ELECTROMAGNETIC AND GRAVITATIONAL WAVE INTERACTIONS

AKASH PATEL
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AKASH PATEL

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Dr. A. Dasgupta Thesis Supervisor	Associate Professor	Ph.D.
Dr. V. Faraoni Thesis Examination Committee Member Bishop's University, Sherbrooke, QC	Professor	Ph.D.
Dr. B. Seyed Mahmoud Thesis Examination Committee Member	Associate Professor	Ph.D.
Dr. K. Vos Chair, Thesis Examination Committee	Associate Professor	Ph.D.

Dedication

I dedicate this work to my family for their support and motivation.

Abstract

With the recent detection of gravitational waves, a new window has opened for studying the universe. Because gravitational waves interact weakly with matter, they can pass through matter without being affected significantly. Due to this, they are very important in the study of the early universe. In this thesis, the interaction of gravitational waves and electromagnetic waves is studied in the Minkowski and de-Sitter spacetime. The explicit form of the perturbations (describing electromagnetic waves) is solved in the presence of a gravitational wave in the Minkowski background. We find a new frequency mode of the perturbed electromagnetic wave and analyze for resonance. The nature of the wave interaction is dependent on the relative direction of propagation of both the waves. For the de-Sitter spacetime background, the inhomogeneous wave equations for the perturbed electromagnetic wave are solved and we find a similar new mode which modulates the electromagnetic wave.

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Chapter 1

Introduction

We wish to understand the origin of the Universe and its evolution using data observed today. According to the Big Bang theory [1], the universe began with the Big Bang in a hot, dense, nearly uniform state approximately 13.8 billion years ago. There was a period of rapid expansion known as cosmic inflation during which the universe expanded and its volume increased by (approximately) a factor of 10^{78} (compared to its initial size). Density waves and gravitational waves were generated due to quantum fluctuations in the matter distribution, which have been magnified by inflation. After inflation (10^{-32}sec), the universe was a hot soup of quarks, gluons, electrons and other elementary particles. As the universe cooled down due to the expansion, quarks clumped into protons and neutrons (10^{-5}s $\sim$ 1s). These protons and neutrons merged together and formed nuclei during the process of nuclear fusion (roughly between 10 sec to 20 min [2]). As the universe cooled down to 10,000°C, the electrons, protons and nuclei combined together to form atoms (mostly hydrogen and helium).

Before the formation of neutral hydrogen (380,000 years after big bang), the universe was opaque due to constant interaction between light and matter (free electrons and protons). Photons started streaming freely after this time and formed the Cosmic Microwave Background (CMB) visible today. E-mode and B-mode polarization in CMB [3] are associated with density waves and gravitational waves respectively. The state of the early universe can be deduced from CMB and CNB (Cosmic Neutrino Background). Since the CMB was produced approximately 380,000 years after the Big Bang, it is very difficult to deduce the

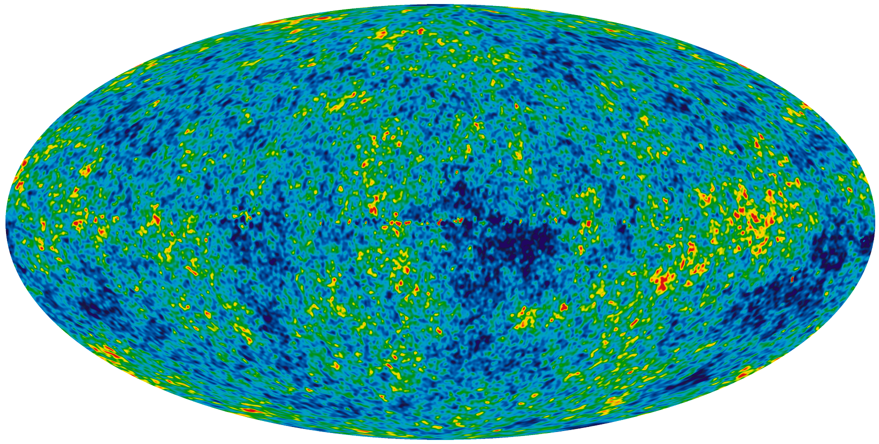

Figure 1.1: The Cosmic Microwave Background radiation
[3]

state of the very early stages of the universe using it. Neutrino decoupling on the other hand took place when the temperature of the universe was approximately 1MeV (approximately 1 second after the Big Bang). So the CNB can indeed help us determine the state of the early universe but neutrinos from CNB have a very low energy (10^{-10} times smaller than current direct detection) due to which it is very difficult to detect CNB and get relevant information about the early universe using CNB.

The recent detection of gravitational waves in February 2016 by LIGO (Laser Interferometer Gravitational-wave Observatory) [4] and Virgo has opened new doors through which relevant observations about the early universe might be made. Gravitational waves can have important information about the very early state of the universe [5] *e.g.* typical frequency, intensity, temperature, *etc*. If the GWB (Gravitational Wave Background or stochastic background) [6, 7] is detected, then it would have a huge impact on early universe cosmology and high energy physics. We intend to study the interaction of gravitational waves and electromagnetic radiation which will lead to the detection of primordial gravity waves.

With our aim being the study primordial gravitational waves, in this work we begin with the study of their interactions with electromagnetic waves in the background of Minkowski spacetime. The behavior of plane-polarized monochromatic electromagnetic wave is studied when it interacts with a weak linearized gravitational wave. The resultant perturbations in the electromagnetic waves are studied at the first order. Different aspects like resonance and the relative direction of propagation of the waves are considered.

In the second part of the work we investigate a similar interaction of the primordial gravitational waves in the background of flat Friedmann–Lemaître-Robertson–Walker (FLRW) spacetime [1, 8]. The FLRW spacetime describes a homogeneous, isotropic, expanding universe. The FLRW metric is the exact solution of the Einstein field equations satisfying above properties.

Electromagnetic waves are fluctuations in the electric and magnetic field which propagate at the speed of light. These waves occur at various wavelengths, producing a spectrum of radiation from radio waves to γ -rays. The Maxwell equations in the curved spacetime [1, 9] describe the behavior of electric and magnetic fields at different points in spacetime, depending on the distribution and motion of charges. The Gauss-Ampere law, in the absence of electric charge, forms the mathematical basis for the electromagnetic waves.

Gravitational waves are disturbances in the curvature of spacetime that propagate outward

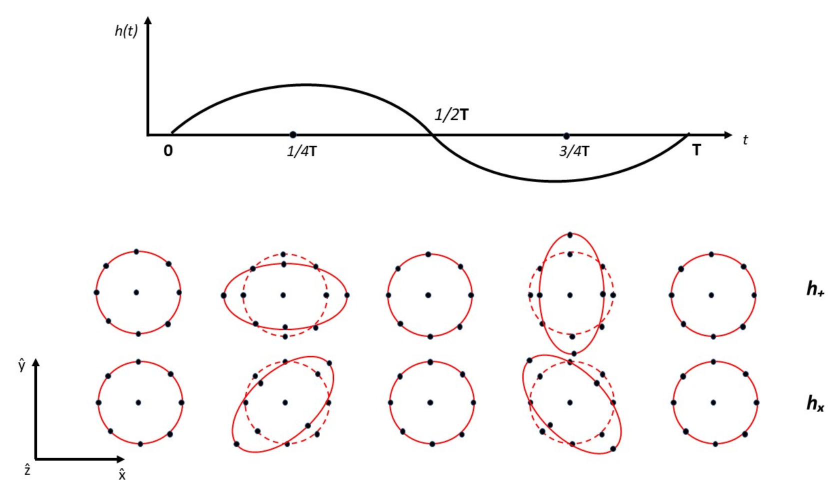

Figure 1.2: Gravitational wave polarization

at the speed of light from the source. Accelerating massive (*e.g.* blackholes, binary neutron stars, supernova) bodies are sources of gravitational waves. These waves were predicted by

Albert Einstein in 1916 on the basis of General Theory of Relativity. They are a form of radiant energy (also known as gravitational radiation), which transports energy away from the accelerated masses, similar to the electromagnetic radiation produced by accelerated charges. Gravitational waves were detected in 2016, a hundred years after its prediction, by the Laser Interferometer Gravitational-Wave Observatory (LIGO) and Virgo [4]. Unlike electromagnetic waves, which are plane polarized, gravitational waves are quadrupole waves and the relevant polarizations are given by two functions. For a gravitational wave propagating in the positive Z-direction, the h_+ polarization stretches and squeezes the XY-plane horizontally and vertically; while the $h_\times$ polarization stretches and squeezes the XY-plane diagonally over time, as can be seen in figure 1.2. For the considered gravitational wave, these polarizations are explicitly given as

$$
\begin{aligned}
h_+ &= A_+ \cos(\omega_g(z-t)) \\
h_\times &= A_\times \cos(\omega_g(z-t)+\delta)
\end{aligned}
$$

where ω_g is the frequency of the gravitational wave and the speed of light $c = 1$ as we work with geometrized units. A_+ and $A_\times$ are the amplitudes of the two polarizations and δ is the phase difference.

The interaction between electromagnetic field and gravitational wave generates perturbations and causes oscillatory patterns in the electromagnetic wave. These new electromagnetic waves frequency modes can be detectable by LIGO and Virgo. The investigation in this direction can help us with the discovery of primordial gravitational waves [10]. The frequency signatures of primordial gravitational waves can be searched in the CMB radiation observed today. This will have a profound impact in our understanding of early universe cosmology and high energy physics as it can be used as a tool to probe the very early state of the universe. These detected primordial gravitational waves may contain information about the density, temperature and intensity of the very early state of the universe [5]. This information can help us develop better theories and models of the universe and deepen our understanding of the young cosmos.

Notable work has been done in a similar direction. One such work studies the resonant conversion of a gravitational wave into an electromagnetic wave and (*vice-versa*) when a static electromagnetic field is present [11]. The considered interaction of gravitational waves and electromagnetic waves are studied too; these include frequency splitting [12], intensity fluctuations [12, 13], deflection of rays [13, 14, 15] and gravitationally induced rotation of the electromagnetic waves polarisation [15, 16, 17, 18, 19]. One of the works takes a similar approach as we have adopted and studies the resonance amplification of electromagnetic waves in the presence of gravitational waves [20].

We have found a nonzero interaction between the electromagnetic waves and the gravitational waves in the background of flat spacetime. When a monochrometic plane-polarized electromagnetic wave (with angular frequency ω_e) and a linearized gravitational wave (with angular frequency ω_g) interact, the resultant frequency mode is found to be $\sqrt{\omega_g^2+\omega_e^2}$. This was expected as the same mode was found in the case of scalar field interaction with gravitational waves [21]. We find null interaction when both waves are propagating in the same direction and a nontrivial interaction when the propagation is parallel but in the opposite direction. This is different from the results of [20]. We further note the absence of any resonance phenomena when frequencies for the waves are the same (*i.e.* $\omega_g = \omega_e$).

The same kind of interaction can also be determined in the background other than Minkowski spacetime. In the later part of the work we investigate the interaction of electromagnetic waves and gravitational waves in the spatially flat FLRW spacetime. The resultant perturbation in the electromagnetic potential components can reveal information about the early state of the universe. In this thesis we derive the explicit form of the perturbations and plot them. The study of the effects of these perturbations on the CMB is a work in progress.

In the chapters to follow, we will give a brief review of general relativity, discuss gravitational waves and the interaction of gravitational waves with electromagnetic waves in the Minkowski and FLRW backgrounds. In chapter two, the important mathematical tools (tensor analysis) which are useful in understanding and working in general relativity are introduced. Then we discuss the nature and description of electromagnetic waves in curved spacetime. In chapter three, a detailed derivation of linearized gravitational waves in the background of Minkowski spacetime is done. The used gauge conditions are discussed. The 'plus' (h_+) and the 'cross'($h_\times$) polarizations of the gravitational waves, and their effects are derived as well. In chapter four,detailed calculations for the interactions of the electromagnetic waves and gravitational waves in the Minkowski spacetime are presented. Cases like $h_\times = 0$ or $h_+ = 0$ and both being nonzero are considered. The resultant first order perturbations in the electromagnetic four potential components and their graphical representation are also shown. In chapter five, we derive the form of electromagnetic waves in the cosmological background. The detailed calculation of the interaction between electromagnetic waves and gravitational waves in the flat-FLRW spacetime is also done in chapter five. The form of the first order perturbations in the electromagnetic vector potential components are found and their graphical representation is also shown. In chapter-6, the results and prospective continuation of the work are discussed briefly.

Chapter 2

General Theory of Relativity

To understand gravitational waves, we first need to understand the general theory of relativity. According to the general theory of relativity, gravity is a geometric property of the four-dimensional spacetime. The gravitational force is the result of spacetime curvature. The Einstein field equations describe how the momentum and energy of matter (and radiation) are related to the curvature of spacetime. In this chapter, we will discuss and derive the mathematical tools that are essential to understand the general theory of relativity. Quantities like four-velocity, four-momentum, number-flux, stress-energy tensor are defined and discussed in the sections to follow. Once all the necessary quantities are defined and explained, we discuss the Einstein field equations as they will be used in chapter 3 for the derivation of linearized gravitational waves.

2.1 Special Theory of Relativity

Since the general theory of relativity is the generalization of the special theory of relativity [22, 23], in this section we will briefly review the special theory of relativity. It is a theory which describes the dynamic relation between space and time, depending on the state of motion of inertial observers.

In classical mechanics space and time are independent of each other, regardless of the motion of observers. The Galilean transformation is used to transform from one inertial reference frame to another. It describes the relationship between the space and time coordinates of two inertial observers (K with coordinates (t,x,y,z) and K' with the coordinates

(t', x', y', z')) travelling with relative velocity v in the X-direction.
The Galilean transformation is given by:

$$\begin{aligned} t' &= t, \\ x' &= x - vt, \\ y' &= y, \\ z' &= z. \end{aligned}$$

For a given phenomenon, the rules of Galilean transformation of classical mechanics do not hold under the implementation of the postulates of the special theory of relativity.
The two postulates of the special theory of relativity are:

1. The laws of physics are invariant (*i.e.* take the same form) in all inertial reference frames.

2. The speed of light is constant (in a vacuum) for all inertial observers.

The transformations derived by implementing these postulates are known as Lorentz transformations. The Lorentz transformations show that inertial observers with relative velocity will not agree about when and where an event occurred. Meaning, contrary to Galilean mechanics, events that occur at some time t for one observer (K) do not necessarily occur at the same time t' for an observer (K') that is moving relative to the observer K. Hence, the notion of 'absolute' space and time as suggested by Galilean transformations is wrong and must be replaced by an observer-dependent dynamic space and time.

For a given reference frame, an event is a happening specified by a definite time and a definite location with respect to that reference frame. Therefore, an event can be thought of as a point, and collection of such events (points) can be thought of as the continuum of spacetime.

Consider an event p in an inertial reference frame K that is given by the coordinates (t, x, y, z). Now consider another inertial reference frame K' which is moving with constant velocity v relative to K in the positive X-direction and has the same coordinate orientation as K. The origins of K and K' coincide at time zero of both reference frames. The coordinates of reference frame K' are denoted by (t', x', y', z'). Being constrained by above postulates, the coordinate values of event p for the frame K' are obtained to be:

$$\begin{aligned} t' &= \gamma\left(t - \frac{vx}{c^2}\right), \\ x' &= \gamma(x - vt), \\ y' &= y, \\ z' &= z, \end{aligned} \tag{2.1}$$

where γ is called the 'Lorentz factor' and is given by $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$. Here c is the speed of light.

The above transformation is known as the 'Lorentz transformation' [9]. These transformations reduce to Galilean transformations when the relative velocity $v << c$, as then $\gamma \approx 1$ and $t >> \frac{vx}{c^2}$.

These transformations predict that the length of a moving object is measured (by other observers) to be shorter (than its rest-frame length) in the direction of motion, a phenomenon known as 'length contraction'. And the elapsed time for moving clocks is longer than the elapsed time for stationary clocks, a phenomenon known as 'time dilation'. In other words, moving clocks run slower with respect to stationary clocks. The Lorentz transformations also indicate that, for moving objects, the addition of velocities is not as simple as suggested by Galilean transformations.

The shortest distance between two points in the Euclidean geometry is a straight line. The 'length' of this shortest distance line remains the same irrespective of the arbitrary choice of coordinates. Hence, we can say that the distance between two points is invariant in Euclidean geometry. Similarly, in spacetime, the shortest invariant distance between two points (events) is given by

$$(\Delta s)^2 = -(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

where s is called the spacetime interval.

The infinitesimal spacetime interval (ds) is then written as

$$(ds)^2 = -(dt)^2 + (dx)^2 + (dy)^2 + (dz)^2.$$

If we take $(x^0, x^1, x^2, x^3) = (t, x, y, z)$, the above can be written as

$$ds^2 = \sum_{\alpha=0}^{3} \sum_{\beta=0}^{3} \eta_{\alpha\beta} dx^\alpha dx^\beta$$
$$\therefore ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta,$$

where in the second step we have used the Einstein summation convention which implies summation over indices that are repeated as superscript and subscript. The $\eta_{\alpha\beta}$ is known as the metric tensor of Minkowski spacetime. The components of the metric tensor (in Cartesian coordinates) are expressed as

$$\eta_{\alpha\beta} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.2}$$

$\eta_{\alpha\beta}$ is known as Minkowski space metric.

Also note that the Lorenz transformations can be written in a compact form as

$$x'^\alpha = \Lambda^\alpha_\beta x^\beta$$

where Λ^{α}_{β} is Lorentz transformation matrix and satisfies $\eta_{\alpha\beta} = \Lambda^{\nu}_{\beta}\Lambda^{\mu}_{\alpha}\eta_{\mu\nu}$. The component form is given by

$$\Lambda^{\alpha}_{\beta} = \begin{bmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where γ is the Lorentz factor. This coordinate transformation can be simply written as (using the chain rule)

$$x'^{\alpha} = \frac{\partial x'^{\alpha}}{\partial x^{\beta}} x^{\beta}. \tag{2.3}$$

Another important invariant quantity in spacetime, which all observers agree upon, is called proper time. Curves in spacetime are often parameterized using the proper time. Proper time along a (time-like) world line is defined as the time that is measured by a clock following that trajectory. Hence, it is independent of coordinates. The proper time between two events in spacetime depends not only on the points (events) but also the curve (world line) connecting them, therefore, on the motion of the clock between the two events. It is analogous to arc length in three-dimensional Euclidean space. It is denoted by τ. Consider an infinitesimal spacetime interval ds on a timelike world-line (trajectory of particle) for an arbitrary Lorentz frame K and an instantaneous rest-frame K' for the same interval.

$$\therefore -c^2 d\tau^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2,$$
$$\therefore d\tau = dt\sqrt{1 - \frac{v^2}{c^2}},$$
$$\therefore d\tau = \frac{dt}{\gamma}.$$

2.2 Tensor analysis

Manifold and Riemannian geometry

A manifold is any surface (space) which can be divided into open sub-spaces, each of which can be mapped to Euclidean space *e.g.* the m dimensional surface of an n dimensional sphere $(m < n)$. It is any set that can be parameterized continuously. The number of independent parameters is the dimension of the manifold, and the parameters themselves are the coordinates of the manifold [1, 9]. A manifold locally looks like Euclidean space. It is smooth and has a certain number of dimensions, but the large scale topology of a manifold maybe very different from Euclidean space *e.g.* a small part of the surface of a torus can be mapped one to one into the plane tangent to it. Hence, a manifold is a space with coordinates that locally look Euclidean but has no distance relation or shape specified. It is a ‘differential manifold’ on which a symmetric tensor field **g** has been singled out to act as the metric. The metric adds structure to the manifold. Thus, choosing different metrics would give different curvature to the manifold. The sum of the diagonal elements of metric is called ‘signature’ of the metric. The metric signature for the special and general theories of relativity is +2. Such a manifold is called pseudo Riemannian manifold. We know thta global Lorentz frames can not exists in a general gravitational field, so we choose any point P on the manifold for which $g_{\mu\nu}(P) = \eta_{\mu\nu}$ for all α, β. It is possible to find coordinates such that, in a neighbourhood of P, the above equations are nearly true:

$$
\begin{aligned}
g_{\mu\nu}(P) &= \eta_{\mu\nu}, \\
\partial_\alpha g_{\mu\nu}(P) &= 0,
\end{aligned}
$$

where ∂_α is used to denote partial derivative with respect to α-th coordinate. Any curved space has a flat space tangent to it at any point and this is the reason why local Lorentz frames exists.

Vectors

Using the Einstein summation convention, the vectors can be represented as

$$\vec{V} = V^\mu \vec{e}_\mu$$

where V^{μ} are vector components and $\vec{e}_{\mu}$ are the basis vectors for a given coordinate system. The Greek indices can take values from 0 to 3. Vectors are usually seen as entities connecting two positions in space. However, we can not have a vector connecting two points on a curved surface as the rules of vector manipulation require space to be flat. To have the notion of vectors on a curved surface, at every point in the space a 'tangent space' is defined as the set of tangent vectors to all curves passing through that point. Vectors on every point in the space lie on these tangent spaces. For a flat space, all the tangent spaces are parallel, but in curved spaces that is not the case. For every vector space, there is a dual vector space, whose elements define a map from the vector space to the field of scalars. For a tangent space, the dual vector space is called a cotangent space and its elements are called one-forms. We can define physical quantites by taking tensor products of m tangent spaces and n cotangent spaces. These are known as $(m\ n)$ tensors. In particular, the vector is a $(1\ 0)$ tensor and the one-form is a $(0\ 1)$ tensor.

One-form

Oneform is a (0 1) tensor, which takes a vector as its argument and results in a scalar. It is denoted by $\tilde{p}$. It is linear in its arguments.

If we have one-form which are defined as $\tilde{a} = \tilde{b} + \tilde{c}$ and $\tilde{d} = \alpha\tilde{a}$ (where α is a scalar), then the linearity property can be seen in the action of the one-form on the vector field such that $\tilde{a}(\vec{V}) = \tilde{b}(\vec{V}) + \tilde{c}(\vec{V})$ and $\tilde{d}(\vec{V}) = \alpha\tilde{a}(\vec{V})$. With above properties, one-form satisfy all the axioms for a vector space. Hence, they are also called dual vector space to distinguish them from the space of all vectors.

The components of any (0 N) tensor are obtained by supplying n number of basis vectors. Hence, the components of one-form $\tilde{p}$ are obtained as

$$p_{\alpha} := \tilde{p}(\vec{e}_{\alpha}).$$

Hence, when a vector is the argument of $\tilde{p}$ we have

$$\tilde{p}(\vec{A}) = \tilde{p}(A^{\alpha}\vec{e}_{\alpha}),$$
$$\therefore \tilde{p}(\vec{A}) = A^{\alpha}\tilde{p}(\vec{e}_{\alpha}),$$
$$\therefore \tilde{p}(\vec{A}) = A^{\alpha}p_{\alpha}.$$

The basis one-forms are denoted as $\tilde{\omega}^{\alpha}$ and are defined as

$$\tilde{\omega}^{\alpha}(\vec{e}_{\beta}) = \delta^{\alpha}_{\beta}.$$

We can see that, $\tilde{\omega}^{\alpha}(\vec{e}_{\beta})$ gives β-th component of α-th basis oneform in a given coordinate system. With these the one-form $\tilde{p}$ is given by

$$\tilde{p} = p_{\alpha}\tilde{\omega}^{\alpha},$$
$$\therefore \tilde{p}(\vec{A}) = p_{\alpha}\tilde{\omega}^{\alpha}(A^{\beta}\vec{e}_{\beta}),$$
$$\therefore \tilde{p}(\vec{A}) = p_{\alpha}A^{\beta}\tilde{\omega}^{\alpha}(\vec{e}_{\beta}),$$
$$\therefore \tilde{p}(\vec{A}) = p_{\alpha}A^{\beta}\delta^{\alpha}_{\beta},$$
$$\therefore \tilde{p}(\vec{A}) = p_{\alpha}A^{\alpha}.$$

Metric tensor

A metric tensor is a symmetric (0 2) tensor, which takes two vectors as its arguments and produces a scalar. It defines the dot product between two vectors. It is linear in its arguments. Meaning,

$$\mathbf{g}(\alpha\vec{A} + \beta\vec{B}, \vec{c}) = \alpha\mathbf{g}(\vec{A}, \vec{C}) + \beta\mathbf{g}(\vec{B}, \vec{C}).$$

For a given space, the metric tensor establishes the distance and angle relation between points.

