CONFORMAL FIELD THEORY AND BLACK HOLE PHYSICS

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Dedication

To my parents, my sister, and Paige R. Ryan.

Abstract

This thesis reviews the use of 2-dimensional conformal field theory applied to gravity, specifically calculating Bekenstein-Hawking entropy of black holes in (2+1) dimensions. A brief review of general relativity, Conformal Field Theory, energy extraction from black holes, and black hole thermodynamics will be given. The Cardy formula, which calculates the entropy of a black hole from the AdS/CFT duality, will be shown to calculate the correct Bekenstein-Hawking entropy of the static and rotating BTZ black holes. The first law of black hole thermodynamics of the static, rotating, and charged-rotating BTZ black holes will be verified.

Acknowledgements

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Chapter 1

Introduction

The holy grail of physics is a grand unified theory, or "the theory of everything". This refers to unifying the four fundamental forces of nature (gravity, the strong nuclear force, the weak nuclear force, and the electromagnetic force) in a theory that is consistent with quantum mechanics. General relativity is a classical theory of gravity that describes interactions of very massive objects, whereas quantum mechanics is a theory that describes interactions of very small packets of energy, or particles. The strong, weak, and electromagnetic forces have been unified in one consistent quantum field theory known as "The Standard Model". However, the standard model is not a grand unification theory because it excludes a description of gravity at very small scales, a theory of quantum gravity.

Black holes are classical solutions of general relativity (Einstein equations), resembling strongly gravitating objects, with so-called event horizons, which prevent matter and radiation entering it from coming out at a later time. Black holes have been observed in nature at the center of galaxies and elsewhere. They behave as one-way membranes, and can hide an indefinite amount of information. However quantum mechanics predicts that black holes can radiate small amounts of energy in the form of blackbody radiation (Hawking radiation) [1] and that they also possess a large amount of entropy (Bekenstein-Hawking entropy) [2]. These pose two problems: first, if a black hole evaporates altogether, the complete destruction of the above information seems to be incompatible with the unitary nature of quantum mechanical evolution (which forbids any information loss). This is known as the black hole information loss problem. Second, what could be the origin of its entropy, in the sense of statistical mechanics? It has been recognized that the two problems may be related, as an understanding of the degrees of freedom responsible for its entropy may enable one to study their dynamics and evolution to determine the exact nature of information loss (if any). Black holes are then ideal theoretical laboratories to test predictions made from possible theories of quantum gravity.

The aim of this thesis is to show that the origin of the degrees of freedom could include conformal invariance. Conformal field theory (CFT) refers to the quantum description of an infinite number of degrees of freedom represented as fields in two dimensions (normally one space and one time), with conformally invariant physics. Conformal symmetry is invariance under any transformation that preserves shape locally, including scale invariance.

Work on conformal field theory developed rapidly during the 1980s and 1990s, after it was realized that conformal invariance restricts the physical behavior of these systems enormously and renders the corresponding mathematical treatment significantly simpler [3–5]. Essentially, the infinite conformal invariance of a theory in 2-dimensions can reduce the infinite degrees of freedom of CFT to a finite number, so that the theories can be solved. Consequently, CFTs have enormous predictive power and contribute to the understanding of these systems. 2-dimensional CFT has been used to study the 2-dimensional Ising model, 2nd-order phase transitions in 2-dimensional systems, and in string theory. Furthermore, their solution is also non-perturbative, and so it is hoped that it may help us toward a non-perturbative solution of other physical field theories with strong coupling (there are many, such as that describing the strong force, quantum chromo-dynamics).

While superstring theory [6], loop quantum gravity [7], etc. have attempted a fundamental understanding of quantum black holes with varying degrees of success, there has been another attempt in the recent past. After recognizing that the degrees of freedom near the horizon may play pivotal roles, and that the dynamics of the latter have an effective scale invariance (due to infinite red-shifts of associated physical quantities), CFT techniques have been applied to understand the microscopic origin of black hole entropy. Indeed, for a class of black holes, it has been demonstrated that the logarithm of degeneracy arising in CFT can account for the entropy [8]. The advantage of this approach lies in its generality (it is not tied to any specific theory of quantum gravity) and related robustness of calculations coming from exact contour integrals on the complex plane.

The idea of conformal invariance playing a role in black hole physics started with the work of Brown and Henneaux [9]. They showed that the asymptotic symmetry group of AdS_3 spacetime, a 3-dimensional vacuum spacetime which is curved by a cosmological constant, is the 2-dimensional conformal group. This idea was extended by Maldacena [10] to conjecture the AdS/CFT correspondence. The ADS/CFT correspondence states that gravity in ADS is dual to a CFT on the boundary.

Brown and Henneaux [9] also showed that the classical canonical realization of this symmetry is given by the Possion bracket algebra of the generators, or the Dirac bracket algebra of the charges, with a central extension given by what is commonly referred to as the Brown and Henneaux central charge:

$$c = \frac{3\ell}{2G}, \tag{1.0.1}$$

where ℓ is the radius of curvature of the AdS spacetime. After quantization, this Poisson algebra becomes the Virasoro algebra with a central extension proportional to the Brown and Henneaux central charge. The Virasoro algebra is the defining algebra of a 2-dimensional quantum conformal field theory.

Cardy [11, 12] used techniques in 2-dimensional conformal field theory to derive the Cardy formula for the asymptotic density of states:

$$\ln \rho(\Delta, \bar{\Delta}) = 2\pi\sqrt{\frac{c\Delta}{6}} + 2\pi\sqrt{\frac{\bar{c}\bar{\Delta}}{6}}, \tag{1.0.2}$$

where ρ is the density of states, $c, \bar{c}$ are the central charges and $\Delta, \bar{\Delta}$ is the energy gap. The Cardy formula gives the entropy of a black hole calculated from the central charge in a conformal field theory. Strominger [13] used the Brown and Henneaux central charge with the Cardy formula to show that black hole entropy obtained agrees with that of the Bekenstein-Hawking entropy of the BTZ black hole, a black hole that reduces to a AdS_3 spacetime far from the origin [14, 15]. Thus, the known Bekenstein-Hawking entropy of the BTZ black hole is recalculated with the assumption of conformal invariance.

This thesis will review this process and calculate the Bekenstein-Hawking entropy for a static BTZ, a rotating BTZ, and a charged rotating BTZ black hole.

Chapter 2 will give a short introduction to general relativity. Einstein's field equations will be defined as well as the spherically symmetric vacuum solution, the Schwarzschild solution. Charged, rotating, and charged-rotating or general black hole solutions will be examined. The ADS solution will also be examined.

In Chapter 3 will present various properties of black holes, mainly the possibility of extracting energy from a black hole, Hawking radiation and the laws of black hole thermodynamics. Formulas for the Hawking temperature and Bekenstein-Hawking entropy will be derived so they can be used later to analyze the BTZ black holes.

Chapter 4 will give a brief introduction to 2-dimensional conformal field theory. First, conformal field theory in any dimension will be defined, and then the very special case of a CFT in 2-dimensions will be reviewed in detail. Conformal generators and their corresponding charges will be defined as well as the operator product expansion, which is used in 2-dimensional CFT to compute correlation functions. The operator product expansion will be used to determine the quantum generators of global conformal symmetries which will be used to derive the Virasoro algebra of a 2-dimensional CFT. A 2-dimensional CFT on a torus parametized by a single parameter, the modular paramter, will also be studied. Modular invariance is a property which refers to tori of different modular parameters being equivalent. By defining a partition function on the torus and using modular invariance, the Cardy formula will be derived explicitly.

The Hamiltonian formulation of general relativity will be covered in Chapter 5. General relativity will be formulated in the Arnowitt, Desser, Misner (ADM) [16], (3 space, 1 time)-from and the ADM-Hamiltonian will be derived. The surface term, which is often left out in most treatments of the ADM-Hamiltonian, will be talked about and given explicitly in the equation of the Hamiltonian generator.

Chapter 6 will cover the conserved charges in general relativity, which will become the eigenvalues for the Virasoro generators in in AdS_3/CFT_2. It will be shown that the surface terms in the ADM-Hamiltonian become the conserved charges in general relativity. The Poisson bracket algebra of the charges will be shown to be isomorphic to the Lie bracket algebra of the symmetry generators. The general algebra with an added central extension, proportional to a central charge c, will also be discussed. It will be shown that the asymptotic symmetry of the asymptotically AdS_3 spacetime is the conformal group in 2-dimensions. The conserved charges of asymptotic AdS_3 spacetime will be calculated as well as the central charge, which turns out to be the Brown-Henneaux central charge.

In Chapter 7 the Hawking temperature and Bekenstein-Hawking entropy for the non-rotating, rotating, and charged rotating BTZ black holes will be calculated. The first law of black hole thermodynamics will also be verified for each case.

Finally, the Conclusion chapter will summarize the major points of this thesis.

Chapter 2

Einstein's field equations and black hole solutions

In this chapter Einstein's field equations will be examined. There will be a brief introduction to general relativity and the essential components of the field equations will all be defined mathematically. Different solutions of the field equations will be examined, vacuum solutions and various black hole solutions. The main resources for this chapter are "Introducing Einstein's Relativity" by Ray D'Invenro [17] and "Spacetime and Geometry - An Introduction to General Relativity" by Sean Carroll [18].

2.1 Conventions and notations

In this thesis Latin indices i,j,k or a,b,c and so on will run over the spatial coordinates (1,2,3 for 4-dimensions or 1,2 for 3-dimensions). Greek indices μ, ν and so on will run over all space-time coordinates with x^0 being the time coordinate. This thesis will employ the +2 signature: for a 4-dimensional Minkowski spacetime $\eta_{11} = \eta_{22} = \eta_{33} = 1$ and $\eta_{00} = -1$. [19]

2.2 Einstein's field equations

Of all of Albert Einstein's great accomplishments, his greatest might be his field equations of general relativity:

$$G_{\mu\nu} + \mathit{\Lambda} g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{2.2.1}$$

Here $g_{\mu\nu}$ is the metric tensor, $G_{\mu\nu}$ is the Einstein tensor ($G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$, with R being the Ricci scaler: $R = g^{\mu\nu} R_{\mu\nu}$), $\mathit{\Lambda}$ is the cosmological constant and $T_{\mu\nu}$ is the energy-momentum tensor. The metric tensor contains all information about the geometry of spacetime and is defined by the line-element squared,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{2.2.2}$$

for the coordinates x^μ. Einstein's most famous equation, $E = mc^2$, states that mass and energy, which make up matter, are different manifestations of the same thing, and that mass can be converted into energy and energy into mass. The energy-momentum tensor contains all information of the distribution of matter and can be determined by differentiating the action, S with respect to the metric,

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}}. \tag{2.2.3}$$

Einstein's field equations relate mass and energy to the curvature of space time. Einstein's field equations are second-order partial differential equations. Solutions to this equation are, most often, very difficult to come by and represent different possible spacetimes.

The Riemann-Christoffel tensor, Riemann tensor, or simply the curvature tensor, is defined as

$$R^\mu{}_{\nu\alpha\beta} = \partial_\alpha \mathit{\Gamma}^\mu{}_{\nu\beta} - \partial_\beta \mathit{\Gamma}^\mu{}_{\nu\alpha} + \mathit{\Gamma}^\mu{}_{\sigma\alpha} \mathit{\Gamma}^\sigma{}_{\nu\beta} - \mathit{\Gamma}^\mu{}_{\sigma\beta} \mathit{\Gamma}^\sigma{}_{\nu\alpha}, \tag{2.2.4}$$

where $\Gamma^{\alpha}{}_{\beta\gamma}$ is the affine connection given by [20]

$$\Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\beta} (\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}). \tag{2.2.5}$$

The affine connection is used to define the co-variant derivative of any tensor:

$$\nabla_\alpha X^{\mu\cdots}_{\nu\cdots} = \partial_\alpha X^{\mu\cdots}_{\nu\cdots} + \Gamma^{\mu}_{\beta\alpha} X^{\beta\cdots}_{\nu\cdots} + \cdots - \Gamma^{\beta}_{\nu\alpha} X^{\mu\cdots}_{\beta\cdots} - \cdots . \tag{2.2.6}$$

The Ricci tensor, $R_{\mu\nu}$, is then defined by contractions of the curvature tensor with the metric:

$$R_{\mu\nu} = g^{\alpha\beta} R^{\mu}{}_{\nu\alpha\beta}, \tag{2.2.7}$$

and the Ricci scalar R can be defined by contracting the Ricci tenor with the metric:

$$R = g^{\mu\nu} R_{\mu\nu}. \tag{2.2.8}$$

The affine connection is a geometrical object that connects nearby tangent spaces. It ensures that the co-variant derivative, a derivative along tangent vectors of a manifold, of a tensor remains a tensor. A necessary and sufficient condition for a manifold to be flat is that the Riemann tensor vanishes [17]. Therefore, since the affine connection is made up of derivatives of the metric, a metric with a vanishing affine connection would imply a vanishing curvature tensor, which would imply flat space: the Minkowski metric is an example.

The problem of finding a solution can be simplified be seeking a vacuum solution, that is $T_{\mu\nu} = 0$. The problem can be simplified further by assuming a zero cosmological constant, $\Lambda = 0$. This admits the flat space family of solutions, the Minkowski metric and it's diffeomorphisms. In 4-dimensional Cartesian coordinates the metric reads

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 \tag{2.2.9}$$

A non-zero cosmological constant allows for curvature in a spacetime in the absence of any matter distributions (dust, fluid, ... etc.) or fields. Even with $T_{\mu\nu} = 0$ (zero matter or fields), the cosmological constant adds energy to the universe. The cosmological constant comes from vacuum energy, which can be thought as coming from of as spontaneous emission and annihilation of virtual particles of the vacuum. The de Sitter (dS) and Anti de Sitter (AdS) metrics are both solutions to Einstein's field equations in vacuum with a non-zero cosmological constant Λ. de Sitter space corresponds to a positive Λ and Anti de Sitter space corresponds to a negative Λ. The AdS spacetime in 3-dimensions will be examined in more detail later.

Geodesics can be thought of as the straightest line paths in curved spacetimes. They describe the motion of inertial test particles [18]. These paths can be determined by solving the geodesic equation

$$\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha{}_{\mu\gamma}\frac{dx^\mu}{d\lambda}\frac{dx^\gamma}{d\lambda} = 0 \tag{2.2.10}$$

for the coordinates of the geodesic $x^\alpha(\lambda)$ [20] parameterized by the affine parameter, λ. Specific geodesics of the sun describe the paths of travel of planets in the solar system.

2.3 Schwarzschild solution and black holes

The metric of the spacetime around the sun, or any spherically symmetric gravitational object, can be approximated by the Schwarzschild solution, quite possibly the most well known and most studied solution to Einstein's vacuum field equations (with zero cosmological constant). In spherical coordinates (t, r, θ, ϕ) it is given by

$$ds^2 = -\left(1 - \frac{2GM}{c^2r}\right)c^2dt^2 + \frac{1}{(1 - \frac{2GM}{c^2r})}dr^2 + r^2(d\phi^2 + \sin^2\phi\, d\phi^2) \tag{2.3.1}$$

The Schwarzschild solution describes the gravitational field outside a spherically symmetric, uncharged, non-rotating mass, M. Because of these conditions, the Schwarzschild solution also describes the most basic of black holes, the Schwarzschild black hole or the static black hole. The Schwarzschild black hole is uncharged and unevolving, that is the metric is independent of time. By examining the metric one can see that the metric coefficients behave strangely at $r = 0$ and $r_s = \frac{2GM}{c^2}$; $g_{tt} = 0$ and g_{rr} becomes infinite. The second is known as the Schwarzschild radius, r_s, and is the 2-dimensional surface known as the event horizon for a Schwarzschild solution (Schwarzschild black hole). Since the Schwarzschild solution is a vacuum solution, for most massive bodies such as the sun or other stars, the interior region, $r \leq r_s$, is not a concern because these bodies extend past r_s. However, this radius does disconnect two coordinate patches of the spacetime, the interior and exterior regions.

Radial null geodesics are paths in spacetime along which massless particles, such as photons, travel. The equations of radial null geodiscs are given by the Euler-Lagrange equations [17]

$$\frac{\partial K}{\partial x^\mu} - \frac{d}{du}\left(\frac{\partial K}{\partial \dot{x}^\mu}\right) = 0, \tag{2.3.2}$$

where u parameterizes the geodesic, the dot represents differentiation with respect to u and

$$K \equiv g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 0. \tag{2.3.3}$$

Consider the class of radial null geodesics of the Schwarzschild spacetime defined by

$$ds^2 = \dot{\theta} = \dot{\phi} = 0, \tag{2.3.4}$$

The variation of the Schwarzchild metric then becomes

$$K = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 \dot{t}^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} \dot{r}^2 = 0 \tag{2.3.5}$$

The Euler-Lagrange equation corresponding to $\mu = 0$ is

$$\frac{d}{du}\left[\left(1 - \frac{2GM}{c^2 r}\right) c^2 \dot{t}\right] = 0, \tag{2.3.6}$$

which after integrating gives

$$\left(1 - \frac{2GM}{c^2 r}\right) c^2 \dot{t} = k, \tag{2.3.7}$$

where k is a constant. Substituting k into (2.3.5) leads to $\dot{r}^2 = k^2/c^2$, or

$$\dot{r} = \pm k/c. \tag{2.3.8}$$

Then the equations of the curves can be found by integrating the equation

$$\frac{dt}{dr} = \frac{dt/du}{dr/du} = \frac{\dot{t}}{\dot{r}}, \tag{2.3.9}$$

using (2.3.7) and (2.3.8). Positive k/c leads to

$$\frac{dt}{dr} = \frac{r}{r - 2GM/c^2}, \tag{2.3.10}$$

which can be integrated to give the ingoing null geodesics [17]

$$ct = -\left(r + \frac{2GM}{c^2} \ln|r - \frac{2GM}{c^2}| + constant\right) \tag{2.3.11}$$

Similarily, a negative k/c leads to the outgoing radial null geodesics are given by

$$ct = r + \frac{2GM}{c^2} \ln|r - \frac{2GM}{c^2}| + constant. \tag{2.3.12}$$

Mapping both geodesics shows the Schwarzschild solution (with θ and ϕ coordinates suppressed) gives Figure 2.1.

It turns out that this singularity is actually a coordinate singularity, since it results from a bad choice of coordinates. By choosing a proper coordinate transformation, the metric can be made regular at $r = r_s$.

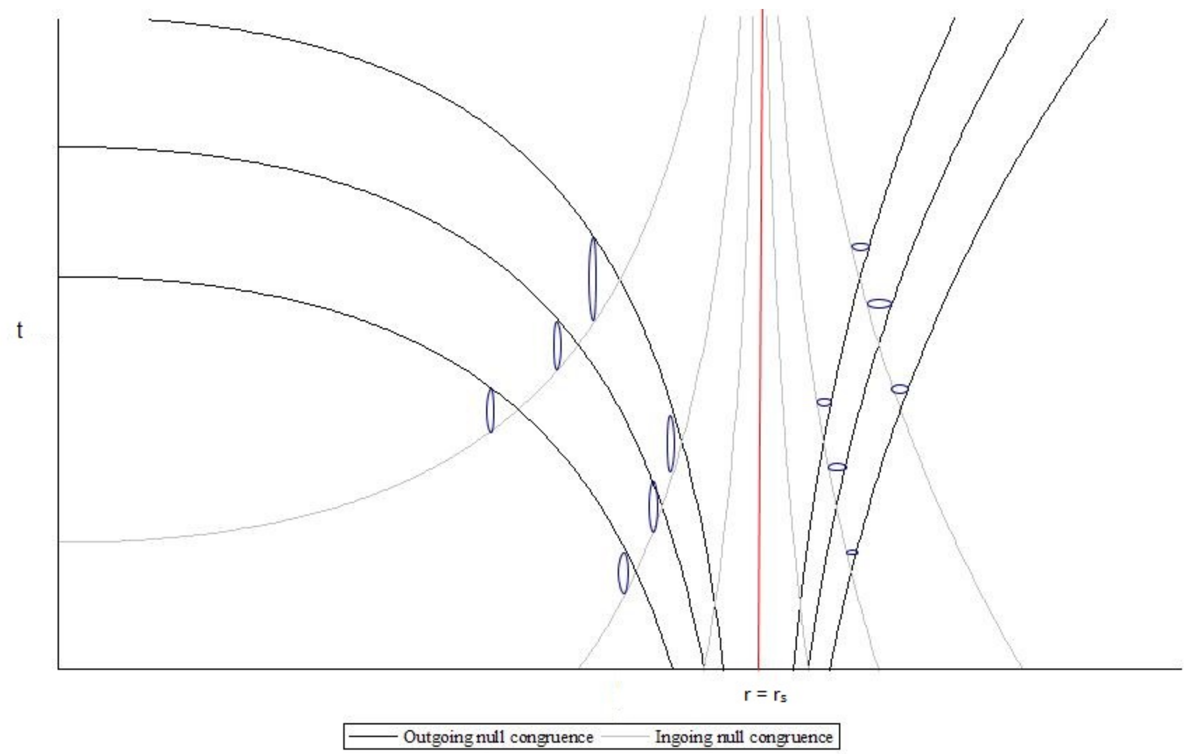

Figure 2.1: The Schwarzschild solution in 2-dimensions (θ and ϕ coordinates are supressed). The lightcones tip inside the event horizon, and thus events which originate inside the event horizon can not be observed by an external observer.

In the interior region the coordinates t and r reverse their character. That is t becomes spacelike and r becomes timelike [17, 18].

By using the Eddington-Finkelstein coordinate transformation

$$t \to \bar{t} = c^2 t + \frac{2GM}{c^2} \ln(r - \frac{2GM}{c^2}) \quad (2.3.13)$$

the metric becomes

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) d\bar{t}^2 + \frac{4GM}{c^2 r} d\bar{t}dr + \frac{1}{(1 - \frac{2GM}{c^2 r})} dr^2 + r^2(d\phi^2 + \sin^2\phi d\phi^2) \quad (2.3.14)$$

The Eddington-Finkelstein metric can also be written in terms of the advanced time parameter using the transformation $v = \bar{t} + r$

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) dv^2 - 2dvdr - r^2(d\theta^2 + \sin^2\phi d\phi^2) \quad (2.3.15)$$

What results is a "tipping" of light-cones in the interior region, as can be seen from the intersection of the null congruences. The light-cones of all particles in the interior region all point towards the $r = 0$ singularity as shown in Figure 2.2.

Light-cones just outside the event horizon are slightly tipped towards the $r = r_s$ singularity [17] (See Fig (2.1). Therefore, particles at or inside the event horizon can not exit. To do so they would have to travel faster than the speed of light, and no such particles have ever been observed. Even light can not escape, hence the name "black hole".

2.4 Charged black holes

Hans Reissner and Gunnar Nordström found a solution to Einstein's field equations, with zero cosmological constant, coupled to electric and magnetic fields and an energy

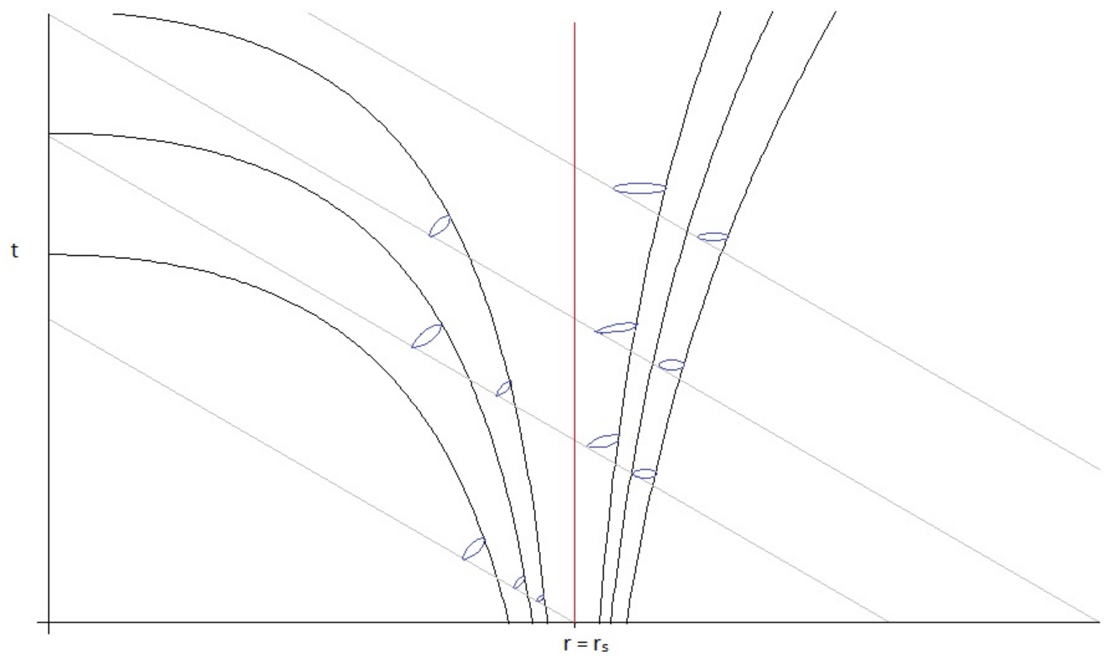

Figure 2.2: The Schwarzchild solution in advanced Eddington-Finkelstein coordinates. Again, the lightcones are tipped inside the event horizon. The darker curved lines are the outgoing null congruences and the lighter straight lines are the ingoing null congruences.

distribution given by the Maxwell energy-momentum tensor. The Maxwell energy-momentum tensor is given by

$$T^{\mu\nu} = \frac{1}{4\pi}[F^{\mu\alpha}F^{\nu}{}_{\alpha} - \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}] \tag{2.4.1}$$

where $F_{\mu\nu}$ is the Maxwell tensor,

$$F^{\mu\nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{bmatrix}. \tag{2.4.2}$$

The Maxwell energy-momentum tensor can be written explicitly in terms of the electric and magnetic fields,

$$T_{\mu\nu} = \begin{bmatrix} \frac{1}{8\pi}(E^2+B^2) & S_x/c & S_y/c & S_z/c \\ S_x/c & -\sigma_{xx} & -\sigma_{xy} & -\sigma_{xz} \\ S_y/c & -\sigma_{yx} & -\sigma_{yy} & -\sigma_{yz} \\ S_z/c & -\sigma_{zx} & -\sigma_{zy} & -\sigma_{zz} \end{bmatrix}, \tag{2.4.3}$$

where $\vec{S}$ is the Poynting vector, which gives the energy flux density of an electromagnetic field, given by

$$\vec{S} = \frac{c}{4\pi}\vec{E} \times \vec{B}, \tag{2.4.4}$$

and

$$\sigma_{ij} = \frac{1}{4\pi}\left(E_iE_j + B_iB_j\right) - \frac{1}{8\pi}\left(E^2 + B^2\right)\delta_{ij}. \tag{2.4.5}$$

The Reissner-Nordström solution is

$$ds^2 = -\left(1 - \frac{2GM}{c^2r} + \frac{\varepsilon_Q^2}{r^2}\right)c^2dt^2 + \frac{1}{(1 - \frac{2GM}{c^2r} + \frac{\varepsilon_Q^2}{r^2})}dr^2 + r^2(d\phi^2 + \sin^2\phi d\phi^2) \tag{2.4.6}$$

where $\varepsilon_Q^2 = \frac{G(Q^2+P^2)}{4\pi\epsilon_0 c^4}$ and Q is the electric charge of the mass and P is the total magnetic charge [17, 18]. Since magnetic monopoles have not been discovered in nature, P, is usually taken to be zero. However, some theoretical models predict the existence of magnetic monopoles and it is also possible that a black hole has a total magnetic charge. This becomes the solution to a spherically symmetric, non-rotating, electrically charged black hole. It is clear to see that there are event horizons at $g^{rr} = 0$, solving for r,

$$r\pm = \frac{GM}{c^2} \pm \sqrt{\frac{G^2M^2}{c^4} - \frac{G(Q^2+P^2)}{4\pi\varepsilon_0 c^4}} \tag{2.4.7}$$

The metric can be analyzed for the three regions:

I. $GM^2 < Q^2 + P^2$: The metric is regular in the coordinates (t, r, θ, ϕ) and t is timelike and r is spacelike; the light-cones are tipped in the positive time direction.

