Q}}(\mathfrak{p}) \leq d_L^{16}$ when d_L is sufficiently large.*

Ahn and Kwon [1] obtained the first explicit exponent for all d_L.

Theorem 1.9. *([1, Theorem 1]) For every finite extension K of $\mathbb{Q}$, every finite Galois extension $L(\neq \mathbb{Q})$ of K with Galois group G and every conjugacy class C of G, there exists an unramified prime ideal $\mathfrak{p}$ of degree one with $\sigma_{\mathfrak{p}} = \mathcal{C}$ which satisfies*

$$N_{K/\mathbb{Q}}(\mathfrak{p}) \leq d_L^{A_1} \tag{1.17}$$

with $A_1 = 12577$.

The best exponent for all d_L is due to Kadiri and Wong [65].

Theorem 1.10. *([65, Theorem 1]) Let L/K be a Galois extension of number fields with Galois group G, and let C be a conjugacy class in G. If $L \neq Q$, then there exists an unramified prime $\mathfrak{p}$ of K, of degree one, such that $\sigma_{\mathfrak{p}} = C$ and $N_{K/\mathbb{Q}}(\mathfrak{p}) \leq d_L^B$ with $B = 310$.*

Note that this result made use of the work of Hasanalizade et al. [54] on $N_K(T)$ in Chapter 2.

1.3.2 Applications of the Chebotarev density theorem

In this subsection we collect several interesting applications of the Chebotarev density theorem.

1. The Chebotarev density theorem can be applied to get information about the structure of the set of prime divisors of the terms $\{U_n\}$ of a second order linear recurrence. Hasse [56] showed that the set of primes $S_V = \{p : p$ is a prime and $p | 2^n + 1$ for some $n \geq 0\}$ has density 2/3. The Lucas sequence $(L_n)_{n\geq 0}$ is defined by $L_0 = 2$, $L_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$ for all $n \geq 0$. Lagarias [69] proved that the set of primes $S_L = \{p : p$ is a prime and $p | L_n$ for some $n \geq 0\}$ has density 2/3. Hasse and Lagarias applied the Chebotarev density theorem to a suitable tower of Kummer fields.

2. Odoni [84] applied Chebotarev density theorem to tackle a conjecture of Krishnamurthy. He proved that for asymptotically one third of primes p the decimal period of $1/p$ is odd.

3. A Carmichael number is a composite number n for which $a^n \equiv a(\text{mod } n)$ for all integers a. Banks et al. [9] applied Chebotarev density theorem to show that for any fixed number field K there are infinitely many Carmichael numbers that are composed solely of primes that split completely in K.

4. Another interesting application of an explicit bound for the least prime ideal in the Chebotarev density theorem is due to Thorner and Zaman [116]. They proved the first explicit, nontrivial and unconditional upper bound for the least prime represented by a positive-definite primitive binary quadratic form. Namely, if $Q(x,y) \in \mathbb{Z}[x,y]$ is a positive-definite primitive binary quadratic form with discriminant D, then there exists a prime $p \nmid D$ represented by $Q(x,y)$ such that $p \ll |D|^{694}$. In particular, if n is a fixed positive integer, there exists a prime $p \nmid n$ represented by $x^2 + ny^2$ such that $p \ll n^{694}$.

5. Let $k \geq 2, N \geq 1$ be integers and let f be a cusp form of weight k for $\Gamma_0(N)$. f has a Fourier expansion at $i\infty$
$$f(z) = \sum_{n\geq 1} a_n e^{2\pi i n z}$$
For simplicity, suppose that all $a_n \in \mathbb{Z}$. For each integral value of a, set
$$\pi_{f,a}(x) = \#\{p \leq x : \ a_p = a\}.$$
If $a = 0$ and f is a CM modular form, then we know that
$$\pi_{f,a} \sim \pi(x)/2.$$

In all other cases, Lang and Trotter conjecture

$$\pi_{f,a}(x) \sim c_{f,a} \begin{cases} x^{1/2}/\log x & \text{if } k = 2 \\ \log\log x & \text{if } k = 3 \\ 1 & \text{if } k \geq 4. \end{cases}$$

Assuming the Riemann Hypothesis for all Artin L-functions and using the Chebotarev density theorem in the effective form given by Lagarias and Odlyzko [71], Serre [104] proved that

$$\pi_{f,a}(x) \ll \begin{cases} x^{7/8}(\log x)^{-1/2} & \text{if } a \neq 0 \\ x^{3/4} & \text{if } a = 0. \end{cases}$$

Assuming the Riemann Hypothesis for all Artin L-functions and improving Serre's original method, Murty, Murty and Saradha [93] obtained the following upper bounds for $\pi_{f,a}$

$$\pi_{f,a}(x) \ll \begin{cases} x^{4/5}(\log x)^{-1/5} & \text{if } a \neq 0 \\ x^{3/4} & \text{if } a = 0. \end{cases}$$

1.4 Counting zeros of zeta functions

The first of our results in this thesis is about counting zeros of zeta functions. We begin with the following definitions of the number of zeros in a box of two zeta functions: the Riemann zeta function $\zeta(s)$ and the Dedekind zeta function $\zeta_K(s)$ attached to a number field K. Let $N(T)$ be the number of zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ in the rectangle $0 < \beta < 1$, $0 < \gamma \leq T$, i.e.

$$N(T) := \left|\{\rho = \beta + i\gamma \in \mathbb{C} |\ \zeta(\rho) = 0,\ 0 < \beta < 1, 0 < \gamma \leq T\}\right| \tag{1.18}$$

In a similar manner we define

$$N_K(T) := \left|\{\rho = \beta + i\gamma \in \mathbb{C} |\ \zeta_K(\rho) = 0,\ 0 < \beta < 1, |\gamma| \leq T\}\right| \tag{1.19}$$

In Chapters 2 and 3 we will obtain explicit asymptotic formulae for $N(T)$ and $N_K(T)$.

1.4.1 Motivation and a brief historical overview

The Chebyshev's psi function is defined as

$$\psi(x) = \sum_{n \leq x} \Lambda(n)$$

where

$$\Lambda(n) = \begin{cases} \log p \text{ if } n = p^k,\ k \geq 0, \\ 0 \text{ otherwise} \end{cases}$$

is von Mangoldt's function. The prime number theorem (PNT) is equivalent to the statement $\psi(x) \sim x$. Recall the explicit formula for $\psi(x)$ in the following form. For all $x, T \geq 2$ we have

$$\psi(x) = x - \sum_{|\gamma| \leq T} \frac{x^\rho}{\rho} + O\Big(\frac{x \log^2 (xT)}{T} + \log x\Big) \tag{1.20}$$

where the summation is over the non-trivial zeros of $\zeta(s)$ with each zero counted according to its multiplicity.

The explicit formula is essential in the proof of PNT. Indeed, it suffices to show that $\sum_{|\gamma| \leq T} \frac{x^\rho}{\rho}$ is small compared to the expected main term x. Since $|x^\rho| = x^\beta$, to bound the latter sum we need to know the size of $N(T)$ and to prove that β is not too close to 1, that there is a zero free region of $\zeta(s)$.

The study of $N(T)$ has a long history. In 1859, in his remarkable 8 pages memoir, Riemann stated without proof that

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T) \tag{1.21}$$

It was von Mangoldt who provided the proof of the Riemann's assertion. In 1895 he established the following result.

For $T > 12$ we have

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log^2 T)$$

and the error term does not exceed $0.34\log^2 T + 1.35\log T + 2.58$.

Later in 1905 Mangoldt was able to reduce the error term in his theorem and obtained the following evaluation.

For $T > 29$ one has

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T)$$

and the error term is less than $0.432\log T + 1.191662\log\log T + 13.07873$.

The constants in bound for the error term were later improved by many mathematicians. For $T \geq T_0$, writing

$$\left|N(T) - \frac{T}{2\pi}\log\left(\frac{T}{2\pi e}\right)\right| \leq C_1\log T + C_2\log\log T + C_3, \tag{1.22}$$

we have the following Table that lists progressive improvements that has been made on the values of C_1, C_2 and C_3.

Explicit bounds for $N(T)$ with specified accuracy have some important applications. For example, they are used in estimations of sums overs zeros of $\zeta(s)$ (see Brent et al. [18]), a conditional explicit result for the prime number theorem in short intervals (see Cully-Hugill and Lee [30]), explicit upper bounds for prime counting functions such as $\pi(x)$ and $\psi(x)$

Table 1.1: Explicit bounds for $N(T)$ in (1.22)

	C_1	C_2	C_3	T_0
Von Mangoldt [123] (1905)	0.4320	1.9167	13.0788	28.5580
Grossmann [48] (1913)	0.2907	1.7862	7.0120	50
Backlund [5] (1918)	0.1370	0.4430	5.2250	200
Rosser [98] (1941)	0.1370	0.4430	2.4630	2
Trudgian [119] (2014)	0.1120	0.2780	3.3850	e
H., Shen and Wong [55]	0.1038	0.2573	9.3675	e

(see Faber and Kadiri [38]) and explicit zero-free regions for the Riemann zeta function (see Mossinghoff et al. [83]).

The first explicit result for $N_K(T)$ was established by Kadiri and Ng [63]. They showed that for $T \geq 1$, one has

$$\left| N_K(T) - \frac{T}{\pi} \log\left(d_K \left(\frac{T}{2\pi e} \right)^{n_K} \right) \right| \leq D_1 (\log d_K + n_K \log T) + D_2 n_K + D_3 \tag{1.23}$$

with admissible triple $(D_1, D_2, D_3) = (0.506, 16.950, 7.663)$, where $n_K = [K : \mathbb{Q}]$ and d_k are the degree and absolute discriminant of K, respectively. This was improved by Trudgian [120] who showed that $(D_1, D_2, D_3) = (0.316, 5.872, 3.655)$ is admissible, but whose arguments are slightly erroneous as pointed out by Bennett et al. [11]. We will fix and also improve it by adapting the arguments developed in the works of Bennett et al., Kadiri and Ng and Trudgian.

The estimate of $N_K(T)$ is crucial for proving effective versions of the Chebotarev density theorem [32] as well as bounding the least prime ideal in the Chebotarev density theorem [65]

In the next subsection we state our main results on $N(T)$ and $N_K(T)$.

1.4.2 Original results

We show in Chapter 2 that

Theorem 1.11 (H., Shen and Wong, [54]). *Given a number field K of degree n_K and with* *absolute discriminant d_K, for any $T \geq 1$, we have*

$$\left| N_K(T) - \frac{T}{\pi} \log\left(d_K \left(\frac{T}{2\pi e} \right)^{n_K} \right) \right| \leq 0.228(\log d_K + n_K \log T) + 23.108 n_K + 4.520. \quad (1.24)$$

The main ingredients of the proof are improved estimates on the Γ-function, an extension of the Rademacher's convexity bound and Backlund's trick.

Following the same arguments as in the proof of Theorem 1.11, in Chapter 3 we study $N(T)$ and prove that

Theorem 1.12 (H., Shen and Wong, [55]). *For any $T \geq e$, we have*

$$\left| N(T) - \frac{T}{2\pi} \log\left(\frac{T}{2\pi e} \right) \right| \leq 0.1038 \log T + 0.2573 \log\log T + 9.3675. \quad (1.25)$$

Note that our result is recovering Trudgian's for $T \geq e^{713}$ and our bound is better than Rosser for $T \geq e^{179}$.

1.5 Explicit results for Diophantine problems

1.5.1 Linear forms in logarithms and Baker's method

In Chapter 4 we apply tools from the transcendence theory namely linear forms in logarithms of algebraic numbers to solve two Diophantine problems.

In 1966-1967, A. Baker in his groundbreaking papers "Linear forms in logarithms of algebraic numbers I, II, III" [6–8] gave an effective lower bound on the absolute value of a nonzero linear form in logarithms of algebraic numbers, that is, for a nonzero expression of the form

$$b_1 \log \alpha_1 + b_2 \log \alpha_2 + \cdots b_n \log \alpha_n,$$

where $\alpha_1, \ldots, \alpha_n$ are algebraic numbers and $b_1, \ldots, b_n$ are integers. This result laid the foundation of a new era of explicit resolution of the exponential Diophantine equations.

For our purpose, we shall give two Baker-type inequalities which are easy to apply. We start with some notations.

Let α be an algebraic number of degree $d \geq 1$ with the minimal polynomial

$$f(X) = a_0 X^d + a_1 X^{d-1} + \cdots + a_d \in \mathbb{Z}[X], \tag{1.26}$$

where $a_0 > 0$ and $(a_0, a_1, \ldots, a_d) = 1$. Now we rewrite (1.26) as

$$f(X) = a_0 \prod_{i=1}^{d} (X - \alpha^{(i)}),$$

where $\alpha = \alpha^{(1)}$. The numbers $\alpha^{(i)}$ are called the *conjugates* of α. The *logarithmic height* of α is defined by

$$h(\alpha) = \frac{1}{d}\big(\log|a_0| + \sum_{i=1}^{d} \log\left(\max\{|\alpha^{(i)}|, 1\}\right)\big).$$

Now let L be a number field with $n_L = [L : \mathbb{Q}]$. Let $\alpha_1, \ldots, \alpha_n$ be nonzero elements of L and let $b_1, \ldots, b_n$ be integers. Put

$$B = \max\{|b_1|, \ldots, |b_n|\}$$

and

$$\Lambda = \prod_{i=1}^{n} \alpha_i^{b_i} - 1.$$

Let $A_1, \ldots, A_n$ be positive real numbers such that

$$A_i \geq \max\{n_L h(\alpha_i), |\log \alpha_i|, 0.16\} \text{ for all } i = 1, \ldots, n.$$

With the above notations, Matveev [79] proved the following theorem.

Theorem 1.13 (Matveev)**.** *If* $\Lambda \neq 0$ *and* L *is real, then*

$$\log \Lambda > -1.4 \cdot 30^{n+3} n^{4.5} n_L^2 (1+\log n_L)(1+\log B) A_1 \cdots A_n.$$

This result is one of our main tools in Chapter 4. When $n = 2$ and α_1, α_2 are positive and multiplicatively independent, we will use a result of Laurent [72]. Namely let in this case A_1 and A_2 be real numbers such that

$$\log A_i \geq \max\left\{ h(\alpha_i), \frac{|\log \alpha_i|}{n_L}, \frac{1}{n_L} \right\} \text{ for } i = 1, 2.$$

Set

$$b' = \frac{b_1}{n_L \log A_2} + \frac{b_2}{n_L \log A_1},$$

where b_1, b_2 are positive integers. Put

$$\Gamma = b_2 \log \alpha_2 - b_1 \log \alpha_1.$$

Note that $\Gamma \neq 0$ since α_1 and α_2 are multiplicatively independent. With the above notation, we cite a result due to Laurent [72].

Theorem 1.14 (Laurent)**.** *Assume that* α_1 *and* α_2 *are positive and multiplicatively independent. Then*

$$\log |\Gamma| \geq -25.2 n_L^4 \left(\max\left\{ \log b' + 0.38, \frac{10}{n_L} \right\} \right)^2 \log A_1 \log A_2.$$

1.5.2 Reduction procedure

Using the lower bounds for the linear forms in logarithms we obtain an upper bound M for the absolute values of the unknowns of a Diophantine equation. In general this bound is too large for exhaustive search of solutions. Its size is directly related to the size of the "numerical constant" that appears in Matveev's theorem which is $1.4 \cdot 30^{n+3} n^{4.5}$ (for the case of two or three logarithms this constant is around 10^{12} and 10^{14} respectively). In order to solve certain types of Diophantine equations it is useful to reduce those bounds. There are many reduction methods in the literature. We will use the reduction lemma based on continued fractions expansions. Let us recall some facts from the theory of continued fractions.

Let α be an irrational number and $(a_n)_{n\geq 0}$ be defined as

$$a_k = \lfloor \alpha_k \rfloor, \ \alpha_{k+1} = \frac{1}{\alpha_k - a_k} \text{ for all } k \geq 0.$$

The integer a_n is called the nth partial quotient of α.

For each $n \geq 0$, put

$$\frac{p_n}{q_n} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}.$$

The rational number $\frac{p_n}{q_n}$ is called the nth convergent of α and is denoted by $[a_0, \ldots, a_n]$. The sequences $\{p_n\}_{n\geq 0}$ and $\{q_n\}_{n\geq 0}$ satisfy

$$\begin{aligned}
p_0 &= a_0, & q_0 &= 1; \\
p_1 &= a_0 a_1 + 1, & q_1 &= a_1; \\
p_k &= a_k p_{k-1} + p_{k-2}, & q_k &= a_k q_{k-1} + q_{k-2}
\end{aligned}$$

One can show that for any irrational α we have $\alpha = \lim_{n\to\infty} \frac{p_n}{q_n}$ and we write $\alpha = [a_0, a_1, \ldots]$. The last expression is known as the *continued fraction* of α.

The next result is a variation of a lemma of Baker and Davenport and is due to Dujella and Petho [34]. For a real number x, we denote by $||x|| = \min\{|x-n| : n \in \mathbb{Z}\}$ the distance from x to the nearest integer.

Lemma 1.15. *Let M be a positive integer and $\frac{p}{q}$ be a convergent of the continued fraction of the irrational γ such that $q > 6M$ and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let $\varepsilon = ||\mu q|| - M||\gamma q||$. If $\varepsilon > 0$, then there is no solution to the inequality*

$$0 < |u\gamma - v + \mu| < AB^{-w}$$

in positive integers u, v and w with

$$u \leq M \text{ and } w \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

Let us explain how we use this lemma. We start with the smallest denominator $q = q_j$ of the jth convergent $\frac{p_j}{q_j}$ of γ that is greater than $6M$. If $\varepsilon = ||\mu q|| - M||\gamma q|| > 0$, we compute the upper bound $w < \frac{\log(Aq/\varepsilon)}{\log B}$. If $\varepsilon < 0$, we consider the denominator q_{j+1} of the $(j+1)$-th convergent $\frac{p_{j+1}}{q_{j+1}}$ instead. Again if ε is positive we compute the respective upper bound for w, otherwise we choose the denominator of the next convergent and we repeat this process until we get a positive ε. The only reason that after several iterations no positive ε occurs is a linear dependence on 1, γ and μ. If such a linear relation exists, then we get an inequality of the form

$$0 < |u'\gamma - v'| < AB^{-w}.$$

For the treatment of such exceptional cases, we use the well-known classical result in the theory of Diophantine approximation due to Legendre.

Lemma 1.16. *Let τ be an irrational number, $\frac{p_0}{q_0}, \frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots$ be all the convergents of the continued fraction of τ, and M be a positive integer. Let N be a nonnegative integer such* *that* $q_N > M$. *Then putting* $a(M) := \max\{a_i :\ i = 0,1,2,\dots,N(M)\}$, *the inequality*

$$\left|\tau - \frac{r}{s}\right| > \frac{1}{(a(M)+2)s^2},$$

holds for all pairs (r,s) *of positive integers with* $0 < s < M$.

1.5.3 Original results

The Fibonacci sequence $(F_n)_{n\geq 0}$ is the binary recurrence sequence defined by $F_0 = 0$, $F_1 = 1$ and

$$F_{n+2} = F_{n+1} + F_n \text{ for all } n \geq 0.$$

In the last few years many mathematicians have contributed to the study of the Diophantine equations involving terms of the binary recurrence sequences.

In 2014, Chern and Cui [26] studied the Fibonacci numbers which are close to a power of 2. More precisely, they found that the only solutions $(F_n, 2^m)$ of the Diophantine inequality

$$|F_n - 2^m| < 2^{m/2} \tag{1.27}$$

are $(1,2)$, $(2,2)$, $(3,2)$, $(3,4)$, $(5,4)$, $(8,8)$, $(13,16)$ and $(34,32)$.

In 2020, Bravo et al. [15] extended the previous work [26] and found all the members of the k-generalized Fibonacci sequence $(F_n^{(k)})_{n\geq -(k-2)}$, $k \geq 2$ which are close to a power of 2.

In Chapter 4 we study the Diophantine inequality

$$|F_n + F_m - 2^a| < 2^{a/2} \tag{1.28}$$

in positive integers n, m and a with $n \geq m$.

In particular, we prove the following main theorem.

Theorem 1.17 (H., [53])**.** *There are exactly 52 solutions* $(n,m,a) \in \mathbb{N}^3$ *to Diophantine inequality* (1.28)*. All solutions satisfy* $n,m \leq 42$ *and* $a \leq 28$*.*

The companion sequence of the Fibonacci sequence is the well-known Lucas sequence $(L_n)_{n\geq 0}$. It satisfies the same recursive relation as the Fibonacci numbers, but with initial conditions $L_0 = 2$ and $L_1 = 1$. As a corollary we also determine all Lucas numbers which are close to a power of 2.

Corollary 1.18. *There are only 9 Lucas numbers which are close to a power of 2. Namely, the solutions* $(n,a) \in \mathbb{N}^2$ *of the inequality*

$$|L_n - 2^a| < 2^{a/2}$$

are $(1,1)$*,* $(2,1)$*,* $(2,2)$*,* $(3,2)$*,* $(4,3)$*,* $(6,4)$*,* $(7,5)$*,* $(10,7)$ *and* $(13,9)$*.*

We proved Theorem 1.17 by following closely the typical strategy as performed in [14], [16], [27]. We used a few times a Baker-type lower bound for non-zero linear forms in logarithms of algebraic numbers due to Matveev to obtain a large upper bound for n. Applying a version of the Baker-Davenport reduction method, we reduced this bound on n.

Recently, Patel and Tripathy [86] have extended our work by considering sums of three Fibonacci numbers close to a power of 2. They found that there are exactly 280 solutions to Diophantine inequality $|F_n + F_m + F_l - 2^a| < 2^{a/2}$ and all solutions satisfy $n,m,l \leq 42$ and $a \leq 28$. One can consider sums of four or more Fibonacci numbers. The challenge is that the reduction process requires a lot of computations and that the number of exceptional cases increases with the number of variables.

We also study the solubility of the exponential Diophantine equation

$$(44m^2+1)^x + (5m^2-1)^y = (7m)^z. \tag{1.29}$$

in positive integers (x,y,z). This is a particular case of a general conjecture due to Terai and Shinsho [114].

Conjecture 1.19. Let m be a positive integer greater than one. Let $a,b,c>1$ be positive integers satisfying $a+b=c^2$. Then the equation

$$(am^2+1)^x+(bm^2-1)^y=(cm)^z$$

has only the positive integer solution $(x,y,z)=(1,1,2)$.

We obtain the following result.

Theorem 1.20 (H., [51])**.** *Let m be a positive integer. When m is odd, we suppose that*

$$m\equiv 2(mod\ 5)\ or\ m\equiv 0,\pm 1,\pm 3(mod\ 7). \tag{1.30}$$

Then equation (1.29) *has only the positive integer solution* $(x,y,z)=(1,1,2)$.

The proof of the above theorem consists of the following steps. First for the cases $m\equiv 2(\text{mod } 5)$ or $m\equiv\pm 1,\pm 3(\text{mod } 7)$ we use some elementary methods such as congruences and properties of the Jacobi symbol to reduce the solution to the case $y=1$ and then apply a lower bound for linear forms in two logarithms due to Laurent [72]. For the case $m\equiv 0(\text{mod } 7)$, we use a result on linear forms in p-adic logarithms due to Bugeaud [20].

1.6 Diophantine equations involving multiplicative functions

In the last chapter we consider two Diophantine problems involving multiplicative functions.

Let $\phi(n)$ denote Euler's totient function which counts the number of positive integers less than or equal to n that are relatively prime to n. In 1958, Schinzel [102] formulated the following conjecture.

Conjecture 1.21. Let k be a fixed positive integer. Then the equation

$$\phi(n) = \phi(n+k) \tag{1.31}$$

holds for infinitely many n.

Note that if p and $2p-1$ are primes that do not divide the even number k, and $n = (2p-1)k$, then n is a solution of (1.31). A special case of the prime k-tuples conjecture asserts that there are infinitely primes p with $2p-1$ prime. Therefore, the prime k-tuples conjecture implies that (1.31) has infinitely many solutions when k is even.

Sierpinski [105] showed that for any k equation (1.31) has at least one solution. This was improved by Schinzel [102], who showed that equation (1.31) has at least two solutions for every positive integer $k \leq 8 \cdot 10^{47}$ and later by Schinzel and Wakulicz [103] for $k \leq 2 \cdot 10^{58}$. Holt extended the upper bound to $1.38 \cdot 10^{26595411}$ for even k.

In 2020, Ford [44] partially solved the above conjecture by proving the following.

Theorem 1.22. *We have*

(a) For any k that is a multiple of 442720643463713815200, Conjecture 1.21 is true.

(b) There is some even $\ell \leq 3570$ such that Conjecture 1.21 is true whenever $\ell | k$.

Makowski [78] considered the equation

$$\phi(n+k) = 2\phi(n). \tag{1.32}$$

He showed that it has at least one solution n for all fixed k. Using elementary methods and explicit bounds for primes in arithmetic progressions (see [10]), we improve this result by proving the following.

Theorem 1.23 (H., [50])**.** *(a) For all positive integers $k \leq 4 \cdot 10^{58}$, Equation* (1.32) *has at least two solutions.*

(b) Moreover, for at least $15 \cdot 10^{55}$ *odd integers* $k \leq 4 \cdot 10^{58}$*, Equation* (1.32) *has at least three solutions.*

Recently, Ferrari and Sillari [40] have improved our result. They showed that Equation (1.32) has at least three solutions for all $k \leq 4 \cdot 10^{58}$.

If 2 in Equation (1.32) is replaced by another integer, very little is known about the solubility of these type of equations. Let A be the set of positive integers k for which the equation $\phi(n+k) = 3\phi(n)$ has at least one solution n. Then for every positive integer N at least half of the integers in $[1,N]$ belong to A. One can prove this by showing that A contains 1 and every multiple of 3 or 4. It is conjectured that the above equation is solvable for all k except $k = 2$.

Another topic that we study in this chapter is a variant of Lehmer's totient problem. In 1932 Lehmer [74] conjectured that if $\phi(n)|n-1$, then n has to be a prime number. A composite positive integer satisfying that divisibility is called a *Lehmer number* or number with the *Lehmer property*. Although this problem has not been settled so far, several partial results are known. Lehmer himself proved that if n has Lehmer property then n is odd, squarefree and has at least seven distinct prime factors. Cohen and Hagis [29], using computational methods, established that $\omega(n) \geq 14$, where $\omega(n)$ denotes the number of distinct prime divisors of n. Burcsi et al. [22] showed that if, additionally, $3|n$, then $\omega(n) \geq 40 \cdot 10^6$ and $n > 10^{36 \cdot 10^7}$. On the other hand, Pomerance [91] proved that every Lehmer number n is less than K^{2^K}, where $K = \omega(n)$. Recently, Burek and Żmija [23] improved this upper bound to $2^{2^K} - 2^{2^{K-1}}$.

In 1998 Subbarao [105] considered the following companions to a Lehmer problem:

$$\psi(n) = 1 + Mn \tag{1.33}$$

and

$$\sigma^*(n) = 1 + Mn, \tag{1.34}$$

where $\sigma^*(n)$ is a multiplicative function given on prime powers by $\sigma^*(p^\alpha) = 1 + p^\alpha$ and $\psi(n)$ is a multiplicative function defined by

$$\psi(n) = n \prod_{p|n} \left(1 + \frac{1}{p}\right).$$

He proposed the following conjecture.

Conjecture 1.24. The congruence $\sigma^*(n) \equiv 1 \pmod{n}$ is possible if and only if n is a prime power and $\psi(n) \equiv 1 \pmod{n}$ if and only if n is a prime.

This problem is also open. Using Grytczuk-Wójtowicz's techniques from the paper [49] and explicit bounds for primes (see [10], [97]), we obtained the lower bounds for $\omega(n)$ and n which are considerably stronger than previously known bounds. The main result is the following theorem.

Theorem 1.25 (H., [52])**.** *Let $M \geq 3$ and n is a composite solution of Equation* (1.34).

(a) If $3|n$, then $\omega(n) = r > \frac{1}{2} \cdot 1578^{AM^2/9}$, where $A = 0.998\ldots$.

(b) If $3 \nmid n$, then $\omega(n) \geq 51^{M/3} - 1$.

As a corollary we obtained the following bounds on the size of n.

Corollary 1.26. *Let $M \geq 3$ and n is a composite solution of Equation* (1.34).

(a) If $3|n$, then $n > (c(AM^2-1)2^{AM^2-1})^{2^{AM^2-1}}$, where $c = (\log 2)/3 = 0.231\ldots$.

(b) If $3 \nmid n$, then $n > (dM3^M)^{3^M}$, where $d = (\log 3)/3 = 0.366\ldots$.

Chapter 2

Counting zeros of Dedekind zeta functions

The results of this chapter have been published in [54].

2.1 Introduction

Given a number field K, the Dedekind zeta function $\zeta_K(s)$ of K is defined by

$$\zeta_K(s) = \sum_{\mathfrak{a}\neq 0} \frac{1}{\mathbb{N}(\mathfrak{a})^s},$$

for $\Re(s) > 1$, where the sum is over non-zero integral ideals of K. It is known that $\zeta_K(s)$ has an analytic continuation to a meromorphic function on $\mathbb{C}$ with only a simple pole at $s = 1$, and its zeros $\rho = \beta + i\gamma$ encode deep arithmetic information of K. For instance, the generalised Riemann hypothesis, asserting that if $\zeta_K(\rho) = 0$ and $\beta \in (0,1)$, then $\beta = \frac{1}{2}$, leads to the strongest form of the prime ideal theorem. A related prominent question is to count the zeros of $\zeta_K(s)$ in the critical strip $0 < \Re(s) < 1$. For $T \geq 0$, we set

$$N_K(T) = \#\{\rho \in \mathbb{C} \mid \zeta_K(\rho) = 0,\ 0 < \beta < 1,\ |\gamma| \leq T\},$$

counted with multiplicity if there are any multiple zeros. The estimate of $N_K(T)$ is crucial for proving effective versions of the Chebotarev density theorem as well as bounding the least prime in the Chebotarev density theorem (see [70, 71]). Moreover, to make these results explicit, it is natural to further require a determination of the implied constants for the estimate of $N_K(T)$.

Adapting the arguments of Backlund [5], McCurley [81], and Rosser [98], in [63], Kadiri and Ng showed that for $T \geq 1$, one has

$$\left| N_K(T) - \frac{T}{\pi}\log\left(d_K\left(\frac{T}{2\pi e}\right)^{n_K}\right)\right| \leq D_1(\log d_K + n_K \log T) + D_2 n_K + D_3, \qquad (2.1)$$

with admissible $(D_1, D_2, D_3) = (0.506, 16.950, 7.663)$, where n_K and d_K are the degree and absolute discriminant of K, respectively; also, D_1 can be taken as small as $(\pi \log 2)^{-1} \approx 0.459$ at expense of larger $D_2 n_K + D_3$. This was improved by Trudgian [120] (not only for Dedekind zeta functions but also for Dirichlet L-functions). In particular, as asserted in [120], the estimate (2.1) is valid with $(D_1, D_2, D_3) = (0.316, 5.872, 3.655)$, and the constant D_1 in (2.1) could be made as small as 0.247 (with larger $D_2 n_K + D_3$). Unfortunately, as pointed out by Bennett, Martin, O'Bryant, and Rechnitzer [11], there is an error in [120] that appears as the ranges of various parameters used in the argument of [120] were not verified properly. In [11], Bennett *et al.* fixed this problem for Dirichlet L-functions.

The objective of this chapter is to prove the following theorem.

Theorem 2.1. *Given a number field K of degree n_K and with absolute discriminant d_K and r_1 real places, for any $T \geq 1$, we have*

$$\left| N_K(T) - \tfrac{T}{\pi}\log\left(d_K\left(\tfrac{T}{2\pi e}\right)^{n_K}\right) + \tfrac{r_1}{4}\right| \leq 0.22737 \log\left(\tfrac{d_K (T+2)^{n_K}}{(2\pi)^{n_K}}\right) + 23.02528 n_K + 4.51954. \qquad (2.2)$$

In addition, writing the right of (2.2) as $C_1 \log\left(\frac{d_K(T+2)^{n_K}}{(2\pi)^{n_K}}\right) + C_2 n_K + C_3$, we have further admissible triples (C_1, C_2, C_3) recorded in Table 2.2 in Section 2.4. Moreover, recalling that for $T \geq T_0$, $\log(T+2) - \log T \leq \log(1 + \frac{2}{T_0})$, from the above theorem and the triangle inequality, we derive the following improved bound for $N_K(T)$.