The components of metric tensor are obtained when the basis vectors (for an arbitrary coordinate system) are taken as arguments.

$$
\begin{aligned}
\mathbf{g}(\vec{e}_\mu, \vec{e}_\nu) &= \vec{e}_\mu \cdot \vec{e}_\nu, \\
\mathbf{g}(\vec{e}_\mu, \vec{e}_\nu) &= g_{\mu\nu}.
\end{aligned}
$$

For Minkowski spacetime (in Cartesian coordinates)

$$\mathbf{g}(\vec{e}_\mu, \vec{e}_\nu) = \eta_{\mu\nu},$$

because $\vec{e}_\mu \cdot \vec{e}_\nu = 0$ for $\mu \neq \nu$, $\vec{e}_0 \cdot \vec{e}_0 = -1$ and $\vec{e}_i . \vec{e}_j = \delta_{ij}$ for Cartesian coordinates. Hence, the dot product between two vectors using metric tensor is defined as

$$
\begin{aligned}
\mathbf{g}(\vec{A}, \vec{B}) &= \mathbf{g}(A^\mu \vec{e}_\mu, B^\nu \vec{e}_\nu), \\
\therefore \mathbf{g}(\vec{A}, \vec{B}) &= A^\mu B^\nu \mathbf{g}(\vec{e}_\mu, \vec{e}_\nu), \\
\therefore \mathbf{g}(\vec{A}, \vec{B}) &= g_{\mu\nu} A^\mu B^\nu.
\end{aligned}
$$

The metric tensor acts as a mapping between one-forms and vectors. Consider $\mathbf{g}(\vec{V},)$, where one vector argument is given and the second vector argument is yet to be supplied. When the second vector argument ($\vec{A}$) is supplied in $\mathbf{g}(\vec{V},)$, it will produce a real number. Hence, $\mathbf{g}(\vec{V},)$ can be seen as a symmetric linear function of vectors producing scalars: a oneform.

$$\mathbf{g}(\vec{V},) := \tilde{V}() := \mathbf{g}(, \vec{V}).$$

When the vector argument $\vec{A}$ is supplied to $\tilde{V}()$, it evaluates $\vec{A}$ to $\vec{V} \cdot \vec{A}$

$$
\begin{aligned}
\tilde{V}(\vec{A}) &:= \mathbf{g}(\vec{V}, \vec{A}), \\
\therefore \tilde{V}(\vec{A}) &= \vec{V} \cdot \vec{A}.
\end{aligned}
$$

The components V_α of the one-form $\tilde{V}$ can be found by supplying the basis vectors $\vec{e}_\alpha$ of the coordinate system.

$$V_\alpha := \tilde{V}(\vec{e}_\alpha) = \vec{V} \cdot \vec{e}_\alpha,$$
$$\therefore V_\alpha = V^\beta(\vec{e}_\beta \cdot \vec{e}_\alpha),$$
$$\therefore V_\alpha = g_{\alpha\beta} V^\beta.$$

As seen in the expression above, the components of the metric tensor $g_{\alpha\beta}$ can be used to lower the index. Similarly, it can be shown that the components of the inverse metric tensor $g^{\alpha\beta}$ can be used to raise the index. (*e.g.* $V^\alpha = g^{\alpha\beta} V_\beta$)

THE STRESS-ENERGY TENSOR

In general relativity, the energy and momentum of a collection of particles (or of a field or of a fluid) act as the source of gravitational field and curvature. It is important to discuss how these quantities are described in a frame independently in terms of tensors.

Four-velocity

The path traced by an object through spacetime is called its world-line. The world-line of an object in spacetime is given by four functions $x^\mu(\tau)$ of proper time τ. The position of this object is given by four-position vector $\vec{x} = x^\mu \vec{e}_\mu$ for a given inertial frame.

Four-velocity $\vec{U}$ can be defined as a four-vector of unit magnitude which is tangent to the world-line of the object. At any point of the world-line, $\vec{U}$ can be given as $\vec{U} = \frac{d\vec{x}}{d\tau}$. Its components are given by

$$U^\mu = \frac{dx^\mu}{d\tau}, \tag{2.4}$$
$$U^0 = \frac{dt}{d\tau} = \gamma, \tag{2.5}$$
$$U^i = \frac{dx^i}{dt}\frac{dt}{d\tau} = \gamma \vec{u}, \tag{2.6}$$

where $\vec{u}$ is the 3-dimensional velocity vector and γ is the Lorentz factor. Hence, $\vec{U}$ can be written as $\vec{U} = \gamma(1, \vec{u})$. We can see that in the object's rest frame $\vec{U} = (1,0,0,0)$ as the spatial velocity is zero.
The magnitude of $\vec{U}$ is obtained by

$$\begin{aligned}
\vec{U} \cdot \vec{U} &= U^{\mu} U^{\nu} \vec{e}_{\mu} \vec{e}_{\nu}, \\
\therefore \vec{U} \cdot \vec{U} &= g_{\mu\nu} \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau}, \\
\therefore \vec{U} \cdot \vec{U} &= -1. \qquad (2.7)
\end{aligned}$$

The second last step follows from the fact that the infinitesimal spacetime interval $ds^2 = -d\tau^2$ in the object's rest frame. The above equality is true for any massive object. It is clear from the definition of four-velocity that it cannot be defined for a photon.

Four-momentum

Four-momentum is the generalization of classical 3-dimensional momentum to 4-dimensional spacetime. It is defined as $\vec{P} = m_0 \vec{U}$, where m_0 is the rest-mass (the mass measured in the rest frame of the particle) of the particle. In any inertial frame it can be written as

$$\begin{aligned}
\vec{P} &= m_0 \vec{U}, \\
\vec{P} &= m_0 \gamma (1, \vec{u}), \\
\vec{P} &= (m, m\vec{u}), \\
\vec{P} &= (E, \vec{p}), \qquad (2.8)
\end{aligned}$$

where $m = m_0 \gamma$ is the relativistic mass, $\vec{p}$ is the 3-D relativistic momentum and E is the energy of the particle.

Number-flux vector $\vec{N}$

A collection of particles, all of which are at rest with respect to some inertial frame is called 'dust'. The number density n is simply the number of particles per unit volume. So n can be written as

$$n = \frac{N}{\Delta x \Delta y \Delta z},$$

where N is the total number of particles contained in the volume $\Delta x \Delta y \Delta z$ from the rest frame. It is easy to see that if a dust has velocity v with respect to some inertial frame, then due to length contraction the number density is given by γn, where γ is the Lorentz factor. The flux across a surface is defined as the number of particles crossing a unit area of that surface in a unit time. It can be shown that the flux across a surface of constant x^μ (normal to x^μ) is given by $\gamma n U^\mu$. Hence, the number-flux vector $\vec{N}$ is defined as $\vec{N} = n\vec{U}$, where n is the number density and $\vec{U}$ is the four-velocity. In an inertial frame $\vec{N}$ can be written as

$$\begin{aligned} \vec{N} &= n\vec{U}, \\ \vec{N} &= n\gamma(1, \vec{u}), \\ \vec{N} &= (n\gamma, n\gamma\vec{u}). \end{aligned} \tag{2.9}$$

From the above equation we can see that the component N^0 is the number density, while the components N^i are the fluxes across the surfaces of constant x^i. Hence, N^0 can be interpreted as the flux across the surface of constant time *i.e.* the number of particles in the spatial volume. Note that $\vec{N} \cdot \vec{N} = -n^2$ as $\vec{U} \cdot \vec{U} = -1$.

Energy Density

Energy density is defined as total amount of energy per unit volume. In the rest frame of dust, the energy of each particle is given by m_0. Therefore the energy density ρ of all the particles is given by $\rho = n m_0$.

If the dust has some velocity with respect to some inertial frame, then the energy density will given by $\gamma^2\rho$ since number density and energy of each particle have a factor γ due to relative velocity.

Stress-energy tensor

The stress-energy tensor is a symmetric $(2\ \ 0)$ tensor which takes two one-forms as its arguments and produces a scalar. It is denoted by $\mathbf{T}$. The components $\mathbf{T}$ are obtained when the basis one-forms $\tilde{d}x^\mu$ and $\tilde{d}x^\nu$ are the arguments.

$$\mathbf{T}(\tilde{d}x^\mu, \tilde{d}x^\nu) = T^{\mu\nu}.$$

The component $T^{\mu\nu}$ represents the flux of the μ-th component of four-momentum P^μ across a surface of constant x^ν. For example, the component T^{00} represents the flux of energy (0-th component of momentum) across the surface of constant time, which is simply the energy density.

Let's briefly consider the meaning of each component.

- T^{00} is the energy density as discussed above.
- T^{0i} is the flux of energy across the surface of constant x^i, therefore it represents the energy flux.
- T^{i0} is the flux of momentum across the surface of constant time, therefore it represents the momentum density.
- T^{ij} is the flux of momentum across the surface of constant x^j, therefore it represents the momentum flux.

Let's consider a general fluid. A fluid is a collection of innumerable particles such that the dynamics of individual particles cannot be followed. Hence, it is called a special kind of continuum whose description can only be given in terms of bulk quantities like number density, energy density, temperature, pressure, *etc*. The bulk (collection of particles) which is large enough so that individual particles do not matter and yet small enough to be homogeneous, is called an element of the fluid.

Each element of the fluid is assigned a value of density, pressure, temperature, *etc*. which may vary for each element. Mathematically, this approximation is expressed by giving each point some value of pressure, density, *etc*. Hence, the fluid can be defined as collection of various field defined at each location and each time.

For a general fluid, T^{0i} can be seen as conduction of heat among the fluid elements, T^{i0} can be interpreted as momentum associated with the energy flux (heat conduction) and T^{ij} represents forces per unit area between adjacent fluid elements, known as stresses. Forces per unit area which are parallel to the interface are called sheares and the forces which are perpendicular to the interface are called pressures.

A perfect fluid is defined as a fluid which has no viscosity and no heat conduction in its rest frame. From this definition we can conclude the following things:

- In the rest frame of the perfect fluid $T^{i0} = T^{0i} = 0$ due to the lack of heat conduction.
- Shear forces due to viscosity is parallel to the interface of the elements. Since a perfect fluid has no viscosity, $T^{ij} = 0$ when $i \neq j$. Hence, $T^{ij} = p\delta^{ij}$, where p is the pressure and δ^{ij} is the identity matrix in 3-dimensions.

From above discussion we can conclude that in the rest frame of perfect fluid, the components of **T** are

$$T^{\mu\nu} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix} \tag{2.10}$$

$$\therefore T^{\mu\nu} = \begin{bmatrix} \rho+p & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix}$$

It is easy to see that above can be expressed as

$$T^{\mu\nu} = (\rho+p)U^{\mu}U^{\nu} + p\eta^{\mu\nu} \tag{2.11}$$

The above equation shows the components of the stress-energy tensor for a perfect fluid. It is frame invariant when written as a tensor equation. Note that the conservation of energy and momentum is described by (in Cartesian coordinates)

$$\partial_{\nu}T^{\mu\nu} = 0. \tag{2.12}$$

This equation is correct only in flat spacetime. To find the correct equation in curved spacetime one needs to replace partial derivative with the covariant derivative as we shall see in the sections to follow.

Derivative of a Vector

If we have a vector $\vec{V} = V^{\alpha}\vec{e}_{\alpha}$, then its derivative is given by

$$\begin{aligned} \frac{\partial\vec{V}}{\partial x^{\beta}} &= \frac{\partial V^{\alpha}}{\partial x^{\beta}}\vec{e}_{\alpha} + V^{\alpha}\frac{\partial\vec{e}_{\alpha}}{\partial x^{\beta}}, \\ \therefore \partial_{\beta}\vec{V} &= \partial_{\beta}V^{\alpha}\vec{e}_{\alpha} + V^{\alpha}\partial_{\beta}\vec{e}_{\alpha}. \end{aligned} \tag{2.13}$$

If we are working in a coordinate system in which the basis vectors are normalized *e.g.* Cartesian coordinates, then $\partial_{\beta}\vec{e}_{\alpha} = 0$. But in any arbitrary coordinate system in which this might not be the case (*e.g.* polar coordinate system) the derivative of a vector is given by

Eq. (2.13).

Christoffel symbols

Because $\partial_\beta \vec{e}_\alpha$ is itself a vector (in eqn. (2.13)), it can be expressed as a linear combination of the basis vectors. If we relabel $\partial_\beta \vec{e}_\alpha$ as $\vec{A}$, then $\vec{A}$ can be written as $\vec{A} = A^\mu \vec{e}_\mu$. Here, the coefficients A^μ is the μ-th coefficient of $\partial_\beta \vec{e}_\alpha$, A^μ also contains the information of which basis vector ($\vec{e}_\alpha$) is being differentiated with respect to what coordinate (x^β).
To indicate all this information, we can denote the coefficients of vector $\partial_\beta \vec{e}_\alpha$ as $\Gamma^\mu_{\alpha\beta}$.

$$\partial_\beta \vec{e}_\alpha = \Gamma^\mu_{\alpha\beta} \vec{e}_\mu. \tag{2.14}$$

These symbols (coefficients) $\Gamma^\mu_{\alpha\beta}$ are called the Christoffel symbols. The Christoffel symbols are symmetric in their lower indices, meaning $\Gamma^\mu_{\alpha\beta} = \Gamma^\mu_{\beta\alpha}$. They can be expressed in terms of derivatives of the metric tensor components as follows

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} (\partial_\nu g_{\beta\mu} + \partial_\mu g_{\beta\nu} - \partial_\beta g_{\mu\nu}). \tag{2.15}$$

This formula can be derived using the definition of the covariant derivative and the compatibility of covariant derivative with respect to the metric. (The covariant derivative of the metric is zero) which we describe next.)

Covariant derivative

Using the Christoffel symbol notation and definition in (2.14), we can rewrite the derivative of a vector in curved geometry as

$$\nabla_\beta \vec{V} = \partial_\beta V^\alpha \vec{e}_\alpha + V^\alpha \partial_\beta \vec{e}_\alpha,$$
$$\therefore \nabla_\beta \vec{V} = \partial_\beta V^\alpha \vec{e}_\alpha + V^\alpha \Gamma^\mu_{\alpha\beta} \vec{e}_\mu,$$
$$\therefore \nabla_\beta \vec{V} = (\partial_\beta V^\alpha + V^\mu \Gamma^\alpha_{\mu\beta}) \vec{e}_\alpha,$$
$$\therefore \nabla_\beta \vec{V} = \nabla_\beta V^\alpha \vec{e}_\alpha.$$

Hence, the covariant derivative of vector components V^α is defined to be

$$\nabla_\beta V^\alpha = \partial_\beta V^\alpha + V^\mu \Gamma^\alpha_{\mu\beta}. \tag{2.16}$$

Using the covariant derivative formula (2.16), the divergence of the vector field V^α is given by

$$\nabla_\alpha V^\alpha = \frac{1}{\sqrt{|g|}} \partial_\alpha (\sqrt{|g|} V^\alpha), \tag{2.17}$$

where $|g|$ is absolute value of the determinant of the metric tensor. Similarly, the covariant derivative of a one-form field p_α is given by

$$\nabla_\beta P_\alpha = \partial_\beta P_\alpha - P_\mu \Gamma^\mu_{\alpha\beta}. \tag{2.18}$$

Some examples of covariant derivative of higher order tensors are as follows [9]

$$\nabla_\beta T_{\mu\nu} = \partial_\beta T_{\mu\nu} - T_{\alpha\nu} \Gamma^\alpha_{\mu\beta} - T_{\mu\alpha} \Gamma^\alpha_{\nu\beta},$$
$$\nabla_\beta A^{\mu\nu} = \partial_\beta A^{\mu\nu} + A^{\alpha\nu} \Gamma^\mu_{\alpha\beta} + A^{\mu\alpha} \Gamma^\nu_{\alpha\beta},$$
$$\nabla_\beta B^\mu_\nu = \partial_\beta B^\mu_\nu + B^\alpha_\nu \Gamma^\mu_{\alpha\beta} - B^\mu_\alpha \Gamma^\alpha_{\nu\beta}.$$

If the arbitrary tensor $T_{\mu\nu}$ is taken to be the metric tensor $g_{\mu\nu}$, then $\nabla_\mu g_{\alpha\beta} = 0$ can be solved to get the above described definition of the Christoffel symbols in terms of the metric tensor components.

Parallel Transport

There are two types of curvatures in general theory of relativity: extrinsic curvature and intrinsic curvature. The extrinsic curvature describes how an m-dimensional surface is embedded into an n-dimensional surface ($n > m$). It describes the curvature of the m-dimensional surface when looked from a higher n-dimensional point of view (bird's eye view). Surface of a cylinder is an example of a surface with an extrinsic curvature. It is a 2-dimensional surface embedded in 3-dimensional space that appears curved when observed from a 3-dimensional point of view. The distance relation of the points of a cylinder is the same as the distance relation of a flat surface when observed from the two dimensional perspective. Parallel lines remain parallel on the surface of a cylinder just like they would on a flat surface. From this 2-dimensional perspective the surface of the cylinder appears analogous to a flat surface. A flat sheet of paper can be converted into a cylinder without tearing or crumpling it, so its intrinsic geometry is that of a plane: it is flat. So, the extrinsic curvature describes how m-dimensional surface points are related to the n-dimensional space.

The intrinsic curvature only considers the relationships amongst the points as observed from the (m -dimensional) surface itself. It does not rely on the notion of a higher-dimensional space. In a surface with intrinsic curvature, the distance between any two points is not the same as of a flat space. Parallel lines would either converge or diverge on a surface with nonzero intrinsic curvature. The general theory of relativity describes this intrinsic curvature of the 4-dimensional spacetime without referring to any higher dimensions. The (2D) surface of a sphere is an example of a surface with an intrinsic curvature.

Consider a triangle with points A, B and C on a flat surface. Now consider a vector at

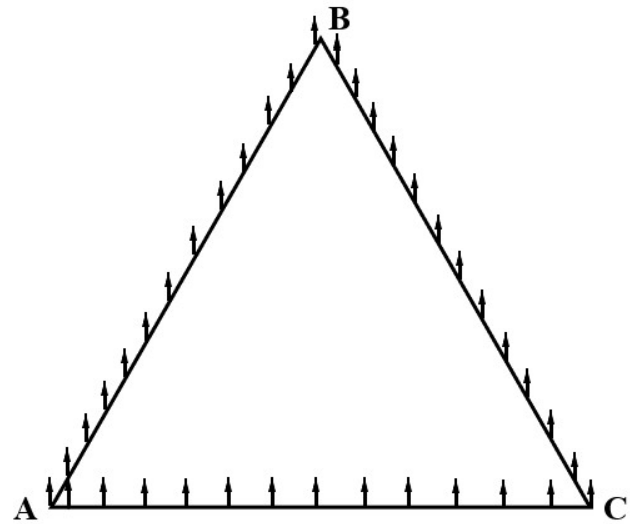

Figure 2.1: Parallel transport of a vector on a flat surface

point A. Let's transport this vector (along the curve) from A to B to C and back to A with the condition that the vector is always tangent to the surface and parallel to itself after any infinitesimal displacement. Such a construction is called parallel-transport. When it returns to point A again, we see (in figure 2.1) that it will be parallel to itself and will have the same magnitude.

Now consider a triangle with points A, B and C on the surface of a sphere such that points A and C are on the equator and point B is on the pole. If we parallel transport a vector along the triangle (curve) ABC and back to A, we see (figure 2.2) that the vector is pointing in some other direction and/or has different magnitude. In other words, the vector is not parallel to itself.

This means that on a curved manifold it is not possible to define a globally parallel vector field. Two vectors can only be compared after parallel-transporting one of them to the other. In other words, they can be compared in the tangent space of a point. And this comparison will depend on the path taken for parallel transport. Therefore, we cannot claim that a cer-

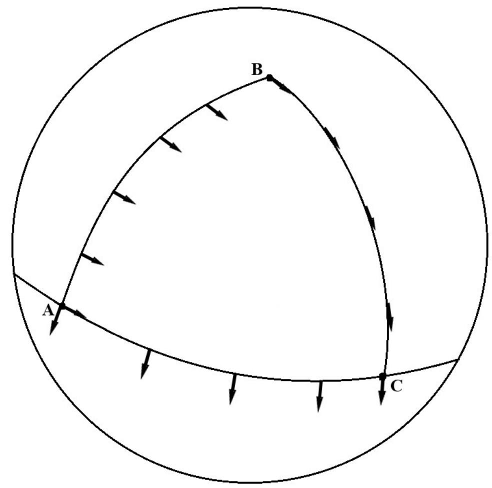

Figure 2.2: Parallel transport of a vector on the surface of a sphere

tain vector at point A is parallel to another vector at B or not.

Let's consider a curve on a curved surface defined by the parameter λ. The tangent vector $\vec{U}$ is then given by

$$\vec{U} = \frac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda}$$

If we have a vector field $\vec{V}$ defined on the curve such that vectors $\vec{V}$ at infinitesimally close points of the curve are parallel and of equal magnitude, then $\vec{V}$ is said to be parallelly transported along the curve. In a locally inertial (flat) coordinate system, the components of $\vec{V}$ must be constant along the curve in the infinitesimal neighbourhood of point **P**.

$$\begin{aligned}
\frac{\mathrm{d}V^{\alpha}}{\mathrm{d}\lambda} &= 0, \\
\therefore \frac{\partial V^{\alpha}}{\partial x^{\beta}}\frac{\mathrm{d}x^{\beta}}{\mathrm{d}\lambda} &= 0, \\
\therefore U^{\beta}\partial_{\beta}V^{\alpha} &= 0, \\
\therefore U^{\beta}\nabla_{\beta}V^{\alpha} &= 0, \\
\therefore \nabla_{\vec{U}}\vec{V} &= 0.
\end{aligned} \tag{2.19}$$

In the fourth step of the above equation we used the fact that $\Gamma^{\alpha}_{\mu\nu} = 0$ at point **P**, hence the partial derivative can be changed as a co-variant derivative. And since it is a tensor equation, it is true in any arbitrary frame.

Geodesic

On a curved surface, a geodesic is analogous to a straight line of Euclid's flat space. It is a curve that is 'as nearly straight as possible' on a curved surface. A straight line in Euclid's geometry is a curve for which the tangent at a point is parallel to the tangent at a previous point. So, a geodesic is essentially a curve that is obtained by parallel transporting the tangent vector. Looking at the definition of parallel transport of a vector, it can be given by

$$\begin{aligned}
\nabla_{\vec{U}}\vec{U} &= 0, \\
\therefore U^{\beta}\nabla_{\beta}U^{\alpha} &= 0, \\
\therefore U^{\beta}\partial_{\beta}U^{\alpha} + \Gamma^{\alpha}_{\mu\beta}U^{\mu}U^{\beta} &= 0, \\
\therefore \frac{\mathrm{d}^2x^{\alpha}}{\mathrm{d}\lambda^2} + \Gamma^{\alpha}_{\mu\beta}\frac{\mathrm{d}^2x^{\mu}}{\mathrm{d}\lambda}\frac{\mathrm{d}x^{\beta}}{\mathrm{d}\lambda} &= 0.
\end{aligned}$$

The above equation is a nonlinear, second-order differential equation for $x^{\alpha}(\lambda)$. It has a unique solution for a given initial position and initial direction. We can get a unique geodesic with different initial conditions.

Riemann Tensor

It is rather difficult to distinguish flat space from curved space just by looking at the metric as the representation of the metric is coordinate dependent. Even in flat space, the metric does not take the simplest form unless one is working with Cartesian coordinates. The Christoffel symbols and their partial derivatives do vanish in flat space when using Cartesian coordinates, but they do not vanish in flat space when represented by any arbitrary coordinates (*e.g.* polar coordinates). This is due to the fact that the Christoffel symbols are not tensors. Even in curved spaces, one can always choose locally inertial coordinates (at some point P) such that the metric is the Minkowski metric and the Christoffel symbols are zero (at that point).

We need a more useful mathematical tool which can be used to distinguish curved spaces from a flat one. Let's use the fact that, on a curved surface (*e.g.* a sphere), when a vector is parallel transported in a closed loop, it is not identical to the original vector when we return to the starting point [24]. Consider a closed loop (see figure 2.3) formed by the coordinate grids $x^\mu, x^\mu + \delta x^\mu, x^\nu$ and $x^\nu + \delta x^\nu$. Consider a vector V_A^λ at point A. Let's say that vector V_A^λ is represented by V_C^λ when parallel transported across ABC. And vector V_A^λ is represented by $\overline{V}_C^\lambda$ when parallel transported across ADC. The difference between V_C^λ and $\overline{V}_C^\lambda$ is denoted as δV^λ (at the point C). δV^λ will tell us whether the space is flat or curved. If δV^λ is zero then the space is flat, otherwise it is curved.

Mathematically, V_C^λ can be represented as $\nabla_\mu \nabla_\nu V^\lambda$ since we are parallel transporting across the coordinate grids. Similarly, $\overline{V}_C^\lambda$ can be represented as $\nabla_\nu \nabla_\mu V^\lambda$.

$$\therefore \delta V^\lambda = \nabla_\mu \nabla_\nu V^\lambda - \nabla_\nu \nabla_\mu V^\lambda,$$

$$\therefore \delta V^\lambda = [\nabla_\mu, \nabla_\nu] V^\lambda.$$

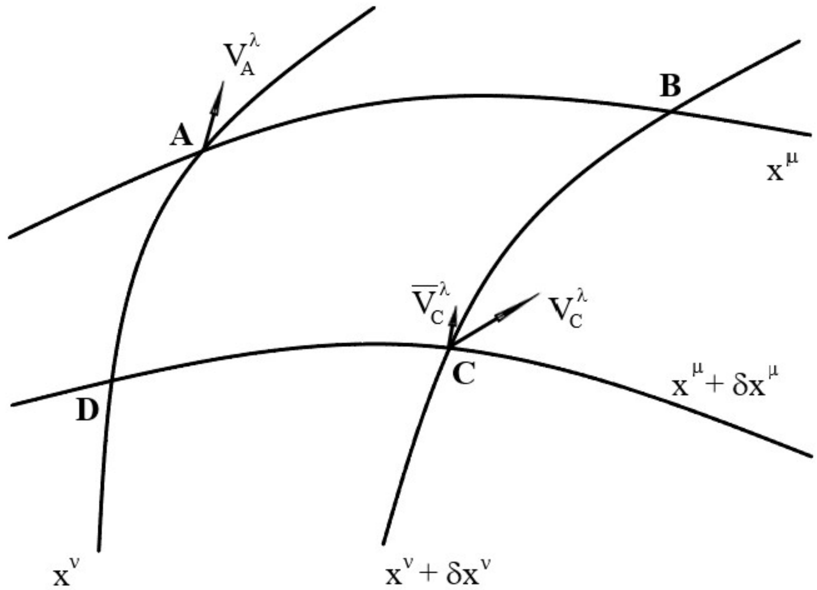

Figure 2.3: Parallel transport of a vector on the path ABC and ADC across the closed loop formed by the coordinate grids

Hence, δV^λ is obtained by finding the commutator of the covariant derivatives ∇_μ and ∇_ν.

$$
\begin{aligned}
[\nabla_\mu, \nabla_\nu]V^\lambda &= \nabla_\mu \nabla_\nu V^\lambda - \nabla_\nu \nabla_\mu V^\lambda \\
&= (\partial_\mu(\nabla_\nu V^\lambda) - \Gamma^\sigma_{\mu\nu}\nabla_\sigma V^\lambda + \Gamma^\lambda_{\mu\sigma}\nabla_\nu V^\sigma) - (\partial_\nu(\nabla_\mu V^\lambda) - \Gamma^\sigma_{\nu\mu}\nabla_\sigma V^\lambda + \Gamma^\lambda_{\nu\sigma}\nabla_\mu V^\sigma), \\
&= (\partial_\mu \Gamma^\lambda_{\nu\alpha} - \partial_\nu \Gamma^\lambda_{\mu\alpha} + \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\nu\alpha} - \Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\mu\alpha})V^\alpha.
\end{aligned}
$$

In last few steps, we reapplied the definition of covariant derivative, relabelled the summation index as to take V^α out of the bracket, removed the same terms with opposite signs and used the fact that the Christoffel symbols are symmetric in their lower index (torsion free condition).