II. $GM^2 > Q^2 + P^2$: The metric has coordinate singularities at r_+ and r_-. At the $r = r_+$ surface t and r change character, that is t is spacelike and r is timelike. The light-cones are tipped towards a decreasing radius. At $r = r_-$ t and r change character again such that t is again timelike and r is spacelike but with the orientation reversed. Upon reaching the $r = r_-$ surface, the light-cones tip back towards $r = r+$, an increasing radius.

III. $GM^2 = Q^2 + P^2$: Known as the "extreme" Reissner-Nordström solution, there is a double event horizon at $r = GM$, that is both horizons coincide at this point. The coordinate r is null-like at $r = GM$ and spacelike in the regions on other side. The coordinate t is timelike on either side of the event horizon, and null like at $r = GM$. The light-cones are tipped towards the positive time direction on either sides of the event horizon. [17, 18]

2.5 Rotating black holes

The rotating black hole solution is a solution to Einstein's vacuum field equations for a massive spinning source. It is given by the Kerr metric:

$$ds^2 = -\left(1 - \frac{2GMr}{c^2\rho^2}\right)c^2dt^2 - \frac{2GMar\sin^2\theta}{c^2\rho^2}(dtd\phi + d\phi dt)$$
$$+\frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}[(r^2+a^2)^2 - a^2\Delta\sin^2\theta]d\phi^2 \tag{2.5.1}$$

where

$$\Delta(r) = r^2 - \frac{2GMr}{c^2} + a^2 \tag{2.5.2}$$

and

$$\rho^2(r,\theta) = r^2 + a^2\cos^2(\theta) \tag{2.5.3}$$

M is the mass of the black hole and a is the angular momentum per unit mass,

$$a = J/M \tag{2.5.4}$$

for the angular momentum, J. With these coordinates it is clear to see that there is a singularity when $g^{rr} = 0$, since $\rho^2 \geq 0$. Setting $\Delta(r) = 0$ and solving for r gives two event horizons for the Kerr black hole at

$$r_\pm = \frac{GM}{c^2} \pm \sqrt{\frac{G^2M^2}{c^4} - a^2} \tag{2.5.5}$$

As a reduces to zero, the metric reduces to the Schwarzschild metric and the two singularities reduce to the $r = 0$ and $r = r_s$ Schwarzschild singularities. At both event horizons (at $r = r_+$ and $r = r_-$) the coordinates t and r are null-like. There

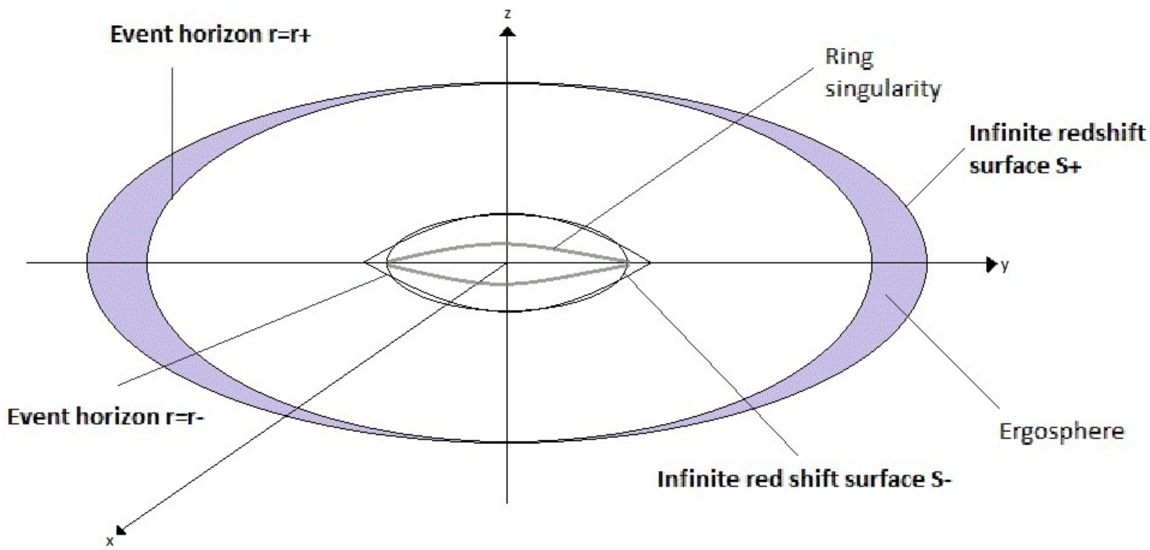

Figure 2.3: The event horizons, ring singularity, Ergosphere and surfaces of infinite redshift for the Kerr black hole.

is a region outside the $r = r_+$ event horizon known as the ergo-sphere. The ergo-sphere is the region of spacetime where particles are free to enter and leave but can not remain stationary. Instead they must move with the direction of rotation of the black hole. The true curvature singularity occurs at $\rho^2(r, \theta) = 0$. For this to occur $r = 0$ and $\theta = \frac{\pi}{2}$ must both be true. In four dimensions $r = 0$ actually sweeps out a disc and with $\theta = \frac{\pi}{2}$ it actually represents a ring. The rotation has "softened" the Schwarzschild singularity, spreading it out over a ring [18]. This is illustrated in Figure 2.3.

2.6 General black holes

A general black hole solution is that of the rotating, charged black hole. It is given by the Kerr-Newman solution, in advanced Eddington-Finkelstein coordinates it is

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2GMr}{c^2\rho^2} + \frac{\varepsilon_0^2}{\rho^2}\right) dv^2 + 2dvdr - \frac{2a}{\rho^2}\left(\frac{2GMr}{c^2} - \varepsilon_0^2\right)\sin^2\theta dv d\bar{\phi} \\ &- 2a\sin^2\theta dr d\bar{\phi} + \rho^2 d\bar{\phi}^2 \\ &+ [(r^2+a^2)^2 - \left(r^2 - \frac{2GMr}{c^2} + a^2 + \varepsilon_0^2\right) a^2 \sin^2\theta] \frac{\sin^2\theta}{\rho^2} d\bar{\phi}^2 \end{aligned} \tag{2.6.1}$$

where $d\bar{\phi} = d\phi + \frac{a}{\Delta}dr$.

2.7 The Anti de Sitter metric

The AdS spacetime can be thought of as a cavity where photons can travel to and reflect off the boundary and come back in a finite time (see Figure 2.4). AdS space is a vacuum solution of Einstein's field equations, with a negative cosmological constant, given by

$$G_{\mu\nu} = -\Lambda g_{\mu\nu}, \tag{2.7.1}$$

where $\Lambda = -d(d-1)/2\ell^2$, d is the dimension of spacetime and ℓ is the radius of curvature of the AdS spacetime. The curvature of spacetime results from the energy of the cosmological constant. Because of these properties, it is a well studied metric by physicists in the field of cosmology and quantum gravity. It has become of great interest to the field of quantum gravity because of the AdS/CFT correspondence, and its applications to string theory. In 3 dimensions the AdS_3 metric in polar coordinates reads,

$$ds^2 = -\left(1 + \frac{r^2}{\ell^2}\right) c^2 dt^2 + \frac{1}{(1+\frac{r^2}{\ell^2})} dr^2 + r^2 d\phi^2. \tag{2.7.2}$$

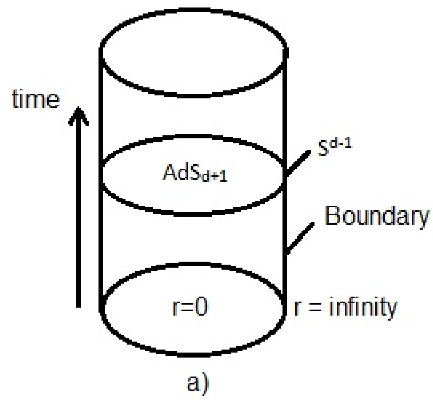

Figure 2.4: a) Penrose diagram for AdS space. b) Massive geodesic is represented by the darker line and the massless geodesic is represented by the lighter line [21].

It is a vacuum solution to Einstein's field equations in three dimensions. The BTZ metric describes a three dimensional black hole ,

$$ds^2 = -\left(-m + \frac{r^2}{\ell^2}\right)c^2dt^2 + \frac{1}{(-m + \frac{r^2}{\ell^2})}dr^2 + r^2d\phi^2 \tag{2.7.3}$$

where $m = \frac{G_3M}{c^2}$ and M denotes the geometrical mass of the spacetime and $G_3 \equiv G$ denotes Newton's constant in three dimensions. The BTZ metric reduces to the AdS$_3$ metric for large r, and as $m \to -1$. This study of the relationship between the BTZ and AdS$_3$ metrics will be of great interest in this thesis. There are also other solutions of AdS black holes. The metrics are given by

$$ds^2 = -f(r)c^2dt^2 + \frac{dr^2}{f(r)} + r^2d\Omega^2 \tag{2.7.4}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ in 4 dimensions and $d\Omega^2 = d\phi^2$ in 3 dimensions. For a Schwarzschild-AdS solution, a solution of (2.7.1),

$$f(r) = 1 - \frac{2GM}{c^2 r} - \frac{\Lambda}{3} r^2. \tag{2.7.5}$$

The Reissner-Nordström-AdS solution is a solution of Einstein's field equations, in the presence of an electromagnetic field,

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} + \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{2.7.6}$$

where $T_{\mu\nu}$ is given by (2.4.1). The Reissner-Nordström-AdS solution is given by (2.7.4) with

$$f(r) = 1 - \frac{2GM}{c^2 r} + \frac{\varepsilon_Q^2}{r^2} - \frac{\Lambda}{3} r^2. \tag{2.7.7}$$

These are some of Einstein's field equations and their solutions of general relativity. With a brief introduction to black hole solutions and how black holes manifest in spacetimes, physical properties of black holes (such as entropy, temperature, radiation, and evaporation) can now be examined.

Chapter 3

Aspects of black holes thermodynamics

After discussing the notion of black hole solutions of Einstein's equations and different black hole solutions, it is natural to discuss the different aspects of black holes. This chapter will give a brief review of energy extraction from black holes (specifically the Penrose process for rotating and charged black holes and superradiance of black holes). It will also give a brief review of black hole thermodynamics and Hawking radiation (a process in which it is proposed that black holes may evaporate). The main references used in this section are [22–25].

3.1 Area theorem

Hawking showed, under general assumptions, that the event horizon surface area of a black hole can never decrease [26]. Consider a null geodesic congruence (the focusing of a bundle of light rays) coming from one side of a spacelike 2-surface. Let the convergence ρ of the congruence be defined as a rate of change of an infinitesimal cross-sectional area $\delta A : \rho = \frac{d}{d\lambda} \ln \delta A$, where λ parametrizes the null geodesics. From this one can get the equation [27]

$$\frac{d}{d\lambda}\rho = \frac{1}{2}\rho^2 + \sigma^2 + R_{\mu\nu}k^\mu k^\nu, \tag{3.1.1}$$

where σ^2 is the square of the shear tensor of the congruence, and k^μ is the tangent vector to the geodesics. This equation is known as the focusing equation, often called the Raychaudhuri equation, or Sach's equation, or Newman-Penrose equation, which relates the null geodesic congruence to the Ricci tensor. The focusing equation shows that an initially converging congruence must reach a "crossing point", where ρ begins to diverge, in a finite λ provided by $R_{\mu\nu}k^\mu k^\nu \geq 0$ [23].

The future event horizon of an asymptotically flat black hole spacetime is defined as the boundary of the past of the future null infinity, that is, the boundary of the points that can communicate with the remote regions of spacetime in the future. Since $R_{\mu\nu}k^\mu k^\nu \geq 0$, and if there are no naked singularities, *the cross sectional area of a future event horizon cannot decrease anywhere* [23].

This is because the focusing equation implies that if the horizon generators are converging, they will reach a crossing point in a finite λ. But such a point cannot lie on the future event horizon because the horizon must be locally tangent to the light cones. And, since the generators cannot leave the horizon, the generators cannot be extended far enough to reach the crossing point; they much reach a singularity [23,26]. Thus, as shall be seen, the area theorem places an upper bound on the total amount of energy that can be extracted from a black hole.

3.2 Energy extraction from black holes

In this section energy extraction from black holes by particles and waves will be examined. For simplicity and pedagogical reasons energy extraction from charged black holes and rotating black holes will be examined separately to give the reader an idea of the basic concepts. The generalization to a rotating, charged black hole (Kerr-Newman black hole) can be made.

3.2.1 Rotating black holes (Penrose process)

Black holes got their name because they represent a region of spacetime where the gravitational potential is so strong that even light can not escape. It came as a big surprise when in 1969 Roger Penrose [28] noted that energy can be extracted from the ergo-sphere of a rotating (Kerr) black hole, in a process known as the Penrose process. Consider a Killing vector field ξ^μ which is spacelike in the ergo-sphere and represents time translation asymptotically at infinity. Then for a test particle with 4-momentum $p^\mu = mu^\mu$, where u^μ is the 4-velocity, the energy

$$E = -p^\mu \xi_\mu \tag{3.2.1}$$

does not need to be positive in the ergo-sphere. Energy can be extracted from the black hole if the particle of negative energy is absorbed by the black hole.

To see this in more detail, consider a particle with 4-momentum p_o^μ and a measured total energy of

$$E_0^\mu = -p_o^\mu \xi_\mu. \tag{3.2.2}$$

Now consider that this particle is falling freely towards a black hole, E_0 will remain constant. As it enters the ergo-sphere it breaks up into two fragments. By local conservation of energy and momentum,

$$p_o^\mu = p_1^\mu + p_2^\mu, \tag{3.2.3}$$

where p_1^μ and p_2^μ are the 4-momentum of the two fragments, and also

$$E_0 = E_1 + E_2. \tag{3.2.4}$$

However, if it was possible that after the breakup of the original particle that one of the daughter particles has negative total energy, $E_1 < 0$, then it can be absorbed into the black hole. If the particle with positive total energy is able to escape the ergo-sphere (in freely falling motion along a geodesic), it should have energy E_2 greater than the energy of the initial particle E_0.

In the case of the Kerr black hole with mass M and $a \neq 0$, it can be explicitly verified that the breakup process can be done so that the second fragment does escape to infinity and that the negative fragment always falls into the black hole [22]. At the end of the process, the second particle has energy $E_0 + |E_1|$, and the black hole has the reduced mass $M - |E_1|$. Therefore, the amount of energy equal to $|E_1|$ has been extracted from the black hole. However, as will be shown, extracting energy from the black hole is self-limiting because the negative energy particles which enter the black hole also carry negative angular momentum, angular momentum opposite to that of the black hole. The angular momentum, $J = Ma$, of the black hole will reduce to zero while M is still finite. However, when $J = 0$, there can be no more extraction of energy because the ergo-sphere no long exists. [22]

The coordinate angular velocity, in the limit $r \to r_+$ of the Kerr metric (2.5.1) is given by

$$\Omega_H = \frac{a}{r_+^2 + a^2}, \tag{3.2.5}$$

and it is related to the Killing field χ^μ by

$$\chi^\mu = (\partial/\partial t)^\mu + \Omega_H (\partial/\partial\phi)^\mu. \tag{3.2.6}$$

(3.2.6) can be thought of as the event horizon of the Kerr black hole rotating with angular velocity Ω_H. χ^μ is future directed null on the horizon. Then for any particle which enters the black hole (including all negative energy particles), what results is

$$0 > p^\mu \chi_\mu = p^\mu(\xi_\mu + \Omega_H \psi_\mu) = -E + \Omega_H L, \tag{3.2.7}$$

where $L = p^\mu \psi_\mu$. Thus we find that

$$L < \frac{E}{\Omega_H} \tag{3.2.8}$$

which verifies the fact that negative energy particles entering the black hole carry negative angular momentum. After absorbing the negative energy particle, the parameters of the Kerr solution are modified by $\delta M = E, \delta J = L$. But (3.2.8) restricts the parameters by

$$\delta J < \frac{\delta M}{\Omega_H}, \tag{3.2.9}$$

which can be rewritten as, [29],

$$\delta M_{irr} > 0, \tag{3.2.10}$$

where the irreducible mass, M_{irr} is defined by

$$M_{irr}^2 = \frac{1}{2}[M^2 + (M^4 - J^2)^{1/2}]. \tag{3.2.11}$$

Inverting (3.2.11) gives

$$\begin{aligned} M^2 &= M_{irr}^2 + \frac{1}{4}\frac{J^2}{M_{irr}^2} \\ &\geqslant M_{irr}^2. \end{aligned} \tag{3.2.12}$$

Therefore, the Penrose process can not reduce the mass of a black hole below the initial value of M_{irr}. Starting with a Kerr black hole of mass M_0 and angular momentum

J_0, the amount of energy that could be extracted from the black hole is $M_0 - M_{irr}$, which can be interpreted as the rotational energy of the black hole. For a maximally rotating black hole, $J_0 = M_0^2$, it represents $\approx 29\%$ of the mass-energy of the black hole [22]. Using the area theorem, the area of the event horizon of the Kerr black hole is given by

$$\begin{aligned} A &= \int_{r=r_+} \sqrt{g_{\theta\theta} g_{\phi\phi}} d\theta d\phi \\ &= \int (r_+^2 + a^2) \sin\theta d\theta d\phi \\ &= 4\pi(r_+^2 + a^2) \\ &= 16\pi M_{irr}^2. \end{aligned} \tag{3.2.13}$$

Thus, from the area theorem, M_{irr} can never decrease, and thus the Penrose process just changes the parameters of the black hole and does not convert it into a naked singularity, so the black hole remains.

3.2.2 Charged black holes

Consider once again the Reissner-Nordström metric (2.4.6),

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r} + \frac{\varepsilon_Q^2}{r^2}\right) c^2 dt^2 + \frac{1}{(1 - \frac{2GM}{c^2 r} + \frac{\varepsilon_Q^2}{r^2})} dr^2 + r^2(d\phi^2 + \sin^2\phi d\phi^2)$$

where $\varepsilon_Q^2 = \frac{G(Q^2+P^2)}{4\pi\epsilon_0 c^4}$. The energy of a particle in this background [22] is given by

$$E = m\sqrt{\left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right)} + e\frac{\varepsilon_Q}{r}, \tag{3.2.14}$$

where m and e are the mass and the charge of the infalling particle. If the charge of the particle is opposite to that of the black hole, then close to the horizon the first term goes to zero and thus the particle has negative energy. Therefore, in this regime,

$$|E| < e\frac{|\varepsilon_Q|}{r_+}. \tag{3.2.15}$$

Now consider two oppositely charged bound particles with total energy E_0 near the horizon of the black hole. One of the particles could have negative energy, by the above argument. The particle with negative energy falls into the black hole and the other escapes. Then, by the conservation of energy

$$E_+ = E_0 + |E_-| \tag{3.2.16}$$

$$M' = M - |E_-| \tag{3.2.17}$$

where E_+ is the energy of the particle that escapes, E_- is the energy of the particle that falls into the black hole, and M' is the final mass of the black hole. The net result is a decrease in the mass of the black hole and a loss of charge with the new charge of the black hole equal to $Q' = Q - e$. The escaping particle will have energy greater than the total energy of the original bound pair of charges. Since for a non-extremal black hole ($\sqrt{G}\varepsilon_Q/r_+ < 1$), (3.2.15) becomes

$$\frac{e}{\sqrt{G}} > |E_+|. \tag{3.2.18}$$

This shows that due to this process, the rate of decrease of charge is greater than the rate of decrease of mass [30].

3.2.3 Superradiance

Superradiant scattering allows energy to be extracted from a black hole in a similar manner as the Penrose process, but using waves instead of particles. If a scalar, electromagnetic, or gravitational wave is incident on a black hole, part of the wave will be absorbed by the black hole, ("transmitted wave"), and part of the wave will escape back to infinity, ("reflected wave") [22]. The transmitted wave will carry positive energy, and the reflected wave will carry less energy than the original incident wave. However, consider a wave of the form $\phi = Re[\phi_0(r,\theta)e^{-i\omega t}e^{im\phi}]$ with

$$0 < \omega < m\Omega_H. \tag{3.2.19}$$

In this case, the transmitted wave will carry negative energy into the black hole, (similar to the negative energy particle in the Penrose process). The reflected wave will carry positive energy to infinity and have a greater amplitude and energy than the original incident wave.

For a rotating black hole this is easily demonstrated by considering the case of a Klein-Gordon scalar field ϕ. The energy-momentum tensor of the Klein-Gordon scalar field is given by

$$T_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \left(\nabla_\lambda \phi \nabla^\lambda \phi + m^2 \phi^2 \right) \tag{3.2.20}$$

and satisfies $\nabla^\mu T_{\mu\nu} = 0$. Contracting the energy-momentum tensor with the timelike Killing field ξ^μ of the Kerr background yields a "energy current"

$$J_\mu = -T_{\mu\nu}\xi^\nu, \tag{3.2.21}$$

which is conserved since $\nabla^\mu J_\mu = -\left(\nabla^\mu T_{\mu\nu}\right)\xi^\nu - T_{\mu\nu}\nabla^\mu \xi^\nu = 0$ [22]. Integrating $\nabla_\mu J^\mu$ over a region of spacetime outside and joined to the horizon using Gauss's law shows that the difference between the incoming and outgoing energies is equal to the integrated flux of J^μ on the horizon. On the horizon the time averaged flux is given by

$$\begin{aligned} \langle J_\mu n^\mu \rangle = -\langle J_\mu \chi^\mu \rangle &= \langle T_{\mu\nu} \chi^\mu \xi^\nu \rangle = \langle \left(\chi^\mu \nabla_\mu \phi\right) \left(\xi^\nu \nabla_\nu \phi\right) \rangle \\ &= \frac{1}{2}\omega(\omega - m\Omega_H)|\phi_0|^2, \end{aligned} \tag{3.2.22}$$

where $n^\mu = -\chi^\mu$ is directed normal to the horizon. Thus, in the frequency range of (3.2.19) the energy flux through the horizon is negative, and hence superradiance is obtained [22]. Similar arguments for the superradiance of electromagnetic waves and gravitational waves in the Kerr background have been shown by Teukolsky and Press [31].

Similarly, for a charged black hole consider the simplest case of a charged scalar field. This field can be solved in the Reissner-Nordström background, (2.4.6), and a relation for the reflection and transmission coefficients , $|R|^2$ and $|T|^2$, can be obtained as [32],

$$1 - |R|^2 = \frac{1}{\kappa}\left(\omega - e\frac{\varepsilon_Q}{r_+}\right)|T|^2, \tag{3.2.23}$$

where κ is the surface gravity given by $(r_+ - r_-)/2r_+^2$. For $\omega < e\frac{\varepsilon_Q}{r_+}, |R|^2$ is greater than 1, meaning that the scalar wave takes energy away from the black hole. Then, the condition for superradiance is [30]

$$m < \omega < e\frac{\varepsilon_Q}{r_+}. \tag{3.2.24}$$

The rate of charge and mass loss for the black hole can be calculated by [30]

$$\frac{d\varepsilon_Q}{dt} = -e\int_m^{e\frac{\varepsilon_Q}{r_+}} |R|^2 d\omega \tag{3.2.25}$$

$$\frac{dM}{dt} = -\int_m^{e\frac{\varepsilon_Q}{r_+}} |R|^2 \omega d\omega. \tag{3.2.26}$$

By (3.2.24),

$$\left|\frac{d\varepsilon_Q}{dt}\right| > \left|\frac{dM}{dt}\right|. \tag{3.2.27}$$

This shows that the rate of decrease of charge is greater than the rate of decrease of mass, as was the case for energy extraction from a charged black hole by particles. Again, this process holds for electromagnetic and gravitational waves.

