EXPLICIT ZERO DENSITY FOR THE RIEMANN ZETA FUNCTION

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Dedication

I dedicate my dissertation to my husband, Alireza whose encouragement and belief in me has been a constant source of strength during the challenges of university and life. I also dedicate it to my family; although my parents are far away in my home country, their love and support continue to inspire me.

Abstract

The Riemann zeta function is a fundamental function in analytical number theory and studying it's zeros is important for understanding the distribution of the prime numbers. In particular, explicit zero density estimates are important for improving estimates of prime counting functions. Kadiri, Lumley and Ng [26] proved an explicit zero density result for for the number of zeros of the Riemann zeta function, $N(\sigma, T_1, T_2)$ with real part greater than σ and imaginary part between T_1 and T_2. In this thesis we improve their proof and provide a general estimate of the form

$$N(\sigma, T_1, T_2) \leq \mathcal{U} T_2^{2a(1-\sigma)+(a'-1)(2\sigma-1)} \log T_2^{2(b+1)(1-\sigma)+(b'+2)(2\sigma-1)+1} + \mathcal{V}(\log T_2)^2.$$

This result can be applied using different bounds for Riemann zeta function on the half line. Specifically, by using the following bound for zeta function on the half line

$$|\zeta(\tfrac{1}{2}+it)|^2 \leq (0.618)^2 T^{\frac{1}{3}} \log^2 T + \zeta(\tfrac{1}{2})^2, \qquad 0 \leq t \leq T,$$

for all $\sigma \in [0.75, 1]$ we will prove

$$N(\sigma, T_1, T_2) \leq 12.4531 T_2^{\frac{8}{3}(1-\sigma)} \log T_2^{(5-2\sigma)} + 3.8689(\log T_2)^2,$$

and by using the following bound for the zeta function on the half line,

$$|\zeta(\tfrac{1}{2}+it)|^2 \leq (66.7)^2 T^{\frac{27}{82}} + \zeta(\tfrac{1}{2})^2, \qquad 0 \leq t \leq T$$

for all $\sigma \in [0.9, 1]$ we will prove

$$N(\sigma, T_1, T_2) \leq 58.946 T_2^{\frac{109}{41}(1-\sigma)} \log T_2^{(2\sigma+1)} + 2.699 (\log T_2)^2.$$

These results improve upon the previous works by Kadiri, Lumley and Ng [26] by revisiting their argument but making a number of changes which result in improvements.

Acknowledgments

I would like to express my deepest gratitude to my supervisor, Dr. Andrew Fiori whose guidance and support have been invaluable throughout my academic journey. His patience, intelligence, and insightful ideas have played a crucial role in my growth as a researcher. I honestly could not have asked for a better mentor and I will always be grateful to have had him as my supervisor.

My heartfelt appreciation goes to Dr. Habiba Kadiri and Dr. Nathan Ng for serving on my thesis committee and providing invaluable feedback and guidance that shaped this research and thesis.

Additionally, I sincerely thank Dr. Wendy Osborn, the chair of my committee, as well as the faculty and staff of the Department of Mathematics at the University of Lethbridge.

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Chapter 1

Introduction

1.1 Introduction and Motivation

The Riemann zeta function, which we denote $\zeta(s)$ is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

when the complex variable s has $\Re(s) > 1$. Riemann showed that $\zeta(s)$ has an analytic continuation to the entire complex plane with a simple pole at $s = 1$. There is a connection between $\zeta(s)$ and the primes which can be seen via the Euler product formula

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p} (1 - \frac{1}{p^s})^{-1},$$

where the product on the right-hand side is over the set of all primes p. By using the Euler product formula we know that $\zeta(s)$ has no zeros for $\sigma > 1$, since for $\sigma > 1$ this infinite product is convergent, and therefore non-zero.

The zeta function satisfies the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin(\frac{\pi s}{2}) \Gamma(1-s) \zeta(1-s),$$

where $\Gamma(s)$ is the gamma function. From $\sin(\frac{\pi s}{2})$ being 0 in the functional equation we can see that $\zeta(s)$ has zeros at negative even integers, $s = -2, -4, -6, ...$ These are called trivial zeros and all non-trivial zeros lie in the open strip $s \in C : 0 < \Re < 1$, which is called the critical strip.

The non-trivial zeros have captured far more attention because their distribution not only is far less understood but, more importantly, their study is intimately connected to the distribution of prime counting functions. For example, the prime number theorem (PNT), established in 1896 by Hadamard and de la Vallée Poussin, by proving that $\zeta(s)$ has no zeros on the line $\sigma = 1$, states that

$$\pi(x) \sim \mathrm{Li}(x),$$

where $\pi(x)$ is the number of primes less than or equal to x and $\mathrm{Li}(x)$ is the logarithmic integral function defined by $\mathrm{Li}(x) = \int_2^x \frac{1}{\log t} dt$. The shape and constant in the error term are determined by what we can prove about the number and location of zeros of zeta function. For example, from [29] an explicit formula and study of the zeros of zeta function lead to the estimate $\pi(x) = \mathrm{Li}(x) + O\left(\frac{x}{\exp(c\sqrt{\log x})}\right)$, for some $c > 0$.

We can deduce properties of the primes from properties of the zeros, and vice versa. So to obtain information about primes, we need to have information about four key things: A numerical verification of Riemann Hypothesis (RH), an explicit zero free region, a formula for counting zeros of zeta function in a rectangle region and explicit zero density result. Because of this connection, a huge amount of work over the past century and a half has been put towards locating the zeros.

1.2 Notations

In this section, a list of the notations used throughout this thesis is provided. The table 1.1 serves as a reference to help readers understand the meaning of the notations.

Table 1.1: List of Symbols and Notations

Symbol/Notation	**Description**
s	Complex number
$\Re$	Real part of the complex number
$\Im$	Imaginary part of the complex number
ρ	Non-trivial zeros of the zeta function which is equal to $\sigma + it$
O	Big-O

1.3 Riemann Hypothesis

The Riemann Hypothesis (RH) is a conjecture about the location of non-trivial zeros of Riemann zeta function. According to this hypothesis all nontrivial zeros of the Riemann zeta function lie on the critical line $\Re(s) = \frac{1}{2}$, and to date still is an open problem. In the absence of a proof, it is extremely important to obtain numerical verification of the Riemann hypothesis which we denote as

$$RH(H_0) : \textbf{all non-trivial zeros } \rho \textbf{ of } \zeta(s) \textbf{ with } |\Im(\rho)| \le H_0 \textbf{ satisfy } \Re(\rho) = \frac{1}{2}. \qquad (1.1)$$

To that end, define H_0 as the largest number for which it is known that all zeroes $\rho = \beta + i\gamma$ with $0 < \gamma \le H_0$ have $\beta = \frac{1}{2}$. Currently, the best published value for H_0 in equation (1.1) is

$$H_0 = 3 \cdot 10^{12},$$

by Platt and Trudgian [35].

1.4 Zero free region

Determining zero-free regions within the critical strip has long been of great interest in number theory. Hadamard and De la Vallée Poussin [41] proved in 1896 that $\zeta(s)$ has no zeros on the vertical line $\sigma = 1$. Moreover, in 1899, De la Vallée Poussin [7] famously proved that there exists a positive constant R such that $\zeta(s)$ is non-zero in the region of the form

$$\sigma \geq 1 - \frac{1}{R \log t}, \tag{1.2}$$

where $s = \sigma + it$ and $t \geq t_0$ for some t_0. We refer to a region of this form as a classical zero-free region of the Riemann zeta-function. An explicit zero-free region for $\zeta(s)$ of this kind is attributed to Mossinghoff–Trudgian [30], who verified (1.2) for $R = 5.573$ and $t \geq 2$. This was established by extending some work of Kadiri [24] by constructing a more favorable trigonometric polynomial, by optimizing some analytic arguments, and by employing the verification of RH up to $3.06 \cdot 10^{10}$ from [34]. By using the verification of RH for $|t| \leq 3.10^{12}$ in [35] and using an improvement to Kadiri's result by Jang and Kwon [21] in 2014, Mossinghoff, Trudgian and Yang [31] improved the classical zero-free region and verified (1.2) for $R = 5.558691$ and $t \geq 2$.

In 1958 Korobov[27] and Vinogradov [43] independently demonstrated that there exists a positive constant $R > 0$, such that $\zeta(s)$ is non-zero in the region $s = \sigma + it$ such that for all $t \geq t_0$ for some t_0,

$$\sigma \geq 1 - \frac{1}{R(\log t)^{\frac{2}{3}}(\log\log t)^{\frac{1}{3}}}. \tag{1.3}$$

The best known zero-free region for $\zeta(s)$ of this kind is attributed to Bellotti [2], who have verified (1.3) for $R = 53.989$ for $t \geq 3$. She also established the zero-free region (1.3) for large t with $R = 48.0718$. The largest known zero-free region in the region $\exp(170.3) \leq t \leq \exp(532141)$ is due to Yang [45]. He proved zeta function has no zeros in the region

$$\sigma > 1 - \frac{\log\log|t|}{21.233\log|t|}, \tag{1.4}$$

for $|t| \geq 3$.

1.5 Counting of Zeros

Within the critical strip, but outside the zero-free region, there exist estimates for the number of non-trivial zeros ρ with $0 < \Im(\rho) \leq T$. For $T \geq 0$, we define

$$N(T) = \#\{\rho \in C : \zeta(\rho) = 0, 0 < \Im(\rho) \leq T, 0 < \Re(\rho) < 1\}. \tag{1.5}$$

In addition to the RH, Riemann conjectured the asymptotic formula for $N(T)$ in the critical strip with $0 \leq \gamma < T$ to be

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T), \tag{1.6}$$

which was proved by Mangoldt in 1905 [44]. There has been a long history about explicit bounds for $N(T)$ which the most recent bound was given by Hasanalizade, Shen and Wong [12]. They proved that for $T \geq e$,

$$|N(T) - \frac{T}{2\pi} \log(\frac{T}{2\pi e})| \leq 0.1038 \log T + 0.2573 \log\log T + 9.3675. \tag{1.7}$$

We consider $N(\sigma, T)$ as the number of zeros of the Riemann zeta function in the region $\sigma \leq \Re(s) \leq 1$ and $0 \leq \Im(s) \leq T$. Trivially we have that $N(\sigma, T) = 0$ for all $T \leq H_0$ and $\sigma > \frac{1}{2}$. In the next section we talk about an explicit bound for $N(\sigma, T)$ and its history.

1.6 Zero Density result

Let $N(\sigma, T)$ denote the number of non-trivial zeros of the Riemann zeta function with real part greater than σ and imaginary part between 0 and T. Explicit upper bounds for $N(\sigma, T)$ commonly referred to as a **zero density** result.

Let $\rho = \Re(s) + i\Im(s)$ denote a non-trivial zero of $\zeta(s)$ lying in the critical strip, $0 < \Re(s) <$

1. Let $\frac{1}{2} < \sigma < 1$ and $T > 0$, we have

$$N(\sigma,T) = \#\{\rho = \beta + i\gamma : \zeta(\rho) = 0, 0 < \Im(s) < T, \sigma < \Re(s) < 1\}. \tag{1.8}$$

There have been several authors who have worked on this topic. The earliest zero-density estimate is due to Bohr and Landau [3] who showed in 1913 that

$$N(\sigma,T) = O\left(\frac{T}{\sigma - \frac{1}{2}}\right),$$

As T grows to infinity. This was improved in 1937 by Ingham [16], who showed

$$N(\sigma,T) = O\left(T^{(2+4c)(1-\sigma)}(\log T)^5\right),$$

Assuming that $\zeta(\frac{1}{2}+it) = O(t^{c+\varepsilon})$. In particular, the Lindelöf hypothesis is a conjecture about the rate of growth of the Riemann zeta function on the critical line. This hypothesis is one of the weaker consequences of Riemann hypothesis, which says that for any $\varepsilon > 0$, $\zeta(\frac{1}{2}+it) = O(t^{\varepsilon})$ and it implies that

$$N(\sigma,T) = O\left(T^{2(1-\sigma)+\varepsilon}\right).$$

Also known as the Density Hypothesis. While the Density Hypothesis remains open, many authors proved different zero density estimates for zeta function in different ranges for σ. The following summary of best known bounds for $\zeta(s)$ is due to Trudgian and Yang [42].

$$N(\sigma,T) = O\left(T^{f(\sigma)+\varepsilon}\right),$$

$$f(\sigma)=\begin{cases}1-(8/7-\delta)(\sigma-1/2), \quad \text{if} \quad 1/2\leq\sigma\leq 1/2+o(1), \quad \text{for any } \delta>0,\\ 3(1-\sigma)/(2-\sigma), \quad 1/2+o(1)<\sigma\leq 3/4,\\ 3(1-\sigma)/(7\sigma-4), \quad 3/4<\sigma<13/17,\\ 6(1-\sigma)/(5\sigma-1), \quad 13/17\leq\sigma<25/32,\\ 49/12-14\sigma/3, \quad 25/32\leq\sigma\leq 11/14,\\ 9(1-\sigma)/(7\sigma-1), \quad 11/14<\sigma<3831/4791,\\ 3(1-\sigma)/2\sigma, \quad 3831/4791\leq\sigma<155/174,\\ 24(1-\sigma)/(30\sigma-11), \quad 155/174\leq\leq 17/18,\\ 4(1-\sigma)/(2\sigma+1), \quad 17/18<\sigma\leq 0.9541...,\\ 6.42(1-\sigma)^{3/2}, \quad \text{if} \quad 0.9541...<\sigma\leq 1.\end{cases}$$

The results themselves are due to Jutila [22], Ingham [18], Ivić [20], Jutila [23], Bourgain [4], Jutila [23], Jutila [23], Ivić [19], Ivić [19] and Heath-Brown [13] respectively.

Making the above bounds for $N(\sigma,T)$ explicit would have important applications as they would allow us to obtain better estimates for prime counting functions including finding prime numbers in short intervals.

Kadiri [25] showed an explicit version of Bohr and Landau's bound. For $\sigma\geq 0.55$ and all $T\geq H_0$ she proved

$$N(0.90,T)\leq c_1T+c_2\log T+c_3,$$

where c_i depend on. σ. We note that with $\sigma=0.90$ this formula simplifies to

$$N(0.90,T)\leq 0.4421T+0.6443\log T+363301.$$

In addition, for T taking a specific value ($H_0=3.061.10^{10}$), she proved

$$N(0.90,H_0)<96.20.$$

Ramaré [39] had proven an explicit version of Ingham's bound. For $\sigma \geq 0.52$ and all $T \geq 2000$, he showed

$$N(\sigma,T) \leq 965(3T)^{\frac{8(1-\sigma)}{3}}(\log T)^{5-2\sigma}+51.5(\log T)^2.$$

We note that with $\sigma = 0.90$ this formula simplifies to

$$N(0.90,T) < 1293.48(\log T)^{\frac{16}{5}}T^{\frac{4}{15}}+51.50(\log T)^2.$$

Which gives the below bound for $T = H_0$:

$$N(0.90,H_0) < 2.1529 \cdot 10^{10}. \tag{1.9}$$

Kadiri, Lumley, and Ng [26] have presented a result that provides a tighter bound for $N(\sigma,T)$. Their result improves upon both Ramaré and Kadiri's estimates by following Ingham's argument but using a more general weight.

Specifically, the bound is given by:

$$N(\sigma,T) \leq C_1(\sigma)(\log T)^{5-2\sigma}T^{\frac{8}{3}(1-\sigma)}+C_2(\sigma)(\log T)^2,$$

for some computable functions $C_1(\sigma)$ and $C_2(\sigma)$. Setting $\sigma = 0.9$, their estimate becomes:

$$N(0.90,T) < 11.499(\log T)^{\frac{16}{5}}T^{\frac{4}{15}}+3.186(\log T)^2,$$

Fiori, Kadiri and Swidinsky [8] used the recent partial numerical verification of the Riemann Hypothesis $H_0 = 3.10^{12}$ from [35] and used the recent sub-convexity bound for zeta function on the half line proven by Hiary, Patel and Yang [15] to present a tighter bound for

$N(\sigma, T)$. For instance for $\sigma = 0.9$, they obtained:

$$N(0.90, T) < 10.8209(\log T)^{\frac{16}{5}} T^{\frac{4}{15}} + 2.8640(\log T)^2. \tag{1.10}$$

The main goal of our thesis is to find better values of $C_1(\sigma)$ and $C_2(\sigma)$. We achieve this by using better bounds on ζ on the half line which will improve the exponents on both the T and $\log T$ terms. We also improve the bounds for some arithmetic sums, which affects both the $C_1(\sigma)$ and $C_2(\sigma)$ term. Furthermore, we provide results which are valid on the intervals. In our thesis, we make use of the recent partial numerical verification of the Riemann Hypothesis $H_0 = 3.10^{12}$ from [35], as well as two different versions of upper bounds for the zeta function on the half-line. Our first main result includes the recent sub-convexity bound for zeta function on the half line proven by Hiary, Patel and Yang [15]. Specifically for $\sigma = 0.9$, we obtain:

$$N(0.90, T) < 7.732(\log T)^{\frac{16}{5}} T^{\frac{4}{15}} + 2.699(\log T)^2.$$

For more details, please see our main Theorem 4.46 and Remark 4.47 and Corollary 4.50. The numerical results for this version are taken from table A.7.

Our second main result includes the recent sub-Weyl bound for zeta function on the half line proven by Patel and Yang [33]. Specifically for $\sigma = 0.9$, we obtain:

$$N(\sigma, T) \leq 60.245(\log T)^{\frac{14}{5}} T^{\frac{109}{410}} + 2.699(\log T)^2.$$

For more details, please see our main Theorem 4.46 and Remark 4.47 and Corollary 4.54. The numerical results for this version are taken from table A.9.

Additionally, my thesis aims to reorganize the proof by breaking it apart into its key elements so that it becomes easier to see how to incorporate other future improvements in machinery used without redoing the whole argument. Moreover, this will more easily allow parts of the argument to be reused in related problems for more general *L*-functions and therefore, it becomes easier to identify areas of improvement.

1.7 Idea of proof

We now explain the main idea of finding an upper bound for $N(\sigma, T)$ in the paper of Kadiri, Lumley and Ng [26]. First, they use Littlewood's zero detection method as the general tool for counting the number of zeros of any analytical functions in a rectangle region. It consists of four different integrals, defined as follows:

$$\begin{aligned} I_1 &:= \int_{T_1}^{T_2} \log|\phi(\sigma_1 + it)|\,dt, \quad I_2 := \int_{\sigma_1}^{\sigma_2} \arg\phi(\tau + iT_2)\,d\tau, \\ I_3 &:= \int_{\sigma_1}^{\sigma_2} \arg\phi(\tau + iT_1)\,d\tau, \quad I_4 := \int_{T_1}^{T_2} \log|\phi(\sigma_2 + it)|\,dt, \end{aligned} \tag{1.11}$$

You can see the details in Section 2.1. Then the goal becomes to determine an upper bound for each integral. In applications I_1 is the largest. Therefore, we describe the key ideas in bounding the first integral in Chapter 2. This includes the smoothing, unsmoothing method, and convexity estimate. I have explained the general form of these methods in Section 2.2. To estimate the fourth integral, as well as parts of the first integral, we use a standard mean value theorem for Dirichlet polynomials, which I have explained in Section 2.3.

Because many of our functions involve Dirichlet series, we shall need bounds on a variety of arithmetic sums. In Chapter 3, we introduce some preliminary lemmas that provide bounds for arithmetic sums.

In Chapter 4 we applied our general methods to the function $\zeta(s)$ and finally we will find explicit bounds for each integral. In the fourth integral, we consider the case $\sigma_2 = \mu$ with $\mu > 1$, and our goal is to find a lower bound for the zeta function in this region. Since finding the lower bound on the integral of the zeta function is challenging, we use the mollifiers introduced in Section 4.1 which can control the value for the zeta function outside of the critical strip. Therefore, the goal becomes replacing $\zeta(s)$ with $\zeta(s) \cdot P(s)$, where $P(s)$ is an entire analytic function.

There are several options for $P(s)$ and what we use is the product of $2-\zeta(s)M(s)$ and $M(s)$, where $M(s)$ is a function which is approximately $\frac{1}{\zeta(s)}$. We call the new function $h(s)$ and since the new function h is easier to work with, more precisely any zeros of the $\zeta(s)$ is zeros of $h(s)$, the focus shifts towards determining an upper bound for $h(s)$ instead of the zeta function itself. We then apply general methods described in Chapter 2 to $h(s)$ and find explicit bounds for the number of zeros of $h(s)$ in a rectangle region R. Then the integrals become:

$$\begin{aligned} I_1 &:= \int_{T_1}^{T_2} \log\left|h\left(\sigma'+it\right)\right| dt, \quad I_2 := \int_{\sigma'}^{\mu} \arg h(\tau+iT_2)\,d\tau, \\ I_3 &:= \int_{\sigma'}^{\mu} \arg h(\tau+iT_1)\,d\tau, \quad I_4 := \int_{T_1}^{T_2} \log|h(\mu+it)|\,dt, \end{aligned} \tag{1.12}$$

where $\mu>1$ is just outside the critical strip and $\sigma'<\sigma$. You can see all details about first integral in Section 4.3 and we provide an explicit upper bound for it in Section 4.3.5. In Section 4.4 you can see the upper bounds for the fourth integral. For the second and third integral, we follow Kadiri, Lumley and Ng's proof [26] and use the general results from [41] by Titchmarsh and [1] by Backlund to find an upper bound for these integrals. This upper bound can be found in Section 4.5. Finally, by combining all the obtained upper bounds for each integral we provide the upper bound for $N(\sigma,T_1,T_2)$ in Section 4.6.

Chapter 2

General Methods

In this Chapter, our goal is to set up the general methods used by Ingham [17]. Our initial step that we describe in Section 2.1 is to introduce the method of Littlewood's classical Lemma as stated in [41] for deducing an upper bound for the number of zeros of a function in a rectangle region. This number is split into evaluating or bounding of four different integrals. Then in Section 2.2, we describe the key idea in bounding the integral of a function when we have a little control over it by shifting where the integral needs to be evaluated. We do this by a smoothing- unsmoothing method, and a convexity theorem. At the end of Section 2.3, we will describe the mean value theorem for Dirichlet polynomials which is crucial for later estimates related to the first and fourth integral.

2.1 Counting zeros of a function in a special rectangle region

There exist many useful tools in complex analysis to count zeros of holomorphic functions inside a specified bounded region. We present here a classic idea of Bohr, Landau, Littlewood, and Titchmarsh as stated in [41] which uses the Residue Theorem, to bound the number of zeros of a holomorphic function in a rectangle region. Let $T_2 > T_1$. For a function ϕ we introduce the number of zeros of ϕ between T_1 and T_2:

$$N_{\phi}(\sigma, T_1, T_2) = \#\{\rho; \phi(\rho) = 0, \sigma < \Re(\rho) < 1, T_1 \le \Im(\rho) \le T_2\}, \tag{2.1}$$

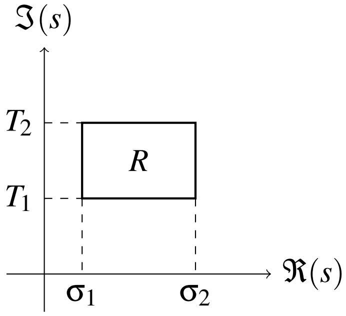
where zeros are counted with multiplicity (poles are counted with negative multiplicity) and zeros with imaginary parts T_1 or T_2 are counted with weight $\frac{1}{2}$. We observe the following relation:

$$N_\phi(\sigma, T_1, T_2) = N_\phi(\sigma, T_2) - N_\phi(\sigma, T_1), \tag{2.2}$$

where $N_\phi(\sigma, T)$ is as defined in the equation (1.8). We now turn to Littlewood's method for counting zeros of the function.

2.1.1 Littlewood's zero detection method

Suppose that $\phi(s)$ is a meromorphic function in and upon the boundary of a rectangle R bounded by the lines $\Im(s) = T_1, \Im(s) = T_2, \Re(s) = \sigma_1$ and $\Re(s) = \sigma_2$ as shown below.



Assume the function is regular and never zero of the side, l, of the rectangle for which $\Re = \sigma_2$ and $\log\phi(s)$ is regular in a neighbourhood containing l. We define $\Phi(s) = \log\phi(s)$ for s on l and for other points s in the rectangle, we define $\Phi(s)$ to be the value obtained from $\log\phi(\sigma_2 + it)$ by continuous variation along the line from $\sigma_2 + it$ to $\sigma + it$ provided the path does not cross a zero or pole of $\phi(s)$; if it does we define

$$\Phi(s) = \frac{1}{2}\lim_{\varepsilon\to 0}(\Phi(\sigma + it + i\varepsilon) + \Phi(\sigma + it - i\varepsilon)). \tag{2.3}$$

Definition 2.1. Let $N_\phi(\sigma, T_1, T_2)$ denote the excess of the number of zeros over the number of the poles of the function ϕ in the region for which $\Re(s) > \sigma$, zeros or poles on $\Im(s) = T_1$ or $\Im(s) = T_2$ counting one-half only.

The following proof is based on [41, page 221, (9.9)].

Lemma 2.2. *Let $\phi(s)$ be a meromorphic function in and upon the boundary of a rectangle R; and regular and never zero of the side, l, of the rectangle for which $\Re(s) = \sigma_2$. Suppose that $\Phi(s) = \log\phi(s)$ where $\Phi(s)$ is defined as above. Then we have*

$$\int_R (\Phi(s))ds = -2\pi i \int_\alpha^\beta N_\phi(\sigma, T_1, T_2)d\sigma, \tag{2.4}$$

The integral on the left is taken round the rectangle in the positive direction.

Proof. Various proofs are possible; the one given here depends on the inversion of the order of a double integration. We may suppose $\Im(s) = T_1$ and $\Im(s) = T_2$ to be free of zeros and poles of $\phi(s)$; It is easily verified that our halving conventions in equation (2.3) and Definition 2.1 ensure the truth of the theorem in the general case. By writing $s = \sigma + it$, we have

$$\int_R \Phi(s)ds = \int_{\sigma_1}^{\sigma_2} \Phi(\sigma + iT_1)d\sigma - \int_{\sigma_1}^{\sigma_2} \Phi(\sigma + iT_2)d\sigma + \int_{T_1}^{T_2} \{\Phi(\sigma_2 + it) - \Phi(\sigma_1 + it)\} idt. \tag{2.5}$$

The last term is equal to

$$\begin{aligned} \int_{T_1}^{T_2} \{\Phi(\sigma_2 + it) - \Phi(\sigma_1 + it)\} idt &= \int_{T_1}^{T_2} \left(\int_{\sigma_1}^{\sigma_2} \frac{\phi'(\sigma + it)}{\phi(\sigma + it)} d\sigma \right) idt \\ &= \int_{\sigma_1}^{\sigma_2} \int_{\sigma + iT_1}^{\sigma + iT_2} \frac{\phi'(s)}{\phi(s)} ds d\sigma. \end{aligned} \tag{2.6}$$

By the theorem of residues:

$$\int_{\sigma + iT_1}^{\sigma + iT_2} \frac{\phi'(s)}{\phi(s)} ds = \left(\int_{\sigma + iT_1}^{\sigma_2 + iT_1} + \int_{\sigma_2 + iT_1}^{\sigma_2 + iT_2} - \int_{\sigma + iT_2}^{\sigma_2 + iT_2} \right) \frac{\phi'(s)}{\phi(s)} ds - 2\pi i N_\phi(\sigma, T_1, T_2). \tag{2.7}$$

Hence

$$\begin{aligned}\int_R \Phi(s)ds = \int_{\sigma_1}^{\sigma_2} \Big(&\Phi(\sigma+iT_1) - \Phi(\sigma+iT_2) + \Phi(\sigma_2+iT_1) - \Phi(\sigma+iT_1) + \Phi(\sigma_2+iT_2) \\ &- \Phi(\sigma_2+iT_1) - \Phi(\sigma_2+iT_2) + \Phi(\sigma+iT_2) - 2\pi i N(\sigma,T_1,T_2)\Big)d\sigma \\ &= -2\pi i \int_{\sigma_1}^{\sigma_2} N_\phi(\sigma,T_1,T_2)d\sigma.\end{aligned} \tag{2.8}$$

□

Now we compare the above number of zeros for our function to its average:

$$N_\phi(\sigma,T_1,T_2) \leq \frac{1}{\sigma-\sigma_1}\int_{\sigma_1}^{\sigma_2}(N_\phi(\sigma',T_2) - N_\phi(\sigma',T_1))d\sigma'. \tag{2.9}$$

We apply Littlewood's classical method to find an equality for $\int_{\sigma_1}^{\sigma_2}(N_\phi(\sigma',T_2) - N_\phi(\sigma',T_1)d\sigma'$ and then have the upper bound for $N_\phi(\sigma,T_1,T_2)$ for $\sigma \in (\sigma_1,\sigma_2]$.

Theorem 2.3. *Let $\phi(s)$ be a meromorphic function in and upon the boundary of a rectangle R bounded by the lines $\Im(s) = T_1$, $\Im(s) = T_2$, $\Re(s) = \sigma_1$ and $\Re(s) = \sigma_2$. Let $\sigma \in (\sigma_1,\sigma_2]$. Then*

$$\begin{aligned}N_\phi(\sigma,T_1,T_2) \leq \frac{1}{2\pi(\sigma-\sigma_1)}\Bigg(&\int_{T_1}^{T_2}\log|\phi(\sigma_1+it)|dt + \int_{\sigma_1}^{\sigma_2}\arg\phi(\sigma'+iT_2)d\sigma' \\ &- \int_{\sigma_1}^{\sigma_2}\arg\phi(\sigma'+iT_1)d\sigma' - \int_{T_1}^{T_2}\log|\phi(\sigma_2+it)|dt\Bigg).\end{aligned} \tag{2.10}$$

Proof. Since equation (2.4) gives $\Phi(s) = \log\phi(s)$, we have

$$\frac{-1}{2\pi i}\int_R \log\phi(s)ds = \int_{\sigma_1}^{\sigma_2} N_\phi(\sigma',T_1,T_2)d\sigma'. \tag{2.11}$$

Given that from equation (2.9) we had $N_\phi(\sigma,T_1,T_2) \leq \frac{1}{\sigma-\sigma_1}\int_{\sigma_1}^{\sigma_2} N_\phi(\sigma',T_1,T_2)d\sigma'$ we conclude:

$$N_\phi(\sigma,T_1,T_2) \leq \frac{1}{\sigma-\sigma_1}\frac{-1}{2\pi i}\int_R \log\phi(s)ds. \tag{2.12}$$

Thus:

$$\begin{aligned} N_\phi(\sigma,T_1,T_2) \leq \frac{1}{\sigma-\sigma_1}\frac{-1}{2\pi i}\Big(& -\int_{T_1}^{T_2}(\log\phi(\sigma_1+it))idt - \int_{\sigma_1}^{\sigma_2}\log\phi(\sigma'+iT_2)d\sigma' \\ & + \int_{\sigma_1}^{\sigma_2}\log\phi(\sigma'+iT_1)d\sigma' + \int_{T_1}^{T_2}(\log\phi(\sigma_2+it))idt\Big). \end{aligned} \tag{2.13}$$

Using $\log\phi(z) = \log|\phi(z)| + i\arg\phi(z)$ we have

$$\begin{aligned} N_\phi(\sigma,T_1,T_2) \leq \frac{1}{2\pi(\sigma-\sigma_1)i}\Big(& \int_{T_1}^{T_2}(\log|\phi(\sigma_1+it)| + i\arg\phi(\sigma_1+it))idt \\ & + \int_{\sigma_1}^{\sigma_2}(\log|\phi(\sigma'+iT_2)| + i\arg\phi(\sigma'+iT_2))d\sigma' \\ & - \int_{\sigma_1}^{\sigma_2}(\log|\phi(\sigma'+iT_1)| + i\arg\phi(\sigma'+iT_1))d\sigma' \\ & - \int_{T_1}^{T_2}(\log|\phi(\sigma_2+it)| + i\arg\phi(\sigma_2+it))idt\Big). \end{aligned} \tag{2.14}$$

We take the real parts, so we have:

$$\begin{aligned} N_\phi(\sigma,T_1,T_2) \leq \frac{1}{2\pi(\sigma-\sigma_1)}\Big(& \int_{T_1}^{T_2}\log|\phi(\sigma_1+it)|dt + \int_{\sigma_1}^{\sigma_2}\arg\phi(\sigma'+iT_2)d\sigma' \\ & - \int_{\sigma_1}^{\sigma_2}\arg\phi(\sigma'+iT_1)d\sigma' - \int_{T_1}^{T_2}\log|\phi(\sigma_2+it)|dt\Big). \end{aligned} \tag{2.15}$$

□

We recall that the integrals in (2.15) are of the same form as the quantities I_1, I_2, I_3, and I_4 defined in the introduction in equation (1.11). A key goal is to minimize the above expression over admissible functions ϕ. We now give an idea of how to estimate the first integral in equation (2.15).

2.2 Bound on a logarithmic integral

In this section, the goal is to find an upper bound for the integral of a logarithmic of a non-negative function in equation (2.15).

Proposition 2.4. *Suppose that f is a non-negative and continuous function, we have*

$$\frac{1}{b-a}\int_a^b \log f(\sigma+it)dt \leq \log\left(\frac{1}{b-a}\int_a^b f(\sigma+it)dt\right), \tag{2.16}$$

where a and b are constants.

Proof. Since log is concave on its domain, by applying the Jensen's inequality we obtain:

$$\frac{1}{b-a}\int_a^b \log f(t)dt \leq \log\left(\frac{1}{b-a}\int_a^b f(t)dt\right). \qquad \square$$

So if we want to find an upper bound for the integral of $\log f(\sigma+it)$, we can find an upper bound for the integral of $f(\sigma+it)$. The idea to bound the integral of $f(\sigma+it)$ is that we can use the convexity bound for integrals to move the problem to each of two boundaries. To use the convexity estimate we first use the smoothing method to smooth the function at σ and therefore rather than bounding $\int_a^b |f(\sigma+it)|dt$ we bound $\int_a^b |f(\sigma+it)g(\sigma+t)|dt$ for some function g. Rather than finding an upper bound for $\int_a^b |f(\sigma+it)g(\sigma+it)|dt$ we bound $\int_a^b |f(\sigma_1+it)g(\sigma_1+it)|dt$ and $\int_a^b |f(\sigma_2+it)g(\sigma_2+it)|dt$ and then for getting bounds at the boundaries we use unsmoothing method to turn problem back to the original integrals at each of the boundaries to find an upper bound for $\int_0^b |f(\sigma_1+it)|dt$ and $\int_0^b |f(\sigma_2+it)|dt$.

Remark 2.5. We want to go from the first step to last step but we can't directly because we have to use the convexity.

Remark 2.6. The hope is that, the bound we can get by interpolating bounds can be better than the bound we could get directly, this happens especially if we have better information about a function at one of the two boundaries and not worse information at the other boundaries, for instance in our context, we have good information about zeta function outside of the critical strip, and, reasonable information at $\frac{1}{2}$ line.

We now detail how each of these steps is carried out.

2.2.1 Smoothing method

The general idea of this method is to replace a bounded sum or integral of a function with an infinite sum or integral of a smoothed version of the function.
To obtain this, we introduce a smooth weight function g and multiply by our function. This makes it easier to analyze the integral of the smoothed function rather than the original one. We choose the weight function g such that for a fixed σ, the function $|g(\sigma+it)|$ approximates the indicator function, and for t large, $g(\sigma+it)$ has rapid decay to 0.
Thus the integral of our function can be approximated by the integral of the product of the function with the weight function.
The following Lemma makes this precise.

Lemma 2.7. *Let f be a continuous function and g be a holomorphic function in the strip $\sigma_1 \leq \Re(s) \leq \sigma_2$. We assume that there exists a positive function $C_2 = C_2(\sigma, T_1, T_2)$ such that g satisfies the following condition on lower bound,*

$$C_2(\sigma, T_1, T_2) = \min_{t \in [T_1, T_2]} g(\sigma + it). \tag{2.17}$$

Then

$$\int_{T_1}^{T_2} |f(\sigma+it)| dt \leq \frac{\int_{-\infty}^{\infty} |g(\sigma+it)||f(\sigma+it)| dt}{C_2(\sigma, T_1, T_2)}. \tag{2.18}$$

Proof. Based on the above remark, we have

$$\begin{aligned} \int_{T_1}^{T_2} |f(\sigma+it)| dt &= \int_{-\infty}^{+\infty} 1_{[T_1,T_2]} |f(\sigma+it)| dt \\ &\leq \frac{\int_{-\infty}^{\infty} |g(\sigma+it)| \, | \, f(\sigma+it) dt}{\min_{t\in[T_1,T_2]} |g(\sigma+it)|} \\ &\leq \frac{\int_{-\infty}^{\infty} |g(\sigma+it)| \, | \, f(\sigma+it)| dt}{C_2(\sigma, T_1, T_2)}. \end{aligned}$$ □

There is a relationship between the bounds on the exact values and bounds on the smoothed values in both directions. In the next Section we will think about how we can bound an integral of a smoothed function from $-\infty$ to ∞. The idea here is using the unsmoothing method.