Since the covariant derivatives ∇_μ and ∇_ν are tensors, their commutator $[\nabla_\mu, \nabla_\nu]$ is also a tensor. Even though the Christoffel symbols are not tensors, $\partial_\mu \Gamma^\lambda_{\nu\alpha} - \partial_\nu \Gamma^\lambda_{\mu\alpha} + \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\nu\alpha} - \Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\mu\alpha}$ is a tensor, it obeys all the tensor transformation laws and is independent of the coordinates.

$$R^{\alpha}_{\beta\mu\nu} = \partial_{\mu}\Gamma^{\alpha}_{\nu\beta} - \partial_{\nu}\Gamma^{\alpha}_{\mu\beta} + \Gamma^{\alpha}_{\mu\sigma}\Gamma^{\sigma}_{\nu\beta} - \Gamma^{\alpha}_{\nu\sigma}\Gamma^{\sigma}_{\mu\beta}. \tag{2.20}$$

$R^{\alpha}_{\beta\mu\nu}$ is called the **Riemann curvature tensor**. It is a (1,3) tensor which measures the curvature of the given space. Unlike the Christoffel symbols, $R^{\alpha}_{\beta\mu\nu}$ does not vanish in local inertial frames.

Using the metric tensor one can write $g_{\alpha\lambda}R^{\lambda}_{\beta\mu\nu} = R_{\alpha\beta\mu\nu}$.

We can contract the first and the third index of $R^{\alpha}_{\beta\mu\nu}$ to define the **Ricci tensor**. It is denoted by $R_{\mu\nu}$.

$$R_{\mu\nu} := R^{\alpha}_{\mu\alpha\nu},$$
$$\therefore R_{\mu\nu} = g^{\alpha\beta}R_{\beta\mu\alpha\nu}.$$

The Ricci tensor ($R_{\mu\nu}$) can be further contracted to define the **Ricci scalar**.

$$R := g^{\mu\nu}R_{\mu\nu},$$
$$\therefore R = g^{\mu\nu}g^{\alpha\beta}R_{\alpha\mu\beta\nu}.$$

2.3 General relativity

The weak equivalence principle:

Freely falling particles move on a timelike geodesic of the spacetime.

The Einstein equivalence principle:

Any local physical experiment not involving gravity will have the same results if performed in a freely falling inertial frame as if it were performed in a flat spacetime of special relativity. [9] Or, in other words, it is possible to choose a locally inertial coordinate system at every spacetime point in a gravitational field such that, within a sufficiently small region of the point, the laws of nature are the same as in special relativity. There is great similarity between the equivalence principle and the appearance of local flatness on a curved Riemannian manifold. Because of this resemblance one may expect that spacetime in general relativity can be described with a pseudo-Riemannian manifold [9].

The Principle of General Covariance:

We will now discuss the principle that is used to generalize valid equations from special relativity to general relativity. It is known as the 'principle of general covariance'. This principle states that a physical equation holds true in all coordinate systems if:

1. The equation holds true in absence of gravitation (*i.e.* it holds true in special relativity).

2. It is a tensor equation (*i.e.* it preserves its form under a general coordinate transformation).

By the equivalence principle, an equation that is correct in a locally inertial coordinate system can be written and then a general coordinate transformation can be made to find the corresponding equation in an arbitrary coordinate system. Using the principle of general covariance, we can find the equation that holds for all coordinate systems in a simple manner. It follows from the equivalence principle by considering any equation that satisfy condition (1) and (2). Since the equation is generally covariant it preserves its form under a general coordinate transformation, so if its form is correct in one coordinate system then it is correct in all coordinate systems. The equivalence principle says that at every point in spacetime there exists locally inertial coordinate systems in which the effects of gravity are absent. Since we assumed that our equation holds in special relativity and therefore holds in these locally inertial systems, it must hold in all coordinate systems.

So to find equations that are correct in a general gravitational field we simply take the valid tensor equations of special relativity and replace all partial derivatives by coordinate appropriate covariant derivatives and the Minkowski spacetime metric $\eta_{\mu\nu}$ by the general metric tensor $g_{\mu\nu}$.

THE EINSTEIN FIELD EQUATIONS

In the above sections we have introduced the mathematical tools which are crucial in understanding how spacetime gets curved due to massive bodies. The Einstein field equations (EFE) are a set of nonlinear equations which describe how the spacetime gets curved [25]. They can be derived from the action principle formulated based on the invariance under general coordinate transformations [25]. They are given by

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu} \tag{2.21}$$

where $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, $g_{\mu\nu}$ is the metric tensor and $T_{\mu\nu}$ is the stress-energy tensor.

The left hand side of the equation is known as the Einstein tensor $G_{\mu\nu}$. It describes how the spacetime curves. All the information of the source (like energy, mass, pressure and electromagnetic field) is contained in the stress-energy tensor $T_{\mu\nu}$. For a given distribution of matter (described by $T_{\mu\nu}$), one can solve the EFE to obtain the metric tensor $g_{\mu\nu}$. Once the metric tensor is found, one can know everything about the given space as it can be used to calculate Christoffel symbols, Riemann tensor, Ricci tensor and Ricci scalar.

Since $R_{\mu\nu}, g_{\mu\nu}$ and $T_{\mu\nu}$ are symmetric in their indices, we have a set of ten independent equations described by EFE. The contracted Bianchi identity for EFE is given by

$$\nabla^{\mu} R_{\mu\nu} = \frac{1}{2} \nabla_{\nu} R. \tag{2.22}$$

But with the consideration of contracted Bianchi identity, we are left with six independent equations which need to be solved for the metric tensor. Once the solution is found, we know how the spacetime is curved. The motion of particles through curved spacetime (in the absence of all the forces except for gravity) is described by the geodesic equation.

2.4 Electromagnetic waves

Maxwell's equations [26] describe the behavior of electric and magnetic fields at different points in spacetime, depending on the distribution and motion of charges. These equations are as follows

$$\begin{aligned}
\vec{\nabla}.\vec{E} &= \frac{\rho}{\varepsilon_0}, && \text{(Gauss's law)}\\
\vec{\nabla}.\vec{B} &= 0, &&\\
\vec{\nabla}\times\vec{E} &= -\frac{\partial\vec{B}}{\partial t}, && \text{(Faraday's law)}\\
\vec{\nabla}\times\vec{B} &= \mu_0\vec{J}+\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} && \text{(Ampere's law)},
\end{aligned}$$

where the vector $\vec{E}$ is the electric field vector, $\vec{B}$ is the magnetic field vector, ρ is the electric charge density and $\vec{J}$ is the electric current density. The constants ε_0 and μ_0 represent the permittivity and the permeability of empty space. We know that

$$\vec{B} = \vec{\nabla}\times\vec{A}, \tag{2.23}$$

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}, \tag{2.24}$$

where $\vec{A}$ is the magnetic potential and ϕ is the electric potential. Now if we transform the magnetic potential as $\vec{A}\to\vec{A}'=\vec{A}+\vec{\nabla}\Lambda$ and the electric potential as $\phi\to\phi'=\phi-\frac{\partial\Lambda}{\partial t}$, then the electric and magnetic field remain unchanged for any scalar function Λ. This can be easily seen by substituting the transformed magnetic and electric potentials in the above equations. A particular choice of the scalar and vector potentials is known as a gauge potential; and such changes in the electric potential ϕ and magnetic potential $\vec{A}$ are known as the gauge transformations.

The Lorenz gauge

In the Lorenz gauge we pick the magnetic vector potential and the scalar electric potential such that $\vec{\nabla}.\vec{A} = -\frac{1}{c^2}\frac{\partial\phi}{\partial t}$ holds.

$$\therefore \vec{\nabla}.\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} = 0. \tag{2.25}$$

In tensor notation above can be written as

$$\partial_\mu A^\mu = 0, \tag{2.26}$$

where the components of the four-vector A^μ are given by $A^\mu = (\frac{\phi}{c}, \vec{A})$. It is known as the electromagnetic four potential.

Maxwell's equations in this gauge become

$$\Box A^\nu = \mu_0 J^\nu, \tag{2.27}$$

where the components of the four-vector J^ν are given by $J^\nu = (c\rho, \vec{j})$. It is known as the four-current (ρ is the charge density).

Note that after the interaction with the gravitational wave, the four-potential A_μ can be divided in two parts, the unperturbed part denoted by $\overline{A}$ and the perturbed part denoted by $\widetilde{A}$.

$$\boxed{A_\mu = \overline{A}_\mu + \widetilde{A}_\mu(h)}. \tag{2.28}$$

Here $\Box\overline{A}_\mu = 0$ is the **unperturbed electromagnetic wave** and $\Box\widetilde{A}_\mu \neq 0$ is the **perturbed electromagnetic wave**. Note that $\overline{A}_\mu \gg \widetilde{A}_\mu$ as the electromagnetic potential perturbations are caused by a weak gravitational wave.

Electromagnetic field strength tensor

It is an anti-symmetric (0 2) tensor with six independent components (electric and magnetic field components). In the covariant component form it is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.29}$$

The contravariant matrix form of it can be given by

$$F^{\mu\nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{bmatrix}. \tag{2.30}$$

Maxwell's equations in tensor form

The tensor form of Gauss's law and Ampere's law can be written as

$$\partial_\nu F^{\mu\nu} = \mu_0 J^\mu. \tag{2.31}$$

Gauss's law for magnetism and Faraday's law can be written as

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0. \tag{2.32}$$

In the absence of sources $J^\mu = (0,0,0,0)$ and eqn. (2.31) will describe electromagnetic waves

$$\partial_\nu F^{\mu\nu} = 0. \tag{2.33}$$

Maxwell's equations in curved geometry

From the 'Minimal Coupling Principle', Maxwell's equations [1] in curved spacetime can be written as

$$\nabla_\nu F^{\mu\nu} = 0;$$

But we know that

$$\nabla_\nu F^{\mu\nu} = \frac{1}{\sqrt{|g|}}\partial_\nu(\sqrt{|g|}F^{\mu\nu}),$$

$$\boxed{\therefore \nabla_\nu F^{\mu\nu} = \partial_\nu F^{\mu\nu} = 0}. \tag{2.34}$$

2.5 Conclusion

The essential ideas of the special theory of relativity were discussed. The Lorentz transformation; and general coordinate transformation notations were introduced. It was shown that the metric tensor **g** defines lengths and angles on the given manifold; and has all the necessary information to describe the manifold. For any arbitrary coordinate system, the components of the metric tensor are given by $g_{\mu\nu} = \vec{e}_\mu \cdot \vec{e}_\nu$. Quantities such as four-velocity(U^μ), four-momentum, number-density, number-flux and energy density were defined in the given framework. The meaning of stress-energy tensor components was discussed in detail and it was shown that in any arbitrary coordinates its components are given by $T^{\mu\nu} = (\rho + p)U^\mu U^\nu + pg^{\mu\nu}$. The notion of differentiation on a curved surface was introduced and the covariant derivative of a vector was defined to be $\nabla_\beta V^\alpha = \partial_\beta V^\alpha + V^\mu \Gamma^\alpha_{\mu\beta}$. With the definition of Christoffel symbols as $\Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}(\partial_\nu g_{\beta\mu} + \partial_\mu g_{\beta\nu} - \partial_\beta g_{\mu\nu})$. Key mathematical tools like parallel transport, geodesic equation, Riemann curvature tensor, Ricci tensor and Ricci scalar were discussed in depth. Using the Einstein equivalence principle, it was established that in the gravitational field at every point a local inertial coordinate system can be chosen such that the laws of physics are the same as in special relativity. The Einstein Field Equations describe how matter and energy curve spacetime.

Chapter 3

Gravitational Waves

3.1 Introduction

Gravitational waves were predicted shortly after Einstein introduced the general theory of relativity in 1916. However, their detection was rather difficult due to experimental and technological limitations. They were recently detected by LIGO and Virgo and announced in February 2016 [4]. These gravitational waves were produced due to a binary blackhole merger. With this detection, a new era of gravitational wave optics has begun. A gravitational wave is a ripple in the fabric of spacetime. It is generally produced due to the acceleration of massive bodies. Because gravitational waves interact weakly with matter, they can be used to observe the events which might not be visible using electromagnetic radiation. There are efforts towards detecting primordial gravitational waves [6, 7, 27]. These are the waves which were produced shortly after the Big-Bang during the process of inflation [28] and due to early astrophysical sources. In this chapter, we will study how the weak gravitational waves can be described mathematically as the perturbation on the background metric. We show a detailed derivation of the form of the metric perturbation $h_{\mu\nu}$. We will only consider the linear terms of the metric perturbation and solve the Einstein field equations in the absence of any matter. The background spacetime is taken to be Minkowski spacetime described by the metric $\eta_{\mu\nu}$. Note that to describe gravitational waves in any other spacetime, we follow a similar procedure as done in this chapter.

3.2 Linearized Einstein field equations

To obtain the linearized Einstein field equations, let's consider a spacetime with modest curvature. One can think of regions of space which are far away from any massive gravitating bodies. The gravity is so weak that the metric $g_{\mu\nu}$ for the given spacetime is 'close' to the metric $\eta_{\mu\nu}$ of Minkowksi spacetime, with small deviations due to curvature. Mathematically, it can be written as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{3.1}$$

where $h_{\mu\nu}$ is a 'small' metric perturbation [9, 29, 30, 27]. Since the metric components can be described in any arbitrary coordinates, there is no natural sense of the norm of a tensor being 'small'. To rectify this, we can require that $|h_{\mu\nu}| \ll 1$ when the metric $\eta_{\mu\nu}$ is described by $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$.

Let's now find the inverse of $h_{\mu\nu}$. We begin with $g^{\mu\nu} = \eta^{\mu\nu} + X^{\mu\nu}$, where $X^{\mu\nu}$ is the unknown inverse of $h_{\mu\nu}$.

$$\begin{aligned}
g_{\mu\nu}g^{\nu\beta} &= (\eta_{\mu\nu} + h_{\mu\nu})(\eta^{\nu\beta} + X^{\nu\beta}), \\
\therefore \delta^{\beta}_{\mu} &= \delta^{\beta}_{\mu} + \eta_{\mu\nu}X^{\nu\beta} + \eta^{\nu\beta}h_{\mu\nu}, \qquad\qquad (\because g_{\mu\nu}g^{\nu\beta} = \eta_{\mu\nu}\eta^{\nu\beta} = \delta^{\beta}_{\mu}) \\
\therefore \eta^{\mu\alpha}\eta_{\mu\nu}X^{\nu\beta} &= -\eta^{\mu\alpha}\eta^{\nu\beta}h_{\mu\nu}, \\
\therefore X^{\alpha\beta} &= -\eta^{\mu\alpha}\eta^{\nu\beta}h_{\mu\nu}.
\end{aligned}$$

Because $|h_{\mu\nu}| \ll 1$, we have neglected the higher order terms of $h_{\mu\nu}$ in the second step.

Now we can write $g^{\mu\nu} = \eta^{\mu\nu} - \eta^{\mu\alpha}\eta^{\nu\beta}h_{\mu\nu}$. Because $h^{\alpha\beta} \equiv \eta^{\mu\alpha}\eta^{\nu\beta}h_{\mu\nu}$, we have

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}. \tag{3.2}$$

This equation is correct up to first order in $h_{\mu\nu}$. Note that the raising and lowering of the indices is not done through the metric tensor $g_{\mu\nu}$, but through $\eta_{\mu\nu}$. This is due to only keeping the linear terms in $h_{\mu\nu}$.

We need to substitute the above definitions of the metric and its inverse to find the linearized Einstein field equations. To do so, let's first find the Christoffel symbols using the definition

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}(\partial_{\nu} g_{\beta\mu} + \partial_{\mu} g_{\beta\nu} - \partial_{\beta} g_{\mu\nu}).$$

By keeping only the terms of linear order, we find that

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} \eta^{\alpha\beta}(\partial_{\nu} h_{\beta\mu} + \partial_{\mu} h_{\beta\nu} - \partial_{\beta} h_{\mu\nu}).$$

Now that the linearized Christoffel symbols have been found, we can find the linearized Ricci tensor by

$$\begin{aligned} R_{\mu\nu} &= \partial_{\alpha}\Gamma^{\alpha}_{\mu\nu} + \partial_{\nu}\Gamma^{\alpha}_{\mu\alpha} + \Gamma^{\alpha}_{\alpha\sigma}\Gamma^{\sigma}_{\nu\mu} - \Gamma^{\alpha}_{\nu\sigma}\Gamma^{\sigma}_{\alpha\mu}, \\ \therefore R_{\mu\nu} &= \frac{1}{2}(\partial_{\nu}\partial^{\alpha} h_{\mu\alpha} + \partial_{\mu}\partial^{\alpha} h_{\nu\alpha} - \partial_{\alpha}\partial^{\alpha} h_{\mu\nu} - \partial_{\nu}\partial_{\mu} h^{\alpha}_{\alpha}) \end{aligned} \tag{3.3}$$

and similarly, the Ricci scalar is given by

$$\begin{aligned} R &= g^{\alpha\beta} R_{\alpha\beta}, \\ \therefore R &= \eta^{\alpha\beta} R_{\alpha\beta}. \qquad (\because |h_{\alpha\beta}| \ll 1) \end{aligned} \tag{3.4}$$

After substituting the above results in the Einstein field equations (2.21), the linearized Einstein equations are found to be

$$\begin{aligned} (\partial_{\nu}\partial^{\alpha} h_{\mu\alpha} + \partial_{\mu}\partial^{\alpha} h_{\nu\alpha} - \partial_{\alpha}\partial^{\alpha} h_{\mu\nu} - \partial_{\nu}\partial_{\mu} h^{\alpha}_{\alpha}) - \eta_{\mu\nu}(\eta^{\alpha\beta} R_{\alpha\beta}) &= 16\pi T_{\mu\nu}, \\ \therefore (\partial_{\nu}\partial^{\alpha} h_{\mu\alpha} + \partial_{\mu}\partial^{\alpha} h_{\nu\alpha} - \partial_{\alpha}\partial^{\alpha} h_{\mu\nu} - \partial_{\nu}\partial_{\mu} h) - \eta_{\mu\nu}[\frac{1}{2}(2\partial^{\alpha}\partial^{\beta} h_{\alpha\beta} - 2\partial^{\alpha}\partial_{\alpha} h)] &= 16\pi T_{\mu\nu}, \\ \therefore \partial_{\nu}\partial^{\alpha} h_{\mu\alpha} + \partial_{\mu}\partial^{\alpha} h_{\nu\alpha} - \partial_{\alpha}\partial^{\alpha} h_{\mu\nu} - \partial_{\nu}\partial_{\mu} h - \eta_{\mu\nu}\partial^{\alpha}\partial^{\beta} h_{\alpha\beta} + \eta_{\mu\nu}\partial^{\alpha}\partial_{\alpha} h &= 16\pi T_{\mu\nu}. \end{aligned} \tag{3.5}$$

To simplify the expression above we can define the trace-reversed form of $h_{\mu\nu}$ as

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h; \tag{3.6}$$

$\bar{h}_{\mu\nu}$ is called trace-reversed of $h_{\mu\nu}$ because $\eta^{\mu\nu}\bar{h}_{\mu\nu} = \bar{h} = -h$. We can also write $h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}$, which can be easily verified.

With this definition of $\bar{h}_{\mu\nu}$, the linearized Einstein field equations can be rewritten as

$$\begin{aligned} -\partial_\alpha\partial^\alpha\bar{h}_{\mu\nu} + \partial_\mu\partial^\alpha\bar{h}_{\nu\alpha} - \eta_{\mu\nu}\partial^\alpha\partial^\beta\bar{h}_{\alpha\beta} &= 16\pi T_{\mu\nu}, \\ \therefore \Box\bar{h}_{\mu\nu} - \partial_\mu\partial^\alpha\bar{h}_{\nu\alpha} + \eta_{\mu\nu}\partial^\alpha\partial^\beta\bar{h}_{\alpha\beta} &= -16\pi T_{\mu\nu}, \end{aligned} \tag{3.7}$$

where $\partial_\alpha\partial^\alpha = -\partial_t^2 + \nabla^2 = \Box$ is called the d'Alembertian operator or the wave operator. If we can simplify the above expression such that the last two terms on the left side of the equality vanish (*i.e.* $\partial^\nu h_{\mu\nu} = 0$), then we will have a wave equation with the source term on the right side of equality. In general relativity, we have the freedom to select a situation appropriate coordinate system or 'gauge'. To better understand if we can choose a coordinate system in which $\partial^\nu\bar{h}_{\mu\nu} = 0$, let us consider an arbitrary infinitesimal coordinate (gauge) transformation of the form

$$x^\alpha \to x'^\alpha = x^\alpha + \xi^\alpha(x^0, x^1, x^2, x^3),$$

where the 'prime' refers to the transformed coordinates for the convenience of the calculation. Here $\xi^\alpha(x^0, x^1, x^2, x^3)$, shortly written as $\xi^\alpha(x^\mu)$, is the function of position that represents an arbitrary infinitesimal displacement four-vector.

To see how this changes the metric perturbations, we need to find the derivative of x'^α and

x^α with respect to each other.

$$\therefore \frac{\partial x'^\alpha}{\partial x^\beta} = \frac{\partial x^\alpha}{\partial x^\beta} + \frac{\partial \xi^\alpha(x^\mu)}{\partial x^\beta},$$
$$\therefore \frac{\partial x'^\alpha}{\partial x^\beta} = \delta^\alpha_\beta + \frac{\partial \xi^\alpha}{\partial x^\beta}. \tag{3.8}$$

Similarly, we can find

$$\begin{aligned}
\frac{\partial x^\alpha}{\partial x'^\beta} &= \frac{\partial x'^\alpha}{\partial x'^\beta} - \frac{\partial \xi^\alpha}{\partial x^\nu}\frac{\partial x^\nu}{\partial x'^\beta}, \qquad (\because x^\alpha = x'^\alpha - \xi^\alpha(x^\mu))\\
\therefore \frac{\partial x^\alpha}{\partial x'^\beta} &= \delta^\alpha_\beta - \frac{\partial \xi^\alpha}{\partial x^\nu}\left[\frac{\partial x'^\nu}{\partial x'^\beta} - \frac{\partial \xi^\nu}{\partial x^\mu}\frac{\partial x^\mu}{\partial x'^\beta}\right],\\
\therefore \frac{\partial x^\alpha}{\partial x'^\beta} &= \delta^\alpha_\beta - \frac{\partial \xi^\alpha}{\partial x^\nu}\left[\delta^\nu_\beta - \frac{\partial \xi^\nu}{\partial x^\mu}\frac{\partial x^\mu}{\partial x'^\beta}\right],\\
\therefore \frac{\partial x^\alpha}{\partial x'^\beta} &= \delta^\alpha_\beta - \frac{\partial \xi^\alpha}{\partial x^\nu}\delta^\nu_\beta + \frac{\partial \xi^\alpha}{\partial x^\nu}\frac{\partial \xi^\nu}{\partial x^\mu}\frac{\partial x^\mu}{\partial x'^\beta},\\
\therefore \frac{\partial x^\alpha}{\partial x'^\beta} &= \delta^\alpha_\beta - \frac{\partial \xi^\alpha}{\partial x^\beta},
\end{aligned} \tag{3.9}$$

where in the last step, the higher order terms of $\xi^\alpha(x^\mu)$ are neglected as $|\xi^\alpha(x^\mu)|$ is very small. Using results above we can write the transformed metric $g'_{\mu\nu}$ as

$$\begin{aligned}
g'_{\mu\nu} &= \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta},\\
\therefore \eta'_{\mu\nu} + h'_{\mu\nu} &= (\delta^\alpha_\mu - \partial_\mu \xi^\alpha)(\delta^\beta_\nu - \partial_\nu \xi^\beta)(\eta_{\alpha\beta} + h_{\alpha\beta}),\\
\therefore \eta_{\mu\nu} + h'_{\mu\nu} &= \delta^\alpha_\mu \delta^\beta_\nu \eta_{\alpha\beta} + \delta^\alpha_\mu \delta^\beta_\nu h_{\alpha\beta} - \delta^\alpha_\mu \eta_{\alpha\beta}\partial_\nu \xi^\beta - \delta^\beta_\nu \eta_{\alpha\beta}\partial_\mu \xi^\alpha \qquad (\because \eta'_{\mu\nu} = \eta_{\mu\nu}),\\
\therefore \eta_{\mu\nu} + h'_{\mu\nu} &= \eta_{\mu\nu} + h_{\mu\nu} - \partial_\nu \xi_\mu - \partial_\mu \xi_\nu,\\
\therefore h'_{\mu\nu} &= h_{\mu\nu} - \partial_\nu \xi_\mu - \partial_\mu \xi_\nu.
\end{aligned} \tag{3.10}$$

In the above calculation, we have neglected the higher order terms of small quantities (*i.e.* $\partial_\mu \xi^\alpha h_{\alpha\nu}$ is neglected).The above equation represents the gauge transformed metric perturbation and is correct to the first order in $h_{\mu\nu}$ and ξ^α.

