.

ASPECTS OF QUANTUM GRAVITY PHENOMENOLOGY

SOUMEN DEB
Bachelor of Science, University of Calcutta, 2008
Master of Science, Indian Institute of Technology Bombay, 2010

A Thesis
Submitted to the School of Graduate Studies
of the University of Lethbridge
in Partial Fulfilment of the
Requirements for the Degree

MASTER OF SCIENCE

Department of Physics and Astronomy
University of Lethbridge
LETHBRIDGE, ALBERTA, CANADA

ASPECTS OF QUANTUM GRAVITY PHENOMENOLOGY

SOUMEN DEB

Date of Defense: January 14, 2015

Dr. Saurya Das Supervisor	Professor	Ph.D.
Dr. Mark Walton Thesis Examination Committee Member	Professor	Ph.D.
Dr. Kent Peacock Thesis Examination Committee Member	Professor	Ph.D.
Dr. Locke Spencer Chair, Thesis Examination Committee	Assistant Professor	Ph.D.

To the bright memory of my father, Dr. Priyaranjan Deb

Abstract

Quantum gravity effects modify the Heisenberg's uncertainty principle to the generalized uncertainty principle (GUP). Earlier work showed that the GUP-induced corrections to the Schrödinger equation, when applied to a non-relativistic particle in a one-dimensional box, led to the quantization of length. Similarly, corrections to the Klein-Gordon and the Dirac equations, gave rise to length, area and volume quantizations. These results suggest a fundamental granular structure of space. This thesis investigates how spacetime curvature and gravity might influence this discreteness of space. In particular, by adding a weak background gravitational field to the above three quantum equations, it is shown that quantization of lengths, areas and volumes continue to hold. Although the nature of this new quantization is quite complex, under proper limits, it reduces to cases without gravity. These results indicate the universality of quantum gravity effects.

Acknowledgements

This work was supported by the Natural Sciences and Engineering Research Council of Canada (NSERC) and the University of Lethbridge.

I am thankful to my supervisor Dr. Saurya Das for his inspiring guidance, constructive criticism, friendly advice and academic as well as non-academic support throughout the research project. I would like to express my gratitude to my committee members Dr. Mark Walton and Dr. Kent Peacock for all the valuable suggestions and comments they provided.

I would also like to thank my family and friends for their help and encouragement.

Contents

List of Figures

Chapter 1

Introduction

1.1 Prologue

Quantum field theory describes the behavior of the fundamental constituent particles and the fields. General relativity, on the other hand, treats one of the fundamental forces, gravity as a derived effect of spacetime curvature and explains the large scale dynamics – from planetary and galactic motions to black hole physics and in general the evolutionary history of the universe. The two theories are successful in their own realms, but they are not really mutually compatible. Einstein's formulation is essentially a deterministic approach. Although it governs the force of gravity, it cannot be applied the same way to explain gravitational field as the Standard Model does to the three other fundamental forces of nature, electromagnetic, strong and weak.

Moreover, the presence of mathematical difficulties like singularities in the Feynmann diagrams, renormalization failure etc. [1] [3] in quantum field theory clearly indicates that a more general formalism is required in order to explain all of the fundamental forces together.

Hawking Radiation [2] can be considered as an example that, despite being an area more relevant to the general relativity, explained better with quantum mechanics in curved spacetime. Rotating and Reissner–Nordström black holes, for example, are expected to emit photons and other particles according to quantum mechanics, the dynamics of which are well comprehended by general relativity. Although a direct signature is yet to be found, this prediction has also been supported by analog gravity experiments [14].

Observational evidence like this along with the technical problems with having two distinct theories suggests a necessity for a successful unification, in other words, a quantum theory of gravity.

There are a few candidates for a successful quantum gravity theory. String theory, Loop quantum gravity and Causal set theory are among the most promising ones. Here is a brief review of these theories.

1. String Theory – this mathematically rigorous theory has a rather simple underlying concept. From the early age of the development of physics, reductionism has always played the driving force of active research. We expect to find simpler things as we go deeper. Macroscopic objects to molecules, molecule to atoms, atom to its constituent particles - reductionism has always worked. Apparently dissimilar forces boil down to four fundamental forces. Problem occurs beyond this point, when a unification of these forces was much sought. Standard model required even many more particles to explain the intrinsic nature of the fundamental forces, and the old reductionism started to fail. At this point, it appeared string theory came up with a much-simplified idea of having all fundamental particles either force carriers (bosons) or that make matter (fermions) as different modes of vibration of the same string. A string can be a closed loop, which typically represents bosons, or open-ended which represents fermions [4].

String theory also introduces the concept of D-branes. A brane is a 2-dimensional membrane or analogous object in lower or higher dimensions. A D-brane or a Dirichlet-brane is a higher dimensional brane such that the two ends of open-ended strings are attached to either one single D-brane or two different D-branes [5]. Clearly this restricts how an open-ended string can vibrate. One of the vibrations can be associated with the gravitational field. In short, string theory appears to solve the problem of merging gravity with standard model, at least theoretically [6].

Problem with string theory is that the predictions are extremely difficult to test. For example, String theory predicts for the existence of 9+1 dimensions. This requires postulating six additional unobserved spatial dimensions which is not quite in agreement with the current experimental evidence.

2. Loop Quantum Gravity(LQG) – the leading alternative to string theory is loop quantum gravity. This approach uses the principles of general relativity as its starting point in an effort to quantize both space and time. The basic consideration of LQG is a granular structure of space which can be viewed as a network of finite quantized loops of size of Planck length. This network is technically known as a spin network, the time evolution of which is called a spin foam. These fine loops are thought to be excited gravitational fields. Unlike string theory, loop quantum gravity does not head for a theory of everything. It mainly aspires to solve the problem of quantum gravity, with having the advantage over string theory by not looking for higher dimensions. A length quantization similar to what we are going to present in this thesis has been shown in LQG [9].

The biggest flaw in loop quantum gravity is that it is not possible to show that a smooth spacetime can be extracted out of a quantized space. Also, like string theory the predictions of LQG are not quite testable yet [7].

3. Causal Set Theory – This approach is based on the assumption that the spacetime is fundamentally discrete and there is a one-to-one map between distinct past and future events [8]. The consequence of the causal set hypothesis is technically known as the dynamics of sequential growth. This theory identifies time as a birth process of consecutive spacetime events, also called the elements of causal set [10].

It is debatable that an initial assumption of discreteness of spacetime in any theory might have a conflict with Lorentz invariance. Causal set is able to address to this problem [11]. Despite being in an early stage of development, causal set theory successfully predicted the fluctuations in the value of the cosmological constant [10].

Although the dynamics has made progress, a complete theory is yet to come.

1.2 Quantum Gravity Phenomenology

1.2.1 Why Phenomenology

People have been working towards quantum gravity for over 70 years. All quantum gravity theories start with assumptions about the structure of spacetime at scales that are extremely small, way beyond the current experimental advancement. Because there is no direct experimental guidance, it is quite natural to try to develop a correct theory based on indirect criteria of conceptual restrictions. Like any other active field, what Quantum Gravity Phenomenology ideally needs is a combination of theory and doable experiments. At the moment, Quantum Gravity Phenomenology (QGP) can be thought of as a combination of all the studies that might contribute to direct or indirect observable predictions [12] [13] and analog models [14] supporting small and large scale structure of spacetime consistent with string theory or any other working formalism of quantum gravity. In this thesis we are more interested in the small scale structure of the spatial dimensions in connection with quantum gravity.

1.2.2 Goals of Quantum Gravity Phenomenology

The first step to identifying the relevant experiments for quantum gravity research would be the identification of the working scale of this new field. String theory suggests the characteristic scale where the quantum properties of spacetime become significant compared to the classical ones is the Planck scale which is $E_p \sim 10^{28} eV$

or the Planck length $\ell_{Pl} \sim 10^{-35}$ m [15]. This is a difficult part of quantum gravity phenomenology, i.e., to find ways to detect this very small scale quantum properties of spacetime. The solution of the quantum gravity problem should also be able to address the quantum picture of particles in the presence of weak as well as strong gravity. In other words, we hope quantum gravity phenomenology will helpful towards grand unification.

The validity of the Equivalence Principle in quantum gravity was first discussed in the mid 1970s with the famous "COW" experiment [16]. Experiments and modification involving the dynamics of matter in earth's gravitational field triggered question on the legitimacy of the Schrödinger equation [17]

$$\left[-\left(\frac{\hbar^2}{2M_I}\right)\nabla^2 + M_G\phi(\vec{r})\right]\psi(t,\vec{r}) = i\hbar\frac{\partial\psi(t,\vec{r})}{\partial t}, \tag{1.1}$$

where M_I denotes inertial mass and M_G denotes gravitational mass, $\phi(\vec{r})$ is gravitational potential.

There are no experiments that suggest the inertial and gravitational masses are different on earth. This might indicate a modification in the Schrödinger equation. String theory suggests a modification in the commutation relation between position and momenta, which leads to a modified Schrödinger's equation [32] as well.

One of the basic aims of quantum gravity phenomenologists is to find a way to test Planck-scale effects of spacetime, which also means providing with boundaries for the theoretical framework (within which a proper quantum theory of gravity is to be developed) and information on what is compatible with experimental data. This is particularly important as the phenomenology is still in its early stage of development. In this thesis, we will focus on one such experimental limit suggested by quantum gravity phenomenology which is also consistent with one of the candidate theories, viz., string theory. We will show how quantum gravity changes the classical idea of the spacetime continuum, by making the space around us discrete.

1.2.3 Uncertainties within Quantum Gravity

Since classical gravity is considered as a derived effect of deformation of spacetime, let us consider the case of distance fuzziness [17], an effect expected within quantum gravity. This also directly relates to the very basic principle of quantum mechanics, i.e., the uncertainty principle. Although the distance operator is affected by inherent uncertainties, usual quantum theory allows us to measure it exactly at the cost of complete obscurity of the conjugate observable (momentum). On the other hand, in the realm of quantum gravity, distance is likely to be subject to uncertainties that are not reducible. This uncertainty is often denoted by $\delta D \geq \ell_{Pl}$, which means the minimum variation in the distance measurement is of the order of Planck length. Some phenomenologists prefer to use a more general version, $\delta D \geq f(D, \ell_{Pl})$, where f is a function such that $f(D, 0) = 0$ [17].

The above idea of QGP-induced uncertainty might also suggest a modification in the usual Heisenberg's uncertainty principle, and incidentally string theory also suggest a similar idea of modified uncertainty principle which in fact goes by the name of Generalized Uncertainty Principle (GUP).

1.3 Generalized Uncertainty Principle from String Theory: Discreteness of Space

In a way, the motivation for quantizing gravity comes from the remarkable success of the quantum theories of the three other fundamental forces of nature and their interactions. Also, if not direct, experimental evidence suggests that gravity can show quantum effects. Analogue gravity experiments are among them [14].

String theory has emerged as the most promising candidate for a quantum theory of gravity. Among the many stringent mathematical results of string theory the one, which is of particular interest and relevant to quantum gravity phenomenology, is a modification of one of the basic principles of quantum mechanics, the uncertainty principle. The form string theory suggests is $\Delta x \geq \frac{\hbar}{\Delta p} + \alpha' \frac{\Delta p}{\hbar}$ [18–25], where $\sqrt{\alpha'} \approx 10^{-32}$cm [1].

Recently proposed doubly special relativity theories (DSRs) suggest a similar modification of position-momentum commutation relation [26–28] which leads to a modification of the uncertainty principle as well. A suggested form of commutator consistent with string theory is [12]

$$[x_i, p_j] = i\hbar \left(\delta_{ij} - \alpha \left(p\delta_{ij} + \frac{p_i p_j}{p} \right) + \alpha^2 (p^2 \delta_{ij} + 3 p_i p_j) \right), \tag{1.2}$$

where $p^2 = \sum_{i=1}^{3} p_i p_i, \alpha = \alpha_0 / M_{Pl} c = \alpha_0 \hbar \ell_{Pl}$,
M_{Pl} = Planck mass, ℓ_{Pl} = Planck length, $M_{Pl}c^2$ = Planck energy. So p can be interpreted as the magnitude of $\vec{p}$.
Then we get a Generalized Uncertainty Principle (GUP) [29–31],

$$\begin{aligned} \Delta x \Delta p &\geq \frac{\hbar}{2} \left[1 - 2\alpha < p > + 4\alpha^2 < p^2 > \right] \\ &\geq \frac{\hbar}{2} \left[1 + \left(\frac{\alpha}{\sqrt{< p^2 >}} + 4\alpha^2 \right) \Delta p^2 + 4\alpha^2 < p >^2 - 2\alpha \sqrt{< p >^2} \right]. \end{aligned} \tag{1.3}$$

Here, the dimensionless parameter α_0 is assumed to be of the order of unity.
Hence, modifying the position and momentum operators accordingly and applying this to a non-relativistic situation where a particle is trapped in a one-dimensional box one can find the GUP-corrected Schrödinger equation. It has been shown that the solution to this new equation gives rise to the result that the length of the box can

assume certain values only [32]. This result suggests that although the space looks smooth to us it the structure of so-called spacetime continuum, at Planck scale, is complex. We will discuss these results in the next section.

As discussed before *(section 1.2.3)*, quantum gravity phenomenology indicates an irremovable uncertainty in distance measurement. String theoretic modified commutation relation of position and momentum operators results in the generalized uncertainty principle which has a similar, but subtler consequence that the apparently continuous-looking space on a very fine scale is actually grainy. One can ask whether this is a sole influence of gravity or a fundamental structure of the spacetime. Now, if we admit the fact that classical gravity is a derived effect of curvature of spacetime caused by mass, we expect to find this discontinuity even in the regions of the universe far from a massive object, if the granular structure of the spatial dimensions is fundamental. The nature of this discreteness may or may not change when the spacetime is no more flat, i.e., in the presence of a gravitational field. In order to investigate that, we use a bottom-to-top approach as the geometry of spacetime is a manifestation of gravity as well. In our analysis *(chap 2 and 3)*, we trap a particle in a box with a gravitational potential inside the box and see if gravity influences the discreteness shown in [32, 33].

1.4 Discreteness in Flat Spacetime

1.4.1 Non-relativistic case

Now we briefly review the solved case of a particle in a box without the influence of gravity [32].

The modified position and momenta operators consistent with Eq.(1.2) and (1.3) are given by,

$$x_i = x_{0i}, \;\; p_i = p_{0i}(1 - \alpha p_0 + 2\alpha^2 p_0^2), \;\; i = 1, .., 3. \tag{1.4}$$

Here, $x_{0i}, \;\; p_{0i}$ satisfy the old canonical commutation relation $[x_{0i}, p_{0i}] = i\hbar\delta_{ij}$, which makes $p_{0i} = -i\hbar\frac{\partial}{\partial x_{0i}}$ the usual momentum (operator) at lower energy and p_i as momentum at higher energy. Like p, p_0 can be defined similar way, given by $p_0^2 = \sum\limits_{i=1}^{3} p_{0i}p_{0i}$ [32].

We see, the α dependent terms in all the above equations are only important when energies are comparable with Planck energy and lengths are comparable with the Planck length.

Following the above prescription, a usual Hamiltonian of the form $H = \frac{p^2}{2m} + V(\overrightarrow{r})$can be written as

$$H = \frac{p_0^2}{2m} + V(\overrightarrow{r}) - \frac{\alpha}{m} p_0^3 + \mathcal{O}(\alpha^2). \tag{1.5}$$

$$\tag{1.6}$$

The extra term in the above Hamiltonian can be viewed as a perturbation caused by Quantum Gravity effects which holds for any classical or quantum system. Now, if we consider a single test particle in one-dimensional box of length L, boundaries being at $x = 0$ and $x = L$, such that $V(\overrightarrow{r}) = V(x) = 0$ inside the box and $V = \infty$ outside, we can write the usual Schrödinger equation $H\psi = E\psi$ in the following form,

$$\frac{d^2}{dx^2}\psi + k_0^2\psi + 2i\alpha\hbar\frac{d^3}{dx^3}\psi = 0, \tag{1.7}$$

where $k_0 = \sqrt{2mE/\hbar^2}$. This is the GUP-corrected version of the Schrödinger equation for a particle in a one-dimensional box.

Let us consider a trial solution of the form $\psi = e^{mx}$. Using this trial solution the above equation becomes

$$m^2 + k_0^2 + 2i\alpha\hbar m^3 = 0 \tag{1.8}$$

It can be shown that this equation has the solution set to the leading order in α given by $m = ik_0^{'}, -ik_0^{''}, i/2\alpha\hbar$, where $k_0^{'} = k_0(1 + k_0\alpha\hbar)$ and $k_0'' = k_0(1 - k_0\alpha\hbar)$ [32]. The general solution to the GUP-corrected Schrödinger equation in flat spacetime is thus given by,

$$\psi = Ae^{ik_0^{'}x} + Be^{-ik_0''x} + Ce^{ix/2\alpha\hbar} \tag{1.9}$$

If we impose the boundary conditions the first two terms, with $k_0^{'} = k_0^{''} = k_0$, lead to the usual quantization of energy. It is to be noted that $\lim_{\alpha\to 0}|C| = 0$ because the last term should drop out in the $\alpha \to 0$ limit. This and making A real by absorbing any phase in ψ, under the boundary condition $\psi(0) = 0$ yield,

$$A + B + C = 0. \tag{1.10}$$

Substituting B from the above in Eq.(1.9),

$$\psi = 2iA\sin(k_0) + C\left[-e^{-ik_0x} + eix/2\alpha\hbar\right] - \alpha\hbar k_0^2 x\left[iCe^{-ik_0x} + 2A\sin(kx)\right] \tag{1.11}$$

The other boundary condition $\psi(L) = 0$ gives,

$$\begin{aligned} 2iA\sin(k_0L) = |C|\left[e^{-i(k_0L+\theta_C)} - e^{i(L/2\alpha\hbar-\theta_0)}\right] + \\ \alpha\hbar k_0^2 L\left[i|C|e^{-i(k_0L+\theta_C)} + 2A\sin(k_0L)\right], \end{aligned} \tag{1.12}$$

where $C = |C|e^{-i\theta_C}$. It is easy to notice that both sides of Eq. (1.12) vanish in the limit $\alpha \longrightarrow 0$, when $k_0 L = n\pi, n$ is an integer and $C = 0$ which in turn means when α is not zero $k_0 L$ must be equal to $n\pi$ plus a small real number ϵ_0, where $\lim_{\alpha \longrightarrow 0} \epsilon_0 = 0$. Also the term containing $\alpha|C|$ on the RHS of Eq.(1.12) has a faster convergence to zero in the same limit compared to $\mathcal{O}(\alpha)$ so it can easily be ignored. Now, collecting real parts of the rest of the equation and considering $\sin(n\pi + \epsilon_0) \approx 0$ we get [32]

$$\cos\left(\frac{L}{2\alpha\hbar} - \theta_C\right) = \cos(k_0 L + \theta_C) = \cos(n\pi + \theta_C + \delta_0) \tag{1.13}$$

which implies

$$\frac{L}{2\alpha\hbar} = \frac{L}{2\alpha_0 \ell_{Pl}} = n\pi + 2q\pi + 2\theta_C \equiv p\pi + 2\theta_C \tag{1.14}$$

$$\frac{L}{2\alpha\hbar} = \frac{L}{2\alpha_0 \ell_{Pl}} = -n\pi + 2q\pi \equiv p\pi, \tag{1.15}$$

where $p \equiv 2q \pm n$ is a natural number.

The above equations clearly show that L is a quantized quantity. This result can be interpreted as the fact that, like the energy of the particle inside the box, the length of the box can assume certain values only. In particular, L has to be in units of $\alpha_0 \ell_{Pl}$.

This indicates that the space, at least in a confined region and without the influence of gravity, is likely to be discrete.

1.4.2 Relativistic one-dimensional case

Further work has shown that this consequence of the GUP can be extended to relativistic scenarios in one, two and three dimensions [33]. There are several reasons for why we need relativistic cases. High energy particles are much more likely to probe the fabric of spacetime near the Planck scale, which means they are necessarily relativistic and ultra-relativistic particles. Also, the fact, that most elementary particles are fermions, replaces the Schrodinger equation with the Dirac equation.

We need the Klein-Gordon equation which is the simplest equivalent of the Schrödinger equation for such relativistic particles. For our one-dimensional box, the Klein-Gordon equation

$$p^2\Psi(t,x) = \left(\frac{E^2}{c^2} - m^2c^2\right)\Psi(t,x). \tag{1.16}$$

It is easy to see that this is identical to the Schrödinger equation, by making the connection: $2mE/\hbar^2 \equiv k_0^2 \to \frac{E^2}{\hbar^2c^2} - \frac{m^2c^2}{\hbar^2}$. By arguing that the quantization of length of the box does not depend on k_0, we can safely deduce that the same would apply to this case as well [33].

For a higher dimensional case, we no longer use the the Klein-Gordon equation for the non-locality of the differential operators [33]. Instead, we use the Dirac equation, the other reason being the fact that most of the fundamental particles are fermions. Thus, it is worthwhile to investigate how the discreteness of space changes under the Dirac equation even if we restrict ourselves to one dimension.
Using the Dirac matrix notations, the GUP-corrected Dirac equation is written as,

$$\begin{aligned} H\psi(\vec{r}) &= (c\vec{\alpha}.\vec{p} + \beta mc^2)\psi(\vec{r}) \\ &= (c\vec{\alpha}.\vec{p_0} - c\alpha(\vec{\alpha}.\vec{p_0})(\vec{\alpha}.\vec{p_0}) + \beta mc^2)\psi(\vec{r}). \\ &= E\psi(\vec{r}). \end{aligned} \tag{1.17}$$

For one spatial dimension, z for example, in position representation, this becomes

$$\left(-i\hbar c\alpha_z \frac{d}{dz} + c\alpha\hbar^2 \frac{d^2}{dz^2} + \beta mc^2\right)\psi(z) = E\psi(z). \tag{1.18}$$

The two linearly independent, positive energy solutions to the above equation are given by [33],

$$\psi_1 = N_1 e^{i\kappa z}\begin{pmatrix} \chi \\ r\sigma_z\chi \end{pmatrix} \tag{1.19}$$

$$\psi_2 = N_2 e^{iz/\alpha\hbar}\begin{pmatrix} \chi \\ \sigma_z\chi \end{pmatrix}, \tag{1.20}$$

where $\kappa = \kappa_0 + \alpha\hbar\kappa_0^2$, κ_0 being the wave number that satisfies $E^2 = (\hbar\kappa_0)^2 + (mc^2)^2$, $r = \frac{\hbar\kappa_0 c}{E+mc^2}$ and $\chi^\dagger\chi = I$.