Corollary 2.2. *Given a number field K of degree n_K and with absolute discriminant d_K, for* *any $T \geq 1$, we have*

$$\left|N_K(T) - \frac{T}{\pi}\log\left(d_K\left(\frac{T}{2\pi e}\right)^{n_K}\right)\right| \leq 0.228(\log d_K + n_K \log T) + 23.108 n_K + 4.520. \quad (2.3)$$

Furthermore, by Table 2.2, writing the right of (2.3) as $D_1(\log d_K + n_K \log T) + D_2 n_K + D_3$, we have the following table of admissible (D_1, D_2, D_3) that not only repair but also improve all triples given in [120, Table 2]. (Note that, for all number fields K, our D_2 and D_3 yield a smaller vlaue of $D_2 n_K + D_3$ than the one given by Trudgian [120].)

Table 2.1: Admissible (D_1, D_2, D_3) in Corollary 2.2 and in [120]

Trudgian [120]					Our improvement				
	$T \geq 1$		$T \geq 10$			$T \geq 1$		$T \geq 10$	
D_1	D_2	D_3	D_2	D_3	D_1	D_2	D_3	D_2	D_3
0.247	8.851	3.024	8.726	2.081	0.245	6.735	4.213	6.449	3.124
0.265	7.521	3.178	7.396	2.101	0.264	5.276	4.082	4.968	3.051
0.282	6.776	3.335	6.651	2.123	0.281	4.478	4.010	4.149	3.012
0.299	6.262	3.494	6.138	2.146	0.296	3.971	3.969	3.622	2.990

The proof of Theorem 2.1 follows closely the arguments of Bennett, Martin, O'Bryant, and Rechnitzer [11], Kadiri and Ng [63], and Trudgian [120], which are an adaption of the methods of Backlund [5], McCurley [81], and Rosser [98]. We also take advantage of the refined estimates for Gamma factors obtained in [11]. Moreover, following the strategy of Bennett *et al.* [11], we extend Rademacher's convexity bound for $\zeta_K(s)$ (cf. Propositions 2.13 and 2.14) that, together with "Backlund's trick" (see Section 2.3.2), plays a central role in improving the leading constants C_1 and D_1. Furthermore, we track all the parameters and related inequalities in a similar manner of Bennett *et al.* [11] to fix the aforementioned error appearing in [120]. Last but not least, we note that we obtain our results by a direct numerical computation (with help from Maple) and that it may be possible to use the "interval analysis" as in [11] to prove an estimate similar to [11, Theorem 1.1]. Nonetheless, since Corollary 2.2 is already as strong as [11, Corollary 1.2], and it is sufficient for most applications, we shall not devote ourselves to do such an interval analysis here.

2.2 The main term and the gamma factor

2.2.1 The main term

Let K be a number field of degree n_K and with absolute discriminant d_K. We let r_1 and r_2 be the numbers of real and complex places, respectively, of K and note that $n_K = r_1 + 2r_2$. We define the completed zeta function $\xi_K(s)$ as

$$\xi_K(s) = s(s-1)d_K^{s/2}\gamma_K(s)\zeta_K(s), \tag{2.4}$$

where

$$\gamma_K(s) = \left(\pi^{-\frac{s+1}{2}}\Gamma\Big(\frac{s+1}{2}\Big)\right)^{r_2}\left(\pi^{-\frac{s}{2}}\Gamma\Big(\frac{s}{2}\Big)\right)^{r_1+r_2}.$$

We recall that $\xi_K(s)$ extends to an entire function of order 1 and satisfies the functional equation

$$\xi_K(s) = \xi_K(1-s). \tag{2.5}$$

As in the introduction, we set

$$N_K(T) = \#\{\rho \in \mathbb{C} \mid \zeta_K(\rho) = 0,\ 0 < \beta < 1,\ |\gamma| \le T\}.$$

To estimate $N_K(T)$, we shall apply the argument principle as follows. For any fixed $\sigma_1 > 1$, we consider the rectangle $\mathcal{R}$ with vertices $\sigma_1 - iT$, $\sigma_1 + iT$, $1-\sigma_1+iT$, and $1-\sigma_1-iT$ (that is away from zeros of $\xi_K(s)$).[1] As $\xi_K(s)$ is entire, it follows from the

[1]Throughout our argument, we will always assume T is away from zeros of $\xi_K(s)$. As shall be seen in Section 2.4, with this assumption, we will prove (2.26) for T away from zeros of $\xi_K(s)$. Nonetheless, if T is the exact height of a zero, we know that $N_K(T) = N_K(T+\varepsilon)$ for all sufficiently small $\varepsilon > 0$ (in other words, $T+\varepsilon$ is away from zeros). Then, by the triangle inequality, applying (2.26) with $T+\varepsilon$, we see that

$$\begin{aligned}
&\left|N_K(T) - \frac{T}{\pi}\log\Big(d_K\Big(\frac{T}{2\pi e}\Big)^{n_K}\Big) + \frac{r_1}{4}\right| \\
&\le \left|N_K(T+\varepsilon) - \frac{T+\varepsilon}{\pi}\log\Big(d_K\Big(\frac{T+\varepsilon}{2\pi e}\Big)^{n_K}\Big) + \frac{r_1}{4}\right| + \left|\frac{T+\varepsilon}{\pi}\log\Big(d_K\Big(\frac{T+\varepsilon}{2\pi e}\Big)^{n_K}\Big) - \frac{T}{\pi}\log\Big(d_K\Big(\frac{T}{2\pi e}\Big)^{n_K}\Big)\right| \\
&\le C_1\log\Big(\frac{d_K(T+\varepsilon+2)^{n_K}}{(2\pi)^{n_K}}\Big) + C_2 n_K + C_3 + \left|\frac{T+\varepsilon}{\pi}\log\Big(d_K\Big(\frac{T+\varepsilon}{2\pi e}\Big)^{n_K}\Big) - \frac{T}{\pi}\log\Big(d_K\Big(\frac{T}{2\pi e}\Big)^{n_K}\Big)\right|.
\end{aligned}$$

Now, taking $\varepsilon \to 0^+$, we conclude that (2.26) is also valid when T is the exact height of a zero.

argument principle that

$$N_K(T) = \frac{1}{2\pi}\Delta_{\mathcal{R}} \arg \xi_K(s).$$

Let $\mathcal{C}$ be the part of the contour of $\mathcal{R}$ in $\Re(s) \geq \frac{1}{2}$ and $\mathcal{C}_0$ be the part of the contour of $\mathcal{R}$ in $\Re(s) \geq \frac{1}{2}$ and $\Im(s) \geq 0$. Since $\overline{\xi_K(s)} = \xi_K(\bar{s})$, the functional equation (3.9) then yields

$$\Delta_{\mathcal{R}} \arg \xi_K(s) = 2\Delta_{\mathcal{C}} \arg \xi_K(s) = 4\Delta_{\mathcal{C}_0} \arg \xi_K(s),$$

which implies that

$$N_K(T) = \frac{2}{\pi}\Delta_{\mathcal{C}_0} \arg \xi_K(s). \tag{2.6}$$

Writing $B = d_K/\pi^{n_K}$, by (3.8), we have

$$\begin{aligned}\Delta_{\mathcal{C}_0} \arg \xi_K(s) &= \Delta_{\mathcal{C}_0} \arg s + \Delta_{\mathcal{C}_0} \arg B^{s/2} \\ &\quad + (r_1 + r_2)\Delta_{\mathcal{C}_0} \arg \Gamma\left(\frac{s}{2}\right) + r_2\Delta_{\mathcal{C}_0} \arg \Gamma\left(\frac{s+1}{2}\right) + \Delta_{\mathcal{C}_0} \arg\left((s-1)\zeta_K(s)\right).\end{aligned} \tag{2.7}$$

It is clear that

$$\begin{aligned}&\Delta_{\mathcal{C}_0} \arg s = \arctan(2T), \\ &\Delta_{\mathcal{C}_0} \arg B^{s/2} = \frac{T}{2}\log B = \frac{T}{2}\log\left(\frac{d_K}{\pi^{n_K}}\right), \\ &\Delta_{\mathcal{C}_0} \arg \Gamma(s) = \Delta_{\mathcal{C}_0}(\Im \log \Gamma(s)) = \Im \log \Gamma\Big(\frac{1}{2} + iT\Big).\end{aligned} \tag{2.8}$$

To control the Gamma factor, we shall appeal for the improved numerical bound established in [11, Sec. 3]. For $a \in \{0,1\}$, we set

$$g_a(T) = \frac{2}{\pi}\Im \log \Gamma\Big(\frac{1}{4} + \frac{a}{2} + i\frac{T}{2}\Big) - \frac{T}{\pi}\log\left(\frac{T}{2e}\right) - \frac{2a-1}{4}.$$

It follows from [11, Proposition 3.2] that for $a \in \{0,1\}$ and $T \geq 5/7$,

$$|g_a(T)| \leq \frac{2-a}{50T}.$$

Hence, setting

$$g_K(T) = (r_1 + r_2)g_0(T) + r_2 g_1(T), \tag{2.9}$$

we then obtain

$$|g_K(T)| \leq \frac{2n_K}{50T} - \frac{r_2}{50T}. \tag{2.10}$$

Now, gathering (2.6), (2.7), (2.8), and (2.9), we obtain

$$N_K(T) = \frac{2}{\pi}\arctan(2T) + g_K(T) + \frac{T}{\pi}\log\left(d_K\left(\frac{T}{2\pi e}\right)^{n_K}\right) - \frac{r_1}{4} + \frac{2}{\pi}\Delta_{C_0}\arg((s-1)\zeta_K(s)). \tag{2.11}$$

Let $\mathcal{C}_1$ denote the vertical line from σ_1 to $\sigma_1 + iT$ and $\mathcal{C}_2$ denote the horizontal line from $\sigma_1 + iT$ to $\frac{1}{2} + iT$. We require the following two estimates.

Lemma 2.3. *For $s = \sigma + it$ with $\sigma > 1$, one has*

$$\frac{\zeta_K(2\sigma)}{\zeta_K(\sigma)} \leq |\zeta_K(s)| \leq \zeta(\sigma)^{n_K},$$

where, as later, $\zeta(s)$ denotes the Riemann zeta function.

Lemma 2.4. *For $\sigma_1 > 1$,*

$$|\Delta_{\mathcal{C}_1}\arg(s-1)\zeta_K(s)| \leq \frac{\pi}{2} + n_K\log\zeta(\sigma_1).$$

Proof. Note that

$$\Delta_{C_1} \arg(s-1)\zeta_K(s) = \Delta_{C_1} \arg(s-1) + \Delta_{C_1} \arg \zeta_K(s) = \arctan\left(\frac{T}{\sigma_1 - 1}\right) + \Delta_{C_1} \arg \zeta_K(s).$$

Now, the lemma follows from the estimate

$$|\Delta_{C_1} \arg \zeta_K(s)| = |\arg \zeta_K(\sigma_1 + iT)| \le |\log \zeta_K(\sigma_1 + iT)| \le \log \zeta_K(\sigma_1) \le n_K \log \zeta(\sigma_1),$$

where the last inequality is due to Lemma 2.3. □

Thus, by Lemma 2.4 and (2.11), we arrive at

$$\left|N_K(T) - \tfrac{T}{\pi} \log\left(d_K \left(\tfrac{T}{2\pi e}\right)^{n_K}\right) + \tfrac{r_1}{4}\right| \le 2 + |g_K(T)| + \tfrac{2n_K}{\pi} \log \zeta(\sigma_1) + \tfrac{2}{\pi} |\Delta_{C_2} \arg((s-1)\zeta_K(s))|. \tag{2.12}$$

2.2.2 Bounding the Gamma factor

For $a \in \{0,1\}$, $0 \le d < 9/2$ and $T \ge 5/7$, we set

$$\mathcal{E}_a(T,d) = \left|\Im \log \Gamma\left(\frac{\sigma + a + iT}{2}\right)\Big|_{\sigma=\frac{1}{2}}^{\frac{1}{2}+d} + \Im \log \Gamma\left(\frac{\sigma + a + iT}{2}\right)\Big|_{\sigma=\frac{1}{2}}^{\frac{1}{2}-d}\right|,$$

and we define

$$\mathcal{E}_K(T,d) = (r_1 + r_2)\mathcal{E}_0(T,d) + r_2 \mathcal{E}_1(T,d). \tag{2.13}$$

Following [11, p. 1463], we let

$$\begin{aligned}
E_a(T,d) &= \frac{2T/3}{(2a+2d+17)^2+4T^2} + \frac{2T/3}{(2a-2d+17)^2+4T^2} - \frac{4T/3}{(2a+17)^2+4T^2} \\
&+ \frac{T}{2}\log\left(1+\frac{(2a+17)^2}{4T^2}\right) - \frac{T}{4}\log\left(1+\frac{(2a+2d+17)^2}{4T^2}\right) - \frac{T}{4}\log\left(1+\frac{(2a-2d+17)^2}{4T^2}\right) \\
&+ \frac{(8+6\pi)/45}{((2a+2d+17)^2+4T^2)^{3/2}} + \frac{(8+6\pi)/45}{((2a-2d+17)^2+4T^2)^{3/2}} + \frac{2(8+6\pi)/45}{((2a+17)^2+4T^2)^{3/2}} \\
&+ \sum_{k=0}^{3}\left(2\arctan\frac{2a+1+4k}{2T} - \arctan\frac{2a+2d+1+4k}{2T} - \arctan\frac{2a-2d+1+4k}{2T}\right) \\
&+ \frac{2a+2d+15}{4}\arctan\frac{2a+2d+17}{2T} + \frac{2a-2d+15}{4}\arctan\frac{2a-2d+17}{2T} \\
&- \frac{2a+15}{2}\arctan\frac{2a+17}{2T}.
\end{aligned}$$

We shall further set

$$E_K(T,d) = (r_1+r_2)E_0(T,d) + r_2E_1(T,d). \tag{2.14}$$

As shown in [11, p. 1462], $\mathcal{E}_a(T,d) \le E_a(T,d)$ for $0 \le d < 9/2$ and $T \ge 5/7$, and thus

$$\mathcal{E}_K(T,d) \le E_K(T,d) \tag{2.15}$$

for $0 \le d < 9/2$ and $T \ge 5/7$. In addition, from [11, Lemma 3.4] and our definition of $E_K(T,d)$, we have the following lemma.

Lemma 2.5. *For $0 \le \delta_1 \le d < 9/2$ and $T \ge 5/7$,*

$$0 < E_K(T,\delta_1) \le E_K(T,d).$$

Furthermore, for $d \in [\frac{1}{4}, \frac{5}{8}]$ and $T \ge 5/7$,

$$\frac{E_K(T,d)}{\pi} \le (r_1+r_2)\frac{640d-112}{1536(3T-1)} + r_2\frac{(640+216)d-112-39}{1536(3T+3-1)} + \frac{n_K}{2^{10}}.$$

2.3 Backlund's trick and the Jensen integral

2.3.1 Introducing the auxiliary function f_N

For the sake of convenience, we shall set $\mathcal{Z}(w) = (w-1)\zeta_K(w)$. In order to analyse the variation of the argument of $\mathcal{Z}(w)$ on $\mathcal{C}_2$, we shall introduce an auxiliary function

$$f_N(s) = \frac{1}{2}\Big(\mathcal{Z}(s+iT)^N + \mathcal{Z}(s-iT)^N\Big)$$

for $N \in \mathbb{N}$. For $\sigma \in \mathbb{R}$, it is clear that

$$\begin{aligned} f_N(\sigma) &= \frac{1}{2}\Big(\mathcal{Z}(\sigma+iT)^N + \mathcal{Z}(\sigma-iT)^N\Big) = \frac{1}{2}\Big(\mathcal{Z}(\sigma+iT)^N + \overline{\mathcal{Z}(\sigma+iT)^N}\Big) \\ &= \mathfrak{Re}(\mathcal{Z}(\sigma+iT)^N). \end{aligned}$$

We need the following definition that measures the variation of the argument of $\mathcal{Z}(w)^N$ on $\mathcal{C}_2$.

Definition 2.6. Let b_N denote the non-negative integer, depending on N, such that

$$b_N \leq \frac{1}{\pi}\Big|\Delta_{\mathcal{C}_2} \arg \mathcal{Z}(w)^N\Big| < b_N + 1.$$

From this definition and the fact that $\arg \mathcal{Z}(w)^N = N \arg \mathcal{Z}(w)$, we immediately obtain

$$\frac{b_N}{N} \leq \frac{1}{\pi}\Big|\Delta_{\mathcal{C}_2} \arg \mathcal{Z}(w)\Big| < \frac{b_N+1}{N}. \tag{2.16}$$

In addition, we have the following lemma concerning the zeros of $f_N(\sigma)$.

Lemma 2.7. *In the notation of Definition 2.6, the function $f_N(\sigma)$ has at least b_N zeros in $[\frac{1}{2}, \sigma_1]$.*

Proof. By Definition 2.6, there are at least b_N different values of σ such that $\frac{1}{2} + \frac{1}{\pi} \arg \mathcal{Z}(\sigma+$

$iT)^N \in \mathbb{Z}$. Thus, for such values of σ, $\mathcal{Z}(\sigma+iT)^N$ is purely imaginary, which means that

$$f_N(\sigma) = \mathfrak{Re}(\mathcal{Z}(\sigma+iT)^N) = 0$$

for at least b_N different values σ. $\square$

We shall also require the following lemma regarding the limiting behaviour of f_N.

Lemma 2.8. *For any $c > 1$, there is an infinite sequence of natural numbers $(N_m)_{m=1}^{\infty}$ such that $f_{N_m}(c) \neq 0$. Moreover, we have*

$$\limsup_{m\to\infty}\Big(-\frac{1}{N_m}\log|f_{N_m}(c)|\Big) \leq \log\Big(\frac{1}{\sqrt{(c-1)^2+T^2}}\frac{\zeta_K(c)}{\zeta_K(2c)}\Big).$$

Proof. Write $\mathcal{Z}(c+iT) = Re^{i\phi}$ for some $R,\phi \in \mathbb{R}$. It is clear that $\mathcal{Z}(c-iT) = Re^{-i\phi}$. Also, as $\mathcal{Z}(c+iT) \neq 0$ for any $c > 1$, we know that $R > 0$. Thus, we have

$$\frac{f_N(c)}{\mathcal{Z}(c+iT)^N} = \frac{1}{2}\Big(1+\frac{\mathcal{Z}(c-iT)^N}{\mathcal{Z}(c+iT)^N}\Big) = \frac{1}{2}(1+e^{-2N\phi i})$$

for any $N \in \mathbb{N}$.

Now, applying Dirichlet's approximation theorem, for any ϕ, there is an infinite sequence of natural numbers $(N_m)_{m=1}^{\infty}$ such that as $m \to \infty$, $-2N_m\phi \to 0$ modulo 2π and $N_m \to \infty$. Thus, $\frac{f_{N_m}(c)}{\mathcal{Z}(c+iT)^{N_m}} \to 1$ as $m \to \infty$, and hence

$$\lim_{m\to\infty}\Big(-\tfrac{1}{N_m}(\log|f_{N_m}(c)|-N_m\log|\mathcal{Z}(c+iT)|)\Big) = \Big(\lim_{m\to\infty}\tfrac{-1}{N_m}\Big)\Big(\lim_{m\to\infty}\log\Big|\tfrac{f_{N_m}(c)}{\mathcal{Z}(c+iT)^{N_m}}\Big|\Big) = 0.$$

Moreover, by the left inequality of Lemma 2.3, we have

$$|\mathcal{Z}(c+iT)| \geq \sqrt{(c-1)^2+T^2}\frac{\zeta_K(2c)}{\zeta_K(c)},$$

which, combined with the above identity, gives

$$
\begin{aligned}
0 &\geq \limsup_{m\to\infty}\Big(-\frac{1}{N_m}\log|f_{N_m}(c)| + \log\Big(\sqrt{(c-1)^2+T^2}\frac{\zeta_K(2c)}{\zeta_K(c)}\Big)\Big)\\
&= \limsup_{m\to\infty}\Big(-\frac{1}{N_m}\log|f_{N_m}(c)|\Big) + \log\Big(\sqrt{(c-1)^2+T^2}\frac{\zeta_K(2c)}{\zeta_K(c)}\Big).
\end{aligned}
$$

Herein, we complete the proof. □

Let $D(c,r)$ be the open disk centred at c with radius r. Let $(N_m)_{m=1}^{\infty}$ be given as in Lemma 2.8. For any $N \in (N_m)_{m=1}^{\infty}$, we set

$$
S_N(c,r) = \frac{1}{N}\sum_{z\in\mathcal{S}_N(D(c,r))}\log\frac{r}{|z-c|},
$$

where $\mathcal{S}_N(D(c,r))$ denotes the set of zeros of $f_N(s)$ in $D(c,r)$. As in [11, Theorem 5.1], we have the following version of Jensen's formula.

Theorem 2.9 (Jensen's formula)**.** *For $c \in \mathbb{C}$ and $r > 0$, if $f_N(c) \neq 0$, then*

$$
S_N(c,r) = -\frac{1}{N}\log|f_N(c)| + \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{N}\log|f_N(c+re^{i\theta})|d\theta.
$$

Applying Jensen's formula and Lemma 2.8, we obtain the following upper bound for $S_N(c,r)$.

Proposition 2.10. *Let c, r, and σ_1 be real numbers such that*

$$
c - r < \frac{1}{2} < 1 < c < \sigma_1 < c + r.
$$

Let $F_{c,r} : [-\pi,\pi] \to \mathbb{R}$ be an even function such that $F_{c,r}(\theta) \geq \frac{1}{N_m}\log|f_{N_m}(c+re^{i\theta})|$. Then we have

$$
\limsup_{m\to\infty} S_{N_m}(c,r) \leq \log\Big(\frac{1}{\sqrt{(c-1)^2+T^2}}\frac{\zeta_K(c)}{\zeta_K(2c)}\Big) + \frac{1}{\pi}\int_0^{\pi} F_{c,r}(\theta)d\theta.
$$

2.3.2 Backlund's trick

We start with the following technical estimate.

Lemma 2.11. *Let* $0 \le d < 1/2$ *and* $T \ge 5/7$*. Then we have*

$$\left|\arg\left((\sigma-1+iT)\zeta_K(\sigma+iT)\right)^N\Big|_{\sigma=\frac{1}{2}}^{\frac{1}{2}+d}\right| \le \left|\arg\left((\sigma-1+iT)\zeta_K(\sigma+iT)\right)^N\Big|_{\sigma=\frac{1}{2}}^{\frac{1}{2}-d}\right| + N\mathcal{E}_K(T,d)+N\frac{\pi}{2},$$

where $\mathcal{E}_K(T,d)$ *is defined as in* (2.13).

Proof. By the functional equation (2.5) and the fact that $\xi_K(s)=\overline{\xi_K(\overline{s})}$, we have

$$\arg\xi_K(\sigma+iT)\Big|_{\sigma=\frac{1}{2}}^{\frac{1}{2}+d} = -\arg\xi_K(\sigma+iT)\Big|_{\sigma=\frac{1}{2}}^{\frac{1}{2}-d}. \tag{2.17}$$

Since

$$\arg(\sigma+iT)+\arg B^{(\sigma+iT)/2} = \arctan\frac{T}{\sigma}+\frac{T}{2}\log B,$$

by (2.4), we have

$$\arg\xi_K(\sigma+iT) = \arctan\frac{T}{\sigma}+\frac{T}{2}\log B+(r_1+r_2)\Im\log\Gamma\Big(\frac{\sigma+iT}{2}\Big)+r_2\Im\log\Gamma\Big(\frac{\sigma+iT+1}{2}\Big) + \arg\left((\sigma+iT-1)\zeta_K(\sigma+iT)\right). \tag{2.18}$$

As we know that for $\pm xy<1$,

$$\arctan x \pm \arctan y = \arctan\frac{x\pm y}{1\mp xy},$$

for $0 \le d < 1/2$, we have

$$\begin{aligned}
&\left|\arctan \frac{T}{\frac{1}{2}+d} - \arctan \frac{T}{\frac{1}{2}} + \arctan \frac{T}{\frac{1}{2}-d} - \arctan \frac{T}{\frac{1}{2}}\right| \\
&= \left|\arctan \frac{\frac{T}{\frac{1}{2}+d} - \frac{T}{\frac{1}{2}}}{1+\frac{T}{\frac{1}{2}+d}\frac{T}{\frac{1}{2}}} + \arctan \frac{\frac{T}{\frac{1}{2}-d} - \frac{T}{\frac{1}{2}}}{1+\frac{T}{\frac{1}{2}-d}\frac{T}{\frac{1}{2}}}\right| \\
&\le \frac{\pi}{2}.
\end{aligned} \tag{2.19}$$

Now, applying the triangle inequality, by (2.17), (2.18), and (2.19), we obtain

$$\begin{aligned}
\left|\arg\Big((\sigma-1+iT)\zeta_K(\sigma+iT)\Big)\Big|_{\sigma=\frac{1}{2}}^{\frac{1}{2}+d}\right| \le{}& \left|\arg\Big((\sigma-1+iT)\zeta_K(\sigma+iT)\Big)\Big|_{\sigma=\frac{1}{2}}^{\frac{1}{2}-d}\right| \\
&+ \mathcal{E}_K(T,d) + \frac{\pi}{2}.
\end{aligned}$$

Recalling that

$$\arg\Big((\sigma-1+iT)\zeta_K(\sigma+iT)\Big)^N\Big|_{\sigma=\frac{1}{2}}^{\frac{1}{2}\pm d} = N\arg\Big((\sigma-1+iT)\zeta_K(\sigma+iT)\Big)\Big|_{\sigma=\frac{1}{2}}^{\frac{1}{2}\pm d},$$

we conclude the proof. □

As argued in [11] and [120], we require the following version of "Backlund's trick".

Proposition 2.12 (Backlund's trick). *Let c and r be real numbers. Set*

$$\sigma_1 = c + \frac{(c-1/2)^2}{r} \quad \text{and} \quad \delta = 2c - \sigma_1 - \frac{1}{2}.$$

If $1 < c < r$ and $0 < \delta < \frac{1}{2}$, then

$$\left|\arg\Big((\sigma+iT-1)\zeta_K(\sigma+iT)\Big)\Big|_{\sigma=\sigma_1}^{1/2}\right| \le \frac{\pi S_N(c,r)}{2\log(r/(c-1/2))} + \frac{E_K(T,\delta)}{2} + \frac{\pi}{N} + \frac{\pi}{2N} + \frac{\pi}{4}.$$

Proof. By the conditions on c and r and the definitions of σ_1 and δ, we know that

$$c-r<\frac{1}{2}-\delta\leq\frac{1}{2}\leq\frac{1}{2}+\delta=2c-\sigma_1\leq c\leq\sigma_1<c+r.$$

As $\log\frac{r}{|z-c|}>0$ for $z\in D(c,r)$, we see that

$$S_N(c,r)=\frac{1}{N}\sum_{z\in\mathcal{S}_N(D(c,r))}\log\frac{r}{|z-c|}\geq\frac{1}{N}\sum_{z\in\mathcal{S}_N((c-r,\sigma_1])}\log\frac{r}{|z-c|}.$$

Recall that by Lemma 2.7, there are at least b_N values of σ satisfying $\sigma\in[1/2,\sigma_1]$ and $f_N(\sigma)=0$, where b_N is defined as in Definition 2.6. For $1\leq k\leq b_N$, we then set δ_k as the smallest non-negative real number such that

$$f_N(1/2+\delta_k)=0\quad\text{and}\quad k-1\leq\frac{1}{\pi}\left|\arg\Big((\sigma+iT-1)\zeta_K(\sigma+iT)\Big)^N\Big|_{\sigma=1/2}^{1/2+\delta_k}\right|. \tag{2.20}$$

Writing $z_k=\frac{1}{2}+\delta_k$, we let x_1 denote the number of z_k with $z_k\in[1/2,1/2+\delta)=[1/2,2c-\sigma_1)$ and let x_2 denote the number of z_k with $z_k\in[2c-\sigma_1,\sigma_1]$. We note that $x_2=b_N-x_1$ and that

$$0\leq\delta_1<\delta_2<\cdots<\delta_{x_1}<\delta\leq\delta_{x_1+1}<\cdots<\delta_{b_N}\leq\sigma_1-1/2.$$

From (2.15), (2.20), and Lemma 2.11, it follows that

$$k-1\leq\frac{1}{\pi}\left|\arg\Big((\sigma-1+iT)\zeta_K(\sigma+iT)\Big)^N\Big|_{\sigma=\frac{1}{2}}^{\frac{1}{2}-\delta_k}\right|+\frac{1}{\pi}NE_K(T,\delta_k)+\frac{N}{2}$$

whenever $1\leq k\leq x_1$ (which implies that $\delta_k<\delta<\frac{1}{2}$).

For each $j\geq 1$, if there exists a k (chosen to be minimal) such that

$$k-1-\frac{1}{\pi}NE_K(T,\delta_k)-\frac{N}{2}\geq j,$$

then f_N has at least j zeros in $[1/2-\delta_k, 1/2)$ since

$$\frac{1}{\pi}\left|\arg\Big((\sigma+iT-1)\zeta_K(\sigma+iT)\Big)^N\Big|_{\sigma=1/2}^{1/2-\delta_k}\right| \geq k-1-\frac{1}{\pi}NE_K(T,\delta_k)-\frac{N}{2}\geq j.$$

For such an instance, we define δ_{-k} as the smallest values of these zeros (to avoid possible repetition), and we shall say that the zero $z_k = 1/2+\delta_k$ has a pair $z_{-k} = 1/2-\delta_{-k}$. We note that $\delta_{-k} \leq \delta_k$ by the construction.

By the same argument as in [11, pp. 1467-1468], we have

$$S_N(c,r) \geq \frac{2b_N - \frac{NE_K(T,\delta)+\frac{N\pi}{2}+\pi}{\pi}}{N}\log\Big(\frac{r}{c-1/2}\Big),$$

and thus

$$\frac{b_N}{N} \leq \frac{S_N(c,r)}{2\log(r/(c-1/2))}+\frac{E_K(T,\delta)}{2\pi}+\frac{1}{4}+\frac{1}{2N},$$

which combined with (2.16) completes the proof. □

2.3.3 Constructing and bounding $F_{c,r}$

We first recall the convexity bound for $\zeta_K(s)$ established by Rademacher [92, Theorem 4].

Proposition 2.13. *Let* $\eta \in (0, \frac{1}{2}]$ *and* $s = \sigma+it$. *If* $-\eta \leq \sigma \leq 1+\eta$, *then one has*

$$|\zeta_K(s)| \leq 3\left|\frac{1+s}{1-s}\right|\Big(d_K\Big(\frac{|1+s|}{2\pi}\Big)^{n_K}\Big)^{\frac{1+\eta-\sigma}{2}}\zeta(1+\eta)^{n_K}.$$

Also, for $\sigma \in [-\frac{1}{2}, 0)$, *one has*

$$|\zeta_K(s)| \leq 3\left|\frac{1+s}{1-s}\right|\Big(d_K\Big(\frac{|1+s|}{2\pi}\Big)^{n_K}\Big)^{\frac{1}{2}-\sigma}\zeta(1-\sigma)^{n_K}. \tag{2.21}$$

We note that the second inequality follows from the first bound by taking $\eta = -\sigma$. Moreover, Rademacher's argument [92] can be used to extend (2.21) for $\sigma < 0$ as follows (cf. [11, Theorem 5.7]). For $x \in \mathbb{R}$, let $[x]$ be the integer closest to x; when there are two integers equally close to x, we shall choose the one closer to 0.