Since our equations are in the trace-reversed form of $h_{\mu\nu}$, let us also find the gauge transformed form of the trace-reversed metric perturbation $\overline{h}_{\mu\nu}$. We have

$$\begin{aligned}
\overline{h}'_{\mu\nu} &= h'_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h', \\
\therefore \overline{h}'_{\mu\nu} &= h_{\mu\nu} - \partial_\nu\xi_\mu - \partial_\mu\xi_\nu - \frac{1}{2}\eta_{\mu\nu}(h - 2\partial_\alpha\xi^\alpha), \\
\therefore \overline{h}'_{\mu\nu} &= h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h - \partial_\nu\xi_\mu - \partial_\mu\xi_\nu + \eta_{\mu\nu}\partial_\alpha\xi^\alpha, \\
\therefore \overline{h}'_{\mu\nu} &= \overline{h}_{\mu\nu} - \partial_\nu\xi_\mu - \partial_\mu\xi_\nu + \eta_{\mu\nu}\partial_\alpha\xi^\alpha. \qquad (3.11)
\end{aligned}$$

The condition that we want Eq. (3.7) to satisfy is

$$\partial^\nu\overline{h}_{\mu\nu} = 0 \qquad \text{(or equivalently } \partial_\nu\overline{h}^{\mu\nu} = 0). \qquad (3.12)$$

The above condition is known as the Lorenz gauge condition and the class of gauges satisfying this condition are known as Lorenz gauges. They are also known as harmonic gauges or Hilbert gauges.

Let us say that the metric perturbation (in some coordinate system) does not satisfy the Lorenz gauge. Is there a way to impose the Lorenz gauge by putting some constraints on ξ^α? To find that, consider a coordinate system that satisfies the Lorenz gauge condition. Therefore, we have

$$\begin{aligned}
\partial^\nu\overline{h}'_{\mu\nu} &= 0, \\
\therefore \partial^\nu\overline{h}_{\mu\nu} - \partial^\nu\partial_\nu\xi_\mu - \partial^\nu\partial_\mu\xi_\nu + \eta_{\mu\nu}\partial^\nu\partial_\alpha\xi^\alpha &= 0, \\
\therefore \partial^\nu\overline{h}_{\mu\nu} - \Box\xi_\mu - \partial_\mu\partial^\nu\xi_\nu + \partial_\mu\partial_\alpha\xi^\alpha &= 0, \\
\therefore \Box\xi_\mu &= \partial^\nu\overline{h}_{\mu\nu}. \qquad (3.13)
\end{aligned}$$

Hence, the Lorenz gauge can be imposed on any metric perturbation $\overline{h}_{\mu\nu}$ by making an infinitesimal coordinate transformation ξ^α with condition that $\Box\xi_\mu = \partial^\nu\overline{h}_{\mu\nu}$. The constraint equation always has solutions for a well behaved source term, as it is the three dimensional inhomogeneous wave equation.

So, if in a certain coordinate system (x^{β}) the $\bar{h}_{\alpha\beta}$ does not satisfy the Lorenz gauge condition, then to find a coordinate system x'^{β} in which the Lorenz gauge is satisfied; solve $\Box\xi_{\beta} = \partial^{\alpha}\bar{h}_{\alpha\beta}$ and make the coordinate transformation $x^{\beta} \to x'^{\beta} = x^{\beta} + \xi^{\beta}$.

Note that the ξ^{μ} is not unique. Any Λ^{μ} satisfying the homogeneous wave equation $\Box\Lambda_{\mu} = 0$ can be simply added to the ξ_{μ} and will still obey the Lorenz gauge condition (*i.e.* $\Box(\xi_{\mu} + \Lambda_{\mu}) = \partial^{\nu}\bar{h}_{\mu\nu}$).

Imposing the Lorenz gauge condition on Eq. (3.7), we get

$$\Box\bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}, \tag{3.14}$$

which is the inhomogeneous wave equation. Here, the metric perturbation $\bar{h}_{\mu\nu}$ propagates on the flat Minkowski background as a wave and is sourced by the energy momentum tensor $T_{\mu\nu}$.

Now let us consider the propagation of gravitational waves when $T_{\mu\nu} = 0$ (*i.e.* in the vacuum)

$$\Box\bar{h}_{\mu\nu} = 0. \tag{3.15}$$

The simplest general solution to above homogeneous wave equation can be given as

$$\begin{aligned} \bar{h}_{\mu\nu} &= A_{\mu\nu} e^{i(-\omega_g t + \vec{k}\cdot\vec{x})} \\ \bar{h}_{\mu\nu} &= A_{\mu\nu} e^{ik_{\alpha}x^{\alpha}}, \end{aligned} \tag{3.16}$$

where the constant $A_{\mu\nu}$ is a symmetric polarization tensor, which contains information about the amplitude and the polarization of the linearized gravitational wave. The x^{α} is simply the position vector components given by (t, x, y, z) and the constant k^{α} represents the four-wave vector components, given by $k^{\alpha} = (\omega_g, k_x, k_y, k_z)$. As one-form components they can be written as $k_\alpha = (-\omega_g, k_x, k_y, k_z)$. The substitution of the general solution yields

$$\begin{aligned}
\partial^\mu \partial_\mu (A_{\mu\nu} e^{ik_\alpha x^\alpha}) &= 0, \\
\therefore \eta^{\mu\nu} \partial_\nu (ik_\alpha \delta^\alpha_\mu e^{ik_\alpha x^\alpha}) &= 0, \\
\therefore -\eta^{\mu\nu} k_\mu k_\nu e^{ik_\alpha x^\alpha} &= 0, \\
\therefore k_\mu k^\mu &= 0, & (3.17) \\
\therefore (\omega_g)^2 &= \delta^{ij} k_i k_j. & (3.18)
\end{aligned}$$

The last result translates into the fact that the gravitational waves travel at the speed of light. The speed of any wave is given by $v = \frac{\omega_g}{|\vec{k}|}$, which is equal to 1 ($c$) using the equation above. Let us now impose the Lorenz gauge condition to the standard solution.

$$\begin{aligned}
\partial^\nu \overline{h}_{\mu\nu} &= 0, \\
\therefore \eta^{\beta\nu} \partial_\beta (A_{\mu\nu} e^{ik_\alpha x^\alpha}) &= 0, \\
\therefore \eta^{\beta\nu} A_{\mu\nu} ik_\beta e^{ik_\alpha x^\alpha} &= 0, \\
\therefore A_{\mu\nu} k^\nu &= 0. & (3.19)
\end{aligned}$$

The above equation shows that the four-wave vector k^ν is orthogonal to the polarization tensor $A_{\mu\nu}$. Due to $A_{\mu\nu}$ being symmetric, it has 10 independent components. But now with the set of four equations (constraints), the number of independent components of $A_{\mu\nu}$ reduces to 6. Let us now try to further reduce the number of independent components of $A_{\mu\nu}$ by doing another gauge transformation.

The constraint $\Box \xi_\mu = \partial^\nu \overline{h}_{\mu\nu}$ put on the infinitesimal displacement vector ξ^α does not exhaust all the freedom to choose coordinates (gauge freedom). We can still make another infinitesimal coordinate transformation $x^\alpha \rightarrow x'^\alpha = x^\alpha + \Lambda^\alpha$ with the constraint that $\Box \Lambda_\mu = 0$. This constraint is imposed by the Lorenz gauge condition. As discussed previously

$\Box(\xi_\mu + \Lambda_\mu) = \partial^\nu \overline{h}_{\mu\nu}$ is valid and consistent.

If we choose the form of Λ_μ as

$$\Lambda_\mu = B_\mu e^{ik_\alpha x^\alpha}, \tag{3.20}$$

where B_μ is a constant and the k_α is the wave-vector used in the standard solution of $\overline{h}_{\mu\nu}$, then it is easy to see that it satisfies $\Box\Lambda_\mu = 0$, due to $k_\nu k^\nu = 0$. As calculated previously, the transformed metric perturbation $h'_{\mu\nu}$ will be given by

$$h'_{\mu\nu} = h_{\mu\nu} - \partial_\nu\Lambda_\mu - \partial_\mu\Lambda_\nu \tag{3.21}$$

and the transformed trace-reversed metric perturbation will be given by

$$\overline{h}'_{\mu\nu} = \overline{h}_{\mu\nu} - \partial_\nu\Lambda_\mu - \partial_\mu\Lambda_\nu + \eta_{\mu\nu}\partial_\alpha\Lambda^\alpha. \tag{3.22}$$

Substituting the form of Λ_μ we obtain

$$\begin{aligned} \overline{h}'_{\mu\nu} &= \overline{h}_{\mu\nu} - \partial_\nu(B_\mu e^{ik_\alpha x^\alpha}) - \partial_\mu(B_\nu e^{ik_\alpha x^\alpha}) + \eta_{\mu\nu}\eta_{\gamma\alpha}\partial_\gamma(B_\alpha e^{ik_\beta x^\beta}), \\ \therefore A'_{\mu\nu}e^{ik_\alpha x^\alpha} &= A_{\mu\nu}e^{ik_\alpha x^\alpha} - ik_\nu B_\mu e^{ik_\alpha x^\alpha} - ik_\mu B_\nu e^{ik_\alpha x^\alpha} + \eta_{\mu\nu}\eta^{\gamma\alpha} ik_\gamma B_\alpha e^{ik_\beta x^\beta}, \\ \therefore A'_{\mu\nu} &= A_{\mu\nu} - ik_\nu B_\mu - ik_\mu B_\nu + \eta_{\mu\nu} ik^\alpha B_\alpha. \end{aligned} \tag{3.23}$$

The above equations show the relation among the polarization tensor components after the transformation ($A'_{\mu\nu}$) and before the transformation ($A_{\mu\nu}$). Now, let us find the relation between B_μ and $A_{\mu\nu}$ such that $A'_{\mu\nu}$ has the simplest form.

If we require that the trace of the transposed polarization tensor components vanish,

$$A'^{\nu}_{\nu} = 0, \tag{3.24}$$

then the relation between B^ν and A^ν_ν is given by

$$\begin{aligned}
\eta^{\mu\nu}A'_{\mu\nu} &= \eta^{\mu\nu}A_{\mu\nu} - \eta^{\mu\nu}ik_\nu B_\mu - \eta^{\mu\nu}ik_\mu B_\nu + \eta^{\mu\nu}\eta_{\mu\nu}ik^\alpha B_\alpha, \\
\therefore A'^\nu_\nu &= A^\nu_\nu - 2ik_\nu B^\nu + 4ik^\alpha B_\alpha, \\
\therefore 0 &= A^\nu_\nu + 2ik_\nu B^\nu, \\
\therefore k_\nu B^\nu &= \frac{i}{2}A^\nu_\nu. \qquad (3.25)
\end{aligned}$$

If we require that

$$A'_{0\nu} = 0, \qquad (3.26)$$

then we have

$$\begin{aligned}
&\therefore A_{0\nu} - ik_\nu B_0 - ik_0 B_\nu + \eta_{0\nu}ik^\alpha B_\alpha = 0, \\
&\therefore A_{0\nu} - ik_\nu B_0 - ik_0 B_\nu - \frac{1}{2}\eta_{0\nu}A^\nu_\nu = 0 \qquad (\because k_\alpha B^\alpha = \frac{i}{2}A^\nu_\nu).
\end{aligned}$$

When the free index $\nu = 0$ in the above equation, then we have

$$\begin{aligned}
A_{00} - 2ik_0B_0 + \frac{1}{2}A^\nu_\nu &= 0, \\
\therefore B_0 = -\frac{i}{2k_0}\left[A_{00} + \frac{1}{2}A^\nu_\nu\right]. \qquad (3.27)
\end{aligned}$$

When the free index ν is spatial, then we have

$$\begin{aligned}
A_{0j} - ik_jB_0 - ik_0B_j &= 0, \\
\therefore A_{0j} - \frac{k_j}{2k_0}\left[A_{00} + \frac{1}{2}A^\nu_\nu\right] - ik_0B_j &= 0, \\
\therefore B_j = -\frac{iA_{0j}}{k_0} + \frac{ik_j}{2(k_0)^2}\left[A_{00} + \frac{1}{2}A^\nu_\nu\right]. \qquad (3.28)
\end{aligned}$$

Using the relations (3.27) and (3.28), we can transform to a coordinate system in which $A'_{0\nu} = 0$ and $A'^\nu_\nu = 0$. After the transformation Λ^μ, we can relabel the $A'_{\mu\nu}$ as $A_{\mu\nu}$.

Before the transformation, $A_{\mu\nu}$ had 6 independent components. Three additional constraints are introduced by the requirement $A_{0\nu} = 0$ and one more constraint is introduced by the required $A^{\nu}_{\nu} = 0$. Hence, $A_{\mu\nu}$ has 2 independent components.
(Note that $A_{0\nu} = 0$ are four different equations, only three constraints are there due to $\nu = 0$ translating to one of the previous constraint $A_{\mu\nu}k^{\nu} = 0$.)
The above used gauge satisfies the following conditions:

- $\partial^{\nu}\overline{h}_{\mu\nu} = 0 \quad (A_{\mu\nu}k^{\nu} = 0)$
- $h_{0\nu} = 0$ ($A_{0\nu} = 0$), meaning all temporal components of $h_{\mu\nu}$ is zero.
- $h^{\nu}_{\nu} = 0$ ($A^{\nu}_{\nu} = 0$), meaning the trace of $h_{\mu\nu}$ is zero.

Such a gauge is known as the **transverse-traceless (TT)** gauge. This name makes sense as the metric perturbation is traceless and is perpendicular to the wave-vector. The notation for the metric perturbation in the TT-gauge is given by $h^{TT}_{\mu\nu}$. Since in the TT-gauge the trace of $\overline{h}^{TT}_{\mu\nu}$ is zero, there is no difference between the trace-reversed metric perturbation $\overline{h}_{\mu\nu}$ and the metric perturbation $h_{\mu\nu}$.

$$\overline{h}^{TT}_{\mu\nu} = h^{TT}_{\mu\nu}. \tag{3.29}$$

Consider a gravitational wave which is propagating in the x^3 direction. The wave-vector components for such a wave is given by

$$k^{\alpha} = (\omega_g, 0, 0, \omega_g). \tag{3.30}$$

This is due to the fact that $k_{\mu}k^{\mu} = 0$. Due to the form of the wave-vector components k^{α}, $A_{\mu\nu}k^{\nu} = 0$ and $A_{0\nu} = 0$, it is easy to conclude that

$$A_{3\nu} = 0. \tag{3.31}$$

This means the only nonzero components are A_{11}, A_{12}, A_{21} and A_{22}. Because $A_{\mu\nu}$ is traceless and symmetric, we can write

$$A_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & A_+ & A_\times & 0 \\ 0 & A_\times & -A_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3.32}$$

The reason for the notation '$+$' and '$\times'$ will become clear when the interaction of gravitational wave with matter particles is explained.

Since we know the form of metric perturbation to be $h^{TT}_{\mu\nu} = A_{\mu\nu}e^{ik_\alpha x^\alpha}$, in the component form it can be written as

$$h_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3.33}$$

where h_+ and $h_\times$ are simply given by

$$h_+ = A_+ \cos(\omega_g z - \omega_g t), \tag{3.34}$$

$$h_\times = A_\times \cos(\omega_g z - \omega_g t + \delta), \tag{3.35}$$

where ω_g is the angular frequency of the gravitational wave and δ is the phase difference.

The metric components $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ can be given by

$$g_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1+h_+ & h_\times & 0 \\ 0 & h_\times & 1-h_+ & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.36}$$

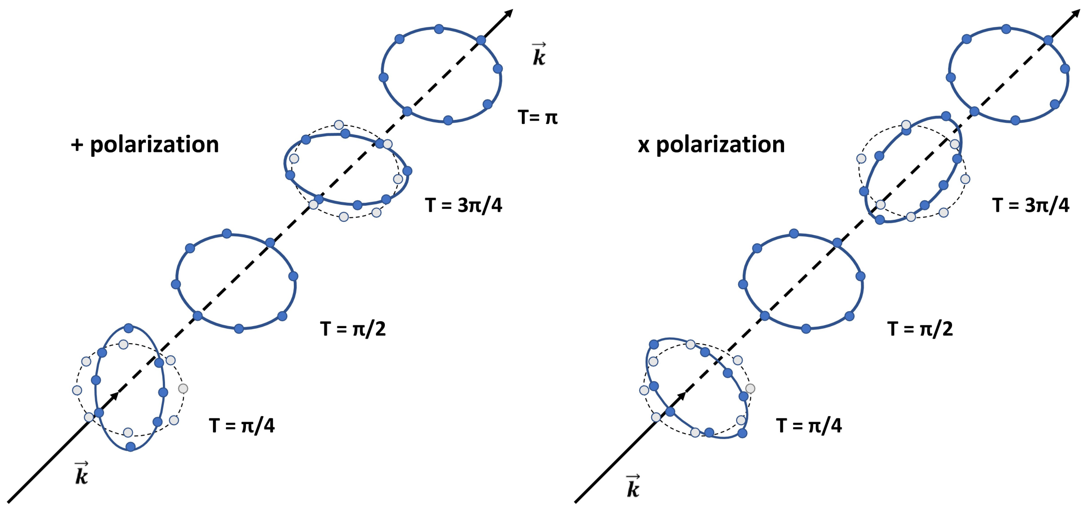

Figure 3.1: The effects of the plus (h_+) and cross ($h_\times$) polarizations of the gravitational wave on matter particles lying on the plane perpendicular to the wave propagation.

Note that $Det(g_{\mu\nu}) = -1 + h_+^2 + h_\times^2 \cong -1$.

The inverse metric was found to be $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$. Hence, it can be written in the component form $g^{\mu\nu}$ as

$$g^{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1-h_+ & -h_\times & 0 \\ 0 & -h_\times & 1+h_+ & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.37}$$

To understand the effects of gravitational waves described by above metric, we need to see how they affect the motion of a collection of particles. We can not use the geodesic equation for a single particle as, to the first order in $h_{\mu\nu}$, we can always find the coordinates in the TT-gauge such that they are stationary [1]. Due to this, we consider a ring of particles lying on a plane perpendicular to the direction of the wave propagation. We consider the relative motion (as described by the geodesic deviation equation) of the particles forming a ring. When the equations of motion are solved, first when $A_+ = 0$ and then when $A_\times = 0$, we find that the particles oscillate back and forth in the shape of a '$+'$ in the former case and in the shape of a '$\times'$ in the later case. A depiction of such a motion is described in figure 3.1.

3.3 Conclusion

The weak metric perturbations $h_{\mu\nu}$ were defined in the background of Minkowski space-time ($\eta_{\mu\nu}$). Quantities like Christoffel symbols, Riemann curvature tensor, Ricci tensor and Ricci scalar were found to the linear order in $h_{\mu\nu}$. Einstein's field equations were solved for the metric with perturbations in the absence of any matter ($T_{\mu\nu} = 0$). Trace reversed metric perturbations $\bar{h}_{\mu\nu}$ were defined to simplify the form of the second order partial differential equation. Simplifying the expression further and choosing the Lorenz gauge yields the homogeneous wave equation in trace-reversed metric perturbations, which is given by $\Box\bar{h}_{\mu\nu} = 0$ in vacuum. The remaining gauge freedom was used by choosing the transverse-traceless (TT) gauge. By solving the homogeneous wave equation using the standard solution and taking the direction of wave propagation to be in the positive Z-direction, it was shown that there are only two independent components of $h^{TT}_{\mu\nu}$. Further calculation showed that the $h^{TT}_{\mu\nu}$ can be divided in two polarizations, the 'plus' (h_+) polarization and the 'cross' ($h_\times$) polarization, as they cause specific oscillations in the matter particles lying in the plane perpendicular to the direction of wave propagation. It was shown that the metric $g_{\mu\nu}$ in Eq. (3.37) describes linearized gravitational waves propagating in the positive Z-direction.

Chapter 4

Interaction of Electromagnetic waves and Gravitational waves

4.1 Introduction

In recent years the gravitational wave detection with LIGO [4] has opened a new window for the study of the Universe. Gravitational waves from distant events are arriving on earth and are being studied and analyzed using the LIGO detector. The Space Laser Interferometer will also be functional in the near future [31]. In this circumstance, it is important to study the interaction of gravitational waves with matter, and predict new observations. The detectors can search for the predictions. Further primordial gravitational waves [7, 27] have not been detected yet and new interactions might shed some light on the stochastic relics from early universe cosmology. Here we analyze and obtain a new perturbation mode (which was previously found for scalar field particles and neutrinos) for electromagnetic waves. This work is very important as electromagnetic wave interactions with gravitational waves are being investigated in the interferometers.

4.2 When $h_\times$ polarization is absent

Now we will study the interaction of gravitational waves with electromagnetic waves by solving for the electromagnetic gauge field in the background metric of the gravitational wave. We have already introduced the equation of motion of the electromagnetic field in chapter 2 equation and Eq. (2.33). As we know the gravitational wave has two different polarizations and for the simplicity of the calculation, we will first consider the case when the cross polarization is absent from the gravitational wave propagating in the positive Z-direction. The metric tensor describing such a gravitational wave will be obtained by substituting $h_\times = 0$ in Eq.(3.37). Hence, the metric tensor $g^{\mu\nu}$ can be given by

$$g^{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1-h_+ & 0 & 0 \\ 0 & 0 & 1+h_+ & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.1}$$

where $h_+ = A_+ \cos(\omega_g z - \omega_g t)$.

To describe the nature of the electromagnetic field in this nearly flat spacetime with perturbations $h_{\mu\nu}$, we will find the correct form of the electromagnetic field strength tensor. We know that the contravariant electromagnetic field strength tensor components can be written as $F^{\alpha\beta} = g^{\alpha\mu} g^{\beta\nu} F_{\mu\nu}$, where the metric $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$. Using this, we can simplify the contravariant electromagnetic field strength tensor as follows:

$$\begin{aligned} F^{\alpha\beta} &= g^{\alpha\mu} g^{\beta\nu} F_{\mu\nu}, \\ \therefore F^{\alpha\beta} &= (\eta^{\alpha\mu} - h^{\alpha\mu})(\eta^{\beta\nu} - h^{\beta\nu}) F_{\mu\nu}, \\ \therefore F^{\alpha\beta} &= \eta^{\alpha\mu}\eta^{\beta\nu} F_{\mu\nu} - \eta^{\alpha\mu} h^{\beta\nu} F_{\mu\nu} - \eta^{\beta\nu} h^{\alpha\mu} F_{\mu\nu}. \end{aligned} \tag{4.2}$$

In the last step of the above calculation, the higher order terms of $h_{\mu\nu}$ is neglected. The above equation describes the correct nature of electromagnetic field for the given spacetime. Now to understand the behaviour of electromagnetic wave, we will find the expression $\partial_\beta F^{\alpha\beta} = 0$ for each free index. We have,

$$\boxed{\partial_\beta F^{\alpha\beta} = \eta^{\alpha\mu}\eta^{\beta\nu}\partial_\beta F_{\mu\nu} - \eta^{\alpha\mu}\partial_\beta (h^{\beta\nu} F_{\mu\nu}) - \eta^{\beta\nu}\partial_\beta (h^{\alpha\mu} F_{\mu\nu}) = 0}. \tag{4.3}$$

Using the definition of $F_{\mu\nu}$ from above, we can simplify each term of the above equation separately.