Using the MIT bag model and imposing boundary conditions on the two wavefunctions, the following relations can be established [33],

$$\kappa L = \delta = \arctan\left(-\frac{\hbar\kappa}{mc}\right) + \mathcal{O}(\alpha), \tag{1.21}$$

$$\frac{L}{\alpha\hbar} = \frac{L}{\alpha_0 \ell_{Pl}} = 2p\pi - \frac{\pi}{2}, \quad p \in \mathbb{N}. \tag{1.22}$$

Eq.(1.21) gives the energy quantization and Eq.(1.22) is the condition for length quantization for relativistic situations. It can be shown that the non-relativistic limit of this equation gives the quantization condition that was obtained from the Schrödinger equation in the previous section.

1.4.3 Relativistic two and three-dimensional cases

As mentioned before, we use the Dirac equation when two and three dimensions are considered. If we define the box under consideration by $0 \leq x_i \leq L_i$, i=1,..,d where d can be 1, 2 or 3 depending the the dimension of the box, and assume the following form of the wavefunction

$$\psi = e^{i\vec{t}.\vec{r}} \begin{pmatrix} \chi \\ r\vec{\rho}.\vec{\sigma}\chi \end{pmatrix}, \tag{1.23}$$

where $\vec{t}$ and $\vec{\rho}$ are two spatial vectors of dimension d and $\chi^\dagger\chi = 1$, the Hamiltonian given by Eq.(1.17) becomes,

$$H\psi = e^{i\vec{t}.\vec{r}} \begin{pmatrix} \left((mc^2 - c\alpha\hbar^2 t^2) + c\hbar(\vec{t}.\vec{\rho} + i\sigma.(\vec{t}\times\vec{\rho}))\right)\chi \\ \left(c\hbar\vec{t} - (mc^2 + c\alpha\hbar^2 t^2)\vec{\rho}\right).\vec{\sigma}\chi \end{pmatrix} \tag{1.24}$$

and the two linearly independent (for a particular spinor χ) and positive energy solutions follow [33],

$$\psi_1 = N_1 e^{i\vec{\kappa}.\vec{r}} \begin{pmatrix} \chi \\ r\hat{\kappa}.\vec{\sigma}\chi \end{pmatrix} \tag{1.25}$$

$$\psi_2 = N_2 e^{i\frac{\hat{q}.\vec{r}}{\alpha\hbar}} \begin{pmatrix} \chi \\ \hat{q}.\vec{\sigma}\chi \end{pmatrix}. \tag{1.26}$$

Note that ψ_2 is the non-perturbative solution comes to existence because of GUP. This new wavefunction gives rise to an additional condition which yields the quantization of length along each direction the box independently [33],

$$k_k L_k = \delta_k = \arctan\left(-\frac{\hbar k_k}{mc}\right) + \mathcal{O}(\alpha) \tag{1.27}$$

$$\frac{|\hat{q}_k| L_k}{\alpha\hbar} = \frac{\hat{q}_k L_k}{\alpha_0 ell_{Pl}} = 2p_k\pi - 2\theta_k, \qquad p_k \in \mathbb{N} \tag{1.28}$$

where k is the index corresponding to the axis of consideration, $\hat{q}_k$ is the kth component of the unit vector $\hat{q}$ along $\vec{\rho}$ and $\theta_k = \arctan(\hat{q}_k)$. Eq.(1.27) gives the energy quantization for d dimensions when we have d such equations for $k = 1, .., d$. Eq.(1.28) yields the length quantization along x_k axis. $|\hat{q}_k| = \frac{n_k}{\sqrt{\sum_{i=1}^{d} n_i^2}}$. If we consider the symmetric case where no direction in space is preferred, $n_1 = n_2 = ... = n_d$, we get $|\hat{q}_k| = \frac{1}{\sqrt{d}}$, $d = 1, 2, 3$; in which case Eq.(1.28) becomes

$$\frac{L_k}{\alpha_0 \ell_{Pl}} = (2p_k\pi - 2\theta_k)\sqrt{d}, \qquad p_k \in \mathbb{N}. \tag{1.29}$$

It is easy to see if we set $d = 1$ (one-dimensional case), $2\theta_k = \arctan(1) = \frac{\pi}{2}$ and the above equation reduces to Eq.(1.22). Moreover, we obtain area (N=2) and volume (N=3) quantization from Eq.(1.29) as below [33],

$$A_N = \prod_{k=1}^{N} \frac{L_k}{\alpha_0 \ell_{Pl}} = d^{N/2} \prod_{k=1}^{N} (2p_k\pi - 2\theta_k), \qquad p_k \in \mathbb{N}. \tag{1.30}$$

Chapter 2

Discreteness of Space from GUP : Non-relativistic Case

2.1 Discreteness of Space in Presence of Gravity

So far, it has been shown the GUP effects imposed on free particles, lead to discreteness of space. Although our test particle was kept in a box, presence of any force field inside the box was not assumed. If we wish to claim that the quantum gravity effects are universal we hope to see the length quantization valid for any situation in presence of any force. Also, the results must not be limited to the Dirac equation, i.e., for fermions only. One must expect to have similar length area and volume quantization in context with bosons as well. In other words, discreteness of space must hold whether or not there is an external field present.Although, in this thesis we restrict ourselves to fermions.

The first step towards this generalization is to consider gravity as the external force field inside our box, since it is the weakest among the four fundamental forces and also gravity is universal. Also, as we have discussed in *sec 1.3*, our particular interest is to find how gravity determines the nature of discreteness. With a gravitational potential present inside the box, we ignore all but the first term of Taylor expansion of the potential, which is linear. This is reasonable because we are interested in the behavior or spacetime fabric near Planck scale and gravitational potential changes only at a very slow rate, over such small distances .

In practice, we often use the gravitational potential energy approximated as $V(h) =$

mgh over a small vertical distance h and the field $E_h = -\frac{1}{m}\frac{\partial dV(h)}{\partial dh} = -g$. This justifies the previous claim of using a linearized potential term as well.

First, we consider a toy model – a particle in a one-dimensional box with a linear potential inside. We do not wish to involve GUP here. This is a case of the usual Schröginger equation with a potential. The intension is to use this solution in the actual problem with GUP.

In order to show discreteness of space in the presence of gravity we start with the simplest case of a particle in a one-dimensional box. We will show that the new solution to the Schrödinger equation will reduce to the usual solution under proper limits.

This is a case without incorporating the GUP. We intend to use this as a model for solving the actual problem. So, the Schrödinger equation has its usual form [34], with a potential term.

2.2 Solution of Schrödinger Equation with a One-dimensional Linear Potential

Let us consider a one-dimensional box of length L with a linear potential inside, which has the form

$$V(x) = \begin{cases} kx & \text{if } 0 \leq x \leq L \\ \infty & \text{otherwise.} \end{cases} \tag{2.1}$$

k is a parameter of unit J/m. Smallness of k is assumed.

The Schrödinger equation governing the motion of a particle of mass m inside the box $(0 \leq x \leq L)$ [34],

$$\frac{d^2\psi(x)}{dx^2} - \frac{2m}{\hbar^2}(kx - E)\psi(x) = 0 \ . \tag{2.2}$$

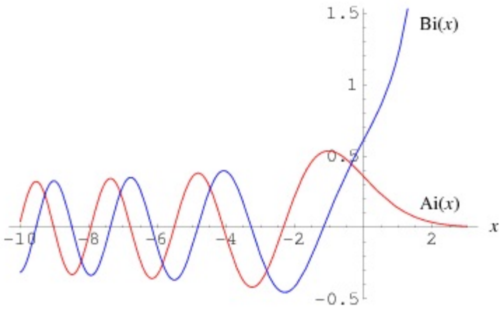

Figure 2.1: Airy functions and the zeroes *[35]*

$\psi(x) = 0$ when $x < 0$ or $x > L$ because the potential outside the box becomes ∞. The above is an Airy equation which has the exact general solution given by [35],

$$\psi(x) = C_1 Ai\left[\frac{\frac{2m}{\hbar^2}(kx - E)}{(\frac{2m}{\hbar^2}k)^{\frac{2}{3}}}\right] + C_2 Bi\left[\frac{\frac{2m}{\hbar^2}(kx - E)}{(\frac{2m}{\hbar^2}k)^{\frac{2}{3}}}\right] , \tag{2.3}$$

where $Ai[u]$ and $Bi[u]$ are Airy functions of the first and second kind respectively. We plan on using this wavefunction for solving the GUP-corrected Schrödinger equation for a particle in a box with a linear potential.

Before proceeding, we need to verify that this wavefunction reduces to the solution corresponding to old Schrödinger equation for an infinite potential well, if we let the potential factor k go to zero.

For a weak potential we can assume : $E > V(x)$

Using the WKB approximation methods, we get

$$\int_0^L p(x)\,dx = n\pi\hbar \; (n = 1, 2, 3, ...), \tag{2.4}$$

where $p(x) \equiv \sqrt{2m[E - V(x)]}$.

This in turn gives the energy as,

$$\begin{aligned} E &= \frac{1}{4}(\sqrt{E_n^0} \pm \sqrt{E_n^0 + ka}) \\ \Rightarrow E_n &= E_n^0 + \frac{1}{2}ka + O\left(\left(\frac{1}{E_n^0}\right)^m\right),\ n = 1, 2, 3, .., \end{aligned} \tag{2.5}$$

where $E_n^0 = \frac{n^2\pi^2\hbar^2}{2ma^2}$.

In order to find the limiting forms of the energy and the wavefunction, we consider the asymptotic form of the Airy functions.

$$\begin{aligned} Ai(-\xi) = \frac{1}{\sqrt{\pi}}\xi^{\frac{-1}{4}}\sin(z + \frac{\pi}{4}) \\ Bi(-\xi) = \frac{1}{\sqrt{\pi}}\xi^{\frac{-1}{4}}\cos(z + \frac{\pi}{4}), \end{aligned} \tag{2.6}$$

when ξ is very large.

Here $z = \frac{2}{3}\xi^{\frac{3}{2}}$ and $\xi \equiv \frac{(\frac{2m}{\hbar^2})^{\frac{1}{3}}(E-kx)}{k^{\frac{2}{3}}}$.

The use of asymptotic forms is justified as ξ is very large in the limit $k \to 0$.

$$\begin{aligned} z &= \frac{2}{3}\left(\frac{(\frac{2m}{\hbar^2})^{\frac{1}{3}}(E-kx)}{k^{\frac{2}{3}}}\right)^{\frac{3}{2}} \\ &\approx \frac{2}{3}\left(\frac{2m}{\hbar^2}\right)^{\frac{1}{2}} E^{\frac{3}{2}}\left(\frac{1}{k} - \frac{3}{2}\frac{x}{E}\right) \qquad \text{(for small } k) , \end{aligned}$$

plugging this into Eq.(2.6) we get,

$$\begin{aligned}
\lim_{k\to 0} Ai(-\xi) &= \lim_{k\to 0} Ai\left(-\left[(\frac{2m}{\hbar^2})^{\frac{1}{3}}\frac{E-kx}{k^{\frac{2}{3}}}\right]\right) \\
&= \frac{1}{\sqrt{\pi}}\left[\left(\frac{2m}{\hbar^2}\right)^{\frac{1}{3}}\frac{E-kx}{k^{\frac{2}{3}}}\right]^{-1/4}\sin\left\{\frac{2}{3}\left(\frac{2m}{\hbar^2}\right)^{\frac{1}{2}}E^{\frac{3}{2}}\left(\frac{1}{k}-\frac{3}{2}\frac{x}{E}\right)+\frac{\pi}{4}\right\} \\
&= H\sin\left\{H_1\left(\frac{1}{k}-\frac{3}{2}\frac{x}{E}\right)+\frac{\pi}{4}\right\}\ , \qquad (2.7)
\end{aligned}$$

where $H=\frac{1}{\sqrt{\pi}}\left[(\frac{2m}{\hbar^2})^{\frac{1}{3}}\frac{E-kx}{k^{\frac{2}{3}}}\right]^{-\frac{1}{4}}$ and $H_1=\frac{2}{3}\left(\frac{2m}{\hbar^2}\right)^{\frac{1}{2}}E^{\frac{3}{2}}$.

Similarly,

$$\lim_{k\to 0} Bi(-\xi) = H\cos\left(H_1\left(\frac{1}{k}-\frac{3}{2}\frac{x}{E}\right)+\frac{\pi}{4}\right). \qquad (2.8)$$

Now, in this limit

$$\begin{aligned}
Ai(-\xi) &= H\sin\left(\left[\frac{H_1}{k}+\frac{\pi}{4}\right]-\frac{3}{2}H_1\frac{x}{E}\right) \\
&= H\sin\left(\frac{H_1}{k}+\frac{\pi}{4}\right)\cos\left(\frac{3}{2}H_1\frac{x}{E}\right)-H\sin\left(\frac{3}{2}H_1\frac{x}{E}\right)\cos\left(\frac{H_1}{k}+\frac{\pi}{4}\right)
\end{aligned}$$

and

$$\begin{aligned}
Bi(-\xi) &= H\cos\left(\left[\frac{H_1}{k}+\frac{\pi}{4}\right]-\frac{3}{2}H_1\frac{x}{E}\right) \\
&= H\cos\left(\frac{H_1}{k}+\frac{\pi}{4}\right)cos\left(\frac{3}{2}H_1\frac{x}{E}\right)+H\sin\left(\frac{H_1}{k}+\frac{\pi}{4}\right)\sin\left(\frac{3}{2}H_1\frac{x}{E}\right)
\end{aligned}$$

$$\begin{aligned}
H &\sim \left(\frac{E-kx}{k^{\frac{2}{3}}}\right)^{-1/4} \\
&= E^{-1/4}k^{\frac{1}{6}}\left(1-\frac{kx}{E}\right)^{-1/4}
\end{aligned}$$

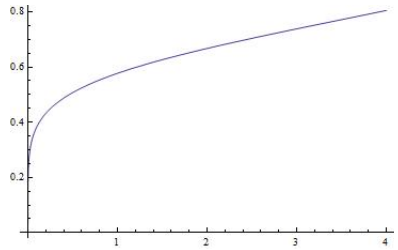

Figure 2.2: H as a function of k (for particles with finite mass and large but finite Energy)

Plugging all these into Eq.(2.3),

$$
\begin{aligned}
\psi(x) &= C_1 Ai\left[-\frac{\left(\frac{2m}{\hbar^2}\right)^{\frac{1}{3}}(kx-E)}{k^{\frac{2}{3}}}\right] + C_2 Bi\left[-\frac{\left(\frac{2m}{\hbar^2}\right)^{\frac{1}{3}}(kx-E)}{k^{\frac{2}{3}}}\right] \\
&= C_1 Ai[-\xi] + C_2 Bi[-\xi] \\
&= C_1\left[H\sin\left(\frac{H_1}{k}+\frac{\pi}{4}\right)\cos\left(\frac{3}{2}H_1\frac{x}{E}\right) - H\sin\left(\frac{3}{2}H_1\frac{x}{E}\right)\cos\left(\frac{H_1}{k}+\frac{\pi}{4}\right)\right] \\
&\quad + C_2\left[H\cos\left(\frac{H_1}{k}+\frac{\pi}{4}\right) cos\left(\frac{3}{2}H_1\frac{x}{E}\right) + H\sin\left(\frac{H_1}{k}+\frac{\pi}{4}\right)\sin\left(\frac{3}{2}H_1\frac{x}{E}\right)\right].
\end{aligned}
\tag{2.9}
$$

Here, $\frac{H_1}{E} = \frac{2}{3}\frac{\sqrt{2mE}}{\hbar}$

$\Rightarrow \psi(x) = A\sin\left(\frac{\sqrt{2mE}}{\hbar}x\right) + B\cos\left(\frac{\sqrt{2mE}}{\hbar}x\right)$.

One can easily identify this is the solution of the Schrödinger equation for a particle in an infinite potential well [34].

Therefore we have shown in the limit $k \to 0$ Eq.(2.3) reduces to the usual wavefunction without any potential, which in turn proves the robustness the solution.

Next, we try to solve the GUP-induced Schrödinger equation with the same potential.

2.3 GUP-corrected Schrödinger Equation with a One-dimensional Linear Potential

Now that we have worked our way through a simple model with a weakly varying gravitational potential, we can use this to a similar situation where we incorporate GUP effects. We use the Hamiltonian given by Eq.(1.5), but in this case $V(x) = kx$ inside the box and $V(x) = \infty$ ouside.

2.4 Solution of the GUP-corrected Schrödinger Equation:

Here. we are going to use the Schrödinger equation given by Eq.(1.7) with the changes caused by this potential $V(\vec{r}) = kx$ inside the box and $V(\vec{r}) = \infty$ outside. Writing the Schrödinger equation with the modified Hamiltonian,

$$2i\alpha\hbar\frac{d^3}{dx^3}\psi + \frac{d^2}{dx^2}\psi + \frac{2m}{\hbar^2}(E - kx)\psi = 0 \tag{2.10}$$

2.4.1 Perturbative Solutions

This is a third order linear differential equation. We hope to have the third solution lead to a similar criterion [32] that would allow us to show the length of the box can only assume some specific values.

Now, the above equation can be thought of as consisting of two parts,

part I: $\frac{d^2}{dx^2}\psi + \frac{2m}{\hbar^2}(E - kx)\psi$

and part II: $2i\alpha\hbar\frac{d^3}{dx^3}\psi$.

We intend to use a trial solution $\psi_1 = \psi_0(E + c\alpha, k, x)$, where the form of ψ_0 is given by Eq.(2.3), and claim that this is the solution we are looking for. It is to be noted that $\psi_0(E, k, x)$ is the solution to part I

$$\begin{aligned}\psi_1 &= \psi_0(E+c\alpha,k,x)\\ &= \psi_0(E,k,x)+c\alpha\frac{d}{dE}\psi_0(E,k,x)\end{aligned} \tag{2.11}$$

Substituting ψ_1 in place of ψ in part I gives $0 + c\alpha$ [terms containing $\frac{d}{dE}\psi_0$].

Since we are not interested in terms containing α^2, we can just use $\psi_1 = \psi_0(E, k, x)$ for part II which in turn yields α[terms containing derivatives of $\psi_0(E, k, x)$].

So, we get

α[terms containing derivatives of $\psi_0(E, k, x)$]$+0 + c\alpha$ [terms containing $\frac{d}{dE}\psi_0$].

In order to have this expression to be zero for a value of c the terms in brackets would have to be either independent of x or of the same leading order terms. In the second case, we would argue that for small values of x we could neglect them and be able to solve for c.

Combining all these along with Eqs.(2.10) and (2.11) we get

$$\begin{aligned}&2i\alpha\hbar\frac{d^3}{dx^3}\psi_0(E,k,x)+2i\alpha\hbar c\alpha\frac{d^3}{dx^3}\left(\frac{d}{dE}\psi_0(E,k,x)\right)+c\alpha\frac{d^2}{dx^2}\left(\frac{d}{dE}\psi_0(E,k,x)\right)\\ &+\frac{2m}{\hbar^2}(E-kx)c\alpha\frac{d}{dE}\psi_0(E,k,x)\\ &=\alpha\left[2i\hbar\frac{d^3}{dx^3}\psi_0(E,k,x)\right]+c\alpha\left[\frac{d^2}{dx^2}\left(\frac{d}{dE}\psi_0(E,k,x)\right)+\frac{2m}{\hbar^2}(E-kx)\frac{d}{dE}\psi_0(E,k,x)\right],\end{aligned} \tag{2.12}$$

where

$$\begin{aligned}\psi_0(E,k,x) &= AAi(-\xi)+BBi(-\xi)\\ &= \frac{A}{\sqrt{\pi}}\xi^{-1/4}\sin\left(\frac{2}{3}\xi^{\frac{3}{2}}+\frac{\pi}{4}\right)+\frac{B}{\sqrt{\pi}}\xi^{-1/4}\cos\left(\frac{2}{3}\xi^{\frac{3}{2}}+\frac{\pi}{4}\right)\end{aligned} \tag{2.13}$$

$\xi = \left(\frac{2m}{\hbar^2}\right)^{\frac{1}{3}} k^{-\frac{2}{3}}(E - kx).$

Now, plugging the derivatives (refer to appendix C) into Eq.(2.12),

$$\alpha\left[2i\hbar\frac{d^3}{dx^3}\psi_0(E,k,x)\right] + c\alpha\left[\frac{d^2}{dx^2}\left(\frac{d}{dE}\psi_0(E,k,x)\right) + \frac{2m}{\hbar^2}(E-kx)\frac{d}{dE}\psi_0(E,k,x)\right]$$
$$= \alpha\left[2i\hbar\left(\frac{d\xi}{dx}\right)^3\frac{d^3\psi_0}{d\xi^3}\right] + c\alpha\left[\left(\frac{d\xi}{dx}\right)^2\frac{d\xi}{dE}\frac{d^3\psi_0}{d\xi^3} + \frac{2m}{\hbar^2}(E-kx)\frac{d\xi}{dE}\frac{\psi_0}{d\xi}\right]. \tag{2.14}$$

For simplicity, we are going to use one of the two independent solutions ψ_0^I and ψ_0^{II} as ψ_0 in Eq.(2.14). Our intention is to check if coefficients of α and $c\alpha$ in the above expression have at least the same leading order terms.