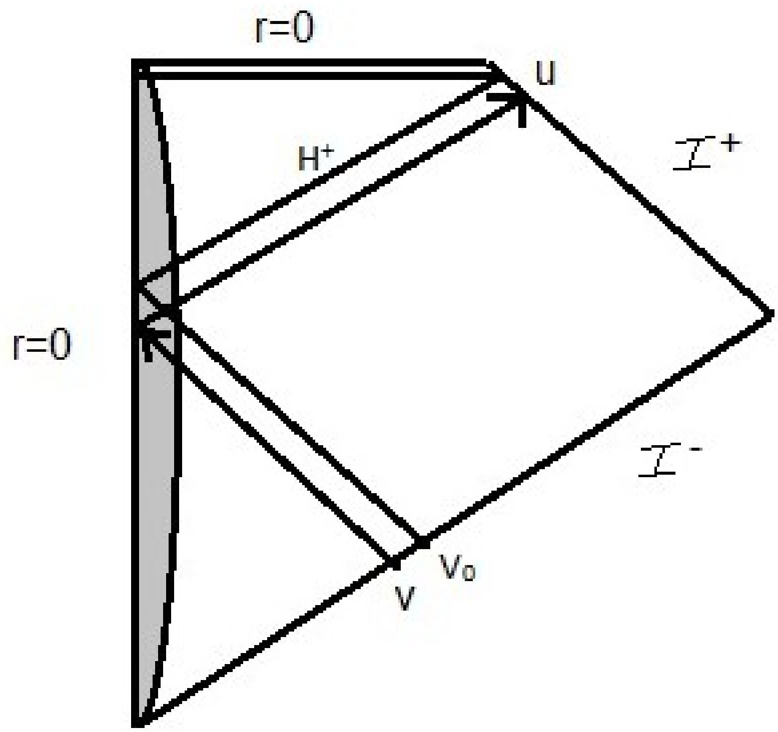

Figure 3.1: The Penrose diagram for a gravitationally collapsed black hole spacetime. The shaded region is the collapsing body. $r = 0$ on the left is the worldline of the center of the collapsing body, $r = 0$ on the top is the curvature singularity. The H^+ is the future even horizon. The ingoing ray with $v < v_0$ from $\mathcal{I}^-$ passes through the body and escapes to $\mathcal{I}^+$ as u is constant. Ingoing rays with $v > v_0$ eventually reach the singularity.

3.3 Hawking radiation

The process by which black holes emit a thermal spectrum of particles is known as the Hawking effect [1,33]. Consider a massless scalar field in the Schwarzschild spacetime. Assume that the black hole is a result of a gravitational collapse sometime in the past. This assumption helps avoid issues of boundary conditions on the past horizon. Also assume that there were no scalar particles present before the collapse began.

For this case, the quantum state is in the vacuum: $|\psi\rangle = |0\rangle_{in}$. The in-modes, $f_{\omega\ell m}$, are pure positive frequency on $\mathcal{I}^-$. Thus, $f_{\omega\ell m} \sim e^{i\omega v}$ as $v \to \infty$, where $v = t + r*$ is the familiar advanced time coordinate. The out-modes, $F_{\omega\ell m}$, are pure positive frequency on $\mathcal{I}^-$. Thus, $F_{\omega\ell m} \sim e^{-i\omega u}$ as $u \to \infty$, where $u = t - r*$ is the familiar retarded time coordinate. To determine the particle creation, the relation between the two sets of modes must be determined. This allows one to calculate the Bogoliubov coefficients, which in turn are used to define the creation and annihilation operators for the in and out regions.

The main interest here is emission of particles at late times, (long after the collapse). This region is dominated by modes which leave $\mathcal{I}^-$ with high frequency, propagate through the collapsing body just before the horizon is formed, then undergo a large redshift on the way out of $\mathcal{I}^+$. Since these modes are extremely high frequency, their propagation maybe described by geometrical optics.

An ingoing ray with $v = \textit{constant}$ passes through the body and emerges as an outgoing ray with $u = \textit{constant}$, where $u = g(v)$ or equivalently, $v = g^{-1}(u) \equiv G(u)$. Then the asymptotic forms for the modes, from the geometrical optics approximation, are

$$f_{\omega\ell m} \sim \frac{Y_{\ell m}(\theta,\phi)}{\sqrt{4\pi\omega}r} \times \begin{cases} e^{-i\omega v}, & \text{on}\, \mathcal{I}^- \\ e^{-i\omega G(u)}, & \text{on}\, \mathcal{I}^+ \end{cases} \tag{3.3.1}$$

and

$$F_{\omega\ell m} \sim \frac{Y_{\ell m}(\theta,\phi)}{\sqrt{4\pi\omega}r} \times \begin{cases} e^{-i\omega u}, & \text{on}\, \mathcal{I}^+ \\ e^{-i\omega g(v)}, & \text{on}\, \mathcal{I}^- \end{cases}, \tag{3.3.2}$$

where $\mathrm{Y}_{\ell m}(\theta,\phi)$ is a spherical harmonic. Hawking [1] gave a ray-tracing argument which led to the result

$$u = g(v) = -4M \ln\left(\frac{v_0 - v}{C}\right), \tag{3.3.3}$$

or

$$v = G(u) = v_0 - Ce^{-u/4M}, \tag{3.3.4}$$

where M is the mass of the black hole, C is a constant, and v_0 is the limiting value for v for rays which pass through the collapsing body before the horizon forms.

In this section this result will be derived using Ford's approach, and will closely follow his work [24] for the explicit case of a thin shell. The inner spacetime of the shell is flat and may be described by the metric

$$ds^2 = -dT^2 + dr^2 + r^2 d\Omega^2. \tag{3.3.5}$$

Then in the interior region, $V = T - r$ and $U = T + r$ are the null coordinates, which are constant on ingoing and on outgoing rays respectively. The shell exterior is described the Schwarzschild metric:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\Omega^2, \tag{3.3.6}$$

where $G = c = 1$. The null coordinates in the exterior region are $v = t - r*$ and $u = t - r*$, where

$$r* = r + 2M \ln\left(\frac{r - 2M}{2M}\right) \tag{3.3.7}$$

is the tortoise coordinate.

Let the history of the shell be described by $r = R(t)$. The metric on the 3-dimensional hypersurface (the shell itself) must be the same as seen from both sides. This leads to the condition

$$1 - \left(\frac{dR}{dT}\right)^2 = -\left(\frac{R-2M}{R}\right)\left(\frac{dt}{dT}\right)^2 + \left(\frac{R-2M}{R}\right)^{-1}\left(\frac{dR}{dT}\right)^2. \tag{3.3.8}$$

The extrinsic curvatures on each side of the hypersurface must match (see, for example, Section 21.13 of [20]), (the extrinsic curvature will be explained in detail in

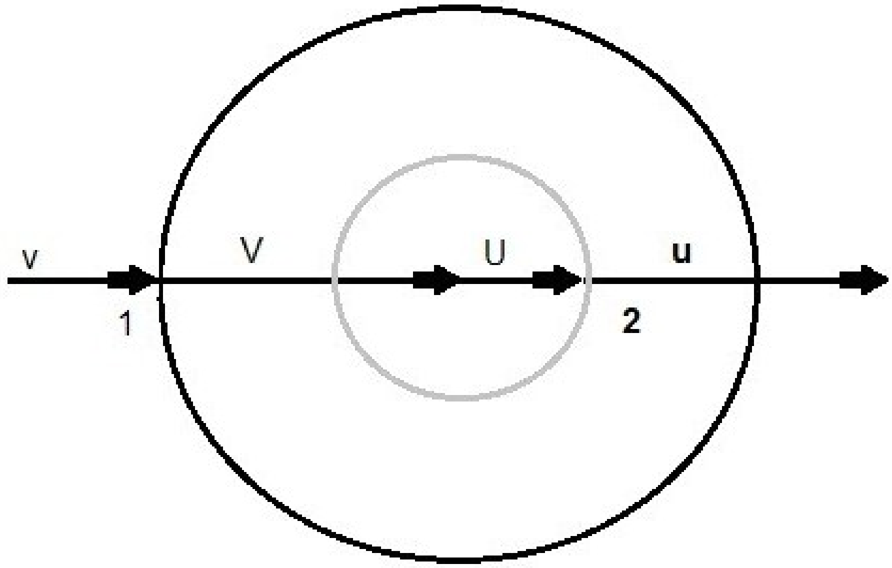

Figure 3.2: A ray entering the collapsing shell at point 1, passes through the center and exits as an outgoing ray at point 2 while the shell is shrinking to that of a shell with a smaller radius.

chapter 5). This leads to the relation of $R(t)$ in terms of the energy-momentum in the shell. For this case however, this relation in not needed and an arbitrary $R(t)$ is assumed.

There are now 3 conditions to be determined: the relation of the null coordinates v and V for the ingoing ray, the relation between U and u for the outgoing ray, and the relation between V and U at the center, (this is illustrated in Figure 3.2).

Suppose that a null ray enters the shell at a radius of R_1, which is larger than $2M$. $R/R-2M$ and dR/dT are both finite and approximately constant at this point, and so dt/dT is approximately constant. Therefore $t \propto T$. And since $r*$ is a linear function of r in the region $r = R_1$, one concludes that

$$V(v) = av + b \tag{3.3.9}$$

in a neighborhood of $v = v_0$, where a and b are constants.

Matching the null coordinates at the center, at $r = 0$ gives

$$U(V) = V \tag{3.3.10}$$

because $V = T - r$ and $U = T + r$.

Consider now the ray exiting the shell. Rays which exit when R is close to $2M$ are of the most interest. Let T_0 be the time at which $R = 2M$. It is worth noting that this occurs as a finite time as seen by an observer inside the shell. Then near $T = T_0$,

$$R(T) \approx \left(\frac{R-2M}{2M}\right)^{-2}\left(\frac{dR}{dT}\right)^2 \approx \frac{(2M)^2}{(T-T_0)^2}, \tag{3.3.11}$$

which implies

$$t \sim -2M \ln\left(\frac{T_0 - T}{B}\right), \quad T \to T_0. \tag{3.3.12}$$

Also, as $T \to T_0$,

$$r* \sim 2M \ln\left(\frac{r-2M}{2M}\right) \sim 2M \ln\left[\frac{A(T_0 - T)}{2M}\right], \tag{3.3.13}$$

and therefore

$$u = t + r* \sim -4M \ln\left(\frac{T_0 - T}{B'}\right), \tag{3.3.14}$$

where B and B' are constants. However, in this limit

$$U = T + r = T + R(T) \sim (1 + A)T + 2M + AT_0. \tag{3.3.15}$$

Combining these results with (3.3.9) and (3.3.10) yields the final result (3.3.3) [24]. This was the explicit calculation for the thin shell. The logarithmic dependence which governs the asymptotic form of $u(v)$ comes from the last step in the sequence of matchings. This step reflects the large redshift which the outgoing rays experience after they have passed through the collapsing body [24]. Consider dividing a spherically symmetric star into a sequence of collapsing shells. Null rays enter and exit each shell. Then, each null coordinate is linearly dependent on the preceding one, (until the null rays exit the last shell). At the exit point of the last shell, the retarded time u in the exterior spacetime (Schwarzschild spacetime) is a logarithmic function of the previous coordinate, and also a logarithmic function of v given by (3.3.3).

The out-modes, when traced back to $\mathcal{I}^-$, have the from

$$F_{\omega\ell m} \sim \begin{cases} e^{-4Mi\omega \ln[(v_0-v)/C]}, & v < v_0 \\ 0, & v > v_0. \end{cases} \tag{3.3.16}$$

Fourier transforming this function allows one to find the Bogoliubov coefficients. The Fourier transform is given by [24]

$$F_{\omega\ell m} = \int_0^\infty d\omega' \Big(\alpha^*_{\omega'\omega\ell m} f_{\omega'\ell m} - \beta_{\omega'\omega\ell m} f^*_{\omega'\ell m} \Big), \tag{3.3.17}$$

where $\alpha_{\omega'\omega\ell m} = \alpha_{\omega'\ell m,\omega\ell m}$ and $\beta_{\omega'\omega\ell m} = \beta_{\omega'\ell -m,\omega\ell m}$. This notation ensures that the dependence of the angular coordinates must be the same in each term in (3.3.17). Thus

$$\alpha^*_{\omega'\omega\ell m} = \frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}} \int_{-\infty}^{v_0} dv e^{i\omega' v} e^{4Mi\omega \ \ln[(v_0-v)/C]}, \tag{3.3.18}$$

and

$$\beta_{\omega'\omega\ell m} = -\frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}} \int_{-\infty}^{v_0} dv e^{-i\omega' v} e^{4Mi\omega \ \ln[(v_0-v)/C]}, \tag{3.3.19}$$

or, equivalently,

$$\alpha^*_{\omega'\omega\ell m} = \frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}} e^{i\omega v_0} \int_0^\infty dv' e^{-i\omega' v'} e^{4Mi\omega \ \ln(v'/C)}, \tag{3.3.20}$$

and

$$\beta_{\omega'\omega\ell m} = -\frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}} e^{i\omega v_0} \int_0^\infty dv' e^{i\omega' v'} e^{4Mi\omega \ \ln(v'/C)}, \tag{3.3.21}$$

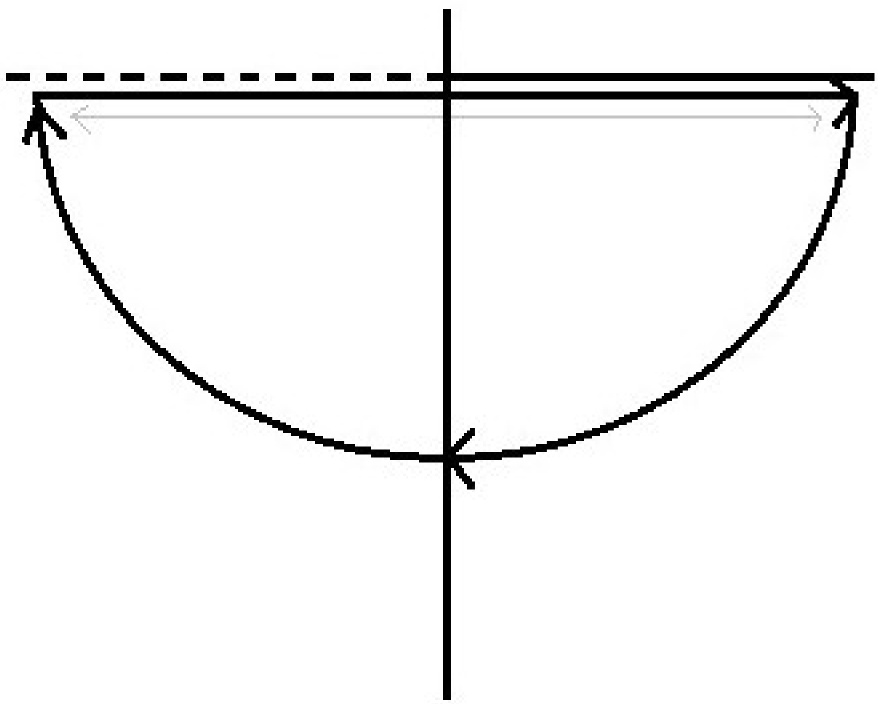

Figure 3.3: The closed contour C of the integration (3.3.22). This integral vanishes which implies that the integrals along the dotted line and the lighter, thin line segments are equal. This implies the first equality in (3.3.23).

where $v' = v_0 - v$.

The two functions above are analytic on the complex plane everywhere but the negative real axis, the branch cut of the logarithm function. Therefore

$$\oint_C dv' e^{-i\omega' v'} \; e^{4Mi\omega \ln (v'/C)} = 0, \tag{3.3.22}$$

where this integration is around the closed contour C (see Figure 3.3). This integral can be expressed as

$$\begin{aligned} \int_0^\infty dv' e^{-i\omega' v'} e^{4Mi\omega \ln (v'/C)} &= -\int_0^\infty dv' e^{i\omega' v'} e^{4Mi\omega \ln (v'/C - i\epsilon)} \\ &= -e^{4\pi M\omega} \int_0^\infty dv' e^{i\omega' v'} e^{4Mi\omega \ln (v'/C)}, \end{aligned} \tag{3.3.23}$$

where in the first step there is a change of variable $v' \to -v'$ applied to (3.3.22). In the second step of (3.3.23) the relation $\ln (-v'/C - i\epsilon) = -i\pi + \ln (v'/C)$. Then comparing (3.3.23) with (3.3.20) and (3.3.21) leads to the result

$$|\alpha_{\omega'\omega\ell m}| = e^{4\pi M\omega} |\beta_{\omega'\omega\ell m}|. \tag{3.3.24}$$

There is an equation relating the in-modes in terms of the out modes,

$$f_j = \sum_k \left(\alpha_{jk} F_k + \beta_{jk} F_k^* \right). \tag{3.3.25}$$

In terms of orthogonality relations between in- and out-modes this relation leads to the condition [24]

$$\sum_k \left(\alpha_{jk} \alpha_{j'k}^* - \beta_{jk} \beta_{j'k}^* \right) = \delta_{jj'} \tag{3.3.26}$$

Then the condition (3.3.26) on (3.3.24) can be written as

$$\sum_{\omega'} \left(|\alpha_{\omega'\omega\ell m}|^2 - |\beta_{\omega'\omega\ell m}|^2 \right) = \sum_{\omega'} \left(e^{8\pi M\omega} - 1 \right) |\beta_{\omega'\omega\ell m}|^2 = 1 \tag{3.3.27}$$

The physical number operator $N_{\omega\ell m}$ counts the particles in the out-region. Adopting the Heisenberg picture of quantum dynamics, the state $|0\rangle_{in}$ remains the state of the system for all time. Then, the mean number of particles created into mode $\omega\ell m$ is given by

$$\langle N_{\omega\ell m}\rangle = {}_{in}\langle 0|b^\dagger_{\omega\ell m} b_{\omega\ell m}|0\rangle_{in} = \sum_{\omega'} |\beta_{\omega'\omega\ell m}|^2, \tag{3.3.28}$$

where $b^\dagger_{\omega\ell m}$ and $b_{\omega\ell m}$ are creation and annhilation opertors in the out-region given by

$$b_{\omega\ell m} = \sum_{\omega'} \left(\alpha_{\omega'\omega\ell m} a_{\omega'} + \beta^*_{\omega'\omega\ell m} a^\dagger_{\omega'}\right) \tag{3.3.29}$$

and $a^\dagger_{\omega'}$ and $a_{\omega'}$ are the creation and annihilation operators in the in-region. If the coefficients $\beta_{\omega'\omega\ell m}$ are non-zero, that is there is a mixing of the positive and negative frequency solutions, then the particles are created by the gravitational field [24].

Comparing (3.3.27) with (3.3.28) gives

$$N_{\omega\ell m} = \sum -\omega' |\beta_{\omega'\omega\ell m}|^2 = \frac{1}{e^{8\pi M\omega} - 1}. \tag{3.3.30}$$

(3.3.30) is the familiar Planck spectrum from statistical mechanics with a temperature of

$$T = \frac{1}{8\pi M}, \tag{3.3.31}$$

which in natural units is the Hawking temperature of the black hole [24],

$$T_H = \frac{\hbar c^3}{8\pi G M k_B}, \tag{3.3.32}$$

where k_B is the Boltzmann constant.

3.4 Black hole thermodynamics

A "black hole" is called that because light rays originating from inside the event horizon can not escape. But light rays may enter the event horizon, which can be thought of as a one-way membrane. Thus things can fall into the black hole and make it bigger but, classically, nothing can leave to make it shrink. This is reminiscent of the second law of thermodynamics of one way, increasing entropy. The size of the black hole, like entropy is always increasing.

The area of the black hole can be used to measure the size of the black hole. For a spherically symmetric black hole with electric charge ε_Q, and angular momentum J, the total surface area of the horizon is given by [34]

$$A = 4\pi r_+^2 \tag{3.4.1}$$

$$A = 4\pi \left[2\,\frac{G^2 M^2}{c^4} - \varepsilon_Q^2 + 2\,\frac{G^2 M^2}{c^4} \left(1 - \frac{c^4\ \varepsilon_Q^2}{G^2 M^2} - \frac{c^4\ J^2}{G^2 M^4} \right)^{1/2} \right], \tag{3.4.2}$$

where $\varepsilon_Q^2 < M^2$ and $J^2 < M^4$.

It is unclear from (3.4.2) that if something effects ε_Q^2, J^2, and M whether the total area increases or decreases. For example, consider energy extraction by the Penrose process. Under the Penrose process for a rotating black hole, J and M both decrease. But examining (3.4.2) shows that when J decreases the total area increases, and when M decreases the total area decreases. The changes in J and M are in competition, but it can be shown that J always wins and the total area always increases.

This analogy between entropy and the area of a black hole, along with Hawking's Area theorem, gives rise to the second law of classical black hole thermodynamics:

$$dA \geq 0, \tag{3.4.3}$$

analogous to the second law of thermodynamics.

There are also laws in classical black hole thermodynamics analogous to the zeroth, first, and third laws of thermodynamics. From (3.4.2) one can obtain,

$$dM = (8\pi)^{-1}\kappa\ dA + \Omega_H\ dJ + \Phi\ d\varepsilon_q \tag{3.4.4}$$

where $(8\pi)^{-1}\kappa \equiv \partial M/\partial A$, etc. (3.4.4) is an expression of the mass-energy conservation, which corresponds to the first law of thermodynamics. Inspecting (3.4.4) and considering the fact that A is thought of as the entropy, then comparing with $T\ dS$, κ is considered to be the temperature. Interestingly, κ can be shown to be constant across the event horizon surface. This is therefore analogous to the zeroth law of thermodynamics, which states that there exists a common temperature parameter for a system in thermodynamic equilibrium [25]. κ is known as the surface gravity of a black hole. The last two terms in (3.4.4) describe the energy extracted from changes in angular momentum and electric charge; Ω_H is the angular velocity and Φ is the electric potential at the event horizon.

If J^2 or ε_Q^2 become large enough, then

$$\frac{J^2}{M^4} + \frac{\varepsilon_Q^2}{M^2} = 1, \tag{3.4.5}$$

then κ vanishes (but A does not). (3.4.5) is analogous to the third law of thermodynamics. A black hole under these conditions is known as an extreme Kerr-Newman black hole. It corresponds to $\kappa = 0$, or absolute zero.

For the simplest black hole, the Schwarzschild black hole ($J = \varepsilon_Q = 0$), (3.4.2) becomes

$$A = 16\pi \frac{G^2 M^2}{c^4} \tag{3.4.6}$$

and since $\kappa \equiv 8\pi \frac{\partial M}{\partial A}$, for the Schwarzschild black hole

$$\kappa = \frac{c^2}{4GM}. \tag{3.4.7}$$

Also, (3.4.4) becomes $dM = \kappa\ dA = T\ dS$. Integrating this expression yields:

$$energy = 2\ entropy\ \times\ temperature, \tag{3.4.8}$$

where the factor of 2 arises because A is proportional to the square of M [25]. Thus, the energy M is finite, and zero temperature implies infinite entropy. However, by rewriting (3.4.6) in the form (3.4.8) gives

$$M = \frac{c^2 A\kappa}{4\pi G} \tag{3.4.9}$$

and the right hand side becomes a product of two finite quantities.

Questions that arise are how can the notion of entropy of a black hole be understood, and why should it be infinite? Entropy can be related to information: the more information about a system, the lower the entropy; the less information about a system, the greater the entropy. A Kerr-Newman black hole is described by three quantities, mass, charge, angular momentum. Therefore since there is minimal information known about a Kerr-Newman black hole, one would expect it to have a very large entropy.

Counting the number of internal degrees of freedom and assigning one bit of information to each gives a crude estimate of the entropy of a black hole. This means calculating the number of particles that go into making up a black hole. However, this number is still arbitrary because one can always choose a large number of particles with low energy: zero rest mass particles like photons or neutrinos with low energy, for example. Then the number of bits, and hence the entropy of a black hole, should be unbounded. This seems to be a problem. However it is rectified when quantum theory is applied to the system.

Using quantum theory, a particle inside the black hole can not be chosen of an arbitrarily small mass because of the relation

$$\begin{aligned} E &= h\nu \\ &= hc/\lambda \\ &= 2\pi\hbar c/\lambda. \end{aligned} \tag{3.4.10}$$

The wavelength, λ must, at least, be less than the size of the black hole if the particle energy E is assumed to be located inside the black hole. Choosing $\lambda \simeq 2GM/c^2$, gives the minimum particle energy of the order of $c^2\hbar/GM$. Thus the maximum number of particles that make up a black hole of mass M is about $GM^2/c^4\hbar$. An estimate of the entropy is then

$$S = \xi k_B \left(\frac{GM^2}{c^4\hbar}\right) \tag{3.4.11}$$

where k_B is the Boltzmann constant and ξ is a number of order unity, to be calculated from a proper, full theory of quantum black holes [25]. Then it is clear in the classical limit, $\hbar \to 0$, the entropy S diverges.

From (3.4.7) and (3.4.9), (3.4.11) can be rewritten as

$$S = \xi \frac{k_B}{16\pi\hbar} A \tag{3.4.12}$$

illustrating that the entropy is indeed proportional to the event horizon surface area of a black hole. Substituting in the known value for the constant ξ, (3.4.12), in natural units, then becomes

$$S = \frac{k_B A c^3}{4G\hbar}. \tag{3.4.13}$$

The Hawking temperature (3.3.32) can also be written as

$$T_H = \frac{\hbar c^3}{8\pi GM\ k_B} = \frac{\hbar c\ \kappa}{2\pi\ k_B}. \tag{3.4.14}$$

3.5 Hawking temperature from a Euclidean metric

As mentioned in [22], the Hawking temperature of black hole can be determined by Euclideanizing a Lorentzian metric. Although the true underlying physics of this method is not clearly understood, it is an effective method when possible, in general for static, spherically symmetric black holes. A brief overview of the calculation, following the work of Horowitz [35], which follows the original work of Gibbons and Hawking [36], will be presented here.

