Discreteness of space from the generalized uncertainty principle

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ABSTRACT

Various approaches to Quantum Gravity (such as String Theory and Doubly Special Relativity), as well as black hole physics predict a minimum measurable length, or a maximum observable momentum, and related modifications of the Heisenberg Uncertainty Principle to a so-called Generalized Uncertainty Principle (GUP). We propose a GUP consistent with String Theory, Doubly Special Relativity and black hole physics, and show that this modifies all quantum mechanical Hamiltonians. When applied to an elementary particle, it implies that the space which confines it must be quantized. This suggests that space itself is discrete, and that all measurable lengths are quantized in units of a fundamental length (which can be the Planck length). On the one hand, this signals the breakdown of the spacetime continuum picture near that scale, and on the other hand, it can predict an upper bound on the quantum gravity parameter in the GUP, from current observations. Furthermore, such fundamental discreteness of space may have observable consequences at length scales much larger than the Planck scale.

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An intriguing prediction of various theories of quantum gravity (such as String Theory) and black hole physics is the existence of a minimum measurable length. This has given rise to the so-called Generalized Uncertainty Principle, or GUP, or equivalently, modified commutation relations between position coordinates and momenta [1]. The recently proposed *Doubly Special Relativity* (or DSR) theories on the other hand (which predict maximum observable momenta), also suggest a similar modification of commutators [2,3]. The commutators which are consistent with String Theory, Black Holes Physics, DSR, *and* which ensure $[x_i, x_j] = 0 = [p_i, p_j]$ (via the Jacobi identity) have the following form [4][1]

$$[x_i, p_j] = i\hbar \left[\delta_{ij} - \alpha \left(p\delta_{ij} + \frac{p_i p_j}{p} \right) + \alpha^2 \left(p^2 \delta_{ij} + 3p_i p_j \right) \right] \quad (1)$$

where $p^2 = \sum_{j=1}^{3} p_j p_j$, $\alpha = \alpha_0/M_{\mathrm{Pl}}c = \alpha_0 \ell_{\mathrm{Pl}}/\hbar$, M_{Pl} = Planck mass, $\ell_{\mathrm{Pl}} \approx 10^{-35}$ m = Planck length, and $M_{\mathrm{Pl}}c^2$ = Planck energy $\approx 10^{19}$ GeV. Eq. (1) gives, in 1-dimension, to $\mathcal{O}(\alpha^2)$

$$\Delta x \Delta p \geqslant \frac{\hbar}{2} \Big[1 - 2\alpha \langle p \rangle + 4\alpha^2 \langle p^2 \rangle \Big]$$
$$\geqslant \frac{\hbar}{2} \left[1 + \left(\frac{\alpha}{\sqrt{\langle p^2 \rangle}} + 4\alpha^2 \right) \Delta p^2 + 4\alpha^2 \langle p \rangle^2 - 2\alpha \sqrt{\langle p^2 \rangle} \right]. \quad (2)$$

Commutators and inequalities similar to (1) and (2) were proposed and derived respectively in [5–8]. These in turn imply a minimum measurable length *and* a maximum measurable momentum (to the best of our knowledge, (1) and (2) are the only forms which imply both)

$$\Delta x \geqslant (\Delta x)_{\min} \approx \alpha_0 \ell_{\mathrm{Pl}}, \quad (3)$$

$$\Delta p \leqslant (\Delta p)_{\max} \approx \frac{M_{\mathrm{Pl}}c}{\alpha_0}. \quad (4)$$

Next, defining [4]

$$x_i = x_{0i}, \qquad p_i = p_{0i}\big(1 - \alpha p_0 + 2\alpha^2 p_0^2\big), \quad (5)$$

with x_{0i}, p_{0j} satisfying the canonical commutation relations $[x_{0i}, p_{0j}] = i\hbar \delta_{ij}$, it can be shown that Eq. (1) is satisfied. Here, p_{0i} can be interpreted as the momentum at low energies (having the standard representation in position space, i.e. $p_{0i} = -i\hbar \partial/\partial x_{0i}$), p_i as that at higher energies, and p_0 as the magnitude of the p_{0i} vector, i.e. $p_0^2 = \sum_{j=1}^{3} p_{0j}p_{0j}$. It is normally assumed that the dimensionless parameter α_0 is of the order of unity, in which case the α dependent terms are important