Proposition 2.14. *Let $s = \sigma + it$ with $\sigma < 0$. Then we have*

$$|\zeta_K(s)| \leq \left(\frac{d_K}{(2\pi)^{n_K}}\right)^{\frac{1}{2}-\sigma} |1+s-[\sigma]|^{n_K(\frac{1}{2}+[\sigma]-\sigma)} \prod_{j=1}^{-[\sigma]} |s+j-1|^{n_K} \zeta(1-\sigma)^{n_K}.$$

Proof. From the functional equation (2.5) we have

$$\begin{aligned}
|\zeta_K(s)| &\leq d_K^{1/2-\sigma} \left|\frac{\gamma_K(1-s)}{\gamma_K(s)}\right| |\zeta_K(1-s)| \\
&= d_K^{1/2-\sigma} \pi^{(\sigma-\frac{1}{2})n_K} \left|\frac{\Gamma(\frac{1}{2}+\frac{1-s}{2})}{\Gamma(\frac{1}{2}+\frac{s}{2})}\right|^{r_2} \left|\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}\right|^{r_1+r_2} |\zeta_K(1-s)|.
\end{aligned}$$

As $\sigma < 0$, by Lemma 2.3, we have $|\zeta_K(1-s)| \leq \zeta(1-\sigma)^{n_K}$. It remains to estimate the ratios of gamma functions. It was obtained in the proof of [11, Theorem 5.7] that for $a, b \in \{0, 1\}$ and $k \in \mathbb{Z}$,

$$\frac{\Gamma(\frac{a}{2}+\frac{1-s}{2})}{\Gamma(\frac{a}{2}+\frac{s}{2})} = \frac{\Gamma(\frac{b}{2}+\frac{1-(s+k)}{2})}{\Gamma(\frac{b}{2}+\frac{s+k}{2})} 2^{-k} \left(\prod_{j=1}^{k}(s+j-1)\right) \frac{\sin(\frac{\pi}{2}(s+k+1-b))}{\sin(\frac{\pi}{2}(s+1-a))}.$$

Setting $a = 0$ and $a = 1$ and taking $b \equiv k \pmod 2$ and $b \equiv k+1 \pmod 2$, respectively, we can make sine factors ± 1. Thus, upon choosing $k = -[\sigma]$ and applying [92, Lemmata 1 and 2] to $\frac{\Gamma(\frac{b}{2}+\frac{1-(s+k)}{2})}{\Gamma(\frac{b}{2}+\frac{s+k}{2})}$, we conclude that

$$\left|\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}\right|^{r_1+r_2} \leq \left(\frac{1}{2}|1+s-[\sigma]|\right)^{(\frac{1}{2}+[\sigma]-\sigma)(r_1+r_2)} 2^{[\sigma](r_1+r_2)} \left(\prod_{j=1}^{-[\sigma]}|s+j-1|\right)^{r_1+r_2}$$

and

$$\left|\frac{\Gamma(\frac{1}{2}+\frac{1-s}{2})}{\Gamma(\frac{1}{2}+\frac{s}{2})}\right|^{r_2} \leq \left(\frac{1}{2}|1+s-[\sigma]|\right)^{(\frac{1}{2}+[\sigma]-\sigma)r_2} 2^{[\sigma]r_2} \left(\prod_{j=1}^{-[\sigma]}|s+j-1|\right)^{r_2}.$$

Collecting above estimates and recalling the fact that $n_K = r_1 + 2r_2$, we obtain the desired result. □

Lemma 2.15. *Let* $\eta \in (0, \frac{1}{2}]$, $s = \sigma + it$, *and* $T > 0$. *If* $\sigma \geq 1 + \eta$, *then we have*

$$\frac{1}{N} \log |f_N(s)| \leq \frac{1}{2} \log((\sigma - 1)^2 + (|t| + T)^2) + n_K \log \zeta(\sigma).$$

If $-\eta \leq \sigma \leq 1 + \eta$, *then we have*

$$\begin{aligned} \frac{1}{N} \log |f_N(s)| \leq \log 3 + \frac{n_K(1+\eta-\sigma)+2}{4} \log((\sigma+1)^2 + (|t|+T)^2) \\ + \frac{1+\eta-\sigma}{2} \log\left(\frac{d_K}{(2\pi)^{n_K}}\right) + n_K \log \zeta(1+\eta). \end{aligned}$$

If $\sigma \leq -\eta$, *then we have*

$$\begin{aligned} \frac{1}{N} \log |f_N(s)| &\leq n_K \log \zeta(1-\sigma) + \frac{1}{2} \log((\sigma-1)^2 + (|t|+T)^2) \\ &+ \frac{1-2\sigma}{2} \log\left(\frac{d_K}{(2\pi)^{n_K}}\right) + \frac{(1-2\sigma+2[\sigma])n_K}{4} \log((1+\sigma-[\sigma])^2 + (|t|+T)^2) \\ &+ \frac{n_K}{2} \sum_{j=1}^{-[\sigma]} \log((\sigma+j-1)^2 + (|t|+T)^2). \end{aligned}$$

Proof. Since $\sigma \geq 1 + \eta > 1$, by Lemma 2.3, we derive

$$\begin{aligned} |f_N(s)| &\leq \frac{1}{2}\left(|s+iT-1|^N |\zeta_K(s+iT)|^N + |s-iT-1|^N |\zeta_K(s-iT)|^N\right) \\ &\leq \left((\sigma-1)^2 + (|t|+T)^2\right)^{\frac{N}{2}} \zeta(\sigma)^{n_K N}. \end{aligned}$$

Now, the first estimate follows from taking logarithms and dividing both sides by N.

Secondly, if $-\eta \leq \sigma \leq 1 + \eta$, then by Proposition 2.13, we see that $|f_N(s)|$ is at most

$$\begin{aligned} &\tfrac{1}{2}\left(3^N |s+iT+1|^N + 3^N |s-iT+1|^N\right)\left(d_K\left(\frac{\sqrt{(\sigma+1)^2+(|t|+T)^2}}{2\pi}\right)^{n_K}\right)^{\frac{(1+\eta-\sigma)N}{2}} \zeta(1+\eta)^{n_K N} \\ &\leq 3^N\left((\sigma+1)^2 + (|t|+T)^2\right)^{\frac{N}{2}}\left(d_K\left(\frac{\sqrt{(\sigma+1)^2+(|t|+T)^2}}{2\pi}\right)^{n_K}\right)^{\frac{(1+\eta-\sigma)N}{2}} \zeta(1+\eta)^{n_K N}. \end{aligned}$$

Again, taking logarithms yields the second bound.

Lastly, for $\sigma \leq -\eta$, it follows from Proposition 2.14 that

$$|f_N(s)| \leq \Big((\sigma-1)^2 + (|t|+T)^2\Big)^{\frac{N}{2}} \Big(\frac{d_K}{(2\pi)^{n_K}}\Big)^{N(\frac{1}{2}-\sigma)} |(1+\sigma-[\sigma])^2 + (|t|+T)^2|^{\frac{(1-2\sigma+2[\sigma])Nn_K}{4}}$$
$$\times \Big(\prod_{j=1}^{-[\sigma]} ((\sigma+j-1)^2 + (|t|+T)^2)\Big)^{\frac{n_K N}{2}} \zeta(1-\sigma)^{n_K N}.$$

We then conclude the proof by taking logarithms. □

Following [11], to proceed further, we introduce some notation and auxiliary functions. We first set

$$L_j(\theta) = \log \frac{(j+c+r\cos\theta)^2 + (|r\sin\theta|+T)^2}{(T+2)^2}.$$

and note that $L_j(\theta)$ is an even function of θ. Moreover, if $\theta \in [0,\pi]$ and $T \geq 5/7$, by the inequality $\log x \leq x-1$, one has $L_j(\theta) \leq \frac{L_j^\star(\theta)}{T+2}$, where

$$L_j^\star(\theta) = 2r\sin\theta - 4 + \frac{7}{19}((j+c+r\cos\theta)^2 + (r\sin\theta-2)^2).$$

In light of the choice of $F_{c,r}(\theta)$ (for Dirichlet L-functions) in [11, Definition 5.10], we shall use the following $F_{c,r}(\theta)$ for $\zeta_K(s)$.

Definition 2.16. For $\theta \in [-\pi,\pi]$, we let $\sigma = c + r\cos\theta$, with $c - r > -\frac{1}{2}$, and $t = r\sin\theta$. For $\sigma \geq 1+\eta$, we define

$$F_{c,r}(\theta) = n_K \log\zeta(\sigma) + \frac{1}{2}L_{-1}(\theta) + \log(T+2).$$

For $-\eta \le \sigma \le 1+\eta$, we define

$$F_{c,r}(\theta) = n_K \log\zeta(1+\eta) + \frac{n_K(1+\eta-\sigma)+2}{4}L_1(\theta) + \frac{n_K(1+\eta-\sigma)+2}{2}\log(T+2)$$
$$+\frac{1+\eta-\sigma}{2}\left(\log\frac{d_K}{(2\pi)^{n_K}}\right)+\log 3.$$

For $\sigma < -\eta$, we define

$$F_{c,r}(\theta) = n_K \log\zeta(1-\sigma) + \frac{1}{2}L_{-1}(\theta) + \log(T+2) + \frac{1-2\sigma}{2}\log\left(\frac{d_K(T+2)^{n_K}}{(2\pi)^{n_K}}\right)$$
$$+\frac{(1-2\sigma+2[\sigma])n_K}{4}L_{1-[\sigma]}(\theta) + \frac{n_K}{2}\sum_{j=1}^{-[\sigma]}L_{j-1}(\theta).$$

We note that $F_{c,r}(\theta)$ is an even function of θ satisfying $F_{c,r}(\theta) \ge \frac{1}{N}\log|f_N(c+re^{i\theta})|$. In order to bound $F_{c,r}(\theta)$, following [11], for $c \in \mathbb{R}$ and $r > 0$, we define

$$\theta_y = \begin{cases} 0 & \text{if } c+r \le y; \\ \arccos\frac{y-c}{r} & \text{if } c-r \le y \le c+r; \\ \pi & \text{if } y \le c-r. \end{cases}$$

For the sake of convenience, we define

$$\kappa_1 = \int_{\theta_{1+\eta}}^{\theta_{-\eta}} \frac{1+\eta-\sigma}{2}d\theta + \int_{\theta_{-\eta}}^{\pi}\frac{1-2\sigma}{2}d\theta,$$

For $J_1, J_2 \in \mathbb{N}$, we shall set

$$\kappa_2(J_1) = \frac{\pi}{4J_1}\left(\log\zeta(c+r) + 2\sum_{j=1}^{J_1-1}\log\zeta\left(c+r\cos\frac{\pi j}{2J_1}\right)\right),$$

and

$$\kappa_3(J_2) = \frac{\pi - \theta_{1-c}}{2J_2}\Big(\log\zeta(1-c+r) + 2\sum_{j=1}^{J_2-1}\log\zeta\Big(1-c-r\cos\Big(\frac{\pi j}{J_2} + \Big(1-\frac{j}{J_2}\Big)\theta_{1-c}\Big)\Big)\Big).$$

In addition, we define

$$\kappa_4 = \frac{1}{4}\int_{\theta_{1+\eta}}^{\theta_{-\eta}}(1+\eta-\sigma)L_1^\star(\theta)d\theta,$$
$$\kappa_5 = \frac{1}{4}\int_{\theta_{-\eta}}^{\theta_{-1/2}}(1-2\sigma)L_1^\star(\theta)d\theta.$$

Similar to [11, Proposition 5.13], we have the following proposition regarding the upper bound of $\int_0^\pi F_{c,r}(\theta)d\theta$.

Proposition 2.17. *Let* c, r, *and* η *be positive real numbers satisfying*

$$-\frac{1}{2} < c-r < -\eta < 1+\eta < c \tag{2.22}$$

and $0 < \eta \le \frac{1}{2}$. *Then for* $T \ge \frac{5}{7}$, *we have*

$$\begin{aligned}\int_0^\pi F_{c,r}(\theta)d\theta \le{}& n_K\int_0^{\theta_{1+\eta}}\log\zeta(\sigma)d\theta + \frac{1}{2(T+2)}\int_0^{\theta_{1+\eta}}L_{-1}^\star(\theta)d\theta + \theta_{1+\eta}\log(T+2)\\ &+ n_K(\log\zeta(1+\eta))(\theta_{-\eta}-\theta_{1+\eta}) + \Big(\log\frac{d_K(T+2)^{n_K}}{(2\pi)^{n_K}}\Big)\kappa_1\\ &+ \frac{n_K}{T+2}\kappa_4 + \frac{1}{2(T+2)}\int_{\theta_{1+\eta}}^{\theta_{-\eta}}L_1^\star(\theta)d\theta + (\theta_{-\eta}-\theta_{1+\eta})\log(3(T+2))\\ &+ n_K\int_{\theta_{-\eta}}^{\pi}\log\zeta(1-\sigma)d\theta + \frac{1}{2(T+2)}\int_{\theta_{-\eta}}^{\pi}L_{-1}^\star(\theta)d\theta + (\pi-\theta_{-\eta})\log(T+2)\\ &+ \frac{n_K}{T+2}\kappa_5.\end{aligned}$$

Proof. We first write

$$\int_0^\pi F_{c,r}(\theta)d\theta = \int_0^{\theta_{1+\eta}}F_{c,r}(\theta)d\theta + \int_{\theta_{1+\eta}}^{\theta_{-\eta}}F_{c,r}(\theta)d\theta + \int_{\theta_{-\eta}}^{\pi}F_{c,r}(\theta)d\theta.$$

By the definition of $F_{c,r}(\theta)$, we have

$$\begin{aligned}\int_0^{\theta_{1+\eta}} F_{c,r}(\theta)d\theta &= n_K\int_0^{\theta_{1+\eta}} \log\zeta(\sigma)d\theta + \frac{1}{2}\int_0^{\theta_{1+\eta}} L_{-1}(\theta)d\theta + \int_0^{\theta_{1+\eta}} \log(T+2)d\theta \\ &\le n_K\int_0^{\theta_{1+\eta}} \log\zeta(\sigma)d\theta + \frac{1}{2(T+2)}\int_0^{\theta_{1+\eta}} L^\star_{-1}(\theta)d\theta + \theta_{1+\eta}\log(T+2).\end{aligned}$$

Secondly, we compute

$$\begin{aligned}\int_{\theta_{1+\eta}}^{\theta_{-\eta}} F_{c,r}(\theta)d\theta =& n_K\int_{\theta_{1+\eta}}^{\theta_{-\eta}} \log\zeta(1+\eta)d\theta + \Big(\log\frac{d_K}{(2\pi)^{n_K}}\Big)\int_{\theta_{1+\eta}}^{\theta_{-\eta}} \frac{1+\eta-\sigma}{2}d\theta \\ &+\log 3\int_{\theta_{1+\eta}}^{\theta_{-\eta}} 1d\theta + \int_{\theta_{1+\eta}}^{\theta_{-\eta}} \frac{n_K(1+\eta-\sigma)+2}{4}L_1(\theta)d\theta \\ &+\log(T+2)\int_{\theta_{1+\eta}}^{\theta_{-\eta}} \frac{n_K(1+\eta-\sigma)+2}{2}d\theta.\end{aligned}\tag{2.23}$$

The first three integrals on the right of (2.23) are

$$n_K(\log\zeta(1+\eta))(\theta_{-\eta}-\theta_{1+\eta}) + \Big(\log\frac{d_K}{(2\pi)^{n_K}}\Big)\int_{\theta_{1+\eta}}^{\theta_{-\eta}} \frac{1+\eta-\sigma}{2}d\theta + (\log 3)(\theta_{-\eta}-\theta_{1+\eta}).$$

As $1+\eta-\sigma\ge 0$ for $\theta\in[\theta_{1+\eta},\theta_{-\eta}]$, it follows that the last two integrals on the right of (2.23) are

$$\begin{aligned}&\frac{n_K}{4}\int_{\theta_{1+\eta}}^{\theta_{-\eta}}(1+\eta-\sigma)L_1(\theta)d\theta + \frac{1}{2}\int_{\theta_{1+\eta}}^{\theta_{-\eta}} L_1(\theta)d\theta + n_K\log(T+2)\int_{\theta_{1+\eta}}^{\theta_{-\eta}}\frac{1+\eta-\sigma}{2}d\theta \\ &+(\theta_{-\eta}-\theta_{1+\eta})\log(T+2) \\ &\le \frac{n_K}{(T+2)}\kappa_4 + \frac{1}{2(T+2)}\int_{\theta_{1+\eta}}^{\theta_{-\eta}} L^\star_1(\theta)d\theta + n_K\log(T+2)\int_{\theta_{1+\eta}}^{\theta_{-\eta}}\frac{1+\eta-\sigma}{2}d\theta \\ &+(\theta_{-\eta}-\theta_{1+\eta})\log(T+2).\end{aligned}$$

Lastly, we have

$$\begin{aligned}\int_{\theta_{-\eta}}^{\pi} F_{c,r}(\theta)d\theta &= n_K \int_{\theta_{-\eta}}^{\pi} \log\zeta(1-\sigma)d\theta + \frac{1}{2}\int_{\theta_{-\eta}}^{\pi} L_{-1}(\theta)d\theta + \int_{\theta_{-\eta}}^{\pi} \log(T+2)d\theta \\ &+ \Big(\log\frac{d_K(T+2)^{n_K}}{(2\pi)^{n_K}}\Big)\int_{\theta_{-\eta}}^{\pi}\frac{1-2\sigma}{2}d\theta + n_K\int_{\theta_{-\eta}}^{\theta_{-\frac{1}{2}}}\frac{1-2\sigma}{4}L_1(\theta)d\theta \\ &+ n_K\sum_{j=1}^{\infty}\int_{\theta_{-j+\frac{1}{2}}}^{\theta_{-j-\frac{1}{2}}}\Big(\frac{1-2\sigma-2j}{4}L_{j+1}(\theta)+\frac{1}{2}\sum_{k=1}^{j}L_{k-1}(\theta)\Big)d\theta.\end{aligned} \tag{2.24}$$

The first four integrals on the right of (2.24) are

$$\begin{aligned}&\le n_K\int_{\theta_{-\eta}}^{\pi}\log\zeta(1-\sigma)d\theta + \frac{1}{2(T+2)}\int_{\theta_{-\eta}}^{\pi} L^{\star}_{-1}(\theta)d\theta + \log(T+2)(\pi-\theta_{-\eta}) \\ &+ \Big(\log\frac{d_K(T+2)^{n_K}}{(2\pi)^{n_K}}\Big)\int_{\theta_{-\eta}}^{\pi}\frac{1-2\sigma}{2}d\theta.\end{aligned}$$

Note that as $-\frac{1}{2} < c-r$, we have $\theta_{-j+\frac{1}{2}} = \theta_{-j-\frac{1}{2}} = \pi$ for $j \ge 1$. Thus, the remaining integral and sum on the the right of (2.24) is

$$n_K\int_{\theta_{-\eta}}^{\theta_{-\frac{1}{2}}}\frac{1-2\sigma}{4}L_1(\theta)d\theta \le \frac{n_K}{T+2}\int_{\theta_{-\eta}}^{\theta_{-\frac{1}{2}}}\frac{1-2\sigma}{4}L_1^{\star}(\theta)d\theta = \frac{n_K}{T+2}\kappa_5.$$

Putting all the estimates together, we complete the proof. □

To control "zeta integrals" in the above proposition, we shall borrow two estimates from [11, Lemmata 5.14 and 5.15] as follows.

Lemma 2.18. *Let c, r and η be positive real numbers, satisfying* (2.22)*, and J_1 and J_2 be positive integers. If $\theta_{1+\eta} \le 2.1$, then for $\sigma = c + r\cos\theta$, one has*

$$\int_0^{\theta_{1+\eta}}\log\zeta(\sigma)d\theta \le \frac{\log\zeta(1+\eta)+\log\zeta(c)}{2}\Big(\theta_{1+\eta}-\frac{\pi}{2}\Big)+\frac{\pi}{4J_1}\log\zeta(c)+\kappa_2(J_1).$$

In addition, assuming further $r > 2c-1$, one has

$$\int_{\theta_{-\eta}}^{\pi} \log\zeta(1-\sigma)d\theta \leq \frac{\log\zeta(1+\eta)+\log\zeta(c)}{2}(\theta_{1-c}-\theta_{-\eta}) + \frac{\pi-\theta_{1-c}}{2J_2}\log\zeta(c)+\kappa_3(J_2).$$

2.4 Completing the proof

Gathering (2.12) and Propositions 2.10 and 2.12, for

$$-\tfrac{1}{2} < c-r < 1-c < -\eta < 0 < \tfrac{1}{4} \leq \delta = 2c-\sigma_1-\tfrac{1}{2} < \tfrac{1}{2} < 1 < 1+\eta < c < \sigma_1 = c+\tfrac{(c-1/2)^2}{r} < c+r,$$

satisfying $\theta_{1+\eta} \leq 2.1$, we have

$$\begin{aligned} &\left|N_K(T)-\frac{T}{\pi}\log\left(d_K\left(\frac{T}{2\pi e}\right)^{n_K}\right)+\frac{r_1}{4}\right| \\ &\leq \frac{5}{2}+|g_K(T)|+\frac{2n_K}{\pi}\log\zeta(\sigma_1)+\frac{\log\left(\frac{1}{\sqrt{(c-1)^2+T^2}}\frac{\zeta_K(c)}{\zeta_K(2c)}\right)}{\log\frac{r}{c-\frac{1}{2}}}+ \\ &+\frac{1}{\pi\log\frac{r}{c-\frac{1}{2}}}\int_0^{\pi}F_{c,r}(\theta)d\theta+\frac{E_K(T,\delta)}{\pi}, \end{aligned} \tag{2.25}$$

where $g_K(T)$ and $E_K(T,\delta)$ are defined as in (2.9) and (2.14), respectively, and

$$\log\frac{\zeta_K(c)}{\zeta_K(2c)} = \int_c^{2c} -\frac{\zeta_K'}{\zeta_K}(\sigma)d\sigma \leq n_K\int_c^{2c} -\frac{\zeta'}{\zeta}(\sigma)d\sigma \leq n_K\log\frac{\zeta(c)}{\zeta(2c)}.$$

Finally, using (2.10), Lemma 2.5, Proposition 2.17, and Lemma 2.18 to bound (2.25) and recalling that $r_1+2r_2 = n_K$, for any $T_0 \geq \frac{5}{7}$, we obtain

$$\left|N_K(T)-\frac{T}{\pi}\log\left(d_K\left(\frac{T}{2\pi e}\right)^{n_K}\right)+\frac{r_1}{4}\right| \leq C_1\log\left(\frac{d_K(T+2)^{n_K}}{(2\pi)^{n_K}}\right)+C_2n_K+C_3 \tag{2.26}$$

whenever $T \geq T_0$, where

$$C_1 = \kappa_1\left(\pi\log\frac{r}{c-\frac{1}{2}}\right)^{-1},$$

$$C_2 = \frac{1}{25T_0} + \frac{2}{\pi}\log\zeta(\sigma_1) + \frac{640\delta-112}{1536(3T_0-1)} + \max\left\{0, \frac{856\delta-151}{1536(3T_0+2)} - \frac{640\delta-112}{1536(3T_0-1)}\right\}$$
$$+ \frac{1}{2^{10}} + \left(\pi\log\frac{r}{c-\frac{1}{2}}\right)^{-1}\left(\frac{\log\zeta(1+\eta)+\log\zeta(c)}{2}\left(\theta_{1+\eta}-\frac{\pi}{2}\right) + \frac{\pi}{4J_1}\log\zeta(c) + \kappa_2(J_1)\right)$$
$$+ \left(\pi\log\frac{r}{c-\frac{1}{2}}\right)^{-1}\left(\frac{\log\zeta(1+\eta)+\log\zeta(c)}{2}(\theta_{1-c}-\theta_{-\eta}) + \frac{\pi-\theta_{1-c}}{2J_2}\log\zeta(c) + \kappa_3(J_2)\right)$$
$$+ \left(\pi\log\frac{r}{c-\frac{1}{2}}\right)^{-1}\left((\log\zeta(1+\eta))(\theta_{-\eta}-\theta_{1+\eta}) + \max\left\{0, \frac{\kappa_4+\kappa_5}{T_0+2}\right\} + \pi\log\frac{\zeta(c)}{\zeta(2c)}\right),$$

$$C_3 = \frac{5}{2} + \left(\pi\log\frac{r}{c-\frac{1}{2}}\right)^{-1}\left(\pi\log\left(1+\frac{2}{T_0}\right) + (\theta_{-\eta}-\theta_{1+\eta})\log 3\right)$$
$$+\max\left\{0, \left(\pi\log\tfrac{r}{c-\frac{1}{2}}\right)^{-1}\left(\tfrac{1}{2(T_0+2)}\left(\int_0^{\theta_{1+\eta}} L^\star_{-1}(\theta)d\theta + \int_{\theta_{1+\eta}}^{\theta_{-\eta}} L^\star_1(\theta)d\theta + \int_{\theta_{-\eta}}^{\pi} L^\star_{-1}(\theta)d\theta\right)\right)\right\}.$$

For $T_0 = 1$ and $T_0 = 10$, choosing $J_1 = 64$ and $J_2 = 39$, via a Maple numerical computation, we have the following table of admissible (C_1, C_2, C_3).

Table 2.2: Choices of parameters (c, r, η) and resulting admissible (C_1, C_2, C_3)

				$T \geq 1$		$T \geq 10$	
c	r	η	C_1	C_2	C_3	C_2	C_3
1.000011314	1.064340602	$4.2826451 \cdot 10^{-6}$	0.22737	23.02528	4.51954	22.97204	3.30668
1.042877508	1.259860485	0.01737451737	0.24493	6.66558	4.21201	6.60397	3.12362
1.079779637	1.410370323	0.03441682600	0.26304	5.22032	4.08149	5.15251	3.05074
1.114294066	1.538391756	0.05247813411	0.28032	4.43521	4.00936	4.36214	3.01124
1.145720440	1.645584376	0.07107039918	0.29590	3.93889	3.96852	3.86136	2.98903

One may find functioning Maple code at `https://arxiv.org/abs/2102.04663`

Chapter 3

Counting zeros of the Riemann zeta function

The results of this chapter have been published in [55].

3.1 Introduction

Let $\zeta(s)$ be the Riemann zeta function defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

for $\Re(s) > 1$, which has an analytic continuation to a meromorphic function on $\mathbb{C}$ with only a simple pole at $s = 1$. The study of zeros of $\zeta(s)$ is an important topic in number theory. In this article, we shall estimate the number of non-trivial zeros $\rho = \beta + i\gamma$, with $0 < \gamma \leq T$, of $\zeta(s)$. For $T \geq 0$, we set

$$N(T) = \#\{\rho \in \mathbb{C} \mid \zeta(\rho) = 0,\ 0 < \beta < 1,\ 0 < \gamma \leq T\}.$$

Before stating our results, we shall note that the study of $N(T)$ has a long history. Indeed, for $T \geq T_0$, writing

$$\left| N(T) - \frac{T}{2\pi} \log\left(\frac{T}{2\pi e}\right) \right| \leq C_1 \log T + C_2 \log\log T + C_3, \tag{3.1}$$

we have the following table summarising the progress that has been made.

Table 3.1: Explicit bounds for $N(T)$ in (3.1)

	C_1	C_2	C_3	T_0
Von Mangoldt [123] (1905)	0.4320	1.9167	13.0788	28.5580
Grossmann [48] (1913)	0.2907	1.7862	7.0120	50
Backlund [5] (1918)	0.1370	0.4430	5.2250	200
Rosser [98] (1941)	0.1370	0.4430	2.4630	2
Trudgian [119] (2014)	0.1120	0.2780	3.3850	e
Corollary 3.2	0.1038	0.2573	9.3675	e

The importance of explicit bounds for $N(T)$ comes from the fact that they are crucial for estimating sums over zeros of $\zeta(s)$, and all the best known bounds for $\pi(x)$ and $\psi(x)$ rely on them (see, e.g., [38]).

In this chapter, we prove the following general result for $N(T)$ with explicit dependence on the given bounds for $\zeta(s)$ on the both $\frac{1}{2}$-line and 1-line.

Theorem 3.1. *Let c, r, η be positive real numbers satisfying*

$$-\tfrac{1}{2} < c-r < 1-c < -\eta < \tfrac{1}{4} \le \delta := 2c-\sigma_1-\tfrac{1}{2} < \tfrac{1}{2} < 1+\eta < \sigma_1 := c+\tfrac{(c-1/2)^2}{r} < c+r$$

and $\theta_{1+\eta} \le 2.1$, where θ_y is defined in (3.28). *Let $c_1, c_2, k_1, k_3 \ge 0$, $k_2 \in [0, \frac{1}{2}]$ and $t_0, t_1 \ge e$ such that for $t \ge t_0$,*

$$|\zeta(1+it)| \le c_1(\log t)^{c_2}, \tag{3.2}$$

and for $t \ge t_1$,

$$|\zeta(\tfrac{1}{2}+it)| \le k_1 t^{k_2}(\log t)^{k_3}. \tag{3.3}$$

Let $T_0 \ge e$ be fixed. Then for any $T \ge T_0$, we have

$$\left| N(T) - \frac{T}{2\pi}\log\left(\frac{T}{2\pi e}\right) + \frac{1}{8} \right| \le C_1 \log T + C_2 \log\log T + C_3, \tag{3.4}$$

where $C_1 = C_1(c, r, \eta; k_2)$, $C_2 = C_2(c, r, \eta; c_2, k_3)$, $C_3 = C_3(c, r, \eta; c_1, c_2, t_0, k_1, k_2, k_3, t_1; T_0)$ are defined in (3.30), (3.31), (3.32), *and* (3.33), *and some admissible values of C_1, C_2, and*

C_3 are recorded in Table 3.2 in Section 3.5.

As a consequence, we obtain an explicit estimate for $N(T)$ as follows.

Corollary 3.2. *For any $T \geq e$, we have*

$$\left| N(T) - \frac{T}{2\pi} \log\left(\frac{T}{2\pi e}\right) \right| \leq 0.1038 \log T + 0.2573 \log\log T + 9.3675. \tag{3.5}$$

Note that $C_1 = 0.1038$ is the smallest value that can be obtained by our argument and computation, and one can make C_2 and C_3 smaller at expense of larger C_1.

Let $S(T) = \frac{1}{\pi}\Delta_L \arg\zeta(s)$, where L denotes the straight line from 2 to $2+iT$ and then to $\frac{1}{2}+iT$. We also have the following theorem concerning the argument of $\zeta(s)$ along the critical line.

Theorem 3.3. *In the notation and assumptions of Theorem 3.1, for any $T \geq T_0$, we have*

$$|S(T)| \leq C_1 \log T + C_2 \log\log T + C_3', \tag{3.6}$$

where $C_1 = C_1(c,r,\eta;k_2)$, $C_2 = C_2(c,r,\eta;c_2,k_3)$, $C_3' = C_3'(c,r,\eta;c_1,c_2,t_0,k_1,k_2,k_3,t_1;T_0)$ are defined in (3.30)*,* (3.31)*,* (3.32)*, and* (3.36)*, and some admissible values of C_1, C_2, and C_3' are recorded in Table 3.2 in Section 3.5.*

A stronger bound for $S(T)$, up to certain given height, can be confirmed using the database of non-trivial zeros of $\zeta(s)$ computed by Platt and made available at [75]. Indeed, nowadays, one has

$$|S(T)| \leq 2.5167 \tag{3.7}$$

for $0 \leq T \leq 30\,610\,046\,000$.[2] Hence, by Theorem 3.3 (with $T_0 = 30\,610\,046\,000$ and Table 3.2) and (3.7), we derive the following explicit bound for $S(T)$.

[2]Recently, Platt and Trudgian [90] verified the Riemann hypothesis for the height up to $3 \cdot 10^{12}$, which would allow one to further bound $S(T)$ for $0 \leq T \leq 3 \cdot 10^{12}$.

Corollary 3.4. *For any $T \geq e$, we have*

$$|S(T)| \leq \min\{0.1038\log T + 0.2573\log\log T + 8.3675,\ \ 0.1095\log T + 0.2042\log\log T + 3.0305\}.$$

We note that this improves the previous best-known explicit bound (for T sufficiently large) due to Platt and Trudgian [88] (see also [117, 119]), who showed that for $T \geq e$,

$$|S(T)| \leq 0.110\log T + 0.290\log\log T + 2.290.$$

The proofs of Theorems 3.2 and 3.3 are based on the work of [11, 54, 119, 120].[3] Compared to the considerations of Dirichlet L-functions in [11,120] and Dedekind zeta functions in [54, 120], we further use subconvexity bounds for $\zeta(\frac{1}{2}+it)$, together with the Phragmén-Lindelöf principle and the functional equation, to obtain a shaper estimate for $\zeta(s)$ in the strip $0 \leq \Re(s) \leq 1$. Also, based on the idea of [11], we refine the bound for $\zeta(s)$ on $\Re(s) < 0$ used in [119] by bounding $\zeta(s)$ over $\Re(s) < -\frac{1}{2}$. Lastly, we note that most numerical computations were performed in Maple.