For the first term of 4.3:

$$\begin{aligned}
\eta^{\alpha\mu}\eta^{\beta\nu}\partial_\beta F_{\mu\nu} &= \eta^{\alpha\mu}\eta^{\beta\nu}\partial_\beta(\partial_\mu A_\nu - \partial_\nu A_\mu), \\
\therefore \eta^{\alpha\mu}\eta^{\beta\nu}\partial_\beta F_{\mu\nu} &= \eta^{\alpha\mu}\partial^\nu(\partial_\mu A_\nu - \partial_\nu A_\mu), \\
\therefore \eta^{\alpha\mu}\eta^{\beta\nu}\partial_\beta F_{\mu\nu} &= \eta^{\alpha\mu}\partial_\mu(\partial^\nu A_\nu) - \eta^{\alpha\mu}\partial_\nu\partial^\nu A_\mu, \\
\therefore \eta^{\alpha\mu}\eta^{\beta\nu}\partial_\beta F_{\mu\nu} &= -\eta^{\alpha\mu}\Box A_\mu \qquad (\because \partial^\nu A_\nu = 0). \quad (4.4)
\end{aligned}$$

For the second term of 4.3:

$$\begin{aligned}
\eta^{\alpha\mu}\partial_\beta(h^{\beta\nu}F_{\mu\nu}) &= \eta^{\alpha\mu}h^{\beta\nu}\partial_\beta F_{\mu\nu} + \eta^{\alpha\mu}F_{\mu\nu}\partial_\beta h^{\beta\nu}, \\
\therefore \eta^{\alpha\mu}\partial_\beta(h^{\beta\nu}F_{\mu\nu}) &= \eta^{\alpha\mu}h^{\beta\nu}\partial_\beta F_{\mu\nu} \qquad (\because \partial_\beta h^{\beta\nu} = 0). \quad (4.5)
\end{aligned}$$

For the third term of 4.3:

$$\begin{aligned}
\eta^{\beta\nu}\partial_\beta(h^{\alpha\mu}F_{\mu\nu}) &= \eta^{\beta\nu}h^{\alpha\mu}\partial_\beta F_{\mu\nu} + \eta^{\beta\nu}F_{\mu\nu}\partial_\beta h^{\alpha\mu}, \\
\therefore \eta^{\beta\nu}\partial_\beta(h^{\alpha\mu}F_{\mu\nu}) &= h^{\alpha\mu}\partial_\mu\partial^\nu A_\nu - h^{\alpha\mu}\partial_\nu\partial^\nu A_\mu + \eta^{\beta\nu}F_{\mu\nu}\partial_\beta h^{\alpha\mu}, \\
\therefore \eta^{\beta\nu}\partial_\beta(h^{\alpha\mu}F_{\mu\nu}) &= -h^{\alpha\mu}\Box A_\mu + \eta^{\beta\nu}F_{\mu\nu}\partial_\beta h^{\alpha\mu} \qquad (\because \partial^\nu A_\nu = 0). \quad (4.6)
\end{aligned}$$

Substituting the results (4.4), (4.5) and (4.6) in Eq. (4.3), we obtain

$$\begin{aligned}
\partial_\beta F^{\alpha\beta} = -\eta^{\alpha\mu}\partial_\nu\partial^\nu A_\mu - \eta^{\alpha\mu}h^{\beta\nu}\partial_\beta F_{\mu\nu} + h^{\alpha\mu}\partial_\nu\partial^\nu A_\mu - \eta^{\beta\nu}F_{\mu\nu}\partial_\beta h^{\alpha\mu} &= 0, \\
\therefore -\eta^{\alpha\mu}\partial_\nu\partial^\nu(\overline{A}_\mu + \widetilde{A}_\mu) - \eta^{\alpha\mu}h^{\beta\nu}\partial_\beta F_{\mu\nu} + h^{\alpha\mu}\partial_\nu\partial^\nu(\overline{A}_\mu + \widetilde{A}_\mu) - \eta^{\beta\nu}F_{\mu\nu}\partial_\beta h^{\alpha\mu} &= 0, \\
\therefore -\eta^{\alpha\mu}\partial_\nu\partial^\nu\widetilde{A}_\mu - \eta^{\alpha\mu}h^{\beta\nu}\partial_\beta F_{\mu\nu} + h^{\alpha\mu}\partial_\nu\partial^\nu\widetilde{A}_\mu - \eta^{\beta\nu}F_{\mu\nu}\partial_\beta h^{\alpha\mu} &= 0, \\
\therefore \eta^{\alpha\mu}\Box\widetilde{A}_\mu + \eta^{\alpha\mu}h^{\beta\nu}\partial_\beta F_{\mu\nu} + \eta^{\beta\nu}F_{\mu\nu}\partial_\beta h^{\alpha\mu} &= 0. \quad (4.7)
\end{aligned}$$

Here, $\overline{A}$ represents the unperturbed electromagnetic four-potential components and $\widetilde{A}$ represents the perturbed electromagnetic four-potential components (note that $|\overline{A}_\mu| \gg |\widetilde{A}_\mu|$). In the second last step, the term $|h^{\alpha\mu}\Box\widetilde{A}_\mu|$ is neglected due to both terms being very small.
In the expression above, α is the only free index which can take values from 0 to 3. Let us find the set of four inhomogeneous wave equations for each value of α. We will neglect the higher order terms of the small quantities during the simplification.
When the free index $\alpha = 0$:

$$\begin{aligned}
\eta^{00}\Box\widetilde{A}_0 + \eta^{00}h^{11}\partial_1 F_{01} + \eta^{00}h^{22}\partial_2 F_{02} = 0,\\
\therefore \Box\widetilde{A}_0 + h^{11}\partial_1(\partial_0 A_1 - \partial_1 A_0) + h^{22}\partial_2(\partial_0 A_2 - \partial_2 A_0) = 0,\\
\therefore \Box\widetilde{A}_0 + h^{11}\partial_0\partial_1(\overline{A}_1 + \widetilde{A}_1) + h^{22}\partial_0\partial_2(\overline{A}_2 + \widetilde{A}_2) - (h^{11}\partial_1^2 + h^{22}\partial_2^2)(\overline{A}_0 + \widetilde{A}_0) = 0,\\
\therefore \Box\widetilde{A}_0 + h^{11}\partial_0\partial_1\overline{A}_1 + h^{22}\partial_0\partial_2\overline{A}_2 - (h^{11}\partial_1^2 + h^{22}\partial_2^2)\overline{A}_0 = 0. \quad (4.8)
\end{aligned}$$

When the free index $\alpha = 1$:

$$\begin{aligned}
\eta^{11}\Box\widetilde{A}_1 + \eta^{11}h^{11}\partial_1 F_{11} + \eta^{11}h^{22}\partial_2 F_{12} + \eta^{00}F_{10}\partial_0 h^{11} + \eta^{33}F_{13}\partial_3 h^{11} = 0,\\
\therefore \Box\widetilde{A}_1 + h^{22}\partial_1\partial_2\overline{A}_2 - h^{22}\partial_2^2\overline{A}_1 + \partial_0 h^{11}(\partial_0\overline{A}_1 - \partial_1\overline{A}_0) + \partial_3 h^{11}(\partial_1\overline{A}_3 - \partial_3\overline{A}_1) = 0. \quad (4.9)
\end{aligned}$$

When the free index $\alpha = 2$:

$$\begin{aligned}
\eta^{22}\Box\widetilde{A}_2 + \eta^{22}h^{11}\partial_1 F_{21} + \eta^{22}h^{22}\partial_2 F_{22} + \eta^{00}F_{20}\partial_0 h^{22} + \eta^{33}F_{23}\partial_3 h^{22} = 0,\\
\therefore \Box\widetilde{A}_2 + h^{11}\partial_2\partial_1\overline{A}_1 - h^{11}\partial_1^2\overline{A}_2 + \partial_0 h^{22}(\partial_0\overline{A}_2 - \partial_2\overline{A}_0) + \partial_3 h^{22}(\partial_2\overline{A}_3 - \partial_3\overline{A}_2) = 0. \quad (4.10)
\end{aligned}$$

When the free index $\alpha = 3$:

$$\begin{aligned}
\eta^{33}\Box\widetilde{A}_3 + \eta^{33}h^{11}\partial_1 F_{31} + \eta^{33}h^{22}\partial_2 F_{32} = 0,\\
\therefore \Box\widetilde{A}_3 + h^{11}\partial_3\partial_1\overline{A}_1 + h^{22}\partial_3\partial_2\overline{A}_2 - (h^{11}\partial_1^2 + h^{22}\partial_2^2)\overline{A}_3 = 0. \quad (4.11)
\end{aligned}$$

To solve the above inhomogeneous wave equations and find the perturbations in the electromagnetic four-potential components, let's consider an electromagnetic wave propagating in the X-direction with angular frequency ω_e. Note that the gravitational wave is propagating in the positive Z-direction.

Consider the electromagnetic wave described by

$$\begin{aligned}
\vec{B} &= B_{0y}\cos(\omega_e x - \omega_e t)\hat{j}, \\
\therefore B_y &= B_{0y}\cos(\omega_e x - \omega_e t), \\
\therefore \partial_3\overline{A}_1 - \partial_1\overline{A}_3 &= B_{0y}\cos(\omega_e x - \omega_e t), \qquad (\because \vec{\nabla}\times\vec{\overline{A}} = \vec{B}) \\
\therefore \partial_1\overline{A}_3 &= -B_{0y}\cos(\omega_e x - \omega_e t) \qquad (\because \partial_3\overline{A}_1 = 0),
\end{aligned} \tag{4.12}$$

where B_{0y} is the amplitude of the magnetic field. Substituting the electromagnetic four-potential components for a plane polarized electromagnetic wave propagating in the positive X-direction in Eq. (4.8), we obtain

$$\boxed{\therefore \Box\widetilde{A}_0 = 0}. \tag{4.13}$$

Substituting the electromagnetic four-potential components in Eq. (4.9), we obtain

$$\begin{aligned}
\Box\widetilde{A}_1 &= -\partial_3 h^{11}\partial_1\overline{A}_3, \\
\therefore \Box\widetilde{A}_1 &= -\partial_3[A_+\cos(\omega_g z - \omega_g t)]\partial_1\overline{A}_3.
\end{aligned}$$

$$\boxed{\therefore \Box\widetilde{A}_1 = -A_+B_{0y}\omega_g\sin(\omega_g z - \omega_g t)\cos(\omega_e x - \omega_e t)}. \tag{4.14}$$

Substituting the electromagnetic four-potential components in Eq. (4.10), we obtain

$$\boxed{\therefore \Box\widetilde{A}_2 = 0}. \tag{4.15}$$

Substituting the electromagnetic four-potential components in Eq. (4.11), we obtain

$$\begin{aligned} \Box\widetilde{A}_3 &= h^{11}\partial_1^2\overline{A}_3, \\ \therefore \Box\widetilde{A}_3 &= [A_+\cos(\omega_g z-\omega_g t)]\partial_1^2\overline{A}_3. \end{aligned}$$

$$\boxed{\therefore \Box\widetilde{A}_3 = A_+ B_{0y}\omega_e\cos(\omega_g z-\omega_g t)\sin(\omega_e x-\omega_e t)}. \tag{4.16}$$

We find when only the h_+ is present, the perturbations are produced in the X and Z directions. The solutions for the above inhomogeneous wave equations are given at the end of section 4.4 in Eq. (4.52).

4.3 When h_+ polarization is absent

Now we will consider the case when the plus polarization is absent from the gravitational wave propagating in the positive Z-direction. The metric tensor describing such a gravitational wave will be obtained by substituting $h_+ = 0$ in Eq. (3.37). Hence, the metric perturbation $h_{\mu\nu}$ and metric tensor $g^{\mu\nu}$ can be given by

$$h_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & h_\times & 0 \\ 0 & h_\times & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{4.17}$$

$$g^{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & -h_\times & 0 \\ 0 & -h_\times & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{4.18}$$

where $h_\times = A_\times \cos(\omega_g z - \omega_g t + \delta)$.

Now to find of perturbation in the electromagnetic-four potentials of the electromagnetic wave, we will substitute the given form of the metric in the Maxwell equations describing wave as shown in Eq. (4.7).

When the free index $\alpha = 0$ in Eq. (4.7) :

$$\therefore \eta^{\alpha\mu}\Box\widetilde{A}_\mu + \eta^{\alpha\mu}h^{\beta\nu}\partial_\beta F_{\mu\nu} + \eta^{\beta\nu}F_{\mu\nu}\partial_\beta h^{\alpha\mu} = 0,$$

$$\therefore \eta^{00}\Box\widetilde{A}_0 + \eta^{00}h^{12}\partial_1 F_{02} + \eta^{00}h^{21}\partial_2 F_{01} = 0,$$

$$\therefore \Box\widetilde{A}_0 + h^{12}\partial_1(\partial_0 A_2 - \partial_2 A_0) + h^{21}\partial_2(\partial_0 A_1 - \partial_1 A_0) = 0,$$

$$\therefore \Box\widetilde{A}_0 + h^{12}\partial_0\partial_1(\widetilde{A}_2 + \overline{A}_2) - h^{12}\partial_1\partial_2(\widetilde{A}_0 + \overline{A}_0),$$

$$+h^{21}\partial_0\partial_2(\widetilde{A}_1 + \overline{A}_1) - h^{21}\partial_1\partial_2(\widetilde{A}_0 + \overline{A}_0) = 0,$$

$$\therefore \Box\widetilde{A}_0 + h^{12}\partial_0\partial_1\overline{A}_2 - h^{12}\partial_1\partial_2\overline{A}_0 + h^{21}\partial_0\partial_2\overline{A}_1 - h^{21}\partial_1\partial_2\overline{A}_0 = 0. \tag{4.19}$$

The higher order terms in the small quantities are neglected in the last step.

When the free index $\alpha = 1$ in Eq. (4.7) :

$$\therefore \eta^{\alpha\mu}\Box\widetilde{A}_\mu + \eta^{\alpha\mu}h^{\beta\nu}\partial_\beta F_{\mu\nu} + \eta^{\beta\nu}F_{\mu\nu}\partial_\beta h^{\alpha\mu} = 0,$$

$$\therefore \eta^{11}\Box\widetilde{A}_1 + \eta^{11}h^{12}\partial_1 F_{12} + \eta^{11}h^{21}\partial_2 F_{11} + \eta^{00}F_{20}\partial_0 h^{12} + \eta^{33}F_{23}\partial_3 h^{12} = 0,$$

$$\therefore \Box\widetilde{A}_1 + h^{12}\partial_1(\partial_1 A_2 - \partial_2 A_1) - (\partial_2 A_0 - \partial_0 A_2)\partial_0 h^{12} + (\partial_2 A_3 - \partial_3 A_2)\partial_3 h^{12} = 0,$$

$$\therefore \Box\widetilde{A}_1 - h^{12}\partial_1\partial_2\overline{A}_1 + h^{12}\partial_1^2\overline{A}_2 - \partial_2\overline{A}_0\partial_0 h^{12} + \partial_0\overline{A}_2\partial_0 h^{12} + \partial_2\overline{A}_3\partial_3 h^{12} - \partial_3\overline{A}_2\partial_3 h^{12} = 0. \tag{4.20}$$

When the free index $\alpha = 2$ in Eq. (4.7) :

$$\therefore \eta^{\alpha\mu}\Box\widetilde{A}_\mu + \eta^{\alpha\mu}h^{\beta\nu}\partial_\beta F_{\mu\nu} + \eta^{\beta\nu}F_{\mu\nu}\partial_\beta h^{\alpha\mu} = 0,$$

$$\therefore \eta^{22}\Box\widetilde{A}_2 + \eta^{22}h^{12}\partial_1 F_{22} + \eta^{22}h^{21}\partial_2 F_{21} + \eta^{00}F_{10}\partial_0 h^{21} + \eta^{33}F_{13}\partial_3 h^{21} = 0,$$

$$\therefore \Box\widetilde{A}_2 + h^{21}\partial_2(\partial_2 A_1 - \partial_1 A_2) - (\partial_1 A_0 - \partial_0 A_1)\partial_0 h^{21} + (\partial_1 A_3 - \partial_3 A_1)\partial_3 h^{21} = 0,$$

$$\therefore \Box\widetilde{A}_2 + h^{21}\partial_2^2\overline{A}_1 - h^{21}\partial_1\partial_2\overline{A}_2 - \partial_1\overline{A}_0\partial_0 h^{21} + \partial_0\overline{A}_1\partial_0 h^{21} + \partial_1\overline{A}_3\partial_3 h^{21} - \partial_3\overline{A}_1\partial_3 h^{21} = 0. \tag{4.21}$$

When the free index $\alpha = 3$ in Eq. (4.7) :

$$\therefore \eta^{\alpha\mu}\Box\widetilde{A}_\mu + \eta^{\alpha\mu}h^{\beta\nu}\partial_\beta F_{\mu\nu} + \eta^{\beta\nu}F_{\mu\nu}\partial_\beta h^{\alpha\mu} = 0,$$

$$\therefore \eta^{33}\Box\widetilde{A}_3 + \eta^{33}h^{12}\partial_1 F_{32} + \eta^{33}h^{21}\partial_2 F_{31} = 0,$$

$$\therefore \Box\widetilde{A}_3 + h^{12}\partial_1(\partial_3 A_2 - \partial_2 A_3) + h^{21}\partial_2(\partial_3 A_1 - \partial_1 A_3) = 0,$$

$$\therefore \Box\widetilde{A}_3 + h^{12}\partial_1\partial_3\overline{A}_2 - h^{12}\partial_1\partial_2\overline{A}_3 + h^{21}\partial_2\partial_3\overline{A}_1 - h^{21}\partial_2\partial_1\overline{A}_3 = 0. \tag{4.22}$$

To solve the above inhomogeneous wave equations and find the perturbations in the electromagnetic four-potential components, let's consider an electromagnetic wave propagating in the positive X-direction with angular frequency ω_e. Note that the gravitational wave is propagating in the positive Z-direction.

Substituting the electromagnetic four-potential components for a plane polarized electromagnetic wave propagating in the positive X-direction in Eq. (4.19), we obtain

$$\therefore \Box\widetilde{A}_0 + h^{12}\partial_0\partial_1\overline{A}_2 + h^{21}\partial_0\partial_2\overline{A}_1 = 0,$$

$$\boxed{\therefore \Box\widetilde{A}_0 = 0}. \tag{4.23}$$

Substituting the electromagnetic four-potential components in Eq. (4.20), we obtain

$$\therefore \Box\widetilde{A}_1 + h^{12}\partial_1^2\overline{A}_2 + \partial_0\overline{A}_2\partial_0 h^{12} = 0,$$

$$\boxed{\therefore \Box\widetilde{A}_1 = 0}. \tag{4.24}$$

Substituting the electromagnetic four-potential components in Eq. (4.21), we obtain

$$\begin{aligned}
&\therefore \Box\widetilde{A}_2 = +\partial_1\overline{A}_0\partial_0 h^{21} - \partial_0\overline{A}_1\partial_0 h^{21} - \partial_1\overline{A}_3\partial_3 h^{21}, \\
&\therefore \Box\widetilde{A}_2 = -\partial_1\overline{A}_3\partial_3 h^{21} \qquad\qquad (\because \partial_1\overline{A}_0 = 0), \\
&\therefore \Box\widetilde{A}_2 = -\partial_1\overline{A}_3\partial_3(A_\times \cos(\omega_g z - \omega_g t + \delta)),
\end{aligned}$$

$$\boxed{\therefore \Box\widetilde{A}_2 = -A_\times \omega_g B_{0y} \sin(\omega_g z - \omega_g t + \delta)\cos(\omega_e x - \omega_e t)}. \tag{4.25}$$

Substituting the electromagnetic four-potential components in Eq. (4.22), we obtain

$$\therefore \Box\widetilde{A}_3 + h^{12}\partial_1\partial_3\overline{A}_2 - h^{12}\partial_1\partial_2\overline{A}_3 + h^{21}\partial_2\partial_3\overline{A}_1 - h^{21}\partial_2\partial_1\overline{A}_3 = 0,$$

$$\boxed{\therefore \Box\widetilde{A}_3 = 0}. \tag{4.26}$$

The solutions for the above inhomogeneous wave equations are calculated in the section 4.4 in Eq. (4.52). We find that the $h_\times$ interacts with only the y-component of the gauge field.

4.4 When both polarization h_+ and $h_\times$ are present

Now we will consider the case when both, the plus and the cross, polarizations are present in the gravitational wave propagating in the positive Z-direction. Hence, the metric perturbation $h_{\mu\nu}$ and metric tensor $g^{\mu\nu}$ can be given by

$$h^{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{4.27}$$

In the above equations $h_+ = A_+ \cos(\omega_g z - \omega_g t)$ and $h_\times = A_\times \cos(\omega_g z - \omega_g t + \delta)$.

As shown in the previous sections, the electromagnetic wave equation in curved geometry is given by Eq. (4.7)

$$\boxed{\therefore \eta^{\alpha\mu}\Box\widetilde{A}_\mu + \eta^{\alpha\mu}h^{\beta\nu}\partial_\beta F_{\mu\nu} + \eta^{\beta\nu}F_{\mu\nu}\partial_\beta h^{\alpha\mu} = 0}.$$

<u>When the free index $\alpha = 0$ in Eq. (4.7)</u>:

$$\begin{aligned}
\therefore \eta^{\alpha\mu}\Box\widetilde{A}_\mu + \eta^{\alpha\mu}h^{\beta\nu}\partial_\beta F_{\mu\nu} + \eta^{\beta\nu}F_{\mu\nu}\partial_\beta h^{\alpha\mu} &= 0, \\
\therefore \eta^{00}\Box\widetilde{A}_0 + \eta^{00}h^{11}\partial_1 F_{01} + \eta^{00}h^{12}\partial_1 F_{02} + \eta^{00}h^{21}\partial_2 F_{01} + \eta^{00}h^{22}\partial_2 F_{02} &= 0, \\
\therefore \Box\widetilde{A}_0 + h^{11}\partial_1(\partial_0 A_1 - \partial_1 A_0) + h^{12}\partial_1(\partial_0 A_2 - \partial_2 A_0) + h^{21}\partial_2(\partial_0 A_1 - \partial_1 A_0)& \\
+ h^{22}\partial_2(\partial_0 A_2 - \partial_2 A_0) &= 0, \\
\therefore \Box\widetilde{A}_0 + h^{11}\partial_0\partial_1(\widetilde{A}_1 + \overline{A}_1) - h^{11}\partial_1^2(\widetilde{A}_0 + \overline{A}_0) + h^{12}\partial_0\partial_1(\widetilde{A}_2 + \overline{A}_2) - h^{12}\partial_1\partial_2(\widetilde{A}_0 + \overline{A}_0)& \\
+ h^{21}\partial_0\partial_2(\widetilde{A}_1 + \overline{A}_1) - h^{21}\partial_1\partial_2(\widetilde{A}_0 + \overline{A}_0) + h^{22}\partial_0\partial_2(\widetilde{A}_2 + \overline{A}_2) - h^{22}\partial_2^2(\widetilde{A}_0 + \overline{A}_0) &= 0, \\
\therefore \Box\widetilde{A}_0 + h^{11}\partial_0\partial_1\overline{A}_1 - h^{11}\partial_1^2\overline{A}_0 + h^{12}\partial_0\partial_1\overline{A}_2 - h^{12}\partial_1\partial_2\overline{A}_0 + h^{21}\partial_0\partial_2\overline{A}_1 - h^{21}\partial_1\partial_2\overline{A}_0& \\
+ h^{22}\partial_0\partial_2\overline{A}_2 - h^{22}\partial_2^2\overline{A}_0 &= 0.
\end{aligned} \tag{4.28}$$

(because $|h^{\alpha\beta}\partial_\mu\partial_\nu\widetilde{A}_\gamma(h)| \approx 0$ due to both terms being function of $h_{\mu\nu}$).