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[1] The results of this article do not depend on this particular form of GUP chosen, and continue to hold for a large class of variants, so long as an $\mathcal{O}(\alpha)$ term is present in the right-hand side of Eq. (1).

only when energies (momenta) are comparable to the Planck energy (momentum), and lengths are comparable to the Planck length. However, we do not impose this condition *a priori*, and note that this may signal the existence of a new physical length scale of the order of $\alpha\hbar = \alpha_0\ell_{\rm Pl}$. Evidently, such an intermediate length scale cannot exceed the electroweak length scale $\sim 10^{17}\ell_{\rm Pl}$ (as otherwise it would have been observed). This implies $\alpha_0 \leqslant 10^{17}$.

Using (5), a Hamiltonian of the form

$$H = \frac{p^2}{2m} + V(\vec{r}) \tag{6}$$

can be written as

$$H = H_0 + H_1 + \mathcal{O}(\alpha^2), \tag{7}$$

where $H_0 = \dfrac{p_0^2}{2m} + V(\vec{r})$ and $\tag{8}$

$$H_1 = -\frac{\alpha}{m}p_0^3. \tag{9}$$

Thus, we see that *any* system with a well defined quantum (or even classical) Hamiltonian H_0, is perturbed by H_1, defined above, near the Planck scale. In other words, Quantum Gravity effects are in some sense universal! The relativistic Dirac equation is modified in a similar way, and is expected to give rise to the main result of this Letter [9], which is unaffected by the inclusion of the $\mathcal{O}(\alpha^2)$ terms in H_1 as well. Phenomenological implications of the GUP in diverse quantum systems have been studied (for example see [10] and references therein).

In this article, we apply the above formalism to a single particle in a box of length L (with boundaries at $x = 0$ and $x = L$), and show that the box length must be quantized. Since this particle can be considered as *test particle* to measure the dimension of the box, this suggests that space itself is quantized, as are all observable lengths. The wavefunction of the particle satisfies the following GUP corrected Schrödinger equation inside the box, where $V(\vec{r}) = 0$ (outside, $V = \infty$ and $\psi = 0$)

$$H\psi = E\psi, \tag{10}$$

or equivalently ($d^n \equiv d^n/dx^n$),

$$d^2\psi + k^2\psi + 2i\alpha\hbar\, d^3\psi = 0, \tag{11}$$

where $k = \sqrt{2mE/\hbar^2}$. A trial solution of the form $\psi = e^{mx}$ yields

$$m^2 + k^2 + 2i\alpha\hbar m^3 = 0, \tag{12}$$

with the following solution set to leading order in α: $m = \{ik', -ik'', i/2\alpha\hbar\}$, where $k' = k(1+k\alpha\hbar)$ and $k'' = k(1-k\alpha\hbar)$. Thus, the general wavefunction to leading order in $\ell_{\rm Pl}$ and α is of the form

$$\psi = Ae^{ik'x} + Be^{-ik''x} + Ce^{ix/2\alpha\hbar}. \tag{13}$$

As is well known, the first two terms (with $k' = k'' = k$) and the boundary conditions $\psi = 0$ at $x = 0, L$ give rise to the standard quantization of energy for a particle in a box, namely $E_n = n^2\pi^2\hbar^2/2mL^2$. However, note the appearance of a new oscillatory term here, with characteristic wavelength $4\pi\alpha\hbar$ and momentum $1/4\alpha = M_{\rm Pl}c/4\alpha_0$ (which is Planckian for $\alpha_0 = \mathcal{O}(1)$). This results in the new quantization mentioned above. Also, as this term should drop out in the $\alpha \to 0$ limit, one must have $\lim_{\alpha\to 0}|C| = 0$. We absorb any phase of A in ψ, such that A is real. The boundary condition

$$\psi(0) = 0 \tag{14}$$

implies

$$A + B + C = 0. \tag{15}$$

Substituting for B in Eq. (13), we get

$$\psi = 2iA\sin(kx) + C[-e^{-ikx} + e^{ix/2\alpha\hbar}] - \alpha\hbar k^2 x[iCe^{-ikx} + 2A\sin(kx)]. \tag{16}$$

