ORIGINS OF THERMALIZATION IN QUANTUM COSMOLOGY

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Dedication

To my family, whose unwavering support sustained me through the cosmic ebbs and flows of this intellectual journey.

Abstract

We aim to provide the effect of accelerated frames in cosmology and identify the origins of thermalization in the evolution of the universe. We begin our discussion by discussing general relativity and cosmology, as well as their successes and failures, which leads to the need for quantum cosmology. We then discuss the canonical formulation of general relativity, which is the basis of quantum cosmology, and its issues. We constructed a wavefunction for the universe whose dynamics are governed by the Wheeler-Dewitt equation. Semiclassical approximations simplify assumptions and approximations that bring the equation closer to a form that can be more easily analyzed. The WKB method is used to approximate the wave function.

We constructed a transformation that is similar to the Rindler transformation motivated by the Klein-Gordon equation in Minkowski spacetime. We performed the Bogoliubov transformation and obtained a result which suggested thermalization. However, we were not using creation and annihilation operators. To interpret this result, we calculated the density matrix and the square of the density matrix to see if the WKB state is a pure or mixed state. The result from the density matrix calculation suggested that the WKB state is a mixed state, which suggested that the result we obtained from the Bogoliubov transformation can be interpreted as thermalization.

Acknowledgments

I would like to express my deepest gratitude to all those who have contributed to completing this thesis in the field of Quantum Cosmology.

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I want to acknowledge the support of my family, Rebecca Oduro and Derrick Baker, for their constant encouragement, patience, and belief in my abilities. Their love and understanding sustained me during the challenging phases of this endeavour.

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I also wish to express my appreciation to the researchers, scholars, and institutions whose work and resources contributed to the foundation of this thesis. Their dedication to advancing the field of Quantum Cosmology has been instrumental in shaping my own understanding.

This thesis is the culmination of the collective efforts, support, and inspiration from those mentioned above and many others who have touched my life in various ways. I am truly grateful for your contributions, and I dedicate this work to the pursuit of knowledge and the quest to unravel the mysteries of the cosmos.

Thank you all for being part of this cosmic journey.

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Chapter 1

Introduction

1.1 The Success of General Relativity and Cosmology

Modern cosmology strives to understand the evolution of the universe, which is a noble goal. If we consider that, unlike many other fields of the natural sciences, we do not have complete control over our inquiry, the discipline's recent success is all the more remarkable. First, our experiments have a particular goal; we can only study one universe and not control the system's starting conditions. Second, since we are limited to a small area within the target range, we can only observe the parts of the universe with matter or radiation visible to astronomical observations. Despite these significant limitations, cosmology has evolved over the past 100 years from a purely mathematical and philosophical field to a quantitative science that makes predictions and compares those predictions with observations.

Its exponential growth has a theoretical and observable basis. After formulating General Relativity in 1915, Einstein, Friedmann, Lemaître, Robertson, and Walker applied the theory to the whole universe in the 1920s and 1930s [32]. It provided the first scientific framework for modelling cosmic evolution. In contrast, information from observations was essential in sorting through the vast array of models the theory had anticipated. Galaxies are moving away from us, and Hubble discovered in 1929 that their velocities are related to their distance. The outcome of this research, the so-called Hubble diagram, gave rise to the first conclusive proof of the expansion of the universe (Hubble 1929) [16].