2.2.2 Unsmoothing method

This method allows to bound $\int_{-\infty}^{\infty}|g(\sigma+it)|\,|f(\sigma+it)|\,dt$ in terms of a finite integral for f, namely of:

$$F(\sigma,T_1,T_2)=\int_{T_1}^{T_2}|f(\sigma+it)|dt. \tag{2.19}$$

When $T_1=0$ we shall denote by $F(\sigma,T)=F(\sigma,0,T)$.

Lemma 2.8. *Let $T>0$, $\sigma_1\le\sigma\le\sigma_2$ and suppose that the function $|f(\sigma+it)|$ is even in t. We assume $F(\sigma,0)=0$ where F defined in equation (2.19) and for any positive α and β, we have* $\lim_{U\to\infty}\left(F(\sigma,U)e^{-\alpha(\frac{U}{T})^{\beta}}\right)=0.$
In addition, we assume that there exists a positive function $C_1=C_1(\sigma,T,\alpha,\beta)$ such that g satisfies:

$$|g(\sigma+it)|\le C_1(\sigma,T,\alpha,\beta)e^{-\alpha(\frac{|t|}{T})^{\beta}} \quad \textit{for all } t. \tag{2.20}$$

Then,

$$\int_{-\infty}^{\infty}|g(\sigma+it)|\,|f(\sigma+it)|\,dt\le 2C_1(\sigma,T,\alpha,\beta)\alpha\beta\int_0^{\infty}x^{\beta-1}e^{-\alpha x^{\beta}}F(\sigma,xT)dx. \tag{2.21}$$

Proof. Based on the assumptions that we have, the function $|f(\sigma+it)|$ is even in t and the bound of g in the equation (2.20) we conclude,

$$\int_{-\infty}^{\infty}|g(\sigma+it)|\,|f(\sigma+it)|\,dt\le 2C_1(\sigma,T,\alpha,\beta)\int_0^{\infty}e^{-\alpha(\frac{t}{T})^{\beta}}|f(\sigma+it)|dt. \tag{2.22}$$

By integrating by parts we have,

$$\begin{aligned}\int_0^\infty e^{-\alpha\left(\frac{t}{T}\right)^\beta}|f(\sigma+it)|dt &= e^{-\alpha\left(\frac{t}{T}\right)^\beta}\cdot\int_0^\infty |f(\sigma+it)|dt - \int_0^\infty -\alpha\beta\left(\frac{t}{T}\right)^\beta e^{-\alpha\left(\frac{t^{\beta-1}}{T^\beta}\right)}F(\sigma,t)dt\\ &= 0+\int_0^\infty \alpha\beta\left(\frac{t}{T}\right)^\beta e^{-\alpha\left(\frac{t^{\beta-1}}{T^\beta}\right)}F(\sigma,t)dt\\ &= \alpha\beta\int_0^\infty \left(\frac{t}{T}\right)^\beta e^{-\alpha\left(\frac{t}{T}\right)^\beta}F(\sigma,t)\frac{dt}{t}.\end{aligned}$$

By changing the variables, $\frac{t}{T}=x$ we have,

$$\begin{aligned}&= \alpha\beta\int_0^\infty x^\beta e^{-\alpha(x)^\beta}F(\sigma,xT)\frac{T}{xT}dx\\ &= \alpha\beta\int_0^\infty x^\beta e^{-\alpha(x)^\beta}F(\sigma,xT)\frac{1}{x}dx\\ &= \alpha\beta\int_0^\infty x^{\beta-1} e^{-\alpha(x)^\beta}F(\sigma,xT)dx.\end{aligned}$$

This combined with (2.22) and we conclude,

$$\int_{-\infty}^\infty |g(\sigma+it)||f(\sigma+it)|dt \le 2C_1(\sigma,T,\alpha,\beta)\alpha\beta\int_0^\infty x^{\beta-1}e^{-\alpha x^\beta}F(\sigma,xT)dx. \qquad \square$$

Unsmoothing is reversing or inverting the process of smoothing. Smoothing allows us to bound $\int_{T_1}^{T_2}|f(\sigma+it)|dt$ in terms of $\int_{-\infty}^{\infty}|g(\sigma+it)||f(\sigma+it)|dt$, whereas unsmoothing allows us to bound $\int_{-\infty}^{\infty}|g(\sigma+it)||f(\sigma+it)|dt$ using bounds on $\int_0^T|f(\sigma+it)|dt$.

Lemma 2.9. *Let $\alpha,\beta,a,a_i,b,b_i,d,d_i>0$ depending on σ, and let A and n be non-negative integers. Suppose that*

$$F(\sigma,t)\le dt^a(\log t)^b\sum_i d_i t^{a_i}(\log t)^{b_i} \quad \textit{for all } t, \tag{2.23}$$

where $\sum_i d_i t^{a_i}(\log t)^{b_i}=1+o(1)$, Then

$$\int_0^\infty x^{\beta-1}e^{-\alpha x^\beta}F(\sigma,xT)dx\le dT^a(\log T)^b C_3, \tag{2.24}$$

where $C_3 = C_3(T, \alpha, \beta, a, a_i, b, b_i, d_i)$ *is given by*

$$C_3 = \sum_i d_i T^{a_i} (\log T)^{b_i} \sum_{k=0}^{b+b_i} \frac{\binom{b+b_i}{k} I_{\alpha,\beta}(\beta + a + a_i - 1, b + b_i - k)}{(\log T)^{b+b_i-k}}, \tag{2.25}$$

with

$$I_{\alpha,\beta}(A, n) = \int_0^\infty x^A e^{-\alpha x^\beta} (\log x)^n dx. \tag{2.26}$$

Remark 2.10. The condition $\sum_i d_i t^{a_i} (\log t)^{b_i} = 1 + o(1)$ implies that up to reordering $d_0 = 1$, $a_0 = 0$ and $b_0 = 0$. Additionally, it implies:

- all $a_i \leq 0$,
- if $a_i = 0$, then $b_i \leq 0$.

Remark 2.11. The constant $C_3(T, \alpha, \beta, a, a_i, b, b_i, d_i) = O(1)$ and is eventually decreasing in T.

Proof. Since, by assumption, we have an upper bound for $F(\sigma, t)$, as given in equation (2.23), we can substitute this bound for $F(\sigma, xT)$ into equation (2.24). This gives us:

$$\int_0^\infty x^{\beta-1} e^{-\alpha x^\beta} F(\sigma, xT) dx < \int_0^\infty x^{\beta-1} e^{-\alpha x^\beta} \left(d(xT)^a (\log xT)^b \sum_i d_i (xT)^{a_i} (\log xT)^{b_i} \right) dx. \tag{2.27}$$

By expanding the logarithm and separating the terms ($\log(xT) = \log x + \log T$), and then Substituting this back into the integral, we get:

$$\int_0^\infty x^{\beta-1} e^{-\alpha x^\beta} \left(d x^a T^a (\log x + \log T)^b \sum_i d_i x^{a_i} T^{a_i} (\log x + \log T)^{b_i} \right) dx. \tag{2.28}$$

Since the summation and the integral are linear, we can interchange them:

$$\int_0^\infty x^{\beta-1} e^{-\alpha x^\beta} F(\sigma, xT) dx < d T^a \sum_i d_i T^{a_i} \int_0^\infty x^{\beta+a+a_i-1} e^{-\alpha x^\beta} (\log x + \log T)^{b+b_i} dx. \tag{2.29}$$

Applying the Binomial Theorem to expand $(\log x+\log T)^{b+b_i}$, we get

$$(\log x+\log T)^{b+b_i}=\sum_{k=0}^{b+b_i}\binom{b+b_i}{k}(\log x)^{b+b_i-k}(\log T)^k.$$

Therefore, we obtain the following upper bound:

$$dT^a\sum_i d_iT^{a_i}\int_0^\infty x^{\beta+a+a_i-1}e^{-\alpha x^\beta}\left(\sum_{k=0}^{b+b_i}\binom{b+b_i}{k}(\log x)^{b+b_i-k}(\log T)^k\right)dx. \tag{2.30}$$

With simplification the above summation, we have $\int_0^\infty x^{\beta-1}e^{-\alpha x^\beta}F(\sigma,xT)dx$ is bounded by:

$$dT^a(\log T)^b\sum_i d_iT^{a_i}(\log T)^{b_i}\left(\sum_{k=0}^{b+b_i}\frac{\binom{b+b_i}{k}I_{\alpha,\beta}(\beta+a+a_i-1,b+b_i-k)}{(\log T)^{b+b_i-k}}\right). \tag{2.31}$$

Recognizing this expression as $C_3(T,\alpha,\beta,a,a_i,b,b_i,d_i)$, we have the desired result

$$\int_0^\infty x^{\beta-1}e^{-\alpha x^\beta}F(\sigma,xT)dx\le dT^a(\log T)^bC_3(T,\alpha,\beta,a,a_i,b,b_i,d_i). \tag{2.32}$$

□

Remark 2.12. In our context, $b_i=0,1,2$ are the main cases we will encounter.

Remark 2.13. In terms of the important role of the constants in Lemma 2.9, the most important one is the case when $k=b+b_i$, which means the main term that we are interested in, is $I_{\alpha,\beta}(\beta+a+a_i-1,0)$ as defined in equation (2.26).

Remark 2.14. By substitution of $\alpha x^\beta=t$, we can express $I_{\alpha,\beta}(\beta+a+a_i-1,0)$ in terms of the Gamma function. Therefore, we can write

$$I_{\alpha,\beta}(\beta+a+a_i-1,0)=\left(\frac{1}{\alpha\beta}\right)\left(\frac{1}{\alpha}\right)^{\frac{a+a_i}{\beta}}\Gamma\left(\frac{a+a_i}{\beta}+1\right). \tag{2.33}$$

Remark 2.15. Note that in the application to the zeta function (Section 4), we do not apply the unsmoothing method immediately after the smoothing method; instead, we use a convexity estimate in between. However, the purpose of applying unsmoothing immediately after smoothing would be to evaluate the quality of the smoothing function g and the process itself. From Lemma 2.9 we can see that both our assumed bound for $F(\sigma,t)$ and the resulting bound for $\int_0^\infty x^{\beta-1}e^{-\alpha x^\beta}F(\sigma,xT)dx$ have the same asymptotic growth but it got worse by a constant, since $\sum_{k=0}^{b+b_i}\frac{\binom{b+b_i}{k}I_{\alpha,\beta}(\beta+a+a_i-1,b+b_i-k)}{(\log T)^{b+b_i-k}}$ in equation (2.24) is $O(1)$, bounded by a constant depending on β,α, b, b_i, a and a_i. In particular, if g were the characteristic function, and if the process were perfect, then we would recover the original bound on $F(\sigma,t)$.

An interesting analysis to do after smoothing and unsmoothing is to illustrate how much worse your answer gets when you first do smoothing and then unsmoothing. This process helps us understand what is the error that arises from the smoothing and unsmoothing methods. If we focus on the main term based on Remark 2.13 the error term is approximately

$$E_\sigma(\alpha,\beta)=\frac{2C_1(\sigma,T_2,\alpha,\beta)}{C_2(\sigma,T_1,T_2)}\alpha\beta\int_0^\infty x^{\beta+a+a_i-1}e^{-\alpha x^\beta}dx. \tag{2.34}$$

For the weight function $g=\left(\frac{s-1}{s}\right)^2e^{\alpha(\frac{s}{T_2})^\beta}$, considered in the paper of Kadiri, Lumley and Ng [26][1], we ultimately have

$$\begin{aligned} C_1(\sigma,T_2,\alpha,\beta)&=e^{\alpha(\frac{\sigma}{T_2})^\beta},\\ C_2(\sigma,T_1,T_2,\alpha)&=\left(1-\frac{1}{T_1}\right)^2e^{\alpha(\frac{\sigma}{T_2})^\beta-\alpha}. \end{aligned} \tag{2.35}$$

With simplification, we obtain

$$\begin{aligned} E_\sigma(\alpha,\beta)&=2e^\alpha\int_0^\infty \alpha\beta x^{\beta-1}e^{-\alpha x^\beta}x^{a+a_i}dx\\ &=2e^\alpha\left(\frac{1}{\alpha}\right)^{\frac{a+a_i}{\beta}}\int_0^\infty e^{-t}t^{\frac{a+a_i}{\beta}}dt. \end{aligned} \tag{2.36}$$

Therefore, the choice of α and β plays a role in determining the size of this error term and also this analysis primarily tests the quality of the weight being used.

---

[1] In [26] the weight they use is ultimately $\left(\frac{s-1}{s}\right)^2e^{2\alpha(\frac{s}{T})^2}$.

For instance, for $\sigma = \frac{1}{2}$ we have a bound of the shape:

$$F(\sigma, T) \leq T^{1+\frac{2}{6}}(\log T)^2.$$

So that we approximate it by setting $a + a_i = \frac{8}{6}$ and obtained the below bound,

$$F(\sigma, T) \leq T^{1+\frac{2}{6}}(\log T)^2 \cdot 2e^{\alpha}\alpha^{-\frac{4}{3\beta}} \int_0^{\infty} e^{-t} t^{\frac{4}{3\beta}} dt.$$

Based on the equation (2.33), we have the following error term:

$$E_{\frac{1}{2}}(\alpha, \beta) = \frac{2e^{\alpha}}{\alpha^{\frac{4}{3\beta}}} \cdot \Gamma\left(\frac{4}{3\beta} + 1\right), \tag{2.37}$$

which we optimize with respect to α and β. We take the logarithm.

$$\log(E_{\frac{1}{2}}(\alpha, \beta)) = \log 2 + \alpha - \frac{4}{3\beta}\log(\alpha) + \log\left(\Gamma\left(\frac{4}{3\beta} + 1\right)\right), \tag{2.38}$$

and then the derivative of this expression with respect to α:

$$\frac{\partial \log(E_{\frac{1}{2}}(\alpha, \beta))}{\partial \alpha} = 1 - \frac{4}{3\beta} \cdot \frac{1}{\alpha}. \tag{2.39}$$

By setting this derivative equal to zero, we find

$$\beta = \tfrac{4}{3\alpha}. \tag{2.40}$$

Substituting this back into the original error term and perform further optimization as needed. We find $E_{\frac{1}{2}}(\alpha, \beta)$ optimal value is:

$$E_{\frac{1}{2}}(\alpha, \tfrac{4}{3\alpha}) = \frac{2e^{\alpha}}{\alpha^{\alpha}} \cdot \Gamma(\alpha + 1). \tag{2.41}$$

Next, we take the logarithm of this new error term:

$$\log(E_{\frac{1}{2}}(\alpha)) = \alpha - \alpha\log(\alpha) + \log(\Gamma(\alpha+1)). \tag{2.42}$$

and then the derivative with respect to α

$$\frac{\partial \log(E_{\frac{1}{2}}(\alpha))}{\partial \alpha} = -\log(\alpha) + \psi(\alpha+1), \tag{2.43}$$

where ψ is the polygamma function defined by $\psi(x) = \frac{\Gamma'}{\Gamma}(x)$. By setting this derivative equal to zero, we find

$$\log(\alpha) = \psi(\alpha+1). \tag{2.44}$$

Once we find the optimal α, we can determine the corresponding optimal β using the relationship $\alpha = \frac{4}{3\beta}$.

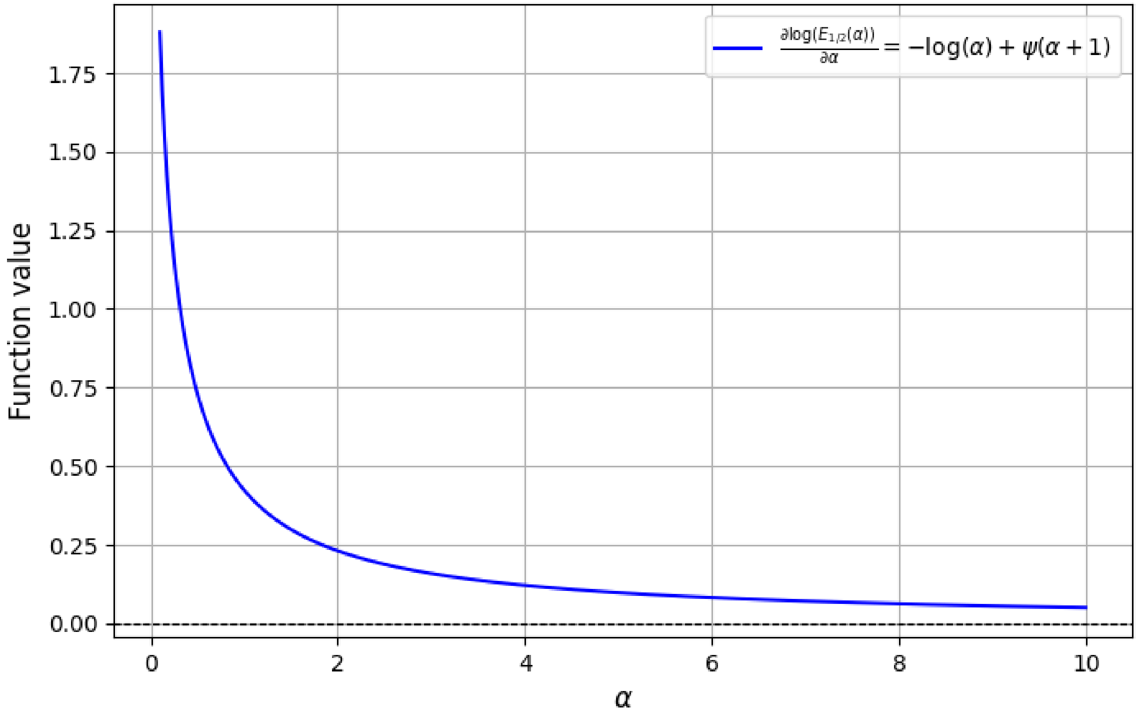


Figure 2.1: Graph of $-\log(\alpha)+\psi(\alpha+1)$.

Remark 2.16. Fram the graph above, we can see that by increasing value for α, $E_{\frac{1}{2}}(\alpha)$ will increase. Hence the smaller α will improve the result and consequently $\beta \to \infty$ in order to optimize the bound.

Remark 2.17. In the application of the zeta function section, we will not only use the smoothing and unsmoothing methods but also employ a convexity estimate. Therefore, the optimal choices for α and β should take this information into account. Therefore, after introducing the convexity estimate and combining all the bounds, we can determine the resulting error term and find the optimal values for α and β. We can then compare them with the optimal α and β obtained from here.

Remark 2.18. The weight functions $g = e^{-\alpha(\frac{t}{T})^{\beta}}$ are well adapted for zeros within $[-T, T]$. However, if we wanted to look only at zeros within $[T_1, T_2]$ we can use this weight function with $T_2 = T$.

Nonetheless, a different weight function might be better, such as $e^{-\alpha\left(\frac{2t-(T_1+T_2)}{(T_2-T_1)}\right)^{\beta}}$.

Remark 2.19. In future work, the unsmoothing process could further be adjusted to separately handle the regions $[-\infty, T_1], [T_1, T_2]$, and $[T_2, \infty]$, by using different bounds on g and $F(\sigma, T)$ for each region. Additionally, if we choose a different weight function g, such as $e^{-\alpha\left(\frac{2t-(T_1+T_2)}{(T_2-T_1)}\right)^{\beta}}$, the decay properties of g could lead to separate bounds for the integrals:

$$\int_{-\infty}^{T_1} |g(\sigma+it)|\,|f(\sigma+it)|\,dt$$
$$\int_{T_1}^{T_2} |g(\sigma+it)|\,|f(\sigma+it)|\,dt$$
$$\int_{T_2}^{\infty} |g(\sigma+it)|\,|f(\sigma+it)|\,dt,$$

which depend on the difference $F(\sigma, T_2) - F(\sigma, T_1)$. This would be good if T_1 and T_2 were relatively close.

In the next section, the goal is to use a convexity estimate to find an upper bound for $\int_{-\infty}^{\infty} |g(\sigma+it)|\,|f(\sigma+it)|\,dt$, where $\sigma_1 \leq \sigma \leq \sigma_2$, given that we already have an upper bound for $\int_{-\infty}^{\infty} |g(\sigma+it)|\,|f(\sigma+it)|\,dt$ at σ_1 and σ_2.

2.2.3 Convexity estimate

In this section, we use some convexity estimate to bound the integrals of the form

$$J(\sigma) = \int_{-\infty}^{\infty} |h(\sigma+it)|^p dt. \tag{2.45}$$

There are several different sources of such bounds in the literature, see for example Hardy-Ingham-Pólya [11] and Abriel [9]. Below is one such statement from [11]:

Theorem 2.20. *Let h be analytic in the interior of the strip* $\sigma_1 < \Re(s) < \sigma_2$ *and continuous on its boundary. Suppose that h satisfies a certain growth condition in the strip* $\sigma_1 < \Re(s) < \sigma_2$*: there exists* $k \in (0, \frac{\pi}{\sigma_2 - \sigma_1})$ *such that*

$$|h(\sigma+it)| = O(e^{e^{k|t|}}), \tag{2.46}$$

If, for any $p > 0$*, the integral*

$$J_h(\sigma) = \int_{-\infty}^{\infty} |h(\sigma+it)|^p dt, \tag{2.47}$$

converges when $\sigma = \sigma_1$ *and when* $\sigma = \sigma_2$*, then* $\log J(\sigma)$ *is a "convex function of* σ*", i.e :*

$$J_h(\sigma) \leq (J_h(\sigma_1))^{\frac{\sigma_2-\sigma}{\sigma_2-\sigma_1}} (J_h(\sigma_2))^{\frac{\sigma-\sigma_1}{\sigma_2-\sigma_1}}. \tag{2.48}$$

It says that the size of the integral in the middle is controlled by the size of the integrals on the sides.

We can explore the application of convexity estimates to bound the integral

$$J_{gf}(\sigma) = \int_{-\infty}^{\infty} |g(\sigma+it)|\,|f(\sigma+it)|\,dt, \tag{2.49}$$

at a given σ assuming we have bounds for $J_{gf}(\sigma_1)$ and $J_{gf}(\sigma_2)$. Namely:

$$J_{gf}(\sigma) \leq \Big(J_{gf}(\sigma_1)\Big)^{\frac{\sigma_2-\sigma}{\sigma_2-\sigma_1}} \Big(J_{gf}(\sigma_2)\Big)^{\frac{\sigma-\sigma_1}{\sigma_2-\sigma_1}}. \tag{2.50}$$

Remark 2.21. When f is not analytic, it may have poles or other singularities that affect the convergence of the integral. In such cases, it may be necessary to select g carefully to eliminate any poles, ensuring that the integral remains well-defined and finite.

In the next section, we will combine all the obtained bounds which leads to an overall bound for $F(\sigma, T_1, T_2)$.

2.2.4 Combination

In this section, the goal is to combine all the obtained bounds to find an upper bound for $F(\sigma, T_1, T_2)$ assuming we have bounds for $F(\sigma_1, T_1, T_2)$ and $F(\sigma_2, T_1, T_2)$ of the form given in equation (2.23). Therefore, we have

Theorem 2.22. *Fix $\sigma_1 < \sigma_2$ and $0 < T_1 < T_2$. Suppose that function $|f(\sigma+it)|$ is even in t and g is a holomorphic function in the strip $\sigma_1 \leq \Re(s) \leq \sigma_2$. Assume there exist $\alpha, \beta > 0$ and positive functions C_1 and C_2 such that for $\sigma_1 \leq \sigma \leq \sigma_2$ we have:*

$$\begin{aligned} &C_2(\sigma, T_1, T_2, \alpha, \beta) \leq \min_{t\in[T_1,T_2]} |g(\sigma+it)|, \\ &\text{and} \quad |g(\sigma+it)| \leq C_1(\sigma, T_2, \alpha, \beta) e^{-\alpha(\frac{|t|}{T_2})^\beta} \quad \text{for all } t \in R. \end{aligned} \tag{2.51}$$

Suppose that there exist $a, a_i, b, b_i, d, d_i > 0$ depending on σ_1 and $a', a_i', b', b_i', d', d_i' > 0$ depending on σ_2 such that, for all t we have

$$\begin{aligned} F(\sigma_1, t) &\leq dt^a (\log t)^b \sum_i d_i t^{a_i} (\log t)^{b_i} \\ F(\sigma_2, t) &\leq d' t^{a'} (\log t)^{b'} \sum_{i'} d_i' t^{a_i'} (\log t)^{b_i'}, \end{aligned} \tag{2.52}$$

where $\sum_i d_i t^{a_i}(\log t)^{b_i} = 1+o(1)$ *and* $\sum_{i'} d'_i t^{a'_i}(\log t)^{b'_i} = 1+o(1)$*. Then*

$$F(\sigma,T_1,T_2) \le C_4 C_5 T_2^{a(\frac{\sigma_2-\sigma}{\sigma_2-\sigma_1})+a'(\frac{\sigma-\sigma_1}{\sigma_2-\sigma_1})} \log T_2^{b(\frac{\sigma_2-\sigma}{\sigma_2-\sigma_1})+b'(\frac{\sigma-\sigma_1}{\sigma_2-\sigma_1})}, \tag{2.53}$$

where $C_4 = C_4(\sigma_1,\sigma,\sigma_2,T_1,T_2,\alpha,\beta)$ *and* $C_5 = C_5(\sigma,T_2,\alpha,\beta,a_i,a'_i,b_i,b'_i,d_i,d'_i)$ *are defined by*

$$\begin{aligned} C_4 &= \alpha\beta\frac{C_1(\sigma_1,T_2,\alpha,\beta)^{(\frac{\sigma_2-\sigma}{\sigma_2-\sigma_1})}C_1(\sigma_2,T_2,\alpha,\beta)^{(\frac{\sigma-\sigma_1}{\sigma_2-\sigma_1})}}{C_2(\sigma,T_1,T_2,\alpha,\beta)}, \\ C_5 &= C_3(\sigma_1,T_2,\alpha,\beta,a,a_i,b,b_i,d_i)^{(\frac{\sigma_2-\sigma}{\sigma_2-\sigma_1})}\cdot C_3(\sigma_2,T_2,\alpha,\beta,a',a'_i,b',b'_i,d'_i)^{(\frac{\sigma-\sigma_1}{\sigma_2-\sigma_1})}, \end{aligned} \tag{2.54}$$

where $C_1(\sigma_1,T_2,\alpha,\beta)$ *and* $C_2(\sigma,T_1,T_2,\alpha,\beta)$ *satisfy the conditions of* (2.51) *and the formula for* $C_3(\sigma_1,T_2,\alpha,\beta,a,a_i,b,b_i,d_i)$ *is defined in equation* (2.25)*.*

Remark 2.23. As C_3 is $O(1)$ and eventually decreasing in T_2 hence so is C_5. One also has that C_5 will be monotonic in σ. In applications, for our choice of g, C_4 will also be decreasing in both T_1 and T_2 and monotonic in σ. See equation (4.18).

Proof. Since by assumption, we have upper bounds for $F(\sigma_1,t)$ and $F(\sigma_2,t)$ as given in equation (2.52), by using equation (2.24), we can substitute these two bounds into equation (2.21). This gives us the following bounds for $\int_{-\infty}^{\infty}|g(\sigma_1+it)||f(\sigma_1+it)|\,dt$ and $\int_{-\infty}^{\infty}|g(\sigma_2+it)||f(\sigma_2+it)|\,dt$:

$$\begin{aligned} &2C_1(\sigma_1,T_2,\alpha,\beta)\alpha\beta d T_2^{a}(\log T_2)^{b} C_3(\sigma_1,T_2,\alpha,\beta,a,a_i,b,b_i,d_i), \\ &2C_1(\sigma_2,T_2,\alpha,\beta)\alpha\beta d' T_2^{a'}(\log T_2)^{b'} C_3(\sigma_2,T_2,\alpha,\beta,a',a'_i,b',b'_i,d'_i). \end{aligned} \tag{2.55}$$

By applying the convexity estimate, for each $\sigma \in [\sigma_1,\sigma_2]$ we obtain the following bound for $\int_{-\infty}^{\infty}|g(\sigma+it)||f(\sigma+it)|dt$:

$$\left(2C_1(\sigma_1,T_2,\alpha,\beta)\alpha\beta d T_2^{a}(\log T_2)^{b}C_3(\sigma_1,T_2,\alpha,\beta,a,a_i,b,b_i,d_i)\right)^{\frac{\sigma_2-\sigma}{\sigma_2-\sigma_1}}\cdot$$
$$\left(2C_1(\sigma_2,T_2,\alpha,\beta)\alpha\beta d' T_2^{a'}(\log T_2)^{b'}C_3(\sigma_2,T_2,\alpha,\beta,a',a_i',b',b_i',d',d_i')\right)^{\frac{\sigma-\sigma_1}{\sigma_2-\sigma_1}}. \tag{2.56}$$

Since $T_1>0$ and f is even in t we have

$$F(\sigma,T_1,T_2)\leq\frac{1}{2}F(\sigma,-T_2,T_2).$$

Hence by applying smoothing method we have

$$\int_{T_1}^{T_2}|f(\sigma+it)|\,dt\leq\frac{1}{2}F(\sigma,-T_2,T_2)\leq\frac{\int_{-\infty}^{\infty}|g(\sigma+it)||f(\sigma+it)|dt}{2C_2(\sigma,T_1,T_2,\alpha,\beta)},$$

and by taking the bound in equation (2.56) we provide an upper bound for $F(\sigma,T_1,T_2)$ as product of the following three terms:

$$\begin{aligned}&\left(2C_1(\sigma_1,T_2,\alpha,\beta)\alpha\beta d T_2^{a}(\log T_2)^{b}C_3(\sigma_1,T_2,\alpha,\beta,a,a_i,b,b_i,d,d_i)\right)^{\frac{\sigma_2-\sigma}{\sigma_2-\sigma_1}}\cdot\\&\left(2C_1(\sigma_2,T_2,\alpha,\beta)\alpha\beta d' T_2^{a'}(\log T_2)^{b'}C_3(\sigma_2,T_2,\alpha,\beta,a',a_i',b',b_i',d',d_i')\right)^{\frac{\sigma-\sigma_1}{\sigma_2-\sigma_1}}\cdot\\&\left(\frac{1}{2C_2(\sigma,T_1,T_2,\alpha,\beta)}\right).\end{aligned} \tag{2.57}$$

Therefore we obtained the desired result:

$$F(\sigma,T_1,T_2)\leq C_4C_5T_2^{a(\frac{\sigma_2-\sigma}{\sigma_2-\sigma_1})+a'(\frac{\sigma-\sigma_1}{\sigma_2-\sigma_1})}(\log T_2)^{b(\frac{\sigma_2-\sigma}{\sigma_2-\sigma_1})+b'(\frac{\sigma-\sigma_1}{\sigma_2-\sigma_1})}. \tag{2.58}$$

□

As I mentioned in the unsmoothing method in Section 2.2.2, in our thesis we will consider $g(s) = \left(\frac{s-1}{s}\right)^2 e^{\alpha(\frac{s}{T})^\beta}$. We ultimately have

$$\begin{aligned} C_1(\sigma, T_2, \alpha, \beta) &= e^{\alpha(\frac{\sigma}{T_2})^\beta}, \\ C_2(\sigma, T_1, T_2, \alpha, \beta) &= \left(1 - \frac{1}{T_1}\right)^2 e^{\alpha(\frac{\sigma}{T_2})^\beta - \alpha}. \end{aligned} \tag{2.59}$$

Remark 2.24. In analyzing the error term, in terms of the constants $C_3(T_2,\alpha,\beta,a,a_i,b,b_i,d_i)$ and $C_3(T_2,\alpha,\beta,a',a'_i,b',b'_i,d'_i)$, the most important case occurs when $k = b + b_i$ and $k = b' + b'_i$.

Using the leading constant, as $T \to \infty$, we will have the following error term, $E_\sigma(\alpha, \beta)$:

$$\frac{\left(e^{\alpha(\frac{\sigma_1}{T})^\beta} \int_0^\infty \alpha\beta x^{\beta-1} e^{-\alpha x^\beta} x^{a+a_i} dx\right)^{(\frac{\sigma_2-\sigma}{\sigma_2-\sigma_1})} \cdot \left(e^{\alpha(\frac{\sigma_2}{T})^\beta} \int_0^\infty \alpha\beta x^{\beta-1} e^{-\alpha x^\beta} x^{a'+a'_i} dx\right)^{(\frac{\sigma-\sigma_1}{\sigma_2-\sigma_1})}}{e^{\alpha(\frac{\sigma}{T})^\beta - \alpha}}. \tag{2.60}$$

With simplification we obtain

$$\begin{aligned} E_\sigma(\alpha, \beta) = \left(\int_0^\infty \alpha\beta x^{\beta-1} e^{-\alpha x^\beta} x^{a+a_i} dx\right)^{(\frac{\sigma_2-\sigma}{\sigma_2-\sigma_1})} \left(\int_0^\infty \alpha\beta x^{\beta-1} e^{-\alpha x^\beta} x^{a'+a'_i} dx\right)^{(\frac{\sigma-\sigma_1}{\sigma_2-\sigma_1})} \\ \left(e^{\alpha((\frac{\sigma_2-\sigma}{\sigma_2-\sigma_1})(\frac{\sigma_1}{T})^\beta + (\frac{\sigma-\sigma_1}{\sigma_2-\sigma_1})(\frac{\sigma_2}{T})^\beta - (\frac{\sigma}{T})^\beta + 1)}\right). \end{aligned} \tag{2.61}$$

Therefore, like the unsmoothing Section 2.2.2, the choice of α and β plays a role in determining the size of the error term and also, we can analyze this error term to understand the best choices for parameters α, β.

The special case that I'm studying is the case that we will perform the analysis using $\sigma_1 = \frac{1}{2}$ and $\sigma_2 = 1 + \frac{\delta_1}{\log X}$ which are of the below shape which appear in the paper of Kadiri, Lumley and Ng:

$$\begin{aligned} F(\sigma_1, T) &\le T^{1+\frac{2}{6}} (\log T)^3, \\ F(\sigma_2, T) &\le (\log T)^2. \end{aligned} \tag{2.62}$$

First of all, by substitution of $\alpha x^{\beta} = t$, we can express the error as the below equation:

$$E_{\sigma}(\alpha,\beta) = \left(\alpha^{-\frac{a+a_i}{\beta}} \int_0^{\infty} e^{-t} t^{\frac{a+a_i}{\beta}} dt\right)^{(\frac{\sigma_2-\sigma}{\sigma_2-\sigma_1})} \left(\alpha^{-\frac{a'+a'_i}{\beta}} \int_0^{\infty} e^{-t} t^{\frac{a'+a'_i}{\beta}} dt\right)^{(\frac{\sigma-\sigma_1}{\sigma_2-\sigma_1})} \left(e^{\alpha((\frac{\sigma_2-\sigma}{\sigma_2-\sigma_1})(\frac{\sigma_1}{T})^{\beta}+(\frac{\sigma-\sigma_1}{\sigma_2-\sigma_1})(\frac{\sigma_2}{T})^{\beta}-(\frac{\sigma}{T})^{\beta}+1)}\right). \tag{2.63}$$

Then with the above bounds for $F(\sigma_1,T)$ and $F(\sigma_2,T)$ from equation (2.62), in the formula $E_{\sigma}(\alpha,\beta)$ we will have $a+a_i = \frac{8}{6}$ and $a'+a'_i = 0$.