3.2 Main term and bounds for gamma factors

We recall the completed Riemann zeta function $\xi(s)$ is defined by

$$\xi(s) = s(s-1)\gamma(s)\zeta(s), \tag{3.8}$$

where

$$\gamma(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right).$$

[3] Regrettably, as pointed out in [11], there is an error appearing in [119,120] (and [88], where the erroneous result of [119] was used) since the ranges of parameters involved in the final formulae were not verified properly; [11, 54] fix this issue for [120]. In a certain degree, the objective of the presented paper is to fix the error occurring in [119].

It is well-known that $\xi(s)$ can be extended to an entire function of order 1, which satisfies the functional equation

$$\xi(s) = \xi(1-s). \tag{3.9}$$

To follow the argument used in [54], it would be simpler to work with the following "symmetric version" of $N(T)$. We introduce

$$N_{\mathbb{Q}}(T) = \#\{\rho \in \mathbb{C} \mid \zeta(\rho) = 0,\ 0 < \beta < 1,\ |\gamma| \leq T\}$$

for $T \geq 0$.[4] Note that $N_{\mathbb{Q}}(T) = 2N(T)$.

Let $\sigma_1 > 1$, and let $\mathcal{R}$ be the rectangle with vertices $\sigma_1 - iT,\ \sigma_1 + iT,\ 1-\sigma_1+iT$, and $1-\sigma_1-iT$ (that is away from zeros of $\xi(s)$). Since $\xi(s)$ is entire, applying the argument principle, we know that

$$N_{\mathbb{Q}}(T) = \frac{1}{2\pi}\Delta_{\mathcal{R}} \arg \xi(s).$$

We let $\mathcal{C}$ be the part of the contour of $\mathcal{R}$ in $\Re(s) \geq \frac{1}{2}$ and $\mathcal{C}_0$ be the part of the contour of $\mathcal{R}$ in $\Re(s) \geq \frac{1}{2}$ and $\Im(s) \geq 0$. From the functional equation (3.9) and the fact that $\overline{\xi(s)} = \xi(\bar{s})$, it follows that

$$\Delta_{\mathcal{R}} \arg \xi(s) = 2\Delta_{\mathcal{C}} \arg \xi(s) = 4\Delta_{\mathcal{C}_0} \arg \xi(s),$$

and thus

$$N_{\mathbb{Q}}(T) = \frac{2}{\pi}\Delta_{\mathcal{C}_0} \arg \xi(s). \tag{3.10}$$

Now, by (3.8), we arrive at

$$\Delta_{\mathcal{C}_0} \arg \xi(s) = \Delta_{\mathcal{C}_0} \arg s + \Delta_{\mathcal{C}_0} \arg \pi^{-s/2} + \Delta_{\mathcal{C}_0} \arg \Gamma\left(\frac{s}{2}\right) + \Delta_{\mathcal{C}_0} \arg\left((s-1)\zeta(s)\right). \tag{3.11}$$

[4]This notation agrees with [54], where we defined $N_K(T)$, the zero-counting function for the Dedekind zeta function of a number field K.

In addition, by a straightforward calculation, we have

$$\begin{aligned}
&\Delta_{\mathcal{C}_0} \arg s = \arctan(2T), \\
&\Delta_{\mathcal{C}_0} \arg \pi^{-s/2} = \frac{T}{2}\log\left(\frac{1}{\pi}\right), \\
&\Delta_{\mathcal{C}_0} \arg \Gamma(s) = \Delta_{\mathcal{C}_0}(\Im \log \Gamma(s)) = \Im \log \Gamma\left(\frac{1}{2} + iT\right).
\end{aligned} \tag{3.12}$$

In order to control the contribution of the gamma factor in (3.11), we set

$$g(T) = \frac{2}{\pi}\Im \log \Gamma\left(\frac{1}{4} + i\frac{T}{2}\right) - \frac{T}{\pi}\log\left(\frac{T}{2e}\right) + \frac{1}{4} \tag{3.13}$$

and recall that by [11, Proposition 3.2], one has

$$|g(T)| \leq \frac{1}{25T} \tag{3.14}$$

for $T \geq 5/7$. Now, gathering (3.10), (3.11), (3.12), and (3.13), we establish

$$N_{\mathbb{Q}}(T) = \frac{2}{\pi}\arctan(2T) + g(T) + \frac{T}{\pi}\log\left(\frac{T}{2\pi e}\right) - \frac{1}{4} + \frac{2}{\pi}\Delta_{\mathcal{C}_0}\arg((s-1)\zeta(s)). \tag{3.15}$$

To control $\Delta_{\mathcal{C}_0}\arg((s-1)\zeta(s))$, we let $\mathcal{C}_1$ be the vertical line from σ_1 to $\sigma_1 + iT$ and $\mathcal{C}_2$ be the horizontal line from $\sigma_1 + iT$ to $\frac{1}{2} + iT$. As

$$\Delta_{\mathcal{C}_1}\arg(s-1)\zeta(s) = \Delta_{\mathcal{C}_1}\arg(s-1) + \Delta_{\mathcal{C}_1}\arg\zeta(s) = \arctan\left(\frac{T}{\sigma_1 - 1}\right) + \Delta_{\mathcal{C}_1}\arg\zeta(s)$$

and for $\sigma_1 > 1$,

$$|\Delta_{\mathcal{C}_1}\arg\zeta(s)| = |\arg\zeta(\sigma_1 + iT)| \leq |\log\zeta(\sigma_1 + iT)| \leq \log\zeta(\sigma_1),$$

we obtain

$$|\Delta_{\mathcal{C}_1}\arg(s-1)\zeta(s)| \leq \frac{\pi}{2} + \log\zeta(\sigma_1). \tag{3.16}$$

Hence, from (3.14), (3.15), and (3.16), it follows that

$$\left| N_{\mathbb{Q}}(T) - \frac{T}{\pi}\log\left(\frac{T}{2\pi e}\right) + \frac{1}{4}\right| \leq 2 + \frac{1}{25T} + \frac{2}{\pi}\log\zeta(\sigma_1) + \frac{2}{\pi}|\Delta_{C_2}\arg((s-1)\zeta(s))|. \quad (3.17)$$

To end this section, we shall borrow some estimates for the gamma function from [11] as follows. For $0 \leq d < 9/2$ and $T \geq 5/7$, we define

$$\mathcal{E}(T,d) = \left| \Im \log\Gamma\left(\frac{\sigma+iT}{2}\right)\Big|_{\sigma=\frac{1}{2}}^{\frac{1}{2}+d} + \Im\log\Gamma\left(\frac{\sigma+iT}{2}\right)\Big|_{\sigma=\frac{1}{2}}^{\frac{1}{2}-d}\right|,$$

As in [11, p. 1463], we set

$$\begin{aligned}
E(T,d) = {} & \frac{2T/3}{(2d+17)^2+4T^2} + \frac{2T/3}{(-2d+17)^2+4T^2} - \frac{4T/3}{17^2+4T^2} \\
& + \frac{T}{2}\log\left(1+\frac{17^2}{4T^2}\right) - \frac{T}{4}\log\left(1+\frac{(2d+17)^2}{4T^2}\right) - \frac{T}{4}\log\left(1+\frac{(-2d+17)^2}{4T^2}\right) \\
& + \frac{(8+6\pi)/45}{((2d+17)^2+4T^2)^{3/2}} + \frac{(8+6\pi)/45}{((-2d+17)^2+4T^2)^{3/2}} + \frac{2(8+6\pi)/45}{(17^2+4T^2)^{3/2}} \\
& + \sum_{k=0}^{3}\left(2\arctan\frac{1+4k}{2T} - \arctan\frac{2d+1+4k}{2T} - \arctan\frac{-2d+1+4k}{2T}\right) \\
& + \frac{2d+15}{4}\arctan\frac{2d+17}{2T} + \frac{-2d+15}{4}\arctan\frac{-2d+17}{2T} - \frac{15}{2}\arctan\frac{17}{2T}.
\end{aligned}$$

It has been shown in [11, p. 1462] that $\mathcal{E}(T,d) \leq E(T,d)$ for $0 \leq d < 9/2$ and $T \geq 5/7$. Moreover, one has the following lemma established in [11, Lemma 3.4].

Lemma 3.5. *For $0 \leq \delta_1 \leq d < 9/2$ and $T \geq 5/7$, one has*

$$0 < E(T,\delta_1) \leq E(T,d).$$

Also, for $d \in [\frac{1}{4},\frac{5}{8}]$ and $T \geq 5/7$, one has

$$\frac{E(T,d)}{\pi} \leq \frac{640d-112}{1536(3T-1)} + \frac{1}{2^{10}}.$$

We remark that the number $5/7$ above is not random but is borrowed directly from [11]. In [11], it is chosen to obtain a proper numerical bound (in order to count the zeros of Dirichlet L-functions). One may replace $5/7$ with an appropriate larger T_0 to get a better result.

3.3 Backlund's trick

In order to estimate $\Delta_{C_2} \arg((s-1)\zeta(s))$, we shall borrow some results from [54]. Define

$$f_N(s) = \frac{1}{2}\left(((s+iT-1)\zeta(s+iT))^N + ((s-iT-1)\zeta(s-iT))^N\right) \tag{3.18}$$

for $N \in \mathbb{N}$. Let $D(c,r)$ be the open disk centred at c with radius r. For any $N \in \mathbb{N}$, we define

$$S_N(c,r) = \frac{1}{N} \sum_{z \in \mathcal{S}_N(D(c,r))} \log \frac{r}{|z-c|},$$

where $\mathcal{S}_N(D(c,r))$ denotes the set of zeros of $f_N(s)$ in $D(c,r)$. In [54, Proposition 3.5], the authors prove the following upper bound for $S_N(c,r)$.

Proposition 3.6. *Let c, r, and σ_1 be real numbers such that*

$$c - r < \frac{1}{2} < 1 < c < \sigma_1 < c + r.$$

Let $F_{c,r} : [-\pi,\pi] \to \mathbb{R}$ be an even function such that $F_{c,r}(\theta) \geq \frac{1}{N_m} \log |f_{N_m}(c+re^{i\theta})|$. Then there is an infinite sequence of natural numbers $(N_m)_{m=1}^{\infty}$ such that

$$\limsup_{m\to\infty} S_{N_m}(c,r) \leq \log\left(\frac{1}{\sqrt{(c-1)^2+T^2}} \frac{\zeta(c)}{\zeta(2c)}\right) + \frac{1}{\pi}\int_0^{\pi} F_{c,r}(\theta)d\theta.$$

To end this section, we recall the following version of Backlund's trick established in [54, Proposition 3.7] (cf. [11, 119, 120]). As explained in [120, Sec. 3], using Backlund's trick, one can track contribution from zeros of $f_N(\sigma)$ in $[1-\sigma_1, \frac{1}{2}]$ whence there are zeros in $[\frac{1}{2}, \sigma_1]$.

Proposition 3.7 (Backlund's trick)**.** *Let c and r be real numbers. Set*

$$\sigma_1 = c + \frac{(c-1/2)^2}{r} \quad \textit{and} \quad \delta = 2c - \sigma_1 - \frac{1}{2}.$$

If $1 < c < r$ *and* $0 < \delta < \frac{1}{2}$*, then*

$$\left| \arg\left((\sigma + iT - 1)\zeta(\sigma + iT)\right)\Big|_{\sigma=\sigma_1}^{1/2} \right| \leq \frac{\pi S_N(c,r)}{2\log(r/(c-1/2))} + \frac{E(T,\delta)}{2} + \frac{\pi}{N} + \frac{\pi}{2N} + \frac{\pi}{4}.$$

3.4 Convexity and subconvexity bounds and $F_{c,r}(\theta)$

3.4.1 Convexity and subconvexity bounds

In light of Propositions 3.6 and 3.7, to estimate (3.17), we shall construct an appropriate $F_{c,r}(\theta)$. We first recall the following version of the Phragmén-Lindelöf principle established by Trudgian [119, Lemma 3].

Proposition 3.8 (Phragmén-Lindelöf principle)**.** *Let* a, b, Q *be real numbers such that* $b > a$ *and* $Q + a > 1$*. Let* $f(s)$ *be a holomorphic function on the strip* $a \leq \Re(s) \leq b$ *such that*

$$|f(s)| < C\exp(e^{k|t|})$$

for some $C > 0$ *and* $0 < k < \frac{\pi}{b-a}$*. Suppose, further, that there are* $A, B, \alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$ *such that* $\alpha_1 \geq \beta_1$ *and*

$$|f(s)| \leq \begin{cases} A|Q+s|^{\alpha_1}(\log|Q+s|)^{\alpha_2} & \textit{for } \Re(s) = a; \\ B|Q+s|^{\beta_1}(\log|Q+s|)^{\beta_2} & \textit{for } \Re(s) = b. \end{cases}$$

Then for $a \leq \Re(s) \leq b$*, one has*

$$|f(s)| \leq \left\{A|Q+s|^{\alpha_1}(\log|Q+s|)^{\alpha_2}\right\}^{\frac{b-\Re(s)}{b-a}} \left\{B|Q+s|^{\beta_1}(\log|Q+s|)^{\beta_2}\right\}^{\frac{\Re(s)-a}{b-a}}.$$

We shall assume that there are $c_1, c_2, k_1, k_3 \geq 0$, $k_2 \in [0, \frac{1}{2}]$ and $t_0, t_1 \geq e$ such that

$$|\zeta(1+it)| \leq c_1(\log t)^{c_2} \tag{3.19}$$

for $t \geq t_0$, and

$$|\zeta(\tfrac{1}{2}+it)| \leq k_1 t^{k_2}(\log t)^{k_3} \tag{3.20}$$

for $t \geq t_1$. Recall that $0 < \eta \leq \frac{1}{2}$. Now, we split our consideration into the following six cases.

(1) Assume $\sigma \geq 1+\eta$. The trivial bound for the zeta function immediately gives

$$|\zeta(s)| \leq \zeta(\sigma). \tag{3.21}$$

(2) Assume $1 \leq \sigma \leq 1+\eta$. From (3.19), it follows that there is $Q_0 > 0$ such that

$$|(1+it-1)\zeta(1+it)| \leq c_1|Q_0+(1+it)|(\log|Q_0+(1+it)|)^{c_2} \tag{3.22}$$

for all t. (We note that Q_0 depends on c_1, c_2, t_0, and it can be computed explicitly.) Also, by (3.21), it is clear that

$$|(1+\eta+it-1)\zeta(1+\eta+it)| \leq \zeta(1+\eta)|Q_0+(1+\eta+it)|.$$

Thus, by Proposition 3.8, for $1 \leq \sigma \leq 1+\eta$,

$$|(s-1)\zeta(s)| \leq (c_1|Q_0+s|(\log|Q_0+s|)^{c_2})^{\frac{1+\eta-\sigma}{\eta}}(\zeta(1+\eta)|Q_0+s|)^{\frac{\sigma-1}{\eta}},$$

which implies

$$|\zeta(s)| \leq \frac{1}{|s-1|}\left(c_1|Q_0+s|(\log|Q_0+s|)^{c_2}\right)^{\frac{1+\eta-\sigma}{\eta}}\left(\zeta(1+\eta)|Q_0+s|\right)^{\frac{\sigma-1}{\eta}}.$$

(3) Let $\frac{1}{2} \leq \sigma \leq 1$. Using (3.20), we deduce

$$|(\tfrac{1}{2}+it-1)\zeta(\tfrac{1}{2}+it)| \leq k_1|Q_1+(\tfrac{1}{2}+it)|^{k_2+1}(\log|Q_1+(\tfrac{1}{2}+it)|)^{k_3}, \tag{3.23}$$

for all t, where $Q_1 > 0$ is a constant depending only on k_1, k_2, k_3, t_1 (which can be computed directly). Now, by (3.22), (3.23), and Proposition 3.8, for $\frac{1}{2} \leq \sigma \leq 1$, we have

$$|\zeta(s)| \leq \frac{1}{|s-1|}\left(k_1|Q_2+s|^{k_2+1}(\log|Q_2+s|)^{k_3}\right)^{2-2\sigma}\left(c_1|Q_2+s|(\log|Q_2+s|)^{c_2}\right)^{2\sigma-1},$$

where $Q_2 = \max\{Q_0, Q_1\}$.

(4) Assume $0 \leq \sigma \leq \frac{1}{2}$. On the one hand, by (3.20), there is $Q_3 > 0$ such that

$$|\zeta(\tfrac{1}{2}+it)| \leq k_1|Q_3+(\tfrac{1}{2}+it)|^{k_2}(\log|Q_3+(\tfrac{1}{2}+it)|)^{k_3} \tag{3.24}$$

for all t. On the other hand, as (3.9) gives

$$\zeta(s) = \pi^{s-\frac{1}{2}}\frac{\Gamma(\frac{1}{2}-\frac{s}{2})}{\Gamma(\frac{s}{2})}\zeta(1-s), \tag{3.25}$$

it follows from (3.19) and the estimate

$$\left|\frac{\Gamma(\frac{1}{2}-\frac{s}{2})}{\Gamma(\frac{s}{2})}\right| \leq \left(\frac{|1+s|}{2}\right)^{\frac{1}{2}-\sigma}, \tag{3.26}$$

for $-\frac{1}{2} \leq \sigma \leq \frac{1}{2}$, that

$$|\zeta(it)| \leq c_1\left(\frac{|1+it|}{2\pi}\right)^{\frac{1}{2}}(\log t)^{c_2}$$

for $t \geq t_0$. Hence, checking small values of t, we can find $Q_4 \geq 1$ such that

$$|\zeta(0+it)| \leq \frac{c_1}{\sqrt{2\pi}}|Q_4+it|^{\frac{1}{2}}(\log|Q_4+it|)^{c_2} \tag{3.27}$$

for all t. Now, together with (3.24), by Proposition 3.8, we then obtain

$$|\zeta(s)| \leq \left(\frac{c_1}{\sqrt{2\pi}}|Q_5+s|^{\frac{1}{2}}(\log|Q_5+s|)^{c_2}\right)^{1-2\sigma}\left(k_1|Q_5+s|^{k_2}(\log|Q_5+s|)^{k_3}\right)^{2\sigma}.$$

where $Q_5 = \max\{Q_3, Q_4\}$.

(5) For $-\eta \leq \sigma \leq 0$, using (3.21), the functional equation (3.25), and the Gamma bound (3.26), as $Q_4 \geq 1$, we obtain

$$|\zeta(-\eta+it)| \leq \left(\frac{|1-\eta+it|}{2\pi}\right)^{\frac{1}{2}+\eta}\zeta(1+\eta) \leq \left(\frac{|Q_4-\eta+it|}{2\pi}\right)^{\frac{1}{2}+\eta}\zeta(1+\eta)$$

for all t. This estimate, combined with (3.27) and Proposition 3.8, then yields

$$|\zeta(s)| \leq \left(\frac{1}{(2\pi)^{\frac{1}{2}+\eta}}\zeta(1+\eta)|Q_4+s|^{\frac{1}{2}+\eta}\right)^{\frac{-\sigma}{\eta}}\left(\frac{c_1}{\sqrt{2\pi}}|Q_4+s|^{\frac{1}{2}}(\log|Q_4+s|)^{c_2}\right)^{\frac{\sigma+\eta}{\eta}}.$$

(6) Finally, for $\sigma \leq -\eta$, we use (3.25) to deduce

$$|\zeta(s)| \leq \pi^{\sigma-\frac{1}{2}}\left|\frac{\Gamma(\frac{1}{2}-\frac{s}{2})}{\Gamma(\frac{s}{2})}\zeta(1-s)\right| \leq \pi^{\sigma-\frac{1}{2}}\left|\frac{\Gamma(\frac{1}{2}-\frac{s}{2})}{\Gamma(\frac{s}{2})}\right|\zeta(1-\sigma).$$

From the proof of [11, Theorem 5.7], it follows that for $a, b \in \{0,1\}$ and $k \in \mathbb{N}$,

$$\frac{\Gamma(\frac{a}{2}+\frac{1-s}{2})}{\Gamma(\frac{a}{2}+\frac{s}{2})} = \frac{\Gamma(\frac{b}{2}+\frac{1-(s+k)}{2})}{\Gamma(\frac{b}{2}+\frac{s+k}{2})}2^{-k}\left(\prod_{j=1}^{k}(s+j-1)\right)\frac{\sin(\frac{\pi}{2}(s+k+1-b))}{\sin(\frac{\pi}{2}(s+1-a))}.$$

Now, for $x \in \mathbb{R}$, we let $[x]$ be the integer closest to x (if there are two integers equally close to x, we then choose the one closer to 0). Note that for $a = 0$ and $b \equiv k \pmod 2$, the sine factors above are ± 1. Thus, upon taking $k = -[\sigma]$ and applying [92, Lemmata 1 and 2] to bound the ratio $\Gamma(\frac{b}{2}+\frac{1-(s+k)}{2})/\Gamma(\frac{b}{2}+\frac{s+k}{2})$, we arrive at

$$\left|\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}\right| \leq \left(\frac{1}{2}|1+s-[\sigma]|\right)^{\frac{1}{2}+[\sigma]-\sigma} 2^{[\sigma]}\left(\prod_{j=1}^{-[\sigma]}|s+j-1|\right),$$

which gives

$$|\zeta(s)| \leq \zeta(1-\sigma)\left(\frac{1}{2\pi}\right)^{\frac{1}{2}-\sigma}(|1+s-[\sigma]|)^{\frac{1}{2}+[\sigma]-\sigma}\left(\prod_{j=1}^{-[\sigma]}|s+j-1|\right).$$

3.4.2 Constructing and estimating $F_{c,r}(\theta)$

Bounding $\frac{1}{N}\log|f_N(s)|$

With the above convexity and subconvexity bounds in hand, we are in a position to bound $\frac{1}{N}\log|f_N(s)|$, where $f_N(s)$ is defined in (3.18). For $\sigma \geq 1+\eta > 1$, we have

$$\begin{aligned}|f_N(s)| &\leq \frac{1}{2}\left(|s+iT-1|^N|\zeta(s+iT)|^N+|s-iT-1|^N|\zeta(s-iT)|^N\right)\\ &\leq ((\sigma-1)^2+(|t|+T)^2)^{\frac{N}{2}}\zeta(\sigma)^N.\end{aligned}$$

Taking logarithms and dividing both sides by N gives

$$\frac{1}{N}\log|f_N(s)| \leq \frac{1}{2}\log((\sigma-1)^2+(|t|+T)^2)+\log\zeta(\sigma).$$

For $1 \leq \sigma \leq 1+\eta$, we have

$$\begin{aligned}|f_N(s)| \leq{}& \left(c_1\sqrt{(Q_0+\sigma)^2+(|t|+T)^2}\left(\log\sqrt{(Q_0+\sigma)^2+(|t|+T)^2}\right)^{c_2}\right)^{\frac{N(1+\eta-\sigma)}{\eta}}\\ &\times\left(\zeta(1+\eta)\sqrt{(Q_0+\sigma)^2+(|t|+T)^2}\right)^{\frac{N(\sigma-1)}{\eta}}.\end{aligned}$$

Taking logarithms of both sides and dividing by N, we obtain

$$\frac{1}{N}\log|f_N(s)| \le \frac{1+\eta-\sigma}{\eta}\log\frac{c_1}{2^{c_2}} + \frac{\sigma-1}{\eta}\log\zeta(1+\eta) + \frac{1}{2}\log\left((Q_0+\sigma)^2+(|t|+T)^2\right)$$
$$+\frac{c_2(1+\eta-\sigma)}{\eta}\log\log\left((Q_0+\sigma)^2+(|t|+T)^2\right).$$

For $\frac{1}{2}\le\sigma\le 1$, from

$$|f_N(s)| \le \left(k_1((Q_2+\sigma)^2+(|t|+T)^2)^{\frac{k_2+1}{2}}\left(\log\sqrt{(Q_2+\sigma)^2+(|t|+T)^2}\right)^{k_3}\right)^{(2-2\sigma)N}$$
$$\times\left(c_1((Q_2+\sigma)^2+(|t|+T)^2)^{\frac{1}{2}}\left(\log\sqrt{(Q_2+\sigma)^2+(|t|+T)^2}\right)^{c_2}\right)^{(2\sigma-1)N},$$

it follows that

$$\frac{1}{N}\log|f_N(s)| \le (2-2\sigma)\log k_1 + (2\sigma-1)\log c_1 - (k_3(2-2\sigma)+c_2(2\sigma-1))\log 2$$
$$+\frac{(2-2\sigma)(k_2+1)+2\sigma-1}{2}\log((Q_2+\sigma)^2+(|t|+T)^2)$$
$$+(k_3(2-2\sigma)+c_2(2\sigma-1))\log\log((Q_2+\sigma)^2+(|t|+T)^2).$$

Assume $0\le\sigma\le\frac{1}{2}$. From

$$|f_N(s)| \le ((\sigma-1)^2+(|t|+T)^2)^{\frac{N}{2}}$$
$$\times\left(\frac{c_1}{\sqrt{2\pi}}((Q_5+\sigma)^2+(|t|+T)^2)^{\frac{1}{4}}\left(\log\sqrt{(Q_5+\sigma)^2+(|t|+T)^2}\right)^{c_2}\right)^{(1-2\sigma)N}$$
$$\times\left(k_1((Q_5+\sigma)^2+(|t|+T)^2)^{\frac{k_2}{2}}\left(\log\sqrt{(Q_5+\sigma)^2+(|t|+T)^2}\right)^{k_3}\right)^{2\sigma N},$$

we derive

$$\begin{aligned}\frac{1}{N}\log|f_N(s)| \leq (1-2\sigma)\log\left(\frac{c_1}{2^{c_2+\frac{1}{2}}\sqrt{\pi}}\right)+2\sigma\log\frac{k_1}{2^{k_3}}+\frac{1}{2}\log\left((\sigma-1)^2+(|t|+T)^2\right)\\ +\frac{1-2\sigma+4k_2\sigma}{4}\log\left((Q_5+\sigma)^2+(|t|+T)^2\right)\\ +(c_2(1-2\sigma)+2k_3\sigma)\log\log\left((Q_5+\sigma)^2+(|t|+T)^2\right).\end{aligned}$$

For $-\eta \leq \sigma \leq 0$, we have

$$\begin{aligned}|f_N(s)| \leq \left(\sqrt{(\sigma-1)^2+(|t|+T)^2}\right)^N\left(\frac{1}{(2\pi)^{\frac{1}{2}+\eta}}\zeta(1+\eta)\left(\sqrt{(Q_4+\sigma)^2+(|t|+T)^2}\right)^{\frac{1}{2}+\eta}\right)^{-\frac{\sigma}{\eta}N}\\ \times\left(\frac{c_1}{\sqrt{2\pi}}\left(\sqrt{(Q_4+\sigma)^2+(|t|+T)^2}\right)^{\frac{1}{2}}\left(\log\sqrt{(Q_4+\sigma)^2+(|t|+T)^2}\right)^{c_2}\right)^{\frac{\sigma+\eta}{\eta}N}.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{1}{N}\log|f_N(s)| \leq -\frac{\sigma}{\eta}\log\frac{1}{(2\pi)^{\frac{1}{2}+\eta}}-\frac{\sigma}{\eta}\log(1+\eta)+\frac{\sigma+\eta}{\eta}\log\frac{c_1}{\sqrt{2\pi}}-\frac{\sigma+\eta}{\eta}c_2\log 2\\ +\frac{1}{2}\log((\sigma-1)^2+(|t|+T)^2)\\ +\left(-\frac{\sigma(1+2\eta)}{4\eta}+\frac{\sigma+\eta}{4\eta}\right)\log((Q_4+\sigma)^2+(|t|+T)^2)\\ +\frac{\sigma+\eta}{\eta}c_2\log\log((Q_4+\sigma)^2+(|t|+T)^2).\end{aligned}$$

Lastly, for $\sigma \leq -\eta$, as

$$\begin{aligned}|f_N(s)| \leq ((\sigma-1)^2+(|t|+T)^2)^{\frac{N}{2}}\left(\frac{1}{2\pi}\right)^{N(\frac{1}{2}-\sigma)}((1+\sigma-[\sigma])^2+(|t|+T)^2)^{\frac{(1-2\sigma+2[\sigma])N}{4}}\\ \times\left(\prod_{j=1}^{-[\sigma]}((\sigma+j-1)^2+(|t|+T)^2)\right)^{\frac{N}{2}}\zeta(1-\sigma)^N,\end{aligned}$$

we obtain

$$\begin{aligned}\frac{1}{N}\log|f_N(s)| \leq \log\zeta(1-\sigma) + \frac{1}{2}\log((\sigma-1)^2 + (|t|+T)^2) \\ + \frac{2\sigma-1}{2}\log 2\pi + \frac{(1-2\sigma+2[\sigma])}{4}\log((1+\sigma-[\sigma])^2 + (|t|+T)^2) \\ + \frac{1}{2}\sum_{j=1}^{-[\sigma]}\log((\sigma+j-1)^2 + (|t|+T)^2).\end{aligned}$$

Constructing and bounding $F_{c,r}(\theta)$

With the above bounds of $\frac{1}{N}\log|f_N(s)|$ in mind, similar to the construction of $F_{c,r}(\theta)$ for Dirichlet L-functions in [11, Definition 5.10] and Dedekind zeta functions in [54], we shall construct $F_{c,r}(\theta)$ for $\zeta(s)$ as follows. (We note that the main difference between our construction and the ones in [11, 54] lies in the range $0 \leq \Re(s) \leq 1+\eta$ as we have sharper bounds for $\zeta(s)$ in this range.)

Similar to [11, 54], we first introduce some auxiliary functions and notation. For $\theta \in [-\pi, \pi]$, we let $\sigma = c + r\cos\theta$, with $c - r > -\frac{1}{2}$, and $t = r\sin\theta$. We define

$$\begin{aligned}L_j(\theta) &= \log\frac{(j+c+r\cos\theta)^2 + (|r\sin\theta|+T)^2}{T^2}, \\ M_j(\theta) &= \log\log((j+c+r\cos\theta)^2 + (|r\sin\theta|+T)^2) - \log\log(T^2).\end{aligned}$$

Now, we give upper bounds for $L_j(\theta)$ and $M_j(\theta)$. From the inequality $\log x \leq x-1$, it follows that

$$L_j(\theta) \leq \frac{(j+c+r\cos\theta)^2 + (|r\sin\theta|+T)^2}{T^2} - 1 = \frac{(j+c+r\cos\theta)^2 + (r\sin\theta)^2}{T^2} + \frac{2r\sin\theta}{T}$$

for $\theta \in [0, \pi]$. Fix $T_0 \geq 1$. For $\theta \in [0, \pi]$, we let

$$L_j^\star(\theta) = \frac{1}{T_0}(j+c+r\cos\theta)^2 + \frac{1}{T_0}(r\sin\theta)^2 + 2r\sin\theta.$$

It is clear that for $T \geq T_0$ and $\theta \in [0,\pi]$,

$$L_j(\theta) \leq \frac{L_j^\star(\theta)}{T}.$$

Similarly, for $T \geq T_0$ and $\theta \in [0,\pi]$, we have

$$\begin{aligned} M_j(\theta) &\leq \frac{\log((j+c+r\cos\theta)^2+(|r\sin\theta|+T)^2)}{\log(T^2)} - 1 \\ &= \frac{1}{\log(T^2)}\left(\log\frac{(j+c+r\cos\theta)^2+(|r\sin\theta|+T)^2}{T^2} + \log(T^2)\right) - 1 \\ &= \frac{1}{\log(T^2)} L_j(\theta) \\ &\leq \frac{L_j^\star(\theta)}{2T\log T}. \end{aligned}$$

We are now in a position to construct $F_{c,r}(\theta)$.