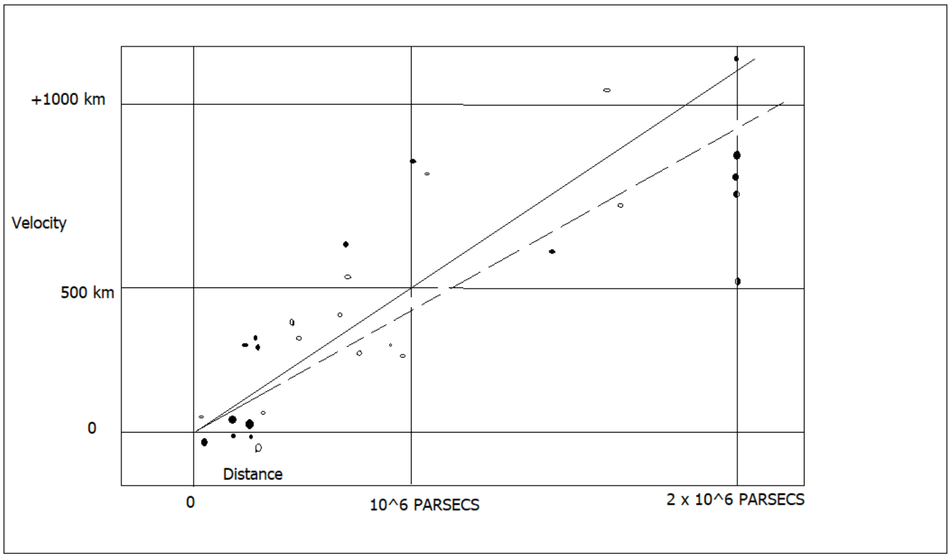

Figure 1.1: The Hubble diagram and the expansion of the Universe [29] [20].

At this time, two schools of thought in the scientific community—the defenders of the steady-state theory, led by Bondi, Gold, and Hoyle, and the proponents of the Big Bang theory, including Gamow and his pupil Alpher—offered different explanations for Hubble's observations. To compensate for the expansion of the universe, steady-state cosmology claims that the universe has always been the same, with galaxies and other cosmic structures constantly forming. In this case, the total mass is not conserved because the average density of the expanding Universe is assumed to be constant. On the other hand, the Big Bang hypothesis supports the principle of conservation of mass by assuming that the expanding Universe must first go through an extremely hot state, which is characteristic of the early stages of its evolution before cooling.

A. Alpher & C. Herman predicted in 1950 that radiation dominated the early universe, a remnant of this radiation which would be visible as a blackbody spectrum around a temperature of 5 Kelvin [1]. In the 1960s, Penzias and Wilson discovered the

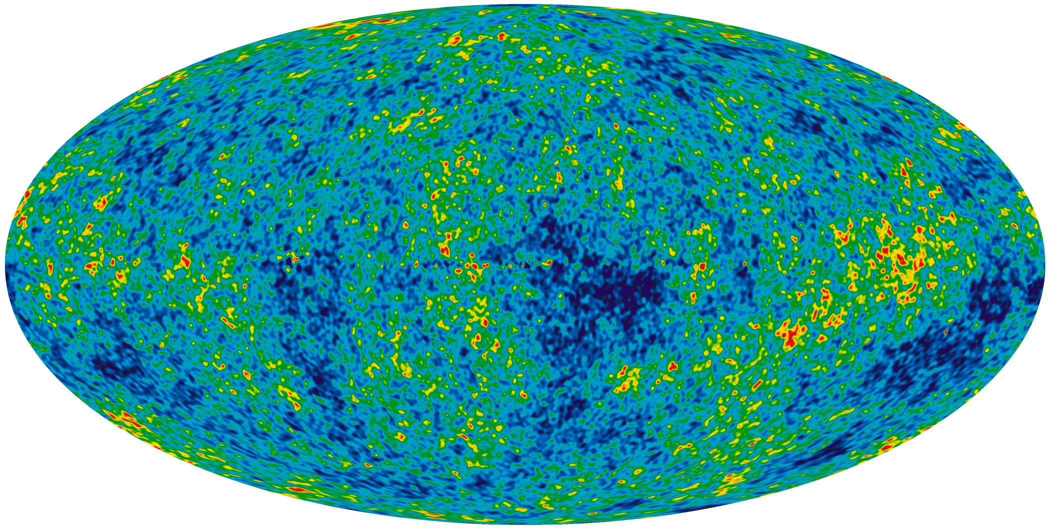

Figure 1.2: The Cosmic Microwave Background Radiation [18].