For the sake of simplicity we will approximate equation (2.63) by $\delta_1 = 0$. Therefore $F(\sigma,T)$, asymptotically is approximately bounded by:

$$T^{\frac{8}{3}(1-\sigma)} (\log T)^{6(1-\sigma)+4(\sigma-\frac{1}{2})} \left(e^{\alpha((2-2\sigma)(\frac{1}{2T})^{\beta}+(2\sigma-1)(\frac{1}{T})^{\beta}-(\frac{\sigma}{T})^{\beta}+1)}\right) \cdot \left(\alpha^{-\frac{4}{3\beta}} \int_0^{\infty} e^{-t} (t)^{\frac{4}{3\beta}} dt\right)^{(2-2\sigma)} \left(\alpha^{-\frac{0}{\beta}} \int_0^{\infty} e^{-t} (t)^{\frac{0}{\beta}} dt\right)^{(2\sigma-1)}. \tag{2.64}$$

By remark 2.14, we can express the error term in terms of the Gamma function. Therefore we have:

$$E_{\sigma}(\alpha,\beta) = \left(e^{\alpha\left((2-2\sigma)\left(\frac{1}{2T}\right)^{\beta}+(2\sigma-1)\left(\frac{1}{T}\right)^{\beta}-\left(\frac{\sigma}{T}\right)^{\beta}+1\right)}\right) \left(\alpha^{-\frac{4}{3\beta}} \Gamma\left(\frac{4}{3\beta}+1\right)\right)^{2-2\sigma}. \tag{2.65}$$

Now we optimize the error term by optimizing α and β. So, similarly to the unsmoothing section, we take the logarithm of the error term:

$$\begin{aligned} \log E_{\sigma}(\alpha,\beta) = {} & \alpha\left((2-2\sigma)\left(\frac{1}{2T}\right)^{\beta} + (2\sigma-1)\left(\frac{1}{T}\right)^{\beta} - \left(\frac{\sigma}{T}\right)^{\beta} + 1\right) \\ & + (2-2\sigma)\left(-\frac{4}{3\beta}\log(\alpha) + \log\Gamma\left(\frac{4}{3\beta}+1\right)\right). \end{aligned} \tag{2.66}$$

Next, we take the derivative of this expression with respect to α:

$$\frac{\partial \log E_{\sigma}(\alpha,\beta)}{\partial \alpha} = (2-2\sigma)\left(\frac{1}{2T}\right)^{\beta} + (2\sigma-1)\left(\frac{1}{T}\right)^{\beta} - \left(\frac{\sigma}{T}\right)^{\beta} + 1 - \frac{4}{3\alpha\beta}(2-2\sigma). \tag{2.67}$$

Setting this derivative equal to zero gives:

$$\alpha = \frac{4(2-2\sigma)}{3\beta\left((2-2\sigma)(\frac{1}{2T})^{\beta} + (2\sigma-1)(\frac{1}{T})^{\beta} - (\frac{\sigma}{T})^{\beta} + 1\right)} \tag{2.68}$$

Remark 2.25. As $\sigma \to 1$, with β fixed, we will have $\alpha \to 0$.

Remark 2.26. As $T \to \infty$ the denominator approaches 3β. Thus, $\alpha \to \frac{4(2-2\sigma)}{3\beta}$.

By using this relationship, as $T \to \infty$, we can substitute $\alpha = \frac{4(2-2\sigma)}{3\beta}$ back into the original error term and perform further optimization as needed. Therefore, we have

$$E_{\sigma}(\beta) = \left(e^{\frac{4(2-2\sigma)}{3\beta}}\right)\left(\left(\frac{4(2-2\sigma)}{3\beta}\right)^{\frac{-4}{3\beta}} \Gamma\left(\frac{4}{3\beta}+1\right)\right)^{2-2\sigma}. \tag{2.69}$$

Again we take logarithm from the error term:

$$\log E_{\sigma}(\beta) = \frac{4(2-2\sigma)}{3\beta} + (2-2\sigma)\left(-\frac{4}{3\beta}\log\frac{4(2-2\sigma)}{3\beta} + \log\Gamma\left(\frac{4}{3\beta}+1\right)\right). \tag{2.70}$$

We then take derivative of this expression with respect to β

$$\frac{\partial \log E_{\sigma}(\beta)}{\partial \beta} = \frac{-12(2-2\sigma)}{9\beta^2} + (2-2\sigma)\left(\frac{12}{9\beta^2}\log\frac{4(2-2\sigma)}{3\beta} + \frac{12}{9\beta^2} - \frac{12}{9\beta^2}\psi\left(\frac{4}{3\beta}+1\right)\right). \tag{2.71}$$

After simplification, we obtain:

$$\frac{\partial \log E_{\sigma}(\beta)}{\partial \beta} = \frac{12(2-2\sigma)}{9\beta^2}\left(\log\frac{4(2-2\sigma)}{3\beta} - \psi\left(\frac{4}{3\beta}+1\right)\right). \tag{2.72}$$

By setting this derivative equal to zero, we have

$$\log\frac{4(2-2\sigma)}{3\beta} = \psi\left(\frac{4}{3\beta}+1\right) \quad \text{or} \quad \sigma = 1. \tag{2.73}$$

We can graph the equation (2.72) with respect to β. To do this, we'll choose $\sigma = 0.9$.

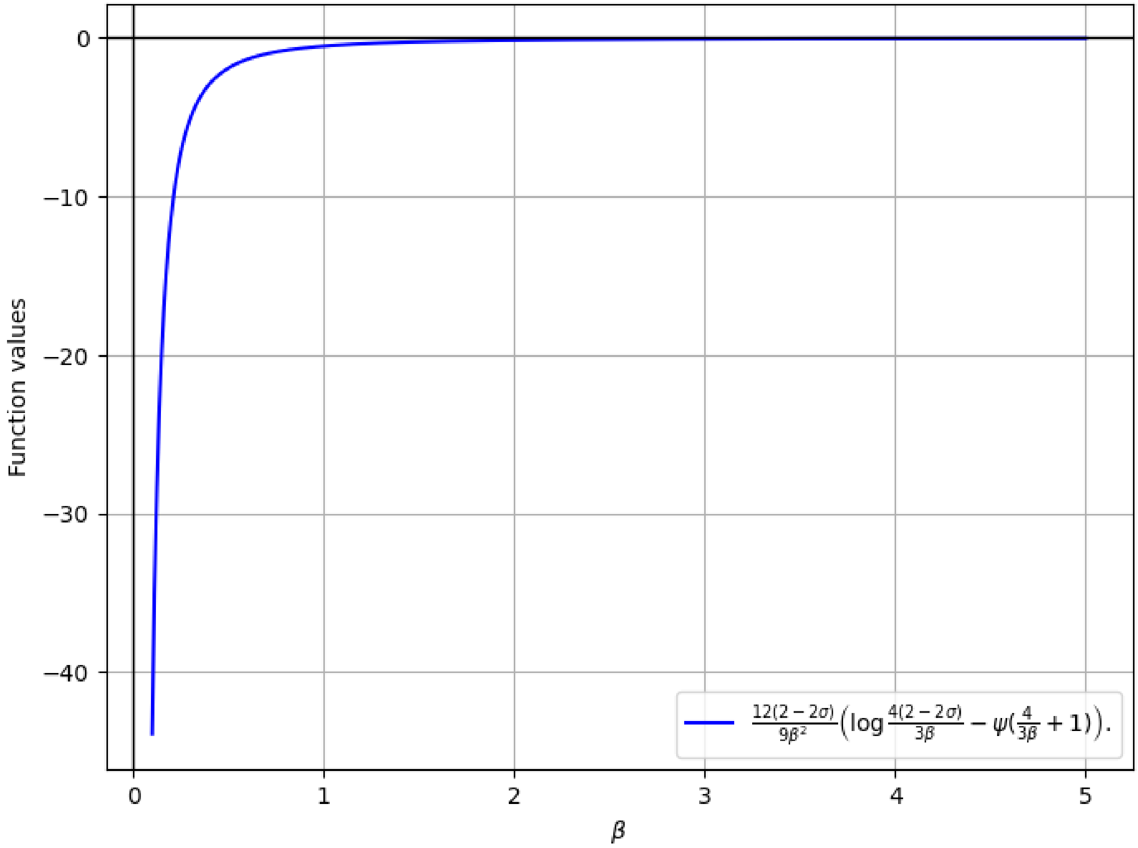


Figure 2.2: Graph of $\frac{12(2-2\sigma)}{9\beta^2}\left(\log\frac{4(2-2\sigma)}{3\beta}-\psi(\frac{4}{3\beta}+1)\right)$ when $\sigma = 0.9$.

Remark 2.27. We see in Figure 2.2, that by increasing β the error term will be reduced. Therefore, taking β large is beneficial, although the computation of C_1 and C_2 becomes more complex.

In principle, the best possible error ratio is 1. However, as the function decreases with β, we get a limit larger than 1. To compare the limits of $E_\sigma(\alpha,\beta)$ as $T \to \infty$ for different values of β, we first consider the case where $\beta = 10$. The corresponding value for $\alpha = 0.02$. Using equation (2.65), we compute the limit of $E_\sigma(\alpha,\beta)$ for $\sigma = 0.9$, yielding:

$$\lim_{T\to\infty} E_\sigma(\alpha,\beta) \approx 1.118.$$

Next, for $\beta = 2$, the corresponding α is 0.13 and the limit becomes:

$$\lim_{T\to\infty} E_\sigma(\alpha,\beta) \approx 1.464.$$

Finally, for $\beta = 1$, the corresponding α is 0.26 and the limit is approximately:

$$\lim_{T\to\infty} E_\sigma(\alpha,\beta) \approx 2.017.$$

2.3 Mean value theorem for Dirichlet polynomials

In this section, we explore a version of the mean value theorem for Dirichlet polynomials, particularly in the form derived by Ramaré [39]. This theorem, is crucial for evaluating the asymptotic behavior of sums involving arithmetic functions and Dirichlet polynomials. We focus on the second moment of Dirichlet polynomials, as this will be useful for our later discussions.

This argument is an explicit version of one of the corollaries of H.L. Montgomery and R.C. Vaughan [28, Corollary 3]. Therefore, we start by presenting this key result from their work [28] about Dirichlet polynomials.

Proposition 2.28. *Let a_n be the coefficients of a Dirichlet polynomial, where $\sum_{n=1}^{\infty} n|a_n|^2 < \infty$. Then*

$$\int_0^\infty \left| \sum_{n=1}^{\infty} a_n n^{-it} \right|^2 dt = \sum_{n=1}^{\infty} |a_n|^2 (T + O(n)), \tag{2.74}$$

where the implied constant in the $O(n)$ term is absolute.

To make this bound explicit, we use the same paper [28] with the improved constant from Preissmann's work [36].

Proposition 2.29. *Let m be a positive integer. Let $\lambda_1, \lambda_2, \ldots$ be a set of distinct real numbers* *and define* $\delta_n = \min_m\{|\lambda_n - \lambda_m| : m \neq n\}$. *If* $a_1, a_2, \ldots \in \mathbb{C}$, *then*

$$\left| \sum_{\substack{m \neq n \\ m,n=1}}^{\infty} \frac{a_m \overline{a_n}}{\lambda_m - \lambda_n} \right| \leq \frac{3}{2}\pi \sum_{n=1}^{\infty} \delta_n^{-1} |a_n|^2. \tag{2.75}$$

In the case $\lambda_m = m$, Montgomery and Vaughan [28] have improved the constant in (2.75).

Proposition 2.30. *Let* m *and* n *be distinct positive integers, and define* $\delta = \min\{|m-n| : \delta > 0\}$. *If* $a_1, a_2, \ldots \in \mathbb{C}$, *then*

$$\left| \sum_{\substack{m \neq n \\ m,n=1}}^{\infty} \frac{a_m \overline{a_n}}{m - n} \right| \leq \pi \delta^{-1} \sum_{n=1}^{\infty} |a_n|^2. \tag{2.76}$$

In 1984, Preissmann [36] improved the constant in (2.75) to πm_0, with

$$m_0 = \sqrt{1 + \frac{2}{3}\sqrt{\frac{6}{5}}}. \tag{2.77}$$

Based on the previous propositions, Ramaré [39, Lemma 6.5], presented the following lemma, which provides a more general bound for the integral of sums involving real-valued sequences.

Lemma 2.31. *Let* (u_n) *be a real-valued sequence for every* $T \geq 0$ *we have*

$$\int_0^T \left| \sum_{n=1}^{\infty} u_n n^{it} \right|^2 dt \leq \sum_{n \geq 1} |u_n|^2 \left(T + \pi m_0 (n+1)\right) \tag{2.78}$$

where m_0 *defined in equation* (2.77).

Proof. Let $P(s) = \sum_{n \geq 1} \frac{u_n}{n^s}$. We can express the integral as follows:

$$\int_0^T |P(it)|^2 dt = \int_0^T \left| \sum_n \frac{u_n}{n^{it}} \right|^2 dt. \tag{2.79}$$

Expanding the square,

$$\begin{aligned}\int_0^T \left|\sum_n \frac{u_n}{n^{it}}\right|^2 dt &= \int_0^T \sum_n \frac{u_n}{n^{it}} \sum_m \frac{\overline{u_m}}{m^{-it}} dt \\ &= \sum_{n,m} u_n \overline{u_m} \int_0^T \left(\frac{m}{n}\right)^{it} dt.\end{aligned}$$

By isolating diagonal terms (where $m = n$) and off diagonal terms (where $m \neq n$), we obtain:

$$\begin{aligned}\sum_{n,m} u_n \overline{u_m} \int_0^T \left(\frac{m}{n}\right)^{it} dt &= T \sum_n |u_n|^2 + \sum_{n \neq m} \frac{u_n \overline{u_m}((\frac{m}{n})^{iT} - 1)}{\log(\frac{m}{n})} \\ &= T \sum_n |u_n|^2 + \sum_{n \neq m} \frac{u_n n^{-iT} \overline{u_m} m^{iT}}{\log(\frac{m}{n})} - \sum_{m \neq n} \frac{u_n \overline{u_m}}{\log(\frac{m}{n})}.\end{aligned} \tag{2.80}$$

Note that (u_n) is real-valued the third summand vanishes identically. We will show the first sum acts as the main term and the second sum as an error term. Thus, we have:

$$\int_0^T \left|\sum_n \frac{u_n}{n^{it}}\right|^2 dt = T \sum_n |u_n|^2 + \sum_{n \neq m} \frac{u_n n^{-iT} \overline{u_m} m^{iT}}{\log(\frac{m}{n})}. \tag{2.81}$$

Given that $\log(\frac{m}{n}) = \log(m) - \log(n)$, we apply (2.75), with $\lambda_m = \log(m)$ and $\lambda_n = \log(n)$. As noted in last paragraph of [28, Page 82], we have $|\log(\frac{m}{n})| \geq \frac{1}{n+1}$ when m and n are distinct positive integers. Using Proposition 2.29 and equation 2.75, with $\delta_n = \frac{1}{n+1}$, we obtain:

$$\begin{aligned}\int_0^T \left|\sum_n \frac{u_n}{n^{it}}\right|^2 dt &\leq T \sum_n |u_n|^2 + \pi m_0 (n+1) |u_n|^2 \\ &\leq \sum_n |u_n|^2 (T + \pi m_0 (n+1))\end{aligned} \tag{2.82}$$

which completes the proof. □

Using Lemma 2.31, we now extend this result to cover integrals over arbitrary intervals $[T_1, T_2]$.

Lemma 2.32. *Let* (u_n) *be a complex-valued sequence. Let* $0 \le T_1 \le T_2$*. Then*

$$\int_{T_1}^{T_2} \left| \sum_{n=1}^{\infty} u_n n^{it} \right|^2 dt \le \sum_{n \ge 1} |u_n|^2 \left(T_2 - T_1 + 2\pi m_0 (n+1) \right). \tag{2.83}$$

Proof. The proof follows the same structure as in Lemma 2.31, except that instead of integrating over $[0, T]$, the integral is taken over $[T_1, T_2]$. So in equation (2.80), in the case of $n = m$, we have $T_2 - T_1$ instead of T and since u_n is complex-valued sequence, for the case $n \neq m$ in equation (2.80), the third sum bounded in the same way as the second term, explaining the factor of 2 in the final bound. □

Chapter 3

Bounds on Arithmetic Sums

In this chapter, we introduce some preliminary lemmas from [39], [5], and [40] that provide bounds for arithmetic sums, which are crucial for later estimates related to the Riemann zeta function. Most of my proofs contain more details than those of Kadiri, Lumley and Ng [26] and Ramaré [39] and some of my proofs, improve the results.

To begin, we introduce the Möbius function denoted $\mu(n)$, which plays a central role in these results:

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n \text{ is not square-free (i.e., } n \text{ has a prime square as a factor)}, \\ (-1)^k & \text{if } n \text{ is square-free with } k \text{ distinct prime factors.} \end{cases} \tag{3.1}$$

Now we start with a fundamental lemma from [5], that provides an asymptotic formula for the count of square-free integers which is required for the subsequent proofs.

Lemma 3.1. *Let $Q(X) = \sum_{n \le X} \mu^2(n)$ denote the number of square-free integers up to X. Then for large X, the function $Q(X)$ satisfies the following asymptotic relation:*

$$Q(X) = \frac{6}{\pi^2} X + O(\sqrt{X}), \tag{3.2}$$

where $\frac{6}{\pi^2}$ represents the density of square-free integers.

Proof. The function $\mu^2(n) = 1$ for square-free n and 0 otherwise. Therefore, the sum $\sum_{n \le X} \mu^2(n)$ counts the square-free integers up to X. Using the Möbius function $\mu(n)$, we can express $Q(X)$ as:

$$Q(X) = \sum_{n \le X} \mu^2(n), \tag{3.3}$$

Or equivalently, by using the fact that $\mu^2(n) = \sum_{d^2|n} \mu(d)$, we can rewrite $Q(X)$ as:

$$Q(X) = \sum_{n \le X} \sum_{d^2|n} \mu(d) \tag{3.4}$$

Reversing the order of summation gives:

$$Q(X) = \sum_{d \le X} \mu(d) \sum_{\substack{n \le X \\ d^2|n}} 1 \tag{3.5}$$

The inner sum, $\sum_{\substack{n \le X \\ d^2|n}} 1$ counts how many integers $n \le X$ are divisible by d^2, which is $\left\lfloor \frac{X}{d^2} \right\rfloor$. Therefore, we have:

$$Q(X) = \sum_{d \le X} \mu(d) \left\lfloor \frac{X}{d^2} \right\rfloor \tag{3.6}$$

Since this summation is zero for $d > \sqrt{X}$, we obtain:

$$Q(X) = \sum_{d \le \sqrt{X}} \mu(d) \left\lfloor \frac{X}{d^2} \right\rfloor \tag{3.7}$$

We can express the floor function, $\left\lfloor \frac{X}{d^2} \right\rfloor$ as

$$\frac{X}{d^2} + O(1).$$

Thus, we approximate the sum $Q(X)$ as:

$$\begin{aligned} Q(X) &= \sum_{d \le \sqrt{X}} \frac{X\mu(d)}{d^2} + O\left(\sum_{d \le \sqrt{X}} 1 \right) = X \sum_{d \le \sqrt{X}} \frac{\mu(d)}{d^2} + O\left(\sqrt{X}\right) \\ &= X \sum_{d} \frac{\mu(d)}{d^2} + O\left(X \sum_{d > \sqrt{X}} \frac{1}{d^2} + \sqrt{X} \right) = \frac{X}{\zeta(2)} + O\left(\sqrt{X}\right). \end{aligned} \tag{3.8}$$

Where $\zeta(2) = \frac{\pi^2}{6}$. Hence, we conclude

$$Q(X) = \frac{6}{\pi^2}X + O\left(\sqrt{X}\right). \tag{3.9}$$

This completes the proof. □

Cohen and Dress [5], in 1988, found the following explicit upper bounds for the remainder term $R(X)$, representing the error in the number of square-free integers up to X. For each $X \geq X_0 \geq 1664$ we have

$$|R(X)| = \left| Q(X) - \frac{6}{\pi^2}X \right| \leq 0.1333\sqrt{X}. \tag{3.10}$$

Using this bound, we can write:

$$\sum_{n \leq X} \mu^2(n) \leq b_1(X_0)X, \tag{3.11}$$

where,

$$b_1(X_0) \leq \frac{6}{\pi^2} + \frac{0.1333}{\sqrt{X_0}}. \tag{3.12}$$

Remark 3.2. Euler's constant is a mathematical constant, usually denoted by γ, defined as the limiting difference between the harmonic series and the natural logarithm, denoted here by log:

$$\gamma = \lim_{n\to\infty}\left(-\log n + \sum_{k=1}^{n} \frac{1}{k} \right).$$

The numerical value of Euler's constant, is: 0.5772....

The following lemma is from [40].

Lemma 3.3. *We have the following asymptotic for the sum of* $\frac{\mu^2(n)}{n}$ *over* $n \leq X$*:*

$$\sum_{n \leq X} \frac{\mu^2(n)}{n} = \frac{6}{\pi^2}\log X + \frac{6}{\pi^2}\left(\gamma - 2\frac{\zeta'(2)}{\zeta(2)}\right) + O\left(\frac{\log X}{\sqrt{X}}\right). \tag{3.13}$$

Proof. To evaluate the sum, we utilize the property that $\sum_{n\le X}\frac{|\mu(n)|}{n}=\sum_{n\le X}\frac{\mu^2(n)}{n}$. Therefore, we have the following calculations:

$$\begin{aligned}
\sum_{n\le X}\frac{\mu^2(n)}{n} &= \sum_{n\le X}\frac{|\mu(n)|}{n}\\
&= \sum_{n\le X}\frac{1}{n}\sum_{k^2|n}\mu(k)\\
&= \sum_{k^2\le X}\mu(k)\sum_{n\le X}\frac{1}{n}\\
&= \sum_{k\le\sqrt{X}}\frac{\mu(k)}{k^2}\sum_{m\le\frac{X}{k^2}}\frac{1}{m}\\
&= \sum_{k\le\sqrt{X}}\frac{\mu(k)}{k^2}\left(\log\frac{X}{k^2}+\gamma+O\left(\frac{k^2}{X}\right)\right).
\end{aligned}\tag{3.14}$$

After simplification, we obtain:

$$(\log X+\gamma)\left(\sum_{k=1}^{\infty}\frac{\mu(k)}{k^2}+O\left(\frac{1}{\sqrt{X}}\right)\right)-2\left(\sum_{k=1}^{\infty}\frac{\mu(k)}{k^2}\log k+O\left(\frac{\log X}{\sqrt{X}}\right)\right)+O\left(\frac{1}{\sqrt{X}}\right).\tag{3.15}$$

Utilizing the convergence of $\sum_{k=1}^{\infty}\frac{\mu(k)}{k^2}=\frac{1}{\zeta(2)}$ and differentiating the series gives us:

$$\begin{aligned}
\sum_{n\le X}\frac{\mu^2(n)}{n} &= (\log X+\gamma)\frac{1}{\zeta(2)}+2\frac{d}{ds}\sum_{k=1}^{\infty}\frac{\mu(k)}{k^s}\bigg|_{s=2}+O\left(\frac{\log(X)}{\sqrt{X}}\right)\\
&= \frac{6}{\pi^2}(\log X+\gamma)+2\frac{d}{ds}\frac{1}{\zeta(s)}\bigg|_{s=2}+O\left(\frac{\log X}{\sqrt{X}}\right)\\
&= \frac{6}{\pi^2}\log X+\frac{6}{\pi^2}\left(\gamma-2\frac{\zeta'(2)}{\zeta(2)}\right)+O\left(\frac{\log X}{\sqrt{X}}\right).
\end{aligned}$$

□

Ramaré [39] found the following explicit upper bounds for the remainder term. Specifically, for $X\ge 1000$, Ramaré showed that the $O\left(\frac{\log(X)}{\sqrt{x}}\right)$ term is bounded by 0.00389 giving:

$$\frac{6}{\pi^2}\log X+1.040\le\sum_{n\le X}\frac{\mu^2(n)}{n}\le\frac{6}{\pi^2}\log X+1.048.\tag{3.16}$$

Consequently with $b_2 = 1.048$ we have

$$\sum_{n\leq X} \frac{\mu^2(n)}{n} - \frac{6}{\pi^2}\log X \leq b_2 \qquad X \geq 1000. \tag{3.17}$$

Recall that we say $f(x) = g(x) + O^*(B)$ for $X > X_0$ to mean that $|f(x) - g(x)| < B$ for $X > X_0$. Here is a lemma from [39, lemma 3.3]:

Lemma 3.4. *For $X \geq 1$, we have*

$$\sum_{n\leq X} \mu^2(n)\frac{\phi(n)}{n^2} = a\log X + b + O^*(0.174), \tag{3.18}$$

where $a = \prod_{p\geq 2}\frac{(p^3-2p+1)}{p^3} = 0.4282 + O^*(10^{-4})$ and

$$\frac{b}{a} = \gamma + \sum_{p\geq 2}\frac{3p-2}{p^3-2p+1}\log p = 2.046 + O^*(10^{-4}). \tag{3.19}$$

The error term can be reduced in specific cases:

$$\begin{aligned} &O^*(0.0533) \quad \textit{for} \quad X \geq 10 \\ &O^*(0.0194) \quad \textit{for} \quad X \geq 48. \end{aligned} \tag{3.20}$$

Remark 3.5. Given that $\frac{\phi(n)}{n^2} < \frac{1}{n}$ this result provides a refinement of the leading term $\frac{6}{\pi^2}\log X$ from earlier lemma. Specifically, the coefficient $\frac{6}{\pi^2}$ is improved by a more precise value 0.4282.

For $n > X$ the function $\lambda_X(n)$ is defined as:

$$\lambda_X(n) = \sum_{\substack{d|n \\ d\leq X}} \mu(d). \tag{3.21}$$

Here is a lemma taken from [39, lemma 5.6].

Lemma 3.6. *Let* $X \geq 10^9$*, we have*

$$\sum_{X<n<5X} \frac{\lambda_X(n)^2}{n^2} \leq \frac{b_3}{X}, \tag{3.22}$$

where $b_3 = 0.605$.

Proof. Since we have $\lambda_X(n) = \sum_{\substack{d|n \\ d\leq X}} \mu(d)$ for $n > X$. Hence it suffices to prove

$$\sum_{X<n<5X} \frac{(\sum_{\substack{d|n \\ d\leq X}} \mu(d))^2}{n^2} \leq \frac{b_3}{X}.$$

We compute separately the contributions arising from $n \in (X, 2X]$ and from $n \in (2X, 5X)$. For $n \in (X, 2X]$, since $n > X$ the divisor n itself is excluded from the sum for $\lambda_X(n)$, and remaining sum over divisors smaller than n must equal $-\mu(n)$, because

$$\sum_{\substack{d|n \\ d\leq X}} \mu(d) = \sum_{d|n} \mu(d) - \mu(n) = 0 - \mu(n) = -\mu(n). \tag{3.23}$$

Therefore, we need to bound

$$S_1 = \sum_{X<n\leq 2X} \frac{\mu^2(n)}{n^2}.$$

We compute this using integration by parts:

$$S_1 = \frac{\sum_{X<n\leq 2X} \mu^2(n)}{(2X)^2} + 2\int_X^{2X} \sum_{X<n\leq t} \mu^2(n) \frac{dt}{t^3}. \tag{3.24}$$

Thus by using equation (3.10), we can find an upper bound for

$$\sum_{X<n\leq 2X} \mu^2(n) \leq \frac{6}{\pi^2} X + 0.1333(1+\sqrt{2})\sqrt{X}. \tag{3.25}$$

Also for $t \in (X,2X]$, we can bound

$$\sum_{X<n\leq t} \mu^2(n) \leq \frac{6}{\pi^2}(t-X) + 0.1333(\sqrt{X}+\sqrt{t}). \tag{3.26}$$

Now we substitute these estimates into the expression for S_1:

$$S_1 \leq \frac{\frac{6}{\pi^2}X + 0.1333(1+\sqrt{2})\sqrt{X}}{(2X)^2} + 2\int_X^{2X} \left(\left(\frac{6}{\pi^2}(t-X) + 0.1333(\sqrt{X}+\sqrt{t}) \right) \frac{dt}{t^3} \right). \tag{3.27}$$

After simplifying the terms, we obtain the upper bound for S_1:

$$\begin{aligned} S_1 &\leq \frac{3}{\pi^2 X} + \frac{0.1333}{X^{3/2}} \left(1 + \sqrt{2} + 1 - 1/4 + \frac{2}{3/2}\left(1 - \frac{1}{2^{3/2}}\right)\right) \\ &\leq \frac{0.304}{X}. \end{aligned} \tag{3.28}$$

For $n \in (2X, 5X)$, we need to analyze the sum $\lambda_X(n) = \sum_{\substack{d|n \\ d\leq X}} \mu(d)$. Here, $\lambda_X(n)$ can take values -1, 0, 1 depending on the contributions of the Möbius function from divisors $d \leq X$. The non-trivial cases is when n is divisible by small primes such as 2 or 3. Specifically, using the fact $\mu(q) = 0$ for non-integer q we have the following cases:

$$\lambda(n)^2 = \begin{cases} (\mu(n)+\mu(n/2))^2 & 2X \leq n < 3X \\ (\mu(n)+\mu(n/2)+\mu(n/3))^2 & 3X \leq n < 4X \\ (\mu(n)+\mu(n/2)+\mu(n/3)+\mu(n/4))^2 & 4X \leq n < 5X \end{cases}$$

We now analyze the behaviour of $\lambda(n)^2$:

- If $4|n$ then $\mu(n) = \mu(n/3) = 0$.
- If $9|n$ then $\mu(n) = \mu(n/2) = \mu(n/4) = 0$
- If $2|n$ but $4 \nmid n$ then $\mu(n) + \mu(n/2) = 0$.

With this we obtain:

$$\lambda(n)^2 = \begin{cases} \mu(n)^2 & X < n \leq 5X, 2 \nmid n, 3 \nmid n \\ 0 & 2X \leq n < 3X, 2|n, 4 \nmid n \\ \mu(n/2)^2 & 2X \leq n < 4X, 4|n, \\ 0 & 4X \leq n \leq 5X, 4|n, 8 \nmid n \\ \mu(n/4)^2 & 4X \leq n \leq 5X, 8|n \\ 0 & 3X \leq n \leq 5X, 2|n, 3 \nmid n, 4 \nmid n \\ \mu(n/3)^2 & 3X \leq n \leq 5X, 2|n, 3|n, 4 \nmid n \\ 0 & 3X \leq n \leq 5X, 3|n, 2 \nmid n, 9 \nmid n \\ \mu(n/3)^2 & 3X \leq n \leq 5X, 9|n, \end{cases}$$

When $\mu(n) \neq 0$, it means that n is not divisible by any square of a prime. In this case, the sum $\lambda_X(n)$ is simple to compute since each divisor contributes either -1 or 1. The Möbius function does not vanish for n or it's divisors $\frac{n}{2}$, $\frac{n}{3}$ and $\frac{n}{4}$, it means that the sum will not have zero terms.

When $\mu(n) = 0$ it means that n contains a squared prime factor, making the Möbius function vanish. In this case, if either $\frac{n}{2}$, $\frac{n}{3}$ and $\frac{n}{4}$ also contains a square of a prime, then their Möbius values will vanish as well. Thus, the contribution from these terms to $\lambda_X(n)$ will be zeros.

Now, to complete the proof, we sum over all $n \in (2X, 5X)$ with $\lambda_X(n) \neq 0$. It is thus enough to bound the following summation

$$S_2 = \sum_{2x<n<5x} 1/n^2.$$

This sum can be estimated as:

$$S_2 \leq \frac{1}{(2X)^2} + \int_{2X}^{5X} \frac{dt}{t^2} = \left(\frac{1}{2} - \frac{1}{5} + \frac{1}{4X}\right)\frac{1}{X}, \tag{3.29}$$

which simplifies to:

$$S_2 \leq \frac{0.301}{X}. \tag{3.30}$$

Finally, combining the results from the two cases, we we get:

$$S_1 + S_2 \leq \frac{0.605}{X} \qquad \square.$$

Remark 3.7. The upper bound in Lemma 3.6 can be improved by considering additional cases for $\lambda_X(n)^2$ more precisely. Specifically, by using the cases that we mentioned in the proof, we obtain the following upper bound for $\sum_{X \leq n < 5X} \frac{\lambda(n)^2}{n^2}$:

$$\sum_{X \leq n < 5X} \frac{\mu(n)^2}{n^2} + \sum_{X/2 \leq n < 5X/8} \frac{\mu(n)^2}{16n^2} + \sum_{X/2 \leq n < X} \frac{\mu(n)^2}{64n^2} + \sum_{X/2 \leq n < 5X/6} \frac{\mu(n)^2}{36n^2} + \sum_{X/3 \leq n < 5X/9} \frac{\mu(n)^2}{81n^2}.$$

In [39, lemma 5.1] Ramaré established the following:

$$\left| \sum_{\substack{n \leq X \\ (n,d)=1}} \frac{\mu(n)}{n^{1+\varepsilon}} \right| \leq 1 + \varepsilon. \tag{3.31}$$

We improve on this bound using the partial summation formula, using the special case of $\varepsilon = 0$. See also [38, Theorem 1.1] and [10, lemma 10.2].

Lemma 3.8. *For any real numbers $X \geq 1$ and $\varepsilon > 0$, and any integer d, we have*

$$\left| \sum_{\substack{n \leq X \\ (n,d)=1}} \frac{\mu(n)}{n^{1+\varepsilon}} \right| \leq 1. \tag{3.32}$$

Proof. Define

$$m_\varepsilon(X) = \sum_{\substack{n \leq X \\ (n,d)=1}} \frac{\mu(n)}{n^{1+\varepsilon}}. \tag{3.33}$$

We proceed using partial summation. Let

$$m_0(X) = \sum_{\substack{n\le X\\(n,d)=1}} \frac{\mu(n)}{n}. \tag{3.34}$$

Applying the partial summation formula, we obtain:

$$m_\varepsilon(X) = m_0(X)X^{-\varepsilon} + \int_1^X \varepsilon m_0(t)t^{-1-\varepsilon}dt. \tag{3.35}$$

Using the bound $m_0(X) \le 1$, we substitute this into (3.35) to obtain:

$$m_\varepsilon(X) \le X^{-\varepsilon} + \int_1^X \varepsilon t^{-1-\varepsilon}dt. \tag{3.36}$$

Next, we evaluate the integral:

$$\int_1^X \varepsilon t^{-1-\varepsilon}dt = \varepsilon\int_1^X t^{-1-\varepsilon}dt = \varepsilon\left(\frac{t^{-\varepsilon}}{-\varepsilon}\right)\Bigg|_1^X = 1 - X^{-\varepsilon}. \tag{3.37}$$

Substituting this result back into (3.36), we find:

$$|m_\varepsilon(X)| \le X^{-\varepsilon} + 1 - X^{-\varepsilon} = 1. \tag{3.38}$$

The proof for the lower bound follows similarly, using $m_0(X) \ge -1$. This completes the proof. □

Here is a lemma from Kadiri, Lumley and Ng [26], with an improvement. Kadiri, Lumley and Ng [26] referred to Ramaré [39] to establish the following inequality

$$\sum_{n\ge 1} \frac{\lambda_X(n)^2}{n^\tau} \le \frac{0.529\tau^2}{\tau-1} e^{\gamma(\tau-1)} \log X. \tag{3.39}$$

We improve this bound by improving the constant 0.529 to 0.471 and removing the τ^2.

Lemma 3.9. *Let* $\tau > 1$ *and* $X \geq X_0 > 48$ *and* γ *denotes Euler's constant. Then*

$$\sum_{n\geq 1} \frac{\lambda_X(n)^2}{n^\tau} \leq \frac{b_4(X_0)}{\tau - 1} e^{\gamma(\tau-1)} \log X. \tag{3.40}$$

where,

$$b_4(X_0) \leq 0.428 + \frac{0.895}{\log X_0} \tag{3.41}$$

In particular, $b_4(10^9) \leq 0.471$.

Proof. We aim to evaluate and bound the sum:

$$G(\tau) = \sum_{n\geq 1} \frac{\lambda_X(n)^2}{n^\tau}. \tag{3.42}$$

So we begin by expanding the square in the sum:

$$G(\tau) = \sum_{n\geq 1} \frac{\left(\sum_{\substack{d|n \\ d\leq X}} \mu(d)\right)^2}{n^\tau}. \tag{3.43}$$

This double sum can be re-written as:

$$G(\tau) = \sum_{d_1,d_2\leq X} \mu(d_1)\mu(d_2) \sum_{\substack{n\geq 1 \\ [d_1,d_2]|n}} \frac{1}{n^\tau}, \tag{3.44}$$

Where $[d_1, d_2]$ is the least common multiple of d_1 and d_2.