Consider a Lorentzian black hole solution of the form

$$ds^2 = -a(r)dt^2 + \frac{dr^2}{b(r)} + c(r)\, r^2 d\Omega^2. \tag{3.5.1}$$

If there is an event horizon at $r = r_0$, then $a(r) \approx a'(r_0)(r-r_0)$ and $b(r) \approx b'(r)(r-r_0)$. Then using a coordinate transformation to Euclidean time, $\tau = it$ and $\rho = 2\sqrt{\frac{(r-r_0)}{b'(r_0)}}$ to give a resulting metric

$$ds^2 = \frac{a'(r_0)\, b'(r_0)}{4}\rho^2\, d\tau^2 + d\rho^2 + c(r_0)\, r_0 d\Omega^2. \tag{3.5.2}$$

Then, to avoid the conical singularity at $\rho = 0$, that is when $b'(r_0) = 0$, τ is identified with the period $4\pi\sqrt{a'(r_0)\, b'(r_0)}$. The Hawking temperature is then given by

$$T_H = \frac{\sqrt{a'(r_0)\, b'(r_0)}}{4\pi}. \tag{3.5.3}$$

(3.5.3) will be used later to determine the Hawking temperature of various BTZ black holes.

Chapter 4

Conformal field theory (in 2 dimensions)

This chapter will be a brief introduction to conformal field theory (CFT) in 2 dimensions. The goal of this chapter is to provide enough theoretical background to help the reader understand why it is such a desirable theory to work with and to introduce a mathematical framework which can be applied with black holes (the Cardy formula). The main resources for this chapter are [4], [5], and [37, 38] to a lesser extent. The conventions of this chapter are the same as previous chapters.

4.1 Conformal field theory

Consider a metric $g_{\mu\nu}$ and line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. The conformal group is a group of coordinate transformations that preserve the metric up to a scale, that is

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x) \tag{4.1.1}$$

Because the scale factor $\Omega(x)$ cancels out in

$$\cos\theta = \frac{v\cdot w}{\sqrt{v^2 w^2}},$$

these transformations also preserve the angle between two vectors v, w (where $v\cdot w = g_{\mu\mu}v^{\mu}w^{\nu}$). Note that the Poincaré group (Lorentz group plus translations) is a subgroup of the conformal group since it leaves the metric invariant ($g'_{\mu\nu} = g_{\mu\nu}$) [4].

In a conformally-invariant field theory (CFT), the action remains conformally invariant, $S[g'_{\mu\nu}(x')] = S[g_{\mu\nu}(x)]$. Varying the action with respect to the metric gives energy-momentum tensor,

$$\begin{aligned} \frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} &= T^{\mu\nu} \\ \frac{1}{\sqrt{-g}}\delta S &= T^{\mu\nu}\delta g_{\mu\nu}. \end{aligned} \tag{4.1.2}$$

Under a conformal transformation $g_{\mu\nu}$ would transform as $g'_{\mu\nu} \to \Omega(x)g_{\mu\nu}$. Taking an infinitesimal expansion of $\Omega(x)$,

$$\begin{aligned} g'_{\mu\nu} &\to (1+\sigma(x))g_{\mu\nu} \\ g'_{\mu\nu} &= g_{\mu\nu} + \sigma(x)g_{\mu\nu}. \end{aligned}$$

Therefore,

$$\delta g_{\mu\nu} = \sigma(x)g_{\mu\nu},$$

where

$$\delta g_{\mu\nu} = g'_{\mu\nu} - g_{\mu\nu},$$

and then (4.1.2) becomes

$$\begin{aligned} \frac{1}{\sqrt{-g}}\delta S &= \sigma(x)T^{\mu\nu}g_{\mu\nu} \\ \frac{1}{\sqrt{-g}}\delta S &= \sigma(x)T^{\mu}{}_{\mu}. \end{aligned} \tag{4.1.3}$$

But since the action is invariant under conformal transformations, this requires the energy-momentum tensor to be traceless, $T^{\mu}{}_{\mu} = 0$ [5].

By examining the infinitesimal coordinate transformation in the Minkowski spacetime, $x^\mu \to x^\mu + \epsilon^\mu$, the infinitesimal generators of the conformal group can be determined. The squared line element transforms as

$$ds^2 \to ds^2 + (\partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu) dx^\mu dx^\nu, \tag{4.1.4}$$

since $g_{\mu\nu}$ transforms like a second rank tensor. For this transformation to remain conformally invariant (to satisfy (4.1.1)), $\partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu$ must be proportional to the original Minkowski metric, $\eta_{\mu\nu}$. Therefore,

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}(\partial \cdot \epsilon)\eta_{\mu\nu} \tag{4.1.5}$$

where tracing both sides of $\eta^{\mu\nu}$ fixes the constant of proportionality. Comparing (4.1.1) with (4.1.5) gives $\Omega(x) = 1 + \frac{2}{d}(\partial \cdot \epsilon)$. The conformal Killing equation[1] describes vector fields which preserve the metric up to a scale. It can be determined from (4.1.5) and is given by

$$(\eta_{\mu\nu}\Box + (d-2)\partial_\mu \partial_\nu)\partial \cdot \epsilon = 0. \tag{4.1.6}$$

This is the conformal Killing equation in any dimension. For $d > 2$ the third derivatives of ϵ must vanish by (4.1.5) and (4.1.6). Therefore, ϵ is at most quadratic in x. Examining the various possibilities for ϵ:

For ϵ zeroth order in x:

$\epsilon^\mu = a^\mu$ which are the ordinary translations independent of x, $(\Omega = 1)$.

There are two cases for ϵ linear in x,:

$\epsilon^\mu = w^\mu{}_\nu x^\nu$ where w is antisymmetric, these are rotations, $(\Omega = 1)$.

[1]Solutions of Killing's equation $\nabla_\mu X_\nu + \nabla_\mu X_\mu = 0$ are vector fields which preserve the metric g. Conformal Killing vectors fields, solutions of (4.1.6), preserve the metric up to a scale.

$\epsilon^\mu = \lambda x^\mu$ these are scale transformations, $(\Omega = \lambda^{-2})$.

When ϵ is quadratic in x,

$\epsilon^\mu = b^\mu x^2 - 2x^\mu b\cdot x$, these are the "special conformal" transformations, $(\Omega(x) = (1 + 2b\cdot x + b^2 x^2)^2)$. The "special conformal" transformations can be thought of as an inversion plus translation followed by an inversion. [4]

The finite conformal transformations are given by

$$x \to x' = x + a \tag{4.1.7}$$

$$x \to x' = \Lambda x \qquad (\Lambda^\mu{}_\nu \in SO(p,q)) \tag{4.1.8}$$

$$x \to x' = \lambda x \tag{4.1.9}$$

$$x \to x' = \frac{x + bx^2}{1 + 2b\cdot x + b^2 x^2} \tag{4.1.10}$$

where (4.1.7) and (4.1.8) are translations and rotations (Poincaré), (4.1.9) are the dilations, and (4.1.10) are the special conformal transformations [4].

4.2 Conformal field theory in 2-dimensions

In 2-dimensions with $\eta_{\mu\nu} = \delta_{\mu\nu}$, (4.1.5) becomes

$$\partial_0 \epsilon_0 = \partial_1 \epsilon_1, \quad \partial_0 \epsilon_1 = -\partial_1 \epsilon_0 \tag{4.2.1}$$

which are the familiar Cauchy-Riemann equations.

Any function $f(z)$ satisfying (4.2.1) is known as an analytic function. To see this, consider ϵ as a function of z, where $z, \bar{z} = x^0 \pm ix^1$ are the complex coordinates, and $\epsilon = \epsilon^0 + i\epsilon^1$, $\bar{\epsilon} = \epsilon^0 - i\epsilon^1$. Then (4.2.1) becomes

$$\partial_z \bar{\epsilon} = 0, \quad \partial_{\bar{z}} \epsilon = 0. \tag{4.2.2}$$

so that $\epsilon = \epsilon(z)$, and $\bar{\epsilon} = \bar{\epsilon}(\bar{z})$

The conformal transformations in two dimensions therefore coincide with the analytic coordinate transformations given by

$$z \to f(z), \quad \bar{z} \to \bar{f}(\bar{z}), \tag{4.2.3}$$

and

$$ds^2 = dz\, d\bar{z} \to \left|\frac{\partial f}{\partial z}\right|^2 dz d\bar{z}, \tag{4.2.4}$$

where $\Omega =\mid \frac{\partial f}{\partial z} \mid^2$. The holomorphic function $f(z)$ can be Laurent expanded to give

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \qquad (n \in \mathbb{Z}), \tag{4.2.5}$$

where a_n are symmetry parameters, and an infinite number of symmetry parameters corresponds to infinite symmetries. The infinite symmetries are what makes 2-dimensional conformal field theory solvable.

The energy-momentum tensor is conserved and traceless in any CFT. When mapped to the complex plane, these equations become:

$$\partial_z T_{\bar{z}\bar{z}} = 0 \longleftrightarrow T_{\bar{z}\bar{z}} = \bar{T}(\bar{z}), \tag{4.2.6}$$

and

$$\bar{\partial}_{\bar{z}} T_{zz} = 0 \longleftrightarrow T_{zz} = T(z). \tag{4.2.7}$$

The generators of symmetry in quantum field theory are conserved charges which come from conserved currents

$$\partial_\mu j^\mu = 0. \tag{4.2.8}$$

For a 2-dimensional CFT, in complex coordinates the conserved current equation is given by

$$\bar{\partial}_{\bar{z}} j(z) + \partial_z \bar{j}(\bar{z}) = 0. \tag{4.2.9}$$

But if the current is holomorphic, the anti-holomorphic part drops out, and

$$\bar{\partial}_{\bar{z}} j(z) = 0. \tag{4.2.10}$$

Using (4.2.7), an infinite number of conserved currents can be generated

$$j(z) = f(z)T(z),$$

the conserved current equation still holds, that is

$$\bar{\partial}_{\bar{z}} j(z) = 0 \tag{4.2.11}$$

Therefore, since there are an infinite number of analytic functions, $f(z)$, then by Noether's theorem there are an infinite number of conserved charges corresponding to an infinite number of symmetries [4].

The infinitesimal transformations of (4.2.3) give

$$z \to z' = z + \epsilon_n(z), \quad \bar{z} \to \bar{z}' = \bar{z} + \bar{\epsilon}_n(\bar{z}) \qquad (n \in \mathbb{Z}),$$

where

$$\epsilon_n(z) = -z^{n+1}, \quad \bar{\epsilon}_n(\bar{z}) = -\bar{z}^{n+1}.$$

The generators of the infinitesimal transformations are

$$\ell_n = -z^{n+1}\partial_z, \quad \bar{\ell}_n = -\bar{z}^{n+1}\partial_{\bar{z}} \qquad (n \in \mathbb{Z}). \tag{4.2.12}$$

The commutation relations of these infinitesimal generators form the local conformal algebra. The algebra is given by

$$[\ell_m, \ell_n] = (m-n)\ell_{m+n}, \quad [\bar{\ell}_m, \bar{\ell}_n] = (m-n)\bar{\ell}_{m+n} \tag{4.2.13}$$

with $[\ell_m, \ell_n] = 0$. This algebra is known as the classical Witt algebra. Since the ℓ_n's commute with the $\bar{\ell}_m$'s, the local conformal algebra is a direct sum of $A \oplus \bar{A}$ of two isomorphic subalgebras with the commutation relations (4.2.13) [4]. The two independent algebras arise naturally because z and $\bar{z}$ are treated as independent coordinates; the conformal group in two dimensions has two independent actions on z and $\bar{z}$. There are two copies of the algebra, (holomorphic and anti-holomorphic corresponding to z and $\bar{z}$, respectively). Therefore, it is often convenient to just deal with the holomorphic sector, and treatment of the anti-holomorphic sector is done in complete analogy.

The group of conformal transformations, which are well-defined and invertible, on the Riemann sphere, define the conformal group in 2-dimensions. The conformal group is generated by the globally defined infinitesimal generators $\{\ell_{-1}, \ell_0, \ell_1\} \cup \{\bar{\ell}_{-1}, \bar{\ell}_0, \bar{\ell}_1\}$. From the finite conformal transformations and (4.2.12) the infinitesimal generators are identified: ℓ_{-1} and $\bar{\ell}_{-1}$ as the generators of translations, $\ell_0 + \bar{\ell}_0$ and $i(\ell_0 - \bar{\ell}_0)$ as the generators of dilatations and rotations, respectively (i.e. generators of translations of r and θ in $z = re^{i\theta}$), and $\ell_1, \bar{\ell}_1$ are generators of the special conformal transformations [4].

The finite form of these transformations is

$$z \to \frac{az+b}{cz+d}, \qquad \bar{z} \to \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}} \tag{4.2.14}$$

where $a, b, c, d \in \mathbf{C}$ and $ad - bc = 1$. This group is the $SL(2,\mathbf{C})/\mathbf{Z}_2 \approx SO(3,1)$, and is also known as the group of projective conformal transformations. The quotient by $\mathbf{Z}_2$ is due to the fact that (4.2.14) is unaffected taking a, b, c, d to minus themselves [4].

The transformations (4.1.7) - (4.1.10) in $SL(2,\mathbf{C})$ become translations: $\begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}$ rotations: $\begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}$

dilatations: $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ special conformal: $\begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix}$,

where $B = a^0 + ia^1$ and $C = b^0 - ib^1$. In higher dimensions there only exists a global conformal group, the local conformal group is unique to 2-dimensions. Even in 2-dimensions the true conformal group is the global conformal group, since the remaining conformal transformations of (4.2.3) do not have global inverses on $\mathbf{C}\cup\infty$.

The global conformal algebra is generated by $\{\ell_{-1}, \ell_0, \ell_1\} \cup \{\bar{\ell}_{-1}, \bar{\ell}_0, \bar{\ell}_1\}$. It is also used to characterize physical states, by working in the basis of eigenstates of the two operators ℓ_0 and $\bar{\ell}_0$ with eigenvalues h and $\bar{h}$, respectively (where h an $\bar{h}$ are real and not complex conjugates of each other). h an $\bar{h}$ are the conformal weights of the state. Since the dilations and rotations are given by $\ell_0 + \bar{\ell}_0$ and $i(\ell_0 - \bar{\ell}_0)$, the scaling dimension, Δ, and the spin, s, are given by $\Delta = \ell_0 + \bar{\ell}_0$ and $s = \ell_0 - \bar{\ell}_0$.

4.3 Primary fields

From (4.2.4) the line element $ds^2 = dz\,d\bar{z}$ transforms under $z \to f(z)$ as

$$ds^2 \to \left(\frac{\partial f}{\partial z}\right)\left(\frac{\partial \bar{f}}{\partial \bar{z}}\right) ds^2, \qquad (4.3.1)$$

where the analytic functions transform as

$$dz \to \left(\frac{\partial f}{\partial z}\right) dz, \quad d\bar{z} \to \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right) d\bar{z}. \qquad (4.3.2)$$

A generalization of this transformation law is given by

$$\Phi(z,\bar{z}) \to \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \Phi(f(z), \bar{f}(\bar{z})) \qquad (4.3.3)$$

where h and $\bar{h}$ are real numbers. This transformation law is similar to the transformation law of tensors, but it differs by the exponent of conformal weights. Any field that transforms under (4.3.3) is known as a primary field of conformal weight $(h, \bar{h})$. All other fields are known as secondary fields.

4.4 Radial quantization and operator product expansions

To examine conformal invariance in 2-dimensional quantum field theory, the Euclidean "space" and "time" coordinates, σ^1 and σ^0 are first mapped onto the complex plane. In Minkowski space the light cone coordinates would be $\sigma^0 \pm \sigma^1$. The complex coordinates in Euclidean space become $\zeta, \bar{\zeta} = \sigma^0 \pm i\sigma^1$. The left-moving and right-moving massless fields in 2-dimensional Minkowski space become Euclidean fields with purely holomorphic and anti-holomorphic dependence on the coordinates. To eliminate any infrared divergences, the space coordinate is compactified, $\sigma^1 \equiv \sigma^1 + 2\pi$. What results is a cylinder in σ^1, σ^0 coordinates [4].

The cylinder can be mapped to the complex plane coordinatized by z using the conformal map $\zeta \to z = e^{\zeta}$, (where $\zeta = \sigma^0 + i\sigma^1$). $\sigma^0 = \pm\infty$, the infinite past and future, are mapped to the points $z = 0, \infty$ on the plane. Equal time surfaces, $\sigma^0 = \text{const}$, become circles on the complex plane, time translation is scaling, and time reversal, $\sigma^0 \to -\sigma^0$, becomes $z \to 1/z^*$.

For a quantum theory of conformal fields on the z-plane, the operators that implement conformal mappings must be realized. Dilations, for example, $z \to e^a z$ are time translations on the cylinder, $\sigma^0 \to \sigma^0 + a$. Because of this, the dilation generator, $\frac{1}{2}(\ell_0 + \bar{\ell}_0)$, on the complex plane can be considered as the Hamiltonian for

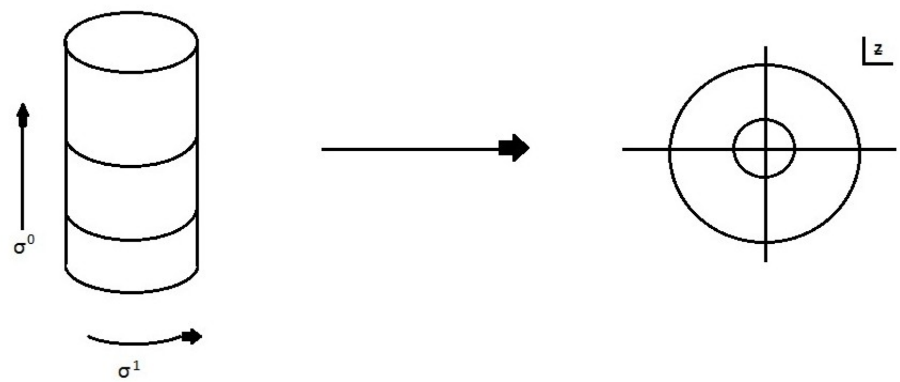

Figure 4.1: Map of cylinder to the plane under the transformation $z = e^{\sigma^0 + i\sigma^1}$.

the system, and the Hilbert space is built up on surfaces of constant radius. This procedure of defining a quantum theory on a plane this way is known as radial quantization [39–41]. Radial quantization for 2-dimensional conformal field theory is useful because it exploits the full power of contour integrals and complex analysis to analyze short distance expansions, conserved charges, etc. [4].

The generators of symmetries can be constructed with the aid of the Noether theorem ([5, 42]). A $d+1$ dimensional quantum theory has a conserved current j^μ associated with an exact symmetry. The conserved current must satisfy $\partial_\mu j^\mu = 0$. Integrating over a fixed time slice gives the conserved charge, Q, that is

$$Q = \int d^d x \; j_0(x). \tag{4.4.1}$$

A consequence of the Noether theorem, extending the Poisson bracket algebra to Lie commutators, is that the infinitesimal variation in any field A is generated by the conserved charge according to $\delta_\epsilon A = \epsilon[Q, A]$ [43]. Local coordinate transformations are generated by charges constructed from the energy-momentum tensor $T_{\mu\nu}$, which has already been shown to be trace-less for conformally invariant theories. The energy-momentum tensor is, in general, symmetric and divergence-free.

Since the holomorphic and anti-holomorphic components of the energy-momentum tensor are generators of symmetries, multiplying T_{zz} by a holomorphic function $f(z)$ results in $T'_{zz} = f(z)T_{zz}$. Then, $\partial_z T'_{zz} = 0$ generates an entirely different symmetry. The holomorphic and anti-holomorphic energy-momentum tensors should generate two independent realizations of the classical Witt algebra (4.2.13). Since T_{zz} can be multiplied by an infinite number of holomorphic functions that still satisfy (4.2.7), there are an infinite number of local symmetries.

Since T and $\bar{T}$ generate local conformal transformations on the z-plane, the conserved charge in radial quantization becomes

$$Q = \frac{1}{2\pi i} \oint \left(dz \; R\Big(T(z)\epsilon(z)\Big) + d\bar{z} \; \Big(\bar{T}(\bar{z})\bar{\epsilon}(\bar{z})\Big)\right). \tag{4.4.2}$$

This line integral is performed over some circle of fixed radius, and by convention, both dz and $d\bar{z}$ integrations are taken in a counter-clockwise direction. In Euclidean space, radial quantization of products of operators $A(z)B(w)$ are defined for $|z| > |w|$. This radial ordering is the familiar time-ordering in quantum mechanics. The radial ordering operation R is defined as

$$R(A(z)B(w)) = \begin{cases} A(z)B(w) & |z| > |w| \\ B(w)A(z) & |z| < |w|. \end{cases} \tag{4.4.3}$$

The variation of any field is given by its commutator with the charge (4.4.2)

$$\delta_{\epsilon,\bar{\epsilon}}\Phi(w,\bar{w}) = \frac{1}{2\pi i} \oint \left(dz \; \epsilon(z) \left[T(z), \Phi(w,\bar{w})\right] + d\bar{z} \; \bar{\epsilon}(\bar{z}) \left[\bar{T}(\bar{z}), \Phi(w,\bar{w})\right]\right). \tag{4.4.4}$$

This can be thought of as an "equal-time" (constant radius) commutator since the integration is over a contour of a specific radius, but the contours can be deformed, as in Figure 4.2, to give a single integration over a single, closed contour.

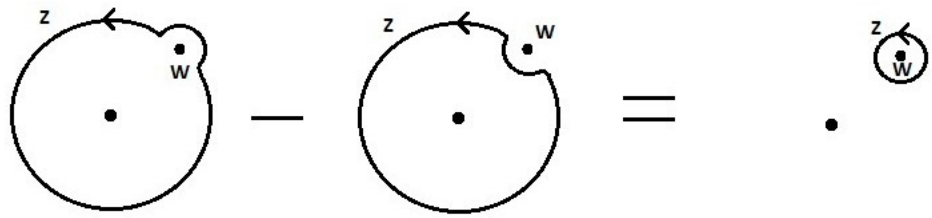

Figure 4.2: Evaluation of "equal-time" commutator on the conformal plane. This shows the difference of the integration over the closed contour drawn tightly around w and the integration over the closed contour around the origin and excluding the point at w. What results is a single contour integral around the point w.

This defines the meaning of the commutators in (4.4.4). The equal-time commutator of a local operator A with the spatial integral of an operator B will become the contour integral of the radially ordered product, $[\int dxB, A]_{E.T.} \to \oint dzR(B(z)A(w))$ [4]. Then rewriting (4.4.4) in the form

$$\begin{aligned}
\delta_{\epsilon,\bar{\epsilon}}\Phi(w,\bar{w}) &= \frac{1}{2\pi i}\left(\oint_{|z|>|w|} - \oint_{|z|<|w|}\right)\Big(dz\ \epsilon(z)R(T(z)\Phi(w,\bar{w})) \\
&\quad + d\bar{z}\,\bar{\epsilon}(\bar{z})R(\bar{T}(\bar{z})\Phi(w,\bar{w}))\Big) \\
&= \frac{1}{2\pi i}\oint\Big(dz\ \epsilon(z)R(T(z)\Phi(w,\bar{w})) + d\bar{z}\ \bar{\epsilon}(\bar{z})R(\bar{T}(\bar{z})\Phi(w,\bar{w}))\Big) \\
&= h\ \partial_w\ \epsilon(w)\Phi(w,\bar{w}) + \epsilon(w)\ \partial_w\Phi(w,\bar{w}) \\
&\quad + \bar{h}\ \bar{\partial}_{\bar{w}}\ \bar{\epsilon}(\bar{w})\Phi(w,\bar{w}) + \bar{\epsilon}(\bar{w})\bar{\partial}_{\bar{w}}\Phi(w,\bar{w})
\end{aligned}$$

where the infinitesimal expansion of $f(z)$, $f(z) = z + \epsilon(z)$ under the transformation (4.3.3), has been substituted in the last line to give the desired result and Cauchy integral formula has been applied. The Cauchy integral formula is given by

$$f^{(n)}(w) = \frac{n!}{2\pi i}\oint_\gamma \frac{f(z)}{(z-w)^{n+1}}dz. \tag{4.4.5}$$

For (4.4.2) to induce the correct infinitesimal conformal transformations, the short distance singularities of T and $\bar{T}$ with Φ should be

$$\begin{aligned} R\Big(T(z)\Phi(w,\bar{w})\Big) &= \frac{h}{(z-w)^2}\Phi(w,\bar{w}) + \frac{1}{z-w}\partial_w\Phi(w,\bar{w}) + \cdots \\ R\Big(\bar{T}(z)\Phi(w,\bar{w})\Big) &= \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\Phi(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}}\Phi(w,\bar{w}) + \cdots \end{aligned}$$

These become the short-distance expansions of radially-ordered operator product expansions (commonly referred to as OPE's) for the holomorphic and anti-holomorphic energy-momentum tensors, T and $\bar{T}$, with the primary field. The symbol R will be dropped from now on, and the operator product expansion will assumed to be radially ordered. The operator product expansions become

$$T(z)\Phi(w,\bar{w}) = \frac{h}{(z-w)^2}\Phi(w,\bar{w}) + \frac{1}{z-w}\partial_w\Phi(w,\bar{w}) + \cdots \tag{4.4.6}$$

$$\bar{T}(z)\Phi(w,\bar{w}) = \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\Phi(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}}\Phi(w,\bar{w}) + \cdots \tag{4.4.7}$$

The quantum energy-momentum tensor can be thought of as being defined by these short distance properties. In quantum field theory, when operators approach one another singularities can occur. The singularities are encoded in operator product expansions of the form

$$A(x)B(x) \sim \sum_i C_i(x-y)O_i(y) \tag{4.4.8}$$

where the O_i's are a complete set of local operators and the C_i's are (singular) numerical coefficients. Quantum field theory traditionally deals with scattering amplitudes between various asymptotic states (free particles). In practice these amplitudes are given by n-point correlations functions [5] and are a perturbative result. OPE's in 2-dimensional conformal field theories are non-perturbative and make use of powerful contour integrals in complex theory. Taking the basis of operators ϕ_i with fixed conformal weight, the normalized 2-point functions become

$$\langle \phi_i(z,\bar{z})\phi_j(w,\bar{w})\rangle = \delta_{ij}\frac{1}{(z-w)^{2h_i}}\frac{1}{(\bar{z}-\bar{w})^{2\bar{h}_i}}. \tag{4.4.9}$$

Higher point correlation functions can be written as ordered product expansions and can be written in terms of 2-point functions using Wick's theorem. Wick's theorem extracts operators from their normal ordering by replacing the operators with their n-point function. n-point correlation functions can be written in terms of singular 2-point ordered product expansions which can be solved using the Cauchy integral formula.

