


Arc-disjoint hamiltonian paths in Cartesian products of directed cycles*

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Abstract

We show that if C_1 and C_2 are directed cycles (of length at least two), then the Cartesian product $C_1 \square C_2$ has two arc-disjoint hamiltonian paths. (This answers a question asked by J. A. Gallian in 1985.) The same conclusion also holds for the Cartesian product of any four or more directed cycles (of length at least two), but some cases remain open for the Cartesian product of three directed cycles. We also discuss the existence of arc-disjoint hamiltonian paths in 2-generated Cayley digraphs on (finite or infinite) abelian groups.

Keywords: Abelian groups, Cayley digraphs, hamiltonian paths.

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1 Introduction

It is easy to see (and well known) that the Cartesian product of any two directed cycles has a hamiltonian path. (See Definition 3.3 for the definition of the Cartesian product.) In 1985, J. A. Gallian (personal communication) asked whether there are two arc-disjoint hamiltonian paths. The main result of this paper establishes that the answer is “yes”:

Theorem 4.4. *If C_1 and C_2 are directed cycles (of length ≥ 2), then the Cartesian product $C_1 \square C_2$ has two arc-disjoint hamiltonian paths.*

In fact, if the lengths of the directed cycles are large, then there are *many* pairs of arc-disjoint hamiltonian paths:

Proposition 4.5. *Let $N(m, n)$ be the number of (unordered) pairs $\{P, P'\}$ of arc-disjoint hamiltonian paths in the Cartesian product of a directed cycle of length m and a directed cycle of length n . If m and n are sufficiently large, then*

$$N(m, n) > \frac{m^2 n^2}{10}.$$

Although the theorem only considers the Cartesian product of precisely two directed cycles, it implies that arc-disjoint hamiltonian paths exist in the Cartesian product of any larger number of directed cycles, except three:

Corollary 5.1. *If C_1, C_2, \dots, C_r are directed cycles (of length ≥ 2), and $r \geq 4$, then the Cartesian product $C_1 \square C_2 \square \dots \square C_r$ has two arc-disjoint hamiltonian paths.*

Thus, $r = 3$ is the only open case of the following conjecture.

Conjecture 1.1. *If C_1, C_2, \dots, C_r are directed cycles (of length ≥ 2), and $r \geq 2$, then the Cartesian product $C_1 \square C_2 \square \dots \square C_r$ has two arc-disjoint hamiltonian paths.*

Although the conjecture has not been proved for all Cartesian products of three directed cycles, we know that it is true in most of these cases:

Proposition 5.2. *Assume $C_1, C_2,$ and C_3 are directed cycles (of length ≥ 2). If either*

- (1) *the Cartesian product of two of the directed cycles has a hamiltonian cycle, or*
- (2) *the lengths of the three directed cycles do not all have the same parity (i.e., if there is a directed cycle of even length and a directed cycle of odd length), or*
- (3) *at least one of the directed cycles has length 2,*

then $C_1 \square C_2 \square C_3$ has two arc-disjoint hamiltonian paths.

Question 1.2. Does there exist a function $f(r) \rightarrow \infty$, such that every Cartesian product of r directed cycles has at least $f(r)$ pairwise arc-disjoint hamiltonian paths?

It is well known that any Cartesian product of two directed cycles is isomorphic to a 2-generated Cayley digraph on an abelian group. (See Definition 3.2 and Example 3.4.) It is natural to expect that Theorem 4.4 can be extended to this setting:

Conjecture 1.3. *If $\{a, b\}$ is a 2-element generating set of a finite abelian group G , and a and b are nontrivial, then $\text{Cay}(G; a, b)$ has two arc-disjoint hamiltonian paths.*

We have some evidence that this is true:

Proposition 1.4. Conjecture 1.3 is true in all cases where either:

- (1) $|G : \langle a - b \rangle|$ is even (see Proposition 6.1), or
- (2) $|G : \langle a \rangle| > 600$ (see Proposition 6.6).

Remark 1.5. Computer calculations verified Conjecture 1.3 for all cases where $|G| \leq 10^4$. (The `sagemath` source code for these computations is available online at <https://arxiv.org/src/2203.11017/anc>.) Although the calculations are not guaranteed to be error-free, they provide additional evidence for the truth of the conjecture.

We also solve the analogous problem for 2-generated Cayley digraphs on infinite abelian groups. The natural analogue of a hamiltonian path in this setting is a *one-way infinite hamiltonian path*, which means a list

$$v_0, v_1, v_2, v_3, \dots,$$

of the vertices of the digraph, such that there is a directed edge from v_i to v_{i+1} for every i . However, it is known (and easy to see) that a 2-generated Cayley digraph on an infinite abelian group never has a one-way infinite hamiltonian path (cf. [13, Theorem 5.1] and [15, Theorem 3.1]), so it certainly does not have two of them (whether arc-disjoint or not). On the other hand, the corresponding natural analogue of a hamiltonian cycle is a *two-way infinite hamiltonian path*, which means a doubly-infinite list

$$\dots, v_{-2}, v_{-1}, v_0, v_1, v_2, \dots$$

of the vertices of the digraph, such that there is a directed edge from v_i to v_{i+1} for every i . It is well known that these can exist, and we determine exactly when there are two of them that are arc-disjoint:

Proposition 7.5. Assume G is an infinite abelian group.

- (1) If there exist $a, b \in G$, such that $\overrightarrow{\text{Cay}}(G; a, b)$ has two arc-disjoint two-way infinite hamiltonian paths, then G is isomorphic to either \mathbb{Z} or $\mathbb{Z} \times \mathbb{Z}_m$, for some $m \geq 2$.
- (2) For $a, b \in \mathbb{Z}$, the Cayley digraph $\overrightarrow{\text{Cay}}(\mathbb{Z}; a, b)$ has two arc-disjoint two-way infinite hamiltonian paths if and only if a and b are odd, and

$$\text{either } \{a, b\} = \{1, -1\} \text{ or } a + b = \pm 2.$$

- (3) For $a, b \in \mathbb{Z} \times \mathbb{Z}_m$, with $m \geq 2$, the Cayley digraph $\overrightarrow{\text{Cay}}(\mathbb{Z} \times \mathbb{Z}_m; a, b)$ has two arc-disjoint two-way infinite hamiltonian paths if and only if either

- (a) $\{a, b\} = \{(1, x), (-1, y)\}$, for some $x, y \in \mathbb{Z}_m$, such that $\langle x + y \rangle = \mathbb{Z}_m$, or
- (b) $m = 2$, $a = (0, 1)$, and $b \in \{\pm 1\} \times \mathbb{Z}_2$, perhaps after interchanging a and b .

This paper is structured in the following manner. Section 2 sets the paper in context by providing a brief review of related results. (None of these results are needed elsewhere in the paper.) Section 3 sets some notation and recalls known results that will be used in later sections. (These include basic properties of the ‘‘arc-forcing’’ subgroup and some work of Curran-Witte [7].) Section 4 studies the Cartesian product of two directed cycles, and Section 5 studies the Cartesian product of more than two directed cycles. Section 6 studies 2-generated Cayley digraphs on finite abelian groups, and Section 7 closes the paper with a discussion of 2-generated Cayley digraphs on infinite abelian groups.

2 Brief review of related results

Several researchers have studied hamiltonian properties of the Cartesian product of two directed cycles. For example, the following result presents three different necessary and sufficient conditions for the existence of a hamiltonian cycle. (We use \vec{C}_n to denote a directed cycle of length n .)

Theorem 2.1. *If $m, n \geq 2$, then the following are equivalent:*

- (1) *The Cartesian product $\vec{C}_m \square \vec{C}_n$ has a hamiltonian cycle.*
- (2) (Rankin, cf. [22]) *There exist $a, b \geq 1$, such that*

$$a + b = \gcd(m, n) \text{ and } \langle (a, b) \rangle = \langle (1, -1) \rangle,$$

where $\langle (x, y) \rangle$ denotes the subgroup generated by (x, y) in the abelian group $\mathbb{Z}_m \times \mathbb{Z}_n$.

- (3) (Trotter-Erdős [24]) *There exist $a, b \geq 1$, such that*

$$a + b = \gcd(m, n) \text{ and } \gcd(a, m) = \gcd(b, n) = 1.$$

- (4) (Curran [25, Theorem 4.3]) *There exist $a, b \geq 1$, such that*

$$\gcd(a, b) = 1 \text{ and } am + bn = mn.$$

Remark 2.2. References to generalizations, such as determining the lengths of all of the directed cycles in $\vec{C}_m \square \vec{C}_n$, can be found in the bibliography of [14].

The same methods also determine whether there are two arc-disjoint hamiltonian cycles:

Theorem 2.3 (Keating [16, Corollary 2.4]). *$\vec{C}_m \square \vec{C}_n$ has two arc-disjoint hamiltonian cycles if and only if there exist $a, b \geq 1$, such that*

$$a + b = \gcd(m, n) \text{ and } \gcd(ab, mn) = 1.$$

Remark 2.4. Theorem 2.3 settles the existence of two arc-disjoint hamiltonian cycles, and the main result of this paper settles the existence of two arc-disjoint hamiltonian paths (see Theorem 4.4).

Mixing the two, one might ask about the existence of a hamiltonian path that is arc-disjoint from a hamiltonian cycle. However, this is not interesting, because the complement of a hamiltonian cycle in $\vec{C}_m \square \vec{C}_n$ must be a union of disjoint directed cycles. Therefore, if the complement contains a hamiltonian path, then it is a hamiltonian cycle.

The undirected case is easier. The starting point here is the easy observation that the Cartesian product of two (undirected) cycles is hamiltonian. In fact, it was proved by A. Kotzig in 1973 that the Cartesian product of two cycles always has two edge-disjoint hamiltonian cycles. In other words, the Cartesian product of two cycles can be decomposed into hamiltonian cycles. This has been generalized to any number of cycles:

Theorem 2.5 (Aubert-Schneider, cf. [2, Theorem A]). *If C_1, C_2, \dots, C_r are undirected cycles (of length at least 3), then the Cartesian product $C_1 \square C_2 \square \dots \square C_r$ can be decomposed into edge-disjoint hamiltonian cycles.*

It is conjectured that this extends to all Cayley graphs on abelian groups:

Conjecture 2.6 (Alspach, cf. [1, Unsolved Problem 4.5, page 454]). *Let G be an abelian group with a symmetric generating set S . If the Cayley graph $\text{Cay}(G; S)$ has no loops, then it can be decomposed into hamiltonian cycles (plus a 1-factor if the valency is odd).*

This conjecture is known to be true for valency 4:

Theorem 2.7 (Bermond-Favaron-Maheo [3]). *If S is a 4-element symmetric generating set of a finite abelian group G (and $0 \notin S$), then $\text{Cay}(G; S)$ has two edge-disjoint hamiltonian cycles.*

Remark 2.8.

- (1) It is not known whether all connected Cayley graphs on nonabelian finite groups have hamiltonian cycles (or hamiltonian paths), but it was proved by D. Bryant and M. Dean [4] that not all of them can be decomposed into hamiltonian cycles.
- (2) The most recent survey on hamiltonian paths and cycles in Cayley graphs and Cayley digraphs seems to be [18].

3 Preliminaries on 2-generated Cayley digraphs

This section starts with a few basic definitions, and then recalls some essential results from the 1980's on hamiltonian paths in 2-generated Cayley digraphs on finite abelian groups.

We use standard terminology and notation from the theory of graphs and digraphs. (In particular, “arc” is synonymous with “directed edge.”) All paths in a digraph are assumed to be *directed* paths.

Notation 3.1. Throughout this section,

$$\{a, b\} \text{ is a 2-element generating set of a finite abelian group } G.$$

We use $o(g)$ to denote the order of an element g of G , and we denote the cardinality of a set S by $\#S$.

Definition 3.2 (cf. [10, page 504]). The *Cayley digraph* $\overrightarrow{\text{Cay}}(G; a, b)$ of G with respect to the generators a and b is the digraph whose vertex set is G and which has a directed edge from v to $v + s$ for every $v \in G$ and $s \in \{a, b\}$.