When the free index $\alpha = 1$ in Eq. (4.7) :

$$\begin{aligned}
\therefore \eta^{\alpha\mu}\Box\widetilde{A}_\mu + \eta^{\alpha\mu}h^{\beta\nu}\partial_\beta F_{\mu\nu} + \eta^{\beta\nu}F_{\mu\nu}\partial_\beta h^{\alpha\mu} = 0,\\
\therefore \eta^{11}\Box\widetilde{A}_1 + \eta^{11}h^{11}\partial_1 F_{11} + \eta^{11}h^{12}\partial_1 F_{12} + \eta^{11}h^{21}\partial_2 F_{11} + \eta^{11}h^{22}\partial_2 F_{12}\\
+\eta^{00}F_{10}\partial_0 h^{11} + \eta^{33}F_{13}\partial_3 h^{11} + \eta^{00}F_{20}\partial_0 h^{12} + \eta^{33}F_{23}\partial_3 h^{12} = 0,\\
\therefore \Box\widetilde{A}_1 - h^{12}\partial_1\partial_2\overline{A}_1 + h^{12}\partial_1^2\overline{A}_2 - h^{22}\partial_2^2\overline{A}_1 + h^{22}\partial_1\partial_2\overline{A}_2 - \partial_1\overline{A}_0\partial_0 h^{11} + \partial_0\overline{A}_1\partial_0 h^{11}\\
+\partial_1\overline{A}_3\partial_3 h^{11} - \partial_3\overline{A}_1\partial_3 h^{11} - \partial_2\overline{A}_0\partial_0 h^{12} + \partial_0\overline{A}_2\partial_0 h^{12} + \partial_2\overline{A}_3\partial_3 h^{12} - \partial_3\overline{A}_2\partial_3 h^{12} = 0.
\end{aligned} \tag{4.29}$$

When the free index $\alpha = 2$ in Eq. (4.7) :

$$\begin{aligned}
\therefore \eta^{\alpha\mu}\Box\widetilde{A}_\mu + \eta^{\alpha\mu}h^{\beta\nu}\partial_\beta F_{\mu\nu} + \eta^{\beta\nu}F_{\mu\nu}\partial_\beta h^{\alpha\mu} = 0,\\
\therefore \eta^{22}\Box\widetilde{A}_2 + \eta^{22}h^{11}\partial_1 F_{21} + \eta^{22}h^{12}\partial_1 F_{22} + \eta^{22}h^{22}\partial_2 F_{22} + \eta^{22}h^{21}\partial_2 F_{21}\\
+\eta^{00}F_{10}\partial_0 h^{21} + \eta^{33}F_{13}\partial_3 h^{21} + \eta^{00}F_{20}\partial_0 h^{22} + \eta^{33}F_{23}\partial_3 h^{22} = 0,\\
\therefore \Box\widetilde{A}_2 + h^{11}\partial_1\partial_2\overline{A}_1 - h^{11}\partial_1^2\overline{A}_2 + h^{21}\partial_2^2\overline{A}_1 - h^{21}\partial_1\partial_2\overline{A}_2 - \partial_1\overline{A}_0\partial_0 h^{21} + \partial_0\overline{A}_1\partial_0 h^{21}\\
+\partial_1\overline{A}_3\partial_3 h^{21} - \partial_3\overline{A}_1\partial_3 h^{21} - \partial_2\overline{A}_0\partial_0 h^{22} + \partial_0\overline{A}_2\partial_0 h^{22} + \partial_2\overline{A}_3\partial_3 h^{22} - \partial_3\overline{A}_2\partial_3 h^{22} = 0.
\end{aligned} \tag{4.30}$$

When the free index $\alpha = 3$ in Eq. (4.7) :

$$\begin{aligned}
\therefore \eta^{\alpha\mu}\Box\widetilde{A}_\mu + \eta^{\alpha\mu}h^{\beta\nu}\partial_\beta F_{\mu\nu} + \eta^{\beta\nu}F_{\mu\nu}\partial_\beta h^{\alpha\mu} = 0,\\
\therefore \eta^{33}\Box\widetilde{A}_3 + \eta^{33}h^{11}\partial_1 F_{31} + \eta^{33}h^{12}\partial_1 F_{32} + \eta^{33}h^{21}\partial_2 F_{31} + \eta^{33}h^{22}\partial_2 F_{32} = 0,\\
\therefore \Box\widetilde{A}_3 + h^{11}\partial_1(\partial_3 A_1 - \partial_1 A_3) + h^{12}\partial_1(\partial_3 A_2 - \partial_2 A_3) + h^{21}\partial_2(\partial_3 A_1 - \partial_1 A_3)\\
+h^{22}\partial_2(\partial_3 A_2 - \partial_2 A_3) = 0,\\
\therefore \Box\widetilde{A}_3 + h^{11}\partial_1\partial_3\overline{A}_1 - h^{11}\partial_1^2\overline{A}_3 + h^{12}\partial_1\partial_3\overline{A}_2 - h^{12}\partial_1\partial_2\overline{A}_3 + h^{21}\partial_2\partial_3\overline{A}_1\\
-h^{21}\partial_2\partial_1\overline{A}_3 + h^{22}\partial_2\partial_3\overline{A}_2 - h^{22}\partial_2^2\overline{A}_3 = 0.
\end{aligned} \tag{4.31}$$

Let's consider an electromagnetic wave propagating in the X-direction with angular frequency ω_e. Note that the gravitational wave is propagating in the Z direction.
Consider the electromagnetic wave described by

$$\vec{B} = B_{0y}\cos(\omega_e x - \omega_e t)\,\hat{j}. \tag{4.32}$$

Then we can use Eq. (4.12) to find the explicit form of the inhomogeneous wave equations. For such an electromagnetic wave, we obtain $\Box\widetilde{A}_\mu$ for each value of the free index μ:

$$\Box\widetilde{A}_0 = 0 \tag{4.33}$$

$$\Box\widetilde{A}_1 = -A_+ B_{0y}\omega_g \sin(\omega_g z - \omega_g t)\cos(\omega_e x - \omega_e t), \tag{4.34}$$

$$\Box\widetilde{A}_2 = A_\times \omega_g B_{0y}\sin(\omega_g z - \omega_g t + \delta)\cos(\omega_e x - \omega_e t), \tag{4.35}$$

$$\Box\widetilde{A}_3 = A_+ B_{0y}\omega_e \cos(\omega_g z - \omega_g t)\sin(\omega_e x - \omega_e t). \tag{4.36}$$

Since we have

$$\Box\widetilde{A}_3 = h^{11}\partial_1^2\overline{A}_3,$$

where h^{11} and $\partial_1^2\overline{A}_3$ are given by

$$h^{11} = A_+\cos(\omega_g(z-t)),$$

$$\partial_1^2\overline{A}_3 = B_{0y}\omega_e\sin(\omega_e(x-t))$$

we find

$$\Box\widetilde{A}_3 = A_+ B_{0y}\omega_e\cos(\omega_g(z-t))\sin(\omega_e(x-t)) = h(\vec{x},t).$$

Using standard results on Cauchy problem and Duhamel's principle [21], one has :-

$$\widetilde{A}_3(\vec{x}) = -\frac{1}{4\pi}\int \frac{h(\vec{x}',t')}{r} d^3x' \tag{4.37}$$

$$= -\frac{A_+ B_{0y}\omega_e}{4\pi}\int \frac{\cos(\omega_g(z'-t'))\sin(\omega_e(x'-t'))}{r} d^3x', \tag{4.38}$$

where

$$r = |\vec{X}' - \vec{X}|,\ t' = t - r \ \text{ and } \ \vec{X}' = \vec{r} + \vec{X}. \tag{4.39}$$

Thus,

$$\begin{aligned}
\widetilde{A}_3(\vec{x}) &= -\frac{A_+ B_{0y}\omega_e}{4\pi}\int \left(\frac{e^{i\omega_g(z'-t')} + e^{-i\omega_g(z'-t')}}{2}\frac{e^{i\omega_e(x'-t')} - e^{-i\omega_e(x'-t')}}{2i}\right)\frac{d^3r}{r} \\
&= -\frac{A_+ B_{0y}\omega_e}{4\pi(4i)}\int \Big(e^{i\omega_g(z'-t')}e^{i\omega_e(x'-t')} - e^{i\omega_g(z'-t')}e^{-i\omega_e(x'-t')} + e^{-i\omega_g(z'-t')}e^{i\omega_e(x'-t')} \\
&\qquad - e^{-i\omega_g(z'-t')}e^{-i\omega_e(x'-t')}\Big)\frac{d^3r}{r} \\
&= -\frac{A_+ B_{0y}\omega_e}{4\pi(4i)}\Big(I_1(\omega_g,\omega_e) - I_1(\omega_g,-\omega_e) + I_1(-\omega_g,\omega_e) - I_1(-\omega_g,-\omega_e)\Big),
\end{aligned} \tag{4.40}$$

where

$$I_1(\omega_g,\omega_e) = \int \frac{e^{i\omega_g(z'-t')}e^{i\omega_e(x'-t')}}{r} d^3X'. \tag{4.41}$$

We simplify

$$I_1(\omega_g,\omega_e) = \int e^{i\omega_g(z+r\cos\theta-t+r)}e^{i\omega_e(x+r\sin\theta\cos\phi-t+r)} r dr \sin\theta d\theta d\phi \tag{4.42}$$

$$= e^{i\omega_g(z-t)}e^{i\omega_e(x-t)}\int e^{i\omega_g(r\cos\theta+r)}e^{i\omega_e(r\sin\theta\cos\phi+r)} r dr \sin\theta d\theta d\phi \tag{4.43}$$

$$= e^{i\omega_g(z-t)}e^{i\omega_e(x-t)} I_0, \tag{4.44}$$

where

$$I_0 = \int_{\phi=0}^{2\pi} e^{i(\omega_g+\omega_e)r} e^{i\omega_g r\cos\theta} e^{i\omega_e r\sin\theta\cos\phi} r dr \sin\theta d\theta d\phi \tag{4.45}$$

$$= \int e^{i(\omega_g+\omega_e)r} e^{i\omega_g r\cos\theta} (2\pi) J_0(\omega_e r\sin\theta) r dr \sin\theta d\theta, \tag{4.46}$$

which is obtained using Gradshteyn and Rhyzik [32]. Making a variable change of $x = \cos\theta$ and using Gradshteyn and Rhyzik [32] gives us,

$$I_0 = 2\pi \int_{x=-1}^{1} e^{i(\omega_g+\omega_e)r} e^{i\omega_g rx} J_0(\omega_e r\sqrt{1-x^2}) dx\ r dr \tag{4.47}$$

$$= 4\pi \int_{r=0}^{t} e^{i(\omega_g+\omega_e)r} \frac{\sin(\sqrt{\omega_g^2+\omega_e^2}r)}{\sqrt{\omega_g^2+\omega_e^2}} dr. \tag{4.48}$$

Making a notation change of $\overline{\omega} = \omega_g + \omega_e$ and $||\vec{\omega}|| = \sqrt{\omega_g^2+\omega_e^2}$ and integrating 4.47 using $\sin t = \frac{e^{it}-e^{-it}}{2i}$ yields

$$I_0 = \frac{4\pi}{||\vec{\omega}||}\frac{1}{2i}\left[e^{i\overline{\omega}t}\left(\frac{e^{i||\vec{\omega}||t}}{i(\overline{\omega}+||\vec{\omega}||)} - \frac{e^{-i||\vec{\omega}||t}}{i(\overline{\omega}-||\vec{\omega}||)}\right) - \frac{1}{i(\overline{\omega}+||\vec{\omega}||)} + \frac{1}{i(\overline{\omega}-||\vec{\omega}||)}\right],$$

which simplifies further to

$$I_0 = \frac{4\pi}{||\vec{\omega}||}\frac{1}{2i}\left[e^{i\overline{\omega}t}\left(\frac{2\overline{\omega}\sin(||\vec{\omega}||t)}{2\omega_g\omega_e} - \frac{2||\vec{\omega}||\cos(||\vec{\omega}||t)}{i2\omega_g\omega_e}\right) + \frac{2||\vec{\omega}||}{i2\omega_g\omega_e}\right] \tag{4.49}$$

$$= \frac{4\pi}{||\vec{\omega}||}\frac{1}{2i}\left[\frac{e^{i\overline{\omega}t}}{\omega_g\omega_e}\left(\overline{\omega}\sin(||\vec{\omega}||t) + i||\vec{\omega}||\cos(||\vec{\omega}||t)\right) + \frac{||\vec{\omega}||}{i\omega_g\omega_e}\right] \tag{4.50}$$

$$= 2\pi\left[-\frac{1}{\omega_g\omega_e} + \frac{e^{i\overline{\omega}t}}{\omega_g\omega_e}\left(\cos(||\vec{\omega}||t) - i\frac{\overline{\omega}}{||\vec{\omega}||}\sin(||\vec{\omega}||t)\right)\right]. \tag{4.51}$$

Substituting this back into Eq. (4.42), we get

$$-\frac{A_+B_{0y}\omega_e}{4\pi(4i)} I_1(\omega_g,\omega_e) = -\frac{A_+B_{0y}}{8i}\frac{e^{i\omega_g z}e^{i\omega_e x}}{\omega_g}\left(\cos(||\vec{\omega}||t) - i\frac{\overline{\omega}}{||\vec{\omega}||}\sin(||\vec{\omega}||t)\right) - \frac{A_+B_{0y}}{8i}\frac{e^{i\omega_g(z-t)}e^{i\omega_e(x-t)}}{\omega_g}.$$

We can find the value of $I_1(\omega_g,\omega_e)-I_1(\omega_g,-\omega_e)+I_1(-\omega_g,\omega_e)-I_1(-\omega_g,-\omega_e)$ using the result above. Similarly, we can find the value of the other components of A_μ in a similar way. The obtained perturbation solution for the x-component is given by

$$
\begin{aligned}
\widetilde{A}_1 = -\frac{B_{0y}A_+}{4\omega_e\sqrt{\omega_g^2+\omega_e^2}}\Big[(\omega_e+\omega_g)\cos(\omega_e x+\omega_g z)\sin(t\sqrt{\omega_e^2+\omega_g^2}) \\
-\sqrt{\omega_g^2+\omega_e^2}\,\sin(\omega_e x+\omega_g z)\cos(t\sqrt{\omega_g^2+\omega_e^2}) \\
+(\omega_e-\omega_g)\cos(\omega_e x-\omega_g z)\sin(t\sqrt{\omega_e^2+\omega_g^2})-\sqrt{\omega_g^2+\omega_e^2}\,\sin(\omega_e x-\omega_g z)\cos(t\sqrt{\omega_e^2+\omega_g^2}) \\
+\Big[\sqrt{\omega_g^2+\omega_e^2}\,\sin(\omega_e x+\omega_g z-t(\omega_g+\omega_e))+\sqrt{\omega_g^2+\omega_e^2}\,\sin(\omega_e x-\omega_g z+t(\omega_e-\omega_g))\Big].
\end{aligned}
\tag{4.52}
$$

The solution for $\widetilde{A}_2$ would be the same as this, except for $A_+\to A_\times$ and $\omega_g z\to\omega_g z+\delta$. The solution for $\widetilde{A}_z$ will also be similar; which we can obtain by using phase shifts of $\omega_e x\to\omega_e x+\pi/2$ and $\omega_g z\to\omega_g z+\pi/2$ and in the coefficient $B_{0y}A_+/(4\omega_e\sqrt{\omega_e^2+\omega_g^2})\to B_{0y}A_+/(4\omega_g\sqrt{\omega_e^2+\omega_g^2})$. It is easy to see that from the pertubed gauge potentials found above, we can find the perturbed $\vec{E}$ and $\vec{B}$ fields. Note that the perturbed wave is not a plane-wave and has propagation in X and Z directions. This can be interpreted as scattering of electromagentic radiation due to gravitational wave. And the perturbed electromagentic wave is no longer plane polarized.

We have found the expected mode in the gauge potential perturbation as it was found for the scalar field interaction in the paper [21]. We have an explicit form of the gauge potential with the frequency of the new modes $\sqrt{\omega_g^2+\omega_e^2}$. We have made a schematic plot of the functional form of the perturbations to show how the mode fluctuates in time and space. We have taken $\omega_g=3$ and $\omega_e=4$ for the schematic plot 4.1, these frequencies are not realistic. For a realistic frequency, the peak frequency of the CMB is 160 GHz. The gravitational waves that can be detected by the LIGO has approximate frequency of 100 Hz.

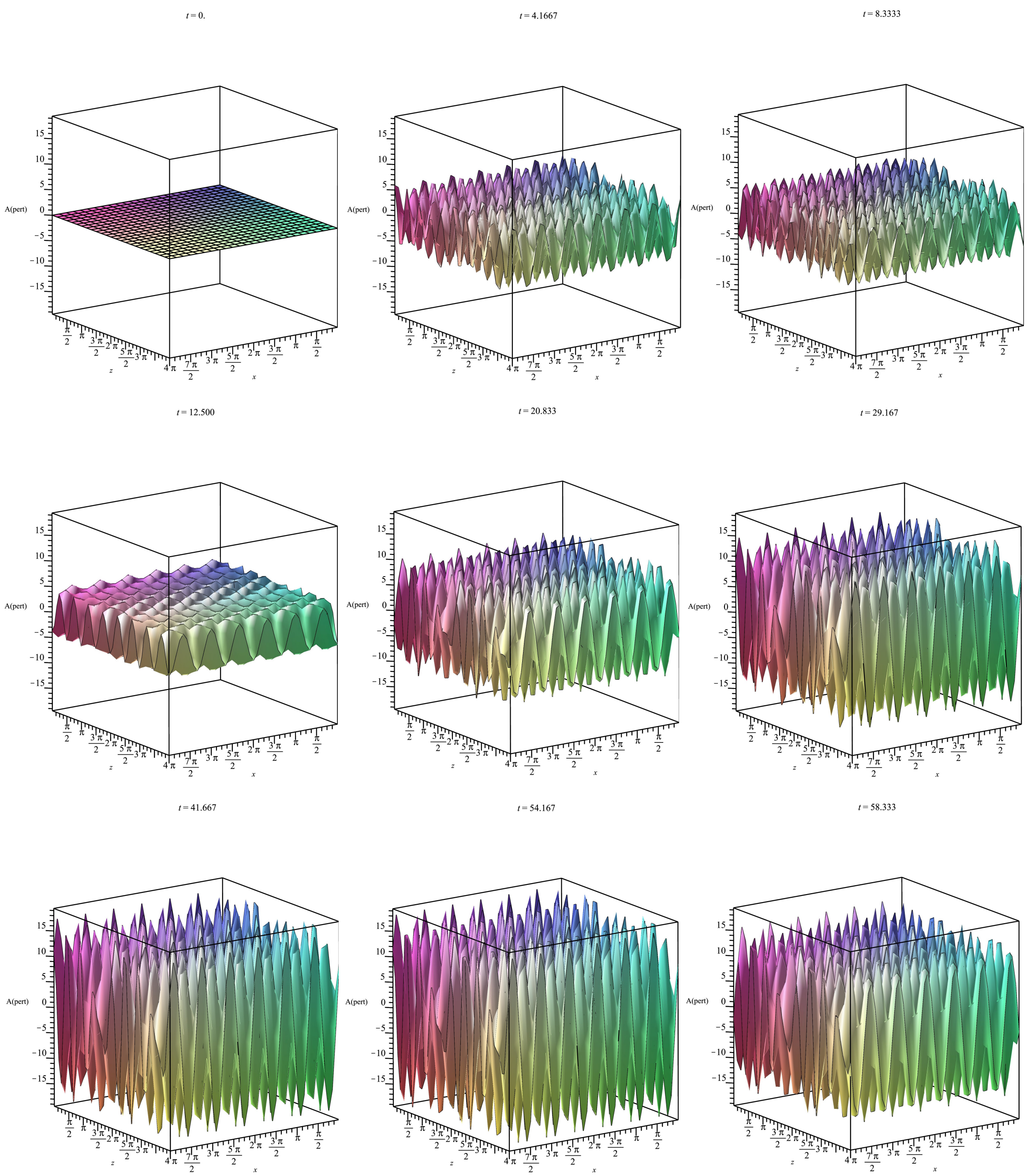

Figure 4.1: The perturbations in the x-component of the electromagnetic four-potential over time. For the purposes of the plot in 4.1 $\omega_g = 3$ and $\omega_e = 4$. The plots are snapshots at 9 different time instants and are not to scale as in a real situation.

Now let's consider an electromagnetic wave propagating in the Z-direction with angular frequency ω_e. Note that the gravitational wave is propagating in the Z direction.
Consider the electromagnetic wave described by

$$\begin{aligned}
\vec{B} &= B_{0y}\cos(\omega_e z - \omega_e t)\hat{j}, \\
\therefore B_y &= B_{0y}\cos(\omega_e z - \omega_e t), \\
\therefore \partial_3\overline{A}_1 - \partial_1\overline{A}_3 &= B_{0y}\cos(\omega_e z - \omega_e t), \qquad (\because \vec{\nabla}\times\vec{\overline{A}} = \vec{B}) \\
\therefore \partial_3\overline{A}_1 &= B_{0y}\cos(\omega_e z - \omega_e t), \qquad (4.53) \\
\therefore \partial_0\overline{A}_1 &= -B_{0y}\cos(\omega_e z - \omega_e t). \qquad (4.54)
\end{aligned}$$

Using the above results we obtain,

$$\Box\widetilde{A}_0 = 0. \qquad (4.55)$$

For the free-index $\alpha = 1$, we have

$$\begin{aligned}
&\Box\widetilde{A}_1 = \partial_3\overline{A}_1\partial_3 h^{11} - \partial_0\overline{A}_1\partial_0 h^{11}, \\
&\therefore \Box\widetilde{A}_1 = -\partial_3\overline{A}_1\partial_3(A_+\cos(\omega_g z - \omega_g t)) + \partial_0\overline{A}_1\partial_0(A_+\cos(\omega_g z - \omega_g t)), \\
&\therefore \Box\widetilde{A}_1 = B_{0y}A_+\omega_g\cos(\omega_e z - \omega_e t)\sin(\omega_g z - \omega_g t)) - B_{0y}A_+\omega_g\cos(\omega_e z - \omega_e t)\sin(\omega_g z - \omega_g t)), \\
&\therefore \Box\widetilde{A}_1 = 0. \qquad (4.56)
\end{aligned}$$

For the free-indices 2 and 3, we have

$$\Box\widetilde{A}_2 = 0, \qquad (4.57)$$

$$\Box\widetilde{A}_3 = 0. \qquad (4.58)$$

Therefore there is zero interaction when the waves are propagating parallel to each other. Now let's consider an electromagnetic wave propagating in the negative Z-direction with angular frequency ω_e. Note that the gravitational wave is propagating in the positive Z-direction.

$$\begin{aligned}
\vec{B} &= B_{0y}\cos(\omega_e z+\omega_e t)\hat{j}, \\
\therefore B_y &= B_{0y}\cos(\omega_e z+\omega_e t), \\
\therefore \partial_3\overline{A}_1-\partial_1\overline{A}_3 &= B_{0y}\cos(\omega_e z+\omega_e t), && (\because \vec{\nabla}\times\vec{\overline{A}}=\vec{B}) \\
\therefore \partial_3\overline{A}_1 &= B_{0y}\cos(\omega_e z+\omega_e t), && (4.59) \\
\therefore \partial_0\overline{A}_1 &= B_{0y}\cos(\omega_e z+\omega_e t). && (4.60)
\end{aligned}$$

Using above results we obtain the value of $\Box\widetilde{A}_\mu$ for each value of μ,

$$\Box\widetilde{A}_0=0. \tag{4.61}$$

For the free-index $\alpha=1$, we have

$$\begin{aligned}
&\Box\widetilde{A}_1=\partial_3\overline{A}_1\partial_3 h^{11}-\partial_0\overline{A}_1\partial_0 h^{11}, \\
&\therefore \Box\widetilde{A}_1=\partial_3\overline{A}_1\partial_3(A_+\cos(\omega_g z-\omega_g t))-\partial_0\overline{A}_1\partial_0(A_+\cos(\omega_g z-\omega_g t)), \\
&\therefore \Box\widetilde{A}_1=-B_{0y}A_+\omega_g\cos(\omega_e z+\omega_e t)\sin(\omega_g z-\omega_g t)-B_{0y}A_+\omega_g\cos(\omega_e z+\omega_e t)\sin(\omega_g z-\omega_g t), \\
&\therefore \Box\widetilde{A}_1=-2B_{0y}A_+\omega_g\cos(\omega_e z+\omega_e t)\sin(\omega_g z-\omega_g t)). && (4.62)
\end{aligned}$$

$$
\begin{aligned}
\therefore \widetilde{A}_1 &= -\frac{A_+ B_{0y}\omega_g}{2}\left[\frac{1}{\overline{\omega}+\tilde{\omega}}\left(\frac{1}{\overline{\omega}}\sin(\overline{\omega}_g t+\overline{\omega}_g z)+\frac{1}{\tilde{\omega}}\sin(\tilde{\omega}_g t+\tilde{\omega}_g z)\right)\right. \\
&+ \frac{1}{\overline{\omega}-\tilde{\omega}}\left(\frac{1}{\overline{\omega}}\sin(-\overline{\omega}t+\overline{\omega}z)-\frac{1}{\tilde{\omega}}\sin(-\tilde{\omega}t+\tilde{\omega}z)\right) \\
&- \frac{1}{(\overline{\omega}+\tilde{\omega})}\left(\frac{1}{\overline{\omega}}\sin(\overline{\omega}z-\tilde{\omega}t)+\frac{1}{\tilde{\omega}}\sin(\tilde{\omega}z-\overline{\omega}t)\right) \\
&+ \left.\frac{1}{\overline{\omega}-\tilde{\omega}}\left(\frac{1}{\tilde{\omega}}\sin(\tilde{\omega}z-\overline{\omega}t)-\frac{1}{\overline{\omega}}\sin(\overline{\omega}z-\tilde{\omega}t)\right)\right],
\end{aligned}
\tag{4.63}
$$

where $\overline{\omega}=\omega_g+\omega_e$, $\tilde{\omega}=\omega_g-\omega_e$ When $\omega_g=\omega_e$ the exact solution gives us

$$
\widetilde{A}_1 = -A_+ B_{0y}\frac{1}{8\omega_g}\left[4(\omega_g t-\sin(2\omega_g t))+\sin(2\omega_g(z+t))+\sin(2\omega_g(z-t))-2\sin(2\omega_g z)\right]. \tag{4.64}
$$

Note that in the above equation, the first term in the bracket is linear in time, so the linearized approximation ($|\overline{A}_\mu| \gg |\widetilde{A}_\mu|$) breaks down as the time grows. However, the perturbed E and B fields are independent from this run-away term. For the free-index $\alpha=2$, we have

$$
\Box\widetilde{A}_2 = \partial_3\overline{A}_1\partial_3 h^{21}-\partial_0\overline{A}_1\partial_0 h^{21}, \tag{4.65}
$$

$$
\Box\widetilde{A}_2 = -2B_{0y}A_\times\omega_g\cos(\omega_e z+\omega_e t)\sin(\omega_g z-\omega_g t). \tag{4.66}
$$

The answer of the above inhomogeneous wave equation can be obtained by changing A_+ to $A_\times$ in the result (4.63).

For the free-index $\alpha=3$, we have

$$
\Box\widetilde{A}_3 = 0. \tag{4.67}
$$

Hence, we see that when the waves are travelling anti-parallel, there is a nonzero interaction; while when they are parallel, there is zero interaction.