Definition 3.9. For $\theta \in [-\pi,\pi]$, we let $\sigma = c + r\cos\theta$, with $c - r > -\frac{1}{2}$, and $t = r\sin\theta$. For $\sigma > 1+\eta$, we define

$$F_{c,r}(\theta) = \frac{1}{2}L_{-1}(\theta) + \log T + \log\zeta(\sigma).$$

For $1 \leq \sigma \leq 1+\eta$, we define

$$\begin{aligned} F_{c,r}(\theta) = \frac{1+\eta-\sigma}{\eta}\log c_1 + \frac{\sigma-1}{\eta}\log\zeta(1+\eta) + \frac{1}{2}L_{Q_0}(\theta) + \log T \\ + \frac{c_2(1+\eta-\sigma)}{\eta}M_{Q_0}(\theta) + \frac{c_2(1+\eta-\sigma)}{\eta}\log\log T. \end{aligned}$$

For $\frac{1}{2} \leq \sigma \leq 1$, we define

$$\begin{aligned} F_{c,r}(\theta) = (2-2\sigma)\log k_1 + (2\sigma-1)\log c_1 + ((2-2\sigma)(k_2+1)+2\sigma-1)\left(\tfrac{L_{Q_2}(\theta)}{2} + \log T\right) \\ + (k_3(2-2\sigma)+c_2(2\sigma-1))(M_{Q_2}(\theta) + \log\log T). \end{aligned}$$

For $0 \le \sigma \le \frac{1}{2}$, we define

$$F_{c,r}(\theta) = (1-2\sigma)\log\frac{c_1}{\sqrt{2\pi}} + 2\sigma\log k_1 + \frac{1}{2}L_{-1}(\theta) + \log T + \\ + \frac{1-2\sigma+4k_2\sigma}{2}\left(\frac{L_{Q_5}(\theta)}{2} + \log T\right) + (c_2(1-2\sigma)+2k_3\sigma)(M_{Q_5}(\theta) + \log\log T).$$

For $-\eta \le \sigma \le 0$, we define

$$F_{c,r}(\theta) = -\frac{\sigma}{\eta}\log\frac{1+\eta}{c_1(2\pi)^\eta} + \log\frac{c_1}{\sqrt{2\pi}} + \frac{1}{2}L_{-1}(\theta) + \log T \\ + \left(-\frac{\sigma(1+2\eta)}{2\eta} + \frac{\sigma+\eta}{2\eta}\right)\left(\frac{L_{Q_4}(\theta)}{2} + \log T\right) + \frac{\sigma+\eta}{\eta}c_2(M_{Q_4}(\theta) + \log\log T).$$

For $\sigma \le -\eta$, we define

$$F_{c,r}(\theta) = \log\zeta(1-\sigma) + \frac{1}{2}L_{-1}(\theta) + \left(1 + \frac{1-2\sigma}{2}\right)\log T - \frac{1-2\sigma}{2}\log 2\pi \\ + \frac{(1-2\sigma+2[\sigma])}{4}L_{1-[\sigma]}(\theta) + \frac{1}{2}\sum_{j=1}^{-[\sigma]} L_{j-1}(\theta).$$

It is clear that $F_{c,r}(\theta)$ is an even function of θ such that $F_{c,r}(\theta) \ge \frac{1}{N}\log|f_N(c+re^{i\theta})|$.

Now, we shall provide an upper bound for $\int_0^\pi F_{c,r}(\theta)d\theta$. In light of work of [11] and [54], for $c \in \mathbb{R}$ and $r > 0$, we define

$$\theta_y = \begin{cases} 0 & \text{if } c+r \le y; \\ \arccos\frac{y-c}{r} & \text{if } c-r \le y \le c+r; \\ \pi & \text{if } y \le c-r. \end{cases} \tag{3.28}$$

Now, we let c, r, and η be positive real numbers satisfying

$$-\frac{1}{2} < c-r < 1-c < -\eta < 1+\eta < c \tag{3.29}$$

and $0 < \eta \leq \frac{1}{2}$. To bound $\int_0^\pi F_{c,r}(\theta) d\theta$, we consider the splitting

$$\int_0^\pi = \int_0^{\theta_{1+\eta}} + \int_{\theta_{1+\eta}}^{\theta_1} + \int_{\theta_1}^{\theta_{\frac{1}{2}}} + \int_{\theta_{\frac{1}{2}}}^{\theta_0} + \int_{\theta_0}^{\theta_{-\eta}} + \int_{\theta_{-\eta}}^{\pi}.$$

Firstly, we have

$$\int_0^{\theta_{1+\eta}} F_{c,r}(\theta) d\theta \leq \log T \int_0^{\theta_{1+\eta}} 1 d\theta + \int_0^{\theta_{1+\eta}} \log \zeta(\sigma) d\theta + \frac{1}{2T} \int_0^{\theta_{1+\eta}} L^\star_{-1}(\theta) d\theta.$$

Secondly, $\int_{\theta_{1+\eta}}^{\theta_1} F_{c,r}(\theta) d\theta$ is bounded above by

$$\begin{aligned}
&\log T \int_{\theta_{1+\eta}}^{\theta_1} 1 d\theta + \frac{c_2}{\eta} \log\log T \int_{\theta_{1+\eta}}^{\theta_1} (1+\eta-\sigma) d\theta + \frac{\log c_1}{\eta} \int_{\theta_{1+\eta}}^{\theta_1} (1+\eta-\sigma) d\theta \\
&+ \frac{\log \zeta(1+\eta)}{\eta} \int_{\theta_{1+\eta}}^{\theta_1} (\sigma - 1) d\theta + \frac{1}{2T} \int_{\theta_{1+\eta}}^{\theta_1} L^\star_{Q_0}(\theta) d\theta + \\
&+ \frac{c_2}{2\eta T \log T} \int_{\theta_{1+\eta}}^{\theta_1} (1+\eta-\sigma) L^\star_{Q_0}(\theta) d\theta.
\end{aligned}$$

A direction calculation shows that

$$\begin{aligned}
&\int_{\theta_1}^{\theta_{\frac{1}{2}}} F_{c,r}(\theta) d\theta \leq \log T \int_{\theta_1}^{\theta_{\frac{1}{2}}} (2-2\sigma)(k_2+1) + 2\sigma - 1 d\theta + \\
&+ \log\log T \int_{\theta_1}^{\theta_{\frac{1}{2}}} k_3(2-2\sigma) + c_2(2\sigma-1) d\theta + \int_{\theta_1}^{\theta_{\frac{1}{2}}} (2-2\sigma) \log k_1 + (2\sigma-1) \log c_1 d\theta \\
&+ \frac{1}{T} \int_{\theta_1}^{\theta_{\frac{1}{2}}} \frac{(2-2\sigma)(k_2+1) + 2\sigma - 1}{2} L^\star_{Q_2}(\theta) d\theta \\
&+ \frac{1}{2T \log T} \int_{\theta_1}^{\theta_{\frac{1}{2}}} (k_3(2-2\sigma) + c_2(2\sigma-1)) L^\star_{Q_2}(\theta) d\theta.
\end{aligned}$$

Also, we can bound $\int_{\theta_{\frac{1}{2}}}^{\theta_0} F_{c,r}(\theta)d\theta$ above by

$$\begin{aligned}
&\log T\int_{\theta_{\frac{1}{2}}}^{\theta_0} 1 d\theta+\frac{\log T}{2}\int_{\theta_{\frac{1}{2}}}^{\theta_0} 1-2\sigma+4k_2\sigma d\theta+\log\log T\int_{\theta_{\frac{1}{2}}}^{\theta_0} c_2(1-2\sigma)+2k_3\sigma d\theta\\
&+\left(\log\frac{c_1}{\sqrt{2\pi}}\right)\int_{\theta_{\frac{1}{2}}}^{\theta_0} 1-2\sigma d\theta+2\log k_1\int_{\theta_{\frac{1}{2}}}^{\theta_0}\sigma d\theta+\frac{1}{2T}\int_{\theta_{\frac{1}{2}}}^{\theta_0} L^{\star}_{-1}(\theta)d\theta\\
&+\frac{1}{4T}\int_{\theta_{\frac{1}{2}}}^{\theta_0}(1-2\sigma+4k_2\sigma)L^{\star}_{Q_5}(\theta)d\theta+\frac{1}{2T\log T}\int_{\theta_{\frac{1}{2}}}^{\theta_0}(c_2(1-2\sigma)+2k_3\sigma)L^{\star}_{Q_5}(\theta)d\theta.
\end{aligned}$$

Moreover, $\int_{\theta_0}^{\theta_{-\eta}} F_{c,r}(\theta)d\theta$ is bounded above by

$$\begin{aligned}
&\log T\int_{\theta_0}^{\theta_{-\eta}} 1-\frac{\sigma(1+2\eta)}{2\eta}+\frac{\sigma+\eta}{2\eta}d\theta+\log\log T\int_{\theta_0}^{\theta_{-\eta}}\frac{\sigma+\eta}{\eta}c_2 d\theta\\
&+\int_{\theta_0}^{\theta_{-\eta}}-\frac{\sigma}{\eta}\log\frac{1+\eta}{c_1(2\pi)^{\eta}}+\log\frac{c_1}{\sqrt{2\pi}}d\theta+\frac{1}{2T}\int_{\theta_0}^{\theta_{-\eta}} L^{\star}_{-1}(\theta)d\theta\\
&+\frac{1}{T}\int_{\theta_0}^{\theta_{-\eta}}\left(-\frac{\sigma(1+2\eta)}{4\eta}+\frac{\sigma+\eta}{4\eta}\right)L^{\star}_{Q_4}(\theta)d\theta+\frac{1}{2T\log T}\int_{\theta_0}^{\theta_{-\eta}}\frac{\sigma+\eta}{\eta}c_2 L^{\star}_{Q_4}(\theta)d\theta.
\end{aligned}$$

Finally, we have

$$\begin{aligned}
\int_{\theta_{-\eta}}^{\pi} F_{c,r}(\theta)d\theta\le{}&\log T\int_{\theta_{-\eta}}^{\pi} 1+\frac{1-2\sigma}{2}d\theta+\int_{\theta_{-\eta}}^{\pi}\log\zeta(1-\sigma)d\theta-\log 2\pi\int_{\theta_{-\eta}}^{\pi}\frac{1-2\sigma}{2}d\theta\\
&+\frac{1}{2T}\int_{\theta_{-\eta}}^{\pi} L^{\star}_{-1}(\theta)d\theta+\frac{1}{T}\int_{\theta_{-\eta}}^{\theta_{-\frac{1}{2}}}\frac{1-2\sigma}{4}L^{\star}_{1}(\theta)d\theta\\
&+\sum_{j=1}^{\infty}\int_{\theta_{-j+\frac{1}{2}}}^{\theta_{-j-\frac{1}{2}}}\left(\frac{1-2\sigma-2j}{4}L_{j+1}(\theta)+\frac{1}{2}\sum_{k=1}^{j}L_{k-1}(\theta)\right)d\theta.
\end{aligned}$$

(We note that from the assumption $-\frac{1}{2}<c-r$ it follows that $\theta_{-j+\frac{1}{2}}=\theta_{-j-\frac{1}{2}}=\pi$ for $j\ge 1$, and thus the last term in the above inequality is equal to zero.)

To end this section, we require the following two estimates from [11, Lemmata 5.14 and 5.15] to control the zeta integrals appearing in the above estimates.

Lemma 3.10. *Let c, r and η be positive real numbers, satisfying* (3.29)*, and J_1 and J_2 be* *positive integers. If* $\theta_{1+\eta} \leq 2.1$*, then for* $\sigma = c + r\cos\theta$*, one has*

$$\int_0^{\theta_{1+\eta}} \log\zeta(\sigma)d\theta \leq \frac{\log\zeta(1+\eta)+\log\zeta(c)}{2}\left(\theta_{1+\eta}-\frac{\pi}{2}\right)+\frac{\pi}{4J_1}\log\zeta(c)+\kappa_1(J_1),$$

where

$$\kappa_1(J_1) = \frac{\pi}{4J_1}\left(\log\zeta(c+r)+2\sum_{j=1}^{J_1-1}\log\zeta\left(c+r\cos\frac{\pi j}{2J_1}\right)\right).$$

In addition, assuming further $r > 2c-1$*, one has*

$$\int_{\theta_{-\eta}}^{\pi} \log\zeta(1-\sigma)d\theta \leq \frac{\log\zeta(1+\eta)+\log\zeta(c)}{2}(\theta_{1-c}-\theta_{-\eta})+\frac{\pi-\theta_{1-c}}{2J_2}\log\zeta(c)+\kappa_2(J_2),$$

where

$$\kappa_2(J_2) = \tfrac{\pi-\theta_{1-c}}{2J_2}\left(\log\zeta(1-c+r)+2\textstyle\sum_{j=1}^{J_2-1}\log\zeta\left(1-c-r\cos\left(\tfrac{\pi j}{J_2}+\left(1-\tfrac{j}{J_2}\right)\theta_{1-c}\right)\right)\right).$$

3.5 Final formulae

In this section, we shall first prove Theorem 3.1.

Proof of Theorem 3.1. Using (3.17) and Propositions 3.6 and 3.7, for

$$\begin{aligned}-\frac{1}{2} < c-r < 1-c < -\eta < 0 < \frac{1}{4} \leq \delta = 2c-\sigma_1-\frac{1}{2} < \frac{1}{2} < 1 < 1+\eta < c < \\ < \sigma_1 = c+\frac{(c-1/2)^2}{r} < c+r,\end{aligned}$$

satisfying $\theta_{1+\eta} \leq 2.1$, we have

$$\begin{aligned}
&\left| N_{\mathbb{Q}}(T) - \frac{T}{\pi}\log\left(\frac{T}{2\pi e}\right) + \frac{1}{4} \right| \\
&\leq \frac{5}{2} + \frac{1}{25T} + \frac{2}{\pi}\log\zeta(\sigma_1) + \frac{1}{\log(r/(c-1/2))}\log\frac{\zeta(c)}{\zeta(2c)} - \frac{1}{\log(r/(c-1/2))}\log T \\
&+ \frac{1}{\pi\log(r/(c-1/2))}\int_0^{\pi} F_{c,r}(\theta)d\theta + \frac{E(T,\delta)}{\pi}.
\end{aligned}$$

Thus, recalling that $N_{\mathbb{Q}}(T) = 2N(T)$ and applying Lemma 3.5 (to bound $E(T,\delta)$) and the estimates from Section 3.4.2 (to bound $\int_0^{\pi} F_{c,r}(\theta)d\theta$), for $T \geq T_0$, we have

$$\left| N(T) - \frac{T}{2\pi}\log\left(\frac{T}{2\pi e}\right) + \frac{1}{8} \right| \leq C_1 \log T + C_2 \log\log T + C_3,$$

where for $j = 1,2,3$,

$$C_j = \frac{\tilde{C}_j}{2\pi\log(r/(c-1/2))}, \tag{3.30}$$

$$\begin{aligned}
\tilde{C}_1 = &\int_{\theta_1}^{\theta_{\frac{1}{2}}} (2-2\sigma)(k_2+1) + 2\sigma - 2d\theta + \frac{1}{2}\int_{\theta_{\frac{1}{2}}}^{\theta_0} 1 - 2\sigma + 4k_2\sigma d\theta \\
&+ \int_{\theta_0}^{\theta_{-\eta}} -\frac{\sigma(1+2\eta)}{2\eta} + \frac{\sigma+\eta}{2\eta}d\theta + \int_{\theta_{-\eta}}^{\pi} \frac{1-2\sigma}{2}d\theta,
\end{aligned} \tag{3.31}$$

$$\begin{aligned}
\tilde{C}_2 = &\frac{c_2}{\eta}\int_{\theta_{1+\eta}}^{\theta_1} (1+\eta-\sigma)d\theta + \int_{\theta_1}^{\theta_{\frac{1}{2}}} k_3(2-2\sigma) + c_2(2\sigma-1)d\theta \\
&+ \int_{\theta_{\frac{1}{2}}}^{\theta_0} c_2(1-2\sigma) + 2k_3\sigma d\theta + \int_{\theta_0}^{\theta_{-\eta}} \frac{\sigma+\eta}{\eta}c_2 d\theta,
\end{aligned} \tag{3.32}$$

$$\tilde{C}_3 = \pi \log(r/(c-1/2))\left(\frac{640\delta - 112}{1536(3T_0-1)} + \frac{1}{2^{10}} + \frac{5}{2} + \frac{1}{25T_0} + \frac{2}{\pi}\log\zeta(\sigma_1)\right) + \pi \log \frac{\zeta(c)}{\zeta(2c)}$$
$$+ \frac{\log c_1}{\eta}\int_{\theta_{1+\eta}}^{\theta_1}(1+\eta-\sigma)d\theta + \frac{\log\zeta(1+\eta)}{\eta}\int_{\theta_{1+\eta}}^{\theta_1}(\sigma-1)d\theta$$
$$+ \int_{\theta_1}^{\theta_{\frac{1}{2}}}(2-2\sigma)\log k_1 + (2\sigma-1)\log c_1 d\theta + \left(\log\frac{c_1}{\sqrt{2\pi}}\right)\int_{\theta_{\frac{1}{2}}}^{\theta_0} 1-2\sigma d\theta +$$
$$+2\log k_1 \int_{\theta_{\frac{1}{2}}}^{\theta_0}\sigma d\theta + \int_{\theta_0}^{\theta_{-\eta}} -\frac{\sigma}{\eta}\log\frac{1+\eta}{c_1(2\pi)^\eta} + \log\frac{c_1}{\sqrt{2\pi}}d\theta - (\log 2\pi)\int_{\theta_{-\eta}}^{\pi}\frac{1-2\sigma}{2}d\theta +$$
$$+ \frac{\log\zeta(1+\eta)+\log\zeta(c)}{2}\left(\theta_{1+\eta} - \frac{\pi}{2}\right) + \frac{\pi}{4J_1}\log\zeta(c) + \kappa_1(J_1) +$$
$$+ \frac{\log\zeta(1+\eta)+\log\zeta(c)}{2}(\theta_{1-c}-\theta_{-\eta}) + \frac{\pi - \theta_{1-c}}{2J_2}\log\zeta(c) + \kappa_2(J_2) + \kappa_3(T_0), \tag{3.33}$$

and $\kappa_3(T_0)$ is equal to

$$\frac{1}{2T_0}\max\Big\{0, \int_0^{\theta_{1+\eta}} L^\star_{-1}(\theta) + \int_{\theta_{1+\eta}}^{\theta_1} L^\star_{Q_0}(\theta)d\theta + \int_{\theta_1}^{\theta_{\frac{1}{2}}}((2-2\sigma)(k_2+1)+2\sigma-1)L^\star_{Q_2}(\theta)d\theta$$
$$+ \int_{\theta_{\frac{1}{2}}}^{\theta_0} L^\star_{-1}(\theta)d\theta + \frac{1}{2}\int_{\theta_{\frac{1}{2}}}^{\theta_0}(1-2\sigma+4k_2\sigma)L^\star_{Q_5}(\theta)d\theta + \int_{\theta_0}^{\theta_{-\eta}} L^\star_{-1}(\theta)d\theta +$$
$$+ \int_{\theta_0}^{\theta_{-\eta}}\left(-\frac{\sigma(1+2\eta)}{2\eta} + \frac{\sigma+\eta}{2\eta}\right)L^\star_{Q_4}(\theta)d\theta + \int_{\theta_{-\eta}}^{\pi} L^\star_{-1}(\theta)d\theta + \int_{\theta_{-\eta}}^{\theta_{-\frac{1}{2}}}\frac{1-2\sigma}{2}L^\star_1(\theta)d\theta\Big\} +$$
$$+ \frac{1}{2T_0\log T_0}\max\Big\{0, \int_{\theta_{1+\eta}}^{\theta_1}\frac{c_2}{\eta}(1+\eta-\sigma)L^\star_{Q_0}(\theta)d\theta + + \int_{\theta_{\frac{1}{2}}}^{\theta_0}(c_2(1-2\sigma)+2k_3\sigma)L^\star_{Q_5}(\theta)d\theta$$
$$+ \int_{\theta_1}^{\theta_{\frac{1}{2}}}(k_3(2-2\sigma)+c_2(2\sigma-1))L^\star_{Q_2}(\theta)d\theta + \int_{\theta_0}^{\theta_{-\eta}}\frac{\sigma+\eta}{\eta}c_2 L^\star_{Q_4}(\theta)d\theta\Big\}.$$

Recall that Patel [87] showed that $|\zeta(1+it)| \le \log t$ for $t \ge 3$ (i.e., in (3.2), $(c_1, c_2, t_0) = (1,1,3)$ is admissible) and that by the work of Hiary [57], $(k_1, k_2, k_3, t_1) = (0.77, \frac{1}{6}, 1, 3)$ is admissible for (3.3).[5] In addition, for $(c_1, c_2, t_0) = (1,1,3)$ and $(k_1, k_2, k_3, t_1) = (0.77, \frac{1}{6}, 1, 3)$,

[5] In fact, Hiary [57] showed that $(k_1, k_2, k_3) = (0.63, \frac{1}{6}, 1)$ was admissible. However, as pointed out by [87], due to an error in [57], one can only take $(k_1, k_2, k_3) = (0.77, \frac{1}{6}, 1)$.

We also note that there are two bounds used in [119, Sec. 5] and [88] that may be no longer valid. On one hand, Trudgian in [119, Sec. 5] used a result from [118] that $|\zeta(1+it)| \le \frac{3}{4}\log t$ for $t \ge 3$. On the other hand, [88, Theorem 1] states that $(k_1, k_2, k_3) = (0.732, \frac{1}{6}, 1)$ is admissible. However, as pointed out by [87], both bounds made use of an incorrect result obtained by Cheng-Graham in [25]. We refer the reader to [87] for a detailed discussion.

we may take

$$(Q_0, Q_1, Q_2, Q_3, Q_4, Q_5) = (1, 1.18, 1.18, 3.9, 2.3, 3.9).$$

Finally, for $T_0 = 30\,610\,046\,000$, choosing $J_1 = 64$ and $J_2 = 39$, we calculate admissible (C_1, C_2, C_3) and record them in Table 3.2. □

Table 3.2: Choices of parameters (c, r, η) and resulting admissible (C_1, C_2, C_3, C_3')

c	r	η	C_1	C_2	C_3	C_3'
1.000011314	1.064340602	$4.2826451 \cdot 10^{-6}$	0.103787	0.257297	9.367419	8.367419
1.025253504	1.182375395	0.009944751381	0.109410	0.204142	4.030486	3.030486
1.035766557	1.229059659	0.014325507360	0.111973	0.189768	3.746756	2.746756

Now, we are in a position to prove Corollary 3.2.

Proof of Corollary 3.2. By Theorem 3.1 (with $T_0 = 30\,610\,046\,000$ and Table 3.2), it is sufficient to verify the corollary for $e \le T \le 30\,610\,046\,000$. We note that by (3.15),

$$S(T) = \frac{1}{\pi}\Delta_{C_0} \arg\zeta(s) = \frac{1}{2}\left(N_{\mathbb{Q}}(T) - \frac{T}{\pi}\log\left(\frac{T}{2\pi e}\right) + \frac{1}{4} - g(T) - 2\right), \tag{3.34}$$

where $g(T)$ is defined as in (3.13). Thus, by (3.7) and (3.14), we obtain

$$\left| N(T) - \frac{T}{2\pi}\log\left(\frac{T}{2\pi e}\right) + \frac{1}{8}\right| \le |S(T)| + \frac{1}{2}|g(T)| + 1 \le 2.5167 + \frac{1}{50e} + 1 \tag{3.35}$$

for $e \le T \le 30\,610\,046\,000$. (We remark that one may apply [17, Lemma 2] to improve (3.35) for $T \ge 2\pi$.) As the quantity on the right of (3.35) clearly is less than $0.1038\log T + 0.2573\log\log T + 9.3675$ for any $T \ge e$. We then conclude the proof by using the triangle inequality. □

To end this section, we shall prove Theorem 3.3.

Proof of Theorem 3.3. Note that

$$S(T) = \frac{1}{\pi}\Delta_{C_0}\arg\zeta(s) = \frac{1}{\pi}\Delta_{C_1}\arg\zeta(s) + \frac{1}{\pi}\Delta_{C_2}\arg(s-1)\zeta(s) - \frac{1}{\pi}\Delta_{C_2}\arg(s-1).$$

We know that

$$|\Delta_{C_1} \arg \zeta(s)| \leq \log \zeta(\sigma_1)$$

and

$$|\Delta_{C_2} \arg(s-1)| = \arctan\left(\frac{\sigma_1 - 1}{T}\right) + \arctan\left(\frac{1}{2T}\right) \leq \arctan\left(\frac{\sigma_1 - 1}{T_0}\right) + \arctan\left(\frac{1}{2T_0}\right)$$

for $T \geq T_0$. From Propositions 3.6 and 3.7, it follows that for

$$-\frac{1}{2} < c - r < 1 - c < -\eta < 0 < \frac{1}{4} \leq \delta = 2c - \sigma_1 - \frac{1}{2} < \frac{1}{2} < 1 < 1 + \eta < c <$$
$$< \sigma_1 = c + \frac{(c-1/2)^2}{r} < c + r,$$

satisfying $\theta_{1+\eta} \leq 2.1$,

$$|S(T)| \leq \frac{1}{4} + \frac{1}{\pi} \log \zeta(\sigma_1) + \frac{1}{2\log(r/(c-1/2))} \log \frac{\zeta(c)}{\zeta(2c)} - \frac{1}{2\log(r/(c-1/2))} \log T$$
$$+ \frac{1}{2\pi \log(r/(c-1/2))} \int_0^{\pi} F_{c,r}(\theta) d\theta + \frac{E(T,\delta)}{2\pi} + \frac{1}{\pi} \arctan\left(\frac{\sigma_1 - 1}{T_0}\right) + \frac{1}{\pi} \arctan\left(\frac{1}{2T_0}\right).$$

Thus, applying Lemma 3.5 (to bound $E(T,\delta)$) and the estimates from Section 3.4.2 (to bound $\int_0^{\pi} F_{c,r}(\theta) d\theta$), for $T \geq T_0$, we have

$$|S(T)| \leq C_1 \log T + C_2 \log\log T + C_3',$$

where

$$C_3' = C_3 - 1 + \frac{1}{\pi} \arctan\left(\frac{\sigma_1 - 1}{T_0}\right) + \frac{1}{\pi} \arctan\left(\frac{1}{2T_0}\right), \tag{3.36}$$

$C_1 = C_1(c,r,\eta;k_2)$, $C_2 = C_2(c,r,\eta;c_2,k_3)$, $C_3 = C_3(c,r,\eta;c_1,c_2,t_0,k_1,k_2,k_3,t_1;T_0)$ are given in (3.30), (3.31), (3.32), and (3.33). In particular, for $T_0 = 30\,610\,046\,000$, we have admissible (C_1, C_2, C_3') recorded in Table 3.2. Finally, applying (3.7), we conclude the proof. □

Chapter 4

Applications of explicit results in logarithms of linear forms to Diophantine problems

In this chapter we apply the methods of transcendence theory (lower bounds for linear forms in two and three logarithms, reduction procedures) to study two Diophantine problems. The results of this chapter have been published in [51, 53].

4.1 Sums of Fibonacci numbers close to a power of 2

4.1.1 Introduction

The Fibonacci sequence $(F_n)_{n\geq 0}$ is the binary recurrence sequence defined by $F_0 = 0$, $F_1 = 1$ and

$$F_{n+2} = F_{n+1} + F_n \text{ for all } n \geq 0.$$

There is a broad literature on the Diophantine equations involving the Fibonacci numbers. Bugeaud, Mignotte and Siksek [21] proved that the only perfect powers among the Fibonacci numbers are $F_0 = 0$, $F_1 = F_2 = 1$, $F_6 = 8$ and $F_{12} = 144$. In particular, 1, 2 and 8 are the only powers of 2 in the Fibonacci sequence. Bravo and Luca [16] studied the Diophantine equation

$$F_n + F_m = 2^a$$

in nonnegative integers n,m and a with $n \geq m$. They showed that the only solutions of the above equation are given by

$$(n,m,a) \in \{(1,1,1),(2,2,1),(3,3,2),(6,6,4),(2,1,1),(4,1,2),(4,2,2),(5,4,3),(7,4,4)\}.$$

Bravo and Bravo [14] determined all the solutions of the Diophantine equation

$$F_n + F_m + F_l = 2^a$$

in nonnegative integers n,m,l and a with $n \geq m \geq l$.

Chim and Ziegler [27] completely solved the Diophantine equations

$$F_{n_1} + F_{n_2} = 2^{a_1} + 2^{a_2} + 2^{a_3}$$

and

$$F_{m_1} + F_{m_2} + F_{m_3} = 2^{t_1} + 2^{t_2}$$

in nonnegative integers $n_1,n_2,m_1,m_2,m_3,a_1,a_2,a_3,t_1,t_2$ with $n_1 \geq n_2$, $a_1 \geq a_2 \geq a_3$, $m_1 \geq m_2 \geq m_3$ and $t_1 \geq t_2$.

We say that an integer n is close to a positive integer m, if it satisfies

$$|n-m| < \sqrt{m}.$$

In 2014, Chern and Cui [26] studied the Fibonacci numbers which are close to a power of 2. More precisely, they found that the only solutions $(F_n, 2^m)$ of the Diophantine inequality

$$|F_n - 2^m| < 2^{m/2} \tag{4.1}$$

are $(1,2)$, $(2,2)$, $(3,2)$, $(3,4)$, $(5,4)$, $(8,8)$, $(13,16)$ and $(34,32)$.

In 2020, Bravo et al. [15] extended the previous work [26] and found all the members of the k-generalized Fibonacci sequence $(F_n^{(k)})_{n\geq -(k-2)}$, $k \geq 2$ which are close to a power of 2, where the k-Fibonacci sequence is given by the recurrence

$$F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)} \text{ for all } n \geq 2$$

with the initial conditions $F_{-(k-2)}^{(k)} = F_{-(k-3)}^{(k)} = \cdots = F_0^{(k)} = 0$ and $F_1^{(k)} = 1$.

In the present chapter, we study the Diophantine inequality

$$|F_n + F_m - 2^a| < 2^{a/2} \tag{4.2}$$

in positive integers n, m and a with $n \geq m$.

In particular, we prove the following main theorem.

Theorem 4.1. *There are exactly 52 solutions $(n,m,a) \in \mathbb{N}^3$ to Diophantine inequality* (4.19). *All solutions satisfy $n \leq 42$ and $a \leq 28$. A list of solutions is given in the appendix.*

As a consequence we find again solutions of the equation $F_n + F_m = 2^a$.

The companion sequence of the Fibonacci sequence is the well-known Lucas sequence $(L_n)_{n\geq 0}$. It satisfies the same recursive relation as the Fibonacci numbers, but with initial conditions $L_0 = 2$ and $L_1 = 1$. Recall the following relation between the Fibonacci numbers and the Lucas numbers

$$L_k = F_{k-1} + F_{k+1} \text{ for all } k \geq 1.$$

As a corollary we determine all Lucas numbers which are close to a power of 2.

Corollary 4.2. *There are only 9 Lucas numbers which are close to a power of 2. Namely,* *the solutions* $(n,a) \in \mathbb{N}^2$ *of the inequality*

$$|L_n - 2^a| < 2^{a/2}$$

are $(1,1)$, $(2,1)$, $(2,2)$, $(3,2)$, $(4,3)$, $(6,4)$, $(7,5)$, $(10,7)$ *and* $(13,9)$.

Recently, Patel and Tripathy [86] have extended our work by considering sums of three Fibonacci numbers close to a power of 2. They found that there are exactly 280 solutions to Diophantine inequality $|F_n + F_m + F_l - 2^a| < 2^{a/2}$ and all solutions satisfy $n,m,l \leq 42$ and $a \leq 28$. One can consider sums of four or more Fibonacci numbers. The challenge is that the reduction process requires a lot of computations and that the number of exceptional cases increases with the number of variables.

We shall prove Theorem 4.1 by following closely the typical strategy as performed in [14], [16], [27]. First, we apply a lower bound for linear forms in logarithms due to Matveev to obtain a large upper bound for n. Applying a version of the Baker-Davenport reduction method we reduce this bound on n.

4.1.2 Preliminary results

First, recall that the Binet formula for Fibonacci numbers

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{4.3}$$

holds for all $n \geq 0$, where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2} = -\frac{1}{\alpha}$ are the roots of the characteristic equation $x^2 - x - 1 = 0$ of the Fibonacci sequence. Moreover, the equality

$$\alpha^{n-2} \leq F_n \leq \alpha^{n-1} \tag{4.4}$$

holds for all $n \geq 1$.

Next we recall a definition and some facts about logarithmic height. Let α be an algebraic number of degree $d \geq 1$ with the minimal polynomial

$$a_d X^d + \cdots + a_1 X + a_0 = a_d \prod_{i}^{d} (X - \alpha_i),$$

where $a_0, a_1, \ldots, a_d$ are relatively prime integers and $\alpha_1, \ldots, \alpha_d$ are the conjugates of α. The absolute logarithmic Weil height of α is defined as

$$h(\alpha) = \frac{1}{d}\Big(\log|a_d| + \sum_{i=1}^{d} \log\left(\max\{|\alpha_i|, 1\}\right)\Big).$$

In particular, if $\alpha = \frac{p}{q}$ is a rational number with $\gcd(p,q) = 1$ and $q > 0$, then $h(\alpha) = \log\max\{|p|, q\}$. The following properties of $h(\alpha)$ are well-known.