Definition 3.3. Recall that the *Cartesian product* $X \square Y$ of two digraphs X and Y is the digraph with vertex set $V(X) \times V(Y)$ where the vertex (x_1, y_1) is joined to (x_2, y_2) by a directed edge if and only if either $x_1 = x_2$ and $(y_1, y_2) \in E(Y)$ or $y_1 = y_2$ and $(x_1, x_2) \in E(X)$.

Example 3.4 (cf. [17, proof of Lemma 1]). The Cartesian product $\vec{C}_m \square \vec{C}_n$ of two directed cycles can be realized as the Cayley digraph $\overrightarrow{\text{Cay}}(\mathbb{Z}_m \times \mathbb{Z}_n; e_1, e_2)$, where $\{e_1, e_2\} = \{(1, 0), (0, 1)\}$ is the standard generating set of the abelian group $\mathbb{Z}_m \times \mathbb{Z}_n$.

Definition 3.5.

- (1) We use the usual terminology that, for $s \in \{a, b\}$, a directed edge of $\overrightarrow{\text{Cay}}(G; a, b)$ is called an s -edge if it is of the form $(v, v + s)$, for some $v \in G$. (So every directed edge is either an a -edge or a b -edge, but not both.)
- (2) Suppose P is a subdigraph of $\overrightarrow{\text{Cay}}(G; a, b)$, and let $s \in \{a, b\}$.
 - (a) We let $\delta_s(P)$ be the number of s -edges in P .
 - (b) We say that a vertex v *travels by* s in P if v is a vertex of outdegree 1 in P , and the out-edge of v is an s -edge [12, page 82].
 - (c) A set of vertices *travels by* s in P if every element of the set travels by s (cf. [12, page 83]).

3.1 Elementary theory of the arc-forcing subgroup

Definition 3.6 ([9, Definition 2.6]). A *spanning quasi-path* in a digraph Γ is a spanning subdigraph P , such that precisely one connected component of P is a directed path, and all other components of P are directed cycles.

In particular, every hamiltonian path is a spanning quasi-path. Next we provide another characterization.

Remark 3.7 (cf. [9, Definition 2.6]). A spanning subdigraph P of Γ is a spanning quasi-path if and only if there exist vertices ι and τ , such that:

- (1) the indegree of ι is 0 in P , but the indegree of all other vertices is 1, in P and,
- (2) the outdegree of τ is 0 in P , but the outdegree of all other vertices is 1 in P .

We call ι the *initial vertex* of P , and call τ the *terminal vertex* of P . (This acknowledges the fact that they are the initial vertex and terminal vertex of the path component of P .)

Definition 3.8. Recall that $G = \langle a, b \rangle$ is a finite abelian group.

- (1) The subgroup $\langle a - b \rangle$ is called the *arc-forcing* subgroup [25, §2.3].
- (2) Suppose τ is the terminal vertex of a spanning quasi-path P of $\overrightarrow{\text{Cay}}(G; a, b)$. Then the coset $\tau + \langle a - b \rangle$ is called the *terminal coset* [20, Definition 5(ii)] and all other cosets are called *non-terminal* cosets. (In the older terminology of Housman [12], $\tau + \langle a - b \rangle$ is called the *special* coset and all other cosets are called *regular* cosets.)

Since no vertex of a spanning quasi-path has indegree 2, it is clear that if a vertex v travels by a , then $v + (a - b)$ cannot travel by b . (Similarly, if v travels by b , then $v - (a - b)$ cannot travel by a .) This observation (which is originally due to R. A. Rankin [22, Lemma 1], in the setting of hamiltonian cycles, rather than hamiltonian paths) has the following consequence:

Lemma 3.9 (Housman, cf. [12, pages 82–83]). *If P is a spanning quasi-path P in $\overrightarrow{\text{Cay}}(G; a, b)$, then:*

- (1) *Each non-terminal coset either travels by a or travels by b .*

(2) If d is the number of vertices in the terminal coset that travel by b , then:

- (a) $\iota = \tau + a + d(a - b) \in \tau + a + \langle a - b \rangle$, and
- (b) if $v = \tau + i(a - b)$ is any vertex of the terminal coset (with $0 \leq i < o(a - b)$), then:
 - v travels by b iff $1 \leq i \leq d$, and
 - v travels by a iff $i > d$.

Remark 3.10 (Housman, cf. [12, proof of Theorem 1]). This implies that a spanning quasi-path in $\overrightarrow{\text{Cay}}(G; a, b)$ is uniquely determined by specifying:

- (1) the terminal vertex,
- (2) how many vertices of the terminal coset travel by b , and
- (3) the non-terminal cosets that travel by b .

Determining whether a spanning quasi-path is a hamiltonian path does not depend on knowing which particular non-terminal cosets travel by b , or which particular vertices of the terminal coset travel by b , but only on the *number* of each:

Proposition 3.11 (Housman, cf. [7, Proposition 6.7]). *Suppose P and P' are spanning quasi-paths in $\overrightarrow{\text{Cay}}(G; a, b)$, such that*

- *the number of non-terminal cosets that travel by b in P is equal to the number of non-terminal cosets that travel by b in P' , and*
- *the number of vertices in the terminal coset that travel by b in P is equal to number of vertices in the terminal coset that travel by b in P' .*

If P is a hamiltonian path, then P' is also a hamiltonian path.

This can be restated in terms of the total number of vertices that travel by b in all of P :

Corollary 3.12. *Assume that P and P' are spanning quasi-paths in $\overrightarrow{\text{Cay}}(G; a, b)$, such that $\delta_b(P) = \delta_b(P')$. If P is a hamiltonian path, then P' is also a hamiltonian path.*

Proof. Let t (resp. t') be the number of non-terminal cosets that travel by b in P (resp. in P'), and let d (resp. d') be the number of vertices of the terminal coset that travel by b in P (resp. in P'). Then $0 \leq d, d' < o(a - b)$, and we have

$$\delta_b(P) = t o(a - b) + d \quad \text{and} \quad \delta_b(P') = t' o(a - b) + d'.$$

Since $\delta_b(P) = \delta_b(P')$ by assumption, we conclude that $t = t'$ and $d = d'$. So Proposition 3.11 tells us that P' is a hamiltonian path. \square

The following construction provides a standard example of a spanning quasi-path in which a particular number t of non-terminal cosets travel by b , and a particular number d of vertices in the terminal coset travel by b . See Figure 1 for some examples.

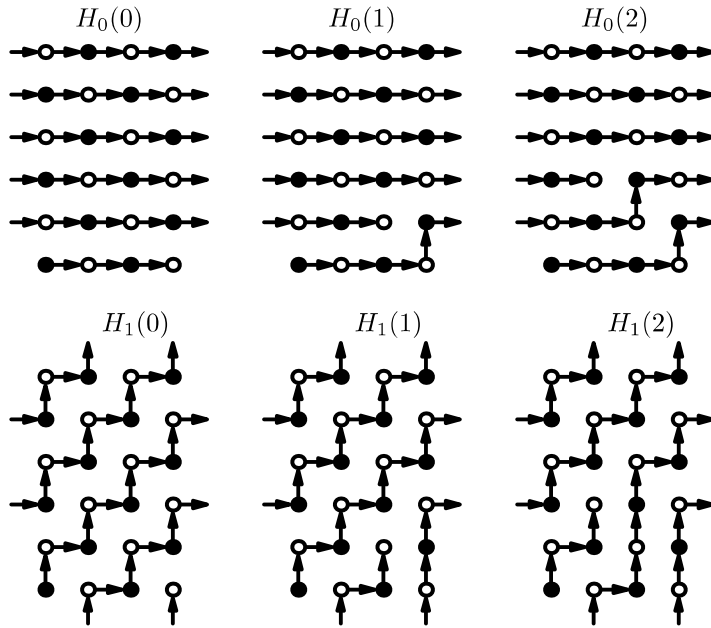


Figure 1: The spanning quasi-path $H_t(d)$ in the Cayley digraph $\text{Ca}\vec{Y}(\mathbb{Z}_4 \times \mathbb{Z}_6; (1, 0), (0, 1))$ for $t \in \{0, 1\}$ and $0 \leq d \leq 2$. (Vertices in the non-terminal coset are black, and vertices in the terminal coset are white.) Of these six subdigraphs, only $H_1(0)$ is a hamiltonian path.

Notation 3.13 ([7, Definition 5.2]). For $0 \leq t < |G : \langle a - b \rangle|$ and $0 \leq d < o(a - b)$, we let $H_t(d)$ be the spanning subdigraph of $\text{Ca}\vec{Y}(G; a, b)$, such that:

- (1) for $0 \leq i < o(a - b)$, the element $-a - i(a - b)$ of the coset $-a + \langle a - b \rangle$:
 - travels by b if $i < d$,
 - has no outedges if $i = d$, and
 - travels by a if $i > d$.
- (2) the cosets $\langle a - b \rangle, a + \langle a - b \rangle, \dots, (t - 1)a + \langle a - b \rangle$ travel by b , and
- (3) all other cosets travel by a .

This is a spanning quasi-path whose initial vertex is 0, and whose terminal vertex is $-a + d(b - a)$. By construction, exactly t non-terminal cosets travel by b , and exactly d vertices in the terminal coset travel by b .

Since t and d are arbitrary in Notation 3.13 (and any non-negative integer can be written in the form $t o(a - b) + d$, with $0 \leq d < o(a - b)$), it follows that there is a spanning quasi-path with any desired number of b -edges (in the allowable range):

Lemma 3.14. For $0 \leq k < |G|$, there is a spanning quasi-path P in $\text{Ca}\vec{Y}(G; a, b)$, such that $\delta_b(P) = k$.

3.2 Some results of Curran-Witte

We now state (without proof) two results from [7]. One is the following theorem. The other is Theorem 3.21, which provides a way to determine whether the subdigraph $H_t(d)$ is a hamiltonian path in $\overrightarrow{\text{Cay}}(G; a, b)$. We will then use this latter result to derive some rather technical information about hamiltonian paths in $\overrightarrow{\text{Cay}}(G; a, b)$.

Theorem 3.15 (Curran-Witte [7, Theorem 9.1]). *If C_1, C_2, \dots, C_r are directed cycles (of length ≥ 2), and $r \geq 3$, then the Cartesian product $C_1 \square C_2 \square \dots \square C_r$ has a hamiltonian cycle.*

The other main result of [7] relies on some additional notation. Recall that a point (x, y) in \mathbb{R}^2 is a *lattice point* if its coordinates are integers (i.e., if $x, y \in \mathbb{Z}$), and that a lattice point (x, y) is *primitive* if $\gcd(x, y) = 1$ (cf. [6, Proposition 7.2, page 36]).

Notation 3.16. Recall that $G = \langle a, b \rangle$ is a finite abelian group. Let:

- (1) $m = o(a)$.
- (2) $n = |G : \langle a \rangle|$, so $|G| = mn$.
- (3) $e \in \mathbb{Z}$, such that $nb = ea$ and $0 \leq e < m$. (A reader who is interested only in Cartesian products of directed cycles can always assume $e = 0$.)
- (4) $T(m, n, e)$ be the closed triangle with vertices $(0, 0)$, $(n, 0)$, and (e, m) .

Notation 3.17 ([7, Notation 3.4]). For $0 \leq t \leq |G : \langle a - b \rangle|$, let

$$X_t = \left(1 - \frac{t}{I}\right)(n, 0) + \frac{t}{I}(e, m) \quad \text{where } I = |G : \langle a - b \rangle|.$$

Note 3.18 ([7, Remark 3.6]). The sequence (X_0, X_1, \dots, X_I) is a list of all the lattice points on the line segment joining $(n, 0)$ and (e, m) . This implies that

$$|G : \langle a - b \rangle| = \gcd(n - e, m).$$

See Figures 2 and 3 for a concrete example of a triangle $T(m, n, e)$, the associated lattice points X_t , and the values defined in the following notation.