In general, the OPE of the energy-momentum tensor with the primary field is given by (4.4.6) and (4.4.7), with conformal dimensions h and $\bar{h}$. The ordered product expansion of the energy-momentum with itself is given by

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{(z-w)}\partial_w T(w) \tag{4.4.10}$$

$$\bar{T}(\bar{z})\bar{T}(\bar{w}) = \frac{\bar{c}/2}{(\bar{z}-\bar{w})^4} + \frac{2}{(\bar{z}-\bar{w})^2}\bar{T}(\bar{w}) + \frac{1}{(\bar{z}-\bar{w})}\partial_{\bar{w}}\bar{T}(\bar{w}) \tag{4.4.11}$$

where c is a constant and turns out to be the central charge. Later it will be shown that modular invariance constrains $c - \bar{c} = 0$. A theory with a Lorentz-invariant, conserved 2-point function requires $c = \bar{c}$. This is equivalent to requiring cancellation of local gravitational anomalies [44], allowing the system to be consistently coupled to 2-dimensional gravity [4].

It is often convenient to define the Laurent expansion of the stress-energy tensor,

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}L_n, \qquad \bar{T}(\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{z}^{-n-2}\bar{L}_n, \tag{4.4.12}$$

in terms of modes L_n (which are also operators themselves) [4]. Inverting the relations gives

$$L_n = \oint \frac{dz}{2\pi i} z^{n+1}T(z), \qquad \bar{L}_n = \oint \frac{d\bar{z}}{2\pi i}\bar{z}^{n+1}\bar{T}(\bar{z}). \tag{4.4.13}$$

A finite subset of the L_n's and $\bar{L}_n$'s are the quantum generators of global conformal symmetry, like the classical generators (4.2.12). L_{-1}, L_0, L_1 close, under commutation, forming a subalgebra of operators. The commutation relations can be determined from the classical Witt algebra (4.2.13):

$$[L_{\pm 1}, L_0] = \pm L_{\pm 1} \quad [L_1, L_{-1}] = 2L_0. \tag{4.4.14}$$

It is easy to see why L_{-1}, L_0, L_1 are generators of global conformal symmetry. From the definition of L_n in (4.4.13), it is clear that in the Cauchy integral L_{-1} and L_0 do not produce any poles; thus the generators are globally defined over the entire complex plane. The generator L_1 also does not produce a pole, however this can be rectified by the conformal transformation $w = 1/z$, which gives the same argument as for the generator L_{-1} with the coordinate w in place of z, and thus L_1 is globally defined. This transformation does not work for $n \geq 2$.

The commutation relations of the operators, and therefore the algebra, can be determined by simply knowing the OPE's between the operators. Consider the holomorphic fields $a(z)$ and $b(w)$ and the integral

$$\oint_w dz\ a(z)b(z) \tag{4.4.15}$$

where the integration is over the contour of counterclockwise circles around w, and radial ordering of fields is assumed, as usual. This integral can be split into two by the difference of two integrations of contours in opposite directions to give the commutator:

$$\begin{aligned} \oint_w dz\ a(z)b(w) &= \oint_{C_1} dz\ a(z)b(w) - \oint_{C_2} dz\ b(w)a(z) & (4.4.16) \\ &= [A, b(w)], & (4.4.17) \end{aligned}$$

where A is given by the contour integral

$$A = \oint a(z)dz, \tag{4.4.18}$$

and C_1 and C_2 are fixed-time contours around the origin.

Then the commutator of two operators, $[A, B]$, each the integral of a holomorphic field, is determined by integrating (4.4.17) over w [5]:

$$[A, B] = \oint_0 dw \oint_w dz\ a(z)b(w), \tag{4.4.19}$$

where

$$A = \oint a(z)dz \quad B = \oint b(z)dz. \tag{4.4.20}$$

For fermions this commutator becomes an anti-commutator. This is again an "equal-time" commutator which is allowed by the subtraction of contours as seen before in Figure 4.2. The commutator (4.4.19) is evaluated by substituting the OPE of $a(z)$ with $b(w)$, of which only terms in $1/(z-w)$ contribute, by the theorem of residues [5].

Everything is now in place to write the algebra of the charges (mode expansions in (4.4.13)) using the relation (4.4.19) and the OPE (4.4.10):

$$\begin{aligned}
[L_n, L_m] &= \frac{1}{(2\pi i)^2}\oint_0 dw\ w^{m+1}\oint_w dz\ z^{n+1}\left\{\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} + reg.\right\} \\
&= \frac{1}{2\pi i}\oint_0 dw\ w^{m+1}\left\{\frac{c}{12}(n+1)n(n-1)w^{n-2}\right. \\
&\qquad\qquad\left. + 2(n+1)w^n T(w) + w^{n+1}\partial_w T(w)\right\} \\
&= \frac{c}{12}n(n^2-1)\delta_{n+m,0} + 2(n+1)L_{m+n} \\
&\qquad\qquad - \frac{1}{2\pi i}\oint_0 dw\ (n+m+2)w^{n+m+1}T(w) \\
&= \frac{c}{12}n(n^2-1)\delta_{n+m,0} + 2(n+1)L_{m+n} + (n-m)L_{m+n},
\end{aligned}$$

where in the first step the contour integral over z is solved using the Cauchy integral formula, and the last term in the third step is solved using integration by parts. This is the Virasoro algebra. It is the quantum extension of the classical Witt algebra.

4.5 Free boson example

The massless, free boson, φ, is the simplest conformal field theory. Its action is given by

$$S = \frac{1}{2} g \int d^2x \; \partial_\mu \varphi \partial^\mu \varphi \tag{4.5.1}$$

where g is a normalization constant. The 2-point function in complex coordinates is given by

$$\langle \varphi(z,\bar{z}) \varphi(w,\bar{w}) \rangle = -\frac{1}{4\pi g} \Big\{ \ln(z-w) + \ln(\bar{z}-\bar{w}) \Big\} + \text{const.} \tag{4.5.2}$$

Taking the derivatives $\partial_z \varphi$ and $\partial_{\bar{z}} \varphi$ separates the 2-point function into holomorphic and anti-holomorphic components:

$$\langle \partial_z \varphi(z,\bar{z}) \partial_w \varphi(w,\bar{w}) \rangle = -\frac{1}{4\pi g} \frac{1}{(z-w)^2} \tag{4.5.3}$$

$$\langle \partial_{\bar{z}} \varphi(z,\bar{z}) \partial_{\bar{w}} \varphi(w,\bar{w}) \rangle = -\frac{1}{4\pi g} \frac{1}{(\bar{z}-\bar{w})^2}. \tag{4.5.4}$$

Focusing just on the holomorphic field from now on, the ordered product expansion of the holomorphic field with itself is given by

$$\partial_z \varphi(z) \partial_z \varphi(w) \sim -\frac{1}{4\pi g} \frac{1}{(z-w)^2}. \tag{4.5.5}$$

The quantum energy-momentum tensor associated with the massless, free boson in complex coordinates is

$$T(z) = -2\pi g : \partial_z \varphi \partial_z \varphi : \tag{4.5.6}$$

where $:\varphi_i \cdots \varphi_j:$ denotes the normal ordering (annihilation operators to the left creation operators to the right) of field operators. The ordered product expansion of $T(z)$ with $\partial_z \varphi$ can be calculated using Wick's theorem:

$$\begin{aligned} T(z)\partial_w\varphi(w) &= -2\pi g : \partial_z\varphi(z)\partial_z\varphi(z) : \partial_w\varphi(w) \\ &\sim -4\pi g : \partial_z\varphi(z)\partial_z\overbrace{\varphi(z) : \partial_w\varphi}(w) \\ &\sim \frac{\partial_z\varphi(z)}{(z-w)^2} \end{aligned}$$

Taylor expanding $\partial_z\varphi(z)$ around w yields

$$T(z)\partial_w\varphi(w) \sim \frac{\partial_w\varphi(w)}{(z-w)^2} + \frac{\partial_w^2\varphi(w)}{(z-w)}. \tag{4.5.7}$$

By comparing with (4.4.6), this shows that $\partial_z\varphi$ is a primary field with conformal dimension $h = 1$. The field φ has no spin and no scaling dimension, however its derivative, $\partial_z\varphi$, has scaling dimension $\Delta = 1$.

Calculating the OPE of the energy-momentum tensor with itself gives

$$\begin{aligned} T(z)T(w) &= 4\pi^2 g^2 : \partial_z\varphi(z)\partial_z\varphi(z) :: \partial_w\varphi(w)\partial_w\varphi(w) : \\ &\sim \frac{1/2}{(z-w)^4} - \frac{4\pi g : \partial_z\varphi(z)\partial_w\varphi(w) :}{(z-w)^2} \\ &\sim \frac{1/2}{(z-w)^4} - \frac{2T(z)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)}. \end{aligned} \tag{4.5.8}$$

By (4.4.10) it is clear that the central charge for a massless, free boson is $c = 1$. It is also clear that the energy-momentum tensor is not a primary field because of the anomalous term $1/(z-w)^4$ which does not appear in (4.4.6) [5]. This anomalous term is of great interest, and will be dealt with later.

4.6 Highest weight representations

Consider the state

$$|h\rangle = \phi(0)|0\rangle, \tag{4.6.1}$$

created by the holomorphic field $\phi(z)$ of weight h. Substituting (4.4.13) into (4.4.6) and then simplifying using the Cauchy integral formula (4.4.5) gives

$$[L_n, \phi(w)] = \oint \frac{dz}{2\pi i} z^{n+1} T(z)\phi(w) = h(n+1)w^n \phi(w) + w^{n+1}\partial_w \phi(w), \tag{4.6.2}$$

so that $[L_n, \phi(0)] = 0, n > 0$. The state $|h\rangle$ then satisfies

$$L_0|h\rangle = h|h\rangle \quad L_n|h\rangle = 0, n > 0. \tag{4.6.3}$$

The anti-holomorphic sector follows similar arguments for the anti-holomorphic operators. In general a state $|h, \bar{h}\rangle$ is created by a primary field $\phi(z, \bar{z})$ of conformal weight $(h, \bar{h})$. $|h, \bar{h}\rangle$ is known as an "in" state. Any state satisfying (4.6.3) is known as a highest weight state. Descendant states are states found by acting on the highest weight states with other quantum operators, $L_{-n_1}...L_{-n_k}|h\rangle$.

The "out" state, $\langle h|$, satisfies

$$\langle h|L_0 = h\langle h| \qquad \langle h|L_n = 0, n < 0 \tag{4.6.4}$$

where $L_n^\dagger = L_{-n}$. $L_0 \pm \bar{L}_0$ are the generators of dilatations and rotations, thus $h \pm \bar{h}$ is identified as the scaling dimension and Euclidean spin of the state. The states $\langle h|L_{n_1} \cdots L_{n_K}$, $(n_i > 0)$ are descendants of the out states [4].

Representations of the Virasoro algebra start with a single primary field and remaining fields in the representation are found by successive operator products with the energy-momentum tensor. All the fields together make up a representation $[\phi_n]$.

If dealing with modes, then descendant fields are found from commuting L_{-n}'s with the primary field. Acting on the vacuum with descendant fields creates descendant states.

Descendant fields $\hat{L}_{-n}\phi$, $n > 0$ can be found from the less singular parts of the OPE of $T(z)$ with a primary field, ϕ,

$$T(z)\phi(w,\bar{w}) \equiv \sum_{n\geq 0}(z-w)^{n-2}\hat{L}_{-n}\phi(w,\bar{w})$$
$$= \frac{1}{(z-w)^2}\hat{L}_0\,\phi + \frac{1}{z-w}\hat{L}_1\,\phi + \hat{L}_2\,\phi + (z-w)\hat{L}_3\,\phi + \cdots \tag{4.6.5}$$

The fields

$$\hat{L}_{-n}\,\phi(w,\bar{w}) = \oint \frac{dz}{2\pi i}\frac{1}{(z-w)^{n-1}}T(z)\phi(w,\bar{w}) \tag{4.6.6}$$

are often denoted as $\phi^{(-n)}$. The conformal weight of the descendant field $\hat{L}_{-n}\,\phi$ is $(h+n,\bar{h})$. By (4.4.6) and (4.4.7) the first 2 descendant fields are $\phi^{(0)} = \hat{L}_0\,\phi = h\phi$ and $\phi^{(-1)} = \hat{L}_{-1}\,\phi = \partial\phi$. $L_n\,\phi = 0$ for all $n > 0$ [4].

The first few descendant fields, ordered by their conformal weights, are

level	dimension	field
0	h	ϕ
1	$h+1$	$\hat{L}_{-1}\phi$
2	$h+2$	$\hat{L}_{-2}\phi$, $\hat{L}_{-1}^2\phi$
3	$h+3$	$\hat{L}_{-3}\phi$, $\hat{L}_{-1}\hat{L}_{-2}\phi$, $\hat{L}_{-1}^3\phi$
$\vdots$	$\vdots$	$\vdots$
N	$h+N$	$P(N)$ fields,

(4.6.7)

where $P(N)$ is the number of states at level N. $P(N)$ is given in terms of the generating function

$$\frac{1}{\prod_{n=1}^{\infty}(1-q^n)} = \sum_{N=0}^{\infty} P(N)q^N, \tag{4.6.8}$$

where $P(0) = 1$. The fields in (4.6.7) arise from repeated short distance expansions of the primary field ϕ with $T(z)$ [4].

This process can be repeated with any other highest weight states. Removing the null vectors results in what is known as the Verma Module. Taking the tensor product of all the Verma Modules obtained from all the different highest weight states, corresponding to all the different primary fields, makes up the Hilbert space of 2-dimensional conformal field theory.

4.7 Central extension and the central charge

It must be determined if the algebra of the operators is the same as that of the infinitesimal generators, the classical Witt algebra,

$$[L_m, L_n] \stackrel{?}{=} (m-n)L_{m+n}. \tag{4.7.1}$$

Acting the commutation relations, (4.7.1), on the vacuum yields:

$$\begin{aligned}
<0|[L_m, L_n]|0> &= <0|(n-m)L_{m+n}|0> \\
<0|[L_{-n}, L_n]|0> &= <0|-2nL_0|0> \qquad \text{where } m=-n \\
<0|L_{-n}L_n - L_nL_{-n}|0> &= -2n <0|L_0|0> \\
-<0|L_nL_{-n}|0> &= -2n <0|L_0|0> \qquad \text{since } L_n|0>=0, (4.6.3) \\
<n|n> &= 2n <0|L_0|0> \qquad \text{since } L_{-n} \text{ is the raising operator}
\end{aligned}$$

Because $L_0|0>=0$, the only unitary solution that exists is the trivial solution. This can be fixed by adding a central extension of the unique form $\frac{c}{12}(n^3-n)\delta_{n+m,0}$, where c is the central charge.

The central charge can not be determined solely from symmetry considerations. It depends on the specific model studied: $c = 1$ for a free boson, $c = \frac{1}{2}$ for a free fermion, for example. When free fields are put together, the central charge becomes the sum of the number of free fields. The central charge is somehow an extensive measure of the number of degrees of freedom of the system [5]. Accordingly, the central charge is related to Casimir energy, the ground state vacuum energy from quantum fluctuations.

Thus the holomorphic and anti-holomorphic commutation relations of the quantum operators becomes

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m,0} \tag{4.7.2}$$

$$[\bar{L}_n, \bar{L}_m] = (n-m)\bar{L}_{n+m} + \frac{\bar{c}}{12}(n^3-n)\delta_{n+m,0} \tag{4.7.3}$$

$$[L_n, \bar{L}_m] = 0. \tag{4.7.4}$$

The algebra obeyed by the L_m in (4.7.2) and the $\bar{L}_m$ in (4.7.3) is again the Virasoro algebra. The conformal field theory realizes two copies of this infinite-dimensional algebra.

4.8 Modular invariance

Consider the mappings of a segment of the z-plane to the cylinder, and the cylinder to the torus. This can be done by identifying two opposite sides of the segment on the z-plane to map the plane to the cylinder, then identifying the two remaining sides to map the cylinder to the torus.

The identifications made in order to achieve this are described by the equations

$$z \sim z + L_1, \qquad z \sim z + iL_2. \tag{4.8.1}$$

The identifications are analytic identifications: $z \sim f(z)$, with f an analytic function. Neither L_1 nor L_2 is a parameter of the torus. Since the z-plane is governed by a

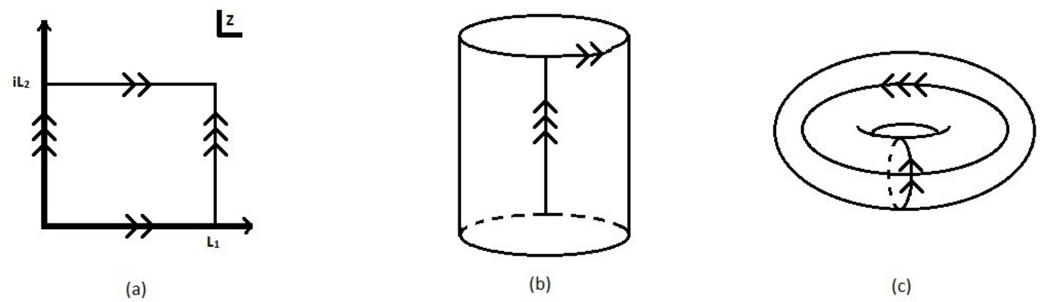

Figure 4.3: (a) A rectangular torus is a rectangular region of the complex z-plane with identifications. (b) Gluing the vertical sides of the rectangular region gives a cylinder. (c) Gluing the horizontal sides as well gives a torus.

conformally invariant theory, there is no change from scaling the coordinate z by a constant. Letting $z' = z/L_1$ gives a new form of (4.8.1):

$$z' \sim z' + 1, \qquad z' \sim z' + iT, \quad T \equiv \frac{L_2}{L_1}. \tag{4.8.2}$$

Thus, the torus is only parameterized by a single parameter, T. Rectangular tori with different T parameters can sometimes be conformally equivalent; tori with parameters T and $1/T$ are conformally equivalent. Therefore, tori with $0 < T \leq 1$ are conformally invariant to tori with $1 \leq T < \infty$. So the moduli space of rectangular tori can be chosen to be the interval $0 < T \leq 1$ or $1 \leq T < \infty$ [45].

Rectangular tori are not the only conformally equivalent tori. Tori can be twisted and identified to produce a cylinder with a twist. Consider the z-plane in Figure 4.4.

The torus is obtained from the identifications

$$z \sim z + \omega_1, \qquad z \sim z + \omega_2 \tag{4.8.3}$$

where ω_1 and ω_2 are both complex numbers. Then by defining

$$\tau \equiv \frac{\omega_2}{\omega_1} \tag{4.8.4}$$

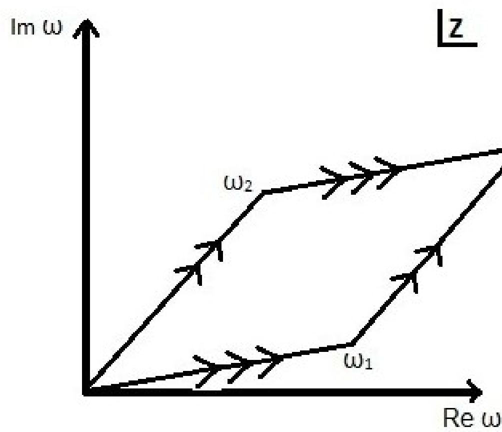

Figure 4.4: A general torus is obtained from the complex z-plane by identifying $z \sim z + \omega_1$ and $z \sim z + \omega_2$.

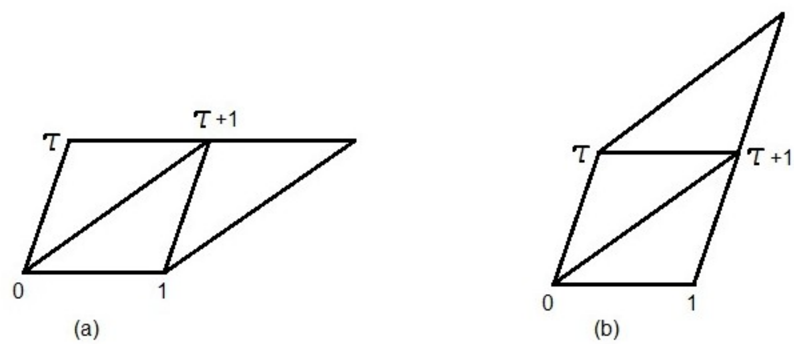

Figure 4.5: The modular transformations: (a) $T : \tau \to \tau + 1$, (b) $U : \tau \to \tau/(\tau + 1)$.

and scaling z by a factor of $1/\omega_1$ the identifications are equivalent to

$$z \sim z + 1 \qquad z \sim z + \tau. \tag{4.8.5}$$

In general a torus is specified by two periods, ω_1, ω_2. By rescaling the coordinates, these 2 periods can be written in terms of 1 and τ, the modular parameter [5, 45]

The group of disconnected diffeomorphisms of the torus is the modular group. It is generated by cutting along either of the non-trivial cycles then regluing after a twist by 2π. The transformation $T : \tau \to \tau + 1$ is generated by cutting along the line of constant σ^0 then regluing. The transformation $U : \tau \to \tau/(\tau+1)$ is generated by cutting along the line of constant σ^1 then regluing. For the modular transformation U, the new modular parameter after coordinate rescaling becomes $\omega \to \omega/(\tau+1)$. The transformations T and U generate a group of transformations, known as the modular group, given by

$$\tau \to \frac{a\tau+b}{c\tau+d} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{Z}), \tag{4.8.6}$$

where $a, b, c, d \in \mathbf{Z}$, and $ad - bc = 1$. Since τ is invariant under change in sign of a, b, c, d in (4.8.6), the modular group is actually $SL(2,\mathbf{Z})/\mathbf{Z}_2$. The generators of the modular group are $S = T^{-1}UT^{-1}$ satisfying $S^2 = (ST)^3 = 1$. S and T can be represented explicitly as

$$T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.8.7}$$

S acts to interchange the boundary conditions in "time" and "space" directions [4]. The S transformation is given by $S : \tau \to \frac{-1}{\tau}$.

Therefore, in the mapping of the complex plane to the torus, the only parameter to consider is the modular parameter τ.

4.9 The partition function

The partition function (or vacuum functional, in Minkowski space-time) is used to describe the statistical properties of a system in thermodynamic equilibrium, such as total energy, free energy, entropy (degrees of freedom), and pressure. The partition function for the canonical ensemble, (a system in thermal contact with the environment and with constant number of constituent particles and temperature, T) is given by

$$Z = \sum_n e^{-\beta E_n}, \tag{4.9.1}$$

where $\beta = \frac{1}{k_B T}$ and k_B is the Boltzmann constant. Since the energies of a system are eigenvalues of its Hamiltonian, one can rewrite (4.9.1) as

$$Z = \mathrm{Tr} e^{-\beta H}. \tag{4.9.2}$$

If H and P are the Hamiltonian and total momentum of the theory, the operator that translates the system parallel to the period ω_2 over a distance a in Euclidean space-time is

$$e^{-\frac{a}{|\omega_2|}\{H\,\mathrm{Im}\,\omega_2 - iP\,\mathrm{Re}\,\omega_2\}}. \tag{4.9.3}$$

If a is considered to be the lattice spacing, (4.9.3) translates from one row of the lattice to the next, but parallel to the period ω_2. If the complete period contains m lattice spacings ($|\omega_2| = ma$) then the partition function is found by taking the trace of (4.9.3) to the m-th power [5]:

$$Z(\omega_1, \omega_2) = \mathrm{Tr}\, e^{-H\,\mathrm{Im}\,\omega_2 - iP\,\mathrm{Re}\,\omega_2}. \tag{4.9.4}$$

H and P can be expressed in terms of Virasoro generators L_0 and $\bar{L}_0$. Treating the torus as identifying the open ends of a finite cylinder of circumference L, the Hamiltonian operator is $H = (2\pi/L)(L_0 + \bar{L}_0 - c/12)$. Here the Virasoro generators are defined on the whole complex plane and the constant term is added to make the vacuum energy density vanish in the $L \to \infty$ limit. Similarly, the momentum operator, (which generates translations along the circumference of the cylinder) is given by $P = (2\pi/L)(L_0 - \bar{L}_0)$. Since ω_1 has been chosen to be real and equal to L, the partition function becomes

$$\begin{aligned} Z(\tau,\bar\tau) &= \mathrm{Tr}\, e^{\pi i\left\{(\tau-\bar\tau)(L_0+\bar L_0-c/12)+(\tau+\bar\tau)(L_0-\bar L_0-\bar c/12)\right\}} \\ &= \mathrm{Tr}\, e^{2\pi i\left\{\tau(L_0-c/24)-\bar\tau(\bar L_0-\bar c/24)\right\}}. \end{aligned} \quad (4.9.5)$$

By defining the parameters

$$q = e^{2\pi i\tau}, \quad \bar q = e^{-2\pi i\bar\tau}, \quad (4.9.6)$$

the partition function can be written as

$$Z(\tau,\bar\tau) = \mathrm{Tr}\, \left(q^{L_0-c/24}\bar q^{\bar L_0-\bar c/24}\right). \quad (4.9.7)$$

This partition function must be modularly invariant. Since τ parameterizes any equivalent tori, consider (4.9.7) under the transformation $T : \tau \to \tau + 1$:

$$\begin{aligned} Z(\tau+1,\bar\tau+1) &= \mathrm{Tr}_{\mathcal{H}_i}\left(q^{L_0-c/24}\, e^{2\pi i(L_0-c/24)}\, \bar q^{\bar L_0-\bar c/24}\, e^{-2\pi i(\bar L_0-\bar c/24}\right) \\ &= \sum_i q^{h_i-c/24}\, \bar q^{\bar h_i-\bar c/24}\, e^{2\pi i(h_i-\bar h_i)}\, e^{2\pi i(\frac{-c+\bar c}{24})}, \end{aligned} \quad (4.9.8)$$

$$(4.9.9)$$

where the trace of the operators L_0 and $\bar L_0$ is taken over the entire Hilbert space to give the eigenvalues h_i and $\bar h_i$. For the vacuum, $h_0 = \bar h_0 = 0$, there are two constraints.