Using $\psi_0 = \psi_0^I$,

$$\alpha\left[2i\hbar\left(\frac{d\xi}{dx}\right)^3\frac{d^3\psi_0}{d\xi^3}\right] + c\alpha\left[\left(\frac{d\xi}{dx}\right)^2\frac{d\xi}{dE}\frac{d^3\psi_0}{d\xi^3} + \frac{2m}{\hbar^2}(E-kx)\frac{d\xi}{dE}\frac{\psi_0}{d\xi}\right]$$
$$= \alpha\left[2i\hbar\left(-\frac{2m}{\hbar^2}\right)k\left(-\frac{1}{\sqrt{\pi}}\left[\frac{3}{4}\xi^{-1/4}\sin\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right) + \xi^{5/4}\cos\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)\right]\right)\right]$$
$$+c\alpha\left[\left(\frac{2m}{\hbar^2}\right)^{2/3}k^{2/3}\left(\frac{2m}{\hbar^2}\right)^{1/3}\frac{-k^{-2/3}}{\sqrt{\pi}}\left(\frac{3}{4}\xi^{-1/4}\sin\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right) + \xi^{5/4}\cos\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)\right)\right.$$
$$\left.+\frac{2m}{\hbar^2}(E-kx)\left(\frac{2m}{\hbar^2}\right)^{1/3}k^{-2/3}\frac{1}{\sqrt{\pi}}\left(-\frac{1}{4}\xi^{-5/4}\sin\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right) + \xi^{1/4}\cos\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)\right)\right]$$

$$
\begin{aligned}
&= \alpha(2i\hbar)\left(\frac{2m}{\hbar^2}\right)\left(\frac{k}{\sqrt{\pi}}\right)\left[\frac{3}{4}\left(\left(\frac{2m}{\hbar^2}\right)^{1/3}k^{-2/3}\right)^{-1/4}(E-kx)^{-1/4}\sin\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)\right]\\
&+\alpha(2i\hbar)\left(\frac{2m}{\hbar^2}\right)\left(\frac{k}{\sqrt{\pi}}\right)\left[\left(\left(\frac{2m}{\hbar^2}\right)^{1/3}k^{-2/3}\right)^{5/4}(E-kx)^{5/4}\cos\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)\right]\\
&-c\alpha\left(\frac{2m}{\hbar^2}\right)\left(\frac{1}{\sqrt{\pi}}\right)\left[\frac{3}{4}\left(\left(\frac{2m}{\hbar^2}\right)^{1/3}k^{-2/3}\right)^{-1/4}(E-kx)^{-1/4}\sin\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)\right]\\
&-c\alpha\left(\frac{2m}{\hbar^2}\right)\left(\frac{1}{\sqrt{\pi}}\right)\left[\left(\left(\frac{2m}{\hbar^2}\right)^{1/3}k^{-2/3}\right)^{5/4}(E-kx)^{5/4}\cos\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)\right]\\
&-\frac{c\alpha}{\sqrt{\pi}}\left[\frac{1}{4}\left(\frac{2m}{\hbar^2}\right)^{11/12}k^{1/6}\,(E-kx)^{-1/4}\sin\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)\right.\\
&\left.-\left(\frac{2m}{\hbar^2}\right)^{17/12}k^{-5/6}\,(E-kx)^{5/4}\cos\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)\right]
\end{aligned}
\tag{2.15}
$$

Since we are interested in the terms containing powers of x, it is to be noted that the above expression would give the same leading order term for both α and $c\alpha$.

Expanding the above expression using Taylor series and collecting coefficients of

x^0 (ignoring terms containing x and higher orders of x),

$$\alpha(2i\hbar)\left(\frac{2m}{\hbar^2}\right)\left(\frac{k}{\sqrt{\pi}}\right)\left[\frac{3}{4}\left(\left(\frac{2m}{\hbar^2}\right)^{1/3}k^{-2/3}\right)^{-1/4}E^{-1/4}\sin\left(\xi_0+\frac{\pi}{4}\right)+\right.$$
$$\left.\left(\left(\frac{2m}{\hbar^2}\right)^{1/3}k^{-2/3}\right)^{5/4}E^{5/4}\cos\left(\xi_0+\frac{\pi}{4}\right)\right]$$
$$=\frac{c\alpha}{\sqrt{\pi}}\frac{2m}{\hbar^2}\left[\frac{3}{4}\left(\left(\frac{2m}{\hbar^2}\right)^{1/3}k^{-2/3}\right)^{-1/4}E^{-1/4}\sin\left(\xi_0+\frac{\pi}{4}\right)+\right.$$
$$\left.\left(\left(\frac{2m}{\hbar^2}\right)^{1/3}k^{-2/3}\right)^{5/4}E^{5/4}\cos\left(\xi_0+\frac{\pi}{4}\right)\right]$$
$$+\frac{c\alpha}{\sqrt{\pi}}\left[\frac{1}{4}\left(\frac{2m}{\hbar^2}\right)^{11/12}k^{1/6}E^{-1/4}\sin\left(\xi_0+\frac{\pi}{4}\right)-\left(\frac{2m}{\hbar^2}\right)^{17/12}k^{-5/6}E^{5/4}\cos\left(\xi_0+\frac{\pi}{4}\right)\right] \tag{2.16}$$

$$\Rightarrow 2i\hbar\left[\frac{3}{4}\left(\frac{2m}{\hbar^2}\right)^{11/12}k^{7/6}E^{-1/4}\sin\left(\xi_0+\frac{\pi}{4}\right)+\left(\frac{2m}{\hbar^2}\right)^{17/12}k^{1/6}E^{5/4}\cos\left(\xi_0+\frac{\pi}{4}\right)\right]$$
$$=c\left(\frac{2m}{\hbar^2}\right)^{11/12}k^{1/6}E^{-1/4}\sin\left(\xi_0+\frac{\pi}{4}\right) \tag{2.17}$$

$$\begin{aligned}\Rightarrow c &= 2i\hbar\frac{\frac{3}{4}\left(\frac{2m}{\hbar^2}\right)^{11/12}k^{7/6}E^{-1/4}\sin\left(\xi_0+\frac{\pi}{4}\right)+\left(\frac{2m}{\hbar^2}\right)^{17/12}k^{1/6}E^{5/4}\cos\left(\xi_0+\frac{\pi}{4}\right)}{\left(\frac{2m}{\hbar^2}\right)^{11/12}k^{1/6}E^{-1/4}\sin\left(\xi_0+\frac{\pi}{4}\right)}\\ &= 2i\hbar\left(\frac{3}{4}k+\frac{E^{3/2}}{\tan(\xi_0+\frac{\pi}{4})}\sqrt{\frac{2m}{\hbar^2}}\right)\end{aligned} \tag{2.18}$$

where,

$$\xi_0=\frac{2}{3}\left(\left(\frac{2m}{\hbar^2}\right)^{1/3}k^{-2/3}E\right)^{3/2}$$

Now we are going to use the actual wavefunction in expression (3.31).

$\psi_0 = C_1\psi_0^I + C_2\psi_0^{II}$

For $C_1\psi_0^I$ part, (3.31) yields,

$$\alpha(2i\hbar)\frac{C_1}{\sqrt{\pi}}\left[\frac{3}{4}\left(\frac{2m}{\hbar^2}\right)^{11/12}k^{7/6}(E-kx)^{-1/4}\sin\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)+\right.$$
$$\left.\left(\frac{2m}{\hbar^2}\right)^{17/12}k^{1/6}(E-kx)^{5/4}\cos\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)\right]$$
$$=(c\alpha)\frac{C_1}{\sqrt{\pi}}\left[\left(\frac{2m}{\hbar^2}\right)^{11/12}k^{1/6}(E-kx)^{-1/4}\sin\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)\right], \tag{2.19}$$

and for $C_1\psi_0^{II}$ part, (3.31) gives,

$$\alpha(2i\hbar)\frac{C_2}{\sqrt{\pi}}\left[\left(\frac{2m}{\hbar^2}\right)^{17/12}k^{1/6}(E-kx)^{5/4}\sin\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)-\right.$$
$$\left.\frac{3}{4}\left(\frac{2m}{\hbar^2}\right)^{11/12}k^{7/6}(E-kx)^{-1/4}\cos\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)\right]$$
$$=c\alpha\frac{C_2}{\sqrt{\pi}}\left[-\left(\frac{2m}{\hbar^2}\right)^{11/12}k^{1/6}(E-kx)^{-1/4}\cos\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)\right].$$

Combining them we get,

$$\alpha(2i\hbar)\frac{3}{4}\left(\frac{2m}{\hbar^2}\right)^{11/12}k^{7/6}(E-kx)^{-1/4}\left[C_1\sin\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)-C_2\cos\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)\right]+$$
$$\alpha(2i\hbar)\frac{3}{4}\left(\frac{2m}{\hbar^2}\right)^{17/12}k^{1/6}(E-kx)^{5/4}\left[C_2\sin\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)+C_1\cos\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)\right]$$
$$=c\alpha\left(\frac{2m}{\hbar^2}\right)^{11/12}k^{1/6}(E-kx)^{-1/4}\left[C_1\sin\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)-C_2\cos\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)\right]. \tag{2.20}$$

Expanding the above equation using Taylor series and collecting coefficients of x^0

(ignoring terms containing x and higher orders of x),

$$\begin{aligned}
&\alpha(2i\hbar)\frac{3}{4}\left(\frac{2m}{\hbar^2}\right)^{11/12} k^{7/6}E^{-1/4}\left[C_1\sin(\xi_0+\frac{\pi}{4})-C_2\cos(\xi_0+\frac{\pi}{4})\right] \\
&+\alpha(2i\hbar)\left(\frac{2m}{\hbar^2}\right)^{17/12} k^{1/6}E^{5/4}\left[C_2\sin(\xi_0+\frac{\pi}{4})-C_1\cos(\xi_0+\frac{\pi}{4})\right] \\
&= c\alpha\left(\frac{2m}{\hbar^2}\right)^{11/12} k^{1/6}E^{-1/4}\left[C_1\sin(\xi_0+\frac{\pi}{4})-C_2\cos(\xi_0+\frac{\pi}{4})\right]
\end{aligned} \tag{2.21}$$

So,

$$\begin{aligned}
c = &\left[(2i\hbar)\frac{3}{4}\left(\frac{2m}{\hbar^2}\right)^{11/12} k^{7/6}E^{-1/4}\left(C_1\sin(\xi_0+\frac{\pi}{4})-C_2\cos(\xi_0+\frac{\pi}{4})\right)+\right. \\
&\left.\alpha(2i\hbar)\left(\frac{2m}{\hbar^2}\right)^{17/12} k^{1/6}E^{5/4}\left(C_2\sin(\xi_0+\frac{\pi}{4})-C_1\cos(\xi_0+\frac{\pi}{4})\right)\right] \div \\
&\left[\left(\frac{2m}{\hbar^2}\right)^{11/12} k^{1/6}E^{-1/4}\left(C_1\sin(\xi_0+\frac{\pi}{4})-C_2\cos(\xi_0+\frac{\pi}{4})\right)\right] \\
&.
\end{aligned} \tag{2.22}$$

Hence, the wavefunction is given by,

$$\begin{aligned}
\psi_1 &= \psi_0(E+c\alpha,k,x) \\
&= \psi_0(E,k,x)+c\alpha\frac{d}{dE}\psi_0(E,k,x) \\
&= \psi_0(E,k,x)+c\alpha\frac{d}{d\xi}\psi_0(E,k,x)\frac{d\xi}{dE}
\end{aligned} \tag{2.23}$$

where $\psi_0(E,k,x)$, $\frac{d}{d\xi}\psi_0(E,k,x)$ and $\frac{d\xi}{dE}$ are given by Eqs.(C.1), (C.8) and (C.3) in order(see appendix C).

It is to be noted here, that the above general solution consists two independent solutions instead of three. The reason is that they are basically coming from a second order differential equation (Eq.(2.2)). We just used them to find a general solution of the actual third order differential equation as each of them satisfies it separately. This perturbative solution is mathematically rigorous and complex to deal with.

In the next few sections, we will find the non-perturbative third solution and develop a method to impose the boundary conditions on the general solution.

2.4.2 Non-perturbative Solution

The solution given in Eq.(2.23) is perturbative and based on the solutions of a second order differential equation.

We assume the form of the third non-perturbative solution of the Eq.(2.10) as $\psi_0^{III} = e^{i\mu x/\ell_{Pl}}$.

If we can find a valid μ that satisfies Eq.(2.10) then we can write the general solution as $\psi(x) = C_1\psi_0^I + C_2\psi_0^{II} + C_3\psi_0^{III}$.

$$\begin{aligned}
\frac{d}{dx}\psi_0^{III} &= \left(\frac{i\mu}{\ell_{Pl}}\right) e^{i\mu x/\ell_{Pl}} = \frac{i\mu}{\ell_{Pl}}\psi_0^{III} \\
\frac{d^2}{dx^2}\psi_0^{III} &= -\left(\frac{\mu}{\ell_{Pl}}\right)^2 \psi_0^{III} \\
\frac{d^3}{dx^3}\psi_0^{III} &= -i\left(\frac{\mu}{\ell_{Pl}}\right)^3 \psi_0^{III}
\end{aligned} \tag{2.24}$$

Plugging the above into Eq.(2.10) we get

$$\begin{aligned}
&2i\alpha\hbar(-i)\left(\frac{\mu^3}{\ell_{Pl}^3}\right) - \frac{\mu^2}{\ell_{Pl}^2} + \frac{2m}{\hbar^2}(E - kx) = 0 \\
&\Rightarrow 2\alpha_0\mu^3 - \mu^2 = 0 \quad \text{(in the limit } \ell_{Pl} \to 0) \\
&\Rightarrow \mu^2(2\alpha_0\mu - 1) = 0 \\
&\Rightarrow \mu = \frac{1}{2\alpha_0}
\end{aligned}$$

Therefore, $\psi_0^{III} = e^{ix/2\alpha_0 \ell_{Pl}} = e^{ix/2\hbar\alpha}$

2.4.3 General Solution

The general solution of the GUP-corrected Schrödinger equation (one-dimension) is given by,

$$
\begin{aligned}
\psi(x) \;=\; & \frac{A}{\sqrt{\pi}}\left[\xi^{-1/4}\sin\left(\frac{2}{3}\xi^{\frac{3}{2}}+\frac{\pi}{4}\right)+\left(\frac{2m}{\hbar^2}\right)^{1/3}k^{-2/3}c\alpha\left(-\frac{1}{4}\xi^{-5/4}\sin\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)+\right.\right.\\
& \left.\left.\xi^{1/4}\cos\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)\right)\right]+\frac{B}{\sqrt{\pi}}\left[\xi^{-1/4}\cos\left(\frac{2}{3}\xi^{\frac{3}{2}}+\frac{\pi}{4}\right)+\right.\\
& \left.\left(\frac{2m}{\hbar^2}\right)^{1/3}k^{-2/3}c\alpha\left(-\xi^{1/4}\sin\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)-\frac{1}{4}\xi^{-5/4}\cos\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)\right)\right]+\\
& Ce^{ix/2\hbar\alpha}
\end{aligned}
\tag{2.25}
$$

The constant C is such that $|C|$ becomes zero in the limit $\alpha \to 0$ as the last term should drop out in this limit. Phase of A can be absorbed in ψ so A can be treated as a real constant.

In the limit $k \to 0$, the above equation should reduce to the general solution of the free particle (no field inside the box) GUP-corrected Schrödinger equation. Using the asymptotic forms of the Airy functions in the same limit and ignoring terms containing higher orders of α we can show that the wavefunction given by Eq.(2.25) becomes,

$$
\begin{aligned}
\psi(x) \;=\; & H\ A_1\sin(\frac{3}{2}H_1\frac{x}{E})+H\ A_2\sin(\frac{3}{2}H_1\frac{x}{E})+(i\alpha\hbar)\left[(\eta_1\ A_1-\eta_2\ A_2)\sin(\frac{3}{2}H_1\frac{x}{E})+\right.\\
& \left.(\eta_1\ A_2+\eta_2\ A_1)\cos(\frac{3}{2}H_1\frac{x}{E})+\right]+Ce^{ix/2\hbar\alpha},
\end{aligned}
\tag{2.26}
$$

where

$$
\begin{aligned}
\eta_1 &= -\frac{3}{8\sqrt{\pi}}\left(\frac{2m}{\hbar^2}\right)^{-1/12} k^{7/6}(E-kx)^{-5/4} \approx -\frac{3}{8\sqrt{\pi}}\left(\frac{2m}{\hbar^2}\right)^{-1/12} k^{7/6}E^{-5/4} \\
\eta_2 &= -\frac{3}{2\sqrt{\pi}}\left(\frac{2m}{\hbar^2}\right)^{5/12} k^{-1/6}(E-kx)^{1/4} \approx -\frac{3}{8\sqrt{\pi}}\left(\frac{2m}{\hbar^2}\right)^{5/12} k^{-1/6}E^{1/4} \\
A_1 &= -A\cos(\frac{H_1}{k}+\frac{\pi}{4}) + B\sin(\frac{H_1}{k}+\frac{\pi}{4}) \\
A_2 &= A\sin(\frac{H_1}{k}+\frac{\pi}{4}) + B\cos(\frac{H_1}{k}+\frac{\pi}{4}) \quad \textit{(see section 2.2)}
\end{aligned}
$$

Also, in the limit $\alpha \to 0$ Eq.(2.10) becomes a second order inhomogeneous differential equation and in that case the solution given by Eq.(2.25) must become $\psi(x)$ containing the Airy functions only, i.e., $\lim_{\alpha\to 0}\psi(x) = \frac{A}{\sqrt{\pi}}\xi^{-1/4}\sin\left(\frac{2}{3}\xi^{\frac{3}{2}}+\frac{\pi}{4}\right) + \frac{B}{\sqrt{\pi}}\xi^{-1/4}\cos\left(\frac{2}{3}\xi^{\frac{3}{2}}+\frac{\pi}{4}\right)$. This implies $\lim_{\alpha\to 0}|C| = 0$

2.5 Boundary Conditions and Length Quantization

Now that we have the wavefunction corresponding to the GUP-induced Schrödinger equation, we can use the boundary conditions $\psi(0) = 0$ and $\psi(L) = 0$. Like the case without a gravitational potential [32], here we hope to find a new condition that would lead to a restriction on the length of the box.

Imposing the boundary condition $\psi(0) = 0$ on Eq.(2.25) we get,

$$\frac{A}{\sqrt{\pi}}\left[\xi_0^{-1/4}\sin\left(\frac{2}{3}\xi_0^{\frac{3}{2}}+\frac{\pi}{4}\right)+\left(\frac{2m}{\hbar^2}\right)^{1/3}k^{-2/3}c\alpha\left(-\frac{1}{4}\xi_0^{-5/4}\sin\left(\frac{2}{3}\xi_0^{3/2}+\frac{\pi}{4}\right)+\right.\right.$$
$$\left.\left.\xi_0^{1/4}\cos\left(\frac{2}{3}\xi_0^{3/2}+\frac{\pi}{4}\right)\right)\right]+\frac{B}{\sqrt{\pi}}\left[\xi_0^{-1/4}\cos\left(\frac{2}{3}\xi_0^{\frac{3}{2}}+\frac{\pi}{4}\right)+\right.$$
$$\left.\left(\frac{2m}{\hbar^2}\right)^{1/3}k^{-2/3}c\alpha\left(-\xi_0^{1/4}\sin\left(\frac{2}{3}\xi_0^{3/2}+\frac{\pi}{4}\right)-\frac{1}{4}\xi_0^{-5/4}\cos\left(\frac{2}{3}\xi_0^{3/2}+\frac{\pi}{4}\right)\right)\right]+C=0 \tag{2.27}$$

Substituting for C in Eq.(2.25),

$$\begin{aligned}\psi(x) \;=\; & \frac{A}{\sqrt{\pi}}\left[\xi^{-1/4}\sin\left(\frac{2}{3}\xi^{\frac{3}{2}}+\frac{\pi}{4}\right)+\left(\frac{2m}{\hbar^2}\right)^{1/3}k^{-2/3}c\alpha\left(-\frac{1}{4}\xi^{-5/4}\sin\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)+\right.\right.\\
&\left.\left.\xi^{1/4}\cos\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)\right)\right]+\frac{B}{\sqrt{\pi}}\left[\xi^{-1/4}\cos\left(\frac{2}{3}\xi^{\frac{3}{2}}+\frac{\pi}{4}\right)+\right.\\
&\left.\left(\frac{2m}{\hbar^2}\right)^{1/3}k^{-2/3}c\alpha\left(-\xi^{1/4}\sin\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)-\frac{1}{4}\xi^{-5/4}\cos\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)\right)\right]\\
&-e^{ix/2\hbar\alpha}\frac{A}{\sqrt{\pi}}\left[\xi_0^{-1/4}\sin\left(\frac{2}{3}\xi_0^{\frac{3}{2}}+\frac{\pi}{4}\right)+\left(\frac{2m}{\hbar^2}\right)^{1/3}k^{-2/3}c\alpha\left(-\frac{1}{4}\xi_0^{-5/4}\sin\left(\frac{2}{3}\xi_0^{3/2}+\right.\right.\right.\\
&\left.\left.\left.\frac{\pi}{4}\right)+\xi_0^{1/4}\cos\left(\frac{2}{3}\xi_0^{3/2}+\frac{\pi}{4}\right)\right)\right]-e^{ix/2\hbar\alpha}\frac{B}{\sqrt{\pi}}\left[\xi_0^{-1/4}\cos\left(\frac{2}{3}\xi_0^{\frac{3}{2}}+\frac{\pi}{4}\right)+\right.\\
&\left.\left(\frac{2m}{\hbar^2}\right)^{1/3}k^{-2/3}c\alpha\left(-\xi_0^{1/4}\sin\left(\frac{2}{3}\xi_0^{3/2}+\frac{\pi}{4}\right)-\frac{1}{4}\xi_0^{-5/4}\cos\left(\frac{2}{3}\xi_0^{3/2}+\frac{\pi}{4}\right)\right)\right].\end{aligned} \tag{2.28}$$