Notation 3.19 (cf. [7, pages 37–38]). For $0 \leq t < |G : \langle a - b \rangle|$:

- (1) Let T_t be the closed triangle with vertices $(0, 0)$, X_t , and X_{t+1} , so $\{T_t \mid 0 \leq t < |G : \langle a - b \rangle|\}$ is a decomposition of the triangle $T(m, n, e)$ into smaller triangles.
- (2) Let $A_t(1), A_t(2), \dots, A_t(f_t)$ be a list of the primitive lattice points in T_t , in order of increasing slope. (So f_t is the number of primitive lattice points in T_t .)
- (3) For $1 \leq k \leq f_t$, let $h_t(k)$ be the number of lattice points that
 - are a positive scalar multiple of $A_t(k)$, and
 - are in the closed triangle T_t , but
 - are *not* on the side of T_t that is opposite $(0, 0)$, that is, *not* on the line segment from X_t to X_{t+1} .

More concretely, we have

$$h_t(k) = \left\lfloor \frac{mn - 1}{m x_k + (n - e) y_k} \right\rfloor \quad \text{where } A_t(k) = (x_k, y_k).$$

(4) For $1 \leq k < f_t$, let

$$u_t(k) = h_t(1) + 2 \sum_{j=2}^k h_t(j).$$

This means that $u_t(1) = h_t(1)$, and $u_t(k) = u_t(k - 1) + 2h_t(k)$ for $2 \leq k < f_t$. Also note that $u_0(1) = h_0(1) = n - 1$.

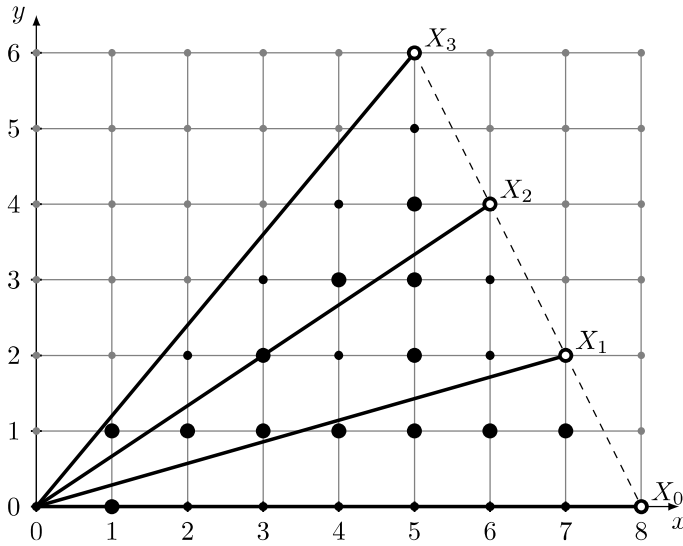


Figure 2: Primitive lattice points in the triangle $T(6, 8, 5)$.

Remark 3.20. For each t , Curran and Witte [7, Notation 3.14] also defined a certain function

$$B_t: \{0, 1, 2, \dots, o(a - b) - 1\} \rightarrow \mathbb{Z} \times \mathbb{Z},$$

such that

$$B_t(d) = (0, 0) \iff d = u_t(k) \text{ for some } k \text{ (with } 1 \leq k < f_t). \quad (*)$$

We will not define the function B_t , because we have no need for it, other than its appearance in the statement of the following theorem, which is the other main result of [7]. (Actually, the statement of [7, Theorem 7.1] only includes $(1) \Leftrightarrow (2)$ of the following result, but $(2) \Leftrightarrow (3)$ is the fundamental property of $B_t(d)$ that is stated in $(*)$ above. All we need is the equivalence $(1) \Leftrightarrow (3)$.)

t	f_t	k	$A_t(k)$	$h_t(k)$	$u_t(k)$
0	6	1	(1, 0)	7	7
		2	(7, 1)	1	$7 + 2 \times 1 = 9$
		3	(6, 1)	1	$9 + 2 \times 1 = 11$
		4	(5, 1)	1	$11 + 2 \times 1 = 13$
		5	(4, 1)	1	$13 + 2 \times 1 = 15$
		6	(7, 2)	0	
1	6	1	(7, 2)	0	0
		2	(3, 1)	2	$0 + 2 \times 2 = 4$
		3	(5, 2)	1	$4 + 2 \times 1 = 6$
		4	(2, 1)	3	$6 + 2 \times 3 = 12$
		5	(5, 3)	1	$12 + 2 \times 1 = 14$
		6	(3, 2)	1	
2	5	1	(3, 2)	1	1
		2	(4, 3)	1	$1 + 2 \times 1 = 3$
		3	(5, 4)	1	$3 + 2 \times 1 = 5$
		4	(1, 1)	5	$5 + 2 \times 5 = 15$
		5	(5, 6)	0	

Figure 3: The functions A_t , h_t , and u_t in the case where $(m, n, e) = (6, 8, 5)$. The primitive lattice points $A_t(k)$ are taken from Figure 2.

Theorem 3.21 (Curran-Witte [7, Theorem 7.1]). *For $0 \leq t < |G : \langle a - b \rangle|$ and $0 \leq d < o(a - b)$, the following are equivalent:*

- (1) $H_t(d)$ is a hamiltonian path.
- (2) $B_t(d) = (0, 0)$.
- (3) $d = u_t(k)$ for some $k \in \{1, 2, \dots, f_t - 1\}$.

Corollary 3.22 (Curran-Witte [7, Remark 8.4]).

$$\#\{ (t, d) \mid H_t(d) \text{ is a hamiltonian path in } \overrightarrow{\text{Cay}}(G; a, b) \}$$

is exactly one less than the number of primitive lattice points in the closed triangle $T(m, n, e)$.

Lemma 3.23 ([7, Proposition 3.13]). *For $0 \leq t < |G : \langle a - b \rangle|$, we have*

$$u_t(f_t - 1) + h_t(f_t) = o(a - b) - 1.$$

This has the following immediate consequences:

Corollary 3.24. *For $0 < t < |G : \langle a - b \rangle|$, we have*

$$u_{t-1}(f_{t-1} - 1) = o(a - b) - 1 - u_t(1).$$

Proof. Note that $A_{t-1}(f_{t-1}) = A_t(1)$ (because this is the primitive lattice point on the edge that T_{t-1} and T_t have in common), so we have $h_{t-1}(f_{t-1}) = h_t(1)$. Since $h_t(1) = u_t(1)$ (by the definition of $u_t(k)$), the desired conclusion is now immediate from Lemma 3.23 (after replacing t with $t - 1$). \square

Corollary 3.25. Assume $0 \leq t, t' < |G : \langle a - b \rangle|$, $1 \leq k < f_t$, and $1 \leq k' \leq f_{t'}$.

- (1) If $o(a - b)$ is odd, then $u_t(k) \equiv u_{t'}(k') \pmod{2}$.
 (2) If $o(a - b)$ is even, then $u_t(k) - u_{t'}(k') \equiv t - t' \pmod{2}$.

Proof. It is immediate from Notation 3.19(4) that the parity of $u_t(k)$ depends only on t . Then we see from Corollary 3.24 that $u_{t-1}(\ast)$ has the same parity as $u_t(\ast)$ if and only if $o(a - b)$ is odd. \square

In the remainder of this section, we will give two consequences of Theorem 3.21. They are stated in [7] only for the special case where $\text{Ca}\vec{y}(G; a, b)$ is the Cartesian product of two directed cycles, but the same arguments apply to the general case. For completeness, we provide proofs.

Proposition 3.26 (Curran-Witte). Assume that m and n are as in Notation 3.16. If $m \geq 2$, then there exists $n' \in \{n - 1, n\}$, such that, for each i in

$$\{n', n' + 2, n' + 4, \dots, n' + 2\lfloor (n - 1)/2 \rfloor\},$$

there is a hamiltonian path P_i in $\text{Ca}\vec{y}(G; a, b)$, such that $\delta_b(P_i) = i$.

Proof (cf. [7, (9.2) on page 67, and Case 3 on page 69]). For convenience, we let $T = T(m, n, e)$. Let $x_1, x_2 \in \mathbb{R}$, such that $(x_1, 1)$ and $(x_2, 2)$ are on the line segment from $(n, 0)$ to (e, m) . (This line segment is the side of T that is opposite the vertex $(0, 0)$.) Then $(x_1, 1)$ is the midpoint of the line segment from $(n, 0)$ to $(x_2, 2)$, so

$$x_1 = \frac{x_2}{2} + \frac{n}{2}.$$

We consider two cases, but the argument is almost the same for both.

Case I. Assume $x_1 \notin \mathbb{Z}$. For convenience, let $x_1^* = \lfloor x_1 \rfloor$, and let

$$I = \{x_1^*, x_1^* - 1, x_1^* - 2, \dots, x_1^* - (\lfloor n/2 \rfloor - 1)\}.$$

For all $x \in I$, it is obvious that $x \leq x_1^* < x_1$. Also, we have

$$x \geq x_1^* - (\lfloor n/2 \rfloor - 1) > (x_1 - 1) - ((n/2) - 1) = \frac{x_2}{2}.$$

So the set $\{(x, 1) \mid x \in I\}$ is contained in the interior of T , and the slope of each point in this set is strictly smaller than the slope of any nonzero lattice point in T that is not in this set, other than the points $(1, 0), (2, 0), \dots, (n, 0)$ that are on the x -axis (and therefore have slope 0). This implies that $(x, 1)$ is in T_0 for all $x \in I$. More precisely, since it is obvious that the lattice point $(x, 1)$ is primitive, we have

$$\{A_0(k) \mid 2 \leq k \leq \lfloor n/2 \rfloor + 1\} = \{(x, 1) \mid x \in I\}.$$

Since $2x > x_2$ for all $x \in I$, this implies $2A_0(k) \notin T$, so

$$h_0(k) = 1 \text{ for } 2 \leq k \leq \lfloor n/2 \rfloor + 1.$$

By the definition of $u_i(k)$, we then have

$$u_0(k) - u_0(k-1) = 2h_0(k) = 2$$

for these values of k , so

$$\begin{aligned} & \{ u_0(k) \mid 1 \leq k \leq \lfloor n/2 \rfloor + 1 \} \\ & = \{ n-1, n+1, n+3, \dots, n-1+2\lfloor n/2 \rfloor \} \end{aligned}$$

Hence, Theorem 3.21 tells us that $H_0(d)$ is a hamiltonian path for all d in this set. Since

$$\delta_b(H_0(d)) = 0 \cdot o(a-b) + d = d,$$

and $2\lfloor n/2 \rfloor \geq 2\lfloor (n-1)/2 \rfloor$, this implies that we may let $n' = n-1$ and $P_i = H_0(i)$.