Cosmic Microwave Background (CMB), which they inadvertently detected as leftover cold radiation from the Universe's early stages; this was crucial and definitive confirmation of the prediction [36]. The discovery of the CMB ended the debate between the steady-state theory and the Big Bang theory in favour of the Big Bang theory. The above description is consistent with a universe containing matter and radiation defined by the Friedmann-Lemaître-Robertson-Walker (FLRW) framework. However, the two missing components in this story are the two main areas of current research. The first item is directly related to the problem presented by Zwicky in the 1930s when he discovered a discrepancy between the mass of the Coma cluster inferred from the virial theorem and the total mass determined from observable galaxies [54]. Zwicky determined that an invisible matter dominates the cluster's virial mass, and Rubin's investigation of the rotating characteristics of galaxies in the 1970s corroborated this in-congruence on galactic scales [42]. The invisible stuff is now referred to as dark matter. Only 20 years ago was the second missing piece discovered. In 1998, two groups reported their observations of Type Ia supernovae, showing that the expansion of the universe is accelerating under the FLRW paradigm [36]. These findings are surprising; if the universe consists only of regular matter (whose density decreases as the universe expands), the rate of expansion should decrease. Dark energy is the exotic energy component fueling the Universe's accelerating expansion. More evidence now points to the presence of dark matter and dark energy components. The CMB deserves mention because it laid the groundwork for precise cosmology over the past 20 years. It turns out that practically all of the information we can glean from the CMB is derived from its microscopic anisotropies, which must exist to explain how the astrophysical sources we observe today could develop. CMB anisotropy was observed by the Cosmic Background Observer (COBE) in 1992, followed by more precise measurements by the Wilkinson Microwave Anisotropy Probe (WMAP) and later by the Planck satellite. Based on the CMB anisotropies, we emerge with the impression of a Universe which is flat (or nearly flat), filled with radiation, which currently accounts for a tiny fraction of the energy budget (total), baryonic matter or ordinary luminous matter, can be explained by the standard model of particle physics, 95% of the energy budget is composed of dark matter and dark energy. Although the nature of these dark components remains a mystery, a standard cosmological model has been deduced from current observations. In this model, dark matter is cold or pressureless, while the cosmological constant provides dark energy. The ΛCDM model is another name for the mainstream cosmological model. This minimal model can reproduce all cosmological observations, including the CMB. The ΛCDM model has made many predictions in recent decades, such as the baryon acoustic oscillations characteristic in 2005 and the CMB polarization in 2002. It was recently discovered that there is some discordance between different data sets. Currently, the most substantial tension is regarding the Hubble rate: Planck collaboration measurements, which support galaxy clustering calculations, show a 3-σ tension with supernova measurements, providing support for higher expansion rates. This disparity has been widely explored in the literature (for example, (Bernal et al. 2016)). Tweaks to the Standard Model, such as dynamic dark energy or changes to the physics of the early universe, have been urged to account for this. Moreover, a minor internal anomaly has been observed in the Planck data sets: the lensing amplitude's contribution to the temperature power spectrum shows a 2-σ tension with the ΛCDM expectation, whereas the 4-point function does not provide a direct estimate of the lensing potential. Although modern cosmology has some anomalies that will eventually be thoroughly investigated, the current cosmology model has passed all experimental tests (including the testing of general relativity near supermassive black holes). Future research in this field has the following objectives in mind:

- Uncovering the nature of its dark components, for which there is currently no convincing physical evidence.
- Extending the existing theory of gravity tests to scales previously unavailable for observation.

1.2 Reasons for Quantum Cosmology

Classical General Relativity and Cosmology are remarkable theories; 'but' they do not answer many physical queries. For example, let's quickly review the questions that the standard "hot big bang" scenario doesn't explain:

- Baryon asymmetry of the Universe, $\eta = \frac{N_B - N_{\overline{B}}}{N_\gamma} \approx 6.01 \times 10^{-10}$ (where N_B, $N_{\overline{B}}$ and N_γ are the number of baryons, antibaryons and photons respectively): The hot Big Bang doesn't explain this exact value, but it is crucial for calculating the abundance of light elements via primordial nucleosynthesis. The photon density decreases as the universe expands and has a value $n_\gamma \approx 20.3T^3K^{-3}cm^{-3}$, where T is the temperature.
- The horizon problem: The fact that the cosmic microwave background radiation is the same everywhere suggests that all parts of the sky must have been in

thermal contact at some point in the past. But in the standard Friedmann-Robertson-Walker (FRW) model, regions separated by more than a few degrees have particle horizons that do not intersect. This means there is no possibility of causal (and therefore thermal) contact in these areas.