Now, the remaining boundary condition $\psi(L) = 0$ implies,

$$e^{iL/2\hbar\alpha} = \frac{f(\xi_L)}{f(\xi_0)}, \tag{2.29}$$

where

$$\begin{aligned} f(\xi_L) \;=\; & \frac{A}{\sqrt{\pi}}\left[\xi_L^{-1/4}\sin\left(\frac{2}{3}\xi_L^{\frac{3}{2}}+\frac{\pi}{4}\right)+\left(\frac{2m}{\hbar^2}\right)^{1/3}k^{-2/3}c\alpha\left(-\frac{1}{4}\xi_L^{-5/4}\sin\left(\frac{2}{3}\xi_L^{3/2}+\frac{\pi}{4}\right)+\right.\right. \\ & \left.\left.\xi_L^{1/4}\cos\left(\frac{2}{3}\xi_L^{3/2}+\frac{\pi}{4}\right)\right)\right]+\frac{B}{\sqrt{\pi}}\left[\xi^{-1/4}\cos\left(\frac{2}{3}\xi_L^{\frac{3}{2}}+\frac{\pi}{4}\right)+\right. \\ & \left.\left(\frac{2m}{\hbar^2}\right)^{1/3}k^{-2/3}c\alpha\left(-\xi_L^{1/4}\sin\left(\frac{2}{3}\xi_L^{3/2}+\frac{\pi}{4}\right)-\frac{1}{4}\xi_L^{-5/4}\cos\left(\frac{2}{3}\xi_L^{3/2}+\frac{\pi}{4}\right)\right)\right], \end{aligned}$$

$$\begin{aligned} f(\xi_0) \;=\; & \frac{A}{\sqrt{\pi}}\left[\xi_0^{-1/4}\sin\left(\frac{2}{3}\xi_0^{\frac{3}{2}}+\frac{\pi}{4}\right)+\left(\frac{2m}{\hbar^2}\right)^{1/3}k^{-2/3}c\alpha\left(-\frac{1}{4}\xi_0^{-5/4}\sin\left(\frac{2}{3}\xi_0^{3/2}+\frac{\pi}{4}\right)+\right.\right. \\ & \left.\left.\xi_0^{1/4}\cos\left(\frac{2}{3}\xi_0^{3/2}+\frac{\pi}{4}\right)\right)\right]+\frac{B}{\sqrt{\pi}}\left[\xi_0^{-1/4}\cos\left(\frac{2}{3}\xi_0^{\frac{3}{2}}+\frac{\pi}{4}\right)+\right. \\ & \left.\left(\frac{2m}{\hbar^2}\right)^{1/3}k^{-2/3}c\alpha\left(-\xi_0^{1/4}\sin\left(\frac{2}{3}\xi_0^{3/2}+\frac{\pi}{4}\right)-\frac{1}{4}\xi_0^{-5/4}\cos\left(\frac{2}{3}\xi_0^{3/2}+\frac{\pi}{4}\right)\right)\right], \end{aligned}$$

and

$$\begin{aligned} \xi_L &= \left(\frac{2m}{\hbar^2}\right)^{1/3}k^{-2/3}(E-kL), \\ \xi_0 &= \left(\frac{2m}{\hbar^2}\right)^{1/3}k^{-2/3}E, \end{aligned}$$

Expanding $f(\xi_L)$ and $f(\xi_0)$ w.r.t. α we get,

$$f(\xi_L) \approx (\frac{2m}{\hbar^2})^{-1/12} k^{1/6} (E-kL)^{-1/4} \left[A \sin\left(\frac{2}{3}\sqrt{\frac{2m}{\hbar^2}} \frac{(E-kL)^{3/2}}{k} + \frac{\pi}{4} \right) + B \cos\left(\frac{2}{3}\sqrt{\frac{2m}{\hbar^2}} \frac{(E-kL)^{3/2}}{k} + \frac{\pi}{4} \right) \right] + \mathcal{O}(\alpha)$$

and

$$f(\xi_0) \approx (\frac{2m}{\hbar^2})^{-1/12} k^{1/6} E^{-1/4} \left[A \sin\left(\frac{2}{3}\sqrt{\frac{2m}{\hbar^2}} \frac{E^{3/2}}{k} + \frac{\pi}{4} \right) + B \cos\left(\frac{2}{3}\sqrt{\frac{2m}{\hbar^2}} \frac{E^{3/2}}{k} + \frac{\pi}{4} \right) \right] + \mathcal{O}(\alpha).$$

Thus,

$$\begin{aligned} e^{iL/2\hbar\alpha} &= \frac{(E-kL)^{-1/4} \left[A \sin\left(\frac{2}{3}\sqrt{\frac{2m}{\hbar^2}} \frac{(E-kL)^{3/2}}{k} + \frac{\pi}{4} \right) + B \cos\left(\frac{2}{3}\sqrt{\frac{2m}{\hbar^2}} \frac{(E-kL)^{3/2}}{k} + \frac{\pi}{4} \right) \right]}{E^{-1/4} \left[A \sin\left(\frac{2}{3}\sqrt{\frac{2m}{\hbar^2}} \frac{E^{3/2}}{k} + \frac{\pi}{4} \right) + B \cos\left(\frac{2}{3}\sqrt{\frac{2m}{\hbar^2}} \frac{E^{3/2}}{k} + \frac{\pi}{4} \right) \right]} \\ &\quad + \mathcal{O}(\alpha) \\ &= \left(1 - \frac{kL}{E}\right)^{-1/4} \frac{A \sin\left(\frac{2}{3}\sqrt{\frac{2m}{\hbar^2}} \frac{(E-kL)^{3/2}}{k} + \frac{\pi}{4} \right) + B \cos\left(\frac{2}{3}\sqrt{\frac{2m}{\hbar^2}} \frac{(E-kL)^{3/2}}{k} + \frac{\pi}{4} \right)}{A \sin\left(\frac{2}{3}\sqrt{\frac{2m}{\hbar^2}} \frac{E^{3/2}}{k} + \frac{\pi}{4} \right) + B \cos\left(\frac{2}{3}\sqrt{\frac{2m}{\hbar^2}} \frac{E^{3/2}}{k} + \frac{\pi}{4} \right)} \\ &\quad + \mathcal{O}(\alpha) \end{aligned} \tag{2.30}$$

Writing $B = |B|e^{i\theta_B}$ we get from the RHS of the above equation,

$$\frac{A \sin\left(\frac{2}{3}\sqrt{\frac{2m}{\hbar^2}} \frac{(E-kL)^{3/2}}{k} + \frac{\pi}{4} \right) + |B|e^{i\theta_B} \cos\left(\frac{2}{3}\sqrt{\frac{2m}{\hbar^2}} \frac{(E-kL)^{3/2}}{k} + \frac{\pi}{4} \right)}{A \sin\left(\frac{2}{3}\sqrt{\frac{2m}{\hbar^2}} \frac{E^{3/2}}{k} + \frac{\pi}{4} \right) + |B|e^{i\theta_B} \cos\left(\frac{2}{3}\sqrt{\frac{2m}{\hbar^2}} \frac{E^{3/2}}{k} + \frac{\pi}{4} \right)}$$

$$= \frac{A \sin\left(\frac{2}{3}\sqrt{\frac{2m}{\hbar^2}} \frac{(E-kL)^{3/2}}{k} + \frac{\pi}{4} \right) + |B|\cos(\theta_B) \cos\left(\frac{2}{3}\sqrt{\frac{2m}{\hbar^2}} \frac{(E-kL)^{3/2}}{k} + \frac{\pi}{4} \right) + i|B|\sin(\theta_B) \cos\left(\frac{2}{3}\sqrt{\frac{2m}{\hbar^2}} \frac{(E-kL)^{3/2}}{k} + \frac{\pi}{4} \right)}{A \sin(\left(\frac{2}{3}\sqrt{\frac{2m}{\hbar^2}} \frac{E^{3/2}}{k} + \frac{\pi}{4} \right)) + |B|\cos(\theta_B) \cos\left(\frac{2}{3}\sqrt{\frac{2m}{\hbar^2}} \frac{E^{3/2}}{k} + \frac{\pi}{4} \right) + i|B|\sin(\theta_B) \cos\left(\frac{2}{3}\sqrt{\frac{2m}{\hbar^2}} \frac{E^{3/2}}{k} + \frac{\pi}{4} \right)}$$

(2.31)

Thus the real part the RHS of Eq.(2.30) is given by,

$$\left[\frac{A^2\sin\left(\frac{2}{3}\sqrt{\frac{2m}{\hbar^2}}\frac{E^{3/2}}{k}+\frac{\pi}{4}\right)+A|B|\cos(\theta_B)\cos\left(\frac{2}{3}\sqrt{\frac{2m}{\hbar^2}}\frac{E^{3/2}}{k}+\frac{\pi}{4}\right)}{\left(A\sin\left(\frac{2}{3}\sqrt{\frac{2m}{\hbar^2}}\frac{E^{3/2}}{k}+\frac{\pi}{4}\right)+|B|\cos(\theta_B)\cos\left(\frac{2}{3}\sqrt{\frac{2m}{\hbar^2}}\frac{E^{3/2}}{k}+\frac{\pi}{4}\right)\right)^2+|B|^2\sin^2(\theta_B)\cos^2\left(\frac{2}{3}\sqrt{\frac{2m}{\hbar^2}}\frac{E^{3/2}}{k}+\frac{\pi}{4}\right)}\right] \times\sin\left(\frac{2}{3}\sqrt{\frac{2m}{\hbar^2}}\frac{(E-kL)^{3/2}}{k}+\frac{\pi}{4}\right)$$

$$+\left[\frac{A|B|\cos(\theta_B)\sin\left(\frac{2}{3}\sqrt{\frac{2m}{\hbar^2}}\frac{E^{3/2}}{k}+\frac{\pi}{4}\right)+|B|^2\cos^2(\theta_B)\cos\left(\frac{2}{3}\sqrt{\frac{2m}{\hbar^2}}\frac{E^{3/2}}{k}+\frac{\pi}{4}\right)+|B|^2\sin^2(\theta_B)\cos^2\left(\frac{2}{3}\sqrt{\frac{2m}{\hbar^2}}\frac{E^{3/2}}{k}+\frac{\pi}{4}\right)}{\left(A\sin\left(\frac{2}{3}\sqrt{\frac{2m}{\hbar^2}}\frac{E^{3/2}}{k}+\frac{\pi}{4}\right)+|B|\cos(\theta_B)\cos\left(\frac{2}{3}\sqrt{\frac{2m}{\hbar^2}}\frac{E^{3/2}}{k}+\frac{\pi}{4}\right)\right)^2+|B|^2\sin^2(\theta_B)\cos^2\left(\frac{2}{3}\sqrt{\frac{2m}{\hbar^2}}\frac{E^{3/2}}{k}+\frac{\pi}{4}\right)}\right] \times\cos\left(\frac{2}{3}\sqrt{\frac{2m}{\hbar^2}}\frac{(E-kL)^{3/2}}{k}+\frac{\pi}{4}\right)$$

$$=A_*\sin\left(\frac{2}{3}\sqrt{\frac{2m}{\hbar^2}}\frac{(E-kL)^{3/2}}{k}+\frac{\pi}{4}\right)+B_*\cos\left(\frac{2}{3}\sqrt{\frac{2m}{\hbar^2}}\frac{(E-kL)^{3/2}}{k}+\frac{\pi}{4}\right)$$

Collecting real terms form both sides of the Eq.(2.30),

$$\cos(L/2\hbar\alpha)=\left(1-\frac{kL}{E}\right)^{-1/4}\left(A_*\sin\left(\frac{2}{3}\sqrt{\frac{2m}{\hbar^2}}\frac{(E-kL)^{3/2}}{k}+\frac{\pi}{4}\right)+B_*\cos\left(\frac{2}{3}\sqrt{\frac{2m}{\hbar^2}}\frac{(E-kL)^{3/2}}{k}+\frac{\pi}{4}\right)\right) \tag{2.32}$$

Following the same argument in *sec 2.2*, the limit $k\to 0$ the RHS the above equation becomes

$$B_1\cos\left(\sqrt{\frac{2mE}{\hbar^2}}L\right)-A_1\sin\left(\sqrt{\frac{2mE}{\hbar^2}}L\right),$$

where $A_1 = H\left(A_* \cos\left(\frac{H_1}{k} + \frac{\pi}{4}\right) - B_* \sin\left(\frac{H_1}{k} + \frac{\pi}{4}\right)\right)$ and
$B_1 = H\left(A_* \sin\left(\frac{H_1}{k} + \frac{\pi}{4}\right) + B_* \cos\left(\frac{H_1}{k} + \frac{\pi}{4}\right)\right)$ *(see page 19 for H, H_1).*
Without loss of generality, letting $A_1 = \sin^{-1}\theta$ and $B_1 = \cos^{-1}\theta$ for an arbitrary θ we can write

$$\cos(L/2\hbar\alpha) = \cos\theta\cos\left(\sqrt{\frac{2mE}{\hbar^2}}L\right) - \sin\theta\sin\left(\sqrt{\frac{2mE}{\hbar^2}}L\right) = \cos\left(\sqrt{\frac{2mE}{\hbar^2}}L + \theta\right)$$

Referring to [32], we know that this implies $\frac{L_0}{2\alpha\hbar} = p\pi$, $p \in \mathbb{N}$, L_0 being the length of the box in flat space-time [32].
Since, L is a perturbation over L_0 we can write,

$$\frac{L}{2\hbar\alpha} = P_0(kL_0) + p\pi. \tag{2.33}$$

It is to be noted that P_0 is a small perturbative term and that $\frac{L_0}{2\alpha\hbar} = p_1\pi$ itself for $p_1 \in \mathbb{N}$. P_0 is a polynomial in kL derived from the RHS of Eq.(2.33).
We can write Eq.(2.35) as ,

$$\frac{L}{2\hbar\alpha} = f(k)p_1\pi + p\pi. \tag{2.34}$$

For each p we have a finite set of p_1 values. As the perturbative term has to be small, the number of p_1 values, for each p, depends on the smallness of $f(k)$. Fig.(3.3) is included to provide a qualitative comparison. We can see we have a fine structure (splitting) of the length quantization when gravity is involved compared to the much simpler shell structure when gravity is not involved. This might remind one of the similarities with the energy quantization of the hydrogen atom.

One might be interested to delve into the possible connection between the two. Consideration of the significance of this apparent coincidence may further suggest investigation of discreteness of space(time). Although the original Heisenberg Un-

Figure 2.3: Comparison between L_0(solid lines) and L (dotted lines)

certainty Principle is restricted to position-momentum commutation and time-energy uncertainty principle has been merely thought of a statistical measure of variance, a more generalized idea of GUP corrected commutation relation involving 4-momentum might give rise to discontinuity of time. This is beyond the scope of our work, but we hope to shed light on this topic in the future.

According to this relation it is evident that there can not exist a single particle inside the box unless the length of the box assumes only certain values. It is also noted from the above relation that the length must be in units of $\alpha_0 l_0$.

So, like the case with flat spacetime, GUP effects lead to length quantization in presence of gravity. Although we have shown it for a particle inside the box under the influence of gravity it can be extended to more general cases. Also, a particle in a box provides a way to measure length in one dimension. This result is sufficient but not limited to one-dimensional or non-relativistic scenarios.

We will discuss the relativistic and higher dimensional counterpart in the next chapter.

Chapter 3

Discreteness of Space from GUP : Relativistic Case

As explained in section 1.4.2, we need a formalism to investigate the modification of discreteness of space in a relativistic situation. The structure of spacetime does not necessarily change depending on relativistic or non-relativistic test particles used to probe it with, but it is quite fair that particles with speeds compared to the speed of light can potentially reveal the structure better compared to less energetic particles. In this chapter, we will have a closer look at the relativistic equivalent of the Schrödinger equation and in particular the modification induced by GUP. Now, the relativistic version of Schrödinger equation is Klein-Gordon equation. First we will derive the GUP-version of the Klein-Gordon equation with a linear potential and then try to solve it to obtain possible length quantization. Notwithstanding it's relative simplicity, Klein-Gordon equation has mathematical difficulties, especially when it comes to dimensions higher than one, it's much easier to resort to a more versatile Dirac equation. In the consecutive sections we will try to solve the Dirac equation in 3-spatial dimensions and hope for getting a similar length quantization as in [33].

3.1 Klein-Gordon Equation in One dimension

The Klein-Gordon (sometimes known as Klein-Gordon-Fock) equation with no field is given by [36],

$$(\hbar^2\Box + m^2c^2)\psi = 0, \tag{3.1}$$

where $\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$, $\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$. It is basically same as Eq.(1.16). In order to find the GUP-corrected Klein-Gordon equation we start from the Lorentz invariant energy momentum equation,

$$\begin{aligned} p_\mu p^\mu &= m^2c^2 \\ \Rightarrow E^2 &= p^2c^2 + m^2c^4 \end{aligned} \tag{3.2}$$

Einstein summation notation is followed here. Now we replace the momentum with the GUP-corrected momentum $p = p_0(1-\alpha p_0)$ and calculate the following quantities,

$$\begin{aligned} &p^2 = p_0^2 - 2\alpha p_0^3 \\ &p_0 \equiv -i\hbar\frac{\partial}{\partial}; E \equiv i\hbar\frac{\partial}{\partial t} \\ &\Rightarrow p_0^2 = -\hbar^2\frac{\partial^2}{\partial x^2}; p_0^3 = i\hbar^3\frac{\partial^3}{\partial x^3}. \end{aligned}$$

Plugging these into Eq.(3.2) we get,

$$-c^2\hbar^2\frac{\partial^2\Psi(x,t)}{\partial x^2} - c^2 2\alpha i\hbar^3\frac{\partial^3\Psi(x,t)}{\partial x^3} + m^2c^4\Psi = -\hbar^2\frac{\partial^2\Psi(x,t)}{\partial t^2}$$

Considering the stationary solutions only,

$$\begin{aligned} &-c^2\hbar^2\frac{\partial^2\Psi(x)}{\partial x^2} - 2c^2\alpha\hbar^3\frac{d^3\Psi(x)}{dx^3} + m^2c^4\Psi = E^2\Psi(x) \\ &\Rightarrow 2i\alpha\hbar\frac{d^3\psi}{dx^3} + \frac{d^2\psi}{dx^2} + \frac{((E/m^2c^2)^2-1)}{\hbar^2}m^2c^2\psi = 0 \end{aligned} \tag{3.3}$$

This is the GUP-corrected one-dimensional Klein-Gordon equation in flat space-time.

If we are to consider a relativistic particle in a one-dimensional box with a linearized potential $V(x) = kx$, we can write the above equation in an equivalent form $E \sim \sqrt{-2i\alpha\hbar\frac{d^3\psi}{dx^3} - \frac{d^2\psi}{dx^2} + m^2c^4} + V(x)$ and then rewrite this as,

$$
\begin{aligned}
& -2i\alpha\hbar\frac{d^3\psi}{dx^3} - \frac{d^2\psi}{dx^2} + m^2c^4\psi = (E - V(x))^2\,\psi \\
& \Rightarrow 2i\alpha\hbar\frac{d^3\psi}{dx^3} + \frac{d^2\psi}{dx^2} + \frac{1}{\hbar^2c^2}\left(E^2 - m^2c^4 - 2Ekx\right)\psi = 0
\end{aligned}
\tag{3.4}
$$

If we can make the following connections between the variables in the above equation and those in equation (2.10): $2i\alpha\hbar\frac{d^3\psi}{dx^3} + \frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E - kx)\psi = 0$

$$
\begin{aligned}
& \frac{2m}{\hbar^2}E \to \frac{1}{\hbar^2c^2}(E^2 - m^2c^4), \\
& \frac{2Ek}{\hbar^2c^2} \to \frac{2mk}{\hbar^2},
\end{aligned}
$$

we get similar length quantization result as in section 2.5.

3.2 Dirac Equation

The three-dimensional version of Klein-Gordon equation suffers from non-locality of differential operators. The term p^2, when GUP is considered, becomes $p^2 = p_0^2 - 2\alpha p_0^3 = -\hbar^2\nabla^2 + 2i\alpha\hbar^3\nabla^3$. Now the second term is $2i\alpha\hbar^3\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)^{3/2}$. Without going into the mathematical details of fractional calculus [37], we can simply use the Dirac equation in order to avoid this problem.

The free particle Dirac equation is given by [39],

$$
i\frac{\partial\Psi}{\partial t} = \left(\beta mc^2 + c\vec{\alpha}.\vec{P}\right)\Psi, \tag{3.5}
$$

where

$$\beta \equiv \gamma^0 = \begin{pmatrix} \mathbf{I}_2 & 0 \\ 0 & -\mathbf{I}_2 \end{pmatrix} \tag{3.6}$$

and

$$\alpha^i \equiv \gamma^0 \gamma^i = \begin{pmatrix} \mathbf{I}_2 & 0 \\ 0 & -\mathbf{I}_2 \end{pmatrix} \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}. \tag{3.7}$$

σ_i, i=1(1)3 for the 3 spatial dimensions, are the Pauli spin matrices and they are given by [38],

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.8}$$

Here $\beta mc^2 + c\vec{\alpha}.\vec{P}$ is the Dirac Hamiltonian with no field. It is to noted that $\vec{\alpha}$ is distinct from the parameter α present in the GUP-corrected quantum mechanical equations.

For an addition of a potential term in the form $V(x) = kx$ we can write the Dirac equation as,

$$i\frac{\partial \Psi}{\partial t} = \left(\beta mc^2 + c\vec{\alpha}.\vec{P} + kx\mathbf{I_4}\right)\Psi. \tag{3.9}$$

Particularly, for one spatial dimension, say z, the GUP-corrected Dirac equation becomes,

$$\left(-ic\hbar\alpha_z \frac{d}{dz} + c\alpha\hbar^2 \frac{d^2}{dz^2} + \beta mc^2 + kz\mathbf{I}_4\right)\psi(z) = E\psi(z). \tag{3.10}$$

Unlike Eq(3.6), the above is an eigenvalue equation.