Case II. Assume $x_1 \in \mathbb{Z}$. This means that $X_1 = (x_1, 1)$. Let

$$I = \{ x_1, x_1 - 1, x_1 - 2, \dots, x_1 - \lfloor (n-1)/2 \rfloor \}.$$

For all $x \in I$, we have

$$x \geq x_1 - \lfloor (n-1)/2 \rfloor > x_1 - (n/2) = \frac{x_2}{2},$$

so, as in the previous case, the points in $\{ (x, 1) \mid x \in I \}$ are in T , and have strictly smaller slope than any other lattice points in T , other than the lattice points on the x -axis. Since $(x_1, 1) = X_1$, this implies $(x, 1)$ is in T_1 . More precisely, we have

$$\{ A_1(k) \mid 1 \leq k \leq \lfloor (n-1)/2 \rfloor + 1 \} = \{ (x, 1) \mid x \in I \}.$$

Since $(x, 1)$ is primitive (and $(x_1, 1)$ is on the boundary of T , not in the interior), this implies that

$$h_0(k) = \begin{cases} 0 & \text{if } k = 1 \\ 1 & \text{otherwise} \end{cases} \quad \text{for } 1 \leq k \leq \lfloor (n-1)/2 \rfloor + 1.$$

Therefore $H_1(d)$ is a hamiltonian path for all d in

$$\{ u_1(k) \mid 1 \leq k \leq \lfloor (n-1)/2 \rfloor + 1 \} = \{ 0, 2, 4, \dots, 2\lfloor (n-1)/2 \rfloor \}.$$

Note that, since the y -coordinate of $X_1 = (x_1, 1)$ is 1, it follows from the definition in Notation 3.17 that $m = |G : \langle a-b \rangle|$, so

$$o(a-b) = \frac{|G|}{|G : \langle a-b \rangle|} = \frac{mn}{m} = n.$$

Therefore

$$\delta_b(H_1(d)) = 1 \cdot o(a-b) + d = 1 \cdot n + d = n + d.$$

Therefore, we may let $n' = n$ and $P_i = H_1(i)$. □

Proposition 3.27 (Curran-Witte, cf. [7, (9.5) on page 68]). *Assume that m , n , and e are as in Notation 3.16. Also assume $m \geq 2$ and $n \geq e$.*

For every integer k , such that $n - 1 \leq k \leq m(n - 1)$, there is a hamiltonian path P in $\overrightarrow{\text{Cay}}(G; a, b)$, such that

$$k \leq \delta_b(P) \leq k + 2 \left\lfloor \frac{mn - 1}{m + n - e} \right\rfloor.$$

Therefore, if $n \geq m + e$, then

$$k \leq \delta_b(P) \leq k + 2 \left\lfloor \frac{n - 1}{2} \right\rfloor.$$

Proof. First, note that if $n \geq m + e$, then

$$\frac{mn - 1}{m + n - e} \leq \frac{mn - 1}{2m} < \frac{mn}{2m} = \frac{n}{2},$$

so

$$\left\lfloor \frac{mn - 1}{m + n - e} \right\rfloor \leq \left\lfloor \frac{n - 1}{2} \right\rfloor.$$

Therefore, the final sentence of the proposition follows from the one that precedes it.

For convenience, let $N = o(a - b)$ and $M = |G|/N - 1$. In this notation, we have

$$\begin{aligned} & \{ \delta_b(P) \mid P \text{ is a hamiltonian path} \} \\ &= \{ u_0(1), u_0(2), \dots, u_0(f_0 - 1), \\ & \quad N + u_1(1), N + u_1(2), \dots, N + u_1(f_1 - 1), \\ & \quad 2N + u_2(1), 2N + u_2(2), \dots, 2N + u_2(f_2 - 1), \\ & \quad \vdots \\ & \quad MN + u_M(1), MN + u_M(2), \dots, MN + u_M(f_M - 1) \}. \end{aligned}$$

Since $u_0(1) = n - 1$ and

$$\begin{aligned} MN + u_M(f_M - 1) &= (|G| - N) + (N - 1 - h_M(f)) \\ &= mn - 1 - h_M(f) \geq mn - m = m(n - 1), \end{aligned}$$

the assertion we wish to prove is that the difference between successive terms in this list is never larger than $2 \lfloor (mn - 1)/(m + n - e) \rfloor + 1$. It therefore suffices to show

$$u_t(k) - u_t(k - 1) \leq 2 \left\lfloor \frac{mn - 1}{m + n - e} \right\rfloor + 1 \text{ for } 2 \leq k < f_t \quad (3.1)$$

and

$$N + u_{t+1}(1) - u_t(f_t - 1) \leq 2 \left\lfloor \frac{mn - 1}{m + n - e} \right\rfloor + 1 \text{ for } 0 \leq t < M. \quad (3.2)$$

To establish (3.1), note that (much as in [7, Case 1 on page 67]) we have

$$\begin{aligned} u_t(k) - u_t(k-1) &= 2h_k \\ &= 2 \left\lfloor \frac{mn-1}{mx_k + (n-e)y_k} \right\rfloor \quad (\text{where } x_k, y_k \in \mathbb{Z}^+) \\ &\leq 2 \left\lfloor \frac{mn-1}{m+n-e} \right\rfloor \quad \left(\begin{array}{l} n-e \geq 0 \\ \text{and } x_k, y_k \geq 1 \end{array} \right). \end{aligned}$$

To establish (3.2), we make three observations. First, note that, by definition, we have $u_{t+1}(1) = h_{t+1}(1)$. Second, we know from Lemma 3.23 that

$$u_t(f_t - 1) = N - 1 - h_t(f).$$

Third, we have $A_t(f_t) = A_{t+1}(1)$. (This is the primitive lattice point on the boundary between T_t and T_{t+1} .) So

$$h_{t+1}(1) = h_t(f) = \left\lfloor \frac{mn-1}{mx_f + (n-e)y_f} \right\rfloor \leq \left\lfloor \frac{mn-1}{m+n-e} \right\rfloor.$$

Therefore

$$\begin{aligned} N + u_{t+1}(1) - u_t(f_t - 1) &= N + h_t(f) - (N - 1 - h_t(f)) \\ &= 2h_t(f) + 1 \\ &\leq 2 \left\lfloor \frac{mn-1}{m+n-e} \right\rfloor + 1. \quad \square \end{aligned}$$

To provide a convenient reference, we now restate Proposition 3.27 for the special case where $\text{Ca}\vec{y}(G; a, b)$ is the Cartesian product of two directed cycles:

Corollary 3.28. *Let $\{e_1, e_2\}$ be the standard generating set of $\mathbb{Z}_m \times \mathbb{Z}_n$, and assume $2 \leq m \leq n$. If $1 \leq k \leq m(n-1)$, then there is a hamiltonian path P in $\text{Ca}\vec{y}(\mathbb{Z}_m \times \mathbb{Z}_n; e_1, e_2)$, such that*

$$k \leq \delta_{e_2}(P) \leq k + 2 \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Proof. We have $e = 0$, so the assumption that $m \leq n$ implies $n \geq m + e$. Therefore, if $k \geq n - 1$, then the desired conclusion is obtained from the final sentence in the statement of Proposition 3.27.

So we may now assume $k < n - 1$. Then, since

$$n - 1 - k \leq n - 2 \leq 2 \lfloor (n - 1)/2 \rfloor,$$

we may let $P = H_0(n-1)$. (Since $u_0(1) = n-1$, we know that $H_0(n-1)$ is a hamiltonian path. We also have $\delta_{e_2}(H_0(n-1)) = n-1$.) \square

4 Cartesian product of two directed cycles

In this section, we show that the Cartesian product of two directed cycles has two arc-disjoint hamiltonian paths. We start with the following lemma.

Lemma 4.1. *Let $\{a, b\}$ be a 2-element generating set of a finite abelian group G , and let k and ℓ' be integers such that*

$$0 \leq k, \ell' < |G| \text{ and } |k - \ell'| \leq 1.$$

Then there exist two arc-disjoint spanning quasi-paths P and P' in $\overrightarrow{\text{Ca}}\overrightarrow{\text{y}}(G; a, b)$, such that $\delta_b(P) = k$ and $\delta_a(P') = \ell'$.

Proof. By Lemma 3.14, there is a spanning quasi-path P , such that $\delta_b(P) = k$. Let ι and τ be the initial vertex and terminal vertex of P , respectively.

Let \overline{P} be the complement of P in $\overrightarrow{\text{Ca}}\overrightarrow{\text{y}}(G; a, b)$. (Thus, a subdigraph of $\overrightarrow{\text{Ca}}\overrightarrow{\text{y}}(G; a, b)$ is arc-disjoint from P if and only if it is contained in \overline{P} .) The vertices that have some a -edge as an out-edge in \overline{P} are precisely the vertices that do not travel by a in P ; this consists of the vertices that travel by b in P , plus the terminal vertex τ . Hence, we have

$$\delta_a(\overline{P}) = \delta_b(P) + 1 = k + 1.$$

Note that \overline{P} is *not* a spanning quasi-path, because the indegree of ι is 2, and the outdegree of τ is 2. (The indegree and outdegree of each of the other vertices is 1.)

Case I. *Assume \overline{P} does not have a directed edge from τ to ι .* If we construct a subdigraph P' of \overline{P} by removing either of the two in-edges of ι and either of the two out-edges of τ , then P' will be a spanning quasi-path. (The initial vertex of the removed in-edge will have outdegree 0, and the terminal vertex of the removed out-edge will have indegree 0.) We can freely decide whether neither, precisely one, or both of the directed edges we remove is an a -edge, so we can construct P' so that $\delta_a(P')$ is any desired element of

$$\{\delta_a(\overline{P}), \delta_a(\overline{P}) - 1, \delta_a(\overline{P}) - 2\} = \{k + 1, k, k - 1\}.$$

Since $|k - \ell'| \leq 1$, this means we can construct P' so that $\delta_a(P')$ is equal to ℓ' , as desired.

Case II. *Assume (τ, ι) is a directed edge of \overline{P} .* Assume, without loss of generality, that (τ, ι) is a b -edge (by interchanging a and b if necessary).

If $\ell' = k - 1$, then $\ell' = \delta_a(\overline{P}) - 2$, so we construct P' by removing two a -edges from \overline{P} : the out-edge of τ that is labelled a , and the in-edge of ι that is labelled a . (These are two different directed edges, because we have assumed that the directed edge (τ, ι) is a b -edge.)

We may now assume that

$$\ell' \in \{k + 1, k\} = \{\delta_a(\overline{P}), \delta_a(\overline{P}) - 1\}.$$

Removing the b -edge (τ, ι) from \overline{P} results in a spanning subdigraph P^* in which the indegree and outdegree of every vertex is 1, so each connected component of P^* is a directed cycle. Hence, removing any directed edge from P^* will result in a spanning quasi-path P' . Thus, if P^* has both an a -edge and a b -edge, then we can choose which type of directed edge to remove, so we can arrange for $\delta_a(P')$ to be either element of

$$\{\delta_a(\overline{P}), \delta_a(\overline{P}) - 1\},$$

so we can certainly arrange for $\delta_a(P')$ to be equal to ℓ' .

Since P^* does have a -edges (such as the out-edge of τ), we may now assume that every directed edge of P^* is an a -edge. This means that $\delta_a(P^*) = |G|$, so

$$k = \delta_a(\overline{P}) - 1 = \delta_a(P^*) - 1 = |G| - 1.$$

Since $\ell' \in \{k+1, k\}$ and $\ell' < |G|$, this implies that $\ell' = k = |G| - 1$. So we can construct the desired spanning quasi-path P' by removing one of the a -edges of P^* . \square

Remark 4.2. It is easy to see that the converse of Lemma 4.1 is true. Namely, if P and P' are arc-disjoint spanning quasi-paths in $\text{Ca}\vec{\gamma}(G; a, b)$, then we have

$$|\delta_b(P) - \delta_a(P')| \leq 1.$$

Proposition 4.3. *Let $\{a, b\}$ be a 2-element generating set of a finite abelian group G , and assume that $\text{Ca}\vec{\gamma}(G; a, b)$ has hamiltonian paths P and P' that satisfy either of the following two equivalent conditions:*

- (1) $|\delta_b(P) - \delta_a(P')| \leq 1$, or
- (2) $|\delta_b(P) + \delta_b(P')| \in \{|G|, |G| - 1, |G| - 2\}$.

Then $\text{Ca}\vec{\gamma}(G; a, b)$ has two arc-disjoint hamiltonian paths.

Proof. First, note that the equivalence of the two conditions is immediate from the observation that $\delta_a(P') = |G| - 1 - \delta_b(P')$. Therefore, we may assume that (1) holds.

Lemma 4.1 tells us that there exist two arc-disjoint spanning quasi-paths P_0 and P'_0 in $\text{Ca}\vec{\gamma}(G; a, b)$, such that $\delta_b(P_0) = \delta_b(P)$ and $\delta_a(P'_0) = \delta_a(P')$. Since $\delta_b(P_0) = \delta_b(P)$ and

$$\delta_b(P'_0) = |G| - 1 - \delta_a(P'_0) = |G| - 1 - \delta_a(P') = \delta_b(P'),$$

we see from Corollary 3.12 that P_0 and P'_0 are hamiltonian paths. Since P_0 and P'_0 are arc-disjoint, this completes the proof. \square

We are now ready to prove the main theorem of the paper.