For the inner sum, we can evaluate the sum, using the Dirichlet series for the Riemann zeta function:

$$\sum_{\substack{n\geq 1 \\ [d_1,d_2]|n}} \frac{1}{n^\tau} = \frac{1}{[d_1,d_2]^\tau} \zeta(\tau). \tag{3.45}$$

Therefore, we can write $G(\tau)$ as:

$$G(\tau) = \zeta(\tau) \sum_{d_1,d_2\leq X} \frac{\mu(d_1)\mu(d_2)}{[d_1,d_2]^\tau}. \tag{3.46}$$

We know that the least common multiple can be written in terms of the greatest common divisor:

$$[d_1,d_2] = \frac{d_1 d_2}{\gcd(d_1,d_2)}. \tag{3.47}$$

Thus, we can rewrite the sum as:

$$\sum_{d_1,d_2\leq X} \frac{\mu(d_1)\mu(d_2)}{[d_1,d_2]^\tau} = \sum_{d_1,d_2\leq X} \frac{\mu(d_1)\mu(d_2)\gcd(d_1,d_2)^\tau}{(d_1 d_2)^\tau}. \tag{3.48}$$

We introduce $\delta = \gcd(d_1,d_2)$, and by letting $d_1 = \delta d_1{}'$ and $d_2 = \delta d_2{}'$ where $\gcd(d_1{}',d_2{}') = 1$, we can rewrite the sum as:

$$\sum_{\delta\leq X} \frac{\mu(\delta)^2}{\delta^\tau} \sum_{\substack{d_1{}',d_2{}'\leq \frac{X}{\delta} \\ \gcd(d_1',d_2')=1}} \frac{\mu(d_1')\mu(d_2')}{(d_1{}'d_2{}')^\tau}, \tag{3.49}$$

since $\mu(\delta d_1{}') = \mu(\delta)\mu(d_1{}')$ and similarly for d_2. So we find the expression for $G(\tau)$:

$$G(\tau) = \zeta(\tau) \sum_{\delta\leq X} \frac{\mu(\delta)^2}{\delta^\tau} \sum_{\substack{d_1{}',d_2{}'\leq \frac{X}{\delta} \\ \gcd(d_1',d_2')=1}} \frac{\mu(d_1')\mu(d_2')}{(d_1{}'d_2{}')^\tau}. \tag{3.50}$$

At this point, we define for any τ the auxiliary function:

$$\phi_\tau(d) = d^\tau \prod_{p|d} (1-p^{-\tau}), \tag{3.51}$$

where the product is taken over all primes dividing d. It satisfies:

$$d^\tau = \sum_{d'|d} \phi_\tau(d'), \tag{3.52}$$

which can be written as $d^\tau = (\mathbb{1} * \phi_\tau)(d)$ where $*$ denotes the Dirichlet convolution.

We will now show that

$$G(\tau)=\zeta(\tau)\sum_{\delta'\leq X}\phi_\tau(\delta')\frac{\mu(\delta')^2}{\delta'^{2\tau}}\left(\sum_{\substack{d_1'\leq\frac{X}{\delta'}\\(d_1',\delta')=1}}\frac{\mu(d_1')}{(d_1')^\tau}\right)\left(\sum_{\substack{d_2'\leq\frac{X}{\delta'}\\(d_2',\delta')=1}}\frac{\mu(d_2')}{(d_2')^\tau}\right). \tag{3.53}$$

The right hand side of (3.53) is easily seen to be

$$\zeta(\tau)\sum_{\delta'\leq X}\phi_\tau(\delta')\frac{\mu(\delta')^2}{\delta'^{2\tau}}\sum_{\substack{d_1',d_2'\leq\frac{X}{\delta'}\\(d_i',\delta')=1}}\frac{\mu(d_1')\mu(d_2')}{(d_1'd_2')^\tau}. \tag{3.54}$$

Now let $d_i=\frac{d_i'}{d}$ and where $\gcd(d_1{}',d_2{}')=d$, we can rewrite the sum as:

$$\zeta(\tau)\sum_{\delta'\leq X}\phi_\tau(\delta')\frac{\mu(\delta')^2}{\delta'^{2\tau}}\sum_{\substack{d\leq\frac{X}{\delta'}\\(d,\delta')=1}}\frac{\mu(d)^2}{d^{2\tau}}\sum_{\substack{d_1,d_2\leq\frac{X}{d\delta'}\\(d_i',d\delta')=1\\\gcd(d_1,d_2)=1}}\frac{\mu(d_1d_2)}{(d_1d_2)^\tau}. \tag{3.55}$$

By letting $\delta=d\delta'$ we obtain:

$$\zeta(\tau)\sum_{\delta}\left(\sum_{\delta'|\delta}\phi_\tau(\delta')\right)\frac{\mu(\delta)^2}{(\delta)^{2\tau}}\sum_{\substack{d_1,d_2\leq\frac{X}{\delta}\\(d_i,\delta)=1\\\gcd(d_1',d_2')=1}}\frac{\mu(d_1d_2)}{(d_1d_2)^\tau}. \tag{3.56}$$

Finally, by equation (3.52), we have $\sum_{\delta'|\delta}\phi_\tau(\delta')=\delta^\tau$. Therefore, we obtain equation (3.50).

Now applying Lemma 3.8 in (3.53), we have the inequality:

$$G(\tau)\leq\zeta(\tau)\sum_{\delta\leq X}\frac{\mu^2(\delta)\phi_\tau(\delta)}{\delta^{2\tau}}. \tag{3.57}$$

Since $\frac{\phi_\tau(\delta)}{\delta^{2\tau}}\leq\frac{\phi_\tau(\delta)}{\delta^2}$, we can apply Lemma 3.4 to obtain:

$$G(\tau)\leq\zeta(\tau)(0.4283\log X+0.8760+0.0194). \tag{3.58}$$

Since by Ramare [39, lemma 5.4] we have the following approximation for $\zeta(s)$ when $s > 1$ is real:

$$\zeta(s) \leq \frac{e^{\gamma(s-1)}}{s-1}. \tag{3.59}$$

By applying this, we obtain:

$$G(\tau) \leq \frac{e^{\gamma(\tau-1)}}{\tau-1}(0.4283\log X + 0.8760 + 0.0194). \tag{3.60}$$

Therefore, by computation for this approximation, we have:

$$\sum_{n\geq 1} \frac{\lambda_X(n)^2}{n^\tau} \leq \frac{b_4(X_0)}{\tau-1} e^{\gamma(\tau-1)} \log X. \qquad \square$$

The following Lemma is a direct consequence of equation (3.40), taking $\tau = 1 + \frac{\delta}{\log X}$.

Lemma 3.10. *Let $\delta > 0$ and $X \geq X_0$ and γ denotes Euler's constant. Then*

$$\sum_{n\geq 1} \frac{\lambda_X(n)^2}{n^{1+\frac{\delta}{\log X}}} \leq \frac{b_4(X_0)}{\delta} e^{\frac{\delta\gamma}{\log X}} (\log X)^2, \tag{3.61}$$

Where $b_4(X_0)$ is as defined in Lemma 3.9.

Proof. By taking $\tau = 1 + \frac{\delta}{\log X}$ and applying the bound for the sum:

$$\sum_{n\geq 1} \frac{\lambda_X(n)^2}{n^\tau} \leq \frac{b_4(X_0)}{\tau-1} e^{\gamma(\tau-1)} \log X, \tag{3.62}$$

Substituting $\tau = 1 + \frac{\delta}{\log X}$ gives us $\tau - 1 = \frac{\delta}{\log X}$. Therefore, we obtain the desired result. $\square$

Lemma 3.11. *let $\delta > 0, X \geq X_0$, and γ denotes Euler's constant. Then*

$$\sum_{n\geq 1} \frac{\lambda_X(n)^2}{n^{2+\frac{2\delta}{\log X}}} \leq \frac{b_4(X_0)}{5\delta e^{\delta}} e^{\frac{\delta(\gamma-\log 5)}{\log X}} \frac{(\log X)^2}{X} + \frac{b_3 e^{-2\delta}}{X}. \tag{3.63}$$

Where $b_4(X_0)$ is as defined in Lemma 3.9.

Proof. We begin by setting $\tau = 2 + \frac{2\delta}{\log X}$ in equation (3.40). By definition of $\lambda_X(n)$, we have $\lambda_X(n)^2 = 0$ when $1 \leq n \leq X$. This allows us to break the sum into two parts, depending on the range of n:

$$\sum_{n\geq 1} \frac{\lambda_X(n)^2}{n^\tau} = \sum_{X<n<5X} \frac{\lambda_X(n)^2}{n^\tau} + \sum_{n\geq 5X} \frac{\lambda_X(n)^2}{n^\tau}.$$

For the range $X < n < 5X$ we have $n \approx X$, so we can factor out powers of X in order to use the previous lemmas. Since $\tau \geq 2$, the following upper bound holds:

$$\sum_{X<n<5X} \frac{\lambda_X(n)^2}{n^\tau} \leq \frac{1}{X^{\tau-2}} \sum_{X<n<5X} \frac{\lambda_X(n)^2}{n^2}. \tag{3.64}$$

Using equation (3.22) in Lemma 3.6 we obtain:

$$\begin{aligned}\sum_{X<n<5X} \frac{\lambda_X(n)^2}{n^\tau} &\leq \frac{1}{X^{\tau-2}} \cdot \frac{b_3}{X} \\ &\leq \frac{b_3 e^{-2\delta}}{X},\end{aligned} \tag{3.65}$$

since $\tau - 2 = \frac{2\delta}{\log X}$ and then we have $X^{\tau-2} = e^{2\delta}$.

Next, for the range $n \geq 5X$ we use the fact that, $n^\tau \geq (5X)^{1+\frac{\delta}{\log X}} n^{1+\frac{\delta}{\log X}}$. Therefore, we have:

$$\sum_{n\geq 5X} \frac{\lambda_X(n)^2}{n^\tau} \leq \frac{1}{(5X)^{1+\frac{\delta}{\log X}}} \sum_{n\geq 5X} \frac{\lambda_X(n)^2}{n^{1+\frac{\delta}{\log X}}}. \tag{3.66}$$

We use equation (3.61) in Lemma 3.10 to bound the second sum. Therefore, we obtain:

$$\sum_{n\geq 5X} \frac{\lambda_X(n)^2}{n^\tau} \leq \frac{1}{(5X)^{1+\frac{\delta}{\log X}}} \frac{b_4}{\delta} e^{\frac{\delta\gamma}{\log X}} (\log X)^2. \tag{3.67}$$

Simplifies to:

$$\sum_{n\geq 5X} \frac{\lambda_X(n)^2}{n^\tau} \leq \frac{1}{5^{1+\frac{\delta}{\log X}}} \frac{b_4}{\delta} e^{\frac{\delta\gamma}{\log X}} \frac{(\log X)^2 e^{-\delta}}{X}. \tag{3.68}$$

By combining the bounds for the two sums, we obtain the desired upper bound:

$$\sum_{n\geq 1} \frac{\lambda_X(n)^2}{n^\tau} \leq \frac{b_4}{5\delta e^\delta} e^{\frac{\delta(\gamma-\log 5)}{\log X}} \cdot \frac{(\log X)^2}{X} + \frac{b_3 e^{-2\delta}}{X}. \qquad \square$$

The function $d(n)$ represents the number of divisors of an integer n, and it is defined as:

$$d(n) = \sum_{d|n} 1, \tag{3.69}$$

where the sum runs over all divisors d of n.

The following lemma from [26], provides an upper bound on a series involving $d(n)$ which will be useful in subsequent analyses.

Lemma 3.12. *Let* $\tau > 1$ *and* $X \geq 1$*, then we have:*

$$\sum_{n\geq X} \frac{d(n)}{n^\tau} \leq E_1(X,\tau), \tag{3.70}$$

where

$$E_1(X,\tau) = \frac{\tau}{X^{\tau-1}} \left(\frac{\log X}{\tau-1} + \frac{1}{(\tau-1)^2} + \frac{\gamma}{\tau-1} + \frac{7}{12\tau X} \right), \tag{3.71}$$

with γ *is the Euler's constant.*

Proof. Let $D(t) = \sum_{n\leq t} d(n)$ be the partial sum of divisors up to t. By applying the partial summation formula to the sum $\sum_{n\geq X} \frac{d(n)}{n^\tau}$, we get:

$$\sum_{n=X}^{\infty} d(n) \cdot \frac{1}{n^\tau} = 0 - \frac{D(X)}{X^\tau} + \tau \int_X^\infty \frac{D(t)}{t^{\tau+1}} dt. \tag{3.72}$$

The term $-\frac{D(X)}{X^\tau}$ is negative, so we have:

$$\sum_{n\geq X} \frac{d(n)}{n^\tau} \leq \tau \int_X^\infty \frac{D(t)}{t^{\tau+1}} dt. \tag{3.73}$$

By using [37, equation 3.1], we have the below inequality:

$$\left|\sum_{n\le t}\frac{1}{n}-\log t-\gamma\right|\le\frac{7}{12t}, \tag{3.74}$$

for $t\ge 1$. Therefore, we use this approximation to find an upper bound for the sum $D(t)$:

$$D(t)=\sum_{n\le t}d(n)=\sum_{ab\le t}1=\sum_{a\le t}\sum_{b\le t/a}1\le t\sum_{a\le t}\tfrac{1}{a}\le t\left(\log t+\gamma+\frac{7}{12t}\right). \tag{3.75}$$

This approximation holds for $t\ge 1$.

Substituting this approximation into the integral, we obtain:

$$\sum_{n\ge X}\frac{d(n)}{n^{\tau}}\le\tau\int_X^{\infty}\frac{t\left(\log t+\gamma+\frac{7}{12t}\right)}{t^{\tau}\cdot t}dt. \tag{3.76}$$

We now separate the integral into three integrals:

$$\tau\left(\int_X^{\infty}\frac{\log t}{t^{\tau}}dt+\gamma\int_X^{\infty}\frac{dt}{t^{\tau}}+\frac{7}{12}\int_X^{\infty}\frac{dt}{t^{\tau+1}}\right). \tag{3.77}$$

For the first integral, by applying integration by parts, we get:

$$\int_X^{\infty}\frac{\log t}{t^{\tau}}dt=\frac{\log X}{(\tau-1)X^{\tau-1}}+\frac{1}{(\tau-1)^2X^{\tau-1}}. \tag{3.78}$$

For the second integral, we have

$$\int_X^{\infty}\frac{dt}{t^{\tau}}=\int_X^{\infty}t^{-\tau}dt=\frac{1}{(\tau-1)X^{\tau-1}}. \tag{3.79}$$

For the last integral, we have

$$\int_X^{\infty}\frac{dt}{t^{\tau+1}}=\int_X^{\infty}t^{-\tau-1}dt=\frac{1}{\tau X^{\tau}}. \tag{3.80}$$

Now, we combine the results of the three integrals:

$$\sum_{n\geq X}\frac{d(n)}{n^{\tau}}\leq\tau\left(\frac{\log X}{(\tau-1)X^{\tau-1}}+\frac{1}{(\tau-1)^2X^{\tau-1}}+\frac{\gamma}{(\tau-1)X^{\tau-1}}+\frac{7}{12\tau X^{\tau}}\right). \tag{3.81}$$

Simplifying the expression, we obtain:

$$\sum_{n\geq X}\frac{d(n)}{n^{\tau}}\leq\frac{\tau}{X^{\tau-1}}\left(\frac{\log X}{(\tau-1)}+\frac{1}{(\tau-1)^2}+\frac{\gamma}{(\tau-1)}+\frac{7}{12\tau X}\right), \tag{3.82}$$

which complete the proof. □

The following lemma is [6, Theorem 2]:

Lemma 3.13. *For $t\geq 2$ we have*

$$\sum_{n\leq t}d(n)^2\leq t\left(\frac{1}{\pi^2}\log^3 t+0.745\log^2 t+0.824\log t+0.461+9.73\log(t)t^{-\frac{1}{4}}\right). \tag{3.83}$$

Furthermore, for $t\geq t_j$ we have

$$\sum_{n\leq t}d(n)^2\leq Kt\log^3 t, \tag{3.84}$$

where one may take $\{K,t_j\}$ to be, among others, $\{\frac{1}{4},433\}$ or $\{1,7\}$.

Lemma 3.14. *Let $\tau>1$ and $X\geq 2$, then we have:*

$$\sum_{n\geq X}\frac{d(n)^2}{n^{\tau}}\leq E_2(X,\tau), \tag{3.85}$$

where

$$E_2(X,\tau) = \frac{1}{X^{\tau-1}}\Bigg(\frac{\frac{1}{\pi^2}\log^3 X + 0.745\log^2 X + 0.824\log X + 0.461}{(\tau-1)} + \frac{\frac{6}{\pi^2}\log X + 1.49}{(\tau-1)^3}$$
$$+ \frac{\frac{3}{\pi^2}\log^2 X + 1.49\log X + 0.824}{(\tau-1)^2} + \frac{\frac{6}{\pi^2}}{(\tau-1)^4} + \frac{9.73X^{\frac{-1}{4}}\log X}{(\tau-\frac{3}{4})}$$
$$+ \frac{9.73X^{-\frac{1}{4}}}{(\tau-\frac{3}{4})^2}\Bigg). \tag{3.86}$$

Proof. This lemma is similar to the previous one: First, we apply partial summation to express the sum in terms of an integral:

$$\sum_{n\geq X}\frac{d(n)^2}{n^\tau} \leq \tau\int_X^\infty \frac{D(t)}{t^{\tau+1}}dt,$$

where $D(t) = \sum_{n\leq t} d(n)^2$. From a result in equation (3.83), we have the following bound:

$$\sum_{n\leq t} d(n)^2 \leq t(\frac{1}{\pi^2}\log^3 t + 0.745\log^2 t + 0.824\log t + 0.461 + 9.73\log(t)t^{-\frac{1}{4}}), \quad \text{for } t \geq 2. \tag{3.87}$$

Using the above bound, we can now substitute this into the integral:

$$\int_X^\infty \frac{\sum_{n\leq t} d(n)^2}{t^{\tau+1}}dt \leq \int_X^\infty \frac{\frac{1}{\pi^2}\log^3 t + 0.745\log^2 t + 0.824\log t + 0.461 + 9.73\log(t)t^{-\frac{1}{4}}}{t^\tau}dt. \tag{3.88}$$

We now separate the integral into five integrals:

$$\Bigg(\frac{1}{\pi^2}\int_X^\infty \frac{\log^3 t}{t^\tau}dt + 0.745\int_X^\infty \frac{\log^2 t}{t^\tau}dt + 0.824\int_X^\infty \frac{\log t}{t^\tau}dt$$
$$+ 0.461\int_X^\infty \frac{1}{t^\tau}dt + 9.73\int_X^\infty \frac{\log t}{t^{\tau+\frac{1}{4}}}dt\Bigg). \tag{3.89}$$

For the first integral, by applying integration by parts, we get:

$$\frac{1}{\pi^2}\int_X^\infty \log^3 t\cdot t^{-\tau}dt = \frac{1}{\pi^2}\left[\frac{\log^3 X}{X^{\tau-1}(\tau-1)} + \frac{3}{\tau-1}\left(\int_X^\infty \log^2 t\cdot t^{-\tau}dt\right)\right]. \tag{3.90}$$

We apply integration by parts again, and obtain

$$\frac{1}{\pi^2}\int_X^\infty \log^3 t \cdot t^{-\tau} dt = \frac{1}{\pi^2}\left[\frac{\log^3 X}{X^{\tau-1}(\tau-1)} + \frac{3}{(\tau-1)}\left(\frac{\log^2 X}{X^{\tau-1}(\tau-1)} + \frac{2}{(\tau-1)}\int_X^\infty \log t \cdot t^{-\tau} dt\right)\right] \tag{3.91}$$

We use equation (3.78) to find the answer for, $\int_X^\infty \log t \cdot t^{-\tau} dt$ and substitute it in the previous equation to obtain the following answer for the first integral:

$$\frac{1}{\pi^2}\left[\frac{\log^3 X}{X^{\tau-1}(\tau-1)} + \frac{3}{(\tau-1)}\left(\frac{\log^2 X}{X^{\tau-1}(\tau-1)} + \frac{2}{(\tau-1)}\left(\frac{\log X}{(\tau-1)X^{\tau-1}} + \frac{1}{(\tau-1)^2 X^{\tau-1}}\right)\right)\right]. \tag{3.92}$$

Simplifying the expression, we obtain:

$$\frac{1}{\pi^2}\left(\frac{\log^3 X}{X^{\tau-1}(\tau-1)} + \frac{3\log^2 X}{X^{\tau-1}(\tau-1)^2} + \frac{6\log X}{X^{\tau-1}(\tau-1)^3} + \frac{6}{X^{\tau-1}(\tau-1)^4}\right). \tag{3.93}$$

For the second integral, by using the integration by parts, we have:

$$0.745\int_X^\infty \frac{\log^2 t}{t^\tau} dt = 0.745\left(\frac{\log^2 X}{X^{\tau-1}(\tau-1)} + \frac{2}{(\tau-1)}\left(\frac{\log X}{(\tau-1)X^{\tau-1}} + \frac{1}{(\tau-1)^2 X^{\tau-1}}\right)\right). \tag{3.94}$$

For the third integral, we have:

$$0.824\int_X^\infty \frac{\log t}{t^\tau} dt = 0.824\left(\frac{\log X}{(\tau-1)X^{\tau-1}} + \frac{1}{(\tau-1)^2 X^{\tau-1}}\right). \tag{3.95}$$

For the fourth integral, we have:

$$0.461\int_X^\infty \frac{1}{t^\tau} dt = \frac{0.461}{(\tau-1)X^{\tau-1}}. \tag{3.96}$$

For the last integral, we obtain:

$$9.73\int_X^\infty \frac{\log t}{t^{\tau+\frac{1}{4}}} dt = 9.73\left(\frac{X^{-\frac{1}{4}}\log X}{(\tau-\frac{3}{4})X^{\tau-1}} + \frac{X^{-\frac{1}{4}}}{(\tau-\frac{3}{4})^2 X^{\tau-1}}\right). \tag{3.97}$$

Now, we combine the results of the five integrals and obtain the following upper bound for our original expression, $\sum_{n\geq X}\frac{d(n)^2}{n^\tau}$:

$$\frac{1}{X^{\tau-1}}\Bigg(\frac{\frac{1}{\pi^2}\log^3 X+0.745\log^2 X+0.824\log X+0.461}{(\tau-1)}+\frac{\frac{3}{\pi^2}\log^2 X+1.49\log X+0.824}{(\tau-1)^2} \\ +\frac{\frac{6}{\pi^2}\log X+1.49}{(\tau-1)^3}+\frac{\frac{6}{\pi^2}}{(\tau-1)^4}+\frac{9.73X^{\frac{-1}{4}}\log X}{(\tau-\frac{3}{4})}+\frac{9.73X^{\frac{-1}{4}}}{(\tau-\frac{3}{4})^2}\Bigg), \tag{3.98}$$

which complete the proof. □

Chapter 4

Application to zeta

In this chapter, we apply all the general methods that we have in Chapter 2 and Chapter 3 to the Riemann zeta function. In Section 4.1, we introduce the mollifiers. Then, in Section 4.2, we apply Littlewood's classical Lemma in Ingham's method to derive an upper bound for the number of zeros of $\zeta(s)$. Finally, we examine how these two approaches complement each other. Next, based on the smoothing 2.2.1, unsmoothing 2.2.2 and convexity Sections 2.2.3 we find an upper bound for the first integral (See Section 4.3.5). In the Section 4.4 we estimate the upper bound for the last integral. Also upper bound for the second and third integral are obtained in Section 4.5. Finally, in Section 4.6 we present our final bounds for the zero density result, $N(\sigma, T_1, T_2)$.

4.1 Mollifiers

In this section, we introduce mollifiers, which will help us to apply the general methods outlined in the Chapter 2 and Chapter 3. This is necessary because direct estimates for the zeta function outside of the critical strip are challenging due to its complexity. By introducing a mollifier, we can reduce errors in these estimations and gain more control over the behavior of the zeta function.

Definition 4.1. Let $X \geq 1$ be a parameter. We define the mollifier as follows:

$$M_X(s) = \sum_{n \leq X} \frac{\mu(n)}{n^s}, \tag{4.1}$$

where $\mu(n)$ is the Möbius function, defined by equation (3.1):

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n \text{ is not square-free(i.e., } n \text{ has a prime square as a factor)}, \\ (-1)^k & \text{if } n \text{ is square-free with } k \text{ distinct prime factors.} \end{cases}$$

These mollifiers were first introduced by Bohr and Landau [3] to reduce or minimize the errors that may occur during the process of making estimations for the zeta function in the critical strip. The key idea is that the Dirichlet series generated by the Möbius function is the inverse of the Riemann zeta function:

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}. \tag{4.2}$$

As a result, for $\Re(s) > 1$, we obtain the following relation.

$$M_X(s)\zeta(s) = 1 + \sum_{n>X} \frac{\lambda_X(n)}{n^s},$$

where

$$\lambda_X(n) = \sum_{\substack{d|n \\ d \le X}} \mu(d).$$

Remark 4.2. For $\Re(s) > 1$, the product $M_X(s)\zeta(s)$ is approximately equal to 1. Thus, when we integrate this expression, because it doesn't oscillate like the zeta function, integrals involving $M_X(s)\zeta(s)$ would be easier to estimate than those involving only $\zeta(s)$.

We now define a new function related to the mollifier:

$$f_X(s) = \zeta(s)M_X(s) - 1. \tag{4.3}$$

The series expansion for $f_X(s)$ is given by:

$$f_X(s) = \frac{1}{\sum_{n\geq 1}\frac{\mu(n)}{n^s}} \cdot \sum_{n\leq X}\frac{\mu(n)}{n^s} - 1 = \sum_{n>X}\left(\sum_{\substack{d|n \\ d\leq X}}\mu(d)\right)n^{-s} = \sum_{n\geq 1}\frac{\lambda_X(n)}{n^s}, \tag{4.4}$$

where

$$\lambda_X(n) = \begin{cases} 0 & \text{if } n \leq X \\ \sum_{\substack{d|n \\ d\leq X}}\mu(d) & \text{if} n > X. \end{cases}$$

Next, we define the function $h(s)$ in terms of $f_X(s)$:

$$h(s) = 1 - f_X(s)^2 = 1 - (\zeta(s)\cdot M_X(s) - 1)^2 = 1 - [(\zeta(s)M_X(s))^2 + 1 - 2\zeta(s)M_X(s)]. \tag{4.5}$$

Simplifying this expression:

$$h(s) = \zeta(s)\cdot[M_X(s)(2 - \zeta(s)M_X(s))]. \tag{4.6}$$

Since any zeros of $\zeta(s)$ is zeros of $h(s)$, our goal becomes to find an upper bound for the number of zeros of the new function $h(s)$ instead of for $\zeta(s)$.

Remark 4.3. The function $h(s)$ has an advantage: outside the critical strip, it is close to 1, making it easier to estimate accurately using the methods in Chapter 2. Since $h(s) \approx 1$, we also have $\log h(s) \approx 0$ and $\log(1 + f(s)^2)$ is a good approximation for $f(s)^2$. Moreover, since $\zeta(s)$ oscillates, its moments tend to be small due to cancellation. Similarly, the integral of $\log h(s)$ is very small, both because the integrand is small and because of cancellation due to oscillation. Furthermore, compared to the product $M_X(s)\zeta(s)$, the new function $h(s)$ involves second moments in certain estimates, which allows us to use the mean value theorem which we saw in Section 2.3.

For counting zeros of a function $h_X(s)$, we refer to equation (2.1) and define:

$$N_h(\sigma, T_1, T_2) = \#\left\{\rho' = \beta' + i\gamma'; h\left(\rho'\right) = 0, \sigma < \beta' < 1, T_1 < \gamma' < T_2\right\}, \tag{4.7}$$

where, as in equation (2.1) zeros are counted with multiplicities. Notice, we have the relation:

$$N_\zeta(\sigma, T_1, T_2) = N_\zeta(\sigma, T_2) - N_\zeta(\sigma, T_1) \leq N_h(\sigma, T_2) - N_h(\sigma, T_1). \tag{4.8}$$

This inequality shows that instead of directly estimating the zeros of $\zeta(s)$, we can estimate the zeros of $h_X(s)$, which is more easy to work with.

In the next section we will focus on counting the number of zeros of $h_X(s)$ within a specific region of the complex plane.

4.2 Counting Zeros of $h_X(s)$ in a Special Rectangle Region

In this section, we apply Littlewood's classical method, as described in Chapter 2, to count the number of zeros of the function $h_X(s)$. By applying the same framework used in Section 2.1, we find an upper bound for the number of zeros of $h_X(s)$, within a specified rectangle region.

Let $\sigma_1 = \sigma'$ and $\sigma_2 = \mu$. Following the ideas introduced in equation (2.9), we compare the number of zeros of $h_X(s)$ to the average, obtaining:

$$N_h(\sigma, T_2) - N_h(\sigma, T_1) \leq \frac{1}{\sigma - \sigma'} \int_{\sigma'}^{\mu} \left(N_h(\tau, T_2) - N_h(\tau, T_1)\right) d\tau, \tag{4.9}$$

where $\mu > 1$ and $\frac{1}{2} < \sigma' < \sigma$ and we take take

Remark 4.4. We follow Kadiri, Lumley and [26] in our choice of σ'. This choice can eventually be justified by considering the optimization that arises when taking a more general shift of the form $\sigma' = \sigma - d$. In such cases one is lead to minimize approximately expressions of the form $\frac{t^{\frac{8}{3}d}}{d}$ which is minimized when $d = \frac{3}{8log(t)}$. More generally, one could consider introducing a parametric dependence on t and σ to fine-tune the bound even further.

To count the number of zeros of $h(s)$, We use Littlewood's classical method over the rectangle R with vertices $\sigma'+iT_1$, $\sigma'+iT_2$, $\mu+iT_2$, $\mu+iT_1$. We apply this method to find an equality for $\int_{\sigma'}^{\mu}(N_h(\tau,T_2)-N_h(\tau,T_1))\,d\tau$ and then based on equation (4.8) obtain the upper bound for $N(\sigma,T_2,T_1)$.

Let $\Phi(s)=\log h(s)$, to find an equality for the integral, $\int_{\sigma'}^{\mu}(N_h(\tau,T_2)-N_h(\tau,T_1))\,d\tau$, we apply Lemma 2.2:

$$\frac{-1}{2\pi i}\int_R \log h(s)ds=\int_{\sigma'}^{\mu}(N_h(\tau,T_2)-N_h(\tau,T_1))\,d\tau. \tag{4.10}$$

Given that for $T_2\geq 10^9$ we have

$$N_\zeta(\sigma,T_2,T_1)\leq N_h(\sigma,T_2)-N_h(\sigma,T_1),$$

and

$$N_h(\sigma,T_2)-N_h(\sigma,T_1)\leq\frac{1}{\sigma-\sigma'}\int_{\sigma'}^{\mu}(N_h(\tau,T_2)-N_h(\tau,T_1))d\tau,$$

we conclude:

$$N_\zeta(\sigma,T_1,T_2)\leq\frac{1}{\sigma-\sigma'}\cdot\frac{-1}{2\pi i}\int_R\log h(s)ds. \tag{4.11}$$

Using Theorem 2.3 and equation (2.10) we derive the following bound for $N_\zeta(\sigma,T_1,T_2)$:

$$\begin{aligned}N_\zeta(\sigma,T_1,T_2)\leq\frac{1}{2\pi(\sigma-\sigma')}\Bigg(&\int_{T_1}^{T_2}\log|h(\sigma'+it)|dt+\int_{\sigma'}^{\mu}\arg h(\tau+iT_2)d\tau\\&-\int_{\sigma'}^{\mu}\arg h(\tau+iT_1)d\tau-\int_{T_1}^{T_2}\log|h(\mu+it)|dt\Bigg).\end{aligned} \tag{4.12}$$

We recall that the integrals in (4.12) are of the same form as the quantities I_1, I_2, I_3, and I_4 defined in the introduction in equation (1.12). So our goal is to find an upper bound for each integral and combine these to obtain an upper bound for $N(\sigma,T_1,T_2)$. As T grows larger the main contribution comes from the first integral $\int_{T_1}^{T_2}\log|h(\sigma'+it)|dt$ and by using improved bounds for zeta function, we will improve the upper bound for the first integral. By improving the bound for this first integral, we aim to refine the overall upper bound. We now give an idea how to estimate the first integral.

4.3 First integral

In this section, our goal is to find an upper bound for the first integral:

$$\left(\int_{T_1}^{T_2} \log|h(\sigma'+it)|dt\right).$$

For finding an upper bound for the integral of $\log|h(\sigma'+it)|$, we can find an upper bound for the integral of $|h(\sigma'+it)|$, since by applying equation (2.16), we have:

$$\frac{1}{T_2-T_1}\int_{T_1}^{T_2} \log(|h(\sigma'+it)|)dt \leq \log\left(\frac{1}{T_2-T_1}\int_{T_1}^{T_2}|h(\sigma'+it)|dt\right). \tag{4.13}$$

Based on the definition of $h(s)$, in (4.5), we have $|h(s)| \leq 1+|f_X(s)|^2$. Therefore, by substituting this inequality in equation (4.13), we obtain:

$$\int_{T_1}^{T_2} \log(|h(\sigma'+it)|)dt \leq (T_2-T_1)\log\left(1+\frac{1}{T_2-T_1}\int_{T_1}^{T_2}|f_X(\sigma'+it)|^2 dt\right). \tag{4.14}$$

For $\sigma \geq \frac{1}{2}$, we denote

$$F_X(\sigma,T) = \int_0^T |f_X(\sigma+it)|^2 dt. \tag{4.15}$$

To resume, we need to find a good bound for $\int_{T_1}^{T_2}|f_X(\sigma'+it)|^2 dt$ which can be expressed as $F_X(\sigma',T_2)-F_X(\sigma',T_1)$. Notably, this difference is bounded above by $F_X(\sigma',T_2)$, allowing us to focus on obtaining an upper bound for $F_X(\sigma,T_2)$.

4.3.1 $F_X(\sigma,T)$

In this section, we apply Theorem 2.22 to find an upper bound for

$$F_X(\sigma,T)=\int_0^T|f(\sigma+it)|^2dt.$$

To do this we must obtain the needed ingredient. First we will take

$$\sigma_1=\frac{1}{2}\quad\text{and}\quad\sigma_2=1+\frac{\delta_1}{\log X},$$

with $\delta_1>0$. We shall use the smoothed weight function g, be as defined in (2.35):

$$g(s)=\left(\frac{s-1}{s}\right)^2e^{\alpha(\frac{s}{T_2})^\beta}. \tag{4.16}$$

Remark 4.5. The term $(\frac{s-1}{s})^2$ in the smoothed weight function $g(s)$ ensures the integral is well-behaved near $s=1$, where the Riemann zeta function has a simple pole. This modification ensures that the multiplication of zeta function and the weight function $g(s)$ is analytic so that we can apply smoothing method.

Let $\alpha>0$ and $\beta=2$. For $\sigma_1\le\sigma\le\sigma_2$, the weight function, g satisfies (2.17) and (2.20):

$$\begin{gathered}C_2(\sigma,T_1,T_2,\alpha,\beta)=\min_{t\in[T_1,T_2]}|g(\sigma+it)|,\qquad\text{where}\qquad C_2\neq0\\|g(\sigma+it)|\le C_1(\sigma,T_2,\alpha,\beta)e^{-\alpha(\frac{|t|}{T_2})^2}\qquad\text{for all }t,\end{gathered} \tag{4.17}$$

with

$$\begin{aligned}C_1(\sigma,T_2,\alpha,\beta)&=e^{\alpha(\frac{\sigma}{T_2})^2},\\C_2(\sigma,T_1,T_2,\alpha,\beta)&=\left(1-\frac{1}{T_1}\right)^2e^{\alpha(\frac{\sigma}{T_2})^2-\alpha}.\end{aligned} \tag{4.18}$$

We will obtain necessary bounds on $F(\sigma_1,T)$ and $F(\sigma_2,T)$ in Sections 4.3.2 and 4.3.3. We have three different versions for $F(\sigma_1,T)$ as stated in Lemma 4.17:

$$F_X(\frac{1}{2},T) \le D_4\left(T^{\frac{1}{3}}\log^2 T+\frac{\omega_7}{\omega_1^2}\right)\left(T+\frac{D_2}{D_1}X\right)(\log X)\left(1+\frac{1}{T^{\frac{1}{6}}\sqrt{D_1\log X}}\right)^2, \tag{4.19}$$

$$F_X(\frac{1}{2},T) \le D_4'\left(T^{\frac{27}{82}}+\frac{\omega_7}{\omega_5^2}\right)\left(T+\frac{D_2}{D_1}X\right)(\log X)\left(1+\frac{1}{\omega_5 T^{\frac{27}{164}}\sqrt{D_1\log X}}\right)^2, \tag{4.20}$$

$$F_X(\frac{1}{2},T) \le D_4''\left(\left(T^{\frac{1}{6}}\log T+\frac{\omega_3}{\omega_2}T^{\frac{1}{6}}-\frac{\omega_4}{\omega_2}\right)^2+\frac{\omega_8}{\omega_2^2}\right)\left(T+\frac{D_2}{D_1}X\right)(\log X)\left(1+\frac{1}{T^{\frac{1}{6}}\sqrt{D_1\log X}}\right)^2. \tag{4.21}$$

We obtain the following bound for $F(\sigma_2,t)$ as stated in Lemma 4.21:

$$F_X(\sigma_2,T) \le \left(D_5(\delta_1)+\frac{D_6(\delta_1)(T+\pi m_0)}{X}\right)(\log X)^2. \tag{4.22}$$

Then we provide a general form of the final bound in Theorem 4.23:

$$F(\sigma',T_1,T_2) \le C_4\tilde{C}_5C_6T_2^{2a(1-\sigma)+(a'-1)(2\sigma-1)}(\log T_2)^{2(b+1)(1-\sigma)+(b'+2)(2\sigma-1)}. \tag{4.23}$$

In the next two sections we will provide details to find an upper bound for $F_X(\frac{1}{2},T)$ and $F_X(1+\frac{\delta_1}{\log X},T)$.