3.3 Solution of Dirac Equation

Rewriting Eq.(3.11) we get,

$$\left(-ic\hbar\alpha_z\frac{d}{dz}+c\alpha\hbar^2\frac{d^2}{dz^2}+\beta mc^2-E+kz\right)\psi(Z)=0. \tag{3.11}$$

3.3.1 Perturbative solution

In order to solve the above equation we develop the following formalism.
The differential operator in Eq.(3.12) can be thought of composed of two components $-ic\hbar\alpha_z\frac{d}{dz}+c\alpha\hbar^2\frac{d^2}{dz^2}+\beta mc^2-E$ and kz. This second component can be considered as a perturbative term as both k and z are small. The solution to the first being already known [33], we can add a small perturbative term to that solution in order to get the complete solution of Eq.(3.12).
Let us use a trial solution of the form $\psi=\psi(\kappa+C_1k)$ which can also be written as $\psi=\psi_1(k=0)+C_1k\frac{d}{d\kappa}\psi_1(k=0)$ to the first order of approximation since a small k perturbative term seems logical to use.
Here, $\psi_1(k=0)=N_1e^{i\kappa z}\begin{pmatrix}\chi\\ r\sigma_z\chi\end{pmatrix}$.
$\kappa=\kappa_0+\alpha\hbar\kappa_0^2$, κ_0 being the wave number that satisfies $E^2=(\hbar\kappa_0)^2+(mc^2)^2, r=\frac{\hbar\kappa_0c}{E+mc^2}$ and $\chi^\dagger\chi=I$. Now, we re-write Eq.(3.12)using the above.

$$\left(-ic\hbar\alpha_z\frac{d}{dz}+c\alpha\hbar^2\frac{d^2}{dz^2}+\beta mc^2-E+kz\right)\left(N_1e^{i\kappa z}\begin{pmatrix}\chi\\ r\sigma_z\chi\end{pmatrix}+C_1kizN_1e^{i\kappa z}\begin{pmatrix}\chi\\ r\sigma_z\chi\end{pmatrix}\right)$$
$$=0 \tag{3.12}$$

As we have discussed above,

$$\left(-ic\hbar\alpha_z\frac{d}{dz}+c\alpha\hbar^2\frac{d^2}{dz^2}+\beta mc^2-E\right)N_1e^{i\kappa z}\begin{pmatrix}\chi\\ r\sigma_z\chi\end{pmatrix}=0.$$

Also, since k is very small, $(kz)\left(kizN_1e^{i\kappa z}\begin{pmatrix}\chi\\ r\sigma_z\chi\end{pmatrix}\right)=0$

In that case, Eq.(3.13) reduces to,

$$\left(-ic\hbar\alpha_z\frac{d}{dz}+c\alpha\hbar^2\frac{d^2}{dz^2}+\beta mc^2-E\right)C_1kizN_1e^{i\kappa z}\begin{pmatrix}\chi\\ r\sigma_z\chi\end{pmatrix}+kzN_1e^{i\kappa z}\begin{pmatrix}\chi\\ r\sigma_z\chi\end{pmatrix}=0$$
$$\Rightarrow C_1\left(-ic\hbar\alpha_z\frac{d}{dz}+c\alpha\hbar^2\frac{d^2}{dz^2}+\beta mc^2-E\right)iz\psi_1=-z\psi_1 \tag{3.13}$$

If we can find a valid C_1 for the above, we can claim ψ_1 is a solution of Eq.(3.13) which in turn means ψ is a solution of Eq. (3.12).

$$C_1\left(c\hbar\alpha_z\frac{d}{dz}(z\psi_1)+ic\alpha\hbar^2\frac{d^2}{dz^2}(z\psi_1)+mc^2\beta(iz\psi_1)-iE(z\psi_1)\right)=-z\psi_1 \tag{3.14}$$

$$\Rightarrow C_1\left((c\hbar+ic\hbar\kappa z)\begin{pmatrix}0&\sigma_z\\ \sigma_z&0\end{pmatrix}+imc^2z\begin{pmatrix}I_2&0\\ 0&-I_2\end{pmatrix}-\left(2c\alpha\kappa\hbar^2+izE+ic\alpha\hbar^2z\right)\begin{pmatrix}I_2&0\\ 0&I_2\end{pmatrix}\right)\psi=-z\psi$$

$$\Rightarrow \begin{pmatrix} \left(imc^2z - 2c\alpha\kappa\hbar^2 - izE - ic\alpha\kappa\hbar^2 z\right) I_2 & c\hbar(1+\kappa z)\sigma_z \\ c\hbar(1+\kappa z)\sigma_z & \left(-imc^2z - 2c\alpha\kappa\hbar^2 - izE - ic\alpha\kappa\hbar^2 z\right) I_2 \end{pmatrix} = -\frac{z}{C_1}\psi_1 \tag{3.15}$$

This is clearly an eigenvalue equation the associated matrix of which must be singular in order to have nontrivial solutions. In other words,

$$\begin{vmatrix} \left(\frac{z}{C_1} + imc^2z - 2c\alpha\kappa\hbar^2 - izE - ic\alpha\kappa\hbar^2 z\right) I_2 & c\hbar(1+i\kappa z)\sigma_z \\ c\hbar(1+i\kappa z)\sigma_z & \left(\frac{z}{C_1} - imc^2z - 2c\alpha\kappa\hbar^2 - izE - ic\alpha\kappa\hbar^2 z\right) I_2 \end{vmatrix} = 0 \tag{3.16}$$

Clearly, this is a characteristic equation in z/C_1, which by expanding the determinant, can be written as,

$$\left(\frac{z}{C_1} - 2ic\alpha\kappa\hbar^2 - iz(E + c\alpha\kappa\hbar^2)\right)^2 - (imc^2z)^2 - c^2\hbar^2(1+i\kappa z)^2 = 0.$$

For small z (which is quite reasonable considering the fact that the dimension we are dealing with is close to the Planck length), the above equation gives,

$$\begin{aligned}
&(\frac{z}{C_1} - 2c\alpha\kappa\hbar)^2 - 2z(\frac{z}{C_1} - 2c\alpha\kappa\hbar)(E + c\alpha\kappa\hbar^2) - \hbar^2c^2 - 2ic^2\hbar^2\kappa z = 0 \\
&\Rightarrow \frac{4c\alpha\kappa\hbar^2 z}{C_1} = 4ic\alpha\kappa\hbar^2 z(E + c\alpha\kappa\hbar^2) - \hbar^2c^2(1 + 2i\kappa z) \\
&\Rightarrow \frac{1}{C_1} = \frac{4i\alpha\kappa z(E + c\alpha\kappa\hbar^2) - (c + 2ic\kappa z)}{4\alpha\kappa z} \\
&\Rightarrow \frac{1}{C_1} = -\frac{c + 2i\alpha\kappa z\left(c(1 - 2\alpha\kappa\hbar^2) - 2E\right)}{4\alpha\kappa z} \\
&\Rightarrow C_1 = -\frac{4\alpha\kappa}{c/z + 2i\alpha\kappa\left(c(1 - 2\alpha\kappa\hbar^2) - 2E\right)}
\end{aligned} \tag{3.17}$$

So, the solution of Eq.(3.12) is given by,

$$\begin{aligned}\psi &= N_1 e^{i\kappa z}\begin{pmatrix}\chi\\ r\sigma_z\chi\end{pmatrix} - \frac{4\alpha\kappa}{c/z + 2i\alpha\kappa\left(c(1-2\alpha\kappa\hbar^2) - 2E\right)} kizN_1 e^{i\kappa z}\begin{pmatrix}\chi\\ r\sigma_z\chi\end{pmatrix}\\ &= \left(1 - \frac{4ik\alpha\kappa z}{c/z + 2i\alpha\kappa\left(c(1-2\alpha\kappa\hbar^2) - 2E\right)}\right) N_1 e^{i\kappa z}\begin{pmatrix}\chi\\ r\sigma_z\chi\end{pmatrix}\end{aligned} \tag{3.18}$$

The above is the perturbative wavefunction corresponding to the GUP-corrected Dirac equation.

3.3.2 Non-perturbative solution

Let us suppose the non-perturbative solution has the form

$$\psi_{NP} = e^{i\mu z/\ell_{Pl}}\begin{pmatrix}\chi\\ \sigma_z\chi\end{pmatrix} \tag{3.19}$$

Now, plugging this solution into Eq.(3.12),

$$\begin{aligned}&\left(-ic\hbar\alpha_z\frac{d}{dz} + c\alpha\hbar^2\frac{d^2}{dz^2} + mc^2\beta - E + kz\right) e^{i\mu z/\ell_{Pl}}\begin{pmatrix}\chi\\ \sigma_z\chi\end{pmatrix} = 0\\ &\Rightarrow \mu(1-\alpha_0\mu) = 0\\ &\Rightarrow \frac{\mu}{\ell_{Pl}} = \frac{1}{\alpha\hbar}\end{aligned}$$

3.4 Boundary Condition and Length Quantization

It is to be noted that GUP-induced Dirac equation with a linearized potential is a second order inhomogeneous differential equation. Also because the matrix associated is of order 4×4, one can expect eight linearly independent solutions. Here, we wish to restrict ourselves to positive energy solutions only. In that case we have four linearly independent solutions given by,

$$\psi_1 = N_1 \left(1 - \frac{4ik\alpha\kappa z}{c/z + 2i\alpha\kappa\left(c(1-2\alpha\kappa\hbar^2) - 2E\right)}\right) e^{i\kappa z} \begin{pmatrix} \chi \\ r\sigma_z\chi \end{pmatrix} \tag{3.20}$$

$$\psi_2 = N_2 e^{iz/\alpha\hbar} \begin{pmatrix} \chi \\ \sigma_z\chi \end{pmatrix} \tag{3.21}$$

where χ is a normalized spinor, meaning $\chi^\dagger\chi = 1$. χ could be chosen as $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for spin up state or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for spin down state or any linear combination of the two.

It is to be noted that similar to the case of GUP-induced Schrödinger equation this non-perturbative solution here should disappear in the limit $\alpha \to 0$ as we should get our old Dirac equation back in the same limit which is essentially a first order differential equation.

Now, if we try to impose the boundary conditions on the wavefunction by letting the gravitational potential go to infinity just outside the box, like we did in the non-relativistic case, the so-called Klein paradox occurs. In other words, the flux of the reflected plane wave on the infinite walls (boundaries) of the box appears higher than that of the incident wave [40].

In order to avoid this situation, we resort to the famous MIT bag model of confined quarks once again [41].

As discussed in [33], the mass of the relativistic particle of interest is considered as a function of z,

$$m(z) = \begin{cases} M & \text{if } z \leq 0 \\ m & 0 \leq z \leq L \\ M & z \geq L, \end{cases} \tag{3.22}$$

where m is the rest mass of the particle and M is a constant. In order to have an equivalent picture as infinite potential walls we will let M eventually grow infinitely large causing the particle trapped inside the box. The advantage of this method is that now we have the opportunity to consider the solution of the GUP-induced Dirac equation separately in three regions, region I associated with $z \leq 0$, II with $0 \leq z \leq L$ and III with $z \geq L$. Now, in all of these three regions the wavefunction should assume the same form of a linear combination of the perturbative and non-perturbative solutions. Although, inside the box, i.e., in region II one might consider an incident and a reflected wave whereas outside the box, it would suffice to consider only one wave travelling outward from the walls. It should be a plane wave travelling left in region I and travelling right in region III.

So, with reference to [33], the wavefunctions in the three regions are given by,

$$\psi_I = A\left(1 + \frac{4ik\alpha\kappa' z}{c/z - 2i\alpha\kappa'\left(c(1+2\alpha\kappa'\hbar^2) - 2E\right)}\right) e^{-i\kappa' z}\begin{pmatrix} \chi \\ -R\sigma_z\chi \end{pmatrix} + Ge^{\frac{iz}{\alpha\hbar}}\begin{pmatrix} \chi \\ \sigma_z\chi \end{pmatrix} \tag{3.23}$$

$$\begin{aligned}\psi_{II} = {} & B\left(1 - \frac{4ik\alpha\kappa z}{c/z + 2i\alpha\kappa\left(c(1-2\alpha\kappa\hbar^2) - 2E\right)}\right) e^{i\kappa z}\begin{pmatrix} \chi \\ r\sigma_z\chi \end{pmatrix} \\ & + C\left(1 + \frac{4ik\alpha\kappa z}{c/z - 2i\alpha\kappa\left(c(1+2\alpha\kappa\hbar^2) - 2E\right)}\right) e^{-i\kappa z}\begin{pmatrix} \chi \\ -r\sigma_z\chi \end{pmatrix} \\ & + Fe^{\frac{iz}{\alpha\hbar}}\begin{pmatrix} \chi \\ \sigma_z\chi \end{pmatrix}\end{aligned}$$

(3.24)

$$\psi_{III} = D\left(1 - \frac{4ik\alpha\kappa^{'}z}{c/z + 2i\alpha\kappa'\left(c(1-2\alpha\kappa'\hbar^2) - 2E\right)}\right) e^{i\kappa^{'}z} \begin{pmatrix} \chi \\ R\sigma_z\chi \end{pmatrix} + He^{\frac{iz}{\alpha\hbar}} \begin{pmatrix} \chi \\ \sigma_z\chi \end{pmatrix}, \tag{3.25}$$

where $\kappa^{'} = \kappa_0^{'} + \alpha\hbar\kappa_0^{'2}$, $E = \sqrt{(\hbar\kappa_0^{'}c)^2 + (Mc^2)^2}$ and $R = \frac{\hbar\kappa_0^{'}c}{E+Mc^2}$. To give the above wavefunctions a little simpler form, let us have

$$\begin{aligned}
\rho_1 &= \left(1 - \frac{4ik\alpha\kappa z}{c/z + 2i\alpha\kappa\left(c(1-2\alpha\kappa\hbar^2) - 2E\right)}\right) \\
\rho_2 &= \left(1 + \frac{4ik\alpha\kappa z}{c/z - 2i\alpha\kappa\left(c(1+2\alpha\kappa\hbar^2) - 2E\right)}\right) \\
\rho_1^{'} &= \left(1 - \frac{4ik\alpha\kappa^{'}z}{c/z + 2i\alpha\kappa'\left(c(1-2\alpha\kappa'\hbar^2) - 2E\right)}\right) \\
\rho_2^{'} &= \left(1 + \frac{4ik\alpha\kappa^{'}z}{c/z - 2i\alpha\kappa'\left(c(1+2\alpha\kappa'\hbar^2) - 2E\right)}\right),
\end{aligned}$$

and re-write the wavefunctions as,

$$\psi_I = A\rho_2{}'e^{-i\kappa^{'}z} \begin{pmatrix} \chi \\ -R\sigma_z\chi \end{pmatrix} + Ge^{\frac{iz}{\alpha\hbar}} \begin{pmatrix} \chi \\ \sigma_z\chi \end{pmatrix} \tag{3.26}$$

$$\psi_{II} = B\rho_1 e^{i\kappa z} \begin{pmatrix} \chi \\ r\sigma_z\chi \end{pmatrix} + C\rho_2 e^{-i\kappa z} \begin{pmatrix} \chi \\ -r\sigma_z\chi \end{pmatrix} + Fe^{\frac{iz}{\alpha\hbar}} \begin{pmatrix} \chi \\ \sigma_z\chi \end{pmatrix} \tag{3.27}$$

$$\psi_{III} = D\rho_1^{'}e^{i\kappa^{'}z} \begin{pmatrix} \chi \\ R\sigma_z\chi \end{pmatrix} + He^{\frac{iz}{\alpha\hbar}} \begin{pmatrix} \chi \\ \sigma_z\chi \end{pmatrix}. \tag{3.28}$$

Following the same line of argument as in [33], we can say that when M is very large,

$E^2 - M^2c^4 < 0$ which means $\kappa_0' = \sqrt{\frac{E^2}{c^2} - M^2c^2}/\hbar$ is imaginary. So, in the limit $M \to \infty$, $\kappa_0' \to i\infty$ and $\kappa' = \frac{i}{\hbar}\sqrt{Mc^2 - \frac{E^2}{c^2}} - \frac{\alpha}{\hbar}(M^2c^2 - \frac{E^2}{c^2})$ is a very large complex number. It follows that $e^{-i\kappa' z} = \left(e^{-\frac{|z|}{\hbar}\sqrt{Mc^2 - \frac{E^2}{c^2}}}\right)\left(e^{-\frac{i\alpha|z|}{\hbar}(M^2c^2 - \frac{E^2}{c^2})}\right) \to 0$ and $e^{i\kappa' z} = \left(e^{-\frac{z}{\hbar}\sqrt{Mc^2 - \frac{E^2}{c^2}}}\right)\left(e^{-\frac{i\alpha z}{\hbar}(M^2c^2 - \frac{E^2}{c^2})}\right) \to 0$ in the limit $M \to \infty$. This is simply because the modulus of each of these complex numbers becomes zero in this limit. Moreover, both ρ_1' and $\rho_2{}' \sim 1 - \mathcal{O}(i/\kappa_0')$ so they become unity in the limit $M \to \infty$.

So, in the limit $M \to \infty$ the terms containing A and D becomes zero. As for the the terms associated with G and H, the fluxes are nonzero [33] which means we have to set $G = 0$ and $H = 0$. Now, in the limit $\alpha \to 0$ and $k \to 0$ Eq.(3.12) becomes the old Dirac equation without any effects of GUP. As we can see from Eq.(3.27), this means ρ_1 and ρ_2 must become unity which is evident. Moreover, this means the term with F must vanish in the limit $\alpha \to 0$, which compels us to choose $F \sim \alpha^s,\ s > 0$. This will also take care of the possible blowup of the exponential term, especially if we let $s \geq 10$ F will decrease reasonably faster than $e^{\frac{1}{\alpha}}$ increases. This lower bound of s is based on a numerical comparison between α and s calculated in *Maple*^{TM} *16*.

Finally, without loss of generality we can choose $B = 1$ like in [33] but the selection of C is to be determined from its relationship with B, which we are about to figure out.

The usual boundary conditions that one would expect for the Schrödinger equation, viz., $\psi_{II}(z = 0) = 0\psi_{II}(z = L)$ do not quite work in this case as they would make the wavefunction inside the box vanish. Alternatively, we use the MIT bag model which is having the outward flux of the Dirac current at the boundaries $z = 0$ and $z = L$

zero [40]. Equivalently one can write [41],

$$i\gamma^3\psi = \psi, \ at \ z = 0 \tag{3.29}$$

$$i\gamma^3\psi = -\psi, \ at \ z = L. \tag{3.30}$$

Using $\psi = \psi_{II}$ in Eq.(3.29) we get,

$$i\gamma^3\psi_{II}(0) = \psi_{II}(0)$$

$$\begin{aligned}
\Rightarrow iB\rho_1 \begin{pmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{pmatrix} \begin{pmatrix} \chi \\ r\sigma_z\chi \end{pmatrix} + iC\rho_2 \begin{pmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{pmatrix} \begin{pmatrix} \chi \\ -r\sigma_z\chi \end{pmatrix} \\
+iF \begin{pmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{pmatrix} \begin{pmatrix} \chi \\ \sigma_z\chi \end{pmatrix} \\
= B\rho_1 \begin{pmatrix} \chi \\ r\sigma_z\chi \end{pmatrix} + C\rho_2 \begin{pmatrix} \chi \\ -r\sigma_z\chi \end{pmatrix} + F \begin{pmatrix} \chi \\ \sigma_z\chi \end{pmatrix}
\end{aligned}$$

Collecting terms,

$$\begin{aligned}
& irB\rho_1 - irC\rho_2 + iF = B\rho_1 + C\rho_2 + F \\
& \Rightarrow ir(\rho_1 B - \rho_2 C) = \rho_1 B + \rho_2 C + F(1-i) \\
& \Rightarrow ir(\rho_1 B - \rho_2 C) = \rho_1 B + \rho_2 C + F\sqrt{2}e^{-i\pi/4} \\
& \Rightarrow \frac{\rho_1 B + \rho_2 C + F^{'}e^{-i\pi/4}}{\rho_1 B - \rho_2 C} = ir, \qquad (3.31)
\end{aligned}$$

where $F^{'} = \sqrt{2}F$.

From the other boundary condition (Eq. (3.30)),

$$iB\rho_1 e^{i\kappa L}\begin{pmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{pmatrix}\begin{pmatrix} \chi \\ r\sigma_z\chi \end{pmatrix} + iC\rho_2 e^{-i\kappa L}\begin{pmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{pmatrix}\begin{pmatrix} \chi \\ -r\sigma_z\chi \end{pmatrix}$$
$$+iFe^{iL/\alpha\hbar}\begin{pmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{pmatrix}\begin{pmatrix} \chi \\ \sigma_z\chi \end{pmatrix}$$
$$= -B\rho_1 e^{i\kappa L}\begin{pmatrix} \chi \\ r\sigma_z\chi \end{pmatrix} - Ce^{-i\kappa L}\rho_2\begin{pmatrix} \chi \\ -r\sigma_z\chi \end{pmatrix} - Fe^{iL/\alpha\hbar}\begin{pmatrix} \chi \\ \sigma_z\chi \end{pmatrix}$$

Collecting terms,

$$irB\rho_1 e^{i\kappa L} - irC\rho_2 e^{-i\kappa L} + iFe^{iL/\alpha\hbar} = -B\rho_1 e^{i\kappa L} - C\rho_2 e^{-i\kappa L} - Fe^{iL/\alpha\hbar}$$
$$\Rightarrow \frac{\rho_1 Be^{i\kappa L} + \rho_2 Ce^{-i\kappa L} + F' e^{i(L/\alpha\hbar+\pi/4)}}{\rho_1 Be^{i\kappa L} - \rho_2 Ce^{-i\kappa L}} = -ir, \tag{3.32}$$

In order to establish a relationship between B and C let us start with Eq.(3.31),

$$\frac{\rho_1 B + \rho_2 C + F' e^{-i\pi/4}}{\rho_1 B - \rho_2 C} = ir$$
$$\Rightarrow C = \frac{\rho_1}{\rho_2}\frac{ir-1}{ir+1}B - F'\frac{e^{-i\pi/4}}{\rho_2(ir+1)}. \tag{3.33}$$

It is to be noted that $\frac{ir-1}{ir+1}$ is unimodular as it can be expressed as
$\frac{ir-1}{ir+1} = \frac{-r^2+1-2ir}{-r^2-1} = \frac{r^2-1}{r^2+1} + i\frac{2r}{r^2+1} = e^{\delta}$ where

$$\delta = tan^{-1}(\frac{2r}{r^2-1}). \tag{3.34}$$

ρ_2 increases with k, the linear potential term, and $\rho_2 \to 1$ as $k \to 0$. So, Eq.(3.33) can be re-written as,

$$C = \frac{|\rho_1|}{|\rho_2|}e^{i(\theta_{\rho_1}-\theta_{\rho_2})}e^{i\delta}B + \mathcal{O}(\alpha), \tag{3.35}$$

and

$$|C| = |B|\frac{|\rho_1|}{|\rho_2|} + \mathcal{O}(\alpha). \tag{3.36}$$

Clearly, in the limit $k \to 0$, $\rho_1 \to 1 = \rho_2 = 1$ and $\theta_{\rho_1} = \theta_{\rho_2} = 0$.