Theorem 4.4. *If C_1 and C_2 are directed cycles (of length ≥ 2), then the Cartesian product $C_1 \square C_2$ has two arc-disjoint hamiltonian paths.*

Proof. Let m and n be the lengths of C_1 and C_2 , so

$$C_1 \square C_2 \cong \text{Ca}\vec{\gamma}(\mathbb{Z}_m \times \mathbb{Z}_n; e_1, e_2),$$

where $\{e_1, e_2\} = \{(1, 0), (0, 1)\}$ is the standard generating set of the abelian group $\mathbb{Z}_m \times \mathbb{Z}_n$. Assume without loss of generality that $m \leq n$. Then Corollary 3.28 provides a hamiltonian path P' , such that

$$mn - n - 2 \left\lfloor \frac{n-1}{2} \right\rfloor \leq \delta_{e_2}(P') \leq mn - n.$$

Since $\delta_{e_1}(P') = mn - 1 - \delta_{e_2}(P')$, we have

$$n - 1 \leq \delta_{e_1}(P') \leq n - 1 + 2 \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Then it is immediate from Proposition 3.26 that there is a hamiltonian path P_i , such that we have $|\delta_{e_2}(P_i) - \delta_{e_1}(P')| \leq 1$. The desired conclusion now follows from Proposition 4.3(1). \square

The above result shows that there is always at least one pair of arc-disjoint hamiltonian paths. In fact, the number of such pairs is large if both directed cycles in the product have large length:

Proposition 4.5 (see Corollary 6.4 below). *Let $N(m, n)$ be the number of (unordered) pairs P, P' of arc-disjoint hamiltonian paths in the Cartesian product of a directed cycle of length m and a directed cycle of length n . If m and n are sufficiently large, then*

$$N(m, n) > \frac{m^2 n^2}{10}.$$

If $\overrightarrow{\text{Cay}}(G; a, b)$ has two arc-disjoint hamiltonian cycles, then removing a directed edge from each of these hamiltonian cycles yields two arc-disjoint hamiltonian paths P and P' . Note that the initial vertex ι' of P' is completely arbitrary, even after the path P has been chosen, because there is no restriction on the directed edge that is removed. (The terminal vertex of P' is also arbitrary, but it is determined by the choice of ι' .) In all other situations, the following observation shows that P' is almost completely determined by the choice of P .

Proposition 4.6. *Suppose P and P' are two arc-disjoint hamiltonian paths in $\overrightarrow{\text{Cay}}(G; a, b)$. Let ι and ι' be the initial vertices of these hamiltonian paths, and let τ and τ' be the terminal vertices. Assume there is no directed edge from τ to ι (or, equivalently by Remark 2.4, that there is no directed edge from τ' to ι'). Then:*

- (1) $\iota' \in \{\tau + a, \tau + b\}$,
- (2) $\tau' \in \{\iota - a, \iota - b\}$,
- (3) ι and τ' are in the same coset of $\langle a - b \rangle$,
- (4) ι' and τ are in the same coset of $\langle a - b \rangle$, and
- (5) there are at most two hamiltonian paths that are arc-disjoint from P .

Proof. Let \overline{P} be the complement of P in $\overrightarrow{\text{Cay}}(G; a, b)$, as in the proof of Lemma 4.1. Note that P' can be obtained from \overline{P} by removing one of the two in-edges of ι and one of the two out-edges of τ . (Since there is no directed edge from τ to ι , the in-edges of ι are distinct from the out-edges of τ .) Thus, we have (1) $\iota' \in \{\tau + a, \tau + b\}$ and (2) $\tau' \in \{\iota - a, \iota - b\}$. Combining these with Lemma 3.9(2)(a) yields (3) and (4).

(5) Since there are only two possible choices for the in-edge of ι to remove, and two possible choices for the out-edge of τ to remove, it is obvious that no more than 4 hamiltonian paths are arc-disjoint from P .

To complete the proof, we show there cannot be more than one hamiltonian path that starts at $\tau + a$ and is arc-disjoint from P . (By symmetry, there is also no more than one that starts at $\tau + b$). Suppose, for a contradiction, that the terminal vertex of P' is $\tau' = \iota - a$ and the terminal vertex of some other arc-disjoint hamiltonian path P'' is $\tau'' = \iota - b$. Then $\tau' - \tau'' = b - a$. Thus, if d is the number of vertices in $\tau' + \langle a - b \rangle$ that travel by b in P' , then the number of vertices in this coset that travel by b in P'' is $d + 1$.

Note that if a non-terminal coset travels by a in P , then it must travel by b in both P' and P'' (since P' and P'' are arc-disjoint from P). Conversely, if a non-terminal coset travels by b in P' or P'' , then it must travel by a in P . Therefore, a non-terminal coset

travels by b in P' if and only if it travels by b in P'' . Hence, if we let t be the number of non-terminal cosets that travel by b in P' , then t is also the number of non-terminal cosets that travel by b in P'' .

Since P' and P'' are hamiltonian paths, we see from Proposition 3.11 that $H_t(d)$ and $H_t(d+1)$ are hamiltonian paths. So Theorem 3.21 tells us that $d = u_t(k')$ and $d+1 = u_t(k'')$, for some k' and k'' . However, it is clear from Notation 3.19(4) that two values of $u_t(*)$ cannot differ by 1. This is a contradiction. \square

The following observation will be used in the proof of Proposition 5.2(2):

Corollary 4.7. *Let $\{a, b\}$ be a 2-element generating set of a finite abelian group G , such that*

- $\text{Ca}\vec{\text{y}}(G; a, b)$ does not have a hamiltonian cycle,
- $|G|$ is even, and
- $|G : \langle a - b \rangle|$ is odd.

If P and P' are two arc-disjoint hamiltonian paths in $\text{Ca}\vec{\text{y}}(G; a, b)$, with initial vertices ι and ι' , and terminal vertices τ and τ' , then

$$\iota' + \tau' = \iota + \tau.$$

Proof. Since $\text{Ca}\vec{\text{y}}(G; a, b)$ is not hamiltonian, there is no directed edge from τ to ι , so we see from Proposition 4.6 that $\iota' \in \{\tau + a, \tau + b\}$ and $\tau' \in \{\iota - a, \iota - b\}$, so

$$(\iota' + \tau') - (\iota + \tau) \in \{0, \pm(a - b)\}.$$

Therefore, since $a - b \notin 2G$ (because $|G|$ is even and $|G : \langle a - b \rangle|$ is odd), it will suffice to show $\iota' + \tau' \equiv \iota + \tau \pmod{2G}$.

Write

$$\delta_b(P) = t o(a - b) + d \text{ and } \delta_b(P') = t' o(a - b) + d',$$

so we see from Theorem 3.21(3) (and Proposition 3.11) that $d = u_t(k)$ and $d' = u_{t'}(k')$ for some k and k' .

Also recall that Lemma 3.9(2)(a) tells us $\tau \in \iota - a + \langle a - b \rangle$. Since (from above) $\tau' \in \{\iota - a, \iota - b\}$, this implies $\tau + \langle a - b \rangle = \tau' + \langle a - b \rangle$, which means that P and P' have the same terminal coset. So they also have the same non-terminal cosets. Since P and P' are arc-disjoint, each non-terminal coset travels by b in either P or P' , but not both. So

$$t + t' = |G : \langle a - b \rangle| - 1.$$

Since $|G : \langle a - b \rangle| - 1$ is even, we conclude that $t \equiv t' \pmod{2}$, so Corollary 3.25(2) tells us that $d \equiv d' \pmod{2}$. We know from Lemma 3.9(2)(a) that

$$\iota - \tau = d(a - b) + a \text{ and } \iota' - \tau' = d'(a - b) + a,$$

so this implies

$$\begin{aligned} \iota' + \tau' &\equiv \iota' - \tau' && \pmod{2G} \\ &= d'(a - b) + a \\ &\equiv d(a - b) + a && \pmod{2G} \\ &= \iota - \tau \\ &\equiv \iota + \tau && \pmod{2G}. \end{aligned} \quad \square$$

5 Cartesian product of more than two directed cycles

Corollary 5.1. *If C_1, C_2, \dots, C_r are directed cycles (of length ≥ 2), and $r \geq 4$, then the Cartesian product $C_1 \square C_2 \square \dots \square C_r$ has two arc-disjoint hamiltonian paths.*

Proof. This argument is very easy (and classical: see the proof of [7, Lemma 9.2], for example). Theorem 3.15 provides a hamiltonian cycle C in the Cartesian product $C_1 \square C_2 \square \dots \square C_{r-1}$. Then $C_1 \square C_2 \square \dots \square C_r$ has a spanning subdigraph that is isomorphic to $C \square C_r$, and Theorem 4.4 provides two arc-disjoint hamiltonian paths in this spanning subdigraph. Hamiltonian paths in a spanning subdigraph are also hamiltonian paths in the entire digraph. \square

We do not know whether the Cartesian product of three directed cycles always has two arc-disjoint hamiltonian paths, but we can prove that the paths exist in most cases:

Proposition 5.2. *Assume C_1, C_2 , and C_3 are directed cycles (of length ≥ 2). If either*

- (1) *the Cartesian product of two of the directed cycles has a hamiltonian cycle, or*
- (2) *the lengths of the three directed cycles do not all have the same parity (i.e., if there is a directed cycle of even length and a directed cycle of odd length), or*
- (3) *at least one of the directed cycles has length 2,*

then $C_1 \square C_2 \square C_3$ has two arc-disjoint hamiltonian paths.

Proof. (1) This uses the same simple argument as the proof of Corollary 5.1. Assume, without loss of generality, that $C_1 \square C_2$ has a hamiltonian cycle C . Then $C \square C_3$ is isomorphic to a spanning subdigraph of $C_1 \square C_2 \square C_3$, and Theorem 4.4 provides two arc-disjoint hamiltonian paths in this spanning subdigraph.

(2) Write

$$C_1 \square C_2 \square C_3 = \text{Ca}\vec{\text{y}}(\mathbb{Z}_m \times \mathbb{Z}_n \times \mathbb{Z}_r; e_1, e_2, e_3),$$

where $\{e_1, e_2, e_3\}$ is the standard generating set of $\mathbb{Z}_m \times \mathbb{Z}_n \times \mathbb{Z}_r$. For $0 \leq i < r$, let X_i be the subdigraph that is induced by $\mathbb{Z}_m \times \mathbb{Z}_n \times \{i\}$, so $X_i \cong \text{Ca}\vec{\text{y}}(\mathbb{Z}_m \times \mathbb{Z}_n; e_1, e_2)$.

By Theorem 4.4, we may let P_0 and P'_0 be two arc-disjoint hamiltonian paths in X_0 . Let ι_0 and ι'_0 be their initial vertices, and let τ_0 and τ'_0 be their terminal vertices. By assumption, we may assume that m and n have opposite parity (by permuting the factors). We may also assume X_0 does not have a directed hamiltonian cycle, for otherwise (1) applies. Then we see from Corollary 4.7 that

$$\tau - \iota' = \tau' - \iota.$$

Therefore, if we let $\Delta := \tau - \iota'$, then

$$\tau = \iota' + \Delta \quad \text{and} \quad \tau' = \iota + \Delta.$$

For $0 \leq i < r$, let P_i and P'_i be the translates of P_0 and P'_0 by $(\Delta i, i)$, so P_i and P'_i are arc-disjoint hamiltonian paths in X_i .

Let (ι_i, i) and (ι'_i, i) be the initial vertices of P_i and P'_i and let (τ_i, i) and (τ'_i, i) be their terminal vertices. Then

$$\begin{aligned} (\tau_i, i) + e_3 &= (\tau + \Delta i, i + 1) = ((\iota' + \Delta) + \Delta i, i + 1) \\ &= (\iota' + \Delta(i + 1), i + 1) = (\iota'_{i+1}, i + 1) \end{aligned}$$

is the initial vertex of P'_{i+1} . This means there is a directed e_3 -edge from the terminal vertex of P_i to the initial vertex of P'_{i+1} . Similarly, since $\tau' = \iota + \Delta$, we see that there is a directed e_3 -edge from the terminal vertex of P'_i to the initial vertex of P_{i+1} .