- The relic problem: Models of the early universe that include phase shifts often produce many topological defects, such as GUT-scale monopoles. If you do the math, you'll find that the density of these relics is so high that, in the normal FRW models, they would be so close to the critical density that the universe should have ended long ago.
- The mystery of the arrow of time: The laws of physics are CPT-invariant. On the other hand, the Second Law of Thermodynamics says there is an arrow of time, which seems to fit the arrow of time that the expansion of the Universe gives us.
- What is the structure of space-time around the Big Bang singularity?
- The origin of density perturbation is a mystery.

The inflationary universe scenario and the Grand Unified Theory (GUT) could answer some of the problems without relying on quantum cosmology. Still, problems like the origin of density perturbation, the arrow of time, and many more require quantum cosmology. The quantum theory of cosmology is not meant to give a full, basic description of the universe. Instead, it is meant to help other theories predict what will happen. It is a structure built on quantized Einstein gravity, also called quantum geometrodynamics, which says the universe is like a closed quantum system. Quantized Einstein gravity is a good approximation to a complete theory of quantum gravity at energies below the Planck scale. This is why this method works. However, quantizing the entire universe is a challenging task because, in General Relativity, matter and space-time are physical quantities. They obey dynamical laws and produce excitations (gravitational waves) that have been recently detected. One must solve mathematical puzzles beyond the simplest models to unify quantum mechanics and General Relativity. Even now, the setting still needs to be fully understood. There are several, sometimes competing, approaches to quantizing gravity: string theory, gauge quantum gravity, and loop quantum gravity, to name a few of the most famous. Quantum theory and general relativity are the primary approaches to investigating the universe. To comprehend quantum cosmology, one must understand quantum mechanics and general relativity. Quantum mechanics describes how matter and energy behave on a small scale. Particles such as electrons and photons can manifest wave-like and particle-like characteristics. Therefore, wave functions can describe their behaviour, which are mathematical functions that assign probabilities to various measurement outcomes. Important to quantum mechanics is the concept of superposition, which asserts that particles can exist simultaneously in multiple states. For instance, an electron can spin clockwise and anticlockwise until it is measured; at this point, it collapses into a definitive state. Werner Heisenberg proposed that specific pairs of physical properties (position and momentum) cannot be known simultaneously with unlimited precision, which implies inherent measurement uncertainties for these properties. The mathematical formalism of quantum mechanics relies on linear operators, represented by matrices or wavefunctions, and the Schrödinger equation. This equation explains the relationship between energy and the wave function of a quantum mechanical system, as well as the evolution of the system over time. Mathematically, it is written as

$$\hat{\mathrm{H}}\psi \ = \ i\hbar\frac{\partial\psi}{\partial t}.(1.1)$$

The $\hat{\mathrm{H}}$ is the Hamiltonian operator, representing the total energy of the system, $\hbar$ is the reduced Planck's constant ($\frac{\mathrm{h}}{2\ \pi}$) and $\frac{\partial\psi}{\partial t}$ is the time derivative of the wavefunction. The solutions to the Schrödinger equation provide information about the probabilities of different outcomes when measured.

The theory of General Relativity has proven to be one of the successful physical theories since its publication in “The Field Equations of Gravitation paper” in 1915 [10]. Albert Einstein’s theory relies on the Equivalence Principle, which states that gravity affects all bodies equally, making it impossible to separate the effects of a gravitational field from those of a uniformly accelerating frame locally and on the independence of physical laws from the reference frame. In light of these assumptions, along with the hypothesis that spacetime is a curved manifold described by a metric tensor $g_{\mu\nu}$, it has been concluded that the geometry of the space-time itself determines the distribution of matter. The Einstein field equations, in particular, specify this relationship as follows:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}, \tag{1.2}$$

where $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, Λ is the cosmological constant, G is Newton’s gravitational constant (Newton’s gravitational constant, as it appears in the Einstein field equation, informs us of the force of gravity), c is the speed of light and $T_{\mu\nu}$ is the stress-energy tensor of matter.

The idea that the universe has a quantum wave function may seem strange. We usually think of cosmology in terms of how the universe works on a big scale.