4.3.2 Bounding $F_X(\frac{1}{2},T)$

In this section we are bounding $F_X(\frac{1}{2},T)$. Based on the definition of $f(s)$ in (4.3) we have,

$$\int_0^T |f(\tfrac{1}{2}+it)|^2dt = \int_0^T |\zeta(\tfrac{1}{2}+it)M_X(\tfrac{1}{2}+it)-1|^2dt. \tag{4.24}$$

Using Minkowski's inequality, $\|x+y\|_p \le \|x\|_p+\|y\|_p$, to obtain:

$$\sqrt{\int_0^T |\zeta(\frac{1}{2}+it)M_X(\frac{1}{2}+it)-1|^2dt} \le \sqrt{\int_0^T |\zeta(\frac{1}{2}+it)M_X(\frac{1}{2}+it)|^2dt}+\sqrt{T}. \tag{4.25}$$

So here we need to bound the second moment of $M_X(\frac{1}{2}+it)$, where M_X is defined in equation (4.1) and also bound the Riemann zeta function on the critical line.

We start with a lemma from [26] to find an upper bound for the integral, $\int_0^T \left|M_X(\frac{1}{2}+it)\right|^2 dt$:

Lemma 4.6. *Let* $T > 0$ *and* $X \geq 10^9$*. Then*

$$\int_0^T \left|M_X(\tfrac{1}{2}+it)\right|^2 dt \leq (D_1 T + D_2 X)(\log X), \tag{4.26}$$

where

$$\begin{aligned} D_1 = D_1(X) &= \frac{6}{\pi^2} + \frac{b_2(X)}{\log(X)} \\ D_2 = D_2(X) &= \frac{\pi m_0 b_1(X)}{\log(X)} + \frac{6m_0}{\pi X} + \frac{\pi m_0 b_2(X)}{X\log(X)}, \end{aligned} \tag{4.27}$$

and the b_1 *and* b_2 *are defined in equation* (3.12) *and* (3.16) *and* m_0 *in equation* (2.77).

Remark 4.7. The coefficients $D_1(X)$ and $D_2(X)$ are decreasing as X becomes large. Specifically, we have $D_1(X) = O(1)$ and $D_2(X) = O(\frac{1}{\log X})$.

Proof. We recall Lemma 2.31 for any real valued sequence u_n, we have

$$\int_0^T \left|\sum_{n=1}^{\infty} u_n n^{it}\right|^2 dt \leq \sum_{n\geq 1} |u_n|^2 (T + \pi m_0(n+1)).$$

We apply this to $M_X(s) = \sum_{n\leq X} \frac{\mu(n)}{n^s}$. Letting $u_n = \frac{\mu(n)}{n^{\frac{1}{2}}}$, we get:

$$\begin{aligned} \int_0^T \left|M_X(\tfrac{1}{2}+it)\right|^2 dt &\leq \sum_{n\leq X} \frac{\mu^2(n)}{n}(T + \pi m_0(n+1)) \\ &\leq (T + \pi m_0)) \sum_{n\leq X} \frac{\mu^2(n)}{n} + m_0\pi \sum_{n\leq X} \mu^2(n). \end{aligned}$$

Since $X \geq 1700$, we apply equation (3.11) to $\pi m_0 \sum_{n\leq X} \mu^2(n)$ and apply equation (3.16) to

$(T+\pi m_0))\sum_{n\leq X}\frac{\mu^2(n)}{n}$. By combining these obtained bounds, we obtain:

$$\begin{aligned}\int_0^T \left|M_X(\tfrac{1}{2}+it)\right|^2 dt &\leq (T+\pi m_0)\left(b_2(X_0)+\frac{6}{\pi^2}\log X\right)+\pi m_0 b_1(X_0)X\\ &\leq Tb_2(X_0)+\frac{6T}{\pi^2}\log X+\pi m_0 b_2(X_0)+\frac{6m_0}{\pi}\log X+\pi m_0 b_1(X_0)X\\ &\leq T(\log X)\left(\frac{b_2(X_0)}{\log X}+\frac{6}{\pi^2}\right)+X\log X\left(\frac{\pi m_0 b_2(X_0)}{X\log X}+\frac{6m_0}{X\pi}+\frac{\pi m_0 b_1(X_0)}{\log X}\right).\end{aligned}$$

So we obtain the following upper bound for $\int_0^T \left|M_X(\frac{1}{2}+it)\right|^2 dt$:

$$(\log X)\left(T\left(\frac{b_2(X_0)}{\log X}+\frac{6}{\pi_2}\right)+X\left(\frac{\pi m_0 b_2(X_0)}{X\log X}+\frac{6m_0}{X\pi}+\frac{\pi m_0 b_1(X_0)}{\log X}\right)\right). \tag{4.28}$$

Hence we obtain:

$$\int_0^T \left|M_X(\tfrac{1}{2}+it)\right|^2 dt \leq (D_1 T+D_2 X)(\log X),$$

where D_1 and D_2 are as defined in equation (4.27), completing the proof. ☐

Now we just need an upper bound for $|\zeta(\frac{1}{2}+it)|^2$. So we record a number of bounds for the zeta function on the half line.

The next lemma is [15, Theorem 1.1] which is an explicit subconvex bound for the Riemann zeta function ζ on the critical line, proven by Hiary, Patel, and Yang [15]:

Lemma 4.8. *Let* $\omega_1 = 0.618$. *Then we have, for all* $t \geq 3$,

$$|\zeta(\tfrac{1}{2}+it)| \leq \omega_1 t^{\frac{1}{6}}\log t. \tag{4.29}$$

Moreover, by [15, Theorem 1.1, part 3.4] Define $\omega_2 = 0.478013$ *and* $\omega_3 = 3.853165$ *and* $\omega_4 = 2.914229$. *Then, for all* $t > 10^{12}$, *we have*

$$|\zeta(\tfrac{1}{2}+it)| \leq \omega_2 t^{\frac{1}{6}}\log t+\omega_3 t^{\frac{1}{6}}-\omega_4. \tag{4.30}$$

In 2021, Patel [32] improved the exponent of t in the equation (4.29) and, more importantly, removed the $\log t$ factor. This bound is referred to sometimes as a sub-Weyl estimate. Next Lemma is [32, Theorem 3.2]:

Lemma 4.9. *For $t \geq 3$, we have*

$$|\zeta(\tfrac{1}{2}+it)| \leq 307.098 t^{\frac{27}{164}}. \tag{4.31}$$

In 2024, Patel and Yang [33] improved the constant 307.098. The next Lemma is [33, Theorem 1.1]:

Lemma 4.10. *Let $\omega_5 = 66.7$. For $t \geq 3$ we have*

$$|\zeta(\tfrac{1}{2}+it)| \leq \omega_5 t^{\frac{27}{164}}. \tag{4.32}$$

Hiary [14], showed an explicit bound for zeta function ζ on the half line for short intervals. Next Lemma is [14, Theorem 1.1]:

Lemma 4.11. *Let $\omega_6 = 1.461$, When $0 \leq t \leq 3$ we have:*

$$|\zeta(\tfrac{1}{2}+it)| \leq \omega_6. \tag{4.33}$$

Hiary indicates, but does not prove, that one can take $\omega_6 = |\zeta(\frac{1}{2})| = 1.4603...$

Proposition 4.12. *We have*

$$\max_{t\in[0,3]} |\zeta(\tfrac{1}{2}+it)|^2 = |\zeta(\tfrac{1}{2})|^2. \tag{4.34}$$

Proof. Hiary's computational results show that for $t \in [\frac{1}{20}, 3]$, the bound on $|\zeta(\frac{1}{2}+it)|$ is already verified using rigorous interval arithmetic. Therefore, it suffices to focus on the interval $t \in [0, \frac{1}{10}]$. Define the function, $f(t) = \zeta(\frac{1}{2}+it)\zeta(\frac{1}{2}-it)$, which is analytic and satisfies $f(t) = |\zeta(\frac{1}{2}+it)|^2$ for $t \in \mathbb{R}$. To determine the maximum of $f(t) = |\zeta(\frac{1}{2}+it)|^2$, we analyze its critical points and second derivative behavior.

The first derivative $f'(t)$ is given by:

$$f'(t) = i\zeta'(\tfrac{1}{2}+it)\zeta(\tfrac{1}{2}-it) - i\zeta(\tfrac{1}{2}+it)\zeta'(\tfrac{1}{2}-it). \tag{4.35}$$

We trivially see $f'(0) = 0$. To confirm that $t = 0$ is indeed the maximum, we will verify that the second derivative is always negative in the interval $[\frac{-1}{10}, \frac{1}{10}]$.

The second derivative is given by

$$f''(t) = -\zeta''(\tfrac{1}{2}+it)\zeta(\tfrac{1}{2}-it) + \zeta'(\tfrac{1}{2}+it)\zeta'(\tfrac{1}{2}-it) + \zeta'(\tfrac{1}{2}+it)\zeta'(\tfrac{1}{2}-it) - \zeta(\tfrac{1}{2}+it)\zeta''(\tfrac{1}{2}-it). \tag{4.36}$$

Using interval arithmetic we verify that $f''(t) \leq -12$ for $t \in [-\frac{1}{10}, \frac{1}{10}]$ so that there is at most one critical point in this interval and it is a maximum. Thus, the maximum of $|\zeta(\frac{1}{2}+it)|^2$ occurs $t = 0$ which completes the proof. □

Remark 4.13. Kadiri, Ng and Lumley [26] used the following bound

$$|\zeta(\frac{1}{2}+it)| \leq 0.63T^{\frac{1}{6}} \log T + (\sqrt{\omega_7} + 1.39) \qquad 0 \leq t \leq T, \tag{4.37}$$

and consequently they used the following bound for the second moment of zeta function:

$$|\zeta(\frac{1}{2}+it)|^2 \leq (0.63)^2 T^{\frac{1}{3}} \log^2 T + 1.26(\sqrt{\omega_7}+1.39)T^{\frac{1}{6}} \log(T) + (\sqrt{\omega_7}+1.39)^2, \quad 0 \leq t \leq T \tag{4.38}$$

We improved this bound through the following lemma.

Lemma 4.14. *Let* $\omega_7 = \zeta(\frac{1}{2})^2$*. For all* $T > 0$ *we have the following bounds for* $|\zeta(\frac{1}{2}+it)|^2$*: By applying the bounds in equation* (4.29)*, we obtain*

$$|\zeta(\tfrac{1}{2}+it)|^2 \leq \omega_1^2 T^{\frac{1}{3}} \log^2 T + \omega_7, \qquad 0 \leq t \leq T \tag{4.39}$$

with ω_1 *is defined in Lemma* (4.8) *and by applying the bounds in equation* (4.32) *we obtain*

$$|\zeta(\tfrac{1}{2}+it)|^2 \leq \omega_5^2 T^{\frac{27}{82}} + \omega_7, \qquad 0 \leq t \leq T \tag{4.40}$$

with ω_5 *is defined in Lemma 4.10.*

Proof. Based on the Proposition 4.12, we get

$$\max_{t\in[0,3]} |\zeta(\tfrac{1}{2}+it)|^2 = \zeta(\tfrac{1}{2})^2. \tag{4.41}$$

Also, by applying Lemma 4.8, we obtain

$$|\zeta(\tfrac{1}{2}+it)|^2 \leq \omega_1^2 t^{\frac{1}{3}} \log^2 t \qquad \text{for} \quad t \geq 3. \tag{4.42}$$

Since these two are positive, so we can add them with each other. Therefore, we have:

$$|\zeta(\tfrac{1}{2}+it)|^2 \leq \omega_1^2 t^{\frac{1}{3}} \log^2 t + \zeta(\tfrac{1}{2})^2. \tag{4.43}$$

Similarly, by applying equation (4.32), we have

$$|\zeta(\tfrac{1}{2}+it)|^2 \leq \omega_5^2 t^{\frac{27}{82}}. \tag{4.44}$$

Since this is positive, so we can add it with equation (4.41), and obtain:

$$|\zeta(\tfrac{1}{2}+it)|^2 \leq \omega_5^2 t^{\frac{27}{82}} + \zeta(\tfrac{1}{2})^2. \tag{4.45}$$

□

Moreover, we can apply equation (4.30) and find the upper bound for $|\zeta(\frac{1}{2}+it)|^2$ for all positive T.

Lemma 4.15. *Let* ω_2, ω_3 *and* ω_4 *be as defined in Lemma 4.8. For any* $T > 0$ *we have the* *following bound for* $|\zeta(\frac{1}{2}+it)|^2$*:*

$$|\zeta(\tfrac{1}{2}+it)|^2 \leq (\omega_2 T^{\frac{1}{6}} \log T + \omega_3 T^{\frac{1}{6}} - \omega_4)^2 + \omega_8 \qquad 0 \leq t \leq T, \tag{4.46}$$

where $\omega_8 = \omega_1^2(10^4)(12\log 10)^2 - (12\omega_2 10^2 \log 10 + \omega_3 10^2 - \omega_4)^2$.

Proof. Using the bounds from equation (4.30), we find that for $t \geq 10^{12}$:

$$|\zeta(\tfrac{1}{2}+it)|^2 \leq (\omega_2 t^{\frac{1}{6}} \log t + \omega_3 t^{\frac{1}{6}} - \omega_4)^2. \tag{4.47}$$

Applying equation (4.42), we need to show that for all $t < 10^{12}$:

$$\omega_1^2 t^{\frac{1}{3}} \log^2 t - (\omega_2 t^{\frac{1}{6}} \log t + \omega_3 t^{\frac{1}{6}} - \omega_4)^2 \leq \omega_8. \tag{4.48}$$

To determine ω_8, we maximize the following expression:

$$\omega_1^2 t^{\frac{1}{3}} \log^2 t - (\omega_2 t^{\frac{1}{6}} \log t + \omega_3 t^{\frac{1}{6}} - \omega_4)^2. \tag{4.49}$$

We differentiate the expression and therefore, we get:

$$\omega_1^2 t^{-\frac{2}{3}}(\frac{1}{3}\log^2 t + 2\log t) - 2(\omega_2 t^{\frac{1}{6}} \log t + \omega_3 t^{\frac{1}{6}} - \omega_4)(\frac{1}{6}\omega_2 t^{-\frac{5}{6}} \log t + \omega_2 t^{-\frac{5}{6}} + \frac{1}{6}\omega_3 t^{-\frac{5}{6}}). \tag{4.50}$$

Simplifying the expression, we obtain:

$$\frac{1}{3} t^{-\frac{2}{3}}(\omega_1^2(\log^2 t + 6\log t) - (\omega_2 \log t + \omega_3 - \omega_4 t^{-\frac{1}{6}})(\omega_2 \log t + 2\omega_2 + \omega_3)). \tag{4.51}$$

We see the function is obviously increasing for $t \geq 2$, and that ω_8 is larger than any value taken on in $[0,2]$ and the maximum occurs at the endpoint, 10^{12}. By substituting 10^{12} into the expression, we obtain:

$$\omega_1^2(10^4)(12\log 10)^2-(12\omega_2 10^2\log 10+\omega_3 10^2-\omega_4)^2. \tag{4.52}$$

Thus, we choose $\omega_8=\omega_1^2(10^4)(12\log 10)^2-(12\omega_2 10^2\log 10+\omega_3 10^2-\omega_4)^2$ which completes the proof. □

Now with the combination of the bounds of the integral $\int_0^T \left|M(\frac{1}{2}+it)\right|^2 dt$ in equation (4.26) and different bounds with different shapes for zeta function on the half line, we obtain different bounds for $F_X(\frac{1}{2},T)$. By applying different bounds in Lemma 4.14 and Lemma 4.15, we obtain the following three versions for bounds for $F_X(\frac{1}{2},T)$:

Remark 4.16. Kadiri, Ng and Lumley [26] used the following bound for $F_X(\frac{1}{2},T)$:

$$D_4\left(T^{\frac{1}{6}}\log T+\frac{\sqrt{\omega_7}+1.39}{0.618}\right)^2\left(T+\frac{D_2}{D_1}X\right)(\log X)\left(1+\frac{1}{\sqrt{\omega_7 D_1\log X}}\right)^2. \tag{4.53}$$

We improved their bound in in two ways: First, using an improved bound on zeta function from equation (4.39). Next by improving the last term, adding $T^{\frac{1}{6}}$ into the dominator we improved the final bound.

Lemma 4.17. *Let $T>0$ and $X\geq 10^9$. Define ω_1, ω_7 and ω_5 as in Lemma 4.14. Using the bound in equation (4.39), we have*

$$F_X(\frac{1}{2},T)\leq D_4\left(T^{\frac{1}{3}}\log^2 T+\frac{\omega_7}{\omega_1^2}\right)\left(T+\frac{D_2}{D_1}X\right)(\log X)\left(1+\frac{1}{T^{\frac{1}{6}}\sqrt{D_1\log X}}\right)^2. \tag{4.54}$$

Using the bound in equation (4.40), we also have

$$F_X(\frac{1}{2},T)\leq D_4'\left(T^{\frac{27}{82}}+\frac{\omega_7}{\omega_5^2}\right)\left(T+\frac{D_2}{D_1}X\right)(\log X)\left(1+\frac{1}{\omega_5 T^{\frac{27}{164}}\sqrt{D_1\log X}}\right)^2. \tag{4.55}$$

Moreover, using the bound in equation (4.46), we also have

$$F_X(\frac{1}{2},T) \leq D''_4\left(\left(T^{\frac{1}{6}}\log T + \frac{\omega_3}{\omega_2}T^{\frac{1}{6}} - \frac{\omega_4}{\omega_2}\right)^2 + \frac{\omega_8}{\omega_2^2}\right)\left(T + \frac{D_2}{D_1}X\right)(\log X)\left(1 + \frac{1}{T^{\frac{1}{6}}\sqrt{D_1\log X}}\right)^2. \tag{4.56}$$

Here D_1 and D_2 are defined in equation (4.27), and

$$D_4 = D_4(X) = D_1\omega_1^2, \quad D'_4 = D'_4(X) = D_1\omega_5^2 \quad \textit{and} \quad D''_4 = D''_4(X) = D_1\omega_2^2. \tag{4.57}$$

Remark 4.18. For large values of T the second bound in equation (4.55) is better than the other two bounds.

Remark 4.19. The coefficients D_4 and D'_4 and D''_4 are decreasing as X becomes large. Specifically, we have $D_4 = 0.246 + o(1)$ and $D'_4 = 2866.887 + o(1)$ and $D''_4 = 0.147 + o(1)$.

Proof. From equation (4.25), we have

$$\sqrt{|F_X(\tfrac{1}{2},T)|} \leq \sqrt{\int_0^T |\zeta(\tfrac{1}{2}+it)M_X(\tfrac{1}{2}+it)|^2 dt} + \sqrt{T}.$$

Applying known bounds for $\zeta(\frac{1}{2}+it)$ and $M_X(\frac{1}{2}+it)$, we proceed by cases. For the first bound, we use equation (4.39) for $|\zeta(\frac{1}{2}+it)|^2$:

$$|\zeta(\tfrac{1}{2}+it)|^2 \leq \omega_1^2 T^{\frac{1}{3}}\log^2 T + \omega_7 \qquad \text{for all} \quad 0 \leq t \leq T. \tag{4.58}$$

For $M_X(\frac{1}{2}+it)$, from equation (4.26), we have:

$$\int_0^T \left|M_X(\tfrac{1}{2}+it)\right|^2 dt \leq (D_1T + D_2X)\,(\log X).$$

Combining these bounds yields:

$$\sqrt{\left|F_X(\tfrac{1}{2},T)\right|} \leq \sqrt{(\omega_1^2 T^{\frac{1}{3}}\log^2 T + \omega_7)\,(D_1T + D_2X)\,(\log X)} + \sqrt{T}.$$

Define:

$$K = (\omega_1^2 T^{\frac{1}{3}} \log^2 T + \omega_7)\,(D_1 T + D_2 X)\,(\log X). \tag{4.59}$$

Then,

$$\left|F_X(\tfrac{1}{2}, T)\right| \leq \left(\sqrt{K} + \sqrt{T}\right)^2 = K\left(1 + \frac{\sqrt{T}}{\sqrt{K}}\right)^2. \tag{4.60}$$

We note that $\omega_1^2 T^{\frac{1}{3}} \log^2 T + \omega_7 > T^{\frac{1}{3}}$ for $T > 0$. So

$$K \geq T^{\frac{1}{3}}\,(D_1 T + D_2 X) \log X \geq D_1 T^{\frac{4}{3}} \log X. \tag{4.61}$$

Since $T > 0$, we have:

$$\frac{K}{T} \geq T^{\frac{1}{3}} D_1 \log X. \tag{4.62}$$

which implies

$$\frac{\sqrt{T}}{\sqrt{K}} \leq \frac{1}{T^{\frac{1}{6}} \sqrt{D_1 \log X}}. \tag{4.63}$$

By factoring out $D_1 \omega_1^2$, we can rewrite the function K as:

$$K = D_1 \omega_1^2 \left(T^{\frac{1}{3}} \log^2 T + \frac{\omega_7}{\omega_1^2}\right)\left(T + \frac{D_2}{D_1} X\right)(\log X). \tag{4.64}$$

Using the bound on $\frac{\sqrt{T}}{\sqrt{K}}$, we obtain:

$$F_X(\frac{1}{2}, T) \leq D_1 \omega_1^2 \left(T^{\frac{1}{3}} \log^2 T + \frac{\omega_7}{\omega_1^2}\right)\left(T + \frac{D_2}{D_1} X\right)(\log X)\left(1 + \frac{1}{T^{\frac{1}{6}} \sqrt{D_1 \log X}}\right)^2. \tag{4.65}$$

Setting $D_4 = D_1 \omega_1^2$, this completes the proof for the first bound (4.54).

For the second bound, we use equation (4.40), which gives:

$$|\zeta(\tfrac{1}{2} + it)|^2 \leq \omega_5^2 T^{\frac{27}{82}} + \omega_7, \qquad 0 \leq t \leq T \tag{4.66}$$

Using the same bound for $M_X(\frac{1}{2}+it)$, we define

$$K' = (\omega_5^2 T^{\frac{27}{82}} + \omega_7)\,(D_1T + D_2X)\,(\log X). \tag{4.67}$$

We observe that $\omega_5^2 T^{\frac{27}{82}} + \omega_7 > \omega_5^2 T^{\frac{27}{82}}$, so we found the below lower bound for K'

$$K' \geq \omega_5^2 T^{\frac{27}{82}}\,(D_1T + D_2X)\log X \geq D_1\omega_5^2 T^{\frac{109}{82}} \log X. \tag{4.68}$$

Therefore, we have the following bound for the second version:

$$F_X(\frac{1}{2},T) \leq D_1\omega_5^2 \left(T^{\frac{27}{82}} + \frac{\omega_7}{\omega_5^2}\right)\left(T + \frac{D_2}{D_1}X\right)(\log X)\left(1 + \frac{1}{\omega_5 T^{\frac{27}{164}}\sqrt{D_1 \log X}}\right)^2. \tag{4.69}$$

This completes the proof for the second bound (4.55).

For the third bound, we use equation (4.46), which gives:

$$|\zeta(\frac{1}{2}+it)|^2 \leq (\omega_2 T^{\frac{1}{6}} \log T + \omega_3 T^{\frac{1}{6}} - \omega_4)^2 + \omega_8 \qquad 0 \leq t \leq T. \tag{4.70}$$

Using the same bound for $M_X(\frac{1}{2}+it)$, we define

$$K'' = ((\omega_2 T^{\frac{1}{6}} \log T + \omega_3 T^{\frac{1}{6}} - \omega_4)^2 + \omega_8)\,(D_1T + D_2X)\,(\log X). \tag{4.71}$$

Obviously, we have the same lower bound for K'' as the bound that K has and therefore we obtain:

$$F_X(\frac{1}{2},T) \leq D_1\omega_2^2\left(\left(T^{\frac{1}{6}}\log T + \frac{\omega_3}{\omega_2}T^{\frac{1}{6}} - \frac{\omega_4}{\omega_2}\right)^2 + \frac{\omega_8}{\omega_2^2}\right)\left(T + \frac{D_2}{D_1}X\right)(\log X)\left(1 + \frac{1}{T^{\frac{1}{6}}\sqrt{D_1 \log X}}\right)^2. \tag{4.72}$$

So this completes the proof for the third bound (4.56). □

We see that $F_X(\frac{1}{2},t) < dt^a(\log t)^b \sum_i d_i t^{a_i}(\log t)^{b_i}$, with

$$d = D_4 \log(X), \quad a = \frac{4}{3}, \quad b = 2,$$

where D_4 is defined in equation (4.42). Below is the table of values of $\sum_i d_i t^{a_i}(\log t)^{b_i}$ by using the first obtained bound in (4.54):

Table 4.1: Values of a_i, b_i, and d_i for $F_X(\frac{1}{2},t) < dt^a(\log t)^b \sum_i d_i t^{a_i}(\log t)^{b_i}$ for equation (4.54).

i	a_i	b_i	d_i
1	0	0	1
2	$-\frac{1}{6}$	0	$\frac{2}{\sqrt{D_1 \log X}}$
3	$-\frac{1}{3}$	0	$\frac{1}{D_1 \log X}$
4	$-\frac{1}{3}$	-2	$\frac{\omega_7}{\omega_1^2}$
5	$-\frac{1}{2}$	-2	$\frac{\omega_7}{\omega_1^2\sqrt{D_1 \log X}}$
6	$-\frac{2}{3}$	-2	$\frac{\omega_7}{2\omega_1^2 D_1 \log X}$
7	-1	0	$\frac{XD_2}{D_1}$
8	$-\frac{7}{6}$	0	$\frac{2XD_2}{D_1\sqrt{D_1 \log X}}$
9	$-\frac{4}{3}$	0	$\frac{XD_2}{D_1^2 \log X}$
10	$-\frac{4}{3}$	-2	$\frac{\omega_7 D_2 X}{2\omega_1^2 D_1}$
11	$-\frac{3}{2}$	-2	$\frac{\omega_7 D_2 X}{\omega_1^2 D_1\sqrt{D_1 \log X}}$
12	$-\frac{5}{3}$	-2	$\frac{\omega_7 D_2 X}{\omega_1^2 D_1^2 \log X}$

where ω_1 and ω_7 are defined in Lemma 4.8 and Lemma 4.14 and the constants D_1 and D_2 are defined in equation (4.27). We see that $F_X(\frac{1}{2},t) < dt^a(\log t)^b \sum_i d_i t^{a_i}(\log t)^{b_i}$ with

$$d = D_4' \log(X), \quad a = \frac{109}{82}, \quad b = 0.$$

where D_4' is defined in equation (4.42). Below is the table of values of $\sum_i d_i t^{a_i}(\log t)^{b_i}$ by using the second obtained bound in (4.55):

Table 4.2: Values of a_i, b_i, and d_i for $F_X(\frac{1}{2},t) < dt^a(\log t)^b \sum_i d_i t^{a_i}(\log t)^{b_i}$ for equation (4.55).

i	a_i	b_i	d_i
1	0	0	1
2	$-\frac{27}{164}$	0	$\frac{2}{\omega_5\sqrt{D_1 \log X}}$
3	$-\frac{27}{82}$	0	$\frac{\omega_7}{\omega_5^2}$
4	$-\frac{27}{82}$	0	$\frac{1}{D_1\omega_5^2 \log X}$
5	$-\frac{81}{164}$	0	$\frac{2\omega_7}{\omega_5^3\sqrt{D_1 \log X}}$
6	$-\frac{109}{164}$	0	$\frac{\omega_7}{\omega_5^4 D_1 \log X}$
7	-1	0	$\frac{XD_2}{D_1}$
8	$-\frac{191}{164}$	0	$\frac{2XD_2}{D_1\sqrt{D_1 \log X}}$
9	$-\frac{109}{82}$	0	$\frac{XD_2\omega_7}{D_1\omega_5^2}$
10	$-\frac{109}{82}$	0	$\frac{D_2X}{2D_1^2\omega_5^2 \log X}$
11	$-\frac{164}{81}$	0	$\frac{2\omega_7 D_2 X}{\omega_5^3 D_1\sqrt{D_1 \log X}}$
12	$-\frac{136}{82}$	0	$\frac{\omega_7 D_2 X}{2\omega_5^4 D_1^2 \log X}$

where ω_1, ω_5 and ω_7 are defined in Lemma 4.8, Lemma 4.10 and Lemma 4.14, and the constants D_1 and D_2 are defined in equation (4.27).

We see that $F_X(\frac{1}{2},t) < dt^a(\log t)^b \sum_i d_i t^{a_i}(\log t)^{b_i}$ with

$$d = D_4'' \log(X), \quad a = \frac{4}{3}, \quad b = 2.$$

Below is the table of values of by using the third obtained bound in (4.56):

4.3.3 Bounding $F_X(1+\frac{\delta_1}{\log X},T)$

In this section we are going to bound $F_X(1+\frac{\delta_1}{\log X},T)$. Based on the definition of $F_X(\sigma,T)$ in equation (4.15) and using the expansion of f_X in equation (4.4), we can find an upper bound for $F_X(1+\frac{\delta_1}{\log X},T)$.

Table 4.3: Values of a_i, b_i, and d_i for $F_X(\frac{1}{2},t) < dt^a(\log t)^b \sum_i d_i t^{a_i}(\log t)^{b_i}$ for equation (4.56).

i	a_i	b_i	d_i
1	0	0	1
2	0	-1	$\frac{2\omega_3}{\omega_2}$
3	0	-2	$\frac{\omega_3^2}{\omega_2^2}$
4	$-\frac{1}{6}$	0	$\frac{2}{\sqrt{D_1 \log(X)}}$
5	$-\frac{1}{6}$	-1	$\frac{4\omega_3}{\omega_2\sqrt{D_1 \log(X)}}$
6	$-\frac{1}{6}$	-1	$-\frac{2\omega_4}{\omega_2}$
7	$-\frac{1}{6}$	-2	$\frac{2\omega_3^2}{\omega_2^2\sqrt{D_1 \log(X)}}$
8	$-\frac{1}{6}$	-2	$-\frac{2\omega_3\omega_4}{\omega_2^2}$
9	$-\frac{1}{3}$	0	$\frac{1}{D_1 \log(X)}$
10	$-\frac{1}{3}$	-1	$-\frac{4\omega_4}{\omega_2\sqrt{D_1 \log(X)}}$
11	$-\frac{1}{3}$	-1	$\frac{2\omega_3}{D_1 \log(X)\omega_2}$
12	$-\frac{1}{3}$	-2	$-\frac{4\omega_3\omega_4}{\omega_2^2\sqrt{D_1 \log(X)}}$
13	$-\frac{1}{3}$	-2	$\frac{\omega_4^2}{\omega_2^2}$
14	$-\frac{1}{3}$	-2	$\frac{\omega_8}{\omega_2^2}$
15	$-\frac{1}{3}$	-2	$\frac{\omega_3^2}{D_1 \log(X)\omega_2^2}$
16	$-\frac{1}{2}$	-1	$-\frac{2\omega_4}{D_1 \log(X)\omega_2}$
17	$-\frac{1}{2}$	-2	$\frac{2\omega_4^2}{\omega_2^2\sqrt{D_1 \log(X)}}$
18	$-\frac{1}{2}$	-2	$\frac{2\omega_8}{\omega_2^2\sqrt{D_1 \log(X)}}$
19	$-\frac{1}{2}$	-2	$-\frac{2\omega_3\omega_4}{D_1 \log(X)\omega_2^2}$
20	$-\frac{2}{3}$	-2	$\frac{\omega_4^2}{D_1 \log(X)\omega_2^2}$
21	$-\frac{2}{3}$	-2	$\frac{\omega_8}{D_1 \log(X)\omega_2^2}$
22	-1	0	$\frac{D_2X}{D_1}$
23	-1	-1	$\frac{2D_2X\omega_3}{D_1\omega_2}$
24	-1	-2	$\frac{D_2X\omega_3^2}{D_1\omega_2^2}$
25	$-\frac{7}{6}$	0	$\frac{2D_2X}{D_1\sqrt{D_1 \log(X)}}$
26	$-\frac{7}{6}$	-1	$\frac{4D_2X\omega_3}{D_1\omega_2\sqrt{D_1 \log(X)}}$
27	$-\frac{7}{6}$	-1	$-\frac{2D_2X\omega_4}{D_1\omega_2}$
28	$-\frac{7}{6}$	-2	$\frac{2D_2X\omega_3^2}{D_1\omega_2^2\sqrt{D_1 \log(X)}}$
29	$-\frac{7}{6}$	-2	$-\frac{2D_2X\omega_3\omega_4}{D_1\omega_2^2}$
30	$-\frac{4}{3}$	0	$\frac{D_2X}{D_1^2 \log(X)}$
31	$-\frac{4}{3}$	-1	$-\frac{4D_2X\omega_4}{D_1\omega_2\sqrt{D_1 \log(X)}}$
32	$-\frac{4}{3}$	-1	$\frac{2D_2X\omega_3}{D_1^2 \log(X)\omega_2}$
33	$-\frac{4}{3}$	-2	$-\frac{4D_2X\omega_3\omega_4}{D_1\omega_2^2\sqrt{D_1 \log(X)}}$
34	$-\frac{4}{3}$	-2	$\frac{D_2X\omega_4^2}{D_1\omega_2^2}$
35	$-\frac{4}{3}$	-2	$\frac{D_2X\omega_8}{D_1\omega_2^2}$
36	$-\frac{4}{3}$	-2	$\frac{D_2X\omega_3^2}{D_1^2 \log(X)\omega_2^2}$
37	$-\frac{3}{2}$	-1	$-\frac{2D_2X\omega_4}{D_1^2 \log(X)\omega_2}$
38	$-\frac{3}{2}$	-2	$\frac{2D_2X\omega_4^2}{D_1\omega_2^2\sqrt{D_1 \log(X)}}$
39	$-\frac{3}{2}$	-2	$\frac{2D_2X\omega_8}{D_1\omega_2^2\sqrt{D_1 \log(X)}}$
40	$-\frac{3}{2}$	-2	$-\frac{2D_2X\omega_3\omega_4}{D_1^2 \log(X)\omega_2^2}$
41	$-\frac{5}{3}$	-2	$\frac{D_2X\omega_4^2}{D_1^2 \log(X)\omega_2^2}$
42	$-\frac{5}{3}$	-2	$\frac{D_2X\omega_8}{D_1^2 \log(X)\omega_2^2}$

Remark 4.20. Kadiri, Lumley and Ng [26] used the following bound

$$F_X(\sigma_2,T) \leq \left(D_5(\delta_1,X) + \frac{D_6(\delta_1,X)(T+\pi m_0)}{X}\right)(\log X)^2, \tag{4.73}$$

where,

$$\begin{aligned} D_5(\delta_1, X) &= \frac{\pi m_0 b_4(X)}{2\delta_1}\left(1+\frac{2\delta_1}{\log X}\right)^2 e^{\frac{2\delta_1\gamma}{\log(X)}} \\ D_6(\delta_1, X) &= \frac{b_4(X)}{5\delta_1 e^{\gamma}}\left(1+\frac{\delta_1}{\log X}\right)^2 + \frac{b_3(X)e^{-2\delta_1}}{(\log(X))^2} \end{aligned} \tag{4.74}$$

We improved their bound by improving both constants, $D_5(\delta_1, X)$ and $D_6(\delta_1, X)$. See the following lemma for improvement:

Lemma 4.21. *Let* $T, \delta_1 > 0$, $X > 10^9$ *and* $\sigma_2 = 1 + \frac{\delta_1}{\log X}$. *Then,*

$$F_X(\sigma_2, T) \le \left(D_5(\delta_1, X) + \frac{D_6(\delta_1, X)(T + \pi m_0)}{X}\right)(\log X)^2, \tag{4.75}$$

where,

$$\begin{aligned} D_5(\delta_1, X) &= \frac{\pi m_0 b_4(X)}{2\delta_1} e^{\frac{2\delta_1\gamma}{\log(X)}} \\ D_6(\delta_1, X) &= \frac{b_4(X)}{5\delta_1 e^{\gamma}} + \frac{b_3(X)e^{-2\delta_1}}{(\log(X))^2}. \end{aligned} \tag{4.76}$$

The $b_3(X)$ *and* $b_4(X)$ *are defined in Lemmas 3.9 and 3.6. The constant* m_0 *is defined in* (2.77) *and* γ *is the Euler's constant.*

Remark 4.22. Both constants $D_5(\delta_1, X)$ and $D_6(\delta_1, X)$ are $O(1)$ and are decreasing in X.