Firstly, modular invariance imposes that $e^{2\pi i(\frac{-c+\bar c}{24})} = 1$ and therefore $\frac{\bar c - c}{24} = \mathbb{Z}$. This leads to the constraint

$$\bar c - c = m\ mod\, 24, \qquad m\epsilon\mathbb{Z}. \quad (4.9.10)$$

Secondly, modular invariance imposes $e^{2\pi i(h_i-\bar h_i)} = 1$, which imposes

$$h_i - \bar h_i = \mathbb{Z}. \quad (4.9.11)$$

Therefore, the partition function is invariant under the transformation T. A similar result would be obtained from (4.9.7) under the transformation U.

The partition function will be used to derive the Cardy formula.

4.10 The Cardy formula

The derivation of the Cardy formula, in this thesis, will closely follow Carlip's [37, 38] derivation. Carlip's derivation exploits the modular invariance of the partition function of a 2-dimensional CFT on a torus. This derivation is for a microcanonical ensemble, an isolated thermodynamical system holding the number of particles, N, volume, V, and total energy, E, constant. The Cardy formula is used to determine the density of states, $\rho(E)$ of a conformal field theory. Then the entropy of the conformal field theory is essentially the logarithm of the density of states [37].

The partition function (4.9.7) on the torus of modulus $\tau = \tau_1 + i\tau_2$ can be written in terms of the density of states of the system:

$$Z(\tau, \bar{\tau}) = \operatorname{Tr} e^{2\pi i \tau L_0} e^{-2\pi i \bar{\tau} \bar{L}_0} = \sum \rho(\Delta, \bar{\Delta}) e^{2\pi i \Delta \tau} e^{-2\pi i \bar{\Delta} \bar{\tau}} \tag{4.10.1}$$

where ρ is the number of states with eigenvalues $\Delta, \bar{\Delta}$ for the Virasoro generators $L_0, \bar{L}_0$.

The basic result of Cardy [11, 12] is that the quantity

$$Z_0(\tau, \bar{\tau}) = \operatorname{Tr} e^{2\pi i (L_0 - \frac{c}{24})\tau} e^{-2\pi i (\bar{L}_0 - \frac{c}{24}\bar{\tau})} = e^{\frac{\pi c}{6}\tau_2} Z_0(\tau, \bar{\tau}) \tag{4.10.2}$$

is modular invariant, under the transformation $\tau \to \frac{-1}{\tau}$ particularly.

Treating τ and $\bar{\tau}$ as complex variables ρ can be extracted from Z by contour integration by

$$\rho(\Delta, \bar{\Delta}) = \frac{1}{(2\pi i)^2} \oint \frac{dq}{q^{\Delta+1}} \frac{d\bar{q}}{\bar{q}^{\bar{\Delta}+1}} Z(q, \bar{q}). \tag{4.10.3}$$

This integration is not over a specific contour since contours can be deformed as needed, however the contour is around the origin since there is a pole at $q = 0, \bar{q} = 0$. For simplicity of notation, the $\bar{\tau}$ dependence shall be suppressed.

By noting that

$$Z(\tau) = e^{\frac{2\pi i c}{24}\tau} Z_0(\tau) \tag{4.10.4}$$

the modular invariance of Z_0 can be exploited to rewrite the contour integral in a form suitable for the saddle-point approximation:

$$\begin{aligned} Z(\tau) &= e^{\frac{2\pi i c}{24}\tau} Z_0\left(\frac{-1}{\tau}\right) \\ &= e^{\frac{2\pi i c}{24}\tau} e^{\frac{2\pi i c}{24}\frac{1}{\tau}} Z_0\left(\frac{-1}{\tau}\right) \end{aligned} \tag{4.10.5}$$

and therefore,

$$\rho(\Delta) = \int d\tau\, e^{-2\pi i \Delta\tau} e^{\frac{2\pi i c}{24}\tau} e^{\frac{2\pi i c}{24}\frac{1}{\tau}} Z_0\left(\frac{-1}{\tau}\right). \tag{4.10.6}$$

With this result, the integral (4.10.3) can be evaluated by a saddle-point approximation. Let Δ_0 be the lowest eigenvalue of L_0 and define

$$\tilde{Z}(\tau) = \sum \rho(\Delta) e^{2\pi i(\Delta-\Delta_0)\tau} = \rho(\Delta_0) + \rho(\Delta_1) e^{2\pi i(\Delta_1-\Delta_0)\tau} + \cdots . \tag{4.10.7}$$

Then using the result (4.10.6) with (4.10.7) gives

$$\rho(\Delta) = \int d\tau\, e^{-2\pi i \Delta\tau} e^{-2\pi i \Delta_0 \frac{1}{\tau}} e^{\frac{2\pi i c}{24}\tau} e^{\frac{2\pi i c}{24}\frac{1}{\tau}} \tilde{Z}\left(\frac{-1}{\tau}\right). \tag{4.10.8}$$

For large τ_2, $\tilde{Z}(-1/\tau)$ approaches $\rho(\Delta_0)$, which is a constant. Therefore, (4.10.7) can be evaluated as long as τ_2 is large at the saddle point. For the saddle-point approximation the integrand must be separated into a rapidly varying phase and a slowly varying prefactor. The integral required should then be of the form

$$I[a,b] = \int d\tau \exp\left[2\pi i a\tau + \frac{2\pi i b}{\tau}\right] f(\tau). \tag{4.10.9}$$

The exponential term is extremal at $\tau_0 = \sqrt{\frac{b}{a}}$. Then, expanding the integral around τ_0 gives

$$I[a,b] \approx \int d\tau \exp\left[4\pi i\sqrt{ab}+\frac{2\pi ib}{\tau_0^3}(\tau-\tau_0)^2\right] f(\tau_0) = \left(-\frac{b}{4a^3}\right)^{1/4} e^{4\pi i\sqrt{ab}} f(\tau_0). \quad (4.10.10)$$

If Δ_0 is small ($\Delta_0 << c$) and Δ is large, the integral (4.10.7) gives

$$\rho(\Delta) = \left(\frac{c}{96\Delta^3}\right)^{1/4} \exp 2\pi\sqrt{\frac{c\Delta}{6}} + \text{higher order terms.} \quad (4.10.11)$$

Taking the natural logarithm of both τ and $\bar{\tau}$ dependence of the exponential term of (4.10.11) yields

$$\ln \rho(\Delta, \bar{\Delta}) = 2\pi\sqrt{\frac{c\Delta}{6}} + 2\pi\sqrt{\frac{\bar{c}\bar{\Delta}}{6}} \quad (4.10.12)$$

which is Cardy's [11, 12] result for the asymptotic density of states, or the standard Cardy formula. The leading term in (4.10.11) gives the leading correction.

Chapter 5

The Hamiltonian formulation of general relativity

In this chapter the ADM Hamiltonian with the surface term, often left out in other treatments, will be derived and the Hamiltonian generators, which will be used to calculate the conserved charges in general relativity, will be defined. The same conventions as previous chapters will be used. The main references for this chapter are [16, 46, 47].

5.1 Einstein's field equations in (1+3)-form

Einstein's field equations were talked about in detail in Chapter 1. They related the curvature of space-time to an energy-momentum. However, there is no notion of time-evolution, or dynamics of these equations. In their famous 1962 paper, Arnowitt, Deser, and Misner [16] developed the Hamiltonian formulation of general relativity. Not only was this an effective formulation to describe the dynamics of general relativity, but with a Hamiltonian formalism, it was also a possible, in principle, to apply the machinery to canonical quantization of gravity. The basic idea of [16] is to isolate the time coordinate in the field equations. This formalism is known as the (1+3) form. The Hamiltonian follows from it.

Consider a scalar field $t(x^\mu)$ such that t defines a family of non-intersecting space-like hypersurfaces. On each hypersurface $\Sigma(t)$ the spatial coordinate y^i is introduced. Then, introducing a congruence of curves that are parameterized by t and that intersect the hypersurfaces $\Sigma(t)$, the two hypersurfaces can be connected. Also, the points on each of the hypersurfaces intersected by the same curve are defined to be given by the same spatial coordinate y^i. Therefore, this procedure introduces a valid 4-dimensional coordinate system $x^\mu = (t, y^i)$ in the spacetime.

The tangent vectors to the surfaces and to the congruence of curves then are

$$e_i^\mu = \frac{\partial x^\mu}{\partial y^i} \tag{5.1.1}$$

$$t^\mu = \frac{\partial x^\mu}{\partial t} \tag{5.1.2}$$

where these act as the projection tetrad to the hypersurface $\Sigma(t)$. Here x^μ is the original spacetime coordinates and y^i are the new spatial coordinates on the hypersurfaces $\Sigma(t)$. The unit normal to the hypersurfaces is given by

$$n_\mu = -N\partial_\mu t, \tag{5.1.3}$$

where N is the scalar function known as the *lapse*. N ensures proper normalization. The time tangent vector t^i can be decomposed into the form

$$t^i = Nn^\mu + N^i e_i^\mu, \tag{5.1.4}$$

where the 3-function N^i form a spatial vector known as the *shift*.

From the coordinate transformation $x^\mu = x^\mu(t, y^i)$, a differential piece of length in a (1+3) coordinate system is given by

$$dx^\mu = t^\mu \; dt + e_i^\mu \; dy^i = (N \; dt)n^\mu + (dy^i + N^i \; dt)e_i^\mu. \tag{5.1.5}$$

Similarly, the original spacetime metric can also be written in the (1+3) form as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -N^2\,dt^2 + h_{ij}(dx^i + N^i\,dt)(dx^j + N^j\,dt), \tag{5.1.6}$$

where h_{ij} is the induced 3-metric on $\Sigma(t)$ given by

$$h_{ij} = g_{\mu\nu}e_i^\mu e_j^\nu = g_{ij}. \tag{5.1.7}$$

(5.1.6) and (5.1.7) give the time-time, time-space, and space-space components of the metric in terms of N, N^i and h_{ij} as $g_{00} = N^iN_i - N^2$; $g_{0i} = N_i$; $g_{ij} = h_{ij}$. From these expressions the contravariant components of the metric can be easily computed to give

$$g^{00} = -N^{-2}, \tag{5.1.8}$$

$$g^{0i} = N^{-2}N^i, \tag{5.1.9}$$

$$g^{ij} = h^{ij} - N^{-2}N^iN^j. \tag{5.1.10}$$

It is just as easy to verify $\sqrt{-g} = N\sqrt{h}$.

Thus spacetime has been split into space and time mathematically by foliating it by a series of spacelike hypersurfaces $\Sigma(t)$ labelled by a coordinate t through a function $t(x^\mu)$ in the spacetime [46]. The induced metric on $\Sigma(t)$ can be expressed in the 4-dimensional notation by

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu, \tag{5.1.11}$$

$$n_0 = -N, \tag{5.1.12}$$

$$n_i = 0, \tag{5.1.13}$$

where n^i is normal to $\Sigma(t)$. This leads the metric components

$$h_{00} = N^iN_i \tag{5.1.14}$$

$$h_{0i} = N_i, \tag{5.1.15}$$

and the spatial components $h_{ij} = g_{ij}$. It should be stressed that even though $h_{\mu\nu}$ is the metric on 3-space, h_{00} and h_{0i} are non-zero because g_{0i} is non-zero [46]. Also, h^{μ}_{ν} acts as a projection tensor onto $\Sigma(t)$.

There are 3 important relations with the normal vector n_{μ} which have been used often to derive many of the upcoming expressions. These are

$$h^{\mu}_{\nu} n^{\nu} = 0, \tag{5.1.16}$$

$$n^{\mu} \nabla_{\rho} n_{\mu} = 0, \tag{5.1.17}$$

$$n_{[\mu} \nabla_{\rho} n_{\nu]} = 0, \tag{5.1.18}$$

where the brackets denote the commutation of indicies. The first relation comes from the definition of the induced metric; the second from differentiating the normalization condition $n^{\mu} n_{\mu} = -1$; the first from the face that n^{μ} is normal to the set of hypersurfaces which foliate spacetime [46].

The covariant 3-space derivative, D_{μ}, is then defined as

$$D_{\mu} X_{\nu} = h^{\rho}_{\mu} h^{\sigma}_{\nu} \nabla_{\rho} X_{\sigma}. \tag{5.1.19}$$

The right hand side is a projection of the 4-dimensional covariant derivative $\nabla_{\mu} X_{\nu}$ onto $\Sigma(t)$ using the projection tensor $h^{\lambda}_{\nu} = \delta^{\lambda}_{\nu} + n^{\lambda} n_{\nu}$ [46], which can be obtained by contracting (5.1.11) with the contravariant 4-dimensional metric $g^{\mu\lambda}$. It will be assumed that D_{μ} acting on scalar functions gives $D_{\mu}\varphi = h^{\nu}_{\mu} \nabla_{\nu} \varphi$. Using these two properties and the standard chain rule, the covariant 3-space derivative of contravariant vectors and second rank covariant tensors can be found (see section 12.2 of [46]). They are given by

$$D_{\mu} V^{\nu} = h^{\nu}_{\lambda} h^{\sigma}_{\mu} \nabla_{\sigma} V^{\lambda}, \tag{5.1.20}$$

$$D_{\mu} T_{\nu\rho} = h^{\sigma}_{\mu} h^{\alpha}_{\nu} h^{\beta}_{\rho} \nabla_{\sigma} T_{\alpha\beta}. \tag{5.1.21}$$

The induced metric and covariant derivative operator intrinsic to the 3-manifold $\Sigma(t)$ are now defined. They describe the local, intrinsic properties of the spatial slices. However, to get a full picture about the structure of spacetime, information about how the spacelike hypersurfaces $\Sigma(t)$ are embedded in the 4-dimensional geometry is needed. Intuition would suggest that this information lies in how the normals to the $\Sigma(t)$ surfaces varies from event to event. This can be quantified by the *extrinsic curvature* defined by

$$K_{\mu\nu} = -h^{\rho}_{\mu} h^{\sigma}_{\nu} \nabla_{\rho} n_{\sigma} = -h^{\rho}_{\mu} \nabla_{\rho} n_{\nu} \tag{5.1.22}$$

The second equality follows from writing $h^{\sigma}_{\nu} = \delta^{\sigma}_{\nu} + n^{\sigma} n_{\nu}$ and using (5.1.17). The first equality illustrates that $K_{\mu\nu}$ carries information about $\nabla_{\mu} n_{\nu}$ projected onto $\Sigma(t)$. It also shows that $K_{\mu\nu} n^{\mu} = K_{\mu\nu} n^{\nu} = 0$. $K^{0\nu} = 0$, since $n_i = 0$, showing that the contravariant components of $K^{\mu\nu}$ are purely spatial [46].

Something that is not obvious is that $K_{\mu\nu}$ is symmetric in its indices. A short proof can be found in either [20, 46]. From this property, the covariant derivative of the normal can be decomposed into tangentials and normals to $\Sigma(t)$:

$$-\nabla_{\mu} n_{\nu} = K_{\mu\nu} + n_{\mu}(n^{\rho} \nabla_{\rho} n_{\nu}) = K_{\mu\mu} + n_{\mu} a_{\nu}. \tag{5.1.23}$$

$K_{\mu\nu}$ and $h_{\mu\nu}$ contain all of the intrinsic and extrinsic information about $\Sigma(t)$, so they should be related to the full Riemann curvature tensor in 4-dimensional spacetime. The 3-dimensional curvature tensor can first be defined, from the standard definition:

$$-{}^{(3)}R^{\mu}_{\nu\rho\sigma} X_{\mu} = D_{\rho} D_{\sigma} X_{\nu} - D_{\sigma} D_{\rho} X_{\nu}. \tag{5.1.24}$$

The 3-dimensional and 4-dimensional are related via the Gauss-Codazzi relations,

$${}^{(3)}R_{\mu\nu\rho\sigma} = h^{\alpha}_{\mu} h^{\beta}_{\nu} h^{\gamma}_{\rho} h^{\delta}_{\sigma} R_{\alpha\beta\gamma\delta} + n_{\theta} n^{\theta} (K_{\mu\rho} K_{\nu\sigma} + K_{\mu\sigma} K_{\nu\rho}). \tag{5.1.25}$$

A derivation for the Gauss-Codazzi relations can be found in either [20, 46]. This result may seem complicated but has a simple geometrical interpretation. The first term is the intrinsic part of the 3-dimensional curvature obtained by projecting the full 4-dimensional curvature onto the 3-surface. The second term comes from the extrinsic properties of the 3-surface embedding into the 4-dimensional space [46, 47].

5.2 The gravitational action in (1+3)-form

The (1+3)-form of general relativity defined in the previous section can now be applied to the standard action principle of general relativity. The standard action in general relativity is given by the Einstein-Hilbert action

$$S = \frac{1}{16\pi G}\int \sqrt{-g}\,(R - 2\Lambda)\,d^4x, \tag{5.2.1}$$

where Λ is the cosmological constant.

(5.2.1) can be recast into (1+3) form by expressing the Ricci scalar R in terms of the extrinsic curvature $K_{\mu\nu}$.

A useful identity, valid for any vector field n^λ, is

$$R_{\mu\nu\rho\sigma}v^\sigma = (\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)v_\rho. \tag{5.2.2}$$

This leads to the result

$$\begin{aligned} R_{\nu\sigma}n^\nu n^\sigma &= g^{\mu\rho}R_{\mu\nu\rho\sigma}n^\nu n^\sigma \\ &= n^\nu\nabla_\mu\nabla_\nu n^\mu - n^\nu\nabla_\nu\nabla_\mu n^\mu \\ &= \nabla_\mu(n^\nu\nabla_\nu n^\mu) - (\nabla_\mu n^\nu)(\nabla_\nu n^\mu) - \nabla_\nu(n^\nu\nabla_\mu n^\mu) + (\nabla_\nu n^\nu)^2 \\ &= \nabla_\alpha(K\,n^\alpha + a^\alpha) - K_{\mu\nu}K^{\mu\nu} + K^\mu_\mu K^\nu_\nu, \end{aligned} \tag{5.2.3}$$

where $K_{\mu\nu}$ is the extrinsic curvature (5.1.22), $K \equiv K^\alpha_\alpha = -\nabla_\alpha n^\alpha$, and $a^\alpha = n^\rho\nabla_\rho n^\alpha$.

Then, from the identity

$$R - 2\Lambda = -Rg_{\mu\nu}n^\mu n^\nu = 2(G_{\mu\nu} - R_{\mu\nu})n^\mu n^\nu \tag{5.2.4}$$

and the Gauss-Codazzi relations (5.1.25), the following equations are obtained:

$$\begin{aligned} R &= {}^{(3)}R - 2\Lambda + K_{\mu\nu}K^{\mu\nu} - K^\mu_\mu K^\nu_\nu - 2\nabla_\alpha(K\,n^\alpha + a^\alpha) \\ &\equiv \mathcal{L}_{ADM} - 2\nabla_\alpha(K\,n^\alpha + a^\alpha), \end{aligned} \tag{5.2.5}$$

where $\mathcal{L}_{ADM} = {}^{(3)}R - 2\Lambda + K_{\mu\nu}K^{\mu\nu} - K^\mu_\mu K^\nu_\nu$ is commonly referred to as the ADM Lagrangian, named after Arnowitt, Deser, and Misner.

The Einstein-Hilbert action now becomes the sum of two terms, one quadratic in time derivative h_{ij}, and a surface term

$$\begin{aligned} S &= \frac{1}{16\pi G}\int_V dt\, d^3x\, N\sqrt{h}\mathcal{L}_{ADM} - \frac{1}{8\pi}\int_V d^4x\,\sqrt{-g}\,\nabla_\alpha(Kn^\alpha - a^\alpha) \\ &\equiv S_{ADM} + S_{surf}. \end{aligned} \tag{5.2.6}$$

Since the surface term will not contribute to the equations of motion [46], it will be ignored for the time being [47].

5.3 The ADM Hamiltonian

The ADM action can be written in terms of the ADM Lagrangian as

$$16\pi G\, S_{ADM} = \int d^4x\,\sqrt{-g}\,\mathcal{L}_{ADM} = \int dt \int d^3x\, N\sqrt{h}\left[{}^{(3)}R - 2\Lambda + K_{\mu\nu}K^{\mu\nu} - K^\mu_\mu K^\nu_\nu\right]. \tag{5.3.1}$$

The ADM Lagrangian can be Legendre transformed into the ADM Hamiltonian. It is worth noting that the ADM Lagrangian is written in the classic form of "kinetic energy minus potential energy". Here, the extrinsic curvature is the kinetic energy and the negative of the intrinsic curvature is the potential energy [48]. The canonical variable for this transformation are N, N^i, and $h_{\mu\nu}$. The conjugate momenta to each canonical variable must first be determined. A quick examination of (5.1.22) reveals that there are no time derivatives of N and N^i. The corresponding conjugate momenta of N and N^i therefore vanish and this will be interpreted as the fact that N and N^i do not represent true degrees of freedom but are undetermined Lagrange multipliers. Later, the variation of the action with respect to N and N^i will be found, which will act as equations of constraint.

Since the time derivative of $h_{\mu\nu}$ does not vanish, it's conjugate momentum can be determined:

$$\begin{aligned} p^{ij} &= \frac{\partial}{\partial \dot{h}_{ij}}(\sqrt{-g}\mathcal{L}_{ADM}) \\ \Rightarrow \quad (16\pi G)p^{ij} &= \frac{\partial K_{ij}}{\partial \dot{h}_{ij}}\frac{\partial}{\partial K_{ij}}(16\pi\mathcal{L}_{ADM}) \\ &= -\sqrt{h}(K^{ij} - K\,h^{ij}) \end{aligned} \tag{5.3.2}$$

Inverting the last relation gives

$$\sqrt{h}\,K^{ij} = -16\pi G\left(p^{ij} - \frac{1}{2}p\,h^{ij}\right) \tag{5.3.3}$$

which will be important to write the Hamiltonian in terms of the proper variables.

The Hamiltonian can be determined by the equation

$$\mathcal{H}_{ADM} = p^{ij}\dot{h}_{ij} - \sqrt{-g}\mathcal{L}_{ADM}. \tag{5.3.4}$$

Working this out explicitly gives

$$
\begin{aligned}
(16\pi G)\mathcal{H}_{ADM} &= \sqrt{h}(K^{ij} - K\,h^{ij})(2N\,K_{ij} - D_j N_i - D_i N_j) \\
&\qquad - ({}^{(3)}R - 2\Lambda + K^{ij}K_{ij} - K^2)\,N\sqrt{h} \\
&= (K^{ij}K_{ij} - K^2 - {}^{(3)}R + 2\Lambda)\,N\sqrt{h} - 2(K^{ij} - K\,h^{ij})\,D_j N_i\sqrt{h} \\
&= (K^{ij}K_{ij} - K^2 - {}^{(3)}R + 2\Lambda)\,N\sqrt{h} - 2D_j\Big[(K^{ij} - K\,h^{ij})\,N_i\Big]\sqrt{h} \\
&\qquad + 2D_j(K^{ij} - K\,h^{ij})\,N_i\sqrt{h}.
\end{aligned}
\tag{5.3.5}
$$

Ignoring the total divergence term, the Hamiltonian density is given by

$$
(16\pi G)\mathcal{H}_{ADM} = \sqrt{h}\Big[N\,(K^{ij}K_{ij} - K^2 - {}^{(3)}R + 2\Lambda) + 2N_i D_j(K^{ij} - K\,h^{ij})\Big], \tag{5.3.6}
$$

Solving for the equations of constraints imposed by N and N^i, by treating them as undetermined Lagrange multipliers, gives

$$
\frac{\partial \mathcal{H}}{\partial N} = 0 = -{}^{(3)}R + 2\Lambda + h^{-1}p^{ij}p_{ij} - \frac{1}{2}h^{-1}p^2 \tag{5.3.7}
$$

$$
\frac{\partial \mathcal{H}}{\partial N_i} = 0 = 2\sqrt{h}\,D_i(\frac{1}{\sqrt{h}}\,p^{ij}). \tag{5.3.8}
$$

Finally, solving for $\mathcal{H}$ and $\mathcal{H}^i$ substituting into (5.3.4) yields

$$
\mathcal{H}_{ADM} = N\mathcal{H} + N_i\mathcal{H}^i + 2\sqrt{h}\,D_i(\frac{1}{\sqrt{h}}\,p^{ij}\,N_j). \tag{5.3.9}
$$

This expression will be used in the next section to help determine the Hamiltonian generators.