4.5 Conclusion

In this chapter we have derived the interaction of the electromagnetic waves with gravitational waves in the background of Minkowski spacetime. Initially we have a plane electromagnetic wave propagating and a perturbation is produced when a weak gravitational wave interacts with it. These perturbations are propagating in time and we find the new mode to have frequency $\sqrt{\omega_g^2+\omega_e^2}$. This mode has been found previously in [20, 33], but we find an explicit form of the perturbations. Since $A^\mu = (\phi, \vec{A})$, one can calculate the components of perturbed electric and magnetic fields using Eq. (2.23) and Eq. (2.24). The perturbed $\vec{E}$ and $\vec{B}$ fields will also have the new mode and new components are generated. It is intresting to note that scattering phenomenon and change from plane wave polarization occur when electromagentic radiation interacts with a gravitational wave. In the anti-parallel case, we find a runaway term in the perturbed gauge potentials, but it does not affect the perturbed electromagnetic field components. We anticipate that this result can be used in the detection of gravitational wave. From our solution we test for resonance. As shown, when both waves are propagating in the same direction, there is zero interaction and hence there is no resonance. When the waves are anti-parallel, we do find nonzero perturbation. Our interpretation of the results are: when there is a gravitational wave, it introduces a new type of fluctuation which exhibits a behavior similar to displacement current if compared with the flat-space Maxwell's equations. By this we mean that the source terms in the inhomogeneous differential equations are generated by the fluctuations in the electromagnetic four-potentials due to the gravitational wave and not by any source charges. If we look at Eq. (4.7), then we see that when the waves are in the same direction, the shape of the initial electromagnetic wave is preserved and there is no perturbation generated. When the waves are in different directions, the shape of the initial wave changes due to the gravitational wave and generates source terms in the Maxwell's equation.

Chapter 5

Interaction in Cosmological background

5.1 Introduction

In the previous chapter we studied the interaction of gravitational waves and electromagnetic waves in the Minkowski background and we anticipate that this will be relevant in the detection of gravitational waves on earth. We are aware that there is an ongoing search for the primordial gravitational waves by Nano-Grav [7]. The primordial gravitational waves were produced during the initial moments after the Big-Bang and also due to the fluctuations in the inflationary era. Today they are remnants in the stochastic background and we try to find them in the B-mode of the cosmic microwave backgroud (CMB). In this chapter, we study the interaction of gravitational waves with the electromagnetic waves in the de-Sitter background with the hope of shedding some new light on the primordial gravitational waves and their imprints on the CMB. We find the inhomogeneous wave equations for the gauge potentials. As in the previous chapter, the electromagnetic wave equations in the presence of the gravitational waves give us source terms for the inhomogeneous wave equations. The explicit form of perturbation in electromagnetic potential is found and the work is in progress towards the effects of these perturbations on the CMB.

5.2 Cosmological metric

The Friedmann–Lemaître–Robertson–Walker (FLRW) metric is an exact solution of Einstein's field equations with the condition that the universe be spatially homogeneous and isotropic over time.

The FLRW metric is given by

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right] \qquad (\text{Here } d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2). \tag{5.1}$$

In above equation $a(t)$ is known as the scale factor and k is the curvature constant, it can have three possible values +1, 0 or -1.

$k = +1$ corresponds to a closed universe. (elliptical geometry).

$k = 0$ corresponds a flat universe. (Euclidean Geometry).

$k = -1$ corresponds to an open universe. (hyperbolic Geometry).

The Einstein field equations in the absence of matter, with the cosmological constant, are given by

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\Lambda g_{\mu\nu}. \tag{5.2}$$

where Λ is called the cosmological constant [34]. For our discussion, we will be considering a a spatially flat universe ($k = 0$). In comoving Cartesian coordinates the flat FLRW metric is given by

$$ds^2 = -dt^2 + a^2(t)\delta_{ij} dx^i dx^j. \tag{5.3}$$

For a metric with the cosmological constant Λ, the scale factor $a(t) = e^{Ht}$ where $H = \sqrt{\frac{\Lambda}{3}}$ and $\Lambda > 0$ Now if we take the change of variables as

$$dt = a(\tau) d\tau$$

where $a(\tau) = -\frac{1}{H\tau}$ and τ is called the conformal time. Note that $a(\tau) \to 0$ when $\tau \to -\infty$ and $a(\tau) = 1$ when $\tau = -\frac{1}{H}$. In the regime we are studying, the conformal time increases and we can redefine the range of τ to be positive in that interval.

We obtain the conformal form of the flat FLRW metric given by

$$ds^2 = a^2(\tau)[-d\tau^2 + \delta_{ij}dx^i dx^j]. \tag{5.4}$$

Hence, the metric components are given by

$$g_{\mu\nu} = a^2(\tau)\eta_{\mu\nu}, \tag{5.5}$$

$$\therefore g^{\mu\nu} = \frac{\eta^{\mu\nu}}{a^2(\tau)}. \tag{5.6}$$

Note that

$$Det\, g_{\mu\nu} = a^8(\tau)$$

List of nonzero Christoffel symbols for the given metric

$$\Gamma^0_{\mu\mu} = \mathcal{H}(\tau), \qquad \Gamma^i_{0i} = \mathcal{H}(\tau), \tag{5.7}$$

where $\mathcal{H}(\tau)$ is the Hubble parameter in conformal time τ and it is given by

$$\mathcal{H}(\tau) = \frac{1}{a(\tau)}\frac{da}{d\tau}. \tag{5.8}$$

Note that μ can take values from 0 to 3, while i can take value from 1 to 3.

5.3 Electromagnetic waves in the cosmological background

In this section we study in detail the propagation of electromagnetic waves in de-Sitter space. These have been found before in [35]. In this thesis we derive the equations and solutions *ab-initio*. This is to facilitate the perturbation calculations which is new and which we formulate in this chapter.

Maxwell's equations in curved geometry

From the 'Minimum Coupling Principle' [1], the Maxwell's equations (describing the electromagentic wave) in curved spacetime can be written as

$$\nabla_\beta F^{\alpha\beta} = 0. \tag{5.9}$$

as discussed in section 2.3. But we know that

$$\nabla_\beta F^{\alpha\beta} = \frac{1}{\sqrt{|g|}}\partial_\beta(\sqrt{|g|}F^{\alpha\beta}) = 0. \tag{5.10}$$

Hence we can further simplify it as

$$\begin{aligned}
\therefore \partial_\beta(\sqrt{|g|}F^{\alpha\beta}) &= 0, \\
\therefore \partial_\beta(\sqrt{a^8}g^{\alpha\mu}g^{\beta\nu}F_{\mu\nu}) &= 0, \\
\therefore \partial_\beta(a^4\eta^{\alpha\mu}a^{-2}\eta^{\beta\nu}a^{-2}F_{\mu\nu}) &= 0, \\
\therefore \eta^{\alpha\mu}\eta^{\beta\nu}\partial_\beta F_{\mu\nu} &= 0.
\end{aligned} \tag{5.11}$$

Lorenz Gauge in curved spacetime

We know that the electromagnetic field has a gauge symmetry and we can fix that using the Lorenz gauge condition. In flat space electromagnetic theory this gauge condition is written as $\vec{\nabla}\cdot\vec{A} = -\frac{1}{c^2}\frac{\partial\phi}{\partial t}$

$$\therefore \vec{\nabla}.\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} = 0$$

(here $\vec{A}$ is the 3-D magnetic vector potential and ϕ is the scalar electric potential)

For Minkowski spacetime the above in tensor notation can be written as

$$\partial_\mu A^\mu = 0 \tag{5.12}$$

(here $A^{\mu} = (\frac{\phi}{c}, \vec{A})$ is known as the electromagnetic four potential.)

Using the 'minimum coupling principle', in curved spacetime the Lorenz gauge can be given as

$$\nabla_{\alpha} A^{\alpha} = 0, \tag{5.13}$$

$$\therefore \nabla_{\alpha} A^{\alpha} = \partial_{\alpha} A^{\alpha} + A^{\mu} \Gamma^{\alpha}_{\mu\alpha} = 0,$$

$$\therefore -\partial_{\alpha} A^{\alpha} = A^{\mu} \Gamma^{\alpha}_{\mu\alpha},$$

$$\therefore -\partial_{\alpha} A^{\alpha} = A^{0} \Gamma^{\alpha}_{0\alpha} + A^{1} \Gamma^{\alpha}_{1\alpha} + A^{2} \Gamma^{\alpha}_{2\alpha} + A^{3} \Gamma^{\alpha}_{3\alpha},$$

$$\therefore -\partial_{\alpha} A^{\alpha} = A^{0} (\Gamma^{0}_{00} + \Gamma^{1}_{01} + \Gamma^{2}_{02} + \Gamma^{3}_{03}),$$

$$\therefore \partial_{\alpha} A^{\alpha} = -4A^{0} \mathcal{H},$$

$$\therefore \partial_{\alpha} A^{\alpha} = 4a^{-2} A_{0} \mathcal{H}, \tag{5.14}$$

where the values of the Christoffel symbols are taken from Eq. (5.7). Also note that

$$\partial_{\alpha} A^{\alpha} = \partial_{\alpha} (g^{\mu\alpha} A_{\mu}),$$

$$\therefore 4a^{-2} A_{0} \mathcal{H} = A_{\mu} \partial_{\alpha} g^{\mu\alpha} + g^{\mu\alpha} \partial_{\alpha} A_{\mu},$$

$$\therefore 4a^{-2} A_{0} \mathcal{H} = 2a^{-2} A_{0} \mathcal{H} - a^{-2} \partial_{0} A_{0} + a^{-2} \eta^{ij} \partial_{i} A_{j},$$

$$\therefore \eta^{ij} \partial_{i} A_{j} = 2A_{0} \mathcal{H} + \partial_{0} A_{0}, \tag{5.15}$$

where in the second step we substitute the equation 5.14.

Now we will find the inhomogeneous wave equations for the components of the electromagnetic gauge potential A_{μ}.

when $\alpha = 1$:

$$\therefore \eta^{\alpha\mu}\eta^{\beta\nu}\partial_\beta F_{\mu\nu} = 0,$$
$$\therefore \eta^{11}\eta^{\beta\nu}\partial_\beta F_{1\nu} = 0,$$
$$\therefore \eta^{0\nu}\partial_0 F_{10} + \eta^{ij}\partial_i F_{1j} = 0$$
$$\therefore -\partial_0(\partial_1 A_0 - \partial_0 A_1) + \eta_{ij}\partial_i(\partial_1 A_j - \partial_j A_1) = 0,$$
$$\therefore \partial_0^2 A_1 - \partial_0\partial_1 A_0 + \partial_1(\eta^{ij}\partial_i A_j) - \eta^{ij}\partial_i\partial_j A_1 = 0,$$
$$\therefore -\Box A_1 + \partial_1(2A_0\mathcal{H} + \partial_0 A_0) - \partial_0\partial_1 A_0 = 0 \qquad (\because 5.15),$$
$$\therefore -\Box A_1 + 2\mathcal{H}\partial_1 A_0 = 0 \qquad (\text{Here } \Box = \partial_\mu\partial^\mu). \tag{5.16}$$

From the symmetry of Eq. (5.11) we can say that

For the free index $\alpha = 2$ the inhomogeneous wave equation is

$$\therefore -\Box A_2 + 2\mathcal{H}\partial_2 A_0 = 0. \tag{5.17}$$

For the free index $\alpha = 3$ the inhomogeneous wave equation is

$$\therefore -\Box A_3 + 2\mathcal{H}\partial_3 A_0 = 0. \tag{5.18}$$

Now we will solve the inhomogeneous wave equations to find the explicit form of the electromagnetic vector potential components A_μ.

when the free index $\alpha = 0$ in Eq. (5.11):

$$
\begin{aligned}
&\therefore \eta^{00}\eta^{\beta\nu}\partial_\beta F_{0\nu} = 0, \\
&\therefore \eta^{00}\partial_0 F_{00} + \eta^{ij}\partial_i F_{0j} = 0 \qquad (\because F_{00} = 0), \\
&\therefore \eta^{ij}\partial_i(\partial_0 A_j - \partial_j A_0) = 0 \\
&\therefore \eta^{ij}\partial_i\partial_0 A_j - \eta^{ij}\partial_i\partial_j A_0 = 0, \\
&\therefore \partial_0(\eta^{ij}\partial_i A_j) - \eta^{ij}\partial_i\partial_j A_0 = 0, \\
&\therefore \partial_0(2A_0\mathcal{H} + \partial_0 A_0) - \eta^{ij}\partial_i\partial_j A_0 = 0, \qquad (\because 5.15) \\
&\therefore 2\mathcal{H}\partial_0 A_0 + 2A_0\dot{\mathcal{H}} + \partial_0^2 A_0 - \eta^{ij}\partial_i\partial_j A_0 = 0 \qquad (\text{Here } \dot{\mathcal{H}} = \frac{d\mathcal{H}}{d\tau}), \\
&\therefore -\Box A_0 + 2\mathcal{H}\partial_0 A_0 + 2A_0\dot{\mathcal{H}} = 0 \qquad (\text{Here } \Box = \partial_\mu\partial^\mu), \\
&\therefore -\Box A_0 - \frac{2}{\tau}\partial_0 A_0 + \frac{2}{\tau^2}A_0 = 0 \qquad (\because \mathcal{H} = -\frac{1}{\tau}).
\end{aligned}
\tag{5.19}
$$

To further simplify the equation let's take the form of $A_0(t,\vec{x})$ as

$$
A_0(\tau,\vec{x}) = A_0(\tau)e^{i\vec{k}.\vec{x}}. \tag{5.20}
$$

Hence the inhomogeneous wave equation (5.19) becomes

$$
\begin{aligned}
&\partial_\tau^2(A_0(\tau)e^{i\vec{k}.\vec{x}}) - \partial_j^2(A_0(\tau)e^{i\vec{k}.\vec{x}}) - \frac{2}{\tau}\partial_0(A_0(\tau)e^{i\vec{k}.\vec{x}}) + \frac{2}{\tau^2}A_0(\tau)e^{i\vec{k}.\vec{x}} = 0, \\
&\therefore [\partial_\tau^2 A_0(\tau) + k^2 A_0 - \frac{2}{\tau}\partial_0 A_0(\tau) + \frac{2}{\tau^2}A_0]e^{i\vec{k}.\vec{x}} = 0, \\
&\therefore \frac{d^2 A_0}{d\tau^2} - \frac{2}{\tau}\frac{dA_0}{d\tau} + (\frac{2}{\tau^2} + k^2)A_0 = 0.
\end{aligned}
\tag{5.21}
$$

The standard form of Bessel's differential equation is given by

$$
x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \alpha^2)y = 0 \tag{5.22}
$$

To get the form of above equation from Eq. (5.21) we will do as follows

Let's take

$$\begin{aligned}
&A_0(\tau) = \tau^n A_0^{'}(\tau),\\
&\therefore \dot{A}_0 = n\tau^{n-1}A_0^{'} + \tau^n \dot{A}_0^{'} \qquad\qquad (\text{where } \dot{A}_0 = \frac{dA_0}{d\tau}),\\
&\therefore \ddot{A}_0 = n(n-1)\tau^{n-2}A_0^{'} + 2n\tau^{n-1}\dot{A}_0^{'} + \tau^n \ddot{A}_0^{'}.
\end{aligned} \tag{5.23}$$

Substituting above in Eq. (5.21) we obtain

$$\therefore n(n-1)\tau^{n-2}A_0^{'} + 2n\tau^{n-1}\dot{A}_0^{'} + \tau^n\ddot{A}_0^{'} - \frac{2}{\tau}n\tau^{n-1}A_0^{'} - \frac{2}{\tau}\tau^n\dot{A}_0^{'} + (\frac{2}{\tau^2} + k^2)\tau^n A_0^{'} = 0,$$
$$\therefore \ddot{A}_0^{'} + (2n-2)\tau^{-1}\ddot{A}_0^{'} + (n^2 - 3n + 2)\tau^{-2}A_0^{'} + k^2 A_0^{'} = 0. \tag{5.24}$$

Comparing with Eq. (5.22) we need

$$2n - 2 = 1,$$
$$\therefore n = \frac{3}{2}.$$

Hence Eq. (5.24) becomes

$$\therefore \ddot{A}_0^{'} + \frac{1}{\tau}\ddot{A}_0^{'} + (-\frac{1}{4\tau^2} + k^2)A_0^{'} = 0. \tag{5.25}$$

Now to further simplify let's take

$$\tau' = s, \tau$$
$$\therefore \frac{d\tau'}{d\tau} = s,$$

where s is a proportionality constant. Hence, Eq. (5.25) becomes

$$\therefore \ddot{A}_0' + \frac{1}{\tau'}\ddot{A}_0' + (-\frac{1}{4\tau'^2} + \frac{k^2}{s^2})A_0' = 0,$$
$$\therefore \tau'^2\ddot{A}_0' + \tau'\ddot{A}_0' + (\tau'^2 - (1/2)^2)A_0' = 0. \qquad (\because s = \pm k \text{ comparing with 5.22}) \tag{5.26}$$

Equation (5.26) has the standard form of Bessel's differential equation and the solution is given by

$$\therefore A_0' = J_{\frac{1}{2}}(\tau'),$$
$$\therefore A_0 = \tau^{\frac{3}{2}} J_{\frac{1}{2}}(\pm k\tau),$$
$$\therefore A_0(\tau) = \left(\frac{2}{\pi k}\right)^{\frac{1}{2}} \tau \sin(k\tau) \qquad \left(\because J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x\right), \tag{5.27}$$
$$\therefore A_0(\tau,\vec{x}) = \left(\frac{2}{\pi k}\right)^{\frac{1}{2}} \tau \sin(k\tau) e^{i\vec{k}.\vec{x}}. \qquad (\because 5.20) \tag{5.28}$$

Substituting the result (5.28) in the inhomogeneous wave equation (5.16) we obtain

$$\therefore \Box A_1(\tau,\vec{x}) = -\frac{2}{\tau}\partial_1 A_0(\tau,\vec{x}),$$
$$\therefore \Box A_1(\tau,\vec{x}) = -2ik_1 \left(\frac{2}{\pi k}\right)^{\frac{1}{2}} \sin(k\tau) e^{i\vec{k}.\vec{x}},$$
$$\therefore A_1(\tau,\vec{x}) = \left[P_1 \sin(k\tau) + Q_1 \cos(k\tau) - \imath k_1 \left(\frac{2}{\pi k^3}\right)^{1/2} \tau \cos(k\tau)\right] e^{i\vec{k}.\vec{x}}. \tag{5.29}$$

Substituting this result (5.28) in inhomogeneous wave equation (5.17) we obtain

$$\therefore \Box A_2(\tau,\vec{x}) = -\frac{2}{\tau}\partial_2 A_0(\tau,\vec{x}),$$
$$\therefore \Box A_2(\tau,\vec{x}) = -2ik_2 \left(\frac{2}{\pi k}\right)^{\frac{1}{2}} \sin(k\tau) e^{i\vec{k}.\vec{x}},$$
$$\therefore A_2(\tau,\vec{x}) = \left[P_2 \sin(k\tau) + Q_2 \cos(k\tau) - \imath k_2 \left(\frac{2}{\pi k^3}\right)^{1/2} \tau \cos(k\tau)\right] e^{i\vec{k}.\vec{x}}. \tag{5.30}$$

Substituting this result (5.28) in inhomogeneous wave equation (5.18) we obtain

$$\therefore \Box A_3(\tau,\vec{x}) = -\frac{2}{\tau}\partial_3 A_0(\tau,\vec{x}),$$
$$\therefore \Box A_3(\tau,\vec{x}) = -2ik_3\left(\frac{2}{\pi k}\right)^{\frac{1}{2}}\sin(k\tau)e^{i\vec{k}.\vec{x}},$$
$$\therefore A_3(\tau,\vec{x}) = \left[P_3\sin(k\tau) + Q_3\cos(k\tau) - \imath k_3\left(\frac{2}{\pi k^3}\right)^{1/2}\tau\cos(k\tau)\right]e^{i\vec{k}.\vec{x}}. \tag{5.31}$$

In the above equations, P_i and Q_i are integration constants. When these constants are zero, $F_{ij} = 0$ and $F_{0i} = -i\frac{k_i}{k}\left(\frac{2}{\pi k}\right)^{\frac{1}{2}}\cos(k\tau)e^{i\vec{k}\cdot\vec{x}}$.

5.4 Perturbation in the conformal background

In this section we will study the perturbation in the magnetic potentials due to gravitational wave in the cosmological background [35, 36, 37, 38]. For this we will solve the Maxwell's equations in the presence of gravitational wave. The weak gravitational wave perturbation in the conformal background is given by

$$g_{\mu\nu} = a^2(\tau)(\eta_{\mu\nu} + h_{\mu\nu}), \tag{5.32}$$
$$\therefore g^{\mu\nu} = \frac{1}{a^2(\tau)}(\eta^{\mu\nu} - h^{\mu\nu}), \tag{5.33}$$

$$g^{\mu\nu} = \frac{1}{a^2(\tau)}\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1-h_+ & -h_\times & 0 \\ 0 & -h_\times & 1+h_+ & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{5.34}$$

Maxwell's equation for electromagentic wave in de-Sitter spacetime with gravitational waves

From the 'Minimal Coupling Principle', the above equation in curved spacetime can be written as

$$\nabla_\nu F^{\mu\nu} = 0$$

But we know that

$$\nabla_\beta F^{\alpha\beta} = \frac{1}{\sqrt{|g|}} \partial_\beta (\sqrt{|g|} F^{\alpha\beta}) = 0,$$

$$\therefore \partial_\beta (\sqrt{a^8} g^{\alpha\mu} g^{\beta\nu} F_{\mu\nu}) = 0,$$
$$\therefore \eta^{\alpha\mu}\eta^{\beta\nu}\partial_\beta F_{\mu\nu} - \eta^{\alpha\mu}h^{\beta\nu}\partial_\beta F_{\mu\nu} - h^{\alpha\mu}\eta^{\beta\nu}\partial_\beta F_{\mu\nu} - \eta^{\beta\nu}F_{\mu\nu}\partial_\beta h^{\alpha\mu} = 0. \quad (5.35)$$

Lorenz gauge

Here we explicitly find the form of the Lorenz gauge condition in the presence of gravitational waves.

$$\partial_\alpha A^\alpha = 4\mathcal{H} A_0 a^{-2},$$
$$\eta^{ij}\partial_i A_j = 2A_0\mathcal{H} + \partial_0 A_0 - h_+(\partial_2 A_2 - \partial_1 A_1) + h_\times F_{12}. \quad (5.36)$$

Now we will find the inhomogeneous wave equations for each value of α in Eq. (5.35).

When the free index $\alpha = 0$:

$$
\begin{aligned}
&\therefore \eta^{00}\eta_{\beta\nu}\partial_\beta F_{0\nu} - \eta^{00}h^{\beta\nu}\partial_\beta F_{0\nu} = 0,\\
&\therefore \eta_{00}\partial_0 F_{00} + \eta^{ij}\partial_i F_{0j} - (h^{11}\partial_1 F_{01} + h^{12}\partial_1 F_{02} + h^{21}\partial_2 F_{01} + h^{22}\partial_2 F_{02}) = 0,\\
&\therefore \eta^{ij}\partial_i(\partial_0 A_j - \partial_j A_0) - (h_+\partial_1 F_{01} + h_\times\partial_1 F_{02} + h_\times\partial_2 F_{01} - h_+\partial_2 F_{02}) = 0,\\
&\therefore \partial_0(\eta^{ij}\partial_i A_j) - \partial_i\partial^i A_0 + [h_+(\partial_2 F_{02} - \partial_1 F_{01}) - h_\times(\partial_1 F_{02} + \partial_2 F_{01})] = 0,\\
&\therefore \partial_0(2A_0\mathcal{H} - h_+\partial_2 A_2 + h_+\partial_1 A_1 + h_\times F_{12} + \partial_0 A_0) -\\
&\qquad \partial_i\partial^i A_0 + h_+(\partial_2 F_{02} - \partial_1 F_{01}) - h_\times(\partial_1 F_{02} + \partial_2 F_{01}) = 0,\\
&\therefore -\Box A_0 + 2\mathcal{H}\partial_0 A_0 + 2A_0\dot{\mathcal{H}} + \partial_0 h_+(\partial_1 A_1 - \partial_2 A_2) + h_+\partial_0(\partial_1 A_1 - \partial_2 A_2) + \partial_0 h_\times F_{12}\\
&\qquad -h_\times\partial_0 F_{12} + h_+(\partial_2 F_{02} - \partial_1 F_{01}) - h_\times\partial_2(F_{01} + F_{02}) = 0,\\
&\therefore \Box\widetilde{A}_0 = -\partial_0 h_+(\partial_1 A_1 - \partial_2 A_2) - h_+\partial_0(\partial_1 A_1 - \partial_2 A_2) - \partial_0 h_\times F_{12}\\
&\qquad +h_\times\partial_0 F_{12} + h_+(\partial_2 F_{02} - \partial_1 F_{01}) - h_\times\partial_2(F_{01} + F_{02}).
\end{aligned}
\tag{5.37}
$$

Because we set all the integral constants of magnetic potentials (describing the electromagnetic wave in this background) to zero, the only nonzero terms on the right hand side will be due to F_{0i} being nonzero.

$$
\begin{aligned}
\therefore \Box\widetilde{A}_0 = &-\partial_0 h_+(\partial_1 A_1 - \partial_2 A_2) - h_+\partial_0(\partial_1 A_1 - \partial_2 A_2)\\
&+h_+(\partial_2 F_{02} - \partial_1 F_{01}) - h_\times\partial_2(F_{01} + F_{02})
\end{aligned}
\tag{5.38}
$$

when the'cross polarization' ($h_\times$) is absent from the gravitational wave, above simplifies to

$$
\Box\widetilde{A}_0 = \partial_0 h_+(\partial_1 A_1 - \partial_2 A_2) + h_+(\partial_1^2 - \partial_2^2)A_0 \tag{5.39}
$$

where we have used the formula for the $h_+ = A_+\omega_g\tau e^{-i\omega_g\tau}\left(1 - \frac{i}{\omega_g\tau}\right)e^{i\omega_g z}$ [27, 39]. From the above it is evident that if the electromagnetic wave is only in the Z-direction, there is no interaction from this term. Note that if the wave vector components $k_1 = k_2$, then the right hand side will vanish and we will have no perturbation in the 0-th component of the gauge potential.