Proof. Given that we have

$$F_X(\sigma, T) = \int_0^T |f_X(\sigma + it)|^2 \, dt.$$

Since we have

$$f_X(s) = \sum_{n > X}\left(\sum_{d|n} \mu(d)\right) n^{-s} = \sum_{n \ge 1} \frac{\lambda_X(n)}{n^s},$$

with

$$\lambda_X(n) = \begin{cases} 0 & \text{if } n \le X, \\ \sum_{\substack{d|n \\ d \le X}} \mu(d) & \text{if } n > X. \end{cases}$$

We can rewrite the function $F_X(\sigma_2, T)$:

$$F_X(\sigma_2, T) = \int_0^T |f_X(\sigma_2 + it)|^2\, dt = \int_0^T \left| \sum_{n \geq 1} \frac{\lambda_X(n)}{n^{\sigma_2 + it}} \right|^2 dt. \tag{4.77}$$

So we need to find an upper bound for the integral $\int_0^T \left| \sum_{n \geq 1} \frac{\lambda_X(n)}{n^{\sigma_2 + it}} \right|^2 dt$. We apply equation (2.78) to this summation. Letting $u_n = \frac{\lambda_X(n)}{n^{\sigma_2}}$, we get:

$$\begin{aligned} \int_0^T \left| \sum \lambda_X(n) n^{-(\sigma_2 + it)} \right|^2 dt &\leq \sum_{n \geq 1} \frac{\lambda_X(n)^2}{n^{2\sigma_2}} (T + \pi m_0 (n+1)) \\ &\leq \sum_{n \geq 1} \frac{\lambda_X(n)^2}{n^{2\sigma_2}} (T + \pi m_0 n + \pi m_0) \\ &\leq \pi m_0 n \sum_{n \geq 1} \frac{\lambda_X(n)^2}{n^{2\sigma_2}} + (T + \pi m_0) \sum_{n \geq 1} \frac{\lambda_X(n)^2}{n^{2\sigma_2}} \\ &\leq \pi m_0 \sum_{n \geq 1} \frac{\lambda_X(n)^2}{n^{2\sigma_2 - 1}} + (T + \pi m_0) \sum_{n \geq 1} \frac{\lambda_X(n)^2}{n^{2\sigma_2}}. \end{aligned} \tag{4.78}$$

For $2\sigma_2 - 1 = 1 + \frac{2\delta_1}{\log X}$ and $2\sigma_2 = 2 + \frac{2\delta_1}{\log X}$, we apply the bounds for arithmetic sums (3.61) and (3.63) to respectively bound the two above sums. Thus,

$$\sum_{n \geq 1} \frac{\lambda_X(n)^2}{n^{1 + 2\frac{\delta_1}{\log X}}} \leq \frac{b_4}{2\delta_1} e^{\frac{2\delta_1 \gamma}{\log X}} (\log X)^2, \tag{4.79}$$

and

$$\sum_{n \geq 1} \frac{\lambda_X(n)^2}{n^{2 + \frac{2\delta_1}{\log x}}} \leq \frac{b_4}{5\delta_1 e^{\delta_1}} e^{\frac{\delta_1(\gamma - \log 5)}{\log X}} \frac{(\log X)^2}{X} + \frac{b_3 e^{-2\delta_1}}{X}. \tag{4.80}$$

Since the function $e^{\frac{\delta_1(\gamma - \log 5)}{\log X}}$ is increasing and bounded by 1, we have:

$$\sum_{n \geq 1} \frac{\lambda_X(n)^2}{n^{2 + \frac{2\delta_1}{\log x}}} \leq \frac{b_4}{5\delta_1 e^{\delta_1}} \frac{(\log X)^2}{X} + \frac{b_3 e^{-2\delta_1}}{X}. \tag{4.81}$$

Equations (4.79) and (4.81), yield the following bound for $\int_0^T \left|\sum_{n\geq 1} \frac{\lambda_X(n)}{n^{\sigma_2+it}}\right|^2 dt$:

$$\pi m_0 \left(\frac{b_4}{2\delta_1} e^{\frac{2\delta_1\gamma}{\log X}} (\log X)^2\right) + (T+\pi m_0)\left(\frac{b_4}{5\delta_1 e^{\delta_1}} \frac{(\log X)^2}{X} + \frac{b_3 e^{-2\delta_1}}{X}\right). \tag{4.82}$$

Simplifying the expression, we obtain:

$$\left[\frac{\pi m_0 b_4}{2\delta_1} e^{\frac{2\delta_1\gamma}{\log X}} + \frac{(T+\pi m_0)}{X}\left(\frac{b_4}{5\delta_1 e^{\delta_1}} + \frac{b_3 e^{-2\delta_1}}{(\log X)^2}\right)\right] (\log X)^2, \tag{4.83}$$

which completes the proof. ☐

We see that $F_X(1+\frac{\delta_1}{\log X}, t) < d' t^{a'} (\log t)^{b'} \sum_i' d_i' t^{a_i'} (\log t)^{b_i'}$ with

$$d' = \frac{D_6(\delta_1, X)\log^2(X)}{X}, \quad a' = 1, \quad b' = 0.$$

where $D_6(\delta_1, X)$ is defined in equation (4.76). Below is the table of values of $\sum_i' d_i' t^{a_i'} (\log t)^{b_i'}$ by using the obtained bound:

Table 4.4: Values of a_i', b_i', and d_i' for $F_X(\frac{1}{2}, t) < d' t^{a'} (\log t)^{b'} \sum_{i'} d_i' t^{a_i'} (\log t)^{b_i'}$ in equation (4.75).

i'	a_i'	b_i'	d_i'
1	0	0	1
2	-1	0	$\frac{D_5(\delta_1,X)X}{D_6(\delta_1,X)}$
3	-1	0	πm_0

4.3.4 Bounding $F_X(\sigma, T_1, T_2)$

In this section our goal is to find an upper bound for $F_X(\sigma, T)$ by the obtained bounds for $F_X(\frac{1}{2}, T)$ and $F_X(1+\frac{\delta_1}{\log X})$.

Theorem 4.23. *Fix T_1 and T_2 with $T_2 > X_0 > 10^9$ and set $X = T_2$. Let $\delta_1, \delta_2 \in (0,1]$. Suppose that $\sigma' = \sigma - \frac{\delta_2}{\log T_2}$ with $\sigma \geq \frac{1}{2} + \frac{\delta_2}{\log T_2}$. Assume the weight function g be defined as in equation (4.16) with upper and lower bounds specified in equation (4.17). Let $\alpha \in (0,1)$ and set $\beta = 2$.*

Suppose the following bounds hold for all t:

$$F_X(\frac{1}{2}, t) \leq d t^a (\log t)^b \sum_i d_i t^{a_i} (\log t)^{b_i}, \tag{4.84}$$

and

$$F_X(1 + \frac{\delta_1}{\log X}, t) \leq d' t^{a'} (\log t)^{b'} \sum_{i'} d'_i t^{a'_i} (\log t)^{b'_i}, \tag{4.85}$$

where $\sum_i d_i t^{a_i} (\log t)^{b_i} = 1 + o(1)$ and $\sum_{i'} d'_i t^{a'_i} (\log t)^{b'_i} = 1 + o(1)$. Then, the following bound holds:

$$F(\sigma', T_1, T_2) \leq C_4 \tilde{C_5} C_6 T_2^{2a(1-\sigma)+(a'-1)(2\sigma-1)} (\log T_2)^{2(b+1)(1-\sigma)+(b'+2)(2\sigma-1)}, \tag{4.86}$$

where $C_4 = C_4(\sigma', T_2, T_1, \alpha, \beta, \delta_1, \delta_2)$ is given by:

$$C_4 = \alpha\beta \frac{C_1(\sigma_1, T_2, \alpha, \beta)^{(\frac{\sigma_2 - \sigma'}{\sigma_2 - \sigma_1})} C_1(\sigma_2, T_2, \alpha, \beta)^{(\frac{\sigma' - \sigma_1}{\sigma_2 - \sigma_1})}}{C_2(\sigma', T_1, T_2, \alpha)}, \tag{4.87}$$

where C_1 and C_2 are defined in equation (4.18).

The constant $\tilde{C_5} = \tilde{C_5}(\sigma', T_2, \alpha, \beta, \delta_1, \delta_2, a, a_i, b, b_i, d_i, a', a'_i, b', b'_i, d'_i)$ is given by

$$\left(\frac{d}{\log T_2} C_3(\sigma_1, T_2, \alpha, \beta, a, a_i, b, b_i, d_i)\right)^{(\frac{\sigma_2 - \sigma'}{\sigma_2 - \sigma_1})} \left(\frac{d' T_2}{(\log T_2)^2} C_3(\sigma_2, T_2, \alpha, \beta, a', a'_i, b', b'_i, d'_i)\right)^{(\frac{\sigma' - \sigma_1}{\sigma_2 - \sigma_1})}, \tag{4.88}$$

where the constant C_3 is defined in equation (2.25) and any of parameters a, a_i, b, b_i, d_i from Tables 4.1, 4.2 and 4.3, and a', a'_i, b', b'_i, d'_i from Table 4.4.

The constant $C_6 = C_6(\sigma', T_2, \delta_1, \delta_2, a, a', b, b')$ is given by

$$C_6 = \frac{T_2^{a(\frac{\sigma_2-\sigma'}{\sigma_2-\sigma_1})+(a'-1)(\frac{\sigma'-\sigma_1}{\sigma_2-\sigma_1})} \log T_2^{(b+1)(\frac{\sigma_2-\sigma'}{\sigma_2-\sigma_1})+(b'+2)(\frac{\sigma'-\sigma_1}{\sigma_2-\sigma_1})}}{T_2^{2a(1-\sigma)+(a'-1)(2\sigma-1)} \log T_2^{2(b+1)(1-\sigma)+(b'+2)(2\sigma-1)}}. \tag{4.89}$$

Proof. We suppose that $X = T_2$ and then apply Theorem 2.22 to obtain the bound on $F(\sigma, T_1, T_2)$. We factor out powers of X and $\log(X)$ to obtain $\tilde{C_5}$ and we replace $\sigma_2 = 1 + \frac{\delta_1}{\log(X)}$ and add the term C_6 to account for the corresponding adjustment to the main term. These three constants will be analyzed in detail in the following propositions. $\square$

Proposition 4.24. *Fix T_1 and T_2 with $T_2 > X_0 > 10^9$ and set $X = T_2$. Let $\delta_1, \delta_2 \in (0,1]$. Suppose that $\sigma' = \sigma - \frac{\delta_2}{\log T_2}$ with $\frac{1}{2} \leq \sigma' < \sigma \leq 1$. Let $\alpha \in (0,1)$ and take $\beta = 2$, then the constant C_4 as defined in equation (4.87) is $O(1)$ and decreasing in T_2, where the constants C_1 and C_2 are defined in equation (4.18).*

In addition, C_4 is concave down at σ and we have:

$$\max_{\sigma \in [\sigma_1, \sigma_2]} C_4(\sigma, T_2, T_1, \alpha, \beta, \delta_1, \delta_2) \leq \hat{C_4}(T_2, T_1, \alpha, \beta, \delta_1) := \left(1 - \frac{1}{T_1}\right)^{-2} \alpha\beta e^{\alpha} \cdot e^{\frac{\alpha}{4(X_0)^2}(\sigma_1+\sigma_2)^2}, \tag{4.90}$$

where $\sigma_1 = \frac{1}{2}$ and $\sigma_2 = 1 + \frac{\delta_1}{\log T_2}$.

Proof. Based on the definition of C_4 in equation (4.87) and C_1 and C_2 in equation (4.18) we have:

$$C_4 \leq \alpha\beta e^{\alpha} \cdot M, \tag{4.91}$$

where,

$$M = \left(1 - \frac{1}{T_1}\right)^{-2} e^{\frac{\alpha}{T_2^2}\left(\sigma_1^2(\frac{\sigma_2-\sigma'}{\sigma_2-\sigma_1})+\sigma_2^2(\frac{\sigma'-\sigma_1}{\sigma_2-\sigma_1})-(\sigma')^2\right)}. \tag{4.92}$$

which is approximately equals to 1 and is decreasing as T_1 and T_2 increase.

We see from the above that C_4 is concave down in σ' and the maximum occurs at $\sigma' = \frac{\sigma_1+\sigma_2}{2}$. By substitution of this critical point in equation (4.92), we obtain the maximum value of

$$\left(1 - \frac{1}{T_1}\right)^{-2} \alpha\beta e^{\alpha} \cdot e^{\frac{\alpha}{4(X_0)^2}(\sigma_1+\sigma_2)^2},$$

which completes the proof. $\square$

Remark 4.25. For instance for large T_1 and T_2 we have

$$\left(1-\frac{1}{T_1}\right)^{-2}\alpha\beta e^{\alpha}\cdot e^{\frac{\alpha}{4(X_0)^2}(\sigma_1+\sigma_2)^2}\approx\alpha\beta e^{\alpha}.$$

Remark 4.26. For $\sigma = 0.9$ and $\alpha = 0.241$ and $\beta = 2$ we have

$$C_4(\sigma', X_0, H_0, \alpha, \beta, \delta_1, \delta_2) \approx 0.6133.$$

Remark 4.27. For any $\sigma_1' > \frac{(\sigma_1+\sigma_2)}{2}$ we have

$$\max_{\sigma\in[\sigma_1',\sigma_2']} C_4(\sigma, X_0, H_0, \alpha, \beta, \delta_1, \delta_2) \le C_4(\sigma_1', X_0, H_0, \alpha, \beta, \delta_1, \delta_2). \tag{4.93}$$

This provides no benefit in our application so we use the simpler formula.

Remark 4.28. Let $\sigma_1 = \frac{1}{2}$ and $\sigma_2 = 1 + \frac{\delta_1}{\log X}$ and $\sigma' = \sigma - \frac{\delta_2}{\log X}$ with $\frac{1}{2} \le \sigma' < \sigma \le 1$, we have the following relationship for the exponents:

$$\frac{\sigma_2-\sigma'}{\sigma_2-\sigma_1} = 1 - \frac{\sigma'-\sigma_1}{\sigma_2-\sigma_1}. \tag{4.94}$$

Proposition 4.29. *Fix T_2 with $T_2 > X_0 > 10^9$ and set $X = T_2$. Let $\delta_1, \delta_2 \in (0,1]$. Suppose that $\sigma' = \sigma - \frac{\delta_2}{\log X}$ with $\sigma \in [\sigma', 1]$ and $\sigma' \ge \frac{1}{2}$. Assume that the constant $\tilde{C}_5$ be as defined in equation (4.88) and that the following holds:*

$$\begin{aligned} A_1 &:= \frac{d}{\log T_2} C_3(\sigma_1, T_2, \alpha, \beta, a, a_i, b, b_i, d_i) > 1, \\ A_2 &:= \frac{d' T_2}{(\log T_2)^2} C_3(\sigma_2, T_2, \alpha, \beta, a', a_i', b', b_i', d_i') > 1. \end{aligned} \tag{4.95}$$

where both A_1 and A_2 are decreasing in T_2 for $T_2 > X_0$ and incorporates parameters, a, a_i, b, b_i, d_i from Tables 4.1, 4.2 and 4.3 , and a', a_i', b', b_i', d_i' from Table 4.4. Then,

- *If* $A_2 < A_1$ *then the constant* $\tilde{C_5}$ *is* $O(1)$ *and decreasing in* T_2*.*

- *If* $A_2 \geq A_1$ *we define a decreasing constant in* T_2*, denoted* $\breve{C}_5(\sigma, T_2, \alpha, \beta, \delta_1, \delta_2)$ *given by*

$$\breve{C}_5(\sigma, T_2, \alpha, \beta, \delta_1, \delta_2) := (A_1)^{(1-\frac{\sigma'-\sigma_1}{\sigma_2-\sigma_1})} (A_2)^{2(\sigma-\sigma_1)}. \tag{4.96}$$

In addition $\tilde{C_5}$ *is monotonic in* σ *and for all* $\sigma_1' > \frac{1}{2} + \frac{\delta_2}{\log T_0}$ *and* $\sigma_2' \leq 1$ *we have*

$$\max_{\sigma \in [\sigma_1', \sigma_2']} \tilde{C_5}(\sigma, T_2, \alpha, \beta, \delta_1, \delta_2) \leq \hat{C_5}([\sigma_1', \sigma_2'], T_0, \alpha, \beta, \delta_1, \delta_2), \tag{4.97}$$

where the function $\hat{C_5}([\sigma_1', \sigma_2'], T_0, \alpha, \beta, \delta_1, \delta_2)$ *is defined as*

$$\hat{C_5}([\sigma_1', \sigma_2'], T_0, \alpha, \beta, \delta_1, \delta_2) = \begin{cases} \breve{C}_5(\sigma_2', T_0, \alpha, \beta, \delta_1, \delta_2) & \textit{if} \quad A_2 \geq A_1 \\ \tilde{C_5}(\sigma_1', T_0, \alpha, \beta, \delta_1, \delta_2) & \textit{if} \quad A_2 < A_1. \end{cases} \tag{4.98}$$

where A_1 *and* A_2 *are defined in equation* (4.95)*.*

Proof. Recall that $\tilde{C_5}(X)$ is defined by equation (4.88) and by Remark 4.28 we have

$$\tilde{C_5} = \Big(A_1\Big)^{(1-\frac{\sigma'-\sigma_1}{\sigma_2-\sigma_1})} \Big(A_2\Big)^{(\frac{\sigma'-\sigma_1}{\sigma_2-\sigma_1})}. \tag{4.99}$$

To analyze whether $\tilde{C_5}$ is decreasing in $T_2 = X$, we define the functions:

$$f_1(X) = A_1(X)^{1-B_1(X)} = \left(\frac{d}{\log X} C_3(\sigma_1, X, \alpha, \beta, a, a_i, b, b_i, d_i)\right)^{(1-\frac{\sigma'-\sigma_1}{\sigma_2-\sigma_1})}$$

and

$$f_2(X) = A_2(X)^{B_1(X)} = \left(\frac{d'X}{(\log X)^2} C_3(\sigma_2, X, \alpha, \beta, a', a_i', b', b_i', d_i')\right)^{(\frac{\sigma'-\sigma_1}{\sigma_2-\sigma_1})}.$$

By using logarithmic differentiation, we have:

$$\begin{aligned} \frac{\partial \log f_1(X)}{\partial X} &= -B_1'(X)\log A_1(X) + (1-B_1(X))\frac{A_1'(X)}{A_1(X)} \\ \frac{\partial \log f_2(X)}{\partial X} &= B_1'(X)\log A_2(X) + B_1(X)\frac{A_2'(X)}{A_2(X)}. \end{aligned} \tag{4.100}$$

Now for both $f_1(X)$ and $f_2(X)$ to be decreasing, we need to have

$$\begin{aligned} & -B_1'(X)\log A_1(X) + (1-B_1(X))\frac{A_1'(X)}{A_1(X)} < 0, \\ & B_1'(X)\log A_2(X) + B_1(X)\frac{A_2'(X)}{A_2(X)} < 0. \end{aligned} \tag{4.101}$$

Since by assumption, both $A_1(X)$ and $A_2(X)$ are decreasing in X, therefore we have $A_1'(X) < 0$ and $A_2'(X) < 0$. Thus, the second terms in $\frac{\partial \log f_1(X)}{\partial X}$ and $\frac{\partial \log f_2(X)}{\partial X}$ is negative. Therefore, it then suffices to show that

$$-B_1'(X)\log A_1(X) + B_1'(X)\log A_2(X) < 0. \tag{4.102}$$

Or equivalently

$$B_1'(X)\Bigg(\log A_2(X) - \log A_1(X)\Bigg) < 0. \tag{4.103}$$

Therefore, first we need to analyze the behavior of $B_1'(X)$. We obtained:

$$B_1'(X) = \frac{\frac{\partial \sigma'}{\partial X}(\sigma_2-\sigma_1) - \frac{\partial \sigma_2}{\partial X}(\sigma'-\sigma_1)}{(\sigma_2-\sigma_1)^2}. \tag{4.104}$$

Since we have

$$\frac{\partial \sigma'}{\partial X} = \frac{\delta_2}{X(\log X)^2}, \quad \text{and} \quad \frac{\partial \sigma_2}{\partial X} = \frac{-\delta_1}{X(\log X)^2}, \tag{4.105}$$

and since $\delta_1, \delta_2 \in (0,1]$ and $\sigma' \geq \frac{1}{2}$, we conclude

$$B_1'(X) > 0. \tag{4.106}$$

By our assumption, we have $A_1(X) > 1$ and $A_2(X) > 1$ and hence $\log A_1(X) > 0$ and $\log A_2(X) > 0$. Therefore, we have two cases here:

- If $A_1(X) > A_2(X)$, therefore we have $\log A_1(X) > \log A_2(X)$, and it makes the derivative (4.103) negative and hence we conclude that $\tilde{C}_5$ is decreasing in T_2. Furthermore, $\tilde{C}_5$ is decreasing in σ, since $A_1(X) > A_2(X)$.

- If $A_1(X) < A_2(X)$, therefore we have $\log A_1(X) < \log A_2(X)$, and it makes the derivative positive.

  To address this, since we have $A_2(X) > 1$ and $\frac{\sigma'-\sigma_1}{\sigma_2-\sigma_1} \leq \frac{\sigma-\sigma_1}{\frac{1}{2}}$, we introduce a bound for $\tilde{C}_5$:

  $$\breve{C}_5(\sigma, T_2, \alpha, \beta, \delta_1, \delta_2) := \left(A_1\right)^{(1-B_1(X))} \left(A_2\right)^{2(\sigma-\sigma_1)}. \tag{4.107}$$

  We verify that $\breve{C}_5(\sigma, T_2, \delta_1, \delta_2)$ is decreasing in T_2 using a similar argument, noting that we easily see that the derivative of $\sigma - \sigma_1$ with respect to T_2 is zero and from equation (4.106), we have $B_1'(X) > 0$. Therefore we have

  $$-B_1'(X) \log A_1(X) < 0. \tag{4.108}$$

  Hence we conclude that $\breve{C}_5$ is decreasing in T_2. Moreover, we easily see that $\tilde{C}_5$ is increasing in σ since $A_1(X) < A_2(X)$.

From these two cases we conclude that $\hat{C}_5$ is decreasing in T_2. □

Remark 4.30. For specific values of $X = T_0 = 3 \cdot 10^{10}$, we have the following

- By using the first version of our obtained bound from Table 4.1, we have $A_1(X) = 1.3471$ and from the Table 4.4, we have $A_2(X) = 6.0848$. Hence we have $A_2(X) > A_1(X)$. Therefore, we have

  $$\max_{\sigma \in [\sigma_1', \sigma_2']} \tilde{C}_5(\sigma, T_2, \alpha, \beta, \delta_1, \delta_2) \leq \hat{C}_5(\sigma_2', T_0, \alpha, \beta, \delta_1, \delta_2), \tag{4.109}$$

where $\hat{C_5}(\sigma_2', T_0, \alpha, \beta, \delta_1, \delta_2) = \breve{C_5}(\sigma_2', T_0, \alpha, \beta, \delta_1, \delta_2)$.

For specific values of $\sigma = 0.9$, $T_0 = X = 3 \cdot 10^{12}$, $\delta_1 = 0.36$, $\delta_2 = 0.35$, and $\alpha = 0.241$ we find $\tilde{C_5} = 4.2161$ and $\breve{C_5} = 4.5592$.

- By using the first version of our obtained bound from Table 4.3, we have $A_1(X) = 1.3121$ and from the Table 4.4, we have $A_2(X) = 6.0848$. Hence we have $A_2(X) > A_1(X)$. Therefore, we have

$$\max_{\sigma \in [\sigma_1', \sigma_2']} \tilde{C_5}(\sigma, T_2, \alpha, \beta, \delta_1, \delta_2) \leq \hat{C_5}(\sigma_2', T_0, \alpha, \beta, \delta_1, \delta_2), \tag{4.110}$$

where $\hat{C_5}(\sigma_2', T_0, \alpha, \beta, \delta_1, \delta_2) = \breve{C_5}(\sigma_2', T_0, \alpha, \beta, \delta_1, \delta_2)$.

For specific values of $\sigma = 0.9$, $T_0 = X = 3 \cdot 10^{12}$, $\delta_1 = 0.36$, $\delta_2 = 0.35$, and $\alpha = 0.241$ we find $\tilde{C_5} = 4.1892$ and $\breve{C_5} = 4.5301$.

- By using the second version of obtained bound from Table 4.2, we have $A_1(X) = 14771.199$ and from Table 4.4 we have $A_2(X) = 6.0848$. Hence we have $A_1(X) > A_2(X)$. Therefore, we have

$$\max_{\sigma \in [\sigma_1', \sigma_2']} \tilde{C_5}(\sigma, T_2, \alpha, \beta, \delta_1, \delta_2) \leq \hat{C_5}(\sigma_1', T_0, \alpha, \beta, \delta_1, \delta_2), \tag{4.111}$$

where $\hat{C_5}(\sigma_1', T_0, \alpha, \beta, \delta_1, \delta_2) = \tilde{C_5}(\sigma_1', T_0, \alpha, \beta, \delta_1, \delta_2)$.

For specific values of $\sigma = 0.9$, $T_0 = X = 3 \cdot 10^{12}$, $\delta_1 = 0.36$, $\delta_2 = 0.35$, and $\alpha = 0.241$ we find $\tilde{C_5} = 40.546$.

Proposition 4.31. *Fix T_2 with $T_2 > X_0 > 10^9$ and set $X = T_2$. Let $\delta_1, \delta_2 \in (0, 1]$. Suppose that $\sigma' = \sigma - \frac{\delta_2}{\log X}$ with $\sigma \in [\sigma', 1]$ and $\sigma' \geq \frac{1}{2}$. Assume that the constant C_6 be as defined in equation (4.89). Then,*

- *If $b - b' > 1$ then the constant C_6 is $O(1)$ and eventually decreasing in T_2.*

- *If* $b-b'<1$ *then the constant* C_6 *is increasing in* T_2 *and hence we define a decreasing constant in* T_2 *in equation* (4.113).

These parameters, a, b *are taken from Tables 4.1, 4.2 and 4.3, and* a', b' *are taken from table 4.4. In addition* C_6 *is monotonic in* σ *and we have follows that*

$$\max_{\sigma\in[\sigma_1',\sigma_2']} C_6(\sigma, T_2, \delta_1, \delta_2, a, a', b, b') \leq \hat{C}_6([\sigma_1', \sigma_2'], T_0, \delta_1, \delta_2, a, a', b, b'), \tag{4.112}$$

where $\hat{C}_6 = \hat{C}_6([\sigma_1', \sigma_2'], T_0, \delta_1, \delta_2, a, a', b, b')$

$$\hat{C}_6 = \begin{cases} C_6(\sigma_2', T_0, \delta_1, \delta_2, a, a', b, b') & b-b'>1 \textit{ and } a-a'>-1 \\ C_6(\sigma_1', T_0, \delta_1, \delta_2, a, a', b, b') & b-b'>1 \textit{ and } a-a'<-1 \\ e^{(1+a-a')(2\delta_1(2\sigma_2'-1)+2\delta_2)} & b-b'\leq 1 \textit{ and } a-a'>-1 \\ e^{(1+a-a')\frac{2\delta_1(2\sigma_1'-1)+2\delta_2}{1+\frac{2\delta_1}{\log T_0}}} & \textit{otherwise.} \end{cases} \tag{4.113}$$

Proof. Recall that C_6 is defined by equation (4.89) and by Remark 4.28 we can simplify the definition of C_6 as the product

$$C_6 = \left(T_2^{a(1-\frac{\sigma'-\sigma_1}{\sigma_2-\sigma_1})+(a'-1)(\frac{\sigma'-\sigma_1}{\sigma_2-\sigma_1})-[2a(1-\sigma)+(a'-1)(2\sigma-1)]} \right) \cdot$$

$$\left(\log T_2^{(b+1)(1-\frac{\sigma'-\sigma_1}{\sigma_2-\sigma_1})+(b'+2)(\frac{\sigma'-\sigma_1}{\sigma_2-\sigma_1})-[2(b+1)(1-\sigma)+(b'+2)(2\sigma-1)]} \right),$$

This simplifies to:

$$\begin{aligned}
C_6 &= T_2^{(1+a-a')(-\frac{\sigma'-\sigma_1}{\sigma_2-\sigma_1}-1)+(1+a-a')(2\sigma)} \log T_2^{(b-b'-1)(-\frac{\sigma'-\sigma_1}{\sigma_2-\sigma_1}-1)+(b-b'-1)(2\sigma)} \\
&= T_2^{(1+a-a')(2\sigma-\frac{\sigma'-\sigma_1}{\sigma_2-\sigma_1}-1)} \log T_2^{(b-b'-1)(2\sigma-\frac{\sigma'-\sigma_1}{\sigma_2-\sigma_1}-1)} \\
&= T_2^{(1+a-a')(\frac{4\sigma\delta_1+2\delta_2-2\delta_1}{\log X+2\delta_1})} \log T_2^{(b-b'-1)(\frac{4\sigma\delta_1+2\delta_2-2\delta_1}{\log X+2\delta_1})} \qquad (4.114)\\
&= \left(T_2^{(1+a-a')} \log T_2^{(b-b'-1)}\right)^{\frac{2\delta_1(2\sigma-1)+2\delta_2}{\log X+2\delta_1}} \\
&= e^{((1+a-a')\log T_2+(b-b'-1)\log\log T_2)(\frac{2\delta_1(2\sigma-1)+2\delta_2}{\log X+2\delta_1})}.
\end{aligned}$$

Denote:

$$g(a,a',b,b',\sigma,T_2) = \big((1+a-a')\log T_2 + (b-b'-1)\log\log T_2\big)\left(\frac{2\delta_1(2\sigma-1)+2\delta_2}{\log T_2+2\delta_1}\right). \qquad (4.115)$$

The behavior of C_6, depends on whether $g(a,a',b,b',\sigma,T_2)$ is increasing or decreasing. Therefore, $\frac{\partial g(\sigma,T_2)}{\partial T_2}$ is given by:

$$\frac{2\delta_1(2\sigma-1)+2\delta_2}{T_2(\log T_2+2\delta_1)^2}\left[(b-b'-1)\left(\frac{2\delta_1}{\log X}-\log\log T_2+1\right)+(1+a-a')2\sigma\right]. \qquad (4.116)$$

Since $\frac{1}{2} < \sigma \leq 1$ and $\delta_1, \delta_2 \in (0,1)$, the fraction outside of the bracket is always positive. Thus, the overall sign depends on the terms inside the brackets:

If $b-b'-1 > 0$ and $1+a-a' < 0$, the first term is negative, and the second term is negative and then the overall sign is negative and hence C_6 is decreasing in T_2.

If $b-b'-1 > 0$ and $1+a-a' > 0$ the first term is negative for large T_2 and the second term is positive. However, the first term dominates eventually so it is decreasing.

If $b-b'-1 < 0$ and $1+a-a' < 0$ the first term is positive for large T_2 and the second term is negative. However, the first term dominates eventually.

If $b-b'-1 < 0$ and $1+a-a' > 0$, the first term is positive, and the second term is positive and then the overall sign is positive and hence C_6 is increasing in T_2.

From this we arrive at the four cases in the definition of $\hat{C}_6$ which essentially come from deleting terms in the exponent in C_6.

Finally, the exponent in the definition of C_6 is obviously linear in σ, hence monotonic. This gives us the four cases. $\square$

Remark 4.32. For specific values of a, a', b, b' from the tables, we have the following:

- For the first and third version of our obtained bound from Table, 4.1, 4.3 we have $a = \frac{4}{3}$ and $b = 2$ and from the Table 4.4, we have $a' = 1$ and $b' = 0$. Therefore we have $b - b' > 1$ and $a - a' > -1$. Hence we are in the first case.

  For specific values of $\sigma = 0.9$, $T_0 = X = 3 \cdot 10^{12}$, $\delta = 0.36$, and $\delta_2 = 0.35$ we find $C_6 = \hat{C}_6 = 6.0813$.

- For the second version from Table 4.2 we have $a = \frac{109}{82}, b = 0$ and similarly from the Table 4.4, we have $a' = 1, b' = 0$. Therefore, we have $b - b' < 1$ and $a - a' > -1$. Hence we are in the second case.

  For specific values of $\sigma = 0.9$, $T_0 = X = 3 \cdot 10^{12}$, $\delta_1 = 0.36$, and $\delta_2 = 0.35$ we find

$$C_6 = 4.523 \quad \text{and} \quad \hat{C}_6 = e^{(1+a-a')(2\delta_1(2\sigma_2'-1)+2\delta_2)} = 5.3272.$$

In both cases we easily see that the functions are now decreasing for $T_2 > 10^9$.

4.3.5 Bounds on the first Integral

We now provide a bound for the first integral. As discussed earlier, it suffices to bound $\int_{T_1}^{T_2} |f_X(\sigma + it)|^2 dt$.

Corollary 4.33. *Fix $\delta_1, \delta_2 > 0$. Let $X = T_2$ and suppose that $T_2 > X_0 > 10^9$ and $T_1 > H_0 > 0$. Let $\alpha > 0$. Let $\sigma_1' > \frac{1}{2} + \frac{\delta_2}{\log T_0}$ and $\sigma_2' < 1$ and parameters, a, b from Tables 4.1, 4.2 and 4.3,* *and* a', b' *from Table 4.4. For any* $\sigma \in [\sigma_1', \sigma_2']$ *we have*

$$\frac{\log T_2}{2\pi\delta_2}\int_{T_1}^{T_2}\log|h_X(\sigma'+it)|dt \le \frac{(T_2-T_1)\log T_2}{2\pi\delta_2}\log\Bigg(1+\frac{\hat{C}_4\hat{C}_5\hat{C}_6}{(T_2-T_1)}T_2^{2a(1-\sigma)+(a'-1)(2\sigma-1)} \\ \log T_2^{2(b+1)(1-\sigma)+(b'+2)(2\sigma-1)}\Bigg). \tag{4.117}$$

and

$$\frac{\log T_2}{2\pi\delta_2}\int_{T_1}^{T_2}\log|h_X(\sigma'+it)|dt \le \Bigg(\frac{\hat{C}_4\hat{C}_5\hat{C}_6}{2\pi\delta_2}T_2^{2a(1-\sigma)+(a'-1)(2\sigma-1)}\cdot \\ \log T_2^{2(b+1)(1-\sigma)+(b'+2)(2\sigma-1)+1}\Bigg), \tag{4.118}$$

where $\hat{C}_4 = \hat{C}_4(T_0, H_0, \alpha, \beta, \delta_1)$ *and* $\hat{C}_5 = \hat{C}_5([\sigma_1', \sigma_2'], T_0, \alpha, \beta, \delta_1, \delta_2)$ *are defined in equations* (4.90) *and* (4.98) *and* $\hat{C}_6 = \hat{C}_6([\sigma_1', \sigma_2'], T_0, \delta_1, \delta_2, a, a', b, b')$ *is defined in equation* (4.113).