5.4 The surface term and the Hamiltonian generator

Going back to the expression (5.2.5), the surface term that arises from integrating the divergence term $\nabla_\alpha(K\,n^\alpha - a^\alpha)$ can now be discussed. This term has one part

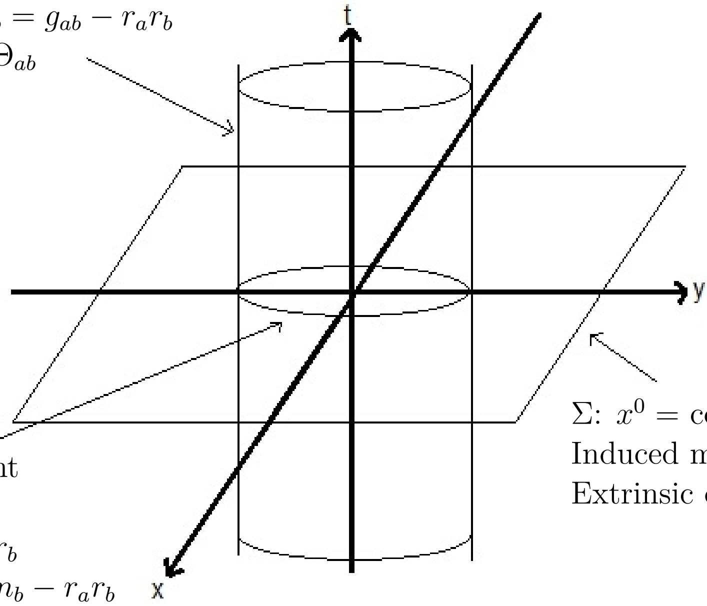

Figure 5.1: The surfaces bounding the region $\mathcal{V}$ that are used to define the action functional. The timelike surfaces $\mathcal{S}$ and spacelike surfaces Σ intersect on a 2-dimensional surface $\mathcal{Q}$ which can be thought of as embedded either in 4-dimensional space or in 3-dimensional space [46]

that depends on K and another which depends on the acceleration a^α of the normal vector field n^α. The second part can be reinterpreted entirely in terms of the extrinsic curvature of the boundary surface.

In order to do this, (5.2.5) is integrated over a four-volume $\mathcal{V}$ bounded by two spacelike hypersurfaces Σ_1 and Σ_2 and a timelike hypersurface $\mathcal{S}$ and a surface at spatial infinity, (see Fig. 5.1). The spacelike hypersurfaces are constant time slices with normals n^μ, and the timelike hypersurface has a normal r^μ orthogonal to n^μ. The induced metric on the spacelike hypersurface Σ is $h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$ and on on the timelike hypersurface $\mathcal{S}$ is $\gamma_{\mu\nu} = g_{\mu\nu} - r_\mu r_\nu$. The surfaces Σ and $\mathcal{S}$ intersect along a 2-dimensional surface $\mathcal{Q}$,with the induced metric $\sigma_{\mu\nu} = h_{\mu\nu} - r_\mu r_\nu = g_{\mu\nu} + n_\mu n_\nu -$

$r_\mu r_\nu$ [46]. Let the hypersurfaces $\Sigma, \mathcal{S}$, and their intersection 2-surface $\mathcal{Q}$ have the corresponding extrinsic curvatures $K_{\mu\nu}, \Theta_{\mu\nu}$ and $q_{\mu\nu}$, then doing the integrals, one obtains [46]

$$\begin{aligned} S = \frac{1}{16\pi G}\int_V d^4x\,\sqrt{-g}\,R = \frac{1}{16\pi G}\int_V d^4x\,\sqrt{-g}\,L_{ADM} - \frac{1}{8\pi G}\int_{\Sigma_1}^{\Sigma_2} d^3x\,\sqrt{h}\,K- \\ - \frac{1}{8\pi G}\int_{\mathcal{S}} dt\,d^2x\,N\,\sqrt{\sigma}\,(r_\mu a^\mu). \end{aligned} \tag{5.4.1}$$

The action (5.4.1) can be rewritten using (5.3.9) [49] as

$$\begin{aligned} S = \int dt\Bigg[\int_{\Sigma_1}^{\Sigma_2} d^{n-1}x\left(p^{\mu\nu}\dot{h}_{\mu\nu} - [N\mathcal{H} + N_\mu\mathcal{H}^\mu]\right) \\ + \int_{\mathcal{S}} d^{n-2}x\sqrt{\sigma}\left(\frac{1}{8\pi G}\,N\Theta - \frac{2}{\sqrt{h}}r_\mu\,p^{\mu\nu}N_\nu\right)\Bigg]. \end{aligned} \tag{5.4.2}$$

Since the Hamiltonian is defined by

$$S = \int dt\left[\int_\Sigma d^{n-1}x\,p^{\mu\nu}\dot{h}_{\mu\nu} - H\right], \tag{5.4.3}$$

(5.4.2) becomes [50, 51],

$$\begin{aligned} H = \int_{\Sigma_1}^{\Sigma_2} d^{n-1}x\left[N\mathcal{H} + N_\mu\mathcal{H}^\mu\right] \\ - \int_{\mathcal{S}} d^{n-2}x\,\sqrt{\sigma}\left[\frac{1}{8\pi G}N\Theta - \frac{2}{\sqrt{h}}r_\mu\,p^{\mu\nu}N_\nu\right] \end{aligned} \tag{5.4.4}$$

The Hamiltonian generator is defined by [49],

$$\begin{aligned} H \equiv \int_{\Sigma_1}^{\Sigma_2} d^{n-1}x\left[\varepsilon\mathcal{H} + \varepsilon_\mu\mathcal{H}^\mu\right] \\ - \int_{\mathcal{S}} d^{n-2}x\,\sqrt{\sigma}\left[\frac{1}{8\pi G}\varepsilon\,\Theta - \frac{2}{\sqrt{h}}r_\mu\,p^{\mu\nu}\varepsilon_\nu\right], \end{aligned} \tag{5.4.5}$$

where the parameters ε and ε^μ are parameters instead of the lapse and shift vector and $\varepsilon^\mu n_\mu = 0$ is assumed, like the shift vector. The Hamiltonian generator can be used to determine the conserved charges at the boundary of an asymptotic spacetime. This will be done in the next chapter for the AdS_3 spacetime.

Chapter 6

Conserved charges in general relativity and asymptotically AdS_3 spacetimes

This chapter shows that the surface terms ii the Hamiltonian generators defined in the previous chapter become the charges in general relativity. The asymptotically AdS_3 spacetime will be defined and its conserved charges will be calculated by adding a counter term to the gravitation action, as introduced by Balasubramanian and Kraus. The central charge on the asymptotic boundary of the spacetime will also be calculated. This main resources for this chapter are [49, 52]

6.1 Algebra of charges

As mentioned in previous chapters, conserved Noether charges are associated to each generator of the symmetries of the action. The Poisson bracket algebra of the Noether charges is isomorphic to the Lie algebra of infinitesimal asymptotic symmetries [9]. Consider the bosonic generators, δ_a's, of a symmetry, whose fine structure constants satisfy the Jacobi identity, ($[\delta_a, \delta_b] = f_{ab}{}^c\delta_c$, and $[\delta_a, [\delta_b, \delta_c]]+[\delta_b, [\delta_c, \delta_a]]+[\delta_c, [\delta_a, \delta_b]] = 0$) and their corresponding Noether charges Q_a's, (the indices here are general). It is assumed that the charges are generators of the algebra, that is when applied to a field ϕ they give

$$\delta_a \phi = \{Q_a, \phi\}, \tag{6.1.1}$$

where the curly braces here imply the Poisson brackets. The condition

$$[\delta_a, \delta_b]\phi = f_{ab}{}^{c}\delta_c\phi, \tag{6.1.2}$$

puts restrictions on the possible Poisson brackets algebra satisfied by the Noether charges [43]. This is easily demonstrated:

$$\begin{aligned}
[\delta_a, \delta_b]\phi &= f_{ab}{}^{c}\delta_c\phi \\
\delta_a\delta_b\phi - \delta_b\delta_a\phi &= f_{ab}{}^{c}\delta_c\phi \\
\delta_a\{Q_b, \phi\} - \delta_b\{Q_a, \phi\} &= f_{ab}{}^{c}\{Q_c, \phi\} \\
\{Q_a, \{Q_b, \phi\}\} - \{Q_b, \{Q_a, \phi\}\} &= f_{ab}{}^{c}\{Q_c, \phi\},
\end{aligned} \tag{6.1.3}$$

where the (6.1.1) has been used in the last two steps. Then using the Jacobi identity yields

$$\{\{Q_a, Q_b\}, \phi\} = f_{ab}{}^{c}\{Q_c, \phi\}. \tag{6.1.4}$$

Thus, the algebra of the charges acting on the field ϕ is

$$\{Q_a, Q_b\} = f_{ab}{}^{c}Q_c, \tag{6.1.5}$$

and the Poisson bracket algebra of the charges is isomorphic to the Lie bracket algebra of the generators.

However, (6.1.5) is not the most general algebra. The most general algebra is one with the presence of a central extension:

$$\{Q_a, Q_b\} = f_{ab}{}^{c}Q_c + \bar{c}\,\Delta_{ab}, \tag{6.1.6}$$

where $\bar{c}$ is a central charge (later will be denoted by c without a bar), and the function Δ_{ab} is antisymmetric in indices a and b [43]. The central extension term commutes with all charges, $\{Q_a, \bar{c}\,\Delta_{ab}\} = 0, \{Q_b, \bar{c}\,\Delta_{ab}\} = 0, \{Q_c, \bar{c}\,\Delta_{ab}\} = 0$, etc.

6.2 Asymptotically AdS_3 spacetime

The asymptotically AdS_3 spacetime is defined by the boundary condition [9]

$$\begin{aligned} g_{tt} &= -\frac{r^2}{\ell^2} + \mathcal{O}(1) \\ g_{tr} &= \mathcal{O}(1/r^3) \\ g_{t\phi} &= \mathcal{O}(1) \\ g_{rr} &= \frac{\ell^2}{r^2} + \mathcal{O}(1/r^4) \\ g_{r\phi} &= \mathcal{O}(1/r^3) \\ g_{\phi\phi} &= r^2 + \mathcal{O}(1) \end{aligned} \tag{6.2.1}$$

The asymptotic symmetry of this spacetime is the coordinate transformation which preserves this boundary condition. This symmetry was determined by Brown and Henneaux [9] by first writing the asymptotic Killing vector ξ^μ $(\mu = t, r, \phi)$ as

$$\begin{aligned} \xi^t &= \ell T(t,\phi) + \frac{\ell^3}{r^2}\bar{T}(t,\phi) + \mathcal{O}(1/r^4), \\ \xi^r &= rR(t,\phi) + \frac{\ell^2}{r}\bar{R}(t,\phi) + \mathcal{O}(1/r^3), \\ \xi^\phi &= \Phi(t,\phi) + \frac{\ell^2}{r^2}\bar{\Phi}(t,\phi) + \mathcal{O}(1/r^4). \end{aligned} \tag{6.2.2}$$

From the definition of the Lie derivative,

$$\begin{aligned} \mathcal{L}_\xi g_{\mu\nu} &\equiv \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \\ &= \xi^\rho \partial_\rho g_{\mu\nu} + g_{\mu\rho}\partial_\nu \xi^\rho + g_{\rho\nu}\partial_\mu \xi^\rho, \end{aligned} \tag{6.2.3}$$

it can be found that $T, \Phi, R, \bar{T}, \bar{\Phi}$ must satisfy

$$\begin{aligned} \ell\partial_t T(t,\phi) &= \partial_\phi \Phi(t,\phi) = -R(t,\phi), \\ \ell\partial_t \Phi(t,\phi) &= \partial_\phi T(t,\phi), \end{aligned} \tag{6.2.4}$$

and

$$\begin{aligned} \bar{T}(t,\phi) &= -\frac{1}{2}\partial_t R(t,\phi), \\ \bar{\Phi}(t,\phi) &= \frac{1}{2}\partial_\phi R(t,\phi). \end{aligned} \tag{6.2.5}$$

However, $\bar{R}(t,\phi)$ can be arbitrary.

From the conditions (6.2.4),

$$\left[\ell^2\partial_t^2 - \partial_\phi^2\right] T(t,\phi) = 0. \tag{6.2.6}$$

There are 4 kinds of solutions for this equation

$$\cos\frac{nt}{\ell}\cos n\phi, \quad \sin\frac{nt}{\ell}\sin n\phi, \quad \sin\frac{nt}{\ell}\cos n\phi, \quad \cos\frac{nt}{\ell}\sin n\phi. \tag{6.2.7}$$

The other functions are completely decided by equations (6.2.4) and (6.2.5) [49]. The solutions are [9]:

A series: $\xi^\mu_{A_n} = \xi^\mu_{A_{-n}}$,

$$\begin{aligned} \xi^t_{A_n} &= \ell\left(1 - \frac{n^2\ell^2}{2r^2}\right)\cos\frac{nt}{\ell}\cos n\phi + \mathcal{O}(1/r^4), \\ \xi^r_{A_n} &= rn\sin\frac{nt}{\ell}\cos n\phi + \mathcal{O}(1/r), \\ \xi^\phi_{A_n} &= -\left(1 + \frac{n^2\ell^2}{2r^2}\right)\sin\frac{nt}{\ell}\sin n\phi + \mathcal{O}(1/r^4). \end{aligned} \tag{6.2.8}$$

B series: $\xi^\mu_{B_n} = \xi^\mu_{B_{-n}}$,

$$\begin{aligned} \xi^t_{B_n} &= \ell\left(1 - \frac{n^2\ell^2}{2r^2}\right)\sin\frac{nt}{\ell}\sin n\phi + \mathcal{O}(1/r^4), \\ \xi^r_{B_n} &= -rn\cos\frac{nt}{\ell}\sin n\phi + \mathcal{O}(1/r), \\ \xi^\phi_{B_n} &= -\left(1 + \frac{n^2\ell^2}{2r^2}\right)\cos\frac{nt}{\ell}\cos n\phi + \mathcal{O}(1/r^4). \end{aligned} \tag{6.2.9}$$

C series: $\xi^{\mu}_{C_n} = \xi^{\mu}_{C_{-n}}$,

$$\begin{aligned}
\xi^t_{C_n} &= \ell\left(1 - \frac{n^2\ell^2}{2r^2}\right)\sin\frac{nt}{\ell}\cos n\phi + \mathcal{O}(1/r^4),\\
\xi^r_{C_n} &= -rn\cos\frac{nt}{\ell}\cos n\phi + \mathcal{O}(1/r),\\
\xi^{\phi}_{C_n} &= \left(1 + \frac{n^2\ell^2}{2r^2}\right)\cos\frac{nt}{\ell}\sin n\phi + \mathcal{O}(1/r^4).
\end{aligned} \tag{6.2.10}$$

D series: $\xi^{\mu}_{D_n} = \xi^{\mu}_{D_{-n}}$,

$$\begin{aligned}
\xi^t_{D_n} &= \ell\left(1 - \frac{n^2\ell^2}{2r^2}\right)\cos\frac{nt}{\ell}\sin n\phi + \mathcal{O}(1/r^4),\\
\xi^r_{D_n} &= rn\sin\frac{nt}{\ell}\sin n\phi + \mathcal{O}(1/r),\\
\xi^{\phi}_{D_n} &= \left(1 + \frac{n^2\ell^2}{2r^2}\right)\sin\frac{nt}{\ell}\cos n\phi + \mathcal{O}(1/r^4).
\end{aligned} \tag{6.2.11}$$

All these vectors make up the conformal group in 2-dimensions, that is, the Lie bracket algebra of two vectors,

$$[\xi_1, \xi_2]^{\mu} = \xi_1^{\nu}\partial_{\nu}\xi_2^{\mu} - \xi_2^{\nu}\partial_{\nu}\xi_1^{\mu}, \tag{6.2.12}$$

is the conformal group algebra. Therefore, the asymptotic symmetry of the asymptotically AdS_3 spacetime enhances from the symmetry of the AdS_3 spacetime, $S0(2,2)$, to the conformal group.

6.3 Conserved charges from the Hamiltonian surface term

This thesis will examine mainly black holes in the asymptotic AdS_3 spacetime. In the last section it was shown how the Poisson bracket algebra of the generators of symmetries is isomorphic to the algebra of the charges. For this thesis, the generator of symmetries is given by the Hamiltonian generator

$$H[\xi] = \int_{\Sigma_1}^{\Sigma_2} d^2x \left[\bar{\xi}^{\perp}\mathcal{H} + \bar{\xi}^{i}\mathcal{H}_i \right] + J[\xi], \tag{6.3.1}$$

(where $i = r, \phi$), which generates the coordinate transformation

$$\xi^t = \frac{1}{N}\bar{\xi}^{\perp}, \qquad \xi^i = -\frac{N^i}{N}\bar{\xi}^{\perp} + \bar{\xi}^i, \tag{6.3.2}$$

where $J[\xi]$ is the surface term. The equations of motion place the constraints $\mathcal{H} = \mathcal{H}^{\mu} = 0$ and therefore the generator becomes only the surface term. The surface term $J[\xi]$ is thus called the *charge*.

6.4 Conserved Charges of asymptotic AdS$_3$ spacetimes

6.4.1 Brown and York stress tensor

Brown and York [53] defined a "quasilocal stress tensor" which is determined from the gravitational action as a functional of the induced boundary metric $\gamma_{\mu\nu}$,

$$T^{\mu\nu} = \frac{2}{\sqrt{-\gamma}}\frac{\delta S_{grav}}{\delta \gamma_{\mu\nu}}. \tag{6.4.1}$$

However, this stress tensor diverges as the boundary is taken to infinity. But a boundary term can always be added to the action without affecting the equations of motion in the bulk. The divergences which appear as the boundary is moved to infinity may be removed by adding local counterterms to the action [52].

The gravitational action, in any dimension d, with the cosmological constant $\Delta =$

$-d(d-1)/2\ell^2$ is given by

$$S = -\frac{1}{16\pi G}\int_{\mathcal{M}} d^{d+1}\sqrt{g}\left(R - \frac{d(d-1)}{\ell^2}\right) - \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^d x\,\sqrt{-\gamma}\Theta + \frac{1}{8\pi G}S_{ct}(\gamma_{\mu\nu}), \tag{6.4.2}$$

where Θ is the extrinsic curvature on the boundary. The second term is the boundary term, expressed in the previous chapter, and the third term is a counterterm added in order to obtain a finite stress tensor.

As seen in Chapter 5, the spacetime metric is written in the ADM decomposition,

$$ds^2 = N^2 dr^2 + \gamma_{\mu\nu}(dx^\mu + N^\mu dr)(dx^\nu + N^\nu dr), \tag{6.4.3}$$

where the $d+1$ dimensional spacetime, $\mathcal{M}$ is foliated into a series of d-dimensional timelike surfaces homeomorphic to the boundary $\partial\mathcal{M}$ where the coordinates x^μ span the timelike surfaces and r is the remaining coordinate. ($\gamma_{\mu\nu}$ is a function of all coordinates). The boundary $\partial\mathcal{M}_r$ is the surface at fixed r, and the interior, or bulk, is denoted $\mathcal{M}_r$.

The quasilocal stress tensor for the $\mathcal{M}_r$ region is calculated from varying the action with respect to the boundary metric $\gamma_{\mu\nu}$. Varying an action produces a bulk term proportional to the equations of motion and a boundary term. Because of the constraints on the equations of motion, only the boundary term contributes. The variation of the action becomes

$$\delta S = \int_{\partial\mathcal{M}_r} d^d x\,\pi^{\mu\nu}\,\delta\gamma_{\mu\nu} + \frac{1}{8\pi G}\int_{\partial\mathcal{M}_r} d^d x\,\frac{\delta S_{ct}}{\delta\gamma_{\mu\nu}}\delta\gamma_{\mu\nu}, \tag{6.4.4}$$

where $\pi^{\mu\nu}$ is the momentum conjugate to $\gamma_{\mu\nu}$ evaluated at the boundary given by

$$\pi^{\mu\nu} = \frac{1}{16\pi G}\sqrt{-\gamma}(\Theta^{\mu\nu} - \Theta\gamma^{\mu\nu}). \tag{6.4.5}$$

The extrinsic curvature, Θ, is given by

$$\Theta^{\mu\nu} = -\frac{1}{2}(\nabla^\mu \hat{n}^\nu + \nabla^\nu \hat{n}^\mu), \tag{6.4.6}$$

where $\hat{n}^\nu$ is an outward pointing normal vector on the boundary. Then, by (6.4.1) the quasilocal stress tensor becomes

$$T^{\mu\nu} = \frac{1}{8\pi G}\left[\Theta^{\mu\nu} - \Theta\gamma^{\mu\nu} + \frac{2}{\sqrt{-\gamma}}\frac{\delta S_{ct}}{\delta\gamma_{\mu\nu}}\right]. \tag{6.4.7}$$

The added S_{ct} is chosen such that it must cancel divergences that arise as $\partial\mathcal{M}_r$ goes to $\partial\mathcal{M}$, the AdS boundary.

6.4.2 Conserved charges from counter term

Balasubramanian and Kraus proposed S_{ct} to be a local functional of the intrinsic geometry of the boundary, chosen to cancel the $\partial\mathcal{M}_r \to \partial\mathcal{M}$ in (6.4.7) [52]. They set $S_{ct} = \int_{\mathcal{M}_r} L_{ct}$ which results in

$$L_{ct} = -\frac{1}{\ell}\sqrt{-\gamma} \Rightarrow T^{\mu\nu} = \frac{1}{8\pi G}\left[\Theta^{\mu\nu} - \Theta\gamma^{\mu\nu} - \frac{1}{\ell}\gamma^{\mu\nu}\right] \tag{6.4.8}$$

for AdS_3,

$$L_{ct} = -\frac{2}{\ell}\sqrt{-\gamma}\left(1 - \frac{\ell^2}{4}R\right) \Rightarrow T^{\mu\nu} = \frac{1}{8\pi G}\left[\Theta^{\mu\nu} - \Theta\gamma^{\mu\nu} - \frac{2}{\ell}\gamma^{\mu\nu} - \ell G^{\mu\nu}\right] \tag{6.4.9}$$

for AdS_4, and

$$L_{ct} = -\frac{3}{\ell}\sqrt{-\gamma}\left(1 - \frac{\ell^2}{12}R\right) \Rightarrow T^{\mu\nu} = \frac{1}{8\pi G}\left[\Theta^{\mu\nu} - \Theta\gamma^{\mu\nu} - \frac{3}{\ell}\gamma^{\mu\nu} - \frac{\ell}{2}G^{\mu\nu}\right] \tag{6.4.10}$$

for AdS_5. All the tensors are functions of the boundary metric $\gamma_{\mu\nu}$ and $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R\gamma_{\mu\nu}$.

To find the conserved charges (mass and angular momentum) of asymptotically AdS geometry, the boundary metric can be written in the ADM form as

$$\gamma_{\mu\nu}dx^\mu dx^\nu = -N_\Sigma^2 dt^2 + \sigma_{ab}(dx^a + N_\Sigma^a dt)(dx^b + N_\Sigma^b dt), \tag{6.4.11}$$

where Σ is a spacelike surface in $\partial\mathcal{M}$ with the metric σ_{ab}. Letting u^μ be the timelike unit vector normal to the surface Σ, then u^μ defines the local flow of time in $\partial\mathcal{M}$. If ξ^μ is a Killing vector generating an isometry on the boundary, the associated conserved charge is given by [53]

$$Q_\xi = \int_\Sigma d^{d-1}x\sqrt{\sigma}(u^\mu\, T_{\mu\nu}\xi^\nu). \tag{6.4.12}$$

The conserved charge corresponding to time translation is the mass of spacetime. By defining the proper energy density,

$$\epsilon = u^\mu u^\nu T_{\mu\nu}, \tag{6.4.13}$$

the mass can be determined by multiplying the ϵ by the lapse N_Σ and integrating:

$$M = \int_\Sigma d^{d-1}x\sqrt{\sigma}N_\Sigma\epsilon. \tag{6.4.14}$$

This definition for the mass coincides with the conserved quantity in (6.4.12) when the timelike Killing vector is $\xi^\mu = N_\Sigma u^\mu$. The momentum can be defined in a similar manner as

$$P_a = \int_\Sigma d^{d-1}x\sqrt{\sigma}j_a, \tag{6.4.15}$$

where

$$j_a = \sigma_{ab}u_\mu T^{a\mu}. \tag{6.4.16}$$

When a is an angular direction, P_a is the corresponding angular momentum [52].

6.4.3 Conserved charges of AdS_3

The Poincaré patch of the AdS_3 spacetime can be written as

$$ds^2 = \frac{\ell^2}{r^2}\,dr^2 + \frac{r^2}{\ell^2}\Big(-dt^2 + dx^2\Big), \tag{6.4.17}$$

(see [15] for embedding of the Poincaré patch in global AdS_3). The normal vector to surfaces of constant r is

$$\hat{n}^\mu = \frac{r}{\ell}\delta^{\mu,r}. \tag{6.4.18}$$

Then, applying (6.4.7) gives

$$\begin{aligned} 8\pi G\,T_{tt} &= -\frac{r^2}{\ell^3} + \frac{2}{\sqrt{-\gamma}}\frac{\delta S_{ct}}{\delta\gamma^{tt}} \\ 8\pi G\,T_{xx} &= \frac{r^2}{\ell} + \frac{2}{\sqrt{-\gamma}}\frac{\delta S_{ct}}{\delta\gamma^{xx}} \\ 8\pi G\,T_{tx} &= \frac{2}{\sqrt{-\gamma}}\frac{\delta S_{ct}}{\delta\gamma^{tx}}. \end{aligned} \tag{6.4.19}$$

Without the S_{ct}, results for physical observables, the mass and momentum, would be divergent. Consider the mass

$$M = \int dx\sqrt{g_{xx}}N_\Sigma u^t u^t T_{tt} = \int dx T_{tt} \approx r^2 \to \infty. \tag{6.4.20}$$

Thus, for the spacetime to have a finite mass density, T_{tt} must be independent of r for large r.

Since one requires that S_{ct} be a local, covariant function of the intrinsic geometry of the boundary, it is essentially unique. The only term that can cancel the divergence in (6.4.20) is $S_{ct} = (-1/\ell)\int\sqrt{-\gamma}$. This yields a divergence free stress tensor: $T_{\mu\nu} = 0$. Higher order counter terms, such as R and R^2, vanish at infinity, due to dimensional analysis. Therefore, the counterterm is completely defined by (6.4.8).