To find a condition on the length of the box, we will consider the other boundary condition. We get from Eq. (3.32),

$$\begin{aligned}
&\rho_1 B e^{i\kappa L} + \rho_2 C e^{-i\kappa L} + F' e^{i(L/\alpha\hbar+\pi/4)} = -ir\rho_1 B e^{i\kappa L} + ir\rho_2 C e^{-i\kappa L} \\
&\Rightarrow \rho_1(ir+1) B e^{i\kappa L} + F' e^{i(L/\alpha\hbar+\pi/4)} = \rho_2(ir-1) C e^{-i\kappa L} \qquad (3.37)
\end{aligned}$$

Now, if we substitute $C = \frac{|\rho_1|}{|\rho_2|} e^{i(\theta_{\rho_1}-\theta_{\rho_2})} e^{i\delta} B$ in Eq.(3.37) we get,

$$\begin{aligned}
\rho_1(ir+1) B e^{i\kappa L} + F' e^{i(L/\alpha\hbar+\pi/4)} &= \rho_2(ir-1)\frac{|\rho_1|}{|\rho_2|} e^{i(\theta_{\rho_1}-\theta_{\rho_2})} e^{-i\kappa L} e^{i\delta} B \\
\Rightarrow B e^{2i\kappa L} + F' \frac{e^{i(L/\alpha\hbar+\pi/4)}}{\rho_1(ir+1)} &= \frac{ir-1}{ir+1} e^{i\delta} B
\end{aligned} \tag{3.38}$$

Choosing $B = 1$ and substituting $e^{i\delta} = \frac{ir-1}{ir+1}$ in Eq.(3.38),

$$e^{2i\kappa L} = \left(\frac{ir-1}{ir+1}\right)^2 - F' \frac{e^{i(L/\alpha\hbar+\pi/4)}}{\rho_1(ir+1)} \tag{3.39}$$

$$\Rightarrow e^{2ikL} = e^{2i\ tan^{-1}\left(\frac{2r}{r^2-1}\right)} - F' \frac{e^{i(L/\alpha\hbar+\pi/4)}}{\rho_1(ir+1)} \tag{3.40}$$

$$\Rightarrow \kappa L = tan^{-1}\left(\frac{2r}{r^2-1}\right) + \mathcal{O}(ln(\alpha)) \tag{3.41}$$

Now, from Eq.(3.37)

$$\begin{aligned} \rho_1(ir+1)Be^{i\kappa L} + F' e^{i(L/\alpha\hbar+\pi/4)} &= \rho_2(ir-1)Ce^{-i\kappa L} \\ \Rightarrow e^{i(L/\alpha\hbar+\pi/4)} &= \frac{\rho_1(ir-1)\left(e^{i(\delta-\kappa L)-e^{i\left(\kappa L-tan^{-1}\left(\frac{2r}{r^2-1}\right)\right)}}\right)}{F'} \end{aligned} \tag{3.42}$$

It follows from Eq.(3.42),

$$\frac{L}{\alpha\hbar} = -\frac{\pi}{4} + arg\left[\frac{\rho_1(ir-1)\left(e^{i(\delta-\kappa L)-e^{i\left(\kappa L-tan^{-1}\left(\frac{2r}{r^2-1}\right)\right)}}\right)}{F'}\right] + 2n\pi,\ n \in \mathbb{N} \tag{3.43}$$

It is to be noted unlike the case without the presence of gravity [33] δ and κL here are not the same as given by Eq.(3.34) and (3.41). They become the same in the limit $k \to 0$.

Thus, a relativistic particle can be trapped in a one-dimensional box, with a linearized gravitational potential inside, only if the length of the box is quantized and the nature of quantization is given by the above equation.

One would expect in the limit $k \to 0$ Eq.(3.39) to reduce to the old quantization rule given by Eq. (27) in [33]. To verify this we begin with Eq. (22) in [33] which on slight algebraic rearrangement gives,

$$e^{-i\pi/4} = \frac{(ir-1)-(ir+1)e^{i\delta}}{F'} \tag{3.44}$$

Now, if we let k go to zero, $\rho \to 1$, $\kappa L \to \delta$. So the RHS of Eq.(3.42) becomes,

$$\begin{aligned}&\frac{\rho_1(ir-1)\left(e^{i(\delta-\kappa L)-e^{i\left(\kappa L - tan^{-1}\left(\frac{2r}{r^2-1}\right)\right)}}\right)}{F'}\\ &= \frac{(ir-1)-(ir+1)e^{i\delta}}{F'}\end{aligned} \tag{3.45}$$

So, comparing Eqs.(3.42),(3.44) and (3.45) we can write (in the limit $k \to 0$),

$$\begin{aligned}e^{i(L/\alpha\hbar+\pi/4)} &= e^{-i\pi/4}\\ \Rightarrow \frac{L}{\alpha\hbar} &= -\frac{\pi}{2}+2n\pi,\ n \in \mathbb{N},\end{aligned} \tag{3.46}$$

which is nothing but the length quatization rule for a relativistic particle in a box without the influence of gravity *(see Eq.(1.22))*.

3.5 Dirac Equation in Three Dimensions

As discussed in section 1.4.3, discreteness of space for two and three dimensions has been shown without gravity. In this section we consider the generalized Dirac equation for three dimensions and extend the result for a box with a slowly varying gravitational potential inside, which can be linearized as before.

Let us consider a box defined by $0 \le x_i \le L_i,\ i = 1(1)d$, d being the dimension of the box, i.e., $d = 1, 2$ or 3. This box can be one two or three-dimensional. Now, we have a linear potential inside as before. Without loss of generality, we can consider the direction in which the potential changes as our x-direction. The Dirac Hamiltonian with a potential term can be written as

$$
\begin{aligned}
H &= \vec{c}\alpha.\vec{p} + \beta mc^2 + V(\vec{r})I \\
&= c\left(\alpha_x p_x + \alpha_y p_y + \alpha_z p_z\right) + \beta mc^2 + kxI \\
&= c\left(\alpha_x p_{0x} + \alpha_y p_{0y} + \alpha_z p_{0z} - \alpha p_{0x}^2 - \alpha p_{0y}^2 - \alpha p_{0z}^2\right) + \beta mc^2 + kxI
\end{aligned}
\tag{3.47}
$$

It is to be noted that we used the GUP-corrected momenta, $p_i = p_0 - \alpha p_0,\ i = 1,..,3,$ where $p_{0i} = -i\hbar\frac{d}{dx_i}$and followed Dirac prescription, i.e., we replaced p_{0i} by $\alpha_i p_{0i}$ and used $\alpha_i^2 = I$.

We are going to use the same ansatz here as in [33] *(see section 1.4.3)* . The Dirac Hamiltonian here is same as that in [33] except for a potential along x-axis. We already know the modified perturbative and non-perturbative wavefunctions under the influence of a one-dimensional linear potential. So, combining Eqs.(3.20), (3.21) in section 3.4 and Eq. (30) and (31) in [33] we can write the two independent wavefunctions corresponding to the above Hamiltonian,

$$
\psi_1 = N_1 \rho_j e^{i\vec{\kappa}.\vec{r}} \begin{pmatrix} \chi \\ r\hat{\kappa}.\vec{\sigma}\chi \end{pmatrix} \tag{3.48}
$$

$$
\psi_2 = N_2 e^{i\frac{\hat{q}.\vec{r}}{\alpha\hbar}} \begin{pmatrix} \chi \\ \hat{q}.\vec{\sigma}\chi \end{pmatrix}, \tag{3.49}
$$

$j = 1, 2$ for wavefronts going rightward and leftward respectively.

$$
\rho_1 = \left(1 - \frac{4ik\alpha\kappa_1 x}{c/x + 2i\alpha\kappa_1\left(c(1 - 2\alpha\kappa\hbar^2) - 2E\right)}\right)
$$
$$
\rho_2 = \left(1 + \frac{4ik\alpha\kappa_1 x}{c/x - 2i\alpha\kappa_1\left(c(1 + 2\alpha\kappa_1\hbar^2) - 2E\right)}\right).
$$

Now, just like we did in section 3.4, here also we distinguish between regions inside and outside the box, but unlike the one-dimensional case, here we consider the wavefunction inside the box only. We impose the boundary conditions on this wavefunction by resorting to the MIT bag model again.

Let us consider the following general form of a wavefunction inside the box going in a particular direction,

$$\Psi = \rho_1^{\epsilon_1 + \frac{1-\epsilon_1}{2}} \rho_2^{\frac{1-\epsilon_1}{2}} e^{i\left(\sum_{i=1}^{d} \epsilon_i \kappa_i x_i + \frac{1-\epsilon_i}{2}\delta_i\right)} \begin{pmatrix} \chi \\ r\sum_{i=1}^{d} \epsilon_i \hat{\kappa_i} \sigma_i \chi \end{pmatrix}, \tag{3.50}$$

where δ_i $(i = 1(1)d)$ and $d = 1, 2, 3$. The highest value of d clearly depends on how many spatial dimensions the box has. There are 2^d possible individual wavefunctions, for all possible combinations with $\epsilon_i(i = 1, .., d) = \pm 1$. The wavefunction inside the box would be a superposition of those 2^d kets and $F\psi_2$,

$$\psi = \begin{pmatrix} \left[\prod_{i=1}^{d} \left(\rho_1^{\delta_{i1}} e^{i\kappa_i x_i} + \rho_2^{\delta_{i1}} e^{-i(\kappa_i x_i - \delta_i)}\right) + F e^{i\frac{\hat{q}.\vec{r}}{\alpha\hbar}}\right]\chi \\ \sum_{j=1}^{d}\left[\prod_{i=1}^{d} \left(\rho_1^{\delta_{i1}} e^{i\kappa_i x_i} + (-1)^{\delta_{ij}} \rho_2^{\delta_{i1}} e^{-i(\kappa_i x_i - \delta_i)}\right) r\hat{\kappa_j} + F e^{i\frac{\hat{q}.\vec{r}}{\alpha\hbar}} \hat{q_j}\right]\sigma_j\chi \end{pmatrix}, \tag{3.51}$$

where δ_{ij} is the usual Kronecker delta. The number of terms in row I is $2^d + 1$ and that in row II is $(2^d + 1) \times d$.

3.6 Boundary Conditions

The MIT bag model boundary conditions are,

$$i\gamma^l \psi = \psi, \ at \ x_l = 0 \tag{3.52}$$

$$i\gamma^l \psi = -\psi, \ at \ x_l = L, \tag{3.53}$$

for $l = 1, .., d$. Like the one-dimensional case, these conditions make sure the flux of the Dirac probability current through any of the six surfaces is zero. This way we can also avoid writing six different wavefunctions for six different regions outside the box.

Now, the above conditions imposed together on the wavefunction given by Eq.(3.51) for any x_l gives,

$$\begin{pmatrix} i\sum_{j=1}^{d}\left[\prod_{i=1}^{d}\left(\rho_1^{\delta_{i1}}e^{i\kappa_i x_i} + (-1)^{\delta_{ij}}\rho_2^{\delta_{i1}}e^{-i(\kappa_i x_i - \delta_i)}\right) r\hat{\kappa_j}\sigma_l\sigma_j + Fe^{i\frac{\hat{q}.\vec{r}}{\alpha\hbar}}\hat{q}_j\sigma_l\sigma_j\right]\chi \\ -i\left[\prod_{i=1}^{d}\left(\rho_1^{\delta_{i1}}e^{i\kappa_i x_i} + \rho_2^{\delta_{i1}}e^{-i(\kappa_i x_i - \delta_i)}\right) + Fe^{i\frac{\hat{q}.\vec{r}}{\alpha\hbar}}\right]\sigma_l\chi \end{pmatrix}$$
$$= \pm \begin{pmatrix} \left[\prod_{i=1}^{d}\left(\rho_1^{\delta_{i1}}e^{i\kappa_i x_i} + \rho_2^{\delta_{i1}}e^{-i(\kappa_i x_i - \delta_i)}\right) + Fe^{i\frac{\hat{q}.\vec{r}}{\alpha\hbar}}\right]\chi \\ \sum_{j=1}^{d}\left[\prod_{i=1}^{d}\left(\rho_1^{\delta_{i1}}e^{i\kappa_i x_i} + (-1)^{\delta_{ij}}\rho_2^{\delta_{i1}}e^{-i(\kappa_i x_i - \delta_i)}\right) r\hat{\kappa_j} + Fe^{i\frac{\hat{q}.\vec{r}}{\alpha\hbar}}\hat{q}_j\right]\sigma_j\chi \end{pmatrix}. \tag{3.54}$$

Equating row I we get,

$$i\sum_{j=1}^{d}\left[\prod_{i=1}^{d}\left(\rho_1^{\delta_{i1}}e^{i\kappa_i x_i} + (-1)^{\delta_{ij}}\rho_2^{\delta_{i1}}e^{-i(\kappa_i x_i - \delta_i)}\right) r\hat{\kappa_j}\sigma_l\sigma_j + Fe^{i\frac{\hat{q}.\vec{r}}{\alpha\hbar}}\hat{q}_j\sigma_l\sigma_j\right]$$
$$= \pm\prod_{i=1}^{d}\left(\rho_1^{\delta_{i1}}e^{i\kappa_i x_i} + \rho_2^{\delta_{i1}}e^{-i(\kappa_i x_i - \delta_i)}\right) + Fe^{i\frac{\hat{q}.\vec{r}}{\alpha\hbar}}$$
$$\Rightarrow i\prod_{i=1}^{d}\left(\rho_1^{\delta_{i1}}e^{i\kappa_i x_i} + (-1)^{\delta_{il}}\rho_2^{\delta_{i1}}e^{-i(\kappa_i x_i - \delta_i)}\right) r\hat{\kappa_l} + iFe^{i\frac{\hat{q}.\vec{r}}{\alpha\hbar}}\hat{q}_j$$
$$+i\sum_{j=1\neq l}^{d}\left[\prod_{i=1}^{d}\left(\rho_1^{\delta_{i1}}e^{i\kappa_i x_i} + (-1)^{\delta_{ij}}\rho_2^{\delta_{i1}}e^{-i(\kappa_i x_i - \delta_i)}\right) r\hat{\kappa_j}\sigma_l\sigma_j + Fe^{i\frac{\hat{q}.\vec{r}}{\alpha\hbar}}\hat{q}_j\sigma_l\sigma_j\right]$$
$$= \pm\prod_{i=1}^{d}\left(\rho_1^{\delta_{i1}}e^{i\kappa_i x_i} + \rho_2^{\delta_{i1}}e^{-i(\kappa_i x_i - \delta_i)}\right) + Fe^{i\frac{\hat{q}.\vec{r}}{\alpha\hbar}}.$$

A little rearrangement yields,

$$
\begin{aligned}
&\prod_{i=1}^{d}\left(\rho_1^{\delta_{i1}}e^{i\kappa_i x_i}+\rho_2^{\delta_{i1}}e^{-i(\kappa_i x_i-\delta_i)}\right)+Fe^{i\frac{\hat{q}.\vec{r}}{\alpha\hbar}}\\
&=\pm\left[i\prod_{i=1}^{d}\left(\rho_1^{\delta_{i1}}e^{i\kappa_i x_i}+(-1)^{\delta_{il}}\rho_2^{\delta_{i1}}e^{-i(\kappa_i x_i-\delta_i)}\right)r\hat{\kappa_l}+iFe^{i\frac{\hat{q}.\vec{r}}{\alpha\hbar}}\hat{q}_j\right.\\
&\left.+i\sum_{j=1\neq l}^{d}\left(\prod_{i=1}^{d}\left(\rho_1^{\delta_{i1}}e^{i\kappa_i x_i}+(-1)^{\delta_{ij}}\rho_2^{\delta_{i1}}e^{-i(\kappa_i x_i-\delta_i)}\right)r\hat{\kappa_j}+Fe^{i\frac{\hat{q}.\vec{r}}{\alpha\hbar}}\hat{q}_j\right)\sigma_l\sigma_j\right],
\end{aligned}
\tag{3.55}
$$

and equating row II,

$$
\begin{aligned}
&-i\left[\prod_{i=1}^{d}\left(\rho_1^{\delta_{i1}}e^{i\kappa_i x_i}+\rho_2^{\delta_{i1}}e^{-i(\kappa_i x_i-\delta_i)}\right)+Fe^{i\frac{\hat{q}.\vec{r}}{\alpha\hbar}}\right]\sigma_l\\
&=\pm\sum_{j=1}^{d}\left[\prod_{i=1}^{d}\left(\rho_1^{\delta_{i1}}e^{i\kappa_i x_i}+(-1)^{\delta_{ij}}\rho_2^{\delta_{i1}}e^{-i(\kappa_i x_i-\delta_i)}\right)r\hat{\kappa_j}+Fe^{i\frac{\hat{q}.\vec{r}}{\alpha\hbar}}\hat{q}_j\right]\sigma_j.
\end{aligned}
$$

Post-multiplying by $i\sigma_l$,

$$
\begin{aligned}
&\prod_{i=1}^{d}\left(\rho_1^{\delta_{i1}}e^{i\kappa_i x_i}+\rho_2^{\delta_{i1}}e^{-i(\kappa_i x_i-\delta_i)}\right)+Fe^{i\frac{\hat{q}.\vec{r}}{\alpha\hbar}}\\
&=\pm\left[i\prod_{i=1}^{d}\left(\rho_1^{\delta_{i1}}e^{i\kappa_i x_i}+(-1)^{\delta_{il}}\rho_2^{\delta_{i1}}e^{-i(\kappa_i x_i-\delta_i)}\right)r\hat{\kappa_l}+iFe^{i\frac{\hat{q}.\vec{r}}{\alpha\hbar}}\hat{q}_l\right.\\
&\left.+i\sum_{j=1\neq l}^{d}\left(\prod_{i=1}^{d}\left(\rho_1^{\delta_{i1}}e^{i\kappa_i x_i}+(-1)^{\delta_{ij}}\rho_2^{\delta_{i1}}e^{-i(\kappa_i x_i-\delta_i)}\right)r\hat{\kappa_j}+Fe^{i\frac{\hat{q}.\vec{r}}{\alpha\hbar}}\hat{q}_j\right)\sigma_j\sigma_l\right].
\end{aligned}
\tag{3.56}
$$

These two equations are almost the same except for the order of multiplication of σ_l

and σ_j in the RHS. Adding Eq. (3.55) and Eq. (3.56),

$$
\begin{aligned}
&2\prod_{i=1}^{d}\left(\rho_1^{\delta_{i1}}e^{i\kappa_i x_i}+\rho_2^{\delta_{i1}}e^{-i(\kappa_i x_i-\delta_i)}\right)+2Fe^{i\frac{\hat{q}.\vec{r}}{\alpha\hbar}}\\
&=\pm 2i\prod_{i=1}^{d}\left(\rho_1^{\delta_{i1}}e^{i\kappa_i x_i}+(-1)^{\delta_{il}}\rho_2^{\delta_{i1}}e^{-i(\kappa_i x_i-\delta_i)}\right)r\hat{\kappa_l}\pm 2iFe^{i\frac{\hat{q}.\vec{r}}{\alpha\hbar}}\hat{q}_l\\
&\pm i\sum_{j=1\neq l}^{d}\left[\left(\prod_{i=1}^{d}\left(\rho_1^{\delta_{i1}}e^{i\kappa_i x_i}+(-1)^{\delta_{ij}}\rho_2^{\delta_{i1}}e^{-i(\kappa_i x_i-\delta_i)}\right)r\hat{\kappa_j}+Fe^{i\frac{\hat{q}.\vec{r}}{\alpha\hbar}}\hat{q}_j\right)(\sigma_l\sigma_j+\sigma_j\sigma_l)\right].
\end{aligned}
\tag{3.57}
$$

Finally, using the anti-commutator $\{\sigma_l,\sigma_j\}=0$ we can write,

$$
\begin{aligned}
&\prod_{i=1}^{d}\left(\rho_1^{\delta_{i1}}e^{i\kappa_i x_i}+\rho_2^{\delta_{i1}}e^{-i(\kappa_i x_i-\delta_i)}\right)+Fe^{i\frac{\hat{q}.\vec{r}}{\alpha\hbar}}\\
&=\pm i\prod_{i=1}^{d}\left(\rho_1^{\delta_{i1}}e^{i\kappa_i x_i}+(-1)^{\delta_{il}}\rho_2^{\delta_{i1}}e^{-i(\kappa_i x_i-\delta_i)}\right)r\hat{\kappa_l}\pm iFe^{i\frac{\hat{q}.\vec{r}}{\alpha\hbar}}\hat{q}_l
\end{aligned}
\tag{3.58}
$$

In order to substitute $x_l=0$ and $x_l=L$ in the above equation we need terms containing (x_l,κ_l,δ_l) and rest of the variables depending on a different i separately. The easiest way to get such terms is to divide Eq.(3.58) by a product of terms that contain all indices but $i=l$. Now the structure of this product depends on whether we want to find a restriction on x_l or on $x_{i\neq l}$.