We take the Green's function from Tsamis and Woodard [40, 41] of the form

$$
\begin{aligned}
G^{\mathrm{ret}}_{\mu\nu}(x,x') =& \frac{\theta(\Delta\tau)}{4\pi}\delta'(y)\frac{\partial y(x,x')}{\partial x^{\mu}}\frac{\partial y(x,x')}{\partial x'^{\nu}} \\
&= -\frac{\theta(\Delta\tau)}{2\pi^2}\mathrm{Lim}_{\varepsilon\to 0}\frac{\varepsilon y}{(y^2+\varepsilon^2)^2}\frac{\partial y(x,x')}{\partial x^{\mu}}\frac{\partial y(x,x')}{\partial x'^{\nu}}
\end{aligned}
\tag{5.40}
$$

where

$$
y(x,x') \equiv H^2 a(\tau)a(\tau')\left(|\vec{x}-\vec{x}'|^2 - (|\tau-\tau'| - i\varepsilon)^2\right)
$$

and we can derive the perturbations using the above Green's function. But when we formulated the integral as in chapter-4, the answer was not obtained in a closed form. So we solve the inhomogeneous differential equations using MAPLE. Now we will find the equations for the other three components and solve them. Setting $\alpha = i$ in the wave equation (5.35), we get:

$$
\begin{aligned}
\eta^{ii}\eta^{\beta\nu}\partial_\beta F_{i\nu} &= \eta^{ii}\eta^{00}\partial_i\partial_0 A_0 - \eta^{ii}\left(\eta^{00}\partial_0^2 + \eta^{jj}\partial_j^2\right)A_i + \eta^{ii}\eta^{jj}\partial_i\partial_j A_j, \\
&= \eta^{ii}\eta^{00}\partial_i\partial_0 A_0 - \eta^{ii}(\Box A_i) + \eta^{ii}\partial_i\left(2A_0\mathcal{H} + \partial_0 A_0 - h_+(\partial_2 A_2 - \partial_1 A_1) + h_\times F_{12}\right).
\end{aligned}
\tag{5.41}
$$

In the second equation we have used the Lorenz gauge condition $\partial_\mu A^\mu = 0$. This gives the following

$$
\begin{aligned}
-\eta^{ii}\Box A_i + \eta^{ii}2\mathcal{H}\partial_i A_0 &= \eta^{ii}\partial_i(h_+(\partial_2 A_2 - \partial_1 A_1)) - \eta^{ii}\partial_i(h_\times F_{12}) + \eta^{ii}h^{\beta\nu}\partial_\beta F_{i\nu}. \\
&+ h^{i\mu}\eta^{\beta\nu}\partial_\beta F_{\mu\nu} + \eta^{\beta\nu}F_{\mu\nu}\partial_\beta h^{i\mu}
\end{aligned}
\tag{5.42}
$$

The above are the inhomogeneous wave equations for the remaining three components of the electromagnetic potential, which can be found using the propagator in Eq. (5.40). However, we solve the inhomogeneous differential equations using MAPLE. In the source term of the equation, we use the F_{0i} as follows. Using the example of the electromagnetic field where the constants P_i, Q_i are zero, we get the non-zero components of the electric field strength to be

$$F_{0i} = -i\frac{k_i}{k}\left(\frac{2}{\pi k}\right)^{1/2}\cos(k\tau)e^{i\vec{k}\cdot\vec{x}}. \tag{5.43}$$

For the 0-th component one has

$$\begin{aligned}\Box\widetilde{A}_0 &= \partial_0 h_+(\partial_1 A_1 - \partial_2 A_2) + h_+(\partial_1^2 - \partial_2^2)A_0 \\ &= A_+ e^{-i\omega_g\tau}(\omega_g\tau)e^{i\omega_g z}(-i\omega_g)(k_1^2 - k_2^2)\left(\frac{2}{\pi k^3}\right)^{1/2}\tau\cos k\tau e^{i\vec{k}\cdot\vec{x}} \\ &\quad - A_+ e^{-i\omega_g\tau}(\omega_g\tau - i)\,e^{i\omega_g z}(k_1^2 - k_2^2)\left(\frac{2}{\pi k}\right)^{1/2}\tau\sin(k\tau)e^{i\vec{k}\cdot\vec{x}} = j_0(\tau, x, y, z,)\end{aligned} \tag{5.44}$$

We assume motivated from the right hand side of the above equation the $\widetilde{A}_\mu(\tau,\vec{x}) = e^{i\vec{\tilde{k}}\cdot\vec{x}}\widetilde{A}_\mu(\tau)$, where $\vec{\tilde{k}} = \vec{k}_g + \vec{k}$. Plugging this we get an equation for $\widetilde{A}_0$ which is a pure function of τ of the form

$$\frac{d^2\widetilde{A}_0}{d\tau^2} - \frac{2}{\tau}\frac{d\widetilde{A}_0}{d\tau} + (\tilde{k}^2 + \frac{2}{\tau^2})\widetilde{A}_0 = -f(\tau) \tag{5.45}$$

where

$$\begin{aligned}f(\tau) &= A_+ e^{-i\omega_g\tau}\omega_g\tau(-i\omega_g)(k_1^2 - k_2^2)\left(\frac{2}{\pi k^3}\right)^{1/2}\tau\cos(k\tau) \\ &\quad - A_+ e^{-i\omega_g\tau}\omega_g\tau\left(1 - \frac{i}{\omega_g\tau}\right)(k_1^2 - k_2^2)\left(\frac{2}{\pi k}\right)^{1/2}\tau\sin(k\tau)\end{aligned} \tag{5.46}$$

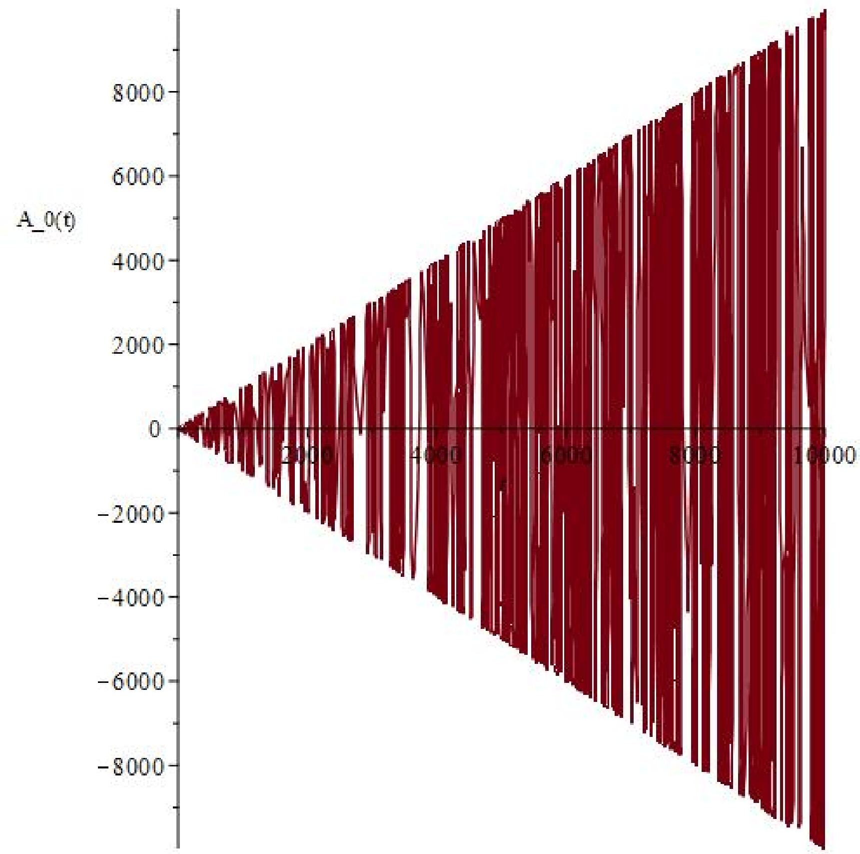

Figure 5.1: The real part of the unperturbed 0-th component of the electromagnetic potential $\overline{A}_0$ as a function of absolute conformal time τ

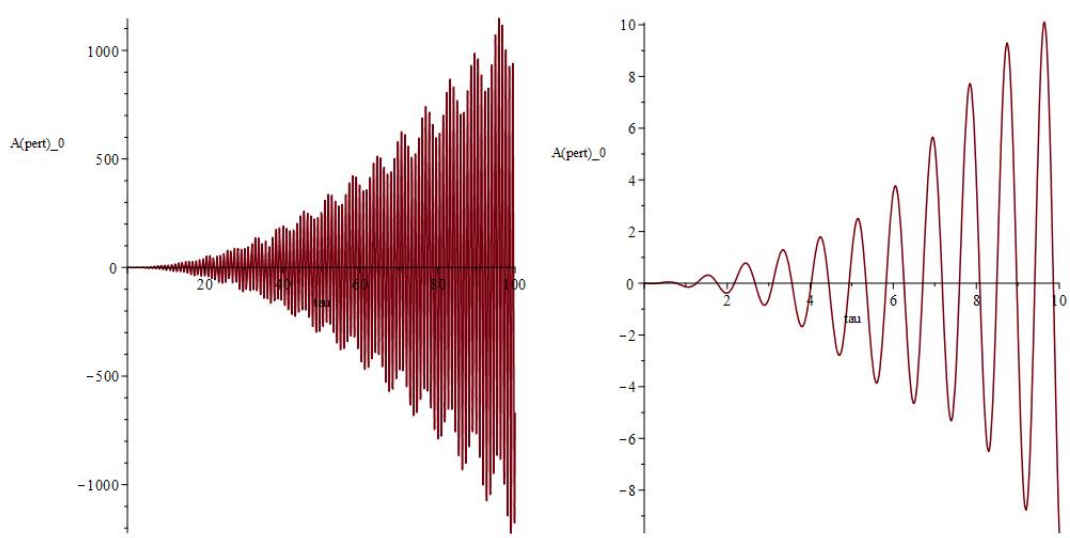

Figure 5.2: The real part of perturbed 0-th component of the electromagnetic potential $\widetilde{A}_0$ as a function of absolute conformal time τ

Using MAPLE we obtain the solution to the above equation as

$$\widetilde{A}_0 = C_1\tau\cos(\widetilde{k}\tau) + C_2\tau\sin(\widetilde{k}\tau) - \frac{\tau}{\widetilde{k}}\left(\sin(\widetilde{k}\tau)\int\frac{\cos(\widetilde{k}\tau)g(\tau)}{\tau}\,d\tau - \cos(\widetilde{k}\tau)\int\frac{\sin(\widetilde{k}\tau)g(\tau)}{\tau}\,d\tau\right) \tag{5.47}$$

We find the re-appearance of the 'new mode' frequencies of $\widetilde{k} = \sqrt{k_g^2 + k^2}$ and the integrals can be obtained using MAPLE. Using MAPLE plot as a function of τ (using $C_1 = C_2 = 0$) we find that the gravitational wave produces modulation over the shape of the unperturbed wave. Note this perturbation is proportional to A_+ and therefore much weaker than the original wave. We expect to study any way to make these modulations detectable, and its effect on the CMB in the near future.

For the plot we use $k_1 = 4, k_2 = 0, \omega_g = 3$ and the frequency of output mode as $\widetilde{k} = 5$. The integral

$$\frac{\tau}{\widetilde{k}}\left(\sin(\widetilde{k}\tau)\int\frac{\cos(\widetilde{k}\tau)g(\tau)}{\tau}\,d\tau - \cos(\widetilde{k}\tau)\int\frac{\sin(\widetilde{k}\tau)g(\tau)}{\tau}\,d\tau\right) \tag{5.48}$$

takes the following form:

$$\begin{aligned}\tilde{A}_0 \sim C_1\tau\cos(5\tau) + C_2\tau\sin(5\tau) - \frac{16}{\sqrt{2\pi}}A_+\tau e^{-3i\tau}\Bigg(\left(\frac{i\tau}{8} + \frac{41}{384}\right)\cos(4\tau) \\ + \frac{(-2i+3\tau)}{32}\sin(4\tau)\Bigg)e^{ik_1x + i\omega_g z}\end{aligned} \tag{5.49}$$

We then plot the real part of the above as a function of time, setting $x = 0; z = 0$ $C_1 = C_2 = 0$ to examine the frequency behaviour in the figure 5.2. We have used the similarity sign as there can be some normalization constants. We then find the mode for the propagation of the $\widetilde{A}_1$ component. The equation for the $\widetilde{A}_1$ is found as

$$\square\widetilde{A}_1 = 2\mathcal{H}\partial_1\widetilde{A}_0 + h_+\partial_1^2 A_1 - h_+\partial_0 F_{01} - F_{01}\partial_0 h_+. \tag{5.50}$$

Using the same method as for the $\widetilde{A}_0$ we find that the time dependent part of the equation can be obtained as:

$$\partial_0^2\widetilde{A}_1 + \tilde{k}^2\widetilde{A}_1 = \frac{2}{\tau}(ik_1)\widetilde{A}_0 + A_+\frac{ik_1}{k}\left(\frac{2}{\pi k}\right)^{1/2}\Big[\left(k_1^2(\omega_g\tau - i) + i\omega_g^2\right)\tau\cos(k\tau) \tag{5.51}$$
$$+ (\omega_g\tau - i)k\sin(k\tau)\Big]e^{-i\omega_g\tau}$$

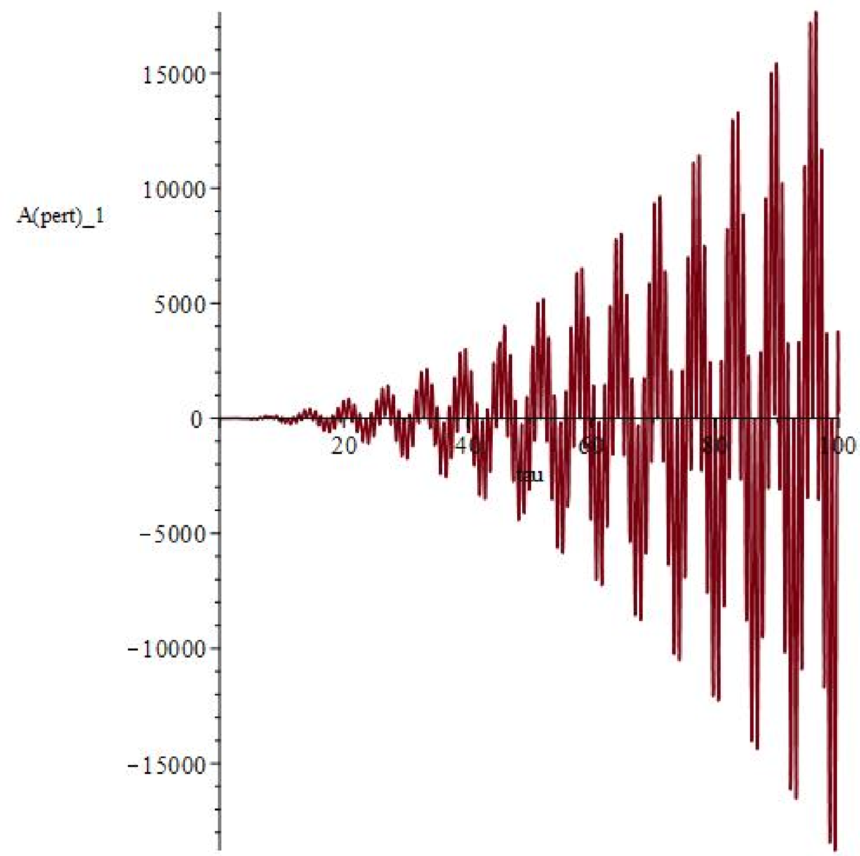

Figure 5.3: The real part perturbed 1st component of the electromagnetic potential $\widetilde{A}_1$ as a function of absolute conformal time τ

As previously we take the example of $k_1 = 4, k_2 = 0, \omega_g = 3$ and find the following solution for $\widetilde{A}_1$ gives

$$\widetilde{A}_1 \sim C_1 \cos(5\tau) + C_2 \sin(5\tau) + \left(\frac{192\tau^2 - 164i\tau - 24}{96\sqrt{2\pi}}\right) \sin(4\tau) e^{-3i\tau} + \frac{1}{\sqrt{2\pi}}\left(\tau - i\frac{1}{24}\right) \cos(4\tau) e^{-3i\tau} \tag{5.52}$$

If we set $k_2 = 0$ then

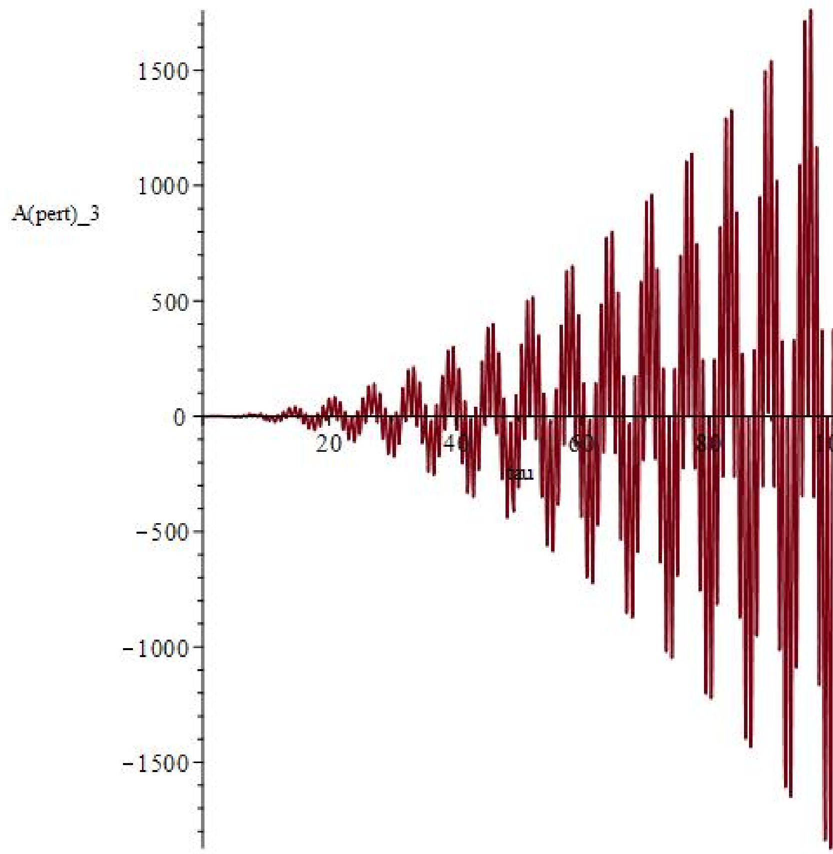

Figure 5.4: The real part of perturbed 3rd component of the electromagnetic potential $\widetilde{A}_3$ as a function of absolute conformal time τ

$$\Box \tilde{A}_2 = 0, \tag{5.53}$$

if we keep k_2 then the equation is similar to that for $\widetilde{A}_1$. Further:

$$\Box\widetilde{A}_3 = \partial_3 h_+(\partial_2 A_2 - \partial_1 A_1). \tag{5.54}$$

We can solve for $\widetilde{A}_3$ easily using MAPLE and we get:

$$\begin{aligned}\widetilde{A}_3 \sim\ & C_1\cos(5\tau) + C_2\sin(5\tau) - \frac{2A_+}{\sqrt{2\pi}}e^{-3i\tau}\left[\left(-\frac{\tau}{2}+\frac{i}{3}\right)\cos(4\tau)\right. \\ & \left. + \left(-\tau^2+\frac{11}{24}+i\tau\right)\sin(4\tau)\right]e^{i3z}e^{4ix}\end{aligned} \tag{5.55}$$

The plots for $\widetilde{A}_1, \widetilde{A}_3$ are in the figures 5.3 and 5.4. We can see that the gravitational wave produces modulations in the form of the waves.

5.5 Conclusion

In this chapter we study the interaction of the gravitational waves with electromagnetic waves in the background of de-Sitter metric. We found the inhomogeneous wave equation for the perturbation in the 0-th component of the gauge field. We find that if the EM wave is in the Z-direction, then there is no perturbation. The form of perturbation in the 0-th component of the electromagnetic potential is found in Eq. (5.49). Similarly, the perturbations $\widetilde{A}_1$ and $\widetilde{A}_2$ are found to be as given in Eq. (5.52) and (5.55) respectively. We find from the graphs that the electromagnetic perturbations are modulated due to the interaction with gravitational waves and we expect to find signatures of this in current observations. We can find the electromagentic field components from the gauge potentials and the scattering of electromagentic radiation (also found in the Minkowski background interaction) occurs due to the interaction with a gravitational wave. We note that the unperturbed gauge potentials increase linearly with conformal time while the perturbed gauge potentials increase quadratically with conformal time τ. This is understood as, the perturbed gauge potentials being the result of interaction between the electromagentic and gravitational wave, both of which are linearly proportional to conformal time.

Chapter 6

Conclusion

Our aim was to analyze the interaction of gravitational waves with electromagnetic waves in the background of Minkowski spacetime and de-Sitter spacetime, so that the perturbations in the electromagnetic four-potential components can be found. The presence of such nonzero perturbations might be detected and used to find the imprints of primordial gravitational waves on the cosmic microwave background radiation observed today.

In chapter 2, the important mathematical and conceptual background was provided in order to explain the fundamental ideas of the general theory of relativity. Relevant mathematical tools like metric tensor, Christoffel symbols, covariant derivative, parallel transport, Riemann curvature tensor, stress-energy, *etc* were introduced. After briefly discussing the equivalence principle, the Einstein field equations were introduced which describe the relation among energy, momentum and curvature of spacetime.

In chapter 3, the metric tensor ($g_{\mu\nu}$) describing the propagation of a weak gravitational wave in the background of Minkowski space was found in Eq. (3.37). Taking the form of the metric tensor as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, the Einstein field equations were solved (to first order in $h_{\mu\nu}$) in the absence of matter; which produced a homogeneous wave equation in the TT-gauge. The explicit form of the metric perturbation $h_{\mu\nu}$ was found and it was shown that it has two independent components, known as h_+ and $h_\times$ polarization.

In chapter 4, we analyzed the interaction of a linearized gravitational wave with a plane polarized monochrometic electromagnetic wave. We find that (to linear order in $h_{\mu\nu}$) this interaction causes perturbations in the electromagnetic potential components A_μ. The perturbed mode of frequency is found to be $\sqrt{\omega_g^2+\omega_e^2}$ as expected, where ω_g and ω_e are the angular frequencies of the gravitational wave and electromagnetic wave, respectively. We note that the nature of wave interaction is dependent on the relative direction of wave propagation. We do not find any resonant amplification when the waves are parallel to each other. The interaction is understood as: The flat-space Maxwell's equations are not valid to describe the nature of electromagnetic field in the presence of a gravitational wave. When the Maxwell's equations are found in the curved background, we find that there are source terms in the inhomogeneous wave equation. This is most clear when we find Maxwell's equations in the curved background (4.7) given by $\eta^{\alpha\mu}\Box\widetilde{A}_\mu = -\eta^{\alpha\mu}h^{\beta\nu}\partial_\beta F_{\mu\nu} - \eta^{\beta\nu}F_{\mu\nu}\partial_\beta h^{\alpha\mu}$. Note that the right hand side of this expression, which acts as a source term, would be zero in the absence of gravitational wave. The perturbation plot of the x-component of $\widetilde{A}_\mu$ at different times is shown in figure 4.1.

In chapter 5, we analyze the interaction of gravitational waves and electromagnetic waves in the background of de-Sitter spacetime. To simplify the expression, we consider the case when the $h_\times$ is absent. The same perturbed mode of frequency is found as in the Minkowski spacetime background. We interpret this perturbation in the same way as previously done. However, the de-Sitter metric does change the nature of perturbation significantly as seen in Eq. (5.49), (5.52) and (5.55) compared to Minkowski spacetime results. The plot for the perturbed 0-th component of electromagnetic potential over the conformal time τ is shown in figure 5.2. Similarly, the plots for the $\widetilde{A}_1$ and $\widetilde{A}_3$ are (respectively) shown in figures 5.3 and 5.4.

We note that in Minkowski and de-Sitter background, scattering of electromagentic radiation occurs when it interacts with a gravitatonal wave. The perturbed radiation is no longer transeversly plane-polarized due to the interaction. When both waves are propagating anti-parallelly (w.r.t each other) in Minkowski background, we find a runaway term in the perturbed gauge potentials which do not affect the electromagnetic wave components. Our results have phase shift (in the electromagentic waves) terms similar to those used by

LIGO and the remaining terms contain the new frequency mode. These perturbations in the electromagnetic wave are tiny in magnitude but might be detectable with technological advancement. In future work, we will be working towards the explicit solution for the perturbed electromagnetic potential when both polarizations of gravitational waves are present in the background of de-Sitter spacetime. We will then try to find their effects on the B-modes of the cosmic microwave background radiation. The search for other physical phenomena in cosmology where these perturbations can be detected is also work in progress.

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