Therefore, if we start with the union of all of the paths P_i for i even and P'_j for j odd, then we can add certain e_3 -edges to construct a hamiltonian path P in $C_1 \square C_2 \square C_3$. Similarly, we can construct a hamiltonian path P' by adding appropriate e_3 -edges to the union of all of the paths P_i for i even and P'_j for j odd. (In the terminology of [7, Definition 9.4], P and P' are “laminated” hamiltonian paths.) It is clear from the construction that these hamiltonian paths are arc-disjoint.

(3) Assume, without loss of generality, that C_1 has length 2. If C_2 has even length, then (since C_1 has length 2) it is easy to see (and well known) that $C_1 \square C_2$ has a hamiltonian cycle, so (1) applies. If C_2 has odd length, then (2) applies. \square

6 2-generated Cayley digraphs on finite abelian groups

Proposition 6.1. Conjecture 1.3 is true in all cases where the arc-forcing subgroup $\langle a - b \rangle$ has even index in G .

Proof. Let $t = |G : \langle a - b \rangle|/2$, and let $d = u_t(1)$, so by Theorem 3.21 we know that $H_t(d)$ is a hamiltonian path. By Corollary 3.24, we have

$$u_{t-1}(f_{t-1} - 1) = o(a - b) - 1 - d,$$

so $H_{t-1}(o(a - b) - 1 - d)$ is also a hamiltonian path. Furthermore,

$$\begin{aligned} \delta_b(H_t(d)) + \delta_b(H_{t-1}(o(a - b) - 1 - d)) \\ &= [t o(a - b) + d] + [(t - 1) o(a - b) + (o(a - b) - 1 - d)] \\ &= 2t o(a - b) - 1 \\ &= |G| - 1. \end{aligned}$$

Therefore, Proposition 4.3(2) provides two arc-disjoint hamiltonian paths. \square

The open cases of Conjecture 1.3 reduce to two types of examples (and $G = \mathbb{Z}_n$ is cyclic in both types):

Proposition 6.2. It suffices to prove Conjecture 1.3 for the following two families of Cayley digraphs:

- (1) $\overrightarrow{\text{CaY}}(\mathbb{Z}_k; a, a + 1)$, where $k, a \in \mathbb{Z}^+$ (and neither a nor $a + 1$ is divisible by k), and
- (2) $\overrightarrow{\text{CaY}}(\mathbb{Z}_k; -a, a + 1)$, where $k, a \in \mathbb{Z}^+$, and k is divisible by $2a + 1$ (and neither a nor $a + 1$ is divisible by k).

Proof. Assume that $\{a, b\}$ is a 2-element generating set of a finite abelian group G . For convenience, let $\mathcal{F} = \langle a - b \rangle$. We may assume that $|G : \mathcal{F}|$ is odd, for otherwise Proposition 6.1 applies. Write $|G : \mathcal{F}| = 2t + 1$ and let $c = ta + tb$.

Case I. Assume that $c + a \neq 0$ and $c + b \neq 0$. Let $a' = c + a$ and $b' = c + b$. Since $a', b' \in \langle a - b \rangle$, and $a' - b' = a - b$, we have

$$\overrightarrow{\text{CaY}}(\mathcal{F}; a', b') \cong \overrightarrow{\text{CaY}}(\mathbb{Z}_k; \ell, \ell + 1),$$

where $k = o(a - b)$ and $b' = \ell(a - b)$. This Cayley digraph is listed in (1), so we may assume that it has two arc-disjoint hamiltonian paths. Thus (by Proposition 3.11), there exist d and d' , such that $H_0(d)$ and $H_0(d')$ are hamiltonian paths in $\overrightarrow{\text{CaY}}(\mathcal{F}; c + a, c + b)$, and (by Remark 4.2) we have $d + d' = o(a - b) - 1 + \epsilon$, where $|\epsilon| \leq 1$. Then $H_t(d)$ and $H_t(d')$ are hamiltonian paths in $\overrightarrow{\text{CaY}}(G; a, b)$ (by the “skewed generators argument,” see the proof of [19, Lemma 2.6]).

Since

$$\begin{aligned} \delta_b(H_t(d)) + \delta_b(H_t(d')) &= (t o(a - b) + d) + (t o(a - b) + d') \\ &= 2t o(a - b) + (d + d') \\ &= (|G : \mathcal{F}| - 1)|\mathcal{F}| + (o(a - b) - 1 + \epsilon) \\ &= |G| - 1 + \epsilon, \end{aligned}$$

we conclude from Proposition 4.3(2) that $\overrightarrow{\text{CaY}}(G; a, b)$ has two arc-disjoint hamiltonian paths.

Case II. Assume that either $c + a = 0$ or $c + b = 0$. Assume without loss of generality that $c + a = 0$. For convenience, let $w = a - b$ and $k = |G|$. We have

$$0 = c + a = (ta + tb) + a = (t + 1)a + tb = (2t + 1)b + (t + 1)w,$$

so $(2t + 1)b = -(t + 1)w$. Also note that

$$o(w) = o(a - b) = \frac{k}{2t + 1}$$

(so k is divisible by $2t + 1$). Hence, as an abelian group, G has the presentation

$$G = \left\langle b, w \mid (2t + 1)b = -(t + 1)w, \frac{k}{2t + 1}w = 0 \right\rangle,$$

so G , b , and w are uniquely determined (up to isomorphism) by k and t , and the assumptions that $|G| = k$, $ta + tb + a = 0$, and $w = a - b$ has order $k/(2t + 1)$. Since $a = b + w$, it is also uniquely determined.

On the other hand, it is clear that letting $G = \mathbb{Z}_k$, $a = 1 + t$, $b = -t$, and $w = 2t + 1$ provides an example (for any k and t , such that k is divisible by $2t + 1$). Hence, we conclude that $\overrightarrow{\text{CaY}}(G; a, b)$ is isomorphic to the Cayley digraph that is listed in (2). \square

It is well known [11, Theorem 459, page 541] that the probability that two random integers are relatively prime is $6/\pi^2 = 0.6079\dots$. This has the following elementary consequence:

Lemma 6.3 ([7, Theorem 8.5]). *If $N(m, n, e)$ is the number of primitive lattice points in the interior of the triangle $T(m, n, e)$, then*

$$\lim_{m, n \rightarrow \infty} \frac{N(m, n, e)}{\text{Area}(T(m, n, e))} = \frac{6}{\pi^2}.$$

Corollary 6.4. *Let $N(G; a, b)$ be the number of (unordered) pairs $\{P, P'\}$ of arc-disjoint hamiltonian paths in $\overrightarrow{\text{CaY}}(G; a, b)$. If $m = o(a)$ and $n = |G|/o(a)$ are sufficiently large, then*

$$N(G; a, b) > \frac{|G|^2}{10}.$$

Proof. Let

$$R = \left\{ t o(a-b) + u_t(k) \mid \begin{array}{l} 0 \leq t < |G| : \langle a-b \rangle, \quad 1 \leq k < f_t, \\ A_t(k) \text{ is in the interior of } T(m, n, e) \end{array} \right\} \\ \subseteq \{1, 2, \dots, |G| - 2\}.$$

Also let

$$R^+ = R \cup (R + 1) \text{ and } R^- = R \cup (R - 1).$$

We claim that $|r - r'| \geq 2$ for all distinct $r, r' \in R$. To see this, first recall that

$$u_t(k) - u_t(k-1) = 2h_t(k) \geq 2.$$

Second, note that, by Corollary 3.24, we have

$$(t o(a-b) + u_t(1)) - ((t-1) o(a-b) + u_{t-1}(f_{t-1} - 1)) = 2u_t(1) + 1.$$

If $A_t(1)$ is in the interior of $T(m, n, e)$, then $h_t(1) \geq 1$. Since $u_t(1) = h_t(1)$, this implies $2u_t(1) + 1 > 2$, which completes the proof of the claim.

From the claim, we see that

$$\#R^+ = \#R^- = 2 \cdot \#R \approx 2 \cdot \frac{6}{\pi^2} \text{Area}(T(m, n, e)) = \frac{6}{\pi^2} |G|.$$

Therefore

$$\begin{aligned} \#(R^+ \cap (|G| - 1 - R^-)) &\geq \#R^+ + \#R^- - |G| \\ &\approx 2 \cdot \frac{6}{\pi^2} |G| - |G| \\ &> 0.2|G|. \end{aligned}$$

For each $r \in R$, we know from the definition of R that there is a hamiltonian path P , such that $\delta_b(P) = r$. Hence, for each r in the above intersection, we have hamiltonian paths P and P' , such that

$$\delta_b(P) \in \{r, r-1\}$$

and

$$\delta_b(P') \in \{|G| - 1 - r, |G| - r\}.$$

Therefore

$$\delta_b(P) + \delta_b(P') \in \{|G|, |G| - 1, |G| - 2\},$$

so we see from (the proof of) Proposition 4.3(2) that these hamiltonian paths can be made arc-disjoint.

This provides more than $0.2|G|$ ordered pairs of arc-disjoint hamiltonian paths. So the number of unordered pairs is more than $0.1|G|$. Furthermore, if $\{P_1, P'_1\}$ and $\{P_2, P'_2\}$ are two such pairs, then we know from the construction that $\delta_b(P_1) \notin \{\delta_b(P_2), \delta_b(P'_2)\}$, so $\{P_1, P'_1\}$ is not a translate of $\{P_2, P'_2\}$. Therefore, since each pair has $|G|$ translates, the total number of unordered pairs of arc-disjoint hamiltonian paths is more than $0.1|G|^2$.

(Actually, there is a slight technical issue that the translates of $\{P, P'\}$ might not all be distinct. However, this can only happen if P' is a translate of P , which implies that $\delta_b(P) = \delta_b(P')$. Since P and P' are arc-disjoint, this determines $\delta_b(P)$ up to an error of at most 1. So this problem is avoided by deleting a small number of values of r that are near $|G|/2$.) \square

The following variant of Lemma 6.3 provides an explicit lower bound on m and n when $6/\pi^2$ is replaced with the smaller constant $1/2$:

Lemma 6.5 (cf. [7, Theorem 8.5]). *Let $N(m, n, e)$ be the number of primitive lattice points in the interior of the triangle $T(m, n, e)$. If $m, n \geq 300$, then*

$$N(m, n, e) > \frac{1}{2} \text{Area}(T(m, n, e)).$$

Sketch of proof (cf. [7, proof of Theorem 8.5]). For every triangle T , let $N(T)$ (resp. $P(T)$) be the number of lattice points (resp. primitive lattice points) that are in the interior of T and are not on the y -axis.

Case I. Assume $-n \leq e \leq n$. Then $\min(|x|, |y|) \leq \min(m, n)$ for all $(x, y) \in T(m, n, e)$, so we have $N(\frac{1}{k}T) = 0$ for all $k > \min(m, n)$. Also, it is elementary to see that

$$\left| N\left(\frac{1}{k}T\right) - \text{Area}\left(\frac{1}{k}T\right) \right| \leq \frac{2 \max(m, n)}{k}$$

for all $k \in \mathbb{Z}^+$. Therefore, letting $A = \text{Area}(T)$ and $\min = \min(m, n)$, we have

$$\begin{aligned} P(T) &= \sum_{k=1}^{\min} \mu(k) N\left(\frac{1}{k}T\right) \\ &= \sum_{k=1}^{\min} \mu(k) \left(\frac{A}{k^2} \pm \frac{2 \max(m, n)}{k} \right) \\ &= A \left(\frac{6}{\pi^2} - \sum_{k>\min} \frac{\mu(k)}{k^2} \right) \pm 2 \max(m, n)(1 + \log \min) \\ &> A \left(\frac{6}{\pi^2} - \frac{1}{\min} \right) - 2 \max(m, n)(1 + \log \min) \\ &= A \left(\frac{6}{\pi^2} - \frac{1}{\min} - \frac{4(1 + \log \min)}{\min} \right). \end{aligned}$$

Since $\min \geq 300$ (in fact, $\min \geq 252$ would suffice), we have

$$\frac{1}{\min} + \frac{4(1 + \log \min)}{\min} < 0.1079,$$

so we conclude that $P(T) > 0.5A$.