We usually think of quantum phenomena in terms of how things work on a very small scale. Still, if the hot Big Bang is the right way to describe the universe, which we can safely assume based on the overwhelming evidence presented in the previous discussion, the universe did start small. There must have been a time when quantum mechanics applied to the whole universe. Quantum systems, like the atom, are often thought of as “small” systems. But if the forces that control the universe are based on quantum mechanics, then the universe itself must be a quantum mechanical system, even if it is on a large scale. Suppose the evolution of the universe was traced back to a point close to the Planck scale. In that case, it seems very likely that the classical explanation of the universe would break down; before the Big Bang, the size of the universe was 10^{-33}cm (Planck length).

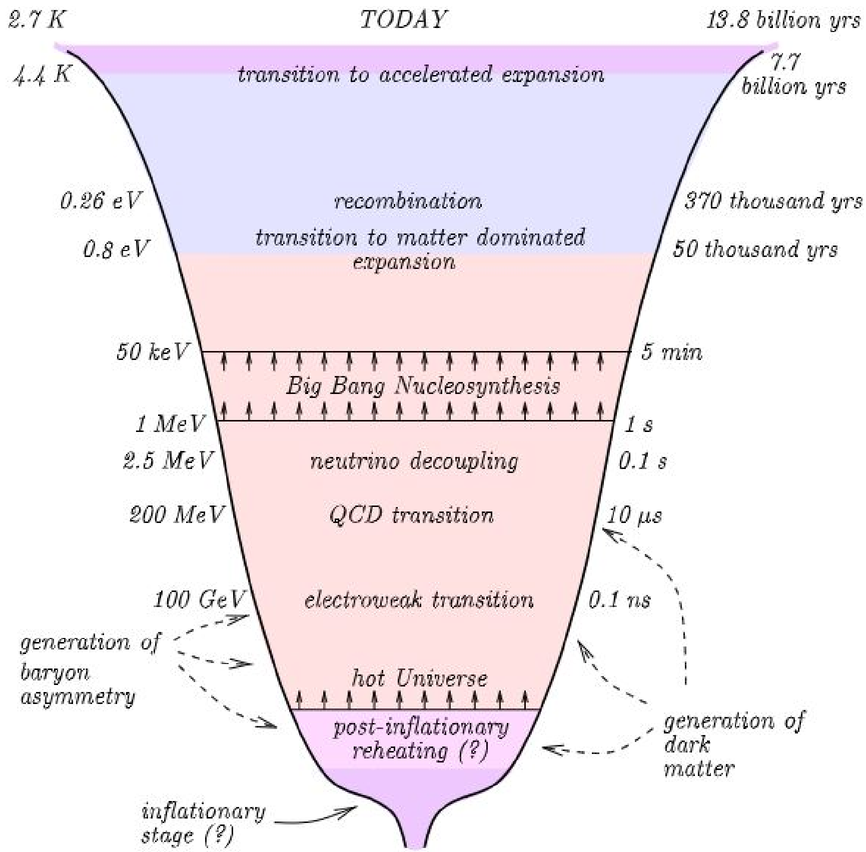

Figure 1.3: The stages of the evolution of the Universe [41].

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Classically, we expect the Big Bang in general relativity, which may not be true. We know that atomic solutions in electrodynamics have no singularities; we have a stable atomic configuration. We expect this to happen in cosmology, known as singularity resolution. In this thesis, we study quantum cosmology using the semiclassical approximation. We inverted a transformation motivated by the Klein-Gordon equation in Minkowski spacetime. We performed Bogoliubov transformation on the WKB wavefunction, and the result suggested a thermalization in the system. Here is a summary of what each chapter will discuss in the thesis:

Chapter 1: This chapter will provide a comprehensive overview of the successes and limitations of General Relativity. It will highlight the shortcomings of the theory, which will set the stage for the necessity of Quantum General Relativity, laying the groundwork for the rest of the thesis.

Chapter 2: In this chapter, we will delve into the foundations of theoretical cosmology. It will introduce the Friedmann equations, fundamental equations that describe the evolution of the universe. We will explore various solutions to these equations, each representing different cosmic eras, establishing them as the backbone of our understanding of the evolving universe.

Chapter 3: In this chapter, we will take you into the world of the Hamiltonian formulation of general relativity, a critical step in connecting classical and quantum physics. We will unveil the Wheeler-DeWitt equation, a key component in the endeavour to unify gravity with quantum mechanics.