The stress-tensor can be now used to reproduce the mass and momentum of a known solution, the rotating BTZ solution [14, 15]. In order to check this, spacetimes of the form

$$ds^2 = \frac{\ell^2}{r^2}\,dr^2 + \frac{r^2}{\ell^2}\Big(-dt^2 + dx^2\Big) + \delta g_{MN} dx^M dx^N \tag{6.4.21}$$

will be studied. To the first order in δg_{MN}, the stress tensor components are

$$\begin{aligned} 8\pi G\, T_{tt} &= \frac{r^4}{2\ell^5} + \frac{\delta g_{xx}}{\ell} - \frac{r}{2\ell}\partial_r \delta g_{xx} \\ 8\pi G\, T_{xx} &= \frac{\delta g_{tt}}{\ell} - \frac{r}{2\ell}\partial_r \delta g_{tt} - \frac{r^4}{2\ell^5}\delta g_{rr} \\ 8\pi G\, T_{tx} &= \frac{1}{\ell}\delta g_{tx} - \frac{r}{2\ell}\partial_r \delta g_{tx}. \end{aligned} \tag{6.4.22}$$

The mass and momentum are:

$$\begin{aligned} M &= \frac{1}{8\pi G}\int dx \left[\frac{r^4}{2\ell^5} + \frac{\delta g_{xx}}{\ell} - \frac{r}{2\ell}\partial_r \delta g_{xx}\right] \\ P &= -\frac{1}{8\pi G}\int dx \left[\frac{1}{\ell}\delta g_{tx} - \frac{r}{2\ell}\partial_r \delta g_{tx}\right]. \end{aligned} \tag{6.4.23}$$

These formulae can now be applied to the spinning BTZ solution:

$$ds^2 = N^2 dt^2 + \rho^2 (d\phi + N^\phi dt)^2 + \frac{r^2}{N^2\rho^2} dr^2, \tag{6.4.24}$$

with

$$\begin{aligned} N^2 &= \frac{r^2(r^2 - r_+^2)}{\ell^2\rho^2}, \qquad & N^\phi &= -\frac{4GJ}{\rho^2}, \\ \rho^2 &= r^2 + 4GM\ell^2 - \frac{1}{2}r_+^2, \qquad & r_+^2 &= 8G\ell\sqrt{M^2\ell^2 - J^2}, \end{aligned} \tag{6.4.25}$$

where ϕ has a period 2π. Expanding for large r yields

$$\delta g_{rr} = \frac{8GM\ell^2}{r^4}, \qquad \delta g_{tt} = 8GM, \qquad \delta g_{t\phi} = -4GJ. \tag{6.4.26}$$

Inserting (6.4.26) into (6.4.23) with $x \to \ell\phi$ and $\int dx \to \ell\int_0^{2\pi} d\phi$ yields the correct mass and momentum for the spinning BTZ solution, $M = M$ and $P_\phi = J$. The spinning BTZ metric reduces to the global AdS_3 metric when $M = \frac{-1}{8G}$ and $J = 0$. When $M = J = 0$, the BTZ metric reduces to a black hole which looks like the Poincaré AdS_3 with an identification of the boundary [52]. However, the time directions of the Poincaré AdS_3 and global AdS_3 coordinates do not agree, give them different definitions of energy. Because of this, the two metrics giving rise to different masses.

6.5 Central charge of asymptotically AdS_3 spacetime

A classical conformal field theory has a traceless stress, (or energy-momentum), tensor. However, when a conformal field theory is defined on a curved 2-dimensional manifold, there is a quantum breaking of macroscopic scale invariance created by the curvature [5]. Therefore the only possible scalar in 2-dimensions for the stress tensor to be proportional to is the Ricci scalar. The stress tensor of a 2-dimensional, $(1+1)$, CFT has a trace anomaly defined as

$$T^{\mu}_{\mu} = -\frac{c}{24\pi}R. \tag{6.5.1}$$

Since the trace anomaly is a quantum effect, the stress tensor is also proportional to the central charge, c.

It must be verified that the quasilocal stress tensor for the AdS_3 spacetime, defined in (6.4.8), has a trace of precisely this form. The mechanism for determining the conformal anomaly from the AdS/CFT correspondence was outlined by Witten [54]. This approach was studied in detail by Henningson and Skenderis [55]. This thesis will follow the approach by Balasubramanian and Kraus [52], which is somewhat different than that of [55].

The trace of the AdS_3 stress tensor (6.4.8) becomes

$$T^{\mu}_{\mu} = -\frac{1}{8\pi G}\Big(\Theta + 2/\ell\Big). \tag{6.5.2}$$

This trace is expressed in terms of the extrinsic curvature. To compare (6.5.2) with (6.5.1) it must be expressed in terms of the intrinsic curvature of the boundary.

Since (6.5.2) is manifestly covariant, the right hand side may be computed in any coordinate system. Choosing the coordinates defined by the metric

$$ds^2 = \frac{\ell^2}{2} dr^2 + \gamma_{\mu\nu} dx^{\mu} dx^{\nu}, \tag{6.5.3}$$

the extrinsic curvature in these coordinates is

$$\Theta_{\mu\nu} = -\frac{r}{2\ell}\partial_r \gamma_{\mu\nu} \tag{6.5.4}$$

Thus, in these coordinates (6.5.2) becomes

$$T^{\mu}_{\mu} = -\frac{1}{8\pi G}\left[-\frac{r}{2\ell}\gamma^{\mu\nu}\partial_r\gamma_{\mu\nu} + \frac{2}{\ell}\right]. \tag{6.5.5}$$

For this calculation, $\gamma_{\mu\nu}$ must be expressed as a power series in $1/r$. Using the Fefferman-Graham expansion [56], the boundary metric is written as

$$\gamma_{\mu\nu} = r^2\gamma^{(0)}_{\mu\nu} + \gamma^{(2)}_{\mu\nu} + \cdots . \tag{6.5.6}$$

Fefferman and Graham also showed that only even powers appear and that the leading term goes as r^2. There are terms with higher powers of $1/r$ as well as logarithmic terms [56], but these will not be needed [52].

The trace of the stress tensor now reads

$$T^{\mu}_{\mu} = -\frac{1}{8\pi G}\frac{1}{\ell r^2}\mathrm{Tr}\Big[(\gamma^{(0)})^{-1}\gamma^{(2)}\Big] + \cdots . \tag{6.5.7}$$

Using Henningson and Skenderis' method of solving Einstein's equation perturbatively [55] gives

$$\mathrm{Tr}\Big[(\gamma^{(0)})^{-1}\gamma^{(2)}\Big] = \frac{\ell^2 r^2}{2} R \tag{6.5.8}$$

where R is the curvature of the metric $\gamma_{\mu\nu}$. Then, inserting (6.5.8) into (6.5.7) in the limit as r goes to infinity yields

$$T^{\mu}_{\mu} = -\frac{\ell}{16\pi G} R. \tag{6.5.9}$$

Comparing with (6.5.1) gives the same result when $c = 3\ell/2G$, which is exactly the Brown and Henneaux central charge [9]

Chapter 7

Thermodynamics of BTZ black hole

The BTZ black hole is a solution of Einstein's field equations of the vacuum with a cosmological constant $\Lambda = -1/\ell^2$. The asymptotically BTZ spacetime is that of the asymptotically AdS_3 where the asymptotic symmetry follows two copies of the Virasoro algebra with central charges

$$c = \bar{c} = \frac{3\ell}{2G}. \tag{7.0.1}$$

As mentioned in Section 4.6, the dilations and rotations are generated by $L_0 + \bar{L}_0$ and $L_0 - \bar{L}_0$. For the BTZ metric the dilations and rotations correspond to the mass, M, and angular momentum, J. In terms of the eigenvalues of the Virasoro generators they are given by

$$M\ell = \Delta + \bar{\Delta}, \qquad J = \Delta - \bar{\Delta}. \tag{7.0.2}$$

Inverting these relations to obtain expressions for eigenvalues only gives

$$\Delta = \frac{M\ell + J}{2}, \qquad \bar{\Delta} = \frac{M\ell - J}{2}. \tag{7.0.3}$$

With this structure in place, the thermodynamics of various BTZ black holes can be examined, specifically the first law of black hole thermodynamics (3.4.4) The surface gravity term, $(8\pi)^{-1}\kappa\ dA$, is written in terms of the temperature, T, and the entropy S as:

$$dM = T_H\ dS + \Omega_H\ dJ + \Phi\ dQ, \tag{7.0.4}$$

where Ω_H is the angular velocity and Φ is the electric potential at the event horizon.

First, the Hawking temperature and black hole entropy for the non-rotating, rotating, and charged-rotating BTZ black holes will be calculated in order to show that the first law of black hole thermodynamics is satisfied.

7.1 The non-rotating BTZ black hole

The solution is given by the metric,

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\phi^2 \tag{7.1.1}$$

with $c^2 = 1$, where

$$f(r) = -8GM + \frac{r^2}{\ell^2} \tag{7.1.2}$$

The event horizon then lies at $r_+^2 = 8GM\ell^2$. Inverting to obtain an expression for M gives

$$M = \frac{r_+^2}{8G\ell^2}, \tag{7.1.3}$$

which can be substituted into (7.0.3) to give the eigenvalues of the Virasoro generators:

$$\Delta = \bar{\Delta} = \frac{r_+^2}{16G\ell}. \tag{7.1.4}$$

The Hawking temperature of the non-rotating BTZ black hole can be calculated using (3.5.3), where here $a'(r_+) = b'(r_+) = f'(r_+) = 2r_+/\ell^2$. The Hawking temperature is then

$$\begin{aligned} T_H &= \frac{\sqrt{(2r_+/\ell^2)^2}}{4\pi} \\ &= \frac{r_+}{2\pi\ell^2}, \end{aligned} \tag{7.1.5}$$

or

$$T_H = \frac{\sqrt{2GM}}{\pi\ell}. \tag{7.1.6}$$

The leading order of the non-rotating BTZ black hole entropy can be calculated using (4.10.12), and is given by

$$S = \frac{2\pi r_+}{4G}, \tag{7.1.7}$$

which is the Bekenstein-Hawking entropy of a non-rotating BTZ black hole. The Bekenstein-Hawking entropy can be written in terms of the mass,

$$S = \frac{\pi\ell}{G}\sqrt{2GM}. \tag{7.1.8}$$

The leading correction using (4.10.11) can be calculated as

$$\rho(\Delta, \bar{\Delta}) = \frac{8G\ell^2}{r_+^3}\, e^{\frac{2\pi r_+}{4G}} + \text{higher order terms.} \tag{7.1.9}$$

And therefore,

$$S = \frac{2\pi r_+}{4G} - 3\ln(r_+) + \text{constant} + \text{higher order terms.} \tag{7.1.10}$$

The entropy can also be written in terms of the mass:

$$S = \frac{2\pi\sqrt{8GM\ell^2}}{4G} - \frac{3}{2}\ln(8GM\ell^2) + \text{constant} + \text{higher order terms.} \tag{7.1.11}$$

The leading order entropy term (7.1.8) and the Hawking temperature (7.1.6) of the non-rotating BTZ black hole can be used to show that the first law of black hole thermodynamics (7.0.4) is satisfied. Using (7.1.8),

$$dS = \frac{\pi\ell}{(2GM)^{1/2}}dM. \tag{7.1.12}$$

For the non-rotating BTZ black hole $J = 0$ and $Q = 0$, therefore (7.0.4) becomes simply

$$dM = T_H dS. \tag{7.1.13}$$

It is clear from (7.1.6) and (7.1.12) that $T_H dS = dM$ and thus the first law of black hole thermodynamics is satisfied.

7.2 The rotating BTZ black hole

The BTZ black hole is a (2+1)-dimensional black hole. The rotating BTZ black hole is given by the metric

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2\left(d\phi + f(\phi)\,dt\right)^2 \tag{7.2.1}$$

with $c^2 = 1$ for simplicity of expressions, where

$$f(r) = -8GM + \frac{r^2}{\ell^2} + \frac{16G^2J^2}{r^2} \tag{7.2.2}$$

and

$$f(\phi) = -\frac{4GJ}{r^2}, \qquad (|J| \leq M\ell). \tag{7.2.3}$$

Similar to how the Schwarzschild solution is asymptotically Minkowski, the BTZ solution is asymptotically AdS_3. The inner and outer event horizons of the rotating BTZ black hole lie at

$$r_\pm^2 = 4GM\ell^2\left\{1 \pm \left[1 - \left(\frac{J}{M\ell}\right)^2\right]^{1/2}\right\}. \tag{7.2.4}$$

Inverting (7.2.4) for expressions of M and J gives

$$M = \frac{r_+^2 + r_-^2}{8G\ell^2} \tag{7.2.5}$$

and

$$J = \frac{r_+ r_-}{4G\ell}. \tag{7.2.6}$$

Substituting the expressions (7.2.5) and (7.2.6) into (7.0.3) and simplifying gives

$$\Delta = \frac{(r_+ + r_-)^2}{16G\ell}, \qquad \bar{\Delta} = \frac{(r_+ - r_-)^2}{16G\ell}. \tag{7.2.7}$$

The Hawking temperature can be calculated as mentioned in section 3.5, where $a'(r_+) = b'(r_+) = f'(r_+) = 2r_+/\ell^2 - 32G^2J^2/r_+^2$. By (3.5.3) the Hawking temperature is then

$$T_H = \frac{r_+}{2\pi\ell^2} - \frac{8G^2J^2}{\pi r_+^3}, \tag{7.2.8}$$

or

$$T_H = \frac{r_+^2 - r_-^2}{2\pi\ell^2 r_+} \tag{7.2.9}$$

by using (7.2.6).

The rotating BTZ black hole entropy, S, can now be calculated using the standard Cardy formula (4.10.12):

$$S = \ln\rho(\Delta, \bar{\Delta}) = 2\pi\sqrt{\frac{c\Delta}{6}} + 2\pi\sqrt{\frac{\bar{c}\bar{\Delta}}{6}}, \tag{7.2.10}$$

where $k_B = 1$. Substituting in expressions (7.0.1) and (7.2.7) and simplifying yields the correct Bekenstein-Hawking entropy

$$S = \frac{2\pi r_+}{4G}. \tag{7.2.11}$$

Using (4.10.11), the logarithmic corrections to the entropy can simply be read off,

$$\rho(\Delta, \bar{\Delta}) = \frac{8G\ell^2}{(r_+^2 - r_-^2)^{3/2}} e^{\frac{2\pi r_+}{4G}} + \text{higher order terms}. \tag{7.2.12}$$

Therefore,

$$\begin{aligned} S &= \frac{2\pi r_+}{4G} - \frac{3}{2}\ln\left(\frac{r_+^2 - r_-^2}{G^2}\right) + \text{constant} + \text{higher order terms} \\ &= \frac{2\pi r_+}{4G} - \frac{3}{2}\ln\left(\frac{2\pi r_+}{G}\right) - \frac{3}{2}\ln\left(T_H\ell\right) + \text{constant} + +\text{higher order terms} \end{aligned} \tag{7.2.13}$$

where the Hawking Temperature T_H is given by

$$T_H = \frac{r_+^2 - r_-^2}{2\pi\ell^2 r_+}. \tag{7.2.14}$$

In terms of M and J, (7.2.11) can be written as

$$S = \frac{2\pi}{4G}\left[4GM\ell^2 + 4G\ell\sqrt{M^2\ell^2 - J^2}\right]^{1/2} \tag{7.2.15}$$

and the leading corrections from (7.2.13) are given as

$$-\frac{3}{4}\ln\left(M^2\ell^2 - J^2\right) + \text{constant} + \cdots . \tag{7.2.16}$$

The first law of black hole thermodynamics for the rotating BTZ black hole can now be examined. Setting $f(r)$ in (7.2.2) equal to 0 and solving for M and J gives,

$$M = \frac{1}{8G}\left(\frac{r_+^2}{\ell^2} + \frac{16G^2J^2}{r_+^2}\right), \tag{7.2.17}$$

and

$$J = r_+\sqrt{\frac{M}{2G} - \frac{r_+^2}{16^2G^2\ell^2}}. \tag{7.2.18}$$

Written in terms of the entropy, (7.2.17) becomes

$$M = \frac{GS^2}{2\pi^2\ell^2} + \frac{\pi^2J^2}{2GS^2}. \tag{7.2.19}$$

Then the differential of M is given by

$$dM = \left(\frac{\partial M}{\partial S}\right)_{J,Q} dS + \left(\frac{\partial M}{\partial J}\right)_{S,Q} dJ, \tag{7.2.20}$$

where

$$\begin{aligned} \left(\frac{\partial M}{\partial S}\right)_{J,Q} = T_H &= \frac{GS}{\pi^2\ell^2} - \frac{\pi^2J^2}{GS^3} \\ &= \frac{r_+}{2\pi\ell^2} - \frac{8G^2J^2}{\pi r_+^3} \end{aligned} \tag{7.2.21}$$

and

$$\left(\frac{\partial M}{\partial J}\right)_{S,Q} = \Omega_H = \frac{\pi^2 J}{GS^2} = \frac{4GJ}{r_+^2}, \tag{7.2.22}$$

which gives the first law of black hole thermodynamics for the rotating BTZ black hole:

$$dM = T_H\, dS + \Omega_H\, dJ. \tag{7.2.23}$$

7.3 The charged rotating BTZ black hole

The charged rotating BTZ black hole is given by the metric

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2\,(d\phi + f(\phi)\,dt)^2 \tag{7.3.1}$$

with $c^2 = 1$ again, for simplicity, where

$$f(r) = -8GM + \frac{r^2}{\ell^2} + \frac{16G^2J^2}{r^2} - \frac{\pi}{2}Q^2\ln(r) \tag{7.3.2}$$

and

$$f(\phi) = -\frac{4GJ}{r^2}, \qquad (|J| \leq M\ell) \tag{7.3.3}$$

and Q is the charge of the black hole. The charged rotating BTZ black hole is not a vacuum solution to Einstein equations, but instead a solution to Einstein's equations in the presence of an electro-magnetic field, with $T_{\mu\nu}$ given by (2.4.1).

The event horizons of the charged rotating BTZ black hole are the roots of (7.3.2). There are three different cases: two distinct roots $r_\pm$, two repeated roots corresponding to a single event horizon that determines an extremal black hole, no real roots corresponding to no event horizon, which implies a naked singularity [57]. Following the work of Akbar, Quevedo, Saifullah, Sánchez, and Taj [57], this thesis will examine the first case of two distinct roots. The analysis of this case is virtually the same as that of the rotating BTZ black hole with the extra charge term in $f(r)$.

The addition of the charge term does not affect the area of the black hole, therefore the Bekenstein-Hawking entropy is the same as that of the rotating BTZ black hole,

$$S = \frac{2\pi r_+}{4G}. \tag{7.3.4}$$

The Hawking temperature of the charged rotating BTZ black hole is found from the familiar method outlined in section 3.5. From (3.5.3), with $a'(r_+) = b'(r_+) =$

$f'(r_+) = 2r_+/\ell^2 - 32G^2J^2/r_+^2 - \pi Q^2/2r_+$ the Hawking temperature is given by

$$T_H = \frac{r_+}{2\pi\ell^2} - \frac{8G^2J^2}{\pi r_+^3} - \frac{Q^2}{8r_+}. \tag{7.3.5}$$

The black hole mass and angular momentum at the exterior event horizon r_+ can be determined by setting $f(r)$ to 0 and solving for M and J to give

$$M = \frac{1}{8G}\left(\frac{r_+^2}{\ell^2} + \frac{16G^2J^2}{r_+^2} - \frac{\pi Q^2}{2}\ln(r_+)\right), \tag{7.3.6}$$

and

$$J = r_+\sqrt{\frac{M}{2G} - \frac{r_+^2}{16^2G^2\ell^2} + \frac{\pi Q^2}{32G^2}\ln(r)}. \tag{7.3.7}$$

(7.3.6) can be written in terms of the entropy:

$$M = \frac{GS^2}{2\pi^2\ell^2} + \frac{\pi^2J^2}{2GS^2} - \frac{\pi Q^2}{16G}\ln\left(\frac{2GS}{\pi}\right). \tag{7.3.8}$$

Then the differential of M is given by

$$dM = \left(\frac{\partial M}{\partial S}\right)_{J,Q} dS + \left(\frac{\partial M}{\partial J}\right)_{S,Q} dJ + \left(\frac{\partial M}{\partial Q}\right)_{S,J} dQ, \tag{7.3.9}$$

where

$$\begin{aligned}\left(\frac{\partial M}{\partial S}\right)_{J,Q} = T_H &= \frac{GS}{\pi^2\ell^2} - \frac{\pi^2J^2}{GS^3} - \frac{\pi Q^2}{16GS} \\ &= \frac{r_+}{2\pi\ell^2} - \frac{8G^2J^2}{\pi r_+^3} - \frac{Q^2}{8r_+},\end{aligned} \tag{7.3.10}$$

$$\left(\frac{\partial M}{\partial J}\right)_{S,Q} = \Omega_H = \frac{\pi^2 J}{GS^2} = \frac{4GJ}{r_+^2}, \tag{7.3.11}$$

and

$$\left(\frac{\partial M}{\partial Q}\right)_{S,J} = \Phi = -\frac{\pi Q}{8G}\ln\left(\frac{2GS}{\pi}\right) = -\frac{\pi Q}{8G}\ln(r_+), \tag{7.3.12}$$

which gives the first law of black hole thermodynamics for the rotating BTZ black hole:

$$dM = T_H\,dS + \Omega_H\,dJ + \Phi\,dQ. \tag{7.3.13}$$

Chapter 8

Conclusion

The aim of this thesis was to give a review of the applications of CFT techniques to Bekenstein-Hawking entropy and to seek possible insights into the origin of the microscopic degrees of freedom that entropy counts.

Einstein's field equations relate the structure of spacetime to the matter distributed throughout it. Energy-momentum (including that of matter) curves spacetime and this curving is what is perceived as gravity. The more dense the matter distribution, the larger the curvature of spacetime and hence the greater the force of gravity. If a matter distribution is very dense, bigger than a critical value, an area of spacetime is refered to as a black hole, where the gravity is so strong that even light is tightly bound to this region.

Black holes have many interesting properties and obey certain laws, the laws of black hole thermodynamics. In brief, the 4 laws of black hole thermodynamics state that a black hole event horizon has a constant surface gravity; change in mass of the black hole is related to change in the area; angular momentum and electric charge of the black hole, the event horizon surface area does not decrease with time; and zero surface gravity for a black hole event horizon is impossible to achieve.

Black holes can radiate particles of various energies in a spectrum similar to black body radiation with a temperature, known as the Hawking temperature. This thermal spectrum of radiation from a black hole is known as Hawking radiation, and allows for a possible mechanism for which black holes may evaporate. Black holes also have have a measurable entropy proportional to their surface area, known as Bekenstein-Hawking entropy.

Black hole thermodynamics involves both classical gravity and quantum mechanics. Studying the origins of Hawking radiation and the origins of the microscopic degrees of freedom which lead to the Bekenstein-Hawking entropy might provide some insight to construct a proper theory of quantum gravity. This thesis reviewed the application of 2-dimensional CFT techniques to BTZ black holes in order to explain the origin of Bekenstein-Hawking entropy.

2-dimensional CFT is a highly symmetric quantum field theory. It has a defining algebra realized by 2 copies of the Virasoro algebra. The Virasoro algebra is quantum extension of the classical Witt algebra with a central extension proportional to the central charge of the theory. By studying the CFT on a torus modular invariance is required, which will impose further restrictions on the CFT. Modular invariance of the partition function can be used to derive the Cardy formula, which relates the logarithm of the density of states of a system, and hence the entropy, to the central charge of the system. In 1998 Maldacena [10] conjectured the AdS/CFT correspondence in which the string theory, a candidate for a quantum theory of gravity, on an AdS spacetime is dual to a CFT on the boundary surface of the spacetime. In this thesis the Cardy formula was used to calculate the Bekenstein-Hawking entropy of BTZ black holes.

The AdS metric is a solution to Einstein's vacuum field equations with a negative cosmological constant. Another solution of these field equations is the BTZ solution. The BTZ black hole is a (2+1) dimensional back hole solution that asymptotically reduces to a AdS_3 spacetime. The BTZ black hole solution can be static, rotating, and more generally, charged and rotating.

The Hamiltonian formulation of general relativity can be used to determine the Hamiltonian generators of conserved charges of gravity, which can be then used to determine the central charge of AdS_3 spacetimes. Because of first class constraints imposed by the equations of motion, the Hamiltonian generator simply becomes the surface term in the Hamiltonian formulation. The surface term then becomes the conserved charge. The Poisson bracket algebra of the charges is isomorphic to the Lie bracket algebra of the generators. The general algebra has a central extension proportional to a central charge. The conserved charges of the rotating BTZ black hole, which lie on the surface of the black hole, become simply the mass, M, and the angular momentum, J. This agrees with the "No-hair" theorem, which states that all black hole solutions are characterized by only the external observables mass, electric charge and angular momentum.

The work of Brown and Henneaux [9] showed that the asymptotically AdS_3 spacetime has an asymptotic symmetry group equivalent to the conformal group in 2 dimensions (an idea which was later generalized to d-dimensions by the Maldacena conjecture) and the central charge equal to the Brown-Henneaux central charge, $c = 3\ell/2G$. Using the central charge and the Cardy formula, the Bekenstein-Hawking entropy of the BTZ black holes can be calculated, as well as the leading order corrections. The

Hawking temperature can be calculated by Euclideanizing the Lorentzian BTZ metrics.

The general BTZ black hole is the charged rotating BTZ black hole. The Bekenstein-Hawking entropy and the Hawking temperature of the charged rotating BTZ black hole were calculated to be

$$S = \frac{2\pi r_+}{4G}, \qquad T_H = \frac{r_+}{2\pi\ell^2} - \frac{8G^2J^2}{\pi r_+^3} - \frac{Q^2}{8r_+}. \tag{8.0.1}$$

The first law of black hole thermodynamics is shown to be satisfied for the charged rotating BTZ black hole.

Applying CFT techniques to the BTZ black hole yields the proper Bekenstein-Hawking entropy, as well as the leading order correction. Furthermore, it offers insight into the origin of the microscopic degrees of freedom, they are related to the energy spectrum corresponding to the Virasoro generators and the central charge of the system. Therefore, conformal invariance must be present in a quantum mechanical description of black hole entropy.

Future work in this field could be to extend the idea of conformal invariance leading to microscopic degrees of freedom to offer insight into the black hole information loss problem. In order to tackle this problem one would first have to explore the role, if any, of conformal invariance to Hawking radiation.

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