Case II. *The general case.* By applying an appropriate element $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ of $\text{SL}(2, \mathbb{Z})$, we may assume $n - m \leq e \leq n$. Since we may assume that Case I does not apply, we then have $n - m \leq e < -n$. The y -axis divides $T(m, n, e)$ into two smaller triangles T_1 and T'_1 . Letting $p = mn/(n - e) \geq n$, we have

- the vertices of T_1 are $(0, 0)$, $(n, 0)$, and $(0, p)$, and
- the vertices of T'_1 are $(0, 0)$, $(0, p)$, and (e, m) .

Since $p \geq n$, we see that Case **I** applies to the triangle T_1 , so

$$P(T_1) > \frac{1}{2} \text{Area}(T_1).$$

Now, rotating T'_1 by 90° clockwise around the origin yields the triangle T''_1 with vertices $(0, 0)$, $(p, 0)$ and $(m, -e)$. Since $p > n$ and $-e > n$, this triangle satisfies the hypotheses of the proposition. Therefore, either Case **I** applies to T''_1 , or the argument in the preceding paragraph divides T''_1 into two triangles T_2 and T'_2 , such that Case **I** applies to T_2 , and T'_2 satisfies the hypotheses of the proposition (after applying an appropriate element of $\text{SL}(2, \mathbb{Z})$). Continuing in this way, we see that $T(m, n, e)$ can be decomposed into a finite union

$$T = T_1 \cup T_2 \cup \dots \cup T_\ell$$

such that Case **I** applies to each T_i (after applying an appropriate element of $\text{SL}(2, \mathbb{Z})$). Therefore, we have $P(T_i) > \frac{1}{2} \text{Area}(T_i)$ for each i , so

$$P(T) \geq \sum_i P(T_i) > \sum_i \frac{1}{2} \text{Area}(T_i) = \frac{1}{2} \text{Area}(T). \quad \square$$

Proposition 6.6. Conjecture 1.3 is true in all cases where we have either $|G : \langle a \rangle| \geq 600$ or $|G : \langle b \rangle| \geq 600$.

Proof. Assume, without loss of generality, that $|G : \langle a \rangle| \geq 600$. We consider two cases.

Case I. Let $m \geq 300$. Then one can see that by Lemma 6.5 we have $2N(m, n, e) > |G|/2$, so the argument in the proof of Corollary 6.4 establishes that the number of pairs of arc-disjoint hamiltonian paths is greater than 0.

Case II. Assume $m < 300$. Since $e < m$ and $n = |G : \langle a \rangle| \geq 600$, this implies $n > m + e$. Hence, we see from Proposition 3.27 that for all k , such that $n - 1 \leq k \leq m(n - 1)$, there is a hamiltonian path P in $\text{Cay}^{\rightarrow}(G; a, b)$, such that

$$k \leq \delta_b(P) \leq k + 2 \left\lfloor \frac{n-1}{2} \right\rfloor.$$

The desired conclusion can now be obtained by combining this with Proposition 3.26, as in the proof of Theorem 4.4. \square

7 2-generated Cayley digraphs on infinite abelian groups

In this section, we study the natural analogue of Conjecture 1.3 for infinite digraphs. This is an addition to the literature on two-way infinite hamiltonian paths in Cayley (di)graphs on infinite abelian groups, which includes:

- Every connected Cayley graph on any countably infinite abelian group has a two-way infinite hamiltonian path [21, Theorem 1].

- A connected, 4-valent, infinite, circulant graph $\text{Cay}(\mathbb{Z}; \pm a, \pm b)$ has two edge-disjoint two-way infinite hamiltonian paths if and only if $a + b$ is even [5, Theorem 9].
- A connected, 4-valent Cayley graph on an infinite abelian group G has two edge-disjoint two-way infinite hamiltonian paths if and only if, for every partition of G into two infinite subsets X and Y , the number of edges from X to Y is either infinite or even [8, Theorem 2].
- Every connected Cayley graph of infinite valency on a finitely generated abelian group can be decomposed into edge-disjoint two-way infinite hamiltonian paths [5, Theorem 8].
- Every finitely generated, infinite abelian group G has an irredundant generating set S , such that $\overrightarrow{\text{Cay}}(G; S)$ has a two-way infinite hamiltonian path [15, Corollary 5.2].
- Assume G is an abelian group that has no cyclic subgroup of finite index, and X is a Cayley digraph on G that has infinite valency. Then X can be decomposed into arc-disjoint two-way infinite hamiltonian paths if and only if X is strongly connected ([16, Theorem 4.1] and [15, proofs of Theorems 5.3 and 6.3]).

The foundation of our results is the observation that the basic theory of the arc-forcing subgroup easily generalizes to the infinite case:

Lemma 7.1 (cf. Lemma 3.9(1)). *Let $\{a, b\}$ be a 2-element subset of a group G (which may be infinite). If P is a spanning subdigraph of $\overrightarrow{\text{Cay}}(G; a, b)$, such that the indegree and outdegree of every vertex is 1, then every right coset of $\langle a - b \rangle$ either travels by a or travels by b .*

This has the following easy consequence, which is the infinite analogue of Theorem 2.1(2):

Proposition 7.2. *Let $\{a, b\}$ be a 2-element generating set of an abelian group G (such that $a \neq b$), and let $I = |G : \langle a - b \rangle|$.*

- (1) *If $\overrightarrow{\text{Cay}}(G; a, b)$ has a two-way infinite hamiltonian path, then $I < \infty$.*
- (2) *Suppose P is a two-way infinite hamiltonian path in $\overrightarrow{\text{Cay}}(G; a, b)$. If k is the number of cosets of $\langle a - b \rangle$ that travel by a , and ℓ is the number of cosets of $\langle a - b \rangle$ that travel by b , then $k + \ell = I$, and $\langle ka + \ell b \rangle = \langle a - b \rangle$.*
- (3) *Conversely, suppose $k + \ell = I < \infty$ and $\langle ka + \ell b \rangle = \langle a - b \rangle$. If P is any spanning subdigraph of $\overrightarrow{\text{Cay}}(G; a, b)$, such that the outdegree of every vertex is 1, and exactly k cosets of $\langle a - b \rangle$ travel by a and exactly ℓ cosets of $\langle a - b \rangle$ travel by b , then P is a two-way infinite hamiltonian path in $\overrightarrow{\text{Cay}}(G; a, b)$.*

Proof (cf. [22, Theorem 4]). (1) Let $P = \dots, v_{-2}, v_{-1}, v_0, v_1, v_2, \dots$ be a two-way infinite hamiltonian path in $\overrightarrow{\text{Cay}}(G; a, b)$. Assume, without loss of generality, that $v_0 = 0$. Then, since $a \equiv b \pmod{\langle a - b \rangle}$, we have $v_i \in ia + \langle a - b \rangle$ for every $i \in \mathbb{Z}$. If $I = \infty$, this implies that each v_i is in a different coset of $\langle a - b \rangle$. Since $\{v_i\}_{i \in \mathbb{Z}}$ is a list of all of the elements of G , we conclude that v_i is the only element of its coset, so each coset has only one element. This means $|\langle a - b \rangle| = 1$, which contradicts the fact that $a \neq b$.

(2) Since every coset of $\langle a-b \rangle$ travels by either a or b (but not both), we have $k + \ell = I$. Now, write $P = \dots, v_{-2}, v_{-1}, v_0, v_1, v_2, \dots$, and assume, without loss of generality, that $v_0 = 0$. Then

$$v_i \in \langle a-b \rangle \iff i \text{ is divisible by } I$$

and

$$v_{jI} = j(ka + \ell b) \text{ for every } j \in \mathbb{Z}.$$

Since $\{v_i\}_{i \in \mathbb{Z}}$ is a list of all of the elements of G , this implies that $\{j(ka + \ell b)\}$ is a list of all of the elements of $\langle a-b \rangle$, which means that $ka + \ell b$ generates $\langle a-b \rangle$.

(3) Let $\dots, v_{-2}, v_{-1}, v_0, v_1, v_2, \dots$ be the path component of P that contains 0 (with $v_0 = 0$). We wish to show that every vertex is in this component.

Suppose $v \in G$. Since

$$\langle a, ka + \ell b \rangle = \langle a, a-b \rangle = \langle a, b \rangle,$$

we have $v \in ia + \langle ka + \ell b \rangle$ for some $i \in \mathbb{Z}$. Since $v_i \equiv ia \pmod{a-b}$, this implies there exists $j \in \mathbb{Z}$, such that

$$v = v_i + j(ka + \ell b) = v_{i+jI}. \quad \square$$

The following observation could easily be proved directly, but we present it as a simple application of Proposition 7.2.

Corollary 7.3. *If $\{e_1, e_2\}$ is the standard generating set of $\mathbb{Z} \times \mathbb{Z}_m$ (and $m \geq 2$), then the Cayley digraph $\overrightarrow{\text{Cay}}(\mathbb{Z} \times \mathbb{Z}_m; e_1, e_2)$ has a unique two-way infinite hamiltonian path, up to translations.*

Proof. (existence) Let $k = 1$ and $\ell = m - 1$. Then

$$k + \ell = m = |\mathbb{Z} \times \mathbb{Z}_m : \langle e_1 - e_2 \rangle|$$

and

$$ke_1 + \ell e_2 = e_1 + (m-1)e_2 = e_1 - e_2,$$

so Proposition 7.2(3) tells us that $\overrightarrow{\text{Cay}}(\mathbb{Z} \times \mathbb{Z}_m; e_1, e_2)$ has a two-way infinite hamiltonian path.

(uniqueness) Let k and ℓ be as in Proposition 7.2(2). Then

$$\langle \langle k, \ell \rangle \rangle = \langle ke_1 + \ell e_2 \rangle = \langle e_1 - e_2 \rangle = \langle \langle 1, -1 \rangle \rangle,$$

so the projection of this subgroup to \mathbb{Z} is surjective. We conclude that $k = 1$. This means that precisely one coset of $\langle a-b \rangle$ travels by e_1 (and all other cosets travel by e_2). Therefore, a two-way infinite hamiltonian path is determined by choosing which coset of $\langle e_1 - e_2 \rangle$ travels by e_1 . Since cosets are translates of each other, this implies that all two-way infinite hamiltonian paths are translates of each other. \square

Corollary 7.4. *Let $\{a, b\}$ be a 2-element generating set of an abelian group G (such that $a \neq b$), and let $I = |G : \langle a-b \rangle|$. The digraph $\overrightarrow{\text{Cay}}(G; a, b)$ has two arc-disjoint two-way infinite hamiltonian paths if and only if $I < \infty$ and there exist $k, \ell \in \{0, 1, \dots, I\}$, such that $k + \ell = I$ and*

$$\langle a-b \rangle = \langle ka + \ell b \rangle = \langle \ell a + kb \rangle.$$

Proof. (\Rightarrow) Let P be a two-way infinite hamiltonian path, and let k and ℓ be as in Proposition 7.2(2), so $k + \ell = I$ and $\langle ka + \ell b \rangle = \langle a - b \rangle$. If P' is a two-way infinite hamiltonian path that is arc-disjoint from P , then the cosets that travel by a in P' are the cosets that travel by b in P , so the number of cosets that travel by a in P' is ℓ , and the number of cosets that travel by b is k . So Proposition 7.2(2) tells us that $\langle \ell a + kb \rangle = \langle a - b \rangle$.