The remaining boundary condition

$$\psi(L) = 0 \tag{17}$$

yields

$$2iA\sin(kL) = |C|[e^{-i(kL+\theta_C)} - e^{i(L/2\alpha\hbar - \theta_C)}] + \alpha\hbar k^2 L[i|C|e^{-i(kL+\theta_C)} + 2A\sin(kL)], \tag{18}$$

where $C = |C|\exp(-i\theta_C)$. Note that both sides of the above equation vanish in the limit $\alpha \to 0$, when $kL = n\pi\ (n \in \mathbb{Z})$ and $C = 0$. Thus, when $\alpha \neq 0$, we must have $kL = n\pi + \epsilon$, where $\epsilon \in \mathbb{R}$ (such that energy eigenvalues E_n remain positive), and $\lim_{\alpha\to 0}\epsilon = 0$. This, along with the previously discussed smallness of $|C|$ ensures that the second line in Eq. (18) above falls off faster than $\mathcal{O}(\alpha)$, and hence can be dropped. Next, equating the real parts of the remaining terms of Eq. (18) (remembering that $A \in \mathbb{R}$), we get

$$\cos\left(\frac{L}{2\alpha\hbar} - \theta_C\right) = \cos(kL + \theta_C) = \cos(n\pi + \theta_C + \epsilon), \tag{19}$$

which implies, to leading order, the following two series of solutions

$$\frac{L}{2\alpha\hbar} = \frac{L}{2\alpha_0\ell_{\rm Pl}} = n\pi + 2q\pi + 2\theta_C \equiv p\pi + 2\theta_C, \tag{20}$$

$$\frac{L}{2\alpha\hbar} = \frac{L}{2\alpha_0\ell_{\rm Pl}} = -n\pi + 2q\pi \equiv p\pi, \tag{21}$$

$p \equiv 2q \pm n \in \mathbb{N}$.

These show that there cannot even be a single particle in the box, unless its length is quantized as above. For other lengths, there is no way to probe or measure the box, even if it exists. Hence, effectively all measurable lengths are quantized in units of $\alpha_0\ell_{\rm Pl}$! We interpret this as space essentially having a discrete nature. Consistency with Eq. (3) requires p to run from 1 in the second case. The minimum length is $\approx \alpha_0\ell_{\rm Pl}$ in each case. Once again, if $\alpha_0 \approx 1$, this fundamental unit is the Planck length. However, current experiments do not rule out discreteness smaller than about a thousandth of a Fermi, thus predicting the previously mentioned bound on α_0.[2] Note that similar quantization of length was shown in the context of Loop Quantum Gravity in [11], albeit following a much more involved analysis, and perhaps under a stronger set of starting assumptions. In general however, we expect our result to emerge from any correct theory of Quantum Gravity. It will be interesting to see whether our result can be generalized to the quantization of areas and volumes, and also to study its possible phenomenological implications. Furthermore, it is plausible that if space has fundamentally a "grainy" structure, the effects may be felt well beyond the Planck scale, e.g. at around 10^{-4} fm, the length scale to be probed at the Large Hadron Collider (similar to Brownian motion observed at scales in excess of 10^5 times the atomic scale). We hope to study such effects and report elsewhere.

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[2] Equating the imaginary parts of (18) yields the auxiliary condition: $\epsilon = -|C|\sin(\theta_C)/A$ and $\epsilon = 0$, for solutions (20) and (21) respectively.

Acknowledgements

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