Proof. From equation (4.12), (4.13) and the definition of F_X in (4.15), we have

$$\frac{\log T_2}{2\pi\delta_2}\int_{T_1}^{T_2}\log(|h_X(\sigma'+it)|)dt \le \frac{(T_2-T_1)\log T_2}{2\pi\delta_2}\log\left(1+\frac{F_X(\sigma', T_1, T_2)}{T_2-T_1}\right). \tag{4.119}$$

By substituting the bound for $F_X(\sigma', T_1, T_2)$ from Theorem 4.23 and applying Propositions 4.24, 4.29 and 4.31 to substitute the maximum for each constant, C_4, $\tilde{C}_5$ and C_6, we obtain the following bound for the first integral:

$$\frac{(T_2-T_1)\log T_2}{2\pi\delta_2}\log\left(1+\frac{\hat{C}_4\hat{C}_5\hat{C}_6}{(T_2-T_1)}T_2^{2a(1-\sigma)+(a'-1)(2\sigma-1)}\log T_2^{2(b+1)(1-\sigma)+(b'+2)(2\sigma-1)}\right). \tag{4.120}$$

Then by applying $\log(1+y) \le y$, we obtain the following upper bound:

$$\frac{\hat{C}_4\hat{C}_5\hat{C}_6}{2\pi\delta_2}T_2^{2a(1-\sigma)+(a'-1)(2\sigma-1)}\log T_2^{2(b+1)(1-\sigma)+(b'+2)(2\sigma-1)+1}. \tag{4.121}$$

which completes the proof. □

Remark 4.34. For specific values of a, a', b, b' from the tables, we have the following:

- For the first and third version of our obtained bound from Table, 4.1, 4.3, we have $a = \frac{4}{3}, b = 2$ and from the Table 4.4, we have $a' = 1, b' = 0$. Therefore we have

$$\frac{\log T_2}{2\pi\delta_2}\int_{T_1}^{T_2}\log|h_X(\sigma'+it)|dt \le \frac{(T_2-T_1)\log T_2}{2\pi\delta_2}\log\left(1+\frac{\hat{C}_4\hat{C}_5\hat{C}_6}{(T_2-T_1)}T_2^{\frac{8}{3}(1-\sigma)}\log T_2^{(2+2(1-\sigma))}\right). \tag{4.122}$$

  and

$$\frac{\log T_2}{2\pi\delta_2}\int_{T_1}^{T_2}\log|h_X(\sigma'+it)|dt \le \left(\frac{\hat{C}_4\hat{C}_5\hat{C}_6}{2\pi\delta_2}T_2^{\frac{8}{3}(1-\sigma)}\log T_2^{(3+2(1-\sigma))}\right). \tag{4.123}$$

- For the second version from Table 4.2 we have $a = \frac{109}{82}, b = 0$ and similarly from the Table 4.4, we have $a' = 1, b' = 0$. Therefore, we have

$$\frac{\log T_2}{2\pi\delta_2}\int_{T_1}^{T_2}\log|h_X(\sigma'+it)|dt \le \frac{(T_2-T_1)\log T_2}{2\pi\delta_2}\log\left(1+\frac{\hat{C}_4\hat{C}_5\hat{C}_6}{(T_2-T_1)}T_2^{\frac{109}{41}(1-\sigma)}\log T_2^{(2\sigma)}\right). \tag{4.124}$$

  and

$$\frac{\log T_2}{2\pi\delta_2}\int_{T_1}^{T_2}\log|h_X(\sigma'+it)|dt \le \left(\frac{\hat{C}_4\hat{C}_5\hat{C}_6}{2\pi\delta_2}T_2^{\frac{109}{41}(1-\sigma)}\log T_2^{(2\sigma+1)}\right). \tag{4.125}$$

4.4 Bounds on the fourth Integral

In this section we need to find an upper bound for $-\int_{T_1}^{T_2}\log|h_X(\mu+it)|dt$ or similarly find an explicit lower bound for $\int_{T_1}^{T_2}\log|h_X(\mu+it)|dt$. For bounding this integral, first we have the below fact here,

$$|f_X(\mu+it)| < 1. \tag{4.126}$$

Lemma 4.35. *For $X > 1000$, and $\mu > 1$ we have*

$$|f_X(\mu+it)| \le E_1(X,\mu),$$

where, $E_1(X,\tau)$ is defined in equation (3.71) and by substituting μ we have:

$$E_1(X,\mu) = \frac{\mu}{X^{\mu-1}}\left(\frac{\log X}{\mu-1} + \frac{1}{(\mu-1)^2} + \frac{\gamma}{\mu-1} + \frac{7}{12\mu X}\right). \tag{4.127}$$

In particular for $\mu = 1+\eta$ with $\eta = \frac{k\log\log X}{\log(X)}$, we have

$$|f_X(\mu+it)| \le E_3(k,X),$$

where $E_3(k,X)$ is defined by the following product:

$$\begin{aligned} E_3(k,X) = &\left(\frac{\log X}{k(\log X)^{k-1}\log\log X} + \frac{1}{(\log X)^{k-1}}\right)\cdot \\ &\left(1+\frac{1}{k\log\log X} + \frac{\gamma}{\log X} + \frac{7k\log\log X}{12X\log X(\log X + k\log\log X)}\right). \end{aligned} \tag{4.128}$$

Proof. By equation (4.3) we have

$$|f_X(s)| \le \sum_{n>X}\frac{|\lambda_X(n)|}{n^{1+\eta}} \le \sum_{n>X}\frac{d(n)}{n^{1+\eta}}, \tag{4.129}$$

and by Lemma 3.12, we obtain

$$|f_X(1+\eta+it)| \le E_3(k,X). \qquad \square$$

Now with using the above Lemma , we can obtain the bound for the fourth integral.

Lemma 4.36. *Suppose $1000 < T_1 < T_2$ and that $\mu = 1+\frac{k\log\log(X)}{\log X}$. Suppose that $k \ge 2$ and assumed the following upper bound for $|f_X(\mu+it)|$:*

$$|f_X(\mu+it)| \le E_3(k,X) < 1. \tag{4.130}$$

Set $X = T_2$. Therefore, we have

$$-\int_{T_1}^{T_2} \log|h_X(\mu+it)|dt \leq \frac{E_5(k,X)}{k\log\log(X)\log(X)^{2k-3}}(\log T_2), \tag{4.131}$$

where $E_5(k,X)$ is defined by

$$E_4(k,X)\Big((X+2\pi m_0)E_2(X,2\mu)+2\pi m_0 E_2(X,2\mu-1)\Big)(k\log\log(X)\log(X)^{2k-4}), \tag{4.132}$$

with E_2 is defined in equation (3.86) and E_4 is defined by

$$E_4(k,X) = -\frac{\log(1-E_3(k,X)^2)}{E_3(k,X)^2}, \tag{4.133}$$

where $E_3(k,X)$ is defined in equation (4.128).

Remark 4.37. $E_4(k,X)$ is $1+o(1)$ and eventually decreasing in X. Moreover, we also have $XE_2(X,2\mu)$ and $E_2(X,2\mu-1)$ are both $\frac{1+o(1)}{\pi^2 k\log\log(X)\log(X)^{2k-4}}$ and eventually decreasing. Hence, $E_5(k,X)$ is $O(1)$ and eventually decreasing in X.

Proof. By our assumption, $X = T_2$ and $T_2 > 1000$. For $\mu = 1+\frac{k\log\log(X)}{\log X}$ and the following assumed upper bound for $|f_X(\mu+it)|$:

$$|f_X(\mu+it)| \leq E_3(k,X) < 1, \tag{4.134}$$

and since $|h_X(\mu+it)| = |1-f_X(\mu+it)^2| \geq 1-|f_X(\mu+it)|^2$, we have

$$-\log|h_X(\mu+it)| \leq -\log(1-|f_X(\mu+it)|^2). \tag{4.135}$$

Since the function $-\frac{\log(1-u^2)}{u^2}$ increases with $u \in (0,1)$, we have

$$-\log(1-|f_X(\mu+it)|^2) \leq E_4(k,X)|f_X(\mu+it)|^2, \tag{4.136}$$

with

$$E_4(k,X) = -\frac{\log(1-E_3(k,X)^2)}{E_3(k,X)^2}. \tag{4.137}$$

By taking integral from both side of equation (4.135) and using the bound in equation (4.136), we have

$$-\int_{T_1}^{T_2} \log|h_X(\mu+it)|dt \leq E_4(k,X)\int_{T_1}^{T_2} |f_X(\mu+it)|^2 dt. \tag{4.138}$$

We apply the Montgomery and Vaughan's mean value theorem and the bound $|\lambda_X(n)| \leq d(n)$ with $\lambda_X(n) = 0$ if $n \leq X$. So we obtain the following bound

$$\begin{aligned}\int_{T_1}^{T_2} |f_X(\mu+it)|^2 dt &\leq \sum_{n\geq 1} \frac{|\lambda_X(n)|^2}{n^{2\mu}}(T_2 - T_1 + 2\pi m_0(n+1)) \\ &\leq \sum_{n>X} \frac{d(n)^2}{n^{2\mu}}(T_2 - T_1 + 2\pi m_0(n+1)) \\ &= (T_2 - T_1 + 2\pi m_0)\sum_{n>X}\frac{d(n)^2}{n^{2\mu}} + 2\pi m_0 \sum_{n>X}\frac{d(n)^2}{n^{2\mu-1}}.\end{aligned} \tag{4.139}$$

We apply Lemma 3.14 to obtain an upper bound for the above sums.

$$\sum_{n>X}\frac{d(n)^2}{n^{2\mu}} \leq E_2(X,2\mu), \tag{4.140}$$

where $E_2(X,\tau)$ is defined in equation (3.86). Similarly, we have:

$$\sum_{n>X}\frac{d(n)^2}{n^{2\mu-1}} \leq E_2(X,2\mu-1), \tag{4.141}$$

where $E_2(X,\tau)$ is defined in equation (3.86). We simplified the above bounds and substitute them into equation (4.139). Therefore, we obtain the following bound:

$$\int_{T_1}^{T_2} |f_X(\mu+it)|^2 dt \leq (T_2 - T_1 + 2\pi m_0)E_2(X,2\mu) + 2\pi m_0 E_2(X,2\mu-1). \tag{4.142}$$

We combine this bound by equation (4.138) and obtain the following bound:

$$-\int_{T_1}^{T_2} \log|h_X(\mu+it)|dt \le E_4(k,X)\Big((T_2-T_1+2\pi m_0)E_2(X,2\mu)+2\pi m_0 E_2(X,2\mu-1)\Big). \tag{4.143}$$

Therefore, since $X = T_2$ we conclude that

$$-\int_{T_1}^{T_2} \log|h_X(\mu+it)|dt \le E_4(k,X)\Big((X+2\pi m_0)E_2(X,2\mu)+2\pi m_0 E_2(X,2\mu-1)\Big). \quad \square$$

Remark 4.38. The condition for choosing $k \ge 2$ is that we need to have $E_3(k,X) < 1$ and also decreasing in X. Therefore, for $k = 2$ and $T_2 = X \ge T_0$ we obtain the following bounds:

- For $T_0 = 3\cdot 10^{12}$, we have $E_5(k,X) = 0.694$ and hence

$$-\int_{T_1}^{T_2} \log|h_X(\mu+it)|dt \le 0.0035 \log T_2. \tag{4.144}$$

- For $T_0 = 3\cdot 10^{100}$, we have $E_5(k,X) = 0.503$ and hence

$$-\int_{T_1}^{T_2} \log|h_X(\mu+it)|dt \le 0.0002 \log T_2. \tag{4.145}$$

4.5 Bounds on the second and third integral

In this section, we need to find an upper bound for the difference of the second and third integral:

$$\int_{\sigma'}^{\mu} \arg h(\tau+iT_2)d\tau - \int_{\sigma'}^{\mu} \arg h(\tau+iT_1)d\tau.$$

First, we obtain an upper bound for arg of a function. The argument we use here is due to Kadiri, Lumley and Ng, [26, Proposition 4.10].

Lemma 4.39. *Let* $\eta > 0$ *. Let* $f(s)$ *be a holomorphic function for* $R(s) \ge -\eta$ *and be real* *for real s. We suppose that there exist positive constants M and m such that*

$$|f(s)| \leq M \qquad R(s) \geq 1+\eta. \tag{4.146}$$

$$|Rf(1+\eta+it)| \geq m > 0 \qquad \textit{for all} \quad t \in R. \tag{4.147}$$

Let $\sigma \in [\frac{1}{2}, 1+\eta]$ [2] *and assume that* U *is not the ordinate of a zero of* $f(s)$*. Then there exist an increasing* $\{N_q\}_{q=1}^{\infty}$ *such that*

$$|\arg f(\sigma+iU)| \leq \frac{\pi}{\log 2}\ell_q + \frac{\pi \log M}{2\log 2} - \frac{\pi \log m}{\log 2} + \frac{\pi}{2} + o_q(1) \tag{4.148}$$

Where

$$\ell_q = \frac{1}{2\pi N_q}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \log\left(\frac{1}{2}\sum_{j=0}^{1} |f(1+\eta+(1+2\eta)e^{i\theta}+(-1)^j iU)|^{N_q}\right) d\theta, \tag{4.149}$$

and $o_q(1)$ *is a term that approaches* 0 *as* $q \to \infty$*.*

We apply Lemma 4.39 to obtain an upper bound for $\arg h(s)$. The following Lemma is due to Kadiri, Lumley and Ng, [26, Lemma 4.12].

Lemma 4.40. *Let* $\eta \in (\eta_0, \frac{1}{2})$*, with* $E_7(X,\eta_0) = 1$ *and* $T_2 > T_0 > 3 \cdot 10^{12}$ *and* $X = T_2$*. Suppose that* $U \geq T_1 \geq 1002$ *and that is* U *is not the ordinate of a zero of* $h(s)$*. Then for all* $\tau \in [\frac{1}{2}, 1+\eta]$, [3] *, we have*

$$|\arg h(\tau+iU)| \leq \frac{(1+2\eta)}{\log 2}\log\left(\frac{E_9(\eta,T_1)}{2\pi}U\right) + \frac{\pi(1+\eta)}{\log 2}(\log X) + \frac{\pi \log E_8(\eta,T_1)}{2\log 2} + \frac{\pi \log E_6(\eta)}{2\log 2} - \frac{\pi\log(1-E_7(X,\eta)^2)}{\log 2} + \frac{\pi}{2}, \tag{4.150}$$

---

[2] In [26], the range was incorrectly stated $(0, 1+\eta]$. This has been corrected $\left[\frac{1}{2}, 1+\eta\right]$.

[3] In [26], the range was incorrectly stated $(0, 1+\eta]$. This has been corrected $\left[\frac{1}{2}, 1+\eta\right]$.

where

$$
\begin{aligned}
E_6(\eta) &= \frac{\zeta(1+\eta)^4}{\zeta(2+2\eta)^2} + \frac{2\zeta(1+\eta)^2}{\zeta(2+2\eta)}, \\
E_7(X,\eta) &= \frac{(1+\eta)(\log X)}{\eta X^{\eta}}\left(1+\frac{1}{\eta\log X}+\frac{\gamma}{\log X}+\frac{7\eta}{12(1+\eta)X\log X}\right), \\
E_8(\eta,T_1) &= \left(1+\frac{2}{3.006\zeta(1+\eta)(T_1)^{1+\eta}}\right)(3.006\zeta(1+\eta))^2, \\
E_9(\eta,T_1) &= \sqrt{\frac{(2+\eta)^2}{T_1^2}+\left(\frac{1+2\eta}{T_1}+1\right)^2}.
\end{aligned}
\tag{4.151}
$$

Proof. The proof is based on Kadiri, Lumley and Ng, [26, Lemma 4.12] except that we relabel the constants defined in equation (4.151) and take $k=1$, [4] in their proof. Additionally, we replace H_0 by T_1 as they play the same role in this argument. In Kadiri, Lumley and Ng [26], the condition $\eta > 0.23622$ is specific to $X = 10^9$, which can be relaxed and can be different for different values of X. Based on the equation (4.150), the real conditions that must be satisfied are $E_7(X,\eta) < 1$ and $E_7(X,\eta)$ decreasing in X which the second condition implied by $\eta > \frac{1}{\log X_0}$. □

Remark 4.41. For $T_1 \geq H_0$ we have following conditions for η:

- For $X = T_2 = 10^9$ the condition is $\eta > 0.23622$,
- For $X = T_2 = 3\cdot 10^{12}$ the condition is $\eta > 0.1876$,
- For $X = T_2 = 3\cdot 10^{100}$ the condition is $\eta > 0.0382$.

By following Kadiri, Lumley and Ng [26, Lemma 4.12] and using Lemma 4.40 now we can bound our argument:

Lemma 4.42. *Let $H_0 \leq T_1$ and $T_2 = X \geq T_0$. Suppose that $\eta \in (\eta_0, \frac{1}{2})$ with $E_7(X,\eta_0) = 1$. Both σ' and μ satisfying $\frac{1}{2} \leq \sigma' < 1 < \mu$,* [5]*. Then we have the following upper bound:*

$$
\left|\int_{\sigma'}^{\mu} \arg h(\tau+iT_2)d\tau - \int_{\sigma'}^{\mu} \arg h(\tau+iT_1)d\tau\right| \leq E_{10}(\eta,T_1,X)(\mu-\sigma')(\log T_2), \tag{4.152}
$$

---

[4] In [26] their k is different than our k.

[5] In Kadiri, Lumley and Ng [26], they used $1+\eta$ as an upper bound for μ

where $E_{10}(\eta, T_1, X)$ *is defined by*

$$\frac{2(1+2\eta)+2\pi(1+\eta)}{\log 2}+\frac{E_{11}(\eta, T_1)}{\log T_0}, \tag{4.153}$$

with

$$\begin{aligned} E_{11}(\eta, T_1, X) = \frac{\pi \log E_8(\eta, T_1)}{\log 2} + \frac{\pi \log E_6(\eta)}{\log 2} - \frac{2\pi \log(1-E_7(X,\eta)^2)}{\log 2} \\ + \pi + \frac{2(1+2\eta)}{\log 2} \log\left(\frac{E_9(\eta, T_1)}{2\pi}\right). \end{aligned} \tag{4.154}$$

Proof. The proof is identical to Kadiri, Lumley and Ng [26] except that we don't have the condition $\mu < 1+\eta$. We note that for $\tau > 1+\eta$ we have the stronger bound, $\arg h(\tau + iU) < \frac{\pi}{2}$ which improves upon the bound given in Lemma 4.40. □

Remark 4.43. Both $E_{10}(\eta, T_1, X)$ and $E_{11}(\eta, T_1, X)$ are decreasing in X and T_1.

Remark 4.44. For specific values of $H_0 = 3 \cdot 10^{12}$ we have

- For $T_0 = X_0 = 3 \cdot 10^{12}$ where $T_2 > T_0$ and $T_1 > H_0$, the optimal choice of η that minimizes the function, $E_{10}(\eta, T_1, X)$ is 0.20341 and hence we obtain $E_{10}(\eta, T_1) = 17.1483$,
- For $T_0 = X_0 = 3 \cdot 10^{100}$ where $T_2 > T_0$ and $T_1 > H_0$, the optimal choice of η that minimizes the function, $E_{10}(\eta, T_1, X)$ is 0.0598 and hence we obtain $E_{10}(\eta, T_1) = 15.9074$.

Remark 4.45. Suppose that $H_0 = 3 \cdot 10^{12}$ and $T_1 \geq H_0$. For different values for σ and T_0, where $T_2 \geq T_0$ we have the following upper bound:

- For $\sigma = 0.9$, $\delta_2 = 0.35$, $k = 2$, $\eta = 0.20341$ and $T_0 = 3 \cdot 10^{12}$ we have

$$\left| \int_{\sigma'}^{\mu} \arg h(\tau + iT_2) d\tau - \int_{\sigma'}^{\mu} \arg h(\tau + iT_1) d\tau \right| \leq 5.9323(\log T_2). \tag{4.155}$$

- For $\sigma = 0.99$, $\delta_2 = 0.35$, $k = 2$, $\eta = 0.20341$ and $T_0 = 3 \cdot 10^{12}$ we have

$$\left| \int_{\sigma'}^{\mu} \arg h(\tau + iT_2) d\tau - \int_{\sigma'}^{\mu} \arg h(\tau + iT_1) d\tau \right| \le 4.3890 (\log T_2). \tag{4.156}$$

- For $\sigma = 0.9$, $\delta_2 = 0.35$, $k = 2$, $\eta = 0.0598$ and $T_0 = 3 \cdot 10^{100}$ we have

$$\left| \int_{\sigma'}^{\mu} \arg h(\tau + iT_2) d\tau - \int_{\sigma'}^{\mu} \arg h(\tau + iT_1) d\tau \right| \le 2.3634 (\log T_2). \tag{4.157}$$

- For $\sigma = 0.99$, $\delta_2 = 0.35$, $k = 2$, $\eta = 0.0598$ and $T_0 = 3 \cdot 10^{100}$ we have

$$\left| \int_{\sigma'}^{\mu} \arg h(\tau + iT_2) d\tau - \int_{\sigma'}^{\mu} \arg h(\tau + iT_1) d\tau \right| \le 0.9317 (\log T_2). \tag{4.158}$$

4.6 Zero density result

In this section, we are able to compile our bounds to obtain an upper bound for zero density result $N(\sigma, T_1, T_2)$:

Theorem 4.46. *Fix $\delta_1, \delta_2 > 0$ and $k = 2$. Let $T_2 > T_0 > 3 \cdot 10^9$ and $T_1 > H_0 > 1002$ and $X = T_2$ and $\mu = 1 + \frac{k \log\log(X)}{\log X}$. Suppose that $\eta \in (\eta_0, \frac{1}{2})$ with $E_7(X, \eta_0) = 1$ and $\alpha > 0$ and parameters, a, b from Tables 4.1, 4.2 or 4.3, and a', b' from Table 4.4. Let $\sigma_1' > \frac{1}{2} + \frac{\delta_2}{\log T_0}$ and $\sigma_2' \le 1$. For any $\sigma \in [\sigma_1', \sigma_2']$ we have*

$$N(\sigma, T_1, T_2) \le (T_2 - T_1)(\log T_2) \log\left(1 + \mathcal{U} \frac{T_2^{2a(1-\sigma)+(a'-1)(2\sigma-1)} (\log T_2)^{2(b+1)(1-\sigma)+(b'+2)(2\sigma-1)}}{(T_2 - T_1)}\right) + \mathcal{V}(\log T_2)^2, \tag{4.159}$$

and

$$N(\sigma, T_1, T_2) \le \mathcal{U} T_2^{2a(1-\sigma)+(a'-1)(2\sigma-1)} (\log T_2)^{2(b+1)(1-\sigma)+(b'+2)(2\sigma-1)+1} + \mathcal{V}(\log T_2)^2, \tag{4.160}$$

where

$$\begin{aligned}\mathcal{U} &= \mathcal{U}(\alpha,\delta_1,\delta_2,\sigma,T_0) = \frac{\hat{C}_4\hat{C}_5\hat{C}_6}{2\pi\delta_2},\\ \mathcal{V} &= \mathcal{V}(\delta_2,\sigma,\eta,k,\mu,T_1,T_2) = \frac{1}{2\pi\delta_2}\left(\frac{E_5(k,X)}{k\log\log(X)\log(X)^{2k-3}} + E_{10}(\eta,H_0,T_0)(\mu-\sigma')\right).\end{aligned} \tag{4.161}$$

where $\hat{C}_4 = \hat{C}_4(T_0,H_0,\alpha,\beta,\delta_1)$, $\hat{C}_5 = \hat{C}_5([\sigma_1',\sigma_2'],T_0,\delta_1,\delta_2)$, $\hat{C}_6 = \hat{C}_6([\sigma_1',\sigma_2'],T_0,\delta_1,\delta_2)$, $E_5(k,X)$ *and* $E_{10}(\eta,H_0,T_0)$ *are respectively defined in equation* (4.90)*,* (4.98)*,* (4.113)*,* (4.132) *and* (4.153)*.*

Proof. From equations (4.12), (4.13) and the definition of F_X in (4.15), we have:

$$\begin{aligned}N(\sigma,T_1,T_2) \le \frac{(T_2-T_1)\log T_2}{2\pi\delta_2}\Bigg(&\log\left(1+\frac{F_X(\sigma',T_1,T_2)}{T_2-T_1}\right)\\ &+\int_{\sigma'}^{\mu}\arg h(\tau+iT_2)d\tau - \int_{\sigma'}^{\mu}\arg h(\tau+iT_1)d\tau - \int_{T_1}^{T_2}\log|h_X(\mu+it)|dt\Bigg).\end{aligned} \tag{4.162}$$

By substituting the bound for $F_X(\sigma',T_1,T_2)$ from Theorem 4.23 and applying Propositions 4.24, 4.29 and 4.31 and also substituting the upper bound for the second and third integral from Lemma 4.42 and applying Lemma 4.36 for the fourth integral, we obtain desired bound:

$$\begin{aligned}N(\sigma,T_1,T_2) \le (T_2-T_1)\log T_2 \log\left(1+\mathcal{U}\frac{T_2^{2a(1-\sigma)+(a'-1)(2\sigma-1)}\log T_2^{2(b+1)(1-\sigma)+(b'+2)(2\sigma-1)}}{(T_2-T_1)}\right)\\ +\mathcal{V}(\log T_2)^2,\end{aligned} \tag{4.163}$$

and then by applying $\log(1+y)\le y$, we obtain the following upper bound:

$$N(\sigma,T_1,T_2) \le \mathcal{U}T_2^{2a(1-\sigma)+(a'-1)(2\sigma-1)}\log T_2^{2(b+1)(1-\sigma)+(b'+2)(2\sigma-1)+1} + \mathcal{V}(\log T_2)^2. \tag{4.164}$$

This completes the proof. □

Remark 4.47. For specific values of a,a',b,b' from the tables, we have the following:

- For the first version of our bound obtained from Table 4.1, we have $a = \frac{4}{3}$, $b = 2$ and from the Table 4.4, we have $a' = 1$, $b' = 0$. Therefore we have

$$N(\sigma, T_1, T_2) \leq (T_2 - T_1) \log T_2 \log \left(1 + \mathcal{U} \frac{T_2^{\frac{8}{3}(1-\sigma)} \log T_2^{(4-2\sigma)}}{(T_2 - T_1)}\right) + \mathcal{V}(\log T_2)^2, \tag{4.165}$$

and

$$N(\sigma, T_1, T_2) \leq \mathcal{U} T_2^{\frac{8}{3}(1-\sigma)} \log T_2^{(5-2\sigma)} + \mathcal{V}(\log T_2)^2. \tag{4.166}$$

Kadiri, Lumley and Ng [26] have presented the following result:

$$N(0.90, T) < 11.499(\log T)^{\frac{16}{5}} T^{\frac{4}{15}} + 3.186(\log T)^2,$$

For specific values of $\sigma = 0.9$, $k = 2$, $\eta = 0.20341$, $T_0 = X = 3 \cdot 10^{12}$, $\delta_1 = 0.36$, $\delta_2 = 0.35$ and $\alpha = 0.241$ we can see how this improves on Kadiri, Lumley and Ng's estimate [26]:

$$N(0.90, T) < 7.733(\log T)^{\frac{16}{5}} T^{\frac{4}{15}} + 2.6993(\log T)^2. \tag{4.167}$$

In addition, you can see the numerical result for this version in Table A.1.

Also we can easily compare our bound with the recent updated bound from Fiori, Kadiri and Swidinsky [8] as well:

$$N(0.90, T) < 10.8209(\log T)^{\frac{16}{5}} T^{\frac{4}{15}} + 2.8640(\log T)^2.$$

- For the second version from Table 4.2 we have $a = \frac{109}{82}, b = 0$ and similarly from the Table 4.4, we have $a' = 1, b' = 0$. Therefore, we have

$$N(\sigma, T_1, T_2) \leq (T_2 - T_1)(\log T_2) \log \left(1 + \mathcal{U} \frac{T_2^{\frac{109}{41}(1-\sigma)} \log T_2^{(2\sigma)}}{(T_2 - T_1)}\right) + \mathcal{V}(\log T_2)^2, \tag{4.168}$$

and

$$N(\sigma,T_1,T_2) \le \mathcal{U} T_2^{\frac{109}{41}(1-\sigma)} \log T_2^{(2\sigma+1)} + \mathcal{V}(\log T_2)^2. \tag{4.169}$$

For specific values of $\sigma = 0.9$, $k = 2$, $\eta = 0.20341$, $T_0 = X = 3\cdot 10^{12}$, $\delta_1 = 0.30$, $\delta_2 = 0.31$ and $\alpha = 0.189$ we obtain for $T_2 \ge T_0$ and $T_1 \ge H_0$ that:

$$N(0.9,T_1,T_2) \le 58.947 T_2^{\frac{109}{410}} \log T_2^{\frac{14}{5}} + 2.6993(\log T_2)^2. \tag{4.170}$$

In addition, you can see the numerical result for this version in table A.3.

- For the third version of our obtained bound from Table 4.3, we have $a = \frac{4}{3}$, $b = 2$ and from the Table 4.4, we have $a' = 1$, $b' = 0$. Therefore we have

$$N(\sigma,T_1,T_2) \le (T_2 - T_1)(\log T_2)\log\left(1 + \mathcal{U}\frac{T_2^{\frac{8}{3}(1-\sigma)}(\log T_2)^{(4-2\sigma)}}{(T_2 - T_1)}\right) + \mathcal{V}(\log T_2)^2, \tag{4.171}$$

and

$$N(\sigma,T_1,T_2) \le \mathcal{U} T_2^{\frac{8}{3}(1-\sigma)} \log T_2^{(5-2\sigma)} + \mathcal{V}(\log T_2)^2. \tag{4.172}$$

For specific values of $\sigma = 0.9$, $k = 2$, $\eta = 0.20341$, $T_0 = X = 3\cdot 10^{12}$, $\delta_1 = 0.36$, $\delta_2 = 0.35$ and $\alpha = 0.240$ we obtain for $T_2 \ge T_0$ and $T_1 \ge H_0$ that:

$$N(0.90,T) < 7.683(\log T)^{\frac{16}{5}} T^{\frac{4}{15}} + 2.6993(\log T)^2, \tag{4.173}$$

In addition, you can see the numerical result for this version in table A.5.

Remark 4.48. We noticed that for $\sigma = 0.9$ the second version, (4.170) is better than the first version, (4.167) for $T_2 \ge 4\cdot 10^{56}$.

Remark 4.49. We note that Remark 4.47 is the bound for $N(\sigma,T_1,T_2)$ at $\sigma = 0.9$. The following results are comparison of the of upper bounds for $N(\sigma,T_1,T_2)$ for different large values of T_0 at $\sigma = 0.99$:

- For specific values of $k = 2$, $\eta = 0.20341$, $T_0 = X = 3 \cdot 10^{12}$, $\delta_1 = 0.35$, $\delta_2 = 0.34$ and $\alpha = 0.146$ by using Table 4.1, we obtain for $T_2 \geq T_0$ and $T_1 \geq H_0$ that

$$N(0.99, T) < 11.912(\log T)^{\frac{151}{50}} T^{\frac{4}{150}} + 1.998(\log T)^2 \tag{4.174}$$

- For specific values of $k = 2$, $\eta = 0.0598$, $T_0 = X = 3 \cdot 10^{100}$, $\delta_1 = 0.36$, $\delta_2 = 0.36$ and $\alpha = 0.091$ by using Table 4.1, we obtain for $T_2 \geq T_0$ and $T_1 \geq H_0$ that

$$N(0.99, T) < 9.497(\log T)^{\frac{151}{50}} T^{\frac{4}{150}} + 0.4239(\log T)^2. \tag{4.175}$$

- For specific values of $k = 2$, $\eta = 0.20341$, $T_0 = X = 3 \cdot 10^{12}$, $\delta_1 = 0.30$, $\delta_2 = 0.31$ and $\alpha = 0.1009$ by using Table 4.2, we obtain for $T_2 \geq T_0$ and $T_1 \geq H_0$ that

$$N(\sigma, T_1, T_2) \leq 16.824 T_2^{\frac{109}{4100}} \log T_2^{\frac{149}{50}} + 1.998(\log T_2)^2. \tag{4.176}$$

- For specific values of $k = 2$, $\eta = 0.0598$, $T_0 = X = 3 \cdot 10^{100}$, $\delta_1 = 0.36$, $\delta_2 = 0.36$ and $\alpha = 0.0.085$ by using Table 4.2 we obtain for $T_2 \geq T_0$ and $T_1 \geq H_0$ that

$$N(\sigma, T_1, T_2) \leq 11.497 T_2^{\frac{109}{4100}} \log T_2^{\frac{149}{50}} + 0.4239(\log T_2)^2. \tag{4.177}$$

- For specific values of $k = 2$, $\eta = 0.20341$, $T_0 = X = 3 \cdot 10^{12}$, $\delta_1 = 0.35$, $\delta_2 = 0.34$ and $\alpha = 0.146$ by using Table 4.3 we obtain for $T_2 \geq T_0$ and $T_1 \geq H_0$ that

$$N(0.99, T) < 11.888(\log T)^{\frac{151}{50}} T^{\frac{4}{150}} + 1.998(\log T)^2 \tag{4.178}$$

- For specific values of $k = 2$, $\eta = 0.0598$, $T_0 = X = 3 \cdot 10^{100}$, $\delta_1 = 0.36$, $\delta_2 = 0.35$ and

$\alpha = 0.091$ by using Table 4.3 we obtain for $T_2 \geq T_0$ and $T_1 \geq H_0$ that

$$N(0.99, T) < 9.386(\log T)^{\frac{151}{50}} T^{\frac{4}{150}} + 0.4239(\log T)^2. \tag{4.179}$$

Using our result from Table 4.1, we can obtain:

Corollary 4.50. *For* $T_2 \geq 3 \cdot 10^{12}$ *and* $\sigma \in [0.75, 1]$ *we have*

$$N(\sigma, T_1, T_2) \leq 12.45321 T_2^{\frac{8}{3}(1-\sigma)} \log T_2^{(5-2\sigma)} + 3.869(\log T_2)^2. \tag{4.180}$$

Proof. The table A.7 provides numerical results for $\sigma \in [0.75, 1]$, showing that for each interval the maximum value of $\mathcal{U}$ occurs at the end points and the maximum value of $\mathcal{V}$ occurs at the start points. □

Using our result from Table 4.3, we can obtain:

Corollary 4.51. *For* $T_2 \geq 3 \cdot 10^{12}$ *and* $\sigma \in [0.75, 1]$ *we have*

$$N(\sigma, T_1, T_2) \leq 12.4351 T_2^{\frac{8}{3}(1-\sigma)} \log T_2^{(5-2\sigma)} + 3.869(\log T_2)^2. \tag{4.181}$$

Proof. The table A.11 provides numerical results for $\sigma \in [0.75, 1]$, showing that for each interval the maximum value of $\mathcal{U}$ occurs at the end points and the maximum value of $\mathcal{V}$ occurs at the start points. □

Remark 4.52. Corollary 4.51 can be compared with the bound obtained by Fiori, Kadiri and Swidinsky [8]. Their result shows for $T \geq 3 \cdot 10^{12}$ and $\sigma \in [0.75, 1]$ they would have:

$$N(\sigma, T) \leq 17.0819 T^{\frac{8}{3}(1-\sigma)} \log T^{(5-2\sigma)} + 4.4648(\log T_2)^2. \tag{4.182}$$

Remark 4.53. Similar results can be obtained for different ranges of σ and different values for T_0. For specific numerical values corresponding to different choices of σ and T_0, see table A.8 and A.12.

Using our result from Table 4.2, we can obtain:

Corollary 4.54. *For* $T_2 \geq 3 \cdot 10^{12}$ *and* $\sigma \in [0.9, 1]$ *we have*

$$N(\sigma, T_1, T_2) \leq 58.947 T_2^{\frac{109}{41}(1-\sigma)} \log T_2^{(2\sigma+1)} + 2.6993(\log T_2)^2, \tag{4.183}$$

Moreover, for $T_2 \geq 3 \cdot 10^{12}$ *and* $\sigma \in [0.99, 1]$ *we have*

$$N(\sigma, T_1, T_2) \leq 14.382 T_2^{\frac{109}{41}(1-\sigma)} \log T_2^{(2\sigma+1)} + 1.9975(\log T_2)^2. \tag{4.184}$$

Proof. The Table A.9 provides numerical results for $\sigma \in [0.75, 1]$, showing that for each interval the maximum value of both $\mathcal{U}$ and $\mathcal{V}$ occurs at the start point. □

Remark 4.55. Similar results can be obtained for different ranges of σ and different values for T_0. For specific numerical values corresponding to different choices of σ and T_0, see Table A.10.

Chapter 5

Conclusions and Future Work

In this thesis, we have proved a general version for the upper bounds for $N(\sigma, T_1, T_2)$ in our main Theorem 4.46. The theorem can be applied with different bounds for zeta function on the half line and in our work we have applied it with three different half line bounds: See Tables 4.1, 4.2 and 4.3. Our result improves upon the previous work by Kadiri, Lumley and Ng [26] and the ways that this bound is improved are:

- In the versions obtained from Tables 4.1 and 4.3 we used the improved sub-convexity bound for zeta function on the half line (Lemma 4.14) and used the updated *RH* verification by Platt-Trudgian. Additionally, we have improved Ramaré's upper bound for $\sum_{n\geq 1} \frac{\lambda_X(n)^2}{n^\tau}$ in Lemma 3.9 and optimized the key parameters, α, δ_1 and δ_2 with selected values provided in Tables A.7 and A.11. There are other small adjustments that improve the result.

  We extend the zero density result to one which is valid on intervals, improving upon the previous work by Fiori, Kadiri and Swidinsky [8]. (See Corollary 4.50 and 4.51 and it's corresponding Tables A.7 and A.11).

- For the version obtained from Table 4.2 we used the sub-Weyl bounds for the zeta function on the half-line by Patel and Yang [33]. This version strengthens the zero density result for larger values of T_2, (See Remark 4.48). Similar to the first version, we extended this version of the zero density result to one which is valid on intervals. (See Corollary 4.54 and table of values in A.9).

Beyond the numerical improvements, an essential contribution of our thesis is the structural reorganization of the proof. By breaking down the argument in to key components and performing the zero density result in a general form, we provide a clear pathway for finding an explicit zero density result for other L-functions. The main requirement to do this for other L-functions is finding an upper bounds on the half line, and analyzing the corresponding arithmetic sums.