3.6.1 Case 1 : Length quantization along x axis

In this case, $l=1$, i.e., $x_l=x$. Dividing Eq.(3.58) by

$f_{\bar{l}}(x_i,\kappa_i,\delta_i) = \prod_{i=1\neq l}^{d} \left(e^{i\kappa_i x_i} + e^{-i(\kappa_i x_i - \delta_i)}\right)$ we get,

$$
\begin{aligned}
&\rho_1 e^{i\kappa_1 x} + \rho_2 e^{-i(\kappa_1 x - \delta_1)} + f_{\bar{l}}^{-1} F e^{i\frac{\hat{q}.\vec{r}}{\alpha\hbar}} \\
&= \pm i\left(\rho_1 e^{i\kappa_1 x} - \rho_2 e^{-i(\kappa_1 x - \delta_1)}\right) r\hat{\kappa}_1 \pm i f_{\bar{l}}^{-1} F e^{i\frac{\hat{q}.\vec{r}}{\alpha\hbar}} \hat{q}_1
\end{aligned} \tag{3.59}
$$

Eq.(3.59), at $x = 0$, yields,

$$
\begin{aligned}
&\rho_1 + \rho_2 e^{i\delta_1} + f_{\bar{l}}^{-1} F = i(\rho_1 - \rho_2 e^{i\delta_1}) r\hat{\kappa_1} + i f_{\bar{l}}^{-1} F \hat{q}_1 \\
&\Rightarrow e^{i\delta_1}(ir\hat{\kappa_1} + 1)\rho_2 = (ir\hat{\kappa_1} - 1)\rho_1 + f_{\bar{l}}^{-1} F' e^{-i\theta_1},
\end{aligned} \tag{3.60}
$$

where $F' = \sqrt{1+\hat{q_1}^2} F$ and $\theta_1 = \tan^{-1}(\hat{q}_1)$. Although we used the index l, $f_{\bar{l}}^{-1}$ is nothing but $f_{\bar{1}}^{-1}$, i.e., evaluated at $x_l = x$.

Eq.(3.59), at $x = L_1$, yields,

$$
\begin{aligned}
&\rho_1 e^{i\kappa_1 L_1} + \rho_2 e^{-i(i\kappa_1 L_1 - \delta_1)} + f_{\bar{l}}^{-1} F e^{i\frac{\hat{q_1} L_1}{\alpha\hbar}} = -i(\rho_1 e^{i\kappa_1 L_1} - \rho_2 e^{-i(\kappa_1 L_1 - \delta_1)}) r\hat{\kappa_1} - \\
&i f_{\bar{l}}^{-1} F e^{i\frac{\hat{q_1} L_1}{\alpha\hbar}} \hat{q_1} \\
&\Rightarrow e^{i(2\kappa_1 L_1 - \delta_1)}(ir\hat{\kappa_1} + 1)\rho_1 = (ir\hat{\kappa_1} - 1)\rho_2 + f_{\bar{l}}^{-1} F' e^{i\theta_1} e^{i\frac{\hat{q_1} L_1}{\alpha\hbar}} e^{i(\kappa_1 L_1 - \delta_1)}.
\end{aligned} \tag{3.61}
$$

It is to be noted that the modulus of $\frac{ir\hat{\kappa_1}-1}{ir\hat{\kappa_1}+1}$ is 1 as it can be expressed as,

$$
\frac{ir\hat{\kappa_1} - 1}{ir\hat{\kappa_1} + 1} = e^{i\tan^{-1}\left(\frac{2r\hat{\kappa_1}}{r^2\hat{\kappa}^2 - 1}\right)}. \tag{3.62}
$$

Using this Eq.(3.60) can be re-written as,

$$
\begin{aligned}
e^{i\delta_1} &= \frac{ir\hat{\kappa_1} - 1}{ir\hat{\kappa_1} + 1}\frac{\rho_1}{\rho_2} + f_{\bar{l}}^{-1} F' e^{-i\theta_1} \frac{1}{\rho_2(ir\hat{\kappa_1} + 1)} \\
&= \frac{\rho_1}{\rho_2} e^{i\tan^{-1}\left(\frac{2r\hat{\kappa_1}}{r^2\hat{\kappa}^2 - 1}\right)} + \mathcal{O}(\alpha)
\end{aligned} \tag{3.63}
$$

It is evident that $|\rho_1| = |\rho_2|$ and

$$\delta_1 = \tan^{-1}\left(\frac{2r\hat{k_1}}{r^2\hat{\kappa}^2 - 1}\right) + \theta_{\rho_1} - \theta_{\rho_2} + \mathcal{O}(\alpha), \tag{3.64}$$

where $\theta_{\rho_1} = arg(\rho_1)$ and $\theta_{\rho_2} = arg(\rho_2)$.

To find a restriction on the length along x-axis we will have to consider the other boundary condition. Eq.(3.61) yields,

$$\begin{aligned}
e^{i(2\kappa_1 L_1 - \delta_1)} &= \frac{ir\hat{\kappa_1} - 1}{ir\hat{\kappa_1} + 1}\frac{\rho_2}{\rho_1} + f_{\bar{l}}^{-1}F' e^{i\theta_1} e^{i\frac{\hat{q_1}L_1}{\alpha\hbar}} e^{i(\kappa_1 L_1 - \delta_1)} \frac{1}{\rho_1(ir\hat{\kappa_1} + 1)} \\
\Rightarrow e^{2i\kappa_1 L_1} &= \left(\frac{ir\hat{\kappa_1} - 1}{ir\hat{\kappa_1} + 1}\right)^2 + f_{\bar{l}}^{-1}F' e^{i\theta_1} e^{i\frac{\hat{q_1}L_1}{\alpha\hbar}} e^{i(\kappa_1 L_1 - \delta_1)} \frac{e^{i\delta_1}}{\rho_1(ir\hat{\kappa_1} + 1)} \\
\Rightarrow e^{2i\kappa_1 L_1} &= e^{2\tan^{-1}\left(\frac{2r\hat{\kappa_1}}{r^2\hat{\kappa_1}^2 - 1}\right)} + \mathcal{O}(\alpha)
\end{aligned} \tag{3.65}$$

Thus,

$$\kappa_1 L_1 = \tan^{-1}\left(\frac{2r\hat{\kappa_1}}{r^2\hat{\kappa_1}^2 - 1}\right) + \mathcal{O}(\alpha) \tag{3.66}$$

Note that in the limit $k \to 0$, $\theta_{\rho_j} = 0$ for $j = 1, 2$ which implies $\delta_1 = \kappa_1 L_1 = \tan^{-1}\left(\frac{2r\hat{\kappa_1}}{r^2\hat{\kappa_1}^2 - 1}\right) + \mathcal{O}(\alpha)$ when the potential vanishes.

Combining Eq.(3.66) and (3.61) we get,

$$\begin{aligned}
f_{\bar{l}}^{-1}F' e^{i(\frac{\hat{q_1}L_1}{\alpha\hbar} + \theta_1)} &= e^{ik_1 L_1}(ir\hat{\kappa_1} + 1)\rho_1 - e^{-i\kappa_1 L_1} e^{i\delta_1}(ir\hat{\kappa_1} - 1)\rho_2 \\
&= \rho_1(ir\hat{\kappa_1} - 1) - \rho_2(ir\hat{\kappa_1} - 1)e^{-i\tan^{-1}\left(\frac{2r\hat{\kappa_1}}{r^2\hat{\kappa_1}^2 - 1}\right)} e^{i\delta_1} \\
&= \rho_1(ir\hat{\kappa_1} - 1) - \rho_2(ir\hat{\kappa_1} + 1)e^{i\delta_1}
\end{aligned}$$

Thus,

$$e^{i(\frac{\hat{q}_1 L_1}{\alpha\hbar}+\theta_1)} = \frac{\rho_1(ir\hat{\kappa}_1 - 1) - \rho_2(ir\hat{\kappa}_1 + 1)e^{i\delta_1}}{F'} f_{\bar{l}} \tag{3.67}$$

It follows from Eq.(3.67),

$$\frac{\hat{q}_1 L_1}{\alpha\hbar} = \frac{\hat{q}_1 L_1}{\alpha_0 \ell_{Pl}} = -\theta_1 + arg\left(\frac{\rho_1(ir\hat{\kappa}_1 - 1) - \rho_2(ir\hat{k}_1 + 1)e^{i\delta_1}}{F'} f_{\bar{l}}\right) + 2n_1\pi,\ n_1 \in \mathbb{N} \tag{3.68}$$

Eq.(3.68) gives the quantization of length along x-axis.

As shown for the one-dimensional Dirac equation, one would expect, in the limit $k \to 0$, Eq.(3.68) to yield the length quantization without the influence of a gravitational potential. In other words, Eq.(3.68) must reduce to Eq.(41) in [33]. Eq.(38) in [33] can be re-written as,

$$e^{-i\theta_1} = \frac{e^{i\delta_1}(ir\hat{\kappa}_1 + 1) - (ir\hat{\kappa}_1 - 1)}{f_{\bar{l}}^{-1} F'}, \tag{3.69}$$

In the limit $k \to 0$ the RHS of Eq.(3.67) yields,

$$\begin{aligned} &\frac{\rho_1(ir\hat{\kappa}_1 - 1) - \rho_2(ir\hat{\kappa}_1 + 1)e^{i\delta_1}}{F'} f_{\bar{l}} \\ &= \frac{1.(ir\hat{\kappa}_1 - 1)\frac{ir\hat{\kappa}_1+1}{ir\hat{\kappa}_1+1} - 1.(ir\hat{\kappa}_1 + 1)\frac{ir\hat{\kappa}_1-1}{ir\hat{\kappa}_1+1}.1}{f_{\bar{l}}^{-1} F'} \\ &= \frac{e^{i\delta_1}(ir\hat{\kappa}_1 + 1) - (ir\hat{\kappa}_1 - 1)}{f_{\bar{l}}^{-1} F'} \end{aligned}$$

Thus, from Eq.(3.69) and (3.67) we get,

$$e^{i(\hat{q}_1 L_1/\alpha\hbar+\theta_1)} = e^{-i\theta_1} \tag{3.70}$$

which implies

$$\frac{\hat{q}_1 L_1}{\alpha\hbar} = \frac{\hat{q}_1 L_1}{\alpha_0 \ell_{Pl}} = -2\theta_1 + 2n_1\pi,\ n_1 \in \mathbb{N}. \tag{3.71}$$

This is identical with Eq.(41) in [33].

3.6.2 Case 2 : Length quantization along y and z axes

In this case, $l \neq 1$, which means we divide Eq.(3.58) by

$$g_{\bar{l}}(x_i, \kappa_i \delta_i) = \prod_{i=1\neq l}^{d} \left(\rho_1^{\delta_{i1}} e^{i\kappa_i x_i} + \rho_2^{\delta_{i1}} e^{-i(\kappa_i x_i - \delta_i)}\right)$$

and get

$$\begin{aligned} & e^{i\kappa_l x} + e^{-i(\kappa_l x - \delta_1)} + g_{\bar{l}}^{-1} F e^{i\frac{\hat{q}.\vec{r}}{\alpha\hbar}} \\ & = \pm i \left(e^{i\kappa_l x} - e^{-i(\kappa_l x - \delta_1)}\right) r\hat{\kappa}_l \pm i g_{\bar{l}}^{-1} F e^{i\frac{\hat{q}.\vec{r}}{\alpha\hbar}} \hat{q}_l \end{aligned} \tag{3.72}$$

Just like case 1, Eq.(3.72), at $x_l = 0$ and $x_l = L_l$ gives,

$$e^{i\delta_l}(ir\hat{\kappa}_l + 1) = (ir\hat{\kappa}_l - 1) + g_{\bar{l}}^{-1} F_l' e^{-i\theta_l}, \tag{3.73}$$

and

$$e^{i(2\kappa_l L_l - \delta_l)}(ir\hat{\kappa}_l + 1) = (ir\hat{\kappa}_l - 1) + g_{\bar{l}}^{-1} F_l' e^{i\theta_l} e^{i\frac{\hat{q}_l L_l}{\alpha\hbar}} e^{i(\kappa_l L_l - \delta_l)}, \tag{3.74}$$

where $F_l' = \sqrt{1 + \hat{q_1}^2} F$ and $\theta_l = \tan^{-1}(\hat{q}_l)$. Also $g_{\bar{l}}$ is assumed to be evaluate at the point on x_i axis. Clearly, $i \neq l$. These equations are much simpler compared to the ones in case 1, and they are practically identical with the case without gravitational potential except for a different function $g_{\bar{l}}$. It is quite easy to show that,

$$\begin{aligned} e^{i\delta_l} &= \frac{ir\hat{\kappa}_l - 1}{ir\hat{\kappa}_l + 1} + g_{\bar{l}}^{-1} F_l' e^{-i\theta_l} \frac{1}{ir\hat{\kappa}_l + 1} \\ &= e^{i\tan^{-1}\left(\frac{2r\hat{\kappa}_l}{r^2\hat{\kappa}_l{}^2 - 1}\right)} + \mathcal{O}(\alpha) \end{aligned}$$

and

$$\begin{aligned} e^{2i\kappa_1 L_1} &= \left(\frac{ir\hat{\kappa}_1 - 1}{ir\hat{\kappa}_1 + 1}\right)^2 + f_{\bar{l}}^{-1} F' e^{i\theta_1} e^{i\frac{\hat{q}_1 L_1}{\alpha\hbar}} e^{i(\kappa_1 L_1 - \delta_1)} \frac{e^{i\delta_1}}{ir\hat{\kappa}_1 + 1} \\ &= e^{2i\tan^{-1}\left(\frac{2r\hat{\kappa}_l}{r^2\hat{\kappa}_l{}^2 - 1}\right)} + \mathcal{O}(\alpha), \end{aligned}$$

which imply

$$\delta_l = \kappa_l L_l = \tan^{-1}\left(\frac{2r\hat{\kappa}_l}{r^2\hat{\kappa}_l{}^2 - 1}\right) + \mathcal{O}(\alpha) \tag{3.75}$$

Now, if we plug $\delta_l = \kappa_l L_l$ into Eq.(3.70) and then compare it with Eq.(3.69), we see the following relation must hold,

$$\frac{\hat{q}_l L_l}{\alpha\hbar} + \theta_l = -\theta_l + 2n_l\pi, \;\; n_l \in \mathbb{N}$$

$$\Rightarrow \frac{\hat{q}_l L_l}{\alpha\hbar} = \frac{\hat{q}_l L_l}{\alpha_0 \ell_{Pl}} = -2\theta_l + 2n_l\pi, \;\; n_l \in \mathbb{N} \tag{3.76}$$

Eq.(3.68) and (3.76) prove length quantization in all directions inside the box. For area and volume quantization we can simply multiply the two equations,

$$A_N = \prod_{l=1}^{N} \frac{\hat{q}_l L_l}{\alpha_0 \ell_{Pl}} = \prod_{l=2}^{N} (2n_l\pi - 2\theta_l) \left(2n_1\pi - \theta_1 + arg\left(\frac{\rho_1(ir\hat{\kappa_1} - 1) - \rho_2(ir\hat{\kappa_1} + 1)e^{i\delta_1}}{F'} f_{\bar{l}} \right) \right), \quad n_l \in \mathbb{N} \tag{3.77}$$

where $N = 2$ and $N = 3$ represent area and volume quantization.

Eq.(3.68), (3.76) and (3.77) show that the space, in presence of weak gravity, is discrete in one, two and three dimensions.

Chapter 4

Conclusions

4.1 Introduction

In this thesis, we have shown if we trap a particle in a one-dimensional box of size close to the Planck length, impose a gravitational potential inside the box and then try to measure the length of the box, the length would appear as a quantized variable in units of $\alpha_0 \ell_{Pl}$ where ℓ_{Pl} is the Planck length. This can be translated as the fact that the discreteness of space holds equally for curved spacetime, as previous work showed space is discontinuous near the Planck-scale in flat spacetime [32, 33]. We have repeated this calculation for one, two and three-dimensional relativistic cases and the discreteness of space seem to exist in every situation. Moreover, the presence of the lengths being proportional to the Planck length in all cases strengthens the claim of the existence of a minimum measurable length.

Being at an early stage of development, one of the main goals of quantum gravity phenomenology (QGP) is to clarify to the prospective theories what is consistent what is not, with the possible experimental search for quantum gravity effects. One of these effects, also present within the scope of string theory, is the non-zero irreducible uncertainties. In terms of the distance measurement, it can not be smaller than the Plank length. The significance of this effect is a modification of momentum, which gives rise to a Planck-scale correction to the uncertainty principle, known as the generalized uncertainty principle (GUP). This correction adds new terms to the quantum mechanical differential equations, which result in new solutions. Finally, these new solutions infer the quantization of the spatial dimensions.

4.2 Summary of the Results

To show the non-relativistic one-dimensional length quantization, we used a linearized potential term, proportional to the distance, with the Schrödinger equation for a particle in a one-dimensional box. We considered the GUP-induced term, containing a third order differential operator, as a small perturbation to the original Schrödinger equation with gravitational potential energy, and followed a perturbative approach to solve this. The result is length quantization similar to the previous work without gravity, but this time we get a fine structure of length quantization which is different than the previous work.

Particle with speed close to c (high energy) are more likely to give better picture of the structure of space near Planck length. Motivated by this fact, work has been done before to show discreteness for a relativistic particle in a one-dimensional box (Klein-Gordon equation) and for more than one dimension (Dirac equation). The justification for using the Dirac equation is evident, as most of the fundamental matter particles are fermions and the easiest equation governs them is the Dirac. Also Dirac equations has its mathematical advantage when it comes to dealing with two and three dimensions. In this thesis, results for flat spacetime have been extended to cases with an additional linear potential term, which we have considered a small perturbation and solved accordingly. The subsequent quantization results for length, area and volume are similar to those in flat spacetime but mathematically more complex. Although, they are transcendental equations which makes it difficult to have explicit formulas but several numerical techniques are available to find precise values for the quantities whenever called for by experimental data.

All of these quantization formulas reduce to the old ones in flat spacetime in the limit the proportionality constant in the potential term goes to zero. This proves the robustness of the results shown in this work.

4.3 Significance of the Results

This work has two main consequences. One, these results, showing discreteness of space, come to existence from the GUP-induced corrections to the Schrödinger, Kein-Gordon and Dirac equations which are essential near Planck-scale. This means the classical notion of so-called spacetime continuum collapses if we look at it through a magnifying glass, powerful enough to see lengths of the order of 10^{-35} m.

Previous work showed discreteness of space in confinement without any force field, which indicates the fundamental nature of space, at a microscopic level, is granular in a region far from a black hole. In this work, this indication becomes even stronger as we have not found any instances, within the scope of this research, where the discreteness does not hold.

We have only shown the discreteness of space in the presence of weak gravity. It surely requires a different approach other than a perturbative one to explore the discreteness close to a massive body. Nevertheless, this work provides with useful techniques to analyze the nature of space near Planck-scale, in slightly curved spacetime. Also, the spatial dimensions quantized as multiples of the Planck length in presence of gravity implies that the claim of having the Planck length as the minimum measurable length may be correct. In other words, we have got a GUP-version of the Heisenberg microscope with maximum resolving power.

The other consequence is the fine structure of length quantization, found in the context of the Schrödinger equation with gravity. This is somewhat similar to the fine structure of energy levels of the hydrogen atom. The significance of this is a complex discrete structure of space which might indicate a fractal discontinuity, addition of an extra field reveals this finer structure. Also, a little more ambitious indication may direct towards the discreteness involving time coordinate. Although this is beyond the scope of current work, we hope to explore this direction in the future.

4.4 Future Directions of Work

We noted in this thesis that the quantization formulas are complex and transcendental. This makes it difficult to study them algebraically. Numerical or graphical methods would be preferable for solving these equations for explicit length. Also, these methods could be used for specific examples of elementary particles in given curved backgrounds.

Future projects might include applications of our discreteness results to various quantum mechanical systems, with the experimental signatures at scales larger than the Planck scale. Quantization expressions for length are unbounded functions of a natural number n. For large n, it may be possible to find experimental evidence of discreteness at macroscopic length scales.

Finally, extension of the method used in this work for arbitrary curved spacetime would be interesting. We have used the first term of Taylor series to describe a linearized potential. Subsequent terms in Taylor series would give rise to a more general curved spacetime. Hence, an arbitrary form of gravitational potential could be analyzed following the same approach. This would still assume a fixed classical background. A complete theory of quantum gravity, once formulated, should be able to address the issues discussed here, with background spacetime which may be fluctuating. In this event we hope that the results derived in this thesis would continue to hold, at least approximately, and in the limit when such fluctuations can be ignored.

Bibliography

[1] Witten, E.(1996, April). Reflections of the fate of spacetime. *Physics Today*, 24-30.

[2] Hawking, S. W.(1975). Particle creation by black holes. *Communications in Mathematical Physics*, 43, 199-220.

[3] Schwarz,J.(1985). *Superstrings: The first 15 years of superstring theory, Vol. 1, 2*. World Scientific Publising Co. Inc.

[4] Becker, M., & Schwarz, J. H. (2007). *String Theory and M-Theory*. Cambridge University Press.

[5] Johnson, C.V.(2003). *D-Branes*. Cambridge University Press.

[6] Zwiebach, B.(2009). *A First course in string theory, 2nd edition*. Cambridge University Press.

[7] Girelli, F., Hinterleitner, F., Major,S. A.(2012).Loop Quantum Gravity Phenomenology: Linking Loops to Observational Physics. *arXiv:1210.145v2[gr-qc].*

[8] Malament, D.(1977). The class of continuous timelike curves determines the topology of spacetime. *Journal of Mathematical Physics, 18(7)*, 1399-1404.

[9] Thiemann, T.(1998). A Length Operator for Canonical Quantum Gravity. *Journal of Mathematical Physics, 39*, 3372-3392[arXiv:gr-qc/9606092].