- $h(\alpha \pm \beta) \leq h(\alpha) + h(\beta) + \log 2$,
- $h(\alpha\beta^{\pm 1}) \leq h(\alpha) + h(\beta)$,
- $h(\alpha^l) = |l| h(\alpha)$, $l \in \mathbb{Z}$.

One of the main tools in this chapter is a widely used estimate on lower bounds for linear forms in logarithms of algebraic numbers due to Matveev [79].

Theorem 4.3. *Let $\gamma_1, \ldots, \gamma_t$ be positive real algebraic numbers in a real algebraic number field K of degree n_K, let $b_1, \ldots, b_t \in \mathbb{Z}$ and*

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1$$

non-zero. Then

$$|\Lambda| > \exp(-1.4 \times 30^{t+3} \times t^{4.5} \times n_K^2 (1 + \log n_K)(1 + \log B) A_1 \ldots A_t)$$

where

$$B \geq \max\{|b_1|, \ldots, |b_t|\}$$

and

$$A_i \geq \max\{n_K h(\gamma_i), |\log \gamma_i|, 0.16\} \text{ for all } i = 1, \ldots, t.$$

During the calculations, we get an upper bound on n which is too large, thus we need to reduce it. To do so, we use some results from the theory of continued fractions and Diophantine approximation. The next lemma is known as the Legendre's criterion

Lemma 4.4. *Let τ be an irrational number, $\frac{p_0}{q_0}, \frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots$ be all the convergents of the continued fraction expansion of τ and M be a positive integer. Let N be a nonnegative integer such that $q_N > M$. Then putting $a(M) := \max\{a_i : i = 0, 1, 2, \ldots, N\}$, the inequality*

$$\left|\tau - \frac{r}{s}\right| > \frac{1}{(a(M)+2)s^2}$$

holds for all pairs (r, s) of positive integers with $0 < s < M$.

At last, we need a variant of the Baker-Davenport lemma, which is due to Dujella and Pethö [34]. For a real number X, we denote by $||X|| = \min\{|x - n|, n \in \mathbb{Z}\}$ the distance from X to the nearest integer.

Lemma 4.5. *Let M be a positive integer, $\frac{p}{q}$ be a convergent of the continued fraction expansion of the irrational number γ such that $q > 6M$ and A, B, μ be some real numbers with $A > 0$ and $B > 1$. Furthermore, let $\varepsilon := ||\mu q|| - M||\gamma q||$. If $\varepsilon > 0$, then there is no solution to the inequality*

$$0 < |u\gamma - v + \mu| < AB^{-w}$$

in positive integers u, v and w with

$$u \leq M \text{ and } w \geq \frac{\log\left(\frac{Aq}{\varepsilon}\right)}{\log B}.$$

4.1.3 Proof

Upper bound for n

First of all, note that if $(n,1,a)$ is a solution of (4.2), then $(n,2,a)$ is also a solution to (4.2) since $F_1 = F_2 = 1$. So from now on we assume that $m \geq 2$. Due to the work of Chern and Cui all solutions of (4.2) with $n = m$ are listed in the appendix. Since $F_n + F_{n-1} = F_{n+1}$, we can assume $n > m+1$ and in particular $n - m \geq 2$.

Proposition 4.6. *There are exactly 52 solutions $(n,m,a) \in \mathbb{N}^3$ to (4.2) with $n \leq 250$. All solutions fulfill $n \leq 42$ and $a \leq 28$. The list of solutions is given in the appendix.*

Proof. The solutions were found by a brute force search using *Mathematica*. □

Because of Proposition 4.6 for the rest of the paper we assume that $n > 250$ and we will show that there exist no solutions with $n > 250$.

Observe that the Binet formula (4.4) yields for $n > 1$ the inequalities

$$0.38\alpha^n < \alpha^n \frac{1-\alpha^{-4}}{\sqrt{5}} \leq F_n = \alpha^n \frac{1-(-1)^n\alpha^{-2n}}{\sqrt{5}} \leq \alpha^n \frac{1-\alpha^{-6}}{\sqrt{5}} < 0.48\alpha^n. \tag{4.5}$$

Due to (4.5) for $n > m > 1$ we have

$$0.38\alpha^n < F_n < F_n + F_m < 0.48\alpha^n + 0.48\alpha^{n-1} < 0.78\alpha^n. \tag{4.6}$$

Let us now establish a relation between n and a. Without loss of generality, we may assume that $a \geq 2$. Combining (4.2) with the right inequality of (4.6), one gets

$$2^{a-1} \leq 2^a - 2^{\frac{a}{2}} < F_n + F_m < 0.78\alpha^n < \alpha^n.$$

On ther hand, combining (4.2) with the left inequality of (4.6), we have

$$0.38\alpha^n < F_n + F_m < 2^a + 2^{\frac{a}{2}} < 2^{a+1}.$$

Thus

$$n\frac{\log\alpha}{\log 2} + \frac{\log 0.38}{\log 2} - 1 < a < n\frac{\log\alpha}{\log 2} + 1, \tag{4.7}$$

where $\frac{\log\alpha}{\log 2} = 0.6942....$ In particular, we have $a < n$.

By the Binet formula, we have

$$\frac{\alpha^n + \alpha^m}{\sqrt{5}} - (F_n + F_m) = \frac{\beta^n + \beta^m}{\sqrt{5}}. \tag{4.8}$$

Taking the absolute values in the above equality and combining it with (4.2), we get

$$\left|\frac{\alpha^n(1+\alpha^{m-n})}{\sqrt{5}} - 2^a\right| < \frac{|\beta|^n + |\beta|^m}{\sqrt{5}} + 2^{a/2} < \frac{1}{3} + 2^{a/2} < 2^{\frac{a}{2}+1}$$

for all $n \geq 4$ and $m \geq 2$. Dividing both sides of the above inequality by 2^a we obtain

$$\left|\frac{\alpha^n(1+\alpha^{m-n})}{2^a\sqrt{5}} - 1\right| < 2^{-\frac{a}{2}+1}. \tag{4.9}$$

In the first application of Matveev's theorem, we take the parameters $t = 3$ and

$$(\gamma_1, b_1) := (2, -a), (\gamma_2, b_2) := (\alpha, n), (\gamma_3, b_3) := \left(\frac{1+\alpha^{m-n}}{\sqrt{5}}, 1\right).$$

The three numbers $\gamma_1, \gamma_2, \gamma_3$ are real, positive and are contained in $K = \mathbb{Q}(\sqrt{5})$, so we can take $n_K = [K : \mathbb{Q}] = 2$.

Note that the left-hand side of (4.9) is not zero, for otherwise we would get the relation

$$2^a\sqrt{5} = \alpha^n + \alpha^m. \tag{4.10}$$

Conjugating the above relation in $\mathbb{Q}(\sqrt{5})$, we get

$$-2^a\sqrt{5} = \beta^n + \beta^m. \tag{4.11}$$

Combining (4.10) and (4.11), we get

$$\alpha^n < \alpha^n + \alpha^m = |\beta^n + \beta^m| \leq |\beta|^n + |\beta|^m < 1$$

which is impossible for positive n. Hence the left-hand side of (4.9) is non-zero.

Since $h(\gamma_1) = \log 2 = 0.6931...$ we can choose $A_1 := 1.4 > n_K h(\gamma_1)$. Further since $h(\gamma_2) = \frac{\log\alpha}{2} = 0.2406...$ we can take $A_2 := 0.5 > n_K h(\gamma_2)$. Let us now estimate $h(\gamma_3)$. We begin by observing that

$$\gamma_3 = \frac{1+\alpha^{m-n}}{\sqrt{5}} < \frac{2}{\sqrt{5}} \text{ and } \gamma_3^{-1} = \frac{\sqrt{5}}{1+\alpha^{m-n}} < \sqrt{5}$$

so that $|\log\gamma_3| < 1$. Next, by the properties of the logarithmic height we have

$$h(\gamma_3) \leq \log\sqrt{5} + |m-n|\left(\frac{\log\alpha}{2}\right) + \log 2 = \log 2\sqrt{5} + (n-m)\left(\frac{\log\alpha}{2}\right).$$

Hence, we can take

$$A_3 := 3 + (n-m)\log\alpha > \max\{2h(\gamma_3), |\log\gamma_3|, 0.16\}.$$

Finally, by recalling that $a < n$, we infer that $\max\{|b_1|,|b_2|,|b_3|\} = n$, so we can take $B := n$. We put $\Lambda_1 := \frac{\alpha^n(1+\alpha^{m-n})}{2^a\sqrt{5}} - 1$. Now Matveev's theorem tells us that the left-hand side of (4.9) is bounded below by

$$\begin{aligned}\log|\Lambda_1| &> -1.4\times 30^6\times 3^{4.5}\times 2^2(1+\log 2)(1+\log n)\times 1.4\times 0.5\times(3+(n-m)\log\alpha)\\ &> -1.4\times 10^{12}\times\log n\times(3+(n-m)\log\alpha)\end{aligned}$$

where we used the fact that the inequality $1+\log n<2\log n$ holds for all $n\geq 3$.

By comparing the above inequality with the right-hand side of (4.9) we get that

$$\left(\frac{a}{2}-1\right)\log 2<1.4\times 10^{12}\times\log n\times(3+(n-m)\log\alpha). \tag{4.12}$$

Let us consider a second linear form in logarithms. To this end, we rewrite (4.8) as follows

$$\frac{\alpha^n}{\sqrt{5}}-(F_n+F_m)=\frac{\beta^n}{\sqrt{5}}-F_m.$$

Again combining the above relation with (4.2), we get

$$\left|\frac{\alpha^n}{\sqrt{5}}-2^a\right|<2^{\frac{a}{2}}+\frac{|\beta|^n}{\sqrt{5}}+F_m<2^{\frac{a}{2}}+\frac{1}{2}+\alpha^m,$$

where we have also used the fact that $|\beta|^n<1/2$ for all $n\geq 2$. Dividing both sides of the above expression by $\frac{\alpha^n}{\sqrt{5}}$ and taking into account that $\alpha>\sqrt{2}$ and $n>m$, we obtain

$$|1-2^a\cdot\alpha^{-n}\cdot\sqrt{5}|<\frac{2^{\frac{a}{2}}\sqrt{5}}{\alpha^n}+\frac{\sqrt{5}}{2\alpha^n}+\frac{\sqrt{5}}{\alpha^{n-m}}<\frac{3\sqrt{5}}{2}\max\{\alpha^{m-n},\alpha^{a-n}\}. \tag{4.13}$$

In a second application of Matveev's theorem, we take the parameters $t=3$ and

$$(\gamma_1,b_1):=(2,a),(\gamma_2,b_2):=(\alpha,-n),(\gamma_3,b_3):=(\sqrt{5},1).$$

The three numbers $\gamma_1,\gamma_2,\gamma_3$ are real, positive and belong to $K=\mathbb{Q}(\sqrt{5})$, so we can take

$n_K = 2$. Put $\Lambda_2 := 1 - 2^a \cdot \alpha^{-n} \cdot \sqrt{5}$. If $\Lambda_2 = 0$, then we have that $2^a = \frac{\alpha^n}{\sqrt{5}}$, so $\alpha^{2n} \in \mathbb{Z}$, which is impossible. Thus $\Lambda_2 \neq 0$. Since $h(\gamma_1) = \log 2 = 0.6931...$ we can choose $A_1 := 1.4 > n_K h(\gamma_1)$. Further since $h(\gamma_2) = \frac{\log \alpha}{2} = 0.2406...$ and $h(\gamma_3) = \log\sqrt{5} = 0.8047...$, it follows that we can take $A_2 := 0.5 > n_K h(\gamma_2)$ and $A_3 := 1.7 > n_K h(\gamma_3)$. Finally, since $a < n$, we deduce that $B := \max\{|b_1|, |b_2|, |b_3|\} = n$.

Then Matveev's theorem tells us that

$$\begin{aligned}\log|\Lambda_2| &> -1.4 \times 30^6 \times 3^{4.5} \times 2^2 (1 + \log 2)(1 + \log n) \times 1.4 \times 0.5 \times 1.7 \\ &> -2.31 \times 10^{12} \log n\end{aligned}$$

Comparing the resulting inequality with (4.13), we get

$$\min\{(n-a)\log\alpha, (n-m)\log\alpha\} < 2.4 \times 10^{12} \log n.$$

Now the argument splits into two cases.

Case 1. $\min\{(n-a)\log\alpha, (n-m)\log\alpha\} = (n-m)\log\alpha$.

In this case we have

$$\begin{aligned}\left(\frac{a}{2} - 1\right)\log 2 &< 1.4 \times 10^{12} \log n (3 + (n-m)\log\alpha) \\ &< 1.4 \times 10^{12} \log n (3 + 2.4 \times 10^{12} \log n) < 3.5 \times 10^{24} \log^2 n.\end{aligned}$$

Combining it with the left inequality of (4.7) and by a calculation in *Mathematica*, we obtain

$$a < 4.5 \times 10^{28} \text{ and } n < 6.6 \times 10^{28}.$$

Case 2. $\min\{(n-a)\log\alpha, (n-m)\log\alpha\} = (n-a)\log\alpha$.

In this case we have

$$(n-a)\log\alpha < 2.4\times 10^{12}\log n.$$

Note that the right inequality of (4.7) yields

$$n-a > n\frac{(1-\log\alpha)}{\log 2}-1.$$

Combining the above inequalities and by a calculation in *Mathematica*, we get

$$a < 2.3\times 10^{14} \text{ and } n < 1.6\times 10^{14}.$$

Thus, in both Case 1 and Case 2, we have

$$a < 4.5\times 10^{28} \text{ and } n < 6.6\times 10^{28}. \tag{4.14}$$

We now need to reduce the above bound for n.

Reduction of the bound

We begin by recalling the following facts. For any non-zero real number x, we have

i) $0 < x < e^x - 1$,

ii) if $x < 0$ and $|e^x - 1| < 1/2$, then $|x| < 2|e^x - 1|$.

We may assume that $n-m > 250$ and $n-a > 250$. We go back to the inequality (4.13). Let

$$z_1 := a\log 2 - n\log\alpha + \log\sqrt{5}.$$

Since we assume that $\min\{n-m, n-a\} > 250$ we get $|e^{z_1}-1| < 1/2$ and by ii) we have

$$|a\log 2 - n\log\alpha + \log\sqrt{5}| < \frac{3\sqrt{5}}{\alpha^{\min\{n-m,n-a\}}}.$$

Dividing both sides of the above inequality by $\log\alpha$, we conclude that

$$0 < \left| a\frac{\log 2}{\log\alpha} - n + \frac{\sqrt{5}}{\log\alpha} \right| < \frac{3\sqrt{5}}{\log\alpha}\cdot\alpha^{-\kappa},$$

where $\kappa = \min\{n-m, n-a\}$. We apply Lemma 4.5 with

$$\gamma := \frac{\log 2}{\log\alpha},\ \mu := \frac{\sqrt{5}}{\log\alpha},\ A := \frac{3\sqrt{5}}{\log\alpha},\ B := \alpha.$$

Clearly γ is an irrational number. We also take $M = 4.5\times 10^{28}$ which is an upper bound for a. We find that the convergent $\frac{p}{q} = \frac{p_{67}}{q_{67}}$ is such that $q > 6M$. By using this we have that $\varepsilon > 0.01038$, therefore either

$$n-m < \frac{\log\left(\frac{3\sqrt{5}q}{0.01038\log\alpha}\right)}{\log\alpha} < 158 \text{ or } n-a < \frac{\log\left(\frac{3\sqrt{5}q}{0.01038\log\alpha}\right)}{\log\alpha} < 158.$$

Thus, we have that either $n-m < 158$ or $n < 213$. The latter case contradicts our assumption that $n > 250$. Inserting the upper bound for $n-m$ into (4.12) we get that $a < 1.3\times 10^{16}$.

Let us now work on the inequality (4.9). We may assume that $a > 28$. By i) and ii) it becomes

$$|a\log 2 - n\log\alpha + \log\eta(n-m)| < \frac{4}{2^{a/2}}$$

where η is defined by $\eta(t) := \sqrt{5}(1+\alpha^{-t})^{-1}$. Again dividing both sides by $\log\alpha$, we obtain that

$$0 < \left| a\frac{\log 2}{\log \alpha} - n + \frac{\log \eta(n-m)}{\log \alpha} \right| < \frac{4}{\log \alpha} \cdot 2^{-a/2} \tag{4.15}$$

Here we take $M = 1.3 \times 10^{16}$ (an upper bound for a) and apply Lemma 4.5 with

$$\gamma := \frac{\log 2}{\log \alpha}, \; \mu := \frac{\eta(n-m)}{\log \alpha}, \; A := \frac{4}{\log \alpha}, \; B := \sqrt{2}$$

for all choices $n-m \in \{1, \ldots, 158\}$ except when $n-m = 2, 6$. With the help of *Mathematica* we find that if (n,m,a) is a possible solution of (4.2) with $n-m \neq 2,6$ then $a \leq 67$ and thus, $n \leq 100$. But this is a contradiction to our assumption that $n > 250$.

Let us now consider the special cases when $n-m = 2$ and 6. Note that we cannot study these cases as before because the parameter μ appearing in Lemma 4.5 is

$$\frac{\log \eta(t)}{\log \alpha} = \begin{cases} 1 & \text{if } t = 2 \\ 3 - \frac{\log 2}{\log \alpha} & \text{if } t = 6 \end{cases}$$

and the corresponding value of ε is always negative. When $n-m = 2$ from (4.15) we get that

$$0 < |a\gamma - (n-1)| < \frac{4}{\log \alpha} \cdot 2^{-a/2} < \frac{4}{\log \alpha} \cdot 2^{-\frac{1}{2}\left(n\frac{\log \alpha}{\log 2} + \frac{\log 0.38}{\log 2} - 1\right)}. \tag{4.16}$$

Recall that $a < 1.3 \times 10^{16}$. Let $[a_0, a_1, a_2, a_3, a_4, \ldots] = [1, 2, 3, 1, 2, \ldots]$ be the continued fraction of γ. A quick search using *Mathematica* reveals that

$$q_{35} < 1.3 \times 10^{16} < q_{36}.$$

Furthermore, $a_M := \max\{a_i : \ i = 0,1,\ldots,36\} = a_{17} = 134$. So, by Lemma 4.4 we have

$$|a\gamma - (n-1)| > \frac{1}{(a_M+2)a}. \tag{4.17}$$

Comparing estimates (4.16) and (4.17) we get that $n < 187$. In a similar manner one can get $n < 187$ in the case when $n - m = 6$. This again contradicts our assumption that $n > 250$.

4.1.4 Appendix

The solutions for Diophantine inequality (4.2) are displayed below

$\|F_1+F_1-2\| < \sqrt{2}$	$\|F_6+F_6-2^4\| < 2^2$	$\|F_{10}+F_7-2^6\| < 2^3$
$\|F_2+F_i-2\| < \sqrt{2}$, $i=1,2$	$\|F_7+F_i-2^4\| < 2^2$, $i=1,2$	$\|F_{11}+F_9-2^7\| < 2^{7/2}$
$\|F_3+F_i-2\| < \sqrt{2}$, $i=1,2$	$\|F_7+F_3-2^4\| < 2^2$	$\|F_{13}+F_6-2^8\| < 2^4$
$\|F_3+F_i-2^2\| < 2$, $i=1,2$	$\|F_7+F_4-2^4\| < 2^2$	$\|F_{13}+F_7-2^8\| < 2^4$
$\|F_3+F_3-2^2\| < 2$	$\|F_7+F_5-2^4\| < 2^2$	$\|F_{13}+F_8-2^8\| < 2^4$
$\|F_4+F_i-2^2\| < 2$, $i=1,2$	$\|F_8+F_6-2^5\| < 2^{5/2}$	$\|F_{13}+F_9-2^8\| < 2^4$
$\|F_4+F_3-2^2\| < 2$	$\|F_8+F_7-2^5\| < 2^{5/2}$	$\|F_{14}+F_{12}-2^9\| < 2^{9/2}$
$\|F_4+F_4-2^3\| < 2^{3/2}$	$\|F_9+F_i-2^5\| < 2^{5/2}$, $i=1,2$	$\|F_{16}+F_6-2^{10}\| < 2^5$
$\|F_5+F_i-2^3\| < 2^{3/2}$, $i=1,2$	$\|F_9+F_3-2^5\| < 2^{5/2}$	$\|F_{16}+F_7-2^{10}\| < 2^5$
$\|F_5+F_3-2^3\| < 2^{3/2}$	$\|F_9+F_4-2^5\| < 2^{5/2}$	$\|F_{16}+F_8-2^{10}\| < 2^5$
$\|F_5+F_4-2^3\| < 2^{3/2}$	$\|F_9+F_9-2^6\| < 2^3$	$\|F_{16}+F_9-2^{10}\| < 2^5$
$\|F_5+F_5-2^3\| < 2^{3/2}$	$\|F_{10}+F_3-2^6\| < 2^3$	$\|F_{16}+F_{10}-2^{10}\| < 2^5$
$\|F_6+F_i-2^3\| < 2^{3/2}$, $i=1,2$	$\|F_{10}+F_4-2^6\| < 2^3$	$\|F_{23}+F_{19}-2^{15}\| < 2^{15/2}$
$\|F_6+F_3-2^3\| < 2^{3/2}$	$\|F_{10}+F_5-2^6\| < 2^3$	$\|F_{42}+F_{29}-2^{28}\| < 2^{14}$
$\|F_6+F_5-2^4\| < 2^2$	$\|F_{10}+F_6-2^6\| < 2^3$	

4.2 On the exponential Diophantine equation $(44m^2+1)^x + (5m^2-1)^y = (7m)^z$

4.2.1 Introduction

Let A, B, C be fixed coprime positive integers with $\min(A, B, C) > 1$. The ternary exponential Diophantine equation

$$A^x + B^y = C^z \tag{4.18}$$

in positive integers x, y, z has been actively studied by many authors. In 1933, Mahler [77] used his p-adic analogue of the Diophantine approximation method of Thue-Siegel to prove that equation (4.18) has only finitely many solutions (x, y, z).

In the last decade, many of the recent works on equation (4.18) concerned the case where

$$A = am^2 + 1,\ B = bm^2 - 1,\ C = cm,$$

and a, b, c are fixed positive integers such that $a + b = c^2$, $2 \nmid c$. Clearly, in this case equation (4.18) always has a solution $(x, y, z) = (1, 1, 2)$.

In 2020 Terai and Shinsho [114] proposed the following conjecture.

Conjecture 4.7. Let m be a positive integer greater than one. Let $a, b, c > 1$ be positive integers satisfying $a + b = c^2$. Then the equation

$$(am^2 + 1)^x + (bm^2 - 1)^y = (cm)^z$$

has only the positive integer solution $(x, y, z) = (1, 1, 2)$.

Although, in general, the above conjecture is widely open, it has been confirmed by several authors under some conditions on m, a, b, c. Below is the non-exhaustive list of some results.

- (N. Terai [109]) $(a,b,c) = (4,5,3)$, $m \leq 20$ or $m \not\equiv 3 \pmod 6$.
- (J.-L. Su and X.-X. Li [107]) $(a,b,c) = (4,5,3)$, $m > 90$, $m \equiv 0 \pmod 3$.
- (C. Bertok [12]) $(a,b,c) = (4,5,3)$, $20 < m \leq 90$.
- (M. Alan [2]) $(a,b,c) = (18,7,5)$, $m \not\equiv 23, 47, 63$ or $87 \pmod{120}$.
- (N. Terai [111]) $(a,b,c) = (4,21,5)$, m satisfies some conditions.
- (N. Terai [110]) $(a,b,c) = (10,15,5)$, for all m.
- (N. Terai and T. Hibino [112]) $(a,b,c) = (12,13,5)$, $m \not\equiv 17, 33 \pmod{40}$.
- (N. Terai and Y. Shinsho [115]) $(a,b,c) = (4,45,7)$, $m \equiv -1 \pmod 3$, $m \equiv 2 \pmod 5$ or $m \equiv \pm 1, \pm 2 \pmod 7$.
- (S. Fei and J. Luo [39]) $(a,b,c) = (28,21,7)$, for all m.
- (N. Terai and T. Hibino [113]) $(a,b,c) = (3p, (p-3)p, p)$, where p is an odd prime with $3 < p < 3784$ and $p \equiv 1 \pmod 4$, $m \not\equiv 0 \pmod 3$, $m \equiv 1 \pmod 4$.
- (E. Kizildere and G. Soydan [67]) $(a,b,c) = (5p, (p-5)p, p)$, where p is an odd prime with $p > 5$ and $p \equiv 3 \pmod 4$, $pm \equiv \pm 1 \pmod 5$.
- (X.-W. Pan [85]) $a \equiv 4$ or $5 \pmod 8$, $m \equiv \pm 1 \pmod c$, $m > 6c^2 \log c$.

In this section we consider the exponential Diophantine equation

$$(44m^2+1)^x + (5m^2-1)^y = (7m)^z. \tag{4.19}$$

We prove the following result.

Theorem 4.8. *Let m be a positive integer. When m is odd, we suppose that*

$$m \equiv 2 (mod\, 5) \text{ or } m \equiv 0, \pm 1, \pm 3 (mod\, 7). \tag{4.20}$$

Then for any m satisfying (4.20) *equation* (4.19) *has only the positive integer solution* $(x,y,z) = (1,1,2)$.

4.2.2 Preliminaries

In this section, we give some lemmas that will be useful for the proof of the main result.

Lemma 4.9. *(Terai-Shinsho [114]) Let r be an odd integer with* $r \geq 3$. *Then the equation*

$$4^x + (r^2 - 4)^y = r^z$$

has only the positive integer solution $(x,y,z) = (1,1,2)$.

Let α be an algebraic number of degree $d \geq 1$ with the minimal polynomial

$$a_0 X^d + +a_1 X^{d-1} + \cdots + a_d = a_0 \prod_{i=1}^{d} (X - \alpha^{(i)}),$$

where the a_is are relatively prime integers with $a_0 > 0$ and $\alpha^{(i)}$s are the conjugates of α. Then the *logarithmic height* of α is defined as

$$h(\alpha) = \frac{1}{d} \big(\log |a_0| + \sum_{i=1}^{d} \log \left(\max\{|\alpha^{(i)}|, 1\} \right) \big).$$

Let α_1, α_2 be two real algebraic numbers with $|\alpha_1|$, $|\alpha_2| \geq 1$ and b_1, b_2 be positive integers. We consider the linear form

$$\Lambda = b_2 \log \alpha_2 - b_1 \log \alpha_1.$$

Let A_1 and A_2 be real numbers greater than 1 with

$$\log A_i \geq \max\left\{h(\alpha_i), \frac{|\log \alpha_i|}{D}, \frac{1}{D}\right\} \ (i = 1, 2),$$

where $D = [\mathbb{Q}(\alpha_1, \alpha_2) : \mathbb{Q}]$. Set

$$b' = \frac{b_1}{D \log A_2} + \frac{b_2}{D \log A_1}.$$

With the above notation, we cite a result due to Laurent [72, Corollary 2] with $m = 10$ and $C_2 = 25.2$. Recall that two nonzero complex numbers α, β are *multiplicatively independent* if the only solution of the equation $\alpha^x \beta^y = 1$ in integers x, y is $x = y = 0$.

Lemma 4.10. *(Laurent [72]) Let* Λ *be given as above with* $\alpha_1 > 1$ *and* $\alpha_2 > 1$. *Suppose that* α_1 *and* α_2 *are multiplicatively independent. Then*

$$\log|\Lambda| \geq -25.2 D^4 \left(\max\left\{\log b' + 0.38, \frac{10}{D}\right\}\right)^2 \log A_1 \log A_2.$$

We will also need a result on linear forms in p-adic logarithms due to Bugeaud. Here we just use a special case $y_1 = y_2 = 1$ in the notation from [20, p.375]. Let p be an odd prime and a_1 and a_2 be non-zero integers prime to p. Let g denote the smallest positive integer such that

$$v_p(a_1^g - 1) \geq 1, \ v_p(a_2^g - 1) \geq 1,$$

where we denote the p-adic valuation by $v_p(\cdot)$. Assume that there exists a real number E such that

$$\frac{1}{p-1} < E \leq v_p(a_1^g - 1).$$

We consider the integer

$$\Lambda = a_1^{b_1} - a_2^{b_2}.$$

Lemma 4.11. *(Bugeaud [20]) Let $A_1 > 1$, $A_2 > 1$ be real numbers such that*

$$\log A_i \geq \max\{\log|a_i|, E\log p\} \ (i = 1, 2)$$

and put

$$b' = \frac{b_1}{\log A_2} + \frac{b_2}{\log A_1}.$$

If a_1 and a_2 are multiplicatively independent then we have the upper estimate

$$v_p(\Lambda) \leq \frac{36.1g}{E^3(\log p)^4}(\max\{\log b' + \log(E\log p) + 0.4, 6E\log p, 5\})^2 \log A_1 \log A_2.$$

The next lemma shows that for a possible solution (x,y,z) of the exponential Diophantine equation $(am^2+1)^x + (bm^2-1)^y = (cm)^z$ there is an upper and lower bound for z depending on $\max\{x,y\}$ and m.

Lemma 4.12. *(Alan [2]) Let a, b, c and $m > 1$ be positive integers such that $a + b = c^2$ and (x,y,z) be a positive integer solution of the exponential Diophantine equation $(am^2+1)^x + (bm^2-1)^y = (cm)^z$. If $M = \max\{x,y\} > 1$, then*

$$\left(2 - \frac{\log\left(\frac{c^2}{\min\left(a, b - \frac{1}{m^2}\right)}\right)}{\log(cm)}\right) M < z < 2M.$$

Finally for the case $m \equiv 0 (\mathrm{mod}\ 7)$ we will require a result from [46] that gives an upper bound for m.

Lemma 4.13. *(Fu-Yang [46]) Let a, b, c, m be positive integers such that $a + b = c^2$, $2|a$,*

$2 \nmid c$, $m > 1$. *If* $c|m$ *and* $m > 36c^3 \log c$, *then* $(am^2+1)^x + (bm^2-1)^y = (cm)^z$ *has only the solution* $(x,y,z) = (1,1,2)$.

4.2.3 Proof

In this section, we give a proof of Theorem 1. The proof follows in a series of lemmas. First we consider the case $m = 1$.

Lemma 4.14. *The equation*

$$45^x + 4^y = 7^z$$

has only the positive integer solution $(x,y,z) = (1,1,2)$.

Proof. This follows from Lemma 4.9 with $r = 7$. □

By Lemma 4.14, we may assume that $m \geq 2$.

Lemma 4.15. *If* (x,y,z) *is a positive integer solution of Equation* (4.19)*, then y is odd.*

Proof. It follows from Equation (4.19) that $z \geq 2$. Taking Equation 4.19 modulo m^2 implies that $1 + (-1)^y \equiv 0 (\text{mod } m^2)$ and hence y is odd. □

Lemma 4.16. *If m is even then Equation* (4.19) *has only the positive integer solution* $(x,y,z) = (1,1,2)$.

Proof. If $z \leq 2$, then $(x,y,z) = (1,1,2)$ is the only positive integer solution of Equation (4.19). Thus we may assume that $z \geq 3$. Taking Equation 4.19 modulo m^3 implies that

$$1 + 44m^2x - 1 + 5m^2y \equiv 0 (\text{mod } m^3).$$

So

$$44x + 5y \equiv 0 (\text{mod } m)$$

which is impossible, since y is odd and m is even. Hence for $z \geq 3$, Equation (4.19) has no positive integer solution when m is even. $\square$

The case m is odd and $m \equiv 2(\textbf{mod } 5)$ or $m \equiv \pm 1, \pm 3(\textbf{mod } 7)$

By Lemma 4.16, we may suppose that m is odd and $m \geq 3$. Let (x, y, z) be a solution of Equation (4.19).