(\Leftarrow) Choose k cosets of $\langle a - b \rangle$. Let P be the spanning subdigraph in which the outdegree of every vertex is 1, and these particular k cosets of $\langle a - b \rangle$ travel by a , and the other ℓ cosets travel by b . Then let P' be the spanning subdigraph in which the outdegree of every vertex is 1, and the k chosen cosets of $\langle a - b \rangle$ travel by b , and the other ℓ cosets travel by a . It is clear from the construction that P' is arc-disjoint from P (since a vertex travels by a in P' if and only if it travels by b in P). Furthermore, we see from Proposition 7.2(3) that P and P' are two-way infinite hamiltonian paths. \square

This can be made much more explicit:

Proposition 7.5. *Assume G is an infinite abelian group.*

- (1) *There exist $a, b \in G$, such that $\text{Ca}\vec{\text{y}}(G; a, b)$ has a two-way infinite hamiltonian path if and only if G is isomorphic to either \mathbb{Z} or $\mathbb{Z} \times \mathbb{Z}_m$, for some $m \geq 2$.*
- (2) *For $a, b \in \mathbb{Z}$, the Cayley digraph $\text{Ca}\vec{\text{y}}(\mathbb{Z}; a, b)$ has two arc-disjoint two-way infinite hamiltonian paths if and only if a and b are odd, and*

$$\text{either } \{a, b\} = \{1, -1\} \text{ or } a + b = \pm 2.$$

- (3) *For $a, b \in \mathbb{Z} \times \mathbb{Z}_m$, with $m \geq 2$, the Cayley digraph $\text{Ca}\vec{\text{y}}(\mathbb{Z} \times \mathbb{Z}_m; a, b)$ has two arc-disjoint two-way infinite hamiltonian paths if and only if either*
 - (a) $\{a, b\} = \{(1, x), (-1, y)\}$, for some $x, y \in \mathbb{Z}_m$, such that $\langle x + y \rangle = \mathbb{Z}_m$, or
 - (b) $m = 2$, $a = (0, 1)$, and $b \in \{\pm 1\} \times \mathbb{Z}_2$, perhaps after interchanging a and b .

Proof. (1) \Leftarrow It is obvious that $\text{Ca}\vec{\text{y}}(\mathbb{Z}; 1, -1)$ has a two-way infinite hamiltonian path. The remaining case is immediate from Corollary 7.3.

(2) \Leftarrow If $a, b \in \{\pm 1\}$, then $\text{Ca}\vec{\text{y}}(\mathbb{Z}; a)$ and $\text{Ca}\vec{\text{y}}(\mathbb{Z}; b)$ are two arc-disjoint two-way infinite hamiltonian paths in $\text{Ca}\vec{\text{y}}(\mathbb{Z}; a, b)$.

We may now assume $a + b = \pm 2$, and $a \neq b$. Since $a + b$ is even, we may write $a - b = 2\ell$, for some nonzero $\ell \in \mathbb{Z}$. Assume without loss of generality that $k > 0$ (by replacing a and b with their negatives if necessary). Letting $k = \ell$, we have

$$k + \ell = 2\ell = a - b = |\mathbb{Z} : \langle a - b \rangle|$$

and

$$\langle ka + \ell b \rangle = \langle \ell a + kb \rangle = \langle \ell(a + b) \rangle = \langle 2\ell \rangle = \langle a - b \rangle,$$

so Corollary 7.4 tells us that $\text{Ca}\vec{\text{y}}(G; a, b)$ has two arc-disjoint two-way infinite hamiltonian paths.

(3)(a) \Leftrightarrow Let $\bar{\cdot} : \mathbb{Z} \times \mathbb{Z}_m \rightarrow \mathbb{Z}$ be the natural projection, and assume, without loss of generality, that $a = (1, x)$ and $b = (-1, y)$. Then $\overline{a - b} = 2$, so $|G : \langle a - b \rangle| = 2m$. Therefore, if we let $k = m + 1$ and $\ell = m - 1$, then

$$k + \ell = 2m = |G : \langle a - b \rangle|$$

and

$$\overline{ka + \ell b} = k\bar{a} + \ell\bar{b} = (m + 1) \cdot 1 + (m - 1) \cdot (-1) = 2,$$

so

$$|G : \langle ka + \ell b \rangle| = 2m = |G : \langle a - b \rangle|,$$

so

$$\langle ka + \ell b \rangle = \langle a - b \rangle.$$

A similar calculation shows $\langle \ell a + kb \rangle = \langle a - b \rangle$. So Corollary 7.4 tells us that $\overrightarrow{\text{Ca}\bar{\text{y}}}(G; a, b)$ has two arc-disjoint two-way infinite hamiltonian paths.

(3)(b) \Leftrightarrow Since $a - b$ is of the form $(\pm 1, *)$, we have $I = 2$. Let $k = \ell = 1$. Then $k + \ell = I$, and, since $a = -a$, we have

$$\langle a - b \rangle = \langle a + b \rangle = \langle ka + \ell b \rangle = \langle \ell a + kb \rangle.$$

So Corollary 7.4 tells us that $\overrightarrow{\text{Ca}\bar{\text{y}}}(G; a, b)$ has two arc-disjoint two-way infinite hamiltonian paths.

(1) \Rightarrow The structure theorem for finitely generated abelian groups [23, 4.2.10] tells us that

$$G \cong \mathbb{Z}^r \times \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_s},$$

where n_{i+1} is a multiple of n_i for $1 \leq i < s$ (and $n_i \geq 2$ for all i).

Since G is infinite, we have $r \geq 1$. On the other hand, we know from Proposition 7.2(2) that $|G : \langle a - b \rangle| < \infty$, so G has a cyclic subgroup of finite index, so $r \leq 1$. We conclude that $r = 1$.

The minimum cardinality of a generating set of G is $r + s$. Since $G = \langle a, b \rangle$, this implies $r + s \leq 2$. Since $r = 1$, we conclude that $s \in \{0, 1\}$. If $s = 0$, then $G \cong \mathbb{Z}$. If $s = 1$, then we may let $m = n_1$.

(2) \Rightarrow , **(3)** \Rightarrow Let $G = \mathbb{Z} \times \mathbb{Z}_m$ for (where $m = 1$ if the group is \mathbb{Z}), let $\bar{\cdot} : \mathbb{Z} \times \mathbb{Z}_m \rightarrow \mathbb{Z}$ be the natural projection, and let \bar{I} be the absolute value of $\bar{a} - \bar{b}$, so $\langle \bar{a} - \bar{b} \rangle = \langle \bar{I} \rangle$.

If $\bar{I} = 0$, then $|G : \langle a - b \rangle| = \infty$, which contradicts the conclusion of Proposition 7.2(1), so we must have $a = b$. Then $G = \langle a, b \rangle = \langle a \rangle \cong \mathbb{Z}$ and $a = b = \pm 1$, so $a + b = \pm 2$. This implies that a and b have the same parity. They cannot both be even (since $\langle a, b \rangle = \mathbb{Z}$), so a and b are odd. Thus, the situation is described in part (2) of the statement of the proposition.

We may now assume $\bar{I} > 0$. Then we claim that $|G : \langle a - b \rangle| = m\bar{I}$. We have

$$|G : \langle a - b \rangle| = \left| \frac{G}{\langle a - b \rangle} \right| = \left| \frac{G}{\langle a - b \rangle + \mathbb{Z}_m} \right| \cdot \left| \frac{\langle a - b \rangle + \mathbb{Z}_m}{\langle a - b \rangle} \right|.$$

Also note that

$$\frac{G}{\langle a - b \rangle + \mathbb{Z}_m} \cong \frac{G/\mathbb{Z}_m}{(\langle a - b \rangle + \mathbb{Z}_m)/\mathbb{Z}_m} \cong \frac{\mathbb{Z}}{\langle \bar{a} - \bar{b} \rangle} = \frac{\mathbb{Z}}{\langle \bar{I} \rangle}$$

has order \bar{I} , and

$$\frac{\langle a-b \rangle + \mathbb{Z}_m}{\langle a-b \rangle} \cong \frac{\mathbb{Z}_m}{\langle a-b \rangle \cap \mathbb{Z}_m} = \frac{\mathbb{Z}_m}{\{0\}} \cong \mathbb{Z}_m$$

has order m . Therefore the claim is proved. By Corollary 7.4, there exist $k, \ell \geq 0$, such that $k + \ell = m\bar{I}$, and

$$\langle ka + \ell b \rangle = \langle \ell a + kb \rangle = \langle a - b \rangle.$$

Therefore

$$\langle k\bar{a} + \ell\bar{b} \rangle = \langle \ell\bar{a} + k\bar{b} \rangle = \langle \bar{a} - \bar{b} \rangle = \langle \bar{I} \rangle,$$

so we may assume

$$k\bar{a} + \ell\bar{b} = \bar{I} \text{ and } \ell\bar{a} + k\bar{b} = \pm\bar{I}.$$

Adding these two equations, we conclude that

$$(k + \ell)(\bar{a} + \bar{b}) \text{ is either } 0 \text{ or } 2\bar{I}.$$

Since $k + \ell = m\bar{I}$, this implies

$$m(\bar{a} + \bar{b}) \in \{0, 2\}.$$

Case I. Assume $m(\bar{a} + \bar{b}) = 0$. This means $\bar{a} = -\bar{b}$. Since $\gcd(\bar{a}, \bar{b}) = 1$, this implies $\bar{a} = \pm 1$ and $\bar{b} = \pm 1$ (so $\bar{I} = 2$). If $G = \mathbb{Z}$, then the situation is described in part (2) of the statement of the proposition.

Therefore, we may assume $G \not\cong \mathbb{Z}$, so $m > 1$. Write $a = (1, x)$ and $b = (-1, y)$ (perhaps after interchanging a and b). Since

$$G = \langle a, b \rangle = \langle a, a + b \rangle = \langle (1, x), (0, x + y) \rangle,$$

it is clear that $\langle x + y \rangle = \mathbb{Z}_m$. So we are in the situation specified by part (3)(a) of the statement of the proposition.

Case II. Let $m(\bar{a} + \bar{b}) = 2$. This immediately implies that $m \in \{1, 2\}$.

If $m = 1$ (and $\bar{a} + \bar{b} = 2$), then $G = \mathbb{Z}$, and we have $a + b = \bar{a} + \bar{b} = 2$. (This implies that a and b have the same parity. They cannot both be even, since $\langle a, b \rangle = G$, so a and b must be odd.) So we are in a situation that is described in part (2) of the statement of the proposition.

Assume now that $m = 2$ (and $\bar{I} = 1$). Then $\bar{a} + \bar{b} = 1$. Note that $\langle 2\bar{a} \rangle \neq \mathbb{Z}$ and $\langle 2\bar{b} \rangle \neq \mathbb{Z}$. Since $\langle k\bar{a} + \ell\bar{b} \rangle = \mathbb{Z}$ and $k + \ell = m\bar{I} = 2$, we conclude that $k = \ell = 1$. This implies

$$\bar{a} + \bar{b} = k\bar{a} + \ell\bar{b} = \bar{I} = 1.$$

Since we also have $\bar{a} - \bar{b} = \pm\bar{I} = \pm 1$, we conclude that $\{\bar{a}, \bar{b}\} = \{0, 1\}$. So we are in the situation that is described in part (3)(b) of the statement of the proposition. \square

Corollary 7.6. $\text{Ca}\vec{\text{y}}(\mathbb{Z} \times \mathbb{Z}_m; e_1, e_2)$ has two arc-disjoint two-way infinite hamiltonian paths if and only if $m = 2$.

Proof. (\Leftarrow) Let $k = \ell = 1$. Then

$$k + \ell = 2 = |\mathbb{Z} \times \mathbb{Z}_2 : \langle e_1 - e_2 \rangle|$$

and

$$\langle ke_1 + \ell e_2 \rangle = \langle \ell e_1 + ke_2 \rangle = \langle e_1 + e_2 \rangle = \langle e_1 - e_2 \rangle,$$

so Corollary 7.4 provides two arc-disjoint two-way infinite hamiltonian paths.

(\Rightarrow) Since $e_2 = (0, 1)$ is obviously not of the form $(\pm 1, *)$, it is clear that the generating set $\{e_1, e_2\}$ is not of the form specified in Proposition 7.5(3)(a), so it must be part (3)(b) of the proposition that applies. So $m = 2$. \square

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