Chapter 4: In this chapter, we will explore the heart of Quantum Cosmology. We will discuss the transformation of the cosmological metric into the language of scale factors and matter terms, a vital step towards providing a quantum description of the cosmos. The derivation of the total Hamiltonian and the application of the WKB solution by Vilenkin will be key milestones within this chapter.

Chapter 5: This chapter will discuss the intriguing world of thermalization in curved spacetime. You will examine the Unruh effect, Bogoliubov transformations, Hawking radiation, and the distinctions between pure and mixed states. These concepts will enrich the understanding of how quantum fields behave in gravitational backgrounds.

Chapter 6: Finally, in this chapter, we will uncover a profound connection between a Rindler-like transformation in cosmology and coordinate transformations in the DeWitt superspace. This revelation will lead to a Bogoliubov transformation of the semi-classical wave function of the Universe, unveiling the presence of entropy production and thermal flows in the universe's evolution. This chapter holds the potential to shed light on the thermodynamic aspects of cosmology and the origins of dark energy and dark matter.

These chapters collectively form a journey through the fundamental aspects of cosmology, quantum gravity, and the intriguing relationship between thermodynamics and the evolution of the universe.

The cosmic microwave background is a thermalized radiation, a relic of the cosmology of the early universe and one of the observational evidence for thermalization. Primordial gravitational waves are also expected to be observed in a stochastic form in the background space-time, and these are relics of gravitational interactions from the Big Bang. We want to solve quantum cosmology and identify a thermal evolution from the earliest time of 10^{-43}s immediately after the Big Bang. We would add matter to the gravitational system and show thermalization in the matter evolution. We expect our results to predict new physics and eventually explain other unknown quantities like dark matter and dark energy.

We used the geometrized unit system, with $\hbar = G = c = 1$ unless otherwise stated.

Chapter 2

Theoretical Cosmology

The Lagrangian formulation of a field theory enables one to derive the field equations from a region ν of the space-time manifold and a scalar function, $L\ (\psi, \partial_\alpha \psi)$. This is known as Lagrangian density, which is a local function of the field variables ψ and their first derivatives. The fields ψ can be of any type; for simplicity, we will only consider tensors of type (k,l) where k is the cotangent space index, and l is the tangent space index. Similar to the Lagrangian formulation of Newtonian mechanics, the action functional $S[\psi]$ is defined as the Lagrangian integral

$$S[\psi] = \int_\nu L(\psi, \partial_\alpha \psi)\ \sqrt{-g}\ d^4x, \tag{2.1}$$

where g is the determinant of the metric tensor $g_{\mu\nu}$, $L(\psi, \partial_\alpha \psi)$ is the Lagrangian density, and $\sqrt{-g}\ d^4x$ is the proper volume element. The field equations are then retrieved by requiring S[ψ] to remain stationary under arbitrary changes in the field (ψ_o). When given a smooth one-parameter family of field configurations, the derivative provides a concise definition of variation

$$\delta\psi[\lambda]\ =\ \frac{d\psi_\lambda}{d\lambda}\ \delta\lambda, \tag{2.2}$$

which we require to vanish since our boundary is an asymptotic flat boundary ∂V

$$\delta\psi_{|\partial V} = 0. \tag{2.3}$$

Let us assume that there exists a smooth tensor field χ of type (k,l) (hence the dual ψ) with an action function equal to

$$S \;=\; \int_{v} d^4x\, \chi\, \psi, \tag{2.4}$$

where the contraction between χ and ψ is implied. Differentiating S with respect to the parameter λ results in

$$\delta S \;=\; \frac{dS}{d\,\lambda}\,\delta\lambda \;=\; \int_{\mathrm{v}} \chi\, \delta\psi\, d^4x. \tag{2.5}$$

The functional derivative is hence the variation of S with respect to ψ about ψ_o.which, because the action is stationary, must vanish:

$$\chi|_{\psi_o} \;=\; 0. \tag{2.6}$$

Using these relationships, we can verify that ψ_o is a solution of the field equations enclosed within the identity (Eq.(2.6)).
In 1915, the variational approach to relativity was explored for the first time by Hilbert and Einstein, which led to the formulation of the gravitational action:

$$S_{E-H} = \frac{1}{16\pi}\int R\,\sqrt{-g}\; d^4x; \tag{2.7}$$

S_{E-H} is the Hilbert term. The only non-trivial scalar function that can