In terms of potential future work there are still several potential sources of improvements:

- Using different mollifiers, which potentially we can save a logarithmic factor,
- Using a more sophisticated weight functions, which could save up to factor of 1.46 on dominant term, that is the $\mathcal{U}$, (as suggested by Remark 2.26.)
- Improving the bounds for the zeta function,
- Refining some of the arithmetic sums, for instance we mentioned one in Remark 3.7.
- Improving the secondary error term for the second and third integrals by several options.

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Appendix A

Calculations

We present calculations illustrating Theorem 4.46, Remark 4.47 and Corollaries 4.50 and 4.54 based on different values of Tables 4.1 and 4.2.

The values in tables were computed using Python with the SymPy and NumPy libraries. The computations were performed with code available on GitHub:

`https://github.com/golnoushfarzan/Thesis`

During the computations, all parameters, α, δ_1, δ_2 and η were optimized. The values presented in the table are rounded and the output values $\mathcal{U}$ and $\mathcal{V}$ are all rounded up.

Table A.1: Sample of values for equations (4.165) and (4.166). These values are for $T_0 = 3 \cdot 10^{12}$ and $H_0 = 3 \cdot 10^{12}$. We use $\mu = 1 + \frac{k \log\log(X)}{\log X}$ with $k = 2$ and η approximately 0.20341. The sub-convexity bound corresponds to table 4.1. In particular for each row for all $T_1 > H_0$ and $T_2 > T_0$ we have $N(\sigma, T_1, T_2) \leq \mathcal{U} T_2^{\frac{8}{3}(1-\sigma)} \log T_2^{(5-2\sigma)} + \mathcal{V}(\log T_2)^2$

σ	α	δ_1	δ_2	$\mathcal{U}$	$\mathcal{V}$
0.750	0.409355	0.382366	0.358985	3.436480	3.868940
0.760	0.398079	0.381098	0.358549	3.637080	3.790960
0.770	0.386811	0.379884	0.358105	3.848100	3.712980
0.780	0.375555	0.378717	0.357650	4.069980	3.635000
0.790	0.364310	0.377590	0.357185	4.303130	3.557020
0.800	0.353077	0.376495	0.356708	4.547990	3.479040
0.810	0.341859	0.375429	0.356220	4.804980	3.401060
0.820	0.330658	0.374386	0.355718	5.074550	3.323090
0.830	0.319477	0.373362	0.355203	5.357110	3.245110
0.840	0.308317	0.372354	0.354673	5.653090	3.167130
0.850	0.297181	0.371354	0.354126	5.962890	3.089150
0.860	0.286072	0.370364	0.353563	6.286900	3.011170
0.870	0.274994	0.369378	0.352982	6.625510	2.933190
0.880	0.263951	0.368393	0.352382	6.979070	2.855210
0.890	0.252947	0.367406	0.351761	7.347900	2.777240
0.900	0.241987	0.366414	0.351117	7.732300	2.699260
0.910	0.231077	0.365414	0.350450	8.132490	2.621280
0.920	0.220223	0.364403	0.349758	8.548700	2.543300
0.930	0.209432	0.363378	0.349037	8.981040	2.465320
0.940	0.198714	0.362337	0.348289	9.429580	2.387340
0.950	0.188078	0.361274	0.347508	9.894320	2.309360
0.960	0.177534	0.360188	0.346693	10.375150	2.231390
0.970	0.167096	0.359076	0.345841	10.871850	2.153410
0.980	0.156779	0.357933	0.344950	11.384100	2.075430
0.990	0.146600	0.356755	0.344016	11.911420	1.997450

Table A.2: Sample of values for equations (4.165) and (4.166). These values are for $T_0 = 3 \cdot 10^{100}$ and $H_0 = 3 \cdot 10^{12}$. We use $\mu = 1 + \frac{k \log\log(X)}{\log X}$ with $k = 2$ and η approximately 0.0598. The sub-convexity bound corresponds to table 4.1. In particular for each row for all $T_1 > H_0$ and $T_2 > T_0$ we have $N(\sigma, T_1, T_2) \leq \mathcal{U} T_2^{\frac{8}{3}(1-\sigma)} \log T_2^{(5-2\sigma)} + \mathcal{V}(\log T_2)^2$.

σ	α	δ_1	δ_2	$\mathcal{U}$	$\mathcal{V}$
0.750	0.359348	0.373249	0.370307	2.903130	2.159860
0.760	0.347236	0.373090	0.370239	3.070630	2.087520
0.770	0.335160	0.372935	0.370169	3.246390	2.015190
0.780	0.323123	0.372782	0.370095	3.430670	1.942850
0.790	0.311128	0.372633	0.370021	3.623740	1.870510
0.800	0.299178	0.372487	0.369944	3.825840	1.798180
0.810	0.287279	0.372342	0.369864	4.037220	1.725840
0.820	0.275433	0.372198	0.369781	4.258090	1.653510
0.830	0.263646	0.372055	0.369695	4.488660	1.581170
0.840	0.251924	0.371912	0.369605	4.729110	1.508830
0.850	0.240272	0.371769	0.369511	4.979580	1.436500
0.860	0.228699	0.371624	0.369414	5.240200	1.364160
0.870	0.217211	0.371477	0.369313	5.511030	1.291830
0.880	0.205819	0.371329	0.369206	5.792100	1.219490
0.890	0.194532	0.371178	0.369096	6.083390	1.147150
0.900	0.183363	0.371027	0.368980	6.384820	1.074820
0.910	0.172323	0.370867	0.368857	6.696220	1.002480
0.920	0.161432	0.370706	0.368729	7.017350	0.930150
0.930	0.150703	0.370540	0.368595	7.347880	0.857810
0.940	0.140161	0.370367	0.368453	7.687400	0.785480
0.950	0.129827	0.370189	0.368303	8.035340	0.713140
0.960	0.119727	0.370004	0.368144	8.391050	0.640800
0.970	0.109892	0.369810	0.367977	8.753740	0.568470
0.980	0.100357	0.369610	0.367798	9.122440	0.496130
0.990	0.091160	0.369400	0.367610	9.496080	0.423800

Table A.3: Sample of values for equations (4.168) and (4.169). These values are for $T_0 = 3 \cdot 10^{12}$ and $H_0 = 3 \cdot 10^{12}$. We use $\mu = 1 + \frac{k \log\log(X)}{\log X}$ with $k = 2$ and η approximately 0.20341. The sub-Weyl bound corresponds to table 4.2. In particular for each row for all $T_1 > H_0$ and $T_2 > T_0$ we have $N(\sigma, T_1, T_2) \leq \mathcal{U} T_2^{\frac{109}{41}(1-\sigma)} \log T_2^{(2\sigma+1)} + \mathcal{V}(\log T_2)^2$

σ	α	δ_1	δ_2	$\mathcal{U}$	$\mathcal{V}$
0.750	0.350851	0.306917	0.321834	427.731850	3.868940
0.760	0.339864	0.306805	0.321702	375.935260	3.790960
0.770	0.328896	0.306689	0.321565	330.287930	3.712980
0.780	0.317947	0.306570	0.321425	290.071730	3.635000
0.790	0.307018	0.306447	0.321280	254.651200	3.557020
0.800	0.296112	0.306319	0.321130	223.464170	3.479040
0.810	0.285234	0.306188	0.320975	196.013420	3.401060
0.820	0.274385	0.306052	0.320815	171.859270	3.323090
0.830	0.263570	0.305911	0.320649	150.612980	3.245110
0.840	0.252792	0.305766	0.320477	131.930970	3.167130
0.850	0.242055	0.305614	0.320299	115.509600	3.089150
0.860	0.231366	0.305456	0.320112	101.080630	3.011170
0.870	0.220728	0.305296	0.319920	88.407130	2.933190
0.880	0.210154	0.305123	0.319719	77.279870	2.855210
0.890	0.199645	0.304944	0.319509	67.514150	2.777240
0.900	0.189211	0.304760	0.319290	58.946940	2.699260
0.910	0.178864	0.304567	0.319061	51.434400	2.621280
0.920	0.168614	0.304365	0.318821	44.849640	2.543300
0.930	0.158474	0.304153	0.318570	39.080770	2.465320
0.940	0.148461	0.303931	0.318306	34.029120	2.387340
0.950	0.138589	0.303698	0.318029	29.607760	2.309360
0.960	0.128882	0.303453	0.317738	25.740080	2.231390
0.970	0.119360	0.303196	0.317431	22.358590	2.153410
0.980	0.110050	0.302925	0.317108	19.403900	2.075430
0.990	0.100983	0.302640	0.316767	16.823700	1.997450

Table A.4: Sample of values for equations (4.168) and (4.169). These values are for $T_0 = 3 \cdot 10^{100}$ and $H_0 = 3 \cdot 10^{12}$. We use $\mu = 1 + \frac{k \log\log(X)}{\log X}$ with $k = 2$ and η approximately 0.0598. The sub-Weyl bound corresponds to table 4.2. In particular for each row for all $T_1 > H_0$ and $T_2 > T_0$ we have $N(\sigma, T_1, T_2) \le \mathcal{U} T_2^{\frac{109}{41}(1-\sigma)} \log T_2^{(2\sigma+1)} + \mathcal{V}(\log T_2)^2$

σ	α	δ_1	δ_2	$\mathcal{U}$	$\mathcal{V}$
0.750	0.351352	0.365426	0.368308	315.111530	2.159860
0.760	0.339313	0.365402	0.368282	276.386650	2.087520
0.770	0.327312	0.365378	0.368256	242.314410	2.015190
0.780	0.315351	0.365352	0.368229	212.346280	1.942850
0.790	0.303433	0.365325	0.368201	185.997340	1.870510
0.800	0.291563	0.365298	0.368172	162.839030	1.798180
0.810	0.279744	0.365270	0.368143	142.492670	1.725840
0.820	0.267981	0.365240	0.368111	124.623760	1.653510
0.830	0.256277	0.365208	0.368084	108.936910	1.581170
0.840	0.244644	0.365178	0.368048	95.171320	1.508830
0.850	0.233083	0.365147	0.368010	83.096790	1.436500
0.860	0.221604	0.365112	0.367975	72.510200	1.364160
0.870	0.210214	0.365076	0.367937	63.232370	1.291830
0.880	0.198923	0.365039	0.367898	55.105310	1.219490
0.890	0.187743	0.365001	0.367857	47.989690	1.147150
0.900	0.176687	0.364960	0.367814	41.762760	1.074820
0.910	0.165767	0.364917	0.367767	36.316330	1.002480
0.920	0.155003	0.364874	0.367720	31.555150	0.930150
0.930	0.144411	0.364827	0.367671	27.395320	0.857810
0.940	0.134015	0.364777	0.367618	23.763010	0.785480
0.950	0.123839	0.364727	0.367562	20.593280	0.713140
0.960	0.113912	0.364675	0.367502	17.828980	0.640800
0.970	0.104265	0.364617	0.367441	15.419900	0.568470
0.980	0.094935	0.364560	0.367372	13.321880	0.496130
0.990	0.085963	0.364495	0.367306	11.496170	0.423800

Table A.5: Sample of values for equations (4.171) and (4.172). These values are for $T_0 = 3 \cdot 10^{12}$ and $H_0 = 3 \cdot 10^{12}$. We use $\mu = 1 + \frac{k \log\log(X)}{\log X}$ with $k = 2$ and η approximately 0.20341. The sub-convexity bound corresponds to table 4.3. In particular for each row for all $T_1 > H_0$ and $T_2 > T_0$ we have $N(\sigma, T_1, T_2) \leq \mathcal{U} T_2^{\frac{8}{3}(1-\sigma)} \log T_2^{(5-2\sigma)} + \mathcal{V}(\log T_2)^2$

σ	α	δ_1	δ_2	$\mathcal{U}$	$\mathcal{V}$
0.750	0.405202	0.382421	0.359032	3.395870	3.868940
0.760	0.394087	0.381157	0.358600	3.595230	3.790960
0.770	0.382980	0.379947	0.358159	3.805060	3.712980
0.780	0.371884	0.378784	0.357708	4.025770	3.635000
0.790	0.360799	0.377662	0.357248	4.257810	3.557020
0.800	0.349727	0.376571	0.356774	4.501620	3.479040
0.810	0.338669	0.375510	0.356292	4.757650	3.401060
0.820	0.327628	0.374472	0.355794	5.026350	3.323090
0.830	0.316607	0.373453	0.355283	5.308140	3.245110
0.840	0.305606	0.372449	0.354758	5.603480	3.167130
0.850	0.294629	0.371456	0.354217	5.912800	3.089150
0.860	0.283679	0.370471	0.353659	6.236500	3.011170
0.870	0.272760	0.369491	0.353084	6.574990	2.933190
0.880	0.261875	0.368513	0.352490	6.928650	2.855210
0.890	0.251028	0.367532	0.351876	7.297830	2.777240
0.900	0.240225	0.366548	0.351239	7.682850	2.699260
0.910	0.229472	0.365554	0.350578	8.084000	2.621280
0.920	0.218772	0.364553	0.349895	8.501510	2.543300
0.930	0.208136	0.363536	0.349183	8.935560	2.465320
0.940	0.197571	0.362505	0.348443	9.386250	2.387340
0.950	0.187086	0.361452	0.347672	9.853620	2.309360
0.960	0.176693	0.360378	0.346867	10.337620	2.231390
0.970	0.166404	0.359277	0.346026	10.838100	2.153410
0.980	0.156232	0.358147	0.345148	11.354780	2.075430
0.990	0.146194	0.356982	0.344226	11.887270	1.997450

Table A.6: Sample of values for equations (4.171) and (4.172). These values are for $T_0 = 3 \cdot 10^{100}$ and $H_0 = 3 \cdot 10^{12}$. We use $\mu = 1 + \frac{k \log\log(X)}{\log X}$ with $k = 2$ and η approximately 0.0598. The sub-convexity bound corresponds to table 4.3. In particular for each row for all $T_1 > H_0$ and $T_2 > T_0$ we have $N(\sigma, T_1, T_2) \leq \mathcal{U} T_2^{\frac{8}{3}(1-\sigma)} \log T_2^{(5-2\sigma)} + \mathcal{V}(\log T_2)^2$.

σ	α	δ_1	δ_2	$\mathcal{U}$	$\mathcal{V}$
0.750	0.359288	0.373786	0.370835	2.318590	2.159860
0.760	0.347180	0.373624	0.370768	2.474240	2.087520
0.770	0.335106	0.373470	0.370697	2.639180	2.015190
0.780	0.323072	0.373318	0.370624	2.813870	1.942850
0.790	0.311080	0.373168	0.370549	2.998720	1.870510
0.800	0.299134	0.373021	0.370471	3.194200	1.798180
0.810	0.287237	0.372876	0.370390	3.400740	1.725840
0.820	0.275394	0.372732	0.370308	3.618770	1.653510
0.830	0.263610	0.372589	0.370221	3.848740	1.581170
0.840	0.251891	0.372445	0.370131	4.091070	1.508830
0.850	0.240243	0.372302	0.370039	4.346170	1.436500
0.860	0.228672	0.372155	0.369940	4.614430	1.364160
0.870	0.217188	0.372010	0.369838	4.896200	1.291830
0.880	0.205798	0.371862	0.369732	5.191810	1.219490
0.890	0.194514	0.371711	0.369621	5.501550	1.147150
0.900	0.183348	0.371557	0.369505	5.825650	1.074820
0.910	0.172312	0.371398	0.369382	6.164270	1.002480
0.920	0.161422	0.371237	0.369254	6.517520	0.930150
0.930	0.150697	0.371069	0.369118	6.885390	0.857810
0.940	0.140158	0.370896	0.368976	7.267790	0.785480
0.950	0.129826	0.370717	0.368826	7.664520	0.713140
0.960	0.119728	0.370532	0.368666	8.075230	0.640800
0.970	0.109896	0.370338	0.368498	8.499420	0.568470
0.980	0.100364	0.370137	0.368319	8.936450	0.496130
0.990	0.091168	0.369925	0.368131	9.385490	0.423800

Table A.7: Sample of values for equations (4.165) and (4.166). These values are for $T_0 = 3\cdot 10^{12}$ and $H_0 = 3\cdot 10^{12}$. We use $\mu = 1+\frac{k\log\log(X)}{\log X}$ with $k=2$ and η approximately 0.20341. The sub-convexity bound corresponds to table 4.1. In particular for each row for all $\sigma \in [\sigma_1', \sigma_2']$, $T_1 > H_0$ and $T_2 > T_0$ we have $N(\sigma, T_1, T_2) \leq \mathcal{U} T_2^{\frac{8}{3}(1-\sigma)} \log T_2^{(5-2\sigma)} + \mathcal{V}(\log T_2)^2$

σ_1'	σ_2'	α	δ_1	δ_2	$\mathcal{U}$	$\mathcal{V}$
0.750	0.760	0.398079	0.381098	0.358549	3.637080	3.868940
0.760	0.770	0.386811	0.379884	0.358105	3.848100	3.790960
0.770	0.780	0.375555	0.378717	0.357650	4.069980	3.712980
0.780	0.790	0.364310	0.377590	0.357185	4.303130	3.635000
0.790	0.800	0.353077	0.376495	0.356707	4.547990	3.557020
0.800	0.810	0.341859	0.375429	0.356220	4.804980	3.479040
0.810	0.820	0.330658	0.374386	0.355718	5.074550	3.401060
0.820	0.830	0.319477	0.373363	0.355203	5.357110	3.323090
0.830	0.840	0.308317	0.372354	0.354673	5.653090	3.245110
0.840	0.850	0.297181	0.371355	0.354126	5.962890	3.167130
0.850	0.860	0.286072	0.370364	0.353563	6.286900	3.089150
0.860	0.870	0.274994	0.369378	0.352982	6.625510	3.011170
0.870	0.880	0.263951	0.368393	0.352382	6.979070	2.933190
0.880	0.890	0.252947	0.367406	0.351761	7.347900	2.855210
0.890	0.900	0.241987	0.366413	0.351117	7.732300	2.777240
0.900	0.910	0.231077	0.365414	0.350450	8.132490	2.699260
0.910	0.920	0.220223	0.364403	0.349758	8.548700	2.621280
0.920	0.930	0.209432	0.363378	0.349037	8.981040	2.543300
0.930	0.940	0.198714	0.362337	0.348289	9.429580	2.465320
0.940	0.950	0.188078	0.361274	0.347508	9.894320	2.387340
0.950	0.960	0.177534	0.360188	0.346693	10.375150	2.309360
0.960	0.970	0.167096	0.359076	0.345841	10.871850	2.231390
0.970	0.980	0.156779	0.357933	0.344950	11.384100	2.153410
0.980	0.990	0.146600	0.356755	0.344016	11.911420	2.075430
0.990	0.991	0.145590	0.356634	0.343920	11.964960	1.997450
0.991	0.992	0.144583	0.356515	0.343824	12.018640	1.989650
0.992	0.993	0.143576	0.356395	0.343727	12.072470	1.981850
0.993	0.994	0.142571	0.356274	0.343630	12.126440	1.974060
0.994	0.995	0.141568	0.356153	0.343533	12.180550	1.966260
0.995	0.996	0.140567	0.356031	0.343434	12.234800	1.958460
0.996	0.997	0.139567	0.355909	0.343336	12.289190	1.950660
0.997	0.998	0.138569	0.355786	0.343236	12.343720	1.942860
0.998	0.999	0.137573	0.355663	0.343137	12.398390	1.935070
0.999	1.000	0.136578	0.355540	0.343037	12.453200	1.927270

Table A.8: Sample of values for equations (4.165) and (4.166). These values are for $T_0 = 3\cdot 10^{100}$ and $H_0 = 3\cdot 10^{12}$. We use $\mu = 1+\frac{k\log\log(X)}{\log X}$ with $k=2$ and η approximately 0.0598. The sub-convexity bound corresponds to table 4.1. In particular for each row for all $\sigma \in [\sigma_1', \sigma_2']$, $T_1 > H_0$ and $T_2 > T_0$ we have $N(\sigma,T_1,T_2) \le \mathcal{U} T_2^{\frac{8}{3}(1-\sigma)} \log T_2^{(5-2\sigma)} + \mathcal{V}(\log T_2)^2$.

σ_1'	σ_2'	α	δ_1	δ_2	$\mathcal{U}$	$\mathcal{V}$
0.750	0.760	0.347236	0.373090	0.370239	3.070630	2.159860
0.760	0.770	0.335160	0.372935	0.370169	3.246390	2.087520
0.770	0.780	0.323123	0.372782	0.370095	3.430670	2.015190
0.780	0.790	0.311128	0.372633	0.370021	3.623740	1.942850
0.790	0.800	0.299178	0.372487	0.369944	3.825840	1.870510
0.800	0.810	0.287279	0.372342	0.369864	4.037220	1.798180
0.810	0.820	0.275433	0.372198	0.369781	4.258090	1.725840
0.820	0.830	0.263646	0.372055	0.369695	4.488660	1.653510
0.830	0.840	0.251924	0.371912	0.369605	4.729110	1.581170
0.840	0.850	0.240272	0.371769	0.369511	4.979580	1.508830
0.850	0.860	0.228699	0.371624	0.369414	5.240200	1.436500
0.860	0.870	0.217211	0.371477	0.369313	5.511030	1.364160
0.870	0.880	0.205819	0.371329	0.369206	5.792100	1.291830
0.880	0.890	0.194532	0.371178	0.369096	6.083390	1.219490
0.890	0.900	0.183363	0.371027	0.368980	6.384820	1.147150
0.900	0.910	0.172323	0.370867	0.368857	6.696220	1.074820
0.910	0.920	0.161432	0.370706	0.368729	7.017350	1.002480
0.920	0.930	0.150703	0.370540	0.368595	7.347880	0.930150
0.930	0.940	0.140161	0.370367	0.368452	7.687400	0.857810
0.940	0.950	0.129827	0.370189	0.368303	8.035340	0.785480
0.950	0.960	0.119727	0.370004	0.368144	8.391050	0.713140
0.960	0.970	0.109892	0.369810	0.367977	8.753740	0.640800
0.970	0.980	0.100357	0.369610	0.367798	9.122440	0.568470
0.980	0.990	0.091160	0.369399	0.367610	9.496080	0.496130
0.990	0.991	0.090260	0.369378	0.367591	9.533670	0.423800
0.991	0.992	0.089364	0.369356	0.367572	9.571300	0.416560
0.992	0.993	0.088472	0.369333	0.367553	9.608950	0.409330
0.993	0.994	0.087584	0.369311	0.367534	9.646650	0.402090
0.994	0.995	0.086700	0.369288	0.367515	9.684370	0.394860
0.995	0.996	0.085820	0.369267	0.367494	9.722120	0.387630
0.996	0.997	0.084943	0.369244	0.367475	9.759910	0.380390
0.997	0.998	0.084072	0.369221	0.367454	9.797720	0.373160
0.998	0.999	0.083204	0.369199	0.367435	9.835550	0.365930
0.999	1.000	0.082340	0.369176	0.367415	9.873410	0.358690

Table A.9: Sample of values for equations (4.168) and (4.169). These values are for $T_0 = 3 \cdot 10^{12}$ and $H_0 = 3 \cdot 10^{12}$. We use $\mu = 1 + \frac{k \log\log(X)}{\log X}$ with $k = 2$ and η approximately 0.20341. The sub-Weyl bound corresponds to table 4.2. In particular for each row for all $\sigma \in [\sigma_1', \sigma_2']$, $T_1 > H_0$ and $T_2 > T_0$ we have $N(\sigma, T_1, T_2) \leq \mathcal{U} T_2^{\frac{109}{41}(1-\sigma)} \log T_2^{(2\sigma+1)} + \mathcal{V}(\log T_2)^2$.

σ_1'	σ_2'	α	δ_1	δ_2	$\mathcal{U}$	$\mathcal{V}$
0.750	0.760	0.349069	0.203710	0.322351	521.416920	3.868940
0.760	0.770	0.338145	0.211061	0.322133	453.844500	3.790960
0.770	0.780	0.327247	0.218322	0.321922	394.641360	3.712980
0.780	0.790	0.316377	0.225495	0.321715	342.830970	3.635000
0.790	0.800	0.305537	0.232580	0.321513	297.540800	3.557020
0.800	0.810	0.294731	0.239572	0.321314	257.992980	3.479040
0.810	0.820	0.283958	0.246487	0.321117	223.495550	3.401060
0.820	0.830	0.273224	0.253308	0.320919	193.434130	3.323090
0.830	0.840	0.262531	0.260042	0.320723	167.264270	3.245110
0.840	0.850	0.251883	0.266690	0.320526	144.504260	3.167130
0.850	0.860	0.241284	0.273251	0.320329	124.728600	3.089150
0.860	0.870	0.230738	0.279727	0.320129	107.562030	3.011170
0.870	0.880	0.220252	0.286115	0.319926	92.674010	2.933190
0.880	0.890	0.209830	0.292417	0.319719	79.773840	2.855210
0.890	0.900	0.199481	0.298633	0.319507	68.606160	2.777240
0.900	0.910	0.189211	0.304761	0.319290	58.946940	2.699260
0.910	0.920	0.179032	0.310799	0.319066	50.599850	2.621280
0.920	0.930	0.168952	0.316751	0.318836	43.393050	2.543300
0.930	0.940	0.158982	0.322612	0.318595	37.176280	2.465320
0.940	0.950	0.149139	0.328382	0.318347	31.818330	2.387340
0.950	0.960	0.139437	0.334061	0.318087	27.204690	2.309360
0.960	0.970	0.129895	0.339646	0.317816	23.235550	2.231390
0.970	0.980	0.120533	0.345136	0.317532	19.824020	2.153410
0.980	0.990	0.111375	0.350529	0.317234	16.894490	2.075430
0.990	0.991	0.102451	0.355824	0.316921	14.381250	1.997450
0.991	0.992	0.101572	0.356348	0.316889	14.150560	1.989650
0.992	0.993	0.100696	0.356871	0.316857	13.923400	1.981850
0.993	0.994	0.099823	0.357393	0.316824	13.699720	1.974060
0.994	0.995	0.098952	0.357914	0.316792	13.479470	1.966260
0.995	0.996	0.098084	0.358434	0.316759	13.262600	1.958460
0.996	0.997	0.097219	0.358953	0.316726	13.049060	1.950660
0.997	0.998	0.096357	0.359471	0.316693	12.838810	1.942860
0.998	0.999	0.095498	0.359987	0.316660	12.631780	1.935070
0.999	1.000	0.094641	0.360503	0.316626	12.427940	1.927270

Table A.10: Sample of values for equations (4.168) and (4.169). These values are for $T_0 = 3 \cdot 10^{100}$ and $H_0 = 3 \cdot 10^{12}$. We use $\mu = 1 + \frac{k \log\log(X)}{\log X}$ with $k = 2$ and η approximately 0.0598. The sub-Weyl bound corresponds to table 4.2. In particular for each row for all $\sigma \in [\sigma_1', \sigma_2']$, $T_1 > H_0$ and $T_2 > T_0$ we have $N(\sigma, T_1, T_2) \le \mathcal{U} T_2^{\frac{109}{41}(1-\sigma)} \log T_2^{(2\sigma+1)} + \mathcal{V}(\log T_2)^2$.

σ_1'	σ_2'	α	δ_1	δ_2	$\mathcal{U}$	$\mathcal{V}$
0.750	0.760	0.351015	0.230516	0.368415	396.383090	2.159860
0.760	0.770	0.338988	0.239602	0.368373	343.930590	2.087520
0.770	0.780	0.327000	0.248671	0.368332	298.066010	2.015190
0.780	0.790	0.315053	0.257724	0.368293	258.015930	1.942850
0.790	0.800	0.303152	0.266760	0.368253	223.089040	1.870510
0.800	0.810	0.291300	0.275778	0.368214	192.668870	1.798180
0.810	0.820	0.279501	0.284779	0.368176	166.206990	1.725840
0.820	0.830	0.267759	0.293763	0.368137	143.216520	1.653510
0.830	0.840	0.256081	0.302729	0.368098	123.266070	1.581170
0.840	0.850	0.244470	0.311676	0.368059	105.974180	1.508830
0.850	0.860	0.232935	0.320605	0.368020	91.004090	1.436500
0.860	0.870	0.221483	0.329517	0.367982	78.059050	1.364160
0.870	0.880	0.210122	0.338407	0.367939	66.877980	1.291830
0.880	0.890	0.198861	0.347278	0.367899	57.231520	1.219490
0.890	0.900	0.187709	0.356129	0.367856	48.918510	1.147150
0.900	0.910	0.176687	0.364960	0.367814	41.762760	1.074820
0.910	0.920	0.165800	0.373770	0.367769	35.610160	1.002480
0.920	0.930	0.155068	0.382559	0.367722	30.326140	0.930150
0.930	0.940	0.144509	0.391325	0.367674	25.793270	0.857810
0.940	0.950	0.134146	0.400068	0.367624	21.909290	0.785480
0.950	0.960	0.124002	0.408788	0.367572	18.585210	0.713140
0.960	0.970	0.114105	0.417484	0.367517	15.743710	0.640800
0.970	0.980	0.104489	0.426155	0.367459	13.317680	0.568470
0.980	0.990	0.095186	0.434800	0.367399	11.248940	0.496130
0.990	0.991	0.086237	0.443418	0.367335	9.487140	0.423800
0.991	0.992	0.085363	0.444278	0.367329	9.326110	0.416560
0.992	0.993	0.084494	0.445138	0.367322	9.167670	0.409330
0.993	0.994	0.083628	0.445998	0.367316	9.011780	0.402090
0.994	0.995	0.082766	0.446857	0.367309	8.858400	0.394860
0.995	0.996	0.081908	0.447717	0.367302	8.707490	0.387630
0.996	0.997	0.081055	0.448575	0.367296	8.559010	0.380390
0.997	0.998	0.080206	0.449434	0.367289	8.412940	0.373160
0.998	0.999	0.079361	0.450292	0.367282	8.269220	0.365930
0.999	1.000	0.078520	0.451151	0.367276	8.127820	0.358690

Table A.11: Sample of values for equations (4.171) and (4.172). These values are for $T_0 = 3 \cdot 10^{12}$ and $H_0 = 3 \cdot 10^{12}$. We use $\mu = 1 + \frac{k \log\log(X)}{\log X}$ with $k = 2$ and η approximately 0.20341. The sub-convexity bound corresponds to table 4.3. In particular for each row for all $\sigma \in [\sigma_1', \sigma_2']$, $T_1 > H_0$ and $T_2 > T_0$ we have $N(\sigma, T_1, T_2) \leq \mathcal{U} T_2^{\frac{8}{3}(1-\sigma)} \log T_2^{(5-2\sigma)} + \mathcal{V}(\log T_2)^2$

σ_1'	σ_2'	α	δ_1	δ_2	$\mathcal{U}$	$\mathcal{V}$
0.750	0.760	0.394087	0.381157	0.358600	3.595230	3.868940
0.760	0.770	0.382980	0.379947	0.358159	3.805060	3.790960
0.770	0.780	0.371885	0.378784	0.357709	4.025770	3.712980
0.780	0.790	0.360799	0.377662	0.357248	4.257810	3.635000
0.790	0.800	0.349727	0.376571	0.356774	4.501620	3.557020
0.800	0.810	0.338669	0.375510	0.356291	4.757650	3.479040
0.810	0.820	0.327628	0.374472	0.355794	5.026350	3.401060
0.820	0.830	0.316607	0.373453	0.355283	5.308140	3.323090
0.830	0.840	0.305606	0.372450	0.354758	5.603480	3.245110
0.840	0.850	0.294629	0.371456	0.354217	5.912800	3.167130
0.850	0.860	0.283679	0.370471	0.353659	6.236500	3.089150
0.860	0.870	0.272760	0.369491	0.353084	6.574990	3.011170
0.870	0.880	0.261875	0.368512	0.352490	6.928650	2.933190
0.880	0.890	0.251028	0.367532	0.351875	7.297830	2.855210
0.890	0.900	0.240225	0.366548	0.351239	7.682850	2.777240
0.900	0.910	0.229471	0.365555	0.350579	8.084000	2.699260
0.910	0.920	0.218772	0.364553	0.349895	8.501510	2.621280
0.920	0.930	0.208136	0.363537	0.349183	8.935560	2.543300
0.930	0.940	0.197571	0.362505	0.348443	9.386250	2.465320
0.940	0.950	0.187086	0.361452	0.347672	9.853620	2.387340
0.950	0.960	0.176693	0.360378	0.346867	10.337620	2.309360
0.960	0.970	0.166403	0.359277	0.346027	10.838100	2.231390
0.970	0.980	0.156232	0.358147	0.345148	11.354780	2.153410
0.980	0.990	0.146194	0.356982	0.344226	11.887270	2.075430
0.990	0.991	0.145199	0.356864	0.344133	11.941370	1.997450
0.991	0.992	0.144206	0.356746	0.344038	11.995620	1.989650
0.992	0.993	0.143213	0.356627	0.343943	12.050020	1.981850
0.993	0.994	0.142222	0.356508	0.343847	12.104580	1.974060
0.994	0.995	0.141233	0.356388	0.343751	12.159280	1.966260
0.995	0.996	0.140246	0.356268	0.343654	12.214130	1.958460
0.996	0.997	0.139260	0.356147	0.343557	12.269130	1.950660
0.997	0.998	0.138276	0.356026	0.343459	12.324280	1.942860
0.998	0.999	0.137293	0.355905	0.343361	12.379580	1.935070
0.999	1.000	0.136314	0.355784	0.343261	12.435020	1.927270

Table A.12: Sample of values for equations (4.171) and (4.172). These values are for $T_0 = 3 \cdot 10^{100}$ and $H_0 = 3 \cdot 10^{12}$. We use $\mu = 1 + \frac{k \log\log(X)}{\log X}$ with $k = 2$ and η approximately 0.0598. The sub-convexity bound corresponds to table 4.3. In particular for each row for all $\sigma \in [\sigma_1', \sigma_2']$, $T_1 > H_0$ and $T_2 > T_0$ we have $N(\sigma, T_1, T_2) \leq \mathcal{U} T_2^{\frac{8}{3}(1-\sigma)} \log T_2^{(5-2\sigma)} + \mathcal{V}(\log T_2)^2$

σ_1'	σ_2'	α	δ_1	δ_2	$\mathcal{U}$	$\mathcal{V}$
0.750	0.760	0.347180	0.373625	0.370768	2.474240	2.159860
0.760	0.770	0.335106	0.373469	0.370697	2.639180	2.087520
0.770	0.780	0.323072	0.373318	0.370623	2.813870	2.015190
0.780	0.790	0.311061	0.373170	0.370548	2.998720	1.942850
0.790	0.800	0.299134	0.373021	0.370471	3.194200	1.870510
0.800	0.810	0.287237	0.372876	0.370391	3.400740	1.798180
0.810	0.820	0.275394	0.372732	0.370308	3.618770	1.725840
0.820	0.830	0.263610	0.372589	0.370221	3.848740	1.653510
0.830	0.840	0.251891	0.372445	0.370131	4.091070	1.581170
0.840	0.850	0.240242	0.372299	0.370038	4.346170	1.508830
0.850	0.860	0.228672	0.372157	0.369940	4.614430	1.436500
0.860	0.870	0.217188	0.372010	0.369839	4.896200	1.364160
0.870	0.880	0.205798	0.371861	0.369732	5.191810	1.291830
0.880	0.890	0.194514	0.371711	0.369621	5.501550	1.219490
0.890	0.900	0.183348	0.371557	0.369505	5.825650	1.147150
0.900	0.910	0.172312	0.371398	0.369381	6.164270	1.074820
0.910	0.920	0.161422	0.371237	0.369254	6.517520	1.002480
0.920	0.930	0.150697	0.371069	0.369118	6.885390	0.930150
0.930	0.940	0.140158	0.370896	0.368976	7.267790	0.857810
0.940	0.950	0.129826	0.370717	0.368826	7.664520	0.785480
0.950	0.960	0.119728	0.370532	0.368667	8.075230	0.713140
0.960	0.970	0.109896	0.370338	0.368499	8.499420	0.640800
0.970	0.980	0.100364	0.370137	0.368320	8.936450	0.568470
0.980	0.990	0.091168	0.369926	0.368132	9.385490	0.496130
0.990	0.991	0.090268	0.369904	0.368113	9.431010	0.423800
0.991	0.992	0.089373	0.369882	0.368093	9.476650	0.416560
0.992	0.993	0.088481	0.369859	0.368075	9.522390	0.409330
0.993	0.994	0.087593	0.369837	0.368055	9.568240	0.402090
0.994	0.995	0.086709	0.369815	0.368035	9.614190	0.394860
0.995	0.996	0.085829	0.369793	0.368016	9.660250	0.387630
0.996	0.997	0.084953	0.369770	0.367996	9.706410	0.380390
0.997	0.998	0.084081	0.369748	0.367976	9.752680	0.373160
0.998	0.999	0.083214	0.369725	0.367956	9.799040	0.365930
0.999	1.000	0.082350	0.369701	0.367936	9.845510	0.358690