[10] Sorkin, R.(2006). Geometry from order: Causal sets. *Einstein online, 2*, 1007.

[11] Henson, J.(2006). The causal set approach to quantum gravity. *arXiv:gr-qc/0601121v2.*

[12] Ali, A. F., Das, S., Vagenas. E. C.(2011). Proposal for testing quantum gravity in the lab. *Physics Review D, 84*, 044013.

[13] Pikovski, I., Vanner, M. R., Aspelmeyer, M., Kim, M. S., Brukner, C.(2012). Probing Planck-scale physics with quantum optics. *Nature Physics 8,* 393397.

[14] Barcelo, C., Liberati, S., Visser, M.(2011). Analogue Gravity. *Living Rev. Relativity, 14*,3.

[15] Shu, F. H.(1982). *The Physical Universe: An Introduction to Astronomy.* CA: University Science Books.

[16] Colella, R., Overhauser, A.W., Werner, S.A.(1975). Observation of Gravitationally Induced Quantum Interference *Physics Review Letters 34*,1472.

[17] Giovanni, A.(2004). Introduction to Quantum-Gravity Phenomenology. *arXiv:gr-qc/0412136v1.*

[18] Amati, D., Ciafaloni, M., Veneziano, G.(1989). Can spacetime be probed below the string size?. *Physics Letters B, 216*(1-2),41-47.

[19] Maggiore, M.(1993). A Generalized Uncertainty Principle in Quantum Gravity.*Physics Letter B, 304*(65),*[arXiv:hep-th/9301067].*

[20] Maggiore, M.(1994). Quantum groups, gravity, and the generalized uncertainty principle. *Physics Review D 49,* 5182.

[21] Maggiore, M.(1993). The algebraic structure of the generalized uncertainty principle. *Physics Letters B, 319*(83)*[arXiv:hep-th/9309034].*

[22] Garay, L. J.(1995). Quantum gravity and minimum length. *International Journal of Modern Physics. A10,*145-156 [arXiv:gr-qc/9403008].

[23] cardigli, F. (1999). Generalized Uncertainty Principle in Quantum Gravity from Micro-Black Hole Gedanken Experiment. *Physics Letters B, 452,*39-44.

[24] Hossenfelder, M., S. Bleicher, Hofmann, S., Ruppert, J., Scherer, S., Stoecker, H.(2003). Signatures in the Planck Regime. *Physics Letters B, 575* 85-99.

[25] Bambi, C., & Urban,F. R.(2008). Natural extension of the Generalised Uncertainty Principle. *Class and Quantum Gravity, 25*:095006.

[26] Magueijo, J. & Smolin, L.(2002). Lorentz invariance with an invariant energy scale. *Physics Review Letters, 88:*190403.

[27] Magueijo, J. & Smolin, L.(2005). String theories with deformed energy momentum relations, and a possible non-tachyonic bosonic string. *Physics Review D, 71*:026010.

[28] Cortes, J.L. & Gamboa, J.(2005). Quantum Uncertainty in Doubly Special Relativity. *Physics Review D, 71*:065015, *[arXiv:hep-th/0405285].*

[29] Kempf, A.(1997). Nonpointlike Particles in Harmonic Oscillators. *Journal of Physics A,30*, 2093.

[30] Kempf, A., Mangano, G., Mann, R.B.(1995). Hilbert space representation of the minimal length uncertainty relation. *Physics Review D 52*,1108.

[31] Brau,F.(1999). Minimal Length Uncertainty Relation and Hydrogen Atom. *Journal of Physics A,32*,7691-7696.

[32] Ali, A.F., Das, S., Vagenas, E.(2009). Discreteness of space from the generalized uncertainty principle. *Physics Letters B, 678*(5),497-499.

[33] Ali, A.F., Das, S., Vagenas, E.(2010). Discreteness of Space from GUP II: Relativistic Wave Equations. *Physics Letters B, 690*(4), 407-412.

[34] Landau, L. D., & Lifschitz, E. M.(1991). *Quantum Mechanics (Non-Relativistic Theory)*, 3rd edition. Oxford, England:Pergamon Press.

[35] Polyanin, A.D., & Zeitsev, V.F.(2003). *Handbook of Exact Solutions for Ordinary Differential Equations.* Chapman and Hall.

[36] Das, A.(2008). *Lectures in Quantum Mechanics.* Singapore: World Scientific Publishing.

[37] Bologna, M.(2005). *Short Introduction to Fractional Calculus.* Universidad de Tarapaca, Arica, Chile. Lecture Notes, 41-54.

[38] Cohen-Tannoudji, C., Diu, B., Laloe, F.(1991).*Quantum Mechanics*, vol. 1. Wiley-VC.

[39] Dirac, P. A. M.(1928). The quantum theory of the electron. *Proceedings of the Royal Society A: Mathematics, Physics and Engineering Science 117*(778): 610.

[40] Alberto, P., Fiolhais, C., Gil, V. M.(1996). Relativistic particle in a box. *European Journal of Physics, 17*, 19-24.

[41] Chodos, A., Thorn,C.B.(1974). Electromagnetic self-energy of confined massless quarks. *Physics Letters B, 53*(4), 359-361.

Appendices

Appendix A

Dimensions and Relative Magnitudes

$$
\begin{aligned}
&[\hbar] = m^2kgs^{-1} \\
&k = mkgs^{-2} \\
&\mu = m^{-1} \\
&\alpha = \ell_{Pl}/\hbar
\end{aligned}
$$

$$
\begin{aligned}
&\frac{k^2m^2x^2}{\mu^6\alpha^2\hbar^6} = \frac{(mkgs^{-2})^2(kg)^2m^2}{(m^{-1})^6(\ell_{Pl}/\hbar)^2(m^2kgs^{-1})^6} = 1 \\
&\frac{km}{\mu^3\hbar^2}\frac{(1+\mu\alpha\hbar)}{\mu^2\alpha^2\hbar^2} \\
&\mu\alpha\hbar = m^{-1}(\ell_{Pl}/\hbar)\hbar = 1 \\
&\frac{km}{\mu^3\hbar^2} = \frac{(mkgs^{-2})kg}{(m^{-3})(m^2kgs^{-1})^2} = 1
\end{aligned}
$$

ϵ can be considered as having the dimension of $1/k$.

$$
\begin{aligned}
&\hbar \sim 10^{-34} m^2 kg s^{-1}\\
&\ell_{Pl} \sim 10^{-35} m\\
&m \sim 10^{-30} kg\\
&x \sim 10\ell_{Pl}\\
&(1) E \sim 10^{-30} \times (3\times 10^8)^2 \approx 10^{-30+17} \approx 10^{-13} kgm^2s^{-2} \qquad (rest\ energy\ of\ an\ electron)\\
&(2) \quad \frac{k^2m^2x^2}{\mu^6\alpha^2\hbar^6} \sim k^2 \frac{10^{-30\times 2}\times 100 l_{pl}^2 \times 10^{-2\times 34}}{8\times 10^{-30\times 3}\times 10^{-13\times 3}\ell_{Pl}^2} \approx k^2 \frac{10^{-60-68+2+90+39}}{8} \approx 10^{-4}k^2\\
&(3) \quad \frac{k^2m^2}{\mu^6\hbar^4}\left(\frac{(1+\mu\alpha\hbar)}{\mu^2\alpha^2\hbar^2}\right)^2\\
&\mu\alpha\hbar \sim \sqrt{\frac{2mE}{\hbar^2}}\frac{\ell_{Pl}}{\hbar}\hbar \approx \sqrt{2\times 10^{-30}\times 10^{-13}}\times 10^{34}\times 10^{-35} \approx 10^{-22.5} \approx 10^{-23}\\
&\frac{k^2m^2}{\mu^6\hbar^4}\left(\frac{(1+\mu\alpha\hbar)}{\mu^2\alpha^2\hbar^2}\right)^2 \sim \frac{k^2 10^{-30\times 2}\times 10^{-34\times 2}}{8\times 10^{-30\times 3}\times 10^{-13\times 3}}\left(\frac{1+10^{-23}}{10^{-23\times 2}}\right)^2 \approx \frac{10^{90+39-60-68+92}}{8} \approx 10^{94}k^2\\
&(4) \quad \frac{2k^2m}{\mu^3\hbar^2}\left(\frac{(1+\mu\alpha\hbar)}{\mu^2\alpha^2\hbar^2}\right) \approx \frac{k^2 10^{-30}\times 10^{-34}}{2^{3/2}\times 10^{-30\times\frac{3}{2}}\times 10^{-13\times\frac{3}{2}}}\left(\frac{1+10^{-23}}{10^{-23\times 2}}\right) \approx 10^{45}k^2\\
&(5) \quad \frac{2k^2mx}{\mu^3\alpha\hbar^3} \sim \frac{k^2 10^{-30}\times 10\ell_{Pl}\times 10^{-34}}{2^{3/2}\times 10^{-30\times 2\times\frac{3}{2}}\times 10^{-13\times\frac{3}{2}}\times \ell_{Pl}} \approx k^2\frac{2}{2^{3/2}}10^{-64+1+45+19.1} \approx k^2.
\end{aligned}
\tag{A.1}
$$

Appendix B

Dimension Check for k and Comparison with α

$$
\begin{aligned}
k^2 &<< \frac{\mu^6\alpha^2\hbar^6}{m^2x^2} \\
dim\left[\frac{\mu^6\alpha^2\hbar^6}{m^2x^2}\right] &= \frac{\left(\frac{1}{m}\right)^6\left(\frac{\ell_{Pl}}{\hbar}\right)^2\hbar^6}{kg^2m^2} \\
&= \frac{m^2\hbar^4}{kg^2m^8} \\
&= \frac{m^2(m^2kgs^{-1})^4}{kg^2m^8} \\
&= \frac{m^{10}kg^2s^{-4}}{m^8} \\
&= \frac{m^4kg^2s^{-4}}{m^2} \\
[k] &= \frac{m^2kgs^{-2}}{m} \\
\left[\frac{Energy}{length}\right] &= \frac{m\left(kg\left(\frac{m}{s^2}\right)\right)}{m}.
\end{aligned}
$$

The underlying assumption of the perturbative approach is that the perturbation must be very small compared to the unperturbed wavefunction, i.e., $k\psi_1 < \psi_0$

$$
\left|\frac{k\psi_1}{\psi_0}\right|^2 = \epsilon^2k^2 + \frac{k^2m^2x^2}{\mu^6\alpha^2\hbar^6} + \frac{k^2m^2}{\mu^6\hbar^4}\left(\frac{(1+\mu\alpha\hbar)}{\mu^2\alpha^2\hbar^2}\right)^2 + \frac{2\epsilon k^2m}{\mu^3\hbar^2}\left(\frac{(1+\mu\alpha\hbar)}{\mu^2\alpha^2\hbar^2}\right)) - \frac{2\epsilon k^2mx}{\mu^3\alpha\hbar^3}.
$$

The term with the highest order of magnitude is $\frac{k^2m^2}{\mu^6\hbar^4}\left(\frac{(1+\mu\alpha\hbar)}{\mu^2\alpha^2\hbar^2}\right)^2$. Setting this as $<< 1$ we get

$$
\begin{aligned}
k^2 << \frac{\mu^6\hbar^4}{m^2}\left(\frac{\mu^2\alpha^2\hbar^2}{(1+\mu\alpha\hbar)}\right)^2 &\approx 10^{-94} \\
&\Rightarrow k << 10^{-47}.
\end{aligned}
$$

Taking $x \sim 10\ell_{Pl}$,

$$\left(\alpha \sim \frac{\alpha_0 \ell_{Pl}}{\hbar} \approx \frac{1 \times 10^{-35}}{10^{-34}} \approx 10^{-1} = 0.1\right).$$

This means k can be considered as smaller than α.

Appendix C

Derivatives

C.1 Computation of some useful quantities

$$\begin{aligned}
\psi_0(E, k.x) &= C_1 Ai(-\xi) + C_2 Bi(-\xi) \\
&= \frac{C_1}{\sqrt{\pi}}\xi^{-1/4}\sin\left(\frac{2}{3}\xi^{\frac{3}{2}} + \frac{\pi}{4}\right) + \frac{C_2}{\sqrt{\pi}}\xi^{-1/4}\cos\left(\frac{2}{3}\xi^{\frac{3}{2}} + \frac{\pi}{4}\right) \qquad \text{(C.1)} \\
&= C_1\psi_0^I + C_2\psi_0^{II},
\end{aligned}$$

where $\xi = \left(\frac{2m}{\hbar^2}\right)^{\frac{1}{3}} k^{-\frac{2}{3}}(E - kx)$.

$$\frac{d\xi}{dx} = \left(\frac{2m}{\hbar^2}\right)^{\frac{1}{3}} k^{-\frac{2}{3}}(-k) = -\left(\frac{2m}{\hbar^2}\right)^{\frac{1}{3}} k^{\frac{1}{3}} \qquad \text{(C.2)}$$

$$\frac{d\xi}{dE} = \left(\frac{2m}{\hbar^2}\right)^{\frac{1}{3}} k^{-\frac{2}{3}} \qquad \text{(C.3)}$$

$$\begin{aligned}
&\frac{d\psi_0^2}{dx^2} = \frac{d}{dx}\left(\frac{d\psi_0}{d\xi}\frac{d\xi}{dx}\right) \\
&= \frac{d}{dx}\left(\frac{d\psi_0}{d\xi}\right)\frac{d\xi}{dx} + \frac{d\psi_0}{d\xi}\frac{d}{dx}\left(\frac{d\xi}{dx}\right) \\
&= \frac{d}{d\xi}\left(\frac{d\psi_0}{d\xi}\right)\frac{d\xi}{dx}\frac{d\xi}{dx} + 0 \\
&= \left(\frac{d\xi}{dx}\right)^2\frac{d}{d\xi}\left(\frac{d\psi_0}{d\xi}\right) \\
&\Rightarrow \frac{d^3\psi_0}{dx^3} = \frac{d}{dx}\left(\left(\frac{d\xi}{dx}\right)^2\frac{d^2\psi_0}{d\xi^2}\right) = \left(\frac{d\xi}{dx}\right)^3\frac{d^3\psi_0}{d\xi^3} \qquad \text{(C.4)}
\end{aligned}$$

$$\frac{d\psi_0}{dE} = \frac{d\psi_0}{d\xi}\frac{d\xi}{dE} \qquad \text{(C.5)}$$

.

$$
\begin{aligned}
\frac{d}{dx}\left(\frac{d\psi_0}{dE}\right) &= \frac{d}{dx}\left(\frac{d\psi_0}{d\xi}\frac{d\xi}{dE}\right) \\
&= \frac{d}{dx}\left(\frac{d\psi_0}{d\xi}\right)\frac{d\xi}{dE} + \frac{d\psi_0}{d\xi}\frac{d}{dx}\left(\frac{d\xi}{dE}\right) \\
&= \frac{d}{d\xi}\left(\frac{d\psi_0}{d\xi}\right)\frac{d\xi}{dx}\frac{d\xi}{dE} + 0 \\
&= \frac{d^2\psi_0}{d\xi^2}\frac{d\xi}{dx}\frac{d\xi}{dE} \\
\Rightarrow \frac{d^2}{dx^2}\left(\frac{d\psi_0}{dE}\right) &= \frac{d}{dx}\left(\frac{d}{dx}\frac{d\psi_0}{dE}\right) \\
&= \frac{d}{dx}\left(\frac{d^2\psi_0}{d\xi^2}\frac{d\xi}{dx}\frac{d\xi}{dE}\right) \\
&= \frac{d}{dx}\left(\frac{d^2\psi_0}{d\xi^2}\right)\frac{d\xi}{dx}\frac{d\xi}{dE} + \frac{d^2\psi_0^2}{d\xi^2}\frac{d}{dx}(\frac{d\xi}{dx})\frac{d\xi}{dE} + \frac{d^2\psi_0^2}{d\xi^2}\frac{d\xi}{dx}\frac{d}{dx}\left(\frac{d\xi}{dE}\right) \\
&= \frac{d}{d\xi}\left(\frac{d^2\psi_0}{d\xi^2}\right)\left(\frac{d\xi}{dx}\right)^2\frac{d\xi}{dE} + 0 + 0 \\
&= \frac{d^3\psi_0}{d\xi^3}\left(\frac{d\xi}{dx}\right)^2\frac{d\xi}{dE} \qquad \text{(C.6)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{d^3}{dx^3}\left(\frac{d\psi_0}{dE}\right) &= \frac{d}{dx}\left(\frac{d^2}{dx^2}\left(\frac{d\psi_0}{dE}\right)\right) \\
&= \frac{d}{dx}\left(\frac{d^3\psi_0}{d\xi^3}\left(\frac{d\xi}{dx}\right)^2\frac{d\xi}{dE}\right) \\
&= \frac{d}{dx}\left(\frac{d^3\psi_0}{d\xi^3}\right)\left(\frac{d\xi}{dx}\right)^2\frac{d\xi}{dE} + 0 + 0 \\
&= \frac{d}{d\xi}\left(\frac{d^3\psi_0}{d\xi^3}\right)\left(\frac{d\xi}{dx}\right)^3\frac{d\xi}{dE} \\
&= \frac{d^4\psi_0}{d\xi^4}\left(\frac{d\xi}{dx}\right)^3\frac{d\xi}{dE} \qquad \text{(C.7)}
\end{aligned}
$$

C.2 Calculation of derivatives required in sec 2.3.1

1.
$$\begin{aligned}\frac{d\psi_0}{d\xi} &= C_1\frac{d\psi_0^I}{d\xi} + C_2\frac{d\psi_0^{II}}{d\xi}\\ &= \frac{C_1}{\sqrt{\pi}}\left[-\frac{1}{4}\xi^{-5/4}\sin\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)+\xi^{1/4}\cos\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)\right]+\\ &\frac{C_2}{\sqrt{\pi}}\left[-\xi^{1/4}\sin\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)-\frac{1}{4}\xi^{-5/4}\cos\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)\right].\end{aligned}$$

2.
$$\begin{aligned}&\frac{d^2\psi_0}{d\xi^2} = \frac{C_1}{\sqrt{\pi}}\left[-\frac{1}{4}\left(-\frac{5}{4}\right)\xi^{-\frac{5}{4}-1}\sin\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)-\right.\\ &\frac{1}{4}\xi^{-\frac{5}{4}+\frac{1}{2}}\cos\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)+\frac{1}{4}\xi^{\frac{1}{4}-1}\cos\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)-\left.\xi^{\frac{1}{4}+\frac{1}{2}}\sin\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)\right]+\\ &\frac{C_2}{\sqrt{\pi}}\left[-\frac{1}{4}\xi^{\frac{1}{4}-1}\sin\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)-\xi^{\frac{1}{4}+\frac{1}{2}}\cos\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)-\frac{1}{4}\left(-\frac{5}{4}\right)\xi^{-\frac{5}{4}-1}\cos\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)+\right.\\ &\left.\frac{1}{4}\xi^{-\frac{5}{4}+\frac{1}{2}}\sin\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)\right]\\ &=\frac{C_1}{\sqrt{\pi}}\left[\frac{5}{16}\xi^{-9/4}\sin\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)-\frac{1}{4}\xi^{-3/4}\cos\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)+\frac{1}{4}\xi^{-3/4}\cos\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)-\right.\\ &\left.\xi^{3/4}\sin\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)\right]+\frac{C_2}{\sqrt{\pi}}\left[-\frac{1}{4}\xi^{-3/4}\sin\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)-\xi^{3/4}\cos\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)+\right.\\ &\left.\frac{3}{16}\xi^{-9/4}\cos\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)+\frac{1}{4}\xi^{-3/4}\sin\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)\right]\end{aligned} \tag{C.8}$$

$$\begin{aligned}&=\frac{C_1}{\sqrt{\pi}}\left[\frac{5}{16}\xi^{-9/4}\sin\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)-\xi^{3/4}\sin\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)\right]+\frac{C_2}{\sqrt{\pi}}\left[-\xi^{3/4}\cos\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)\right]\\ &=-\frac{C_1}{\sqrt{\pi}}\left[\xi^{3/4}\sin\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)\right]-\frac{C_2}{\sqrt{\pi}}\left[\xi^{3/4}\cos\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)\right]\end{aligned} \tag{C.9}$$

3.
$$\begin{aligned}\frac{d^3\psi_0}{d\xi^3} &= -\frac{C_1}{\sqrt{\pi}}\left[\frac{3}{4}\xi^{\frac{3}{4}-1}\sin\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)+\xi^{\frac{3}{4}+\frac{1}{2}}\cos\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)\right]-\\ &\frac{C_2}{\sqrt{\pi}}\left[\frac{3}{4}\xi^{\frac{3}{4}-1}\cos\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)-\xi^{\frac{3}{4}+\frac{1}{2}}\sin\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)\right]\\ &= -\frac{C_1}{\sqrt{\pi}}\left[\frac{3}{4}\xi^{-1/4}\sin\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)+\xi^{5/4}\cos\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)\right]+\\ &\frac{C_2}{\sqrt{\pi}}\left[-\frac{3}{4}\xi^{-1/4}\cos\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)+\xi^{5/4}\sin\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)\right]\end{aligned}$$

4. $$\frac{d^4\psi_0}{d\xi^4} = \frac{C_1}{\sqrt{\pi}}\left[\left(\frac{3}{16}\xi^{-5/4} + \xi^{7/4}\right)\sin\left(\frac{2}{3}\xi^{3/2} + \frac{\pi}{4}\right) - 2\xi^{1/4}\cos\left(\frac{2}{3}\xi^{3/2} + \frac{\pi}{4}\right)\right] + \frac{C_2}{\sqrt{\pi}}\left[\left(\frac{3}{16}\xi^{-5/4} + \xi^{7/4}\right)\cos\left(\frac{2}{3}\xi^{3/2} + \frac{\pi}{4}\right) + 2\xi^{1/4}\cos\left(\frac{2}{3}\xi^{3/2} + \frac{\pi}{4}\right)\right] \tag{C.10}$$