Lemma 4.17. *If m is odd and $m \equiv 2(mod\ 5)$ or $m \equiv \pm 1, \pm 3(mod\ 7)$, then $y = 1$ and x is odd.*

Proof. We follow closely an argument in [115]. We first show that x is odd and z is even.

a) $m \equiv 2(\text{mod } 5)$. Taking Equation (4.19) modulo 5 implies that

$$2^x + (-1)^y \equiv (-1)^z (\text{mod } 5).$$

Since y is odd, we have $2^x \equiv 1 + (-1)^z (\text{mod } 5)$. This shows that z is even. Then $1 = \left(\frac{2}{5}\right)^{x-1} = (-1)^{x-1}$, where $\left(\frac{*}{*}\right)$ denotes the Jacobi symbol. Hence x is odd.

b) $m \equiv \pm 1, \pm 3(\text{mod } 7)$. Since $m^2 \equiv 1(\text{mod } 7)$ or $m^2 \equiv 2(\text{mod } 7)$, taking Equation (4.19) modulo 7 implies that

$$3^x + 4^y \equiv 0(\text{mod } 7) \text{ or } 5^x + 2^y \equiv 0(\text{mod } 7),$$

that is, $\left(\frac{3}{7}\right)^x = \left(\frac{-4^y}{7}\right)$ or $\left(\frac{5}{7}\right)^x = \left(\frac{-2^y}{7}\right)$. This shows that x is odd. In these cases, we have that $\left(\frac{5m^2-1}{44m^2+1}\right) = 1$ and $\left(\frac{7m}{44m^2+1}\right) = -1$. Indeed,

$$\left(\frac{5m^2-1}{44m^2+1}\right) = \left(\frac{49m^2}{44m^2+1}\right) = 1$$

and

$$\left(\frac{7m}{44m^2+1}\right) = \left(\frac{7}{44m^2+1}\right)\left(\frac{m}{44m^2+1}\right) = \left(\frac{44m^2+1}{7}\right)\left(\frac{44m^2+1}{m}\right) = \left(\frac{2m^2+1}{7}\right)$$
$$= -1,$$

since $m^2 \equiv 1, 2 \pmod 7$. Hence z is even from Equation (4.19).

Suppose that $y \geq 2$. Taking Equation (4.19) modulo 8 implies that

$$5^x \equiv (7m)^z \equiv 1 \pmod 8$$

so x is even, but this contradicts the fact that x is odd as seen from above. Hence, $y = 1$. □

From Lemma 4.17, it follows that $y = 1$ and x is odd. If $x = 1$, then clearly $z = 2$. From now on, we may suppose that $x \geq 3$. Thus our theorem is reduced to solving Pillai's equation

$$c^z - a^x = b \tag{4.21}$$

with $x \geq 3$, where $a = 44m^2 + 1$, $b = 5m^2 - 1$ and $c = 7m$.

Next we obtain a lower bound for x.

Lemma 4.18. *If (x, y, z) is a positive integer solution of Equation* (4.21)*, then $x \geq \frac{1}{44}(m^2 - 5)$.*

Proof. Since $x \geq 3$, from Equation (4.21) we get

$$(7m)^z = (44m^2+1)^x + 5m^2 - 1 \geq (44m^2+1)^3 + 5m^2 - 1 > (7m)^3.$$

Hence $z \geq 4$.Taking Equation (4.21) modulo m^4 implies that

$$1 + 44m^2x + 5m^2 - 1 \equiv 0 (\text{mod } m^4),$$

so $44x + 5 \equiv 0 (\text{mod } m^2)$ and the lemma follows. □

Lemma 4.19. *If* (x, y, z) *is a positive integer solution of Equation* (4.21)*, then* $x < 2521 \log 7m$*.*

Proof. The proof follows the argument as in [2], [112], [115]. Without loss of generality we may assume that $z > 2$. Consider the linear form of two logarithms

$$\Lambda = z \log c - x \log a.$$

Using the inequality $\log(1 + t) < t$ for $t > 0$, we have

$$0 < \Lambda = \log\left(\frac{c^z}{a^x}\right) = \log\left(1 + \frac{b}{a^x}\right) < \frac{b}{a^x}. \tag{4.22}$$

Hence, we get

$$\log \Lambda < \log b - x \log a. \tag{4.23}$$

On the other hand, it follows from Lemma 4.10 that

$$\log \Lambda \geq -25.2 (\max\{\log b' + 0.38, 10\})^2 \log a \log c, \tag{4.24}$$

where

$$b' = \frac{x}{\log c} + \frac{z}{\log a}.$$

Observe that $a^{x+1} - c^z = a(c^z - b) - c^z = (a-1)c^z - ab > 44m^2 \cdot 49m^2 - (44m^2 + 1)(5m^2 -$

$1) > 0$, since $z > 2$. Thus $b' < \frac{2x+1}{\log c}$. Put $M = \frac{x}{\log c}$. Now combining (4.23) and (4.24) we obtain

$$x \log a < \log b + 25.2 \left(\max \left\{ \log \left(2M + \frac{1}{\log c} \right) + 0.38, 10 \right\} \right)^2 \log a \log c.$$

Since $\frac{\log b}{\log a \log c} < 1$ and $\log c = \log 7m > 2$ for $m \geq 3$, we can rewrite the above inequality as

$$M < 1 + 25.2 (\max \{ \log (2M + 0.5) + 0.38, 10 \})^2.$$

If $\log(2M+0.5)+0.38 > 10$, then $M \geq 7532$. But the inequality $M < 1+25.2(\log(2M+0.5) + 0.38)^2$ implies $M \leq 1867$. Therefore, $\max\{\log(2M+0.5)+0.38, 10\} = 10$ yields $M < 2521$ and hence $x < 2521 \log 7m$. $\square$

We are now ready to proof Theorem 1. It follows from Lemmas 4.18, 4.19 that

$$\frac{1}{44}(m^2 - 5) < 2521 \log 7m.$$

Hence we obtain $m \leq 990$. From (4.22) have the inequality

$$\left| \frac{\log a}{\log c} - \frac{z}{x} \right| < \frac{b}{x a^x \log c}.$$

Since $\frac{a^x \log c}{b} > \frac{44 m^2 x}{5 m^2} > 2x$, we get that

$$\left| \frac{\log a}{\log c} - \frac{z}{x} \right| < \frac{1}{2x^2},$$

which implies that $\frac{z}{x}$ is a convergent in the simple continued fraction expansion to $\frac{\log a}{\log c}$. Let $\frac{z}{x} = \frac{p_k}{q_k}$, where $\frac{p_k}{q_k}$ is the k-th convergent of the simple continued fraction expansion to $\frac{\log a}{\log c}$.

Note that $q_k \le x$, since $(p_k, q_k) = 1$. It follows then that

$$\frac{1}{q_k^2(a_{k+1}+2)} < \left|\frac{\log a}{\log c} - \frac{p_k}{q_k}\right| < \frac{b}{xa^x \log c} < \frac{b}{q_k a^{q_k} \log c},$$

where a_{k+1} is the $(k+1)$-st partial quotient to $\frac{\log a}{\log c}$. Thus q_k and a_{k+1} satisfy the inequality

$$a_{k+1} + 2 > \frac{a^{q_k} \log c}{bq_k}. \tag{4.25}$$

Finally, using a computer program we checked that there do not exist any convergent $\frac{p_k}{q_k}$ of $\frac{\log a}{\log c}$ satisfying (4.25) when $q_k < 2521 \log 7m$ in the range $3 \le m \le 990$.

In view of the above proof, we have proved the following:

Proposition 4.20. *Let m be a positive integer with $m \ge 3$. Put $a = 44m^2 + 1$, $b = 5m^2 - 1$ and $c = 7m$. Then Pillai's equation*

$$c^z - a^x = b$$

has no positive integer solutions x, z with $x \ge 3$.

The case m is odd and $7|m$

From Lemma 4.13, we may assume that $m \le 24028$ when $m \equiv 0 (\mathrm{mod}\ 7)$.

Lemma 4.21. *Let (x,y,z) be a positive integer solution of Equation (4.19). Suppose that $m \equiv 0(mod\ 7)$. Then the only positive integer solution of Equation (4.19) is $(x,y,z) = (1,1,2)$.*

Proof. Obviously $(1,1,2)$ is the only solution of Equation (4.19) for $M = \max\{x,y\} = 1$. Suppose that $M > 1$. From Lemma 4.12 for $m \ge 7$ we have that

$$1.35M < \left(2 - \frac{\log\left(\frac{49}{4}\right)}{\log 49}\right) M < z < 2M.$$

Hence $z \geq 3$. Taking Equation (4.19) modulo m, we see that y is odd. Here we apply Lemma 4.11. For this we set $p = 7$, $a_1 = 44m^2 + 1$, $a_2 = 1 - 5m^2$, $b_1 = x$, $b_2 = y$, and

$$\Lambda := (44m^2 + 1)^x - (1 - 5m^2)^y.$$

Then we may take $g = 1$, $E = 2$, $A_1 = 44m^2 + 1$, $A_2 = 5m^2 - 1$. We get

$$2z \leq \frac{36.1}{8(\log 7)^4}(\max\{\log b' + \log(2\log 7) + 0.4, 12\log 7, 5\})^2 \log(44m^2 + 1)\log(5m^2 - 1),$$

where

$$b' = \frac{x}{\log(5m^2 - 1)} + \frac{y}{\log(44m^2 + 1)}.$$

Suppose that $z \geq 4$. We will show that this leads to a contradiction. Taking Equation (4.19) modulo m^4, we find

$$44x + 5y \equiv 0 (\mathrm{mod}\ m^2).$$

Then $M \geq \frac{m^2}{49}$. Since $z > \left(2 - \frac{\log(\frac{49}{4})}{\log(7m)}\right)M$ and $b' < \frac{M}{\log m}$ we have that

$$\begin{aligned} 2\left(2 - \frac{\log(\frac{49}{4})}{\log(7m)}\right)M \leq \frac{36.1}{8(\log 7)^4}&\left(\max\{\log\left(\frac{M}{\log m}\right) + \log(2\log 7) + 0.4, 12\log 7\}\right)^2 \\ &\times \log(44m^2 + 1)\log(5m^2 - 1). \end{aligned} \tag{4.26}$$

Let

$$h = \max\{\log\left(\frac{M}{\log m}\right) + \log(2\log 7) + 0.4, 12\log 7\}.$$

Suppose that $\log\left(\frac{M}{\log m}\right) + \log(2\log 7) + 0.4 \geq 12\log 7$. Then the inequality $\log M \geq$

$12\log 7 - \log(2\log 7) - 0.4$ implies that $M \geq 2383998120$. On the other hand, from Inequality (4.26) we have that

$$2M \leq 0.32(\log M + 1.76)^2 \log(44 \cdot 24028^2 + 1)\log(5 \cdot 24028^2 - 1)$$

which implies that $M < 10061$, a contradiction. Hence $h = 12\log 7$ and therefore from Inequality (4.26) we get

$$\frac{2m^2}{49}\left(2 - \frac{\log\left(\frac{49}{4}\right)}{\log(7m)}\right) \leq 172\log(44m^2+1)\log(5m^2-1).$$

This yields that $m \leq 795$. Hence $M \leq \frac{172\log(44m^2+1)\log(5m^2-1)}{2\left(2-\frac{\log\left(\frac{49}{4}\right)}{\log(7m)}\right)}$ and therefore all x, y, z are bounded. Using a program in Maple we found that there is no (m, x, y, z) under consideration satisfying Equation (4.19). We conclude $z \leq 3$. In this case, one can easily show that $(x, y, z) = (1, 1, 2)$. Thus there is no positive integer solution of Equation (4.19) other than $(x, y, z) = (1, 1, 2)$ when $7|m$. □

4.2.4 Example

In this section we verify that when $1 \leq m \leq 18$ equation (4.19) has only the positive integer solution $(x, y, z) = (1, 1, 2)$.

Example 4.22. Let m be a positive integer with $1 \leq m \leq 18$. Then the Diophantine equation

$$(44m^2+1)^x + (5m^2-1)^y = (7m)^z$$

has only the positive integer solution $(x, y, z) = (1, 1, 2)$.

Proof. It follows from Theorem 4.8 that the above equation has only the positive integer solution $(x, y, z) = (1, 1, 2)$ in all cases $1 \leq m \leq 18$ except for the following two cases $m =$

5, 9:

$$(a)\ (3\cdot 367)^x + (4\cdot 31)^y = 35^z$$

$$(b)\ (5\cdot 23\cdot 31)^x + (4\cdot 101)^y = 63^z$$

(a) Case 1: $y = 1$. Then it follows from Proposition 4.20 that the equation

$$35^z - (3\cdot 367)^x = 124$$

has only the positive integer solution $z = 2, x = 1$.

Case 2: $y \geq 2$. Then taking (a) modulo 8 implies that $5^x \equiv 3^z \pmod 8$. By $(\frac{124}{367}) = 1$ and $(\frac{35}{367}) = -1$, we see that z is even and hence x is even, say $z = 2Z$, $x = 2X$. Then

$$(35^Z + (3\cdot 367)^X)(35^Z - (3\cdot 367)^X) = (4\cdot 31)^y.$$

Thus we have the following two systems of equations:

$$\begin{cases} 35^Z \pm (3\cdot 367)^X = 2^{2y-1} \\ 35^Z \mp (3\cdot 367)^X = 2\cdot 31^y \end{cases}$$

or

$$\begin{cases} 35^Z \pm (3\cdot 367)^X = 2^{2y-1}\cdot 31^y \\ 35^Z \mp (3\cdot 367)^X = 2. \end{cases}$$

We deal with the first system. Adding these two equations yields

$$35^Z = 2^{2y-2} + 31^y.$$

Taking the above equation modulo 5 implies that $(-1)^{y-1} + 1 \equiv 0 \pmod 5$ and y is even, which is impossible.

Now we consider the second system. Adding these two equations yields

$$35^Z = 2^{2y-2}31^y + 1$$

Again taking the above equation modulo 5 we see that y is even, which is impossible.

(b) Case 1: $y = 1$. Then it follows from Proposition 4.20 that the equation

$$63^z - (5 \cdot 23 \cdot 31)^x = 404$$

has only the positive integer solution $z = 2, x = 1$.

Case 2: $y \geq 2$. Then taking (b) modulo 8 implies that $5^x \equiv (-1)^z (\bmod 8)$. By $(\frac{404}{23}) = 1$ and $(\frac{63}{23}) = -1$, we see that z is even and hence x is even, say $z = 2Z$, $x = 2X$. Then

$$(63^Z + (5 \cdot 23 \cdot 31)^X)(63^Z - (5 \cdot 23 \cdot 31)^X) = (4 \cdot 101)^y.$$

Thus we have the following two systems of equations:

$$\begin{cases} 63^Z \pm (5 \cdot 23 \cdot 31)^X = 2^{2y-1} \\ 35^Z \mp (5 \cdot 23 \cdot 31)^X = 2 \cdot 101^y \end{cases}$$

or

$$\begin{cases} 63^Z \pm (5 \cdot 23 \cdot 31)^X = 2^{2y-1} \cdot 101^y \\ 63^Z \mp (5 \cdot 23 \cdot 31)^X = 2. \end{cases}$$

We deal with the first system. Adding these two equations yields

$$63^Z = 2^{2y-2} + 101^y.$$

Taking the above equation modulo 3 implies that y is odd. On the other hand, taking the same equation modulo 7, yields $4^{y-1} \equiv 4^y \pmod 7$, which is impossible.

Now we consider the second system.Adding these two equations yields

$$63^Z = 2^{2y-2}101^y + 1$$

Again taking the above equation modulo 3 we see that y is odd. On the other hand, taking the same equation modulo 31, yields $1 \equiv 2^{5y-2} + 1 \pmod{31}$, which is impossible. $\square$

Chapter 5

Diophantine equations involving multiplicative functions

5.1 On the equation $\phi(n+k) = 2\phi(n)$

The results of this chapter have been published in [50, 52].

5.1.1 Introduction

W. Sierpinski [105] proved that the equation $\phi(n+k) = \phi(n)$ has at least one solution for every fixed positive integer k, where $\phi(n)$ denotes Euler's phi-function. A. Makowski [78] proved the analogous result for the equation

$$\phi(n+k) = 2\phi(n). \tag{5.1}$$

The proof is easy: if $(k,6) = 1$, we can take $n = 2k$; if k is even, $k = 2^l u$ ($l \geq 1$, u odd), we may take $n = k$; if k is odd and $3|k$, $k = 6l+3$, take $n = 2l+1$.

In 1959, A. Schinzel and A. Wakulicz [103] showed that for each fixed $k \leq 2 \cdot 10^{58}$ there are at least two solutions to $\phi(n+k) = \phi(n)$. The next theorem establishes the result of the same spirit for Equation (5.1).

Theorem 5.1. *(a) For all integers $k \leq 4 \cdot 10^{58}$, Equation* (5.1) *has at least two solutions.*

(b) Moreover, for at least $15 \cdot 10^{55}$ odd integers $k \leq 4 \cdot 10^{58}$, Equation (5.1) *has at least three solutions.*

Following K.Ford [44], Hypothesis $\mathcal{P}(a,b)$ will refer to the statement that there are infinitely many $n \in \mathbb{N}$ such that both $an+1$ and $bn+1$ are primes. Note that $\mathcal{P}(a,b)$ is just a particular case of Dickson's prime k-tuple conjecture.

There is an intimate connection between Hypothesis $\mathcal{P}(a,b)$ and the following conjecture for certain integers k.

Hypothesis Φ_k. Equation (5.1) has infinitely many solutions $n \in \mathbb{N}$.

If k is odd and divisible by 3, and both $2n+1$ and $3n+1$ are primes greater than k, then

$$\phi(3(2n+1)k) = \phi(2(3n+1)k+k) = 2\phi(2(3n+1)k).$$

Thus Hypothesis $\mathcal{P}(2,3)$ implies Φ_k for every $3|k$, k odd.

Our next lemma generalizes this observation.

Lemma 5.2. *Let k be an odd integer, divisible by* 3*. Suppose j and $2j+k$ have the same prime divisors (thus j is odd), $g=(j,2j+k)$ and for some positive integer r, $\frac{2j}{g}r+1$ and $\frac{2j+k}{g}r+1$ are primes that do not divide j. Then $n=2j(\frac{2j+k}{g}r+1)$ is a solution of* (5.1)*. In other words, $\mathcal{P}\left(\frac{2j}{(j,2j+k)}, \frac{2j+k}{(j,2j+k)}\right)$ implies Φ_k for $3|k$, k odd.*

Proof. We get

$$\phi(n) = \phi\Big(2j\Big(\frac{2j+k}{g}r+1\Big)\Big) = \frac{2j+k}{g}r\phi(j)$$

and

$$\phi(n+k) = \phi\Big((2j+k)\Big(\frac{2j}{g}r+1\Big)\Big) = \frac{2j}{g}r\phi(2j+k).$$

Since by assumption j and $2j+k$ have the same prime factors, it follows that

$$\phi(j)(2j+k) = j\phi(2j+k),$$

which yields the desired result. $\square$

In order to proceed further we need some terminology. We say that a collection of distinct linear forms

$$L_i(x) = a_i x + b_i,\ (1 \le i \le k),\ a_i, b_i \in \mathbb{Z},\ a_i > 0$$

is admissible if for every prime p there exists $x_p \in \mathbb{Z}$ such that $p \nmid \prod_{i=1}^{k} L_i(x_p)$. We say that $DHL^*(k;m)$ holds if, for any admissible set of k linear forms $(a_1 n + b_1, ..., a_k n + b_k)$ there exist distinct $i_1, ..., i_m \in \{1, ..., k\}$ such that there are infinitely many r with m numbers $a_{i_1} r + b_{i_1}, ..., a_{i_m} r + b_{i_m}$ simultaneously prime (see [44]).

Assuming the Elliot-Halberstam conjecture, Maynard [80] showed that $DHL^*(5;2)$ is true. A generalized version of the Elliot-Halberstam conjecture implies $DHL^*(3;2)$ holds.

Theorem 5.3. *(a) If $DHL^*(5;2)$ holds then Φ_k is true for all odd k with $3255|k$.*

(b) If $DHL^*(4;2)$ holds then Φ_k is true for all odd k with $105|k$.

(c) If $DHL^*(3;2)$ holds then Φ_k is true for all odd k with $21|k$.

5.1.2 Proofs

Proof of Theorem 5.1. (a) We consider two cases: for even values of k and for odd values of k.

For even values of k, we will apply a modified version of Schinzel's argument [102]. Let $k = 2^l r$, r is odd, $l \ge 1$. We use the notation $a|^*b$ to indicate that each prime divisor of a is a prime divisor of b. Assume that the sequence of primes $3 = p_1 < p_2 < ... < p_m$ satisfies the conditions

1. $(p_i - 2)|p_1 p_2 ... p_{i-1},\ (2 \le i \le m)$,

2. $(p_i - 1)|^* 2 p_1 p_2 ... p_{i-1},\ (2 \le i \le m)$.

Suppose that $p_1 p_2 ... p_m \nmid r$ and let p_j be the smallest prime in the sequence that does not divide r. Then $(r, p_j) = (p_j, 2) = 1$ and either $p_j = 3$ or $p_1 p_2 ... p_{j-1}|r$ $(j \ge 2)$. From (1), (2), $p_j - 2|r$ and $p_j - 1|^* 2r$. Since $(p_j - 1, p_j - 2) = 1$, then $p_j - 1|^* \frac{2r}{p_j-2}$ and

$$\phi\Big(\frac{2^{l+1}r}{p_j-2}(p_j-1)\Big) = (p_j-1)\phi\Big(\frac{2^{l+1}r}{p_j-2}\Big) = 2^l\phi(p_j)\phi\Big(\frac{r}{p_j-2}\Big) = 2\phi\Big(\frac{2^l p_j r}{p_j-2}\Big).$$

As $\frac{p_j}{p_j-2} > 1$, $n = \frac{2^l r p_j}{p_j-2}$ is a solution of (5.1) distinct from that given by Makowski's result. The sequence of primes 3, 5, 7, 17, 19, 37, 97, 113, 257, 401, 487, 631, 971, 1297, 1801, 19457, 22051, 28817, 65537, 157303, 160001 satisfies the required conditions and $3 \cdot 5 \cdot 7 \cdots 160001 > 2 \cdot 10^{58}$. This argument works unless r is a multiple of $p_1 \cdots p_m$, and so for all $r < 2 \cdot 10^{58}$. Note that Holt [58] showed that it is impossible to extend the above list of primes. Therefore, increasing the upper bound would require a different method.

Let k be an odd integer. Observe that if $F_n = 2^{2^n} + 1$ is the Fermat's prime number such that $(2^{2^n} + 1, k) = 1$, then $x = 2^{2^n}k$ is a new solution of the equation $\phi(x+k) = 2\phi(x)$. There are only five known Fermat's primes: $F_0 = 3, F_1 = 5, F_2 = 17, F_3 = 257, F_4 = 65537$. Now every odd integer k is either coprime to all F_n's or divisible by exactly m distinct F_n's ($0 \le n \le 4$, $1 \le m \le 5$). Clearly this defines the partition of all odd integers. If $(k, F_0F_1F_2F_3F_4) = 1$, then we get four new solutions: $x_j = 2^{2^j}k$, $1 \le j \le 4$. Otherwise, suppose $i_1, i_2, ..., i_m \in \{0,1,2,3,4\}$ are distinct such that $F_{i_1}F_{i_2}...F_{i_m}|k$. Choose $i_t \neq 0$, $i_t \notin \{i_1, i_2, ..., i_m\}$. This gives a solution $x = 2^{2^{i_t}}k$ in addition to Makowski's result. There are two exceptional cases when such a choice does not yield a new solution of this form. If $5 \cdot 17 \cdot 257 \cdot 65537|k$, then choose $l \in \mathbb{N}$ such that $4l+1$ and $5l+1$ are both primes relatively prime to k. Then $x = 4k(5l+1)$ is a new solution. If $3 \cdot 5 \cdot 17 \cdot 257 \cdot 65537|k$, then choose $m \in \mathbb{N}$ such that $2m+1$ and $3m+1$ are both primes relatively prime to k. Then $x = 2k(3m+1)$ is an additional solution.

Maple computations show that we can take, for example,

$$l = 2^3 \cdot 3^2 \cdot 5^2 \cdot 229 \cdot 13643373661321 \cdot 1214436755047 \cdot 1464182919280833707$$

$$m = 2 \cdot 7 \cdot 17 \cdot 37 \cdot 148140750636881 \cdot 2606307747466367058840819067669.$$

For such choice of l and m, we have $5\cdot 17\cdot 257\cdot 65537\cdot(5l+1) > 4\cdot 10^{58}$ and $3\cdot 5\cdot 17\cdot 257\cdot 65537\cdot(3m+1) > 4\cdot 10^{58}$. Hence for all odd $k \leq 4\cdot 10^{58}$, Equation (5.1) has at least two solutions.

(b) If k is odd, $k\cdot 2^n+1$ is prime for some $n \geq 1$, then

$$\phi(k^2\cdot 2^n) = 2^{n-1}\phi(k^2) = 2^{n-1}k\phi(k),$$
$$\phi(k^2\cdot 2^n+k) = \phi(k(k\cdot 2^n+1)) = 2^n k\phi(k).$$

Let $p \equiv 3 (\mathrm{mod}\ 4)$ and $k = \frac{p-1}{2}$. Then $k\cdot 2+1$ is prime. If $p \equiv 13 (\mathrm{mod}\ 280)$ and $k = \frac{p-1}{4}$, then $k\cdot 2^2+1$ is prime while $k\cdot 2+1$ is composite (see [60]). Using explicit bounds for $\pi(x;q,a)$ (Bennett et al. [10]) we get

$$\pi(4\cdot 10^{58};4,3) > \frac{4\cdot 10^{58}}{\phi(4)\log(4\cdot 10^{58})},$$
$$\pi(4\cdot 10^{58};280,13) > \frac{4\cdot 10^{58}}{\phi(280)\log(4\cdot 10^{58})}.$$

From the above estimates we infer that for at least $\frac{(2+\frac{1}{24})\cdot 10^{58}}{\log(4\cdot 10^{58})} > 15\cdot 10^{55}$ of all odd $k \leq 4\cdot 10^{58}$, Equation (5.1) has at least three solutions. $\square$

Remark. Note that if the hypotheses $\mathcal{P}(2,3)$ and $\mathcal{P}(4,5)$ are true, then for a positive proportion of all odd $k \in \mathbb{N}$, Equation (5.1) has at least three solutions. The result follows from a theorem of Erdös and Odlyzko [36]: there exists a positive, effectively computable constant c_1 such that if $N(x)$ is the number of odd positive integers $k \leq x$ such that $k\cdot 2^n+1$ is prime for some positive n, then $N(x) \geq c_1 x$ for $x \geq 1$.

Let $a < b$ be integers such that

$$a = 2^l u,\ b = 2^s t,\ l > s, s \geq 0,\ u,t \text{ odd}. \tag{5.2}$$

Define

$$a_1 = \frac{a}{2^{k_1}(a,b)},\ b_1 = \frac{b}{(a,b)},\ k_1 = l-s,\ \kappa(a,b) = (b_1 - 2^{k_1}a_1)\prod_{p|a_1b_1} p. \tag{5.3}$$

The next lemma is a modified version of [44, Lemma 3].

Lemma 5.4. *Asssume $\mathcal{P}(a,b)$, where a,b are of the form* (5.2). *Then Φ_k holds for every $k = v\kappa(a,b)$, v odd.*

Proof. Observe that $\mathcal{P}(a,b) \Rightarrow \mathcal{P}(2^{k_1}a_1, b_1)$. Let $s_1 = \prod_{p|a_1b_1} p$ and suppose that $r > max(2^{k_1}a_1, b_1)$ is such that $2^{k_1}a_1r+1$ and b_1r+1 are both primes. Let $v \in \mathbb{N}$ be odd. Note that we can choose r so large that $(2^{k_1}a_1r+1, v) = (b_1r+1, v) = 1$. Set

$$m_1 = b_1vs_1(2^{k_1}a_1r+1),\ m_2 = 2^{k_1}a_1vs_1(b_1r+1).$$

Then

$$\phi(m_1) = \phi(b_1vs_1(2^{k_1}a_1r+1)) = 2^{k_1}a_1rb_1\phi(vs_1),$$
$$\phi(m_2) = \phi(2^{k_1}a_1vs_1(b_1r+1)) = 2^{k_1-1}b_1ra_1\phi(vs_1).$$

It follows that $\phi(m_1) = 2\phi(m_2)$ and $m_1 - m_2 = (b_1 - 2^{k_1}a_1)vs_1 = v\kappa(a,b)$. □

Proof of Theorem 5.3. (a) Let $\{a_1,a_2,a_3,a_4,a_5\} = \{2^4, 2^3\cdot 3, 2^2\cdot 7, 2\cdot 3\cdot 5, 31\}$. If $DHL^*(5;2)$ holds, then for some i,j with $1 \le i,j \le 5$, $\mathcal{P}(a_i,a_j)$ is true. We compute

$$lcm\{\kappa(a_i,a_j) : 1 \le i,j \le 5\} = 3\cdot 5\cdot 7\cdot 31.$$

Thus by Lemma 5.4, Φ_k is true for all odd multiples of 3255.

(b) This is the same as the proof of (a), but take $\{a_1,a_2,a_3,a_4\} = \{2^3, 2^2\cdot 3, 2\cdot 7, 3\cdot 5\}$ if $DHL^*(4;2)$ holds.

(c) Take $\{a_1,a_2,a_3\} = \{2^2, 2\cdot 3, 7\}$ if $DHL^*(3;2)$ holds. □

5.2 On the equation $\sigma^*(n) = 1 + Mn$

5.2.1 Introduction

We say that d is a *unitary* divisor of n if $d|n$ and $\left(d, \frac{n}{d}\right) = 1$. Let $\sigma^*(n) = \sum_{d|n,\ (d,\frac{n}{d})=1} d$ be the sum of all unitary divisors of n. If $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$ where p_i are distinct primes and $\alpha_i > 0$ for all i, then it is easy to see that

$$\sigma^*(n) = \prod_{i=1}^{r}(1 + p_i^{\alpha_i})$$

and that $\sigma^*(n)$ is a multiplicative function.

The *Dedekind psi function* $\psi(n)$ is a multiplicative function given by

$$\psi(n) = n \prod_{p|n} \left(1 + \frac{1}{p}\right).$$

Let $M \geq 1$ be a positive integer. Subbarao [108] studied the Diophantine equations

$$\psi(n) = 1 + Mn \tag{5.4}$$

and

$$\sigma^*(n) = 1 + Mn. \tag{5.5}$$

He formulated the following conjecture.

Conjecture 5.5. The congruence $\sigma^*(n) \equiv 1 \pmod n$ is possible if and only if n is a prime power. In particular, for arbitrary distinct primes $p_1, \dots, p_r$, $r \geq 1$,

$$(p_1 + 1) \dots (p_r + 1) \equiv 1 \pmod{p_1 \dots p_r} \text{ if and only if } r = 1.$$

Equivalently, $\psi(n) \equiv 1 \pmod n$ if and only if n is a prime.

No counterexample to Conjecture 1 is known although there are some partial results. For $M = 1, 2, 3, \ldots$ define

$$T_M = \{n : \psi(n) = 1 + Mn\},\ \mathcal{T} = \bigcup_{M\geq 2} T_M,$$
$$T_M^* = \{n : \sigma^*(n) = 1 + Mn\},\ \mathcal{T}^* = \bigcup_{M\geq 2} T_M^*.$$

Note that $T_M \subset T_M^*$. Subbarao's conjecture is that $\mathcal{T}$ and $\mathcal{T}^*$ are empty.

In the sequel we assume that $M > 1$. We recall that a number is said to be powerful if each exponent in its canonical representation is at least 2. Subbarao himself obtained the following results:

i) If $n \in T_M^*$, then n is not a powerful number, M is odd ≥ 3 and $\omega(n) \geq 16$, $n > 10^{20}$, where $\omega(n)$ denotes the number of distinct prime factors of n;

ii) If $n \in T_M^*$ with $\omega(n) = r$, then

$$n < (r-1)^{2^r - 1}; \tag{5.6}$$

iii) If $n \in T_M$ and $3|n$, then we have

$$\omega(n) > 2557,\ n > 5.9 \cdot 10^{10766}, \tag{5.7}$$

(stated without proof);

iv) If $n \in T_M$ and $3 \nmid n$, then we have

$$\omega(n) \geq 123. \tag{5.8}$$

Our first theorem improves the upper bound in Inequality (5.6).

Theorem 5.6. *If $n \in T_M^*$, then*

$$n \leq 2^{2^r-1} - 2^{2^{r-1}-1}, \tag{5.9}$$

where r denotes the number of distinct prime divisors of n.

Using Grytczuk-Wójtowicz's techniques from the paper [49] we obtain the lower bounds for $\omega(n)$ and n for $n \in T_M^*$ with $M \geq 3$, which are also valid for $n \in T_M$ with $M > 3$; thereby improving Inequalities (5.7)-(5.8).

The main result is the following theorem.

Theorem 5.7. *Let $M \geq 3$ and $n \in T_M^*$.*

(a) If $3|n$, then $\omega(n) > \frac{1}{2} \cdot 1578^{AM^2/9}$, where $A = 0.998\ldots$.

(b) If $3 \nmid n$, then $\omega(n) \geq 51^{M/3} - 1$.

The corollary below is an immediate consequence of Theorem 5.7 and Robin's inequality (see [96, Théorème 6]) that for every positive integer n we have

$$n > \left(\frac{r \log r}{3}\right)^r, \tag{5.10}$$

where $r = \omega(n)$. Note that for $M \geq 3$ we have the following rough bounds: $\frac{1}{2} \cdot 1578^{AM^2/9} > 2^{AM^2-1}$ and $51^{M/3} - 1 > 3^M$.

Corollary 5.8. *Let $M \geq 3$ and $n \in T_M^*$.*

(a) If $3|n$, then $n > (c(AM^2-1)2^{AM^2-1})^{2^{AM^2-1}}$, where $c = 0.231\ldots = \log 2/3$.

(b) If $3 \nmid n$, then $n > (dM3^M)^{3^M}$, where $d = 0.366\ldots = \log 3/3$.

Thus, if $3|n$ and $M \geq 5$, we have that $\omega(n) > 3.7 \cdot 10^8$ and $n > 3.9 \cdot 10^{10^8}$ (compare with Inequality (5.7)). If $3 \nmid n$ and $M \geq 5$, then $\omega(n) > 700$ and $n > 10^{643}$ (compare with Inequality (5.8)).

Using Inequality (5.9) we obtain the following analogue of [49, Theorem 2].

Theorem 5.9. *Let $\mathcal{P} = \{P_1, P_2 \ldots\}$, where $P_i < P_{i+1}$ for all $i \geq 1$ denote the set of all odd prime numbers. For every integer $k \geq 2$ there exists an infinite subset $\mathcal{P}(k)$ of the set $\mathcal{P}$ such that*

(a) for every pairwise distinct primes $p_1, p_2, \ldots, p_k \in \mathcal{P}(k)$ and $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{N}$ the number $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$ does not fulfill Equation (5.5);

(b) $\mathcal{P}(k)$ is maximal with respect to inclusion.

(Note that, since $\omega(n) \geq 16$ in general, we have $\mathcal{P}(k) = \mathcal{P}$ for $k \leq 15$.)

5.2.2 Preliminaries

In this section we shall collect some preliminary results.

Lemma 5.10. *Let $n \in T_M^*$, $M > 1$. If $p|n$ and $q^\beta + 1 \equiv 0 \pmod p$, then q^β cannot be a unitary divisor of n. In particular, if $p|n$, $q + 1 \equiv 0 \pmod p$, and q^β is a unitary divisor of n, then β cannot be odd.*

Proof. Given $p|n$ and $q^\beta + 1 \equiv 0 \pmod p$, if $q^\beta || n$, then also $\sigma^*(q^\beta) = q^\beta + 1 | \sigma^*(n)$, so that $p|\sigma^*(n)$. Thus, $p|(n, \sigma^*(n))$, leading to a contradiction. Hence, the lemma is proven. $\square$

The next lemma due to Goto [47] plays an important role in the proof of the Theorem 5.6.

Lemma 5.11. *Let r, a, b be positive integers. If integers $x_1, \ldots, x_r$ satisfy $1 \leq x_1 < x_2 < \ldots < x_r$ and*

$$1 \leq \prod_{i=1}^{r-1} \left(1 + \frac{1}{x_i}\right) < \frac{a}{b} \leq \prod_{i=1}^{r} \left(1 + \frac{1}{x_i}\right), \tag{5.11}$$

then it follows that

$$\prod_{i=1}^{r} x_i \leq (b+1)^{2^r - 1} - (b+1)^{2^{r-1} - 1} \tag{5.12}$$

with equality if and only if $x_i = m_i$ for $1 \le i \le r$, where

$$m_i = \begin{cases} (b+1)^{2^{i-1}} \; i = 1, \ldots, r-1, \\ (b+1)^{2^{i-1}} - 1 \; i = r. \end{cases}$$

5.2.3 Proofs

Proof of Theorem 5.6. Let $n \in T_M^*$ with $\omega(n) = r$. Write $n = N_1 \cdots N_r$ where $N_1, \ldots, N_r$ denote prime powers satisfying $N_i < N_j$, $(N_i, N_j) = 1$ for $i < j$. Then

$$\frac{\sigma^*(n) - 1}{n} < \frac{\sigma^*(n)}{n} = \prod_{i=1}^{r} \left(1 + \frac{1}{N_i}\right).$$

Moreover,

$$1 < \prod_{i=1}^{r-1} \left(1 + \frac{1}{N_i}\right) < \prod_{i=1}^{r} \left(1 + \frac{1}{N_i}\right) - \frac{1}{N_1 \cdots N_r} = \frac{\sigma^*(n) - 1}{n}.$$

Thus,

$$1 < \prod_{i=1}^{r-1} \left(1 + \frac{1}{N_i}\right) < \frac{\sigma^*(n) - 1}{n} < \prod_{i=1}^{r} \left(1 + \frac{1}{N_i}\right).$$

Hence, the Inequality (5.11) is satisfied for $x_i = N_i$, $a = \frac{\sigma^*(n)-1}{n}$, $b = 1$. From Inequality (5.12) we get

$$n = N_1 \cdots N_r \le 2^{2^r - 1} - 2^{2^{r-1} - 1}$$

and the theorem follows. $\square$

Proof of Theorem 5.7. From Equation (5.5) we get that if $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r} \in T_M^*$, then

$$M = \frac{\sigma^*(n) - 1}{n} < \frac{\sigma^*(n)}{n} = \prod_{i=1}^{r} \left(1 + \frac{1}{p_i^{\alpha_i}}\right) < \prod_{i=1}^{r} \left(1 + \frac{1}{p_i}\right)$$

and hence

$$\log M < \sum_{i=1}^{r} \log\left(1+\frac{1}{p_i}\right). \tag{5.13}$$

Part (a). From Lemma 5.10 it follows that $p_i \equiv 1 \pmod 6$ or α_i is even for $i \geq 2$. We define the set $\mathcal{A} := \{3\} \cup \{p \in \mathcal{P} : \ p \equiv 1 \pmod 6\} \cup \{p^2 : \ p \equiv 5 \pmod 6\} = \{a_1, a_2, \ldots\}$, where $a_j < a_{j+1}$ for $j = 1, 2, \ldots$.

Put

$$\alpha(k) = \sum_{i=1}^{k} \log\left(1+\frac{1}{a_i}\right),$$

where $k \geq 16$. Thus,

$$\log M < \alpha(r). \tag{5.14}$$

Let k_0 be the least integer k with $\log 3 \leq \alpha(k)$. Since $\alpha(805) = 1.098538\ldots < \log 3 = 1.098612\ldots < 1.098613\ldots = \alpha(806)$, we obtain that $k_0 = 806$. Now from Inequality (5.14) it follows that for $n \in T_M^*$ with $M \geq 3$ we have $\omega(n) = r \geq 806$.

Let $Q' = \{Q_1', Q_2', \ldots\}$ denote the set of all primes $\equiv 1 \pmod 6$ with $Q_i' < Q_{i+1}'$ and $Q^* = \{Q_1^*, Q_2^*, \ldots\}$ denote the set of all primes $\equiv 5 \pmod 6$ with $Q_i^* < Q_{i+1}^*$ for $i = 1, 2, \ldots$. Since $i \geq 806$, then $a_i \equiv 1 \pmod 6$ and $a_i \geq 13441 = Q_{790}'$ or $a_i = p^2$ with $p \equiv 5 \pmod 6$ a prime and $p \geq 131 = Q_{16}^*$. Hence,

$$\begin{aligned}\log M <& \alpha(805) + \sum_{i=806}^{r} \log\left(1+\frac{1}{a_i}\right) < \log 3 + \sum_{i=806}^{r} \frac{1}{a_i} \\ &< \log 3 + \sum_{j=790}^{r} \frac{1}{Q_i'} + \sum_{i=16}^{\infty} \frac{1}{Q_j^{*2}}.\end{aligned}$$

Using the estimate $Q_n', Q_n^* > 2n\log(2n)$ for $Q_n', Q_n^* > 198$, which follows from [10, Corollary 1.6], we get

$$\begin{aligned}\log M &< \log 3 + \sum_{j=16}^{23} \frac{1}{Q_j^{*2}} + \sum_{j=790}^{r} \frac{1}{2j\log(2j)} + \sum_{j=24}^{\infty} \frac{1}{4j^2\log^2(2j)} \\ &< \log 3 + 0.00081 + \int_{789}^{r} \frac{dx}{2x\log(2x)}\end{aligned}$$

Here we used the facts that the first sum is finite, second is convergent and $\sum_{k=m+1}^{n} f(k) \leq \int_m^n f(x)dx$ for $f(x)$ monotone, non-increasing on $m \leq x \leq \infty$.

Hence, $\log M < \log 3 + 0.00081 + \frac{1}{2}\log\log(2r) - \frac{1}{2}\log\log 1578$. Therefore $\omega(n) = r > \frac{1}{2} \cdot 1578^{AM^2/9}$ where $A = 0.998\ldots$.

Part (b). Let $Q = \{Q_1, Q_2, \ldots\}$ denote the set of all primes with $Q_i < Q_{i+1}$ for $i = 1, 2, \ldots$. We have in Inequality (5.13) that $p_1 \geq 5 = Q_3$. Put

$$\beta(k) = \sum_{i=1}^{k} \log\left(1 + \frac{1}{Q_{i+2}}\right),$$

where $k \geq 16$. Now from Inequality (5.13) we obtain

$$\log M < \beta(r). \tag{11'}$$

Since $\beta(49) = 1.09651\ldots < \log 3 < 1.10069\ldots = \beta(50)$, from Inequality (11′) it follows that for $n \in T_M^*$ with $M \geq 3$ we have $\omega(n) = r \geq 50$. Thus,

$$\begin{aligned}\log M &< \beta(49) + \sum_{i=50}^{r} \log\left(1 + \frac{1}{Q_{i+2}}\right) < \log 3 + \sum_{i=50}^{r} \frac{1}{Q_{i+2}} \\ &< \log 3 + \int_{49}^{r} \frac{dx}{(x+2)\log(x+2)},\end{aligned}$$

as

$$Q_m > m\log m \text{ for } m \geq 1.$$

(see [97]). Hence, $\log M < \log 3 + \log\log(r+2) - \log\log 51$ i.e., $\omega(n) = r \geq 51^{M/3} - 1$, as claimed. $\square$

Proof of Theorem 5.9. The theorem immediately follows from Theorem 4.8 since we can take $\mathcal{P}(k)$ to be the set of primes larger than $2^{2^k-1} - 2^{2^{k-1}-1}$. Indeed, n cannot fulfil $\sigma^*(n) = 1 + Mn$ if any prime factor of n is larger than $2^{2^k-1} - 2^{2^{k-1}-1}$.

The existence of a maximal (with respect to inclusion) set $\mathcal{P}(k)$ follows from Kuratowski-Zorn's lemma. $\square$

Bibliography

[1] J.-H. Ahn and S.-H. Kwon. An explicit upper bound for the least prime ideal in the Chebotarev density theorem. *Ann. Inst. Fourier (Grenoble)*, 69:1411–1458, 2019.

[2] M. Alan. On the exponential Diophantine equation $(18m^2+1)^x+(7m^2-1)^y=(5m)^z$. *Turkish J. Math.*, 42(4):1990–1999, 2018.

[3] C. Axler. New estimates for some functions defined over primes. *Integers*, 18(A52):21 pp., 2018.

[4] E. Bach and J. Sorenson. Explicit bounds for primes in residue classes. *Math. Comp.*, 65(216):1717–1735, 1996.

[5] R. Backlund. Über die Nullstellen der Riemannschen Zetafunction. *Acta Math.*, 41:374–375, 1916.

[6] A. Baker. Linear forms in the logarithms of algebraic numbers, i. *Mathematika*, 13:204–216, 1966.

[7] A. Baker. Linear forms in the logarithms of algebraic numbers, ii. *Mathematika*, 14:102–107, 1967.

[8] A. Baker. Linear forms in the logarithms of algebraic numbers, iii. *Mathematika*, 14:220–228, 1967.

[9] W. Banks, A. Güloğlu, and A. Yeager. Carmichael meets Chebotarev. *Canad. Math. Bull.*, 56(4):695–708, 2013.

[10] M.A. Bennett, G. Martin, K. O'Bryant, and A. Rechnitzer. Explicit bounds for primes in arithmetic progressions. *Illinois J. Math.*, 62(1-4):427–532, 2018.

[11] M.A. Bennett, G. Martin, K. O'Bryant, and A. Rechnitzer. Counting zeros of Dirichlet l-functions. *Math. Comp.*, 90(329):1455–1482, 2021.

[12] C. Bertok. The complete solution of the Diophantine equation $(4m^2+1)^x+(3m^2-1)^y=(5m)^z$. *Period. Math. Hung.*, 72:37–42, 2016.

[13] S. Bhattarcharjee, A.B. Dixit, and D. Saikia. An effective bound on Generalized Diophantine m-tuples. *https://arxiv.org/abs/2209.15120*, 2022.

[14] E.F. Bravo and J.J. Bravo. Powers of 2 as sums of three Fibonacci numbers. *Lith. Math. J.*, 55(3):301–311, 2015.

[15] J.J. Bravo, C.A. Gomez, and J.L. Herrera. k-Fibonacci numbers close to a power of 2. *Quaest. Math.*, 44(12):1681–1690, 2021.

[16] J.J. Bravo and L. Luca. On the Diophantine equation $f_n + f_m = 2^a$. *Quaest. Math.*, 39(3):391–400, 2016.

[17] R. P. Brent, D. J. Platt, and T.S. Trudgian. Accurate estimation of sums over zeros of the Riemann zeta-function. *Math. Comp.*, 90(332):2923—-2935, 2021.

[18] R. P. Brent, D. J. Platt, and T.S. Trudgian. The mean square of the error term in the prime number theorem. *J. Number Theory*, 238:740–762, 2022.

[19] S. Broadbent, H. Kadiri, A. Lumley, N. Ng, and K. Wilk. Sharper bounds for the Chebyshev function $\theta(x)$. *Math. Comp.*, 90(331):2281—-2315, 2021.

[20] Y. Bugeaud. Linear forms in p-adic logarithms and the Diophantine equation $(x^n - 1)/(x-1) = y^q$. *Math. Proc. Camb. Phil. Soc.*, 127:373–381, 1999.

[21] Y. Bugeaud, M. Mignotte, and S. Siksek. Classical and modular approaches to exponential Diophantine equations. i. fibonacci and lucas perfect powers. *Ann. of Math.*, 163(3):969–1018, 2006.

[22] P. Burcsi, S. Czirbusz, and G. Farkas. Computational investigation of Lehmer's totient problem. *Ann. Univ. Sci. Budapest. Sect. Comput.*, 35:43–49, 2011.

[23] D. Burek and B. Żmija. A new upper bound for numbers with the Lehmer property and its application to repunit numbers. *Int. J. Number Theory*, 15(7):1463–1468, 2019.

[24] J. Büthe. An analytic method for bounding $\psi(x)$. *Math. Comp.*, 87(312):1991–2009, 2018.

[25] Y. F. Cheng and S. W. Graham. Explicit estimates for the Riemann zeta function. *Rocky Mountain J. Math.*, 34(4):1261–1280, 2004.

[26] S. Chern and A. Cui. Fibonacci numbers close to a power of 2. *Fibonacci Quart.*, 52(4):344–348, 2014.

[27] K.C. Chim and V. Ziegler. On Diophantine equations involving sums of Fibonacci numbers and powers of 2. *Integers*, 18(A99):30pp, 2018.

[28] Y.J. Choie, N. Lichiardopol, P. Moree, and P. Solé. On Robin's criterion for the Riemann hypothesis. *J. Théor. Nombres Bordeaux*, 19(2):357–372, 2007.

[29] G.L. Cohen and P. Hagis. On the number of prime factors of n if $\phi(n)|n-1$. *Nieuw. Arch. Wisk. (3)*, 28(2):177–185, 1980.

[30] E. Cully-Hugill and E.S. Lee. Explicit interval estimates for prime numbers. *Math. Comp.*, 91(336):1955–1970, 2021.

[31] J. Dalton and O. Trifonov. Representing positive integers as a sum of a squarefree number and a small prime. *https://arxiv.org/abs/2301.12585*, 2023.

[32] S. Das. An explicit version of Chebotarev's density theorem. *MSc. Thesis, University of Lethbridge*, page 143 pp., 2020.

[33] A. Dujella. On the size of Diophantine m-tuples. *Math. Proc. Cambridge Philos. Soc.*, 132(1):23–33, 2002.

[34] A. Dujella and A. Pethö. A generalization of a theorem of Baker and Davenport. *Quart. J. Math. Oxford Ser. (2)*, 49(195):291–306, 1998.

[35] A. Dusart. Inégalités explicites pour $\psi(x)$, $\theta(x)$, $\pi(x)$ et les nombres premiers. *C. R . Math. Acad. Sci. Soc. R . Can.*, 21(2):53–59, 1999.

[36] P. Erdős and A.M. Odlyzko. On the density of odd integers of the form $(p-1)2^{-n}$ and related questions. *J. Number Theory*, 11(2):257–263, 1979.

[37] A. Erman. Briefwechsel zwischen W.Olbers und F.W.Bessel. *Avenarius u. Mendelssohn*, 1852.

[38] L. Faber and H. Kadiri. New bounds for $\psi(x)$. *Math. Comp.*, 84(293):1339–1357, 2015.

[39] S. Fei and J. Luo. A note the exponential Diophantine equation $(rlm^2+1)^x+(r(r-l)m^2-1)^y=(rm)^z$. *Bull. Braz. Math. Soc. (N.S.)*, 53(4):1499–1517, 2022.

[40] M Ferrari and L. Sillari. On the minimal number of solutions of the equation $\phi(n+k)=m\phi(n)$, $m=1,2$. *J. Integer Seq.*, 26(no. 1):10 pp, 2023.

[41] A. Fiori. Lower bounds for the least prime in Chebotarev. *Algebra Number Theory*, 13(9):2199–2203, 2019.

[42] A. Fiori, H. Kadiri, and J. Swidinsky. Sharper bounds for Chebyshev $\psi(x)$. *J. Math. Anal. Appl.*, 527(2):Paper No. 127426, 2023.

[43] A. Fiori, H. Kadiri, and J. Swidinsky. Sharper bounds for the error term in the prime number theorem. *Res. Number Theory*, 9(3):Paper No. 63, 2023.

[44] K. Ford. Solutions of $\phi(n)=\phi(n+k)$ and $\sigma(n)=\sigma(n+k)$. *Int. Math. Res. Not. IMRN*, 2022(5):3561—-3570, 2022.

[45] F.J. Francis and E.S. Lee. Additive representations of natural numbers. *Integers*, 22(A14):10 pp., 2022.

[46] R. Fu and H. Yang. On the exponential Diophantine equation $(am^2+1)^x+(bm^2-1)^y=(cm)^z$ with $c|m$. *Period. Math. Hung.*, 75:143–149, 2017.

[47] T. Goto. Upper bounds for unitary perfect numbers and unitary harmonic numbers. *Rocky Mountain J. Math.*, 37(5):1557–1576, 2007.

[48] J. Grossmann. über die Nullstellen der Riemannschen Zeta-Funktion und der Dirichletschen l-Funktionen. *PhD thesis, Georg-August-Universität Göttingen*, 1913.

[49] A. Grytczuk and M. Wóytowicz. On a Lehmer problem concerning Euler's totient function. *Proc. Japan Acad. Ser. A*, 79:136–138, 2003.

[50] E. Hasanalizade. On the equation $\phi(n+k) = 2\phi(n)$. *Integers*, 21(A68):5 pp., 2021.

[51] E. Hasanalizade. A note on the exponential Diophantine equation $(44m^2+1)^x + (5m^2-1)^y = (7m)^z$. *preprint*, 2023.

[52] E. Hasanalizade. On the equation $\sigma^*(n) = 1+mn$. *Integers*, 23(A31):6 pp., 2023.

[53] E. Hasanalizade. Sums of Fibonacci numbers close to a power of 2. *Bol. Soc. Mat. Mex. (3)*, 29(1):11 pp., 2023.

[54] E. Hasanalizade, Q. Shen, and P.-J. Wong. Counting zeros of Dedekind zeta functions. *Math. Comp.*, 91(333):277–293, 2021.

[55] E. Hasanalizade, Q. Shen, and P.-J. Wong. Counting zeros of the Riemann zeta function. *J. Number Theory*, 235:219–241, 2022.

[56] H. Hasse. Über die Dichte der Primzahlen p, für die eine vorgegebene ganzrationale Zahl $a \neq 0$ von geraden bxw. ungerader Ordnung mod p ist. *Math. Ann.*, 166:19–23, 1966.

[57] G. A. Hiary. An explicit van der Corput estimate for $\zeta(1/2+it)$. *Indag. Math. (N.S.)*, 27(2):524–533, 2016.

[58] J. Holt. The minimal number of solutions to $\phi(n) = \phi(n+k)$. *Math. Comp.*, 72(244):2059–2061, 2003.

[59] B. Hough. Solution of the minimum modulus problem for covering systems. *Ann. of Math. (2)*, 181(1):361–382, 2015.

[60] G. Jaeschke. On the smallest k such that all $k \cdot 2^n + 1$ are composite. *Math. Comp.*, 40(161):257–263, 1983.

[61] D. R. Johnston and A. Yang. Some explicit estimates for the error term in the prime number theorem. *J. Math. Anal. Appl.*, 527(2):23 pp., 2023.

[62] H. Kadiri. Explicit zero-free regions for Dedekind zeta functions. *Int. J. Number Theory*, 8(1):125–147, 2012.

[63] H. Kadiri and N. Ng. Explicit zero density theorems for Dedekind zeta functions. *J. Number Theory*, 132:748–775, 2012.

[64] H. Kadiri, N. Ng, and P.-J. Wong. The least prime ideal in the Chebotarev density theorem. *Proceedings of the American Mathematical Society*, 147:2289–2303, 2019.

[65] H. Kadiri and P.-J. Wong. Primes in the Chebotarev density theorem for all number fields (with an appendix by Andrew Fiori). *J. Number Theory*, 241:700–737, 2022.

[66] P. Kinlaw, M. Kobayashi, and C. Pomerance. On the equation $\phi(n) = \phi(n+1)$. *Acta Arith.*, 196(1):69–92, 2020.

[67] E. Kizildere and G. Soydan. Estimates of trigonometric sums and their applications. *Honam Math. J.*, 42(1):139–150, 2020.

[68] N. M. Korobov. Estimates of trigonometric sums and their applications. *Uspehi Mat. Nauk*, 13(4 (82)):185–192, 1958.

[69] J.C. Lagarias. The set of primes dividing the Lucas numbers has density $2/3$. *Pacific J. Math.*, 118:449–461, 1985.

[70] J.C. Lagarias, H.L. Montgomery, and A.M. Odlyzko. A bound for the least prime ideal in the Chebotarev density theorem. *Invent. Math.*, 54:271–296, 1979.

[71] J.C. Lagarias and A.M. Odlyzko. Effective versions of the Chebotarev density theorem. *Algebraic number fields: L-functions and Galois properties (Proc. Sympos., Univ. Durham, Durham, pp. 409–464. Academic Press, London*, 1977.

[72] M. Laurent. Linear forms in two logarithms and interpolation determinants *ii*. *Acta Arith.*, 133:325–348, 2008.

[73] E.S. Lee. On an explicit zero-free region for the Dedekind zeta-function. *J. Number Theory*, 224:307–322, 2021.

[74] D.H. Lehmer. On Euler's totient function. *Bull. Amer. Math, Soc.*, 38:745–751, 1932.

[75] The LMFDB Collaboration. The L-functions and modular forms database. 2021. [Online; accessed 28 January 2021].

[76] A. Lumley. Explicit results on primes. *MSc. Thesis, University of Lethbridge*, page 132 pp., 2014.

[77] K. Mahler. Zur Approximation algebraischer Zahlen I: Über den grössten Primteiler binärer Formen. *Math. Ann.*, 107:691–730, 1933.

[78] A. Makowski. On the equation $\phi(n+k) = 2\phi(n)$. *Elem. Math.*, 29:13, 1974.

[79] E.M. Matveev. An explicit lower bound for a homogeneous rational linear form in the logarithms of algebraic numbers, ii. *Izv. Ross. Akad. Nauk Ser. Mat.*, 64(6):125–180, 2000.

[80] J. Maynard. Dense clusters of primes in subsets. *Compos. Math.*, 152(7):1517–1554, 2016.

[81] K. S. McCurley. Explicit estimates for the error term in the prime number theorem for arithmetic progressions. *Math. Comp.*, 42(165):265–285, 1984.

[82] P. Moree. Diophantine equations of Erdős-Moser type. *Bull. Austral. Math. Soc.*, 53:281–292, 1996.

[83] M. J. Mossinghoff, T. Trudgian, and A. Yang. Explicit zero-free regions for the Riemann zeta function. *preprint*, 2022.

[84] R.W.K. Odoni. A conjecture of Krishnamurthy on decimal periods and some allied problems. *J. Number Theory*, 13:303–319, 1981.

[85] X.-W. Pan. A note on the exponential Diophantine equation $(am^2+1)^x+(bm^2-1)^y=(cm)^z$. *Colloq. Math.*, 149(2):265–273, 2017.

[86] B. K. Patel and B. P. Tripathy. Sums of three fibonacci numbers close to a power of 2. *Integers*, 23(A4):13 pp., 2023.

[87] D. Patel. An explicit upper bound for $|\zeta(1+it)|$. *Indag. Math.*, 33(5):1012–1032, 2022.

[88] D. J. Platt and T. S. Trudgian. An improved explicit bound on $|\zeta(\frac{1}{2}+it)|$. *J. Number Theory*, 147:842–851, 2015.

[89] D. J. Platt and T. S. Trudgian. On the first sign change of $\theta(x)-x$. *Math. Comp.*, 85(299):1539–1547, 2016.

[90] D. J. Platt and T. S. Trudgian. The Riemann hypothesis is true up to $3\cdot 10^{12}$. *Bull. London Math. Soc.*, 53(3):792—-797, 2021.

[91] C. Pomerance. On composite n for which $\phi(n)|n-1$, ii. *Pacific J. Math.*, 69(2):177–186, 1977.

[92] H. Rademacher. On the Phragmén-Lindelöf theorem and some applications. *Math. Z.*, 72:192–204, 1959/1960.

[93] M. Ram Murty, V. Kumar Murty, and N. Saradha. Modular forms and the Chebotarev density theorem. *Amer. J. Math.*, 110:253–281, 1988.

[94] O. Ramare and R. Rimely. Primes in arithmetic progressions. *Math. Comp.*, 65(213):397—-425, 1996.

[95] B. Riemann. Über die Anzahl der Primzahlen unter einer gegebener Größe. *Monatsber. Kgl. Preuß. Akad. Wiss. Berlin*, pages 671–680, 1860.

[96] G. Robin. Estimation de la fonction Tchebychef θ sur le k-ième nombre premier et grandes valeurs de la fonction $\omega(n)$ nombre de diviseurs premiers de n. *Acta Arith.*, 42:367–389, 1983.

[97] J.B. Rosser. The n-the prime is greater than $n\log n$. *Proc. London Math. Soc.*, 45:21–44, 1938.

[98] J.B. Rosser. Explicit bounds for some functions of prime numbers. *Amer. J. Math.*, 63:211–232, 1941.

[99] J.B. Rosser and L. Schoenfeld. Approximate formulas for some functions of prime numbers. *Illinois. J. Math.*, 6:64–94, 1962.

[100] J.B. Rosser and L. Schoenfeld. Sharper bounds for Chebyshev functions $\theta(x)$ and $\psi(x)$ i. *Math. Comp.*, 29:243–269, 1975.

[101] J.C. Saunders. Diophantine equations involving the Euler totient function. *J. Number Theory*, 209:347–358, 2020.

[102] A. Schinzel. Sur l'équation $\phi(x+k) = \phi(x)$. *Acta Arith.*, 4:181–184, 1958.

[103] A. Schinzel and A. Wakulicz. Sur l'équation $\phi(x+k) = \phi(x)$, ii. *Acta Arith.*, 5:425–426, 1959.

[104] J.-P. Serre. Quelques applications du theoreme de densite de Chebotarev. *Publ. Math. IHES*, 54:123–201, 1982.

[105] W. Sierpinski. Sur une propriété de la fonction $\phi(n)$. *Publ. Math. Devrecen.*, 4:184–185, 1956.

[106] H. Stark. Some effective cases of the Brauer-Siegel theorem. *Invent. Math.*, 23:135–152, 1974.

[107] J. Su and X. Li. The exponential Diophantine equation $(4m^2+1)^x + (3m^2-1)^y = (5m)^z$. *Abstract and Applied Analysis*, 2014(2):1–5, 2014.

[108] M.V. Subbarao. A companion to a Lehmer problem. *Publ. Math. Debrecen*, 52(3-4):683–698, 1998.

[109] N. Terai. The exponential Diophantine equation $(4m^2+1)^x + (3m^2-1)^y = (5m)^z$. *Int. J. Algebra*, 6:1135—-1146, 2012.

[110] N. Terai. On the exponential Diophantine equation $a^x + b^y = c^z$. *Proceedings of the 11th Fukuoka Symposium on Number Theory*, pages 117–127, 2017.

[111] N. Terai. On the exponential Diophantine equation $(4m^2+1)^x + (21m^2-1)^y = (5m)^z$. *Annales Mathematicae et Informaticae*, 52:243—-253, 2020.

[112] N. Terai and T. Hibino. On the exponential Diophantine equation $(12m^2+1)^x + (13m^2-1)^y = (5m)^z$. *Int. J. Algebra*, 9(6):261–272, 2015.

[113] N. Terai and T. Hibino. On the exponential Diophantine equation $(3pm^2+1)^x + (p(p-3)m^2-1)^y = (pm)^z$. *Period. Math. Hungarica*, 74(2):227–234, 2017.

[114] N. Terai and Y. Shinsho. On the exponential Diophantine equation $(3m^2+1)^x + (qm^2-1)^y = (rm)^z$. *SUT J. Math.*, 56(2):147–158, 2020.

[115] N. Terai and Y. Shinsho. On the exponential Diophantine equation $(4m^2+1)^x + (45m^2-1)^y = (7m)^z$. *Int. J. Algebra*, 15(4):233–241, 2021.

[116] J. Thorner and A. Zaman. An explicit bound for the least prime ideal in the Chebotarev density theorem. *Algebra Number Theory*, 11:1135–1197, 2017.

[117] T. S. Trudgian. An improved upper bound for the argument of the Riemann zeta-function on the critical line. *Math. Comp.*, 81(278):1053–1061, 2012.

[118] T. S. Trudgian. A new upper bound for $|\zeta(1+it)|$. *Bull. Aust. Math. Soc.*, 89(2):259–264, 2014.

[119] T.S. Trudgian. An improved upper bound for the argument of the Riemann zeta-function on the critical line II. *J. Number Theory*, 134:280–292, 2014.

[120] T.S. Trudgian. An improved upper bound for the error in the zero-counting formulae for Dirichlet l-functions and Dedekind zeta-functions. *Math. Comp.*, 84(293):1439–1450, 2015.

[121] T.S. Trudgian. Updating the error term in the prime number theorem. *Ramanujan J.*, 39(2):225–234, 2016.

[122] I.M. Vinogradov. A new estimate of the function $\zeta(1+it)$. *Izv. Akad. Nauk SSSR. Ser. Mat.*, 22:161–164, 1958.

[123] H.C.F. von Mangoldt. Zur Verteilung der Nullstellen der Riemannschen Funktion $\xi(t)$. *Math. Ann.*, 60(1):1–19, 1905.

[124] D. Zagier. The first 50 million prime numbers. *The Mathematical Intelligencer*, 1:7–19, 1977.

[125] A. Zaman. Bounding the least prime ideal in the Chebotarev density theorem. *Funct. Approx. Comment. Math.*, 57(1):115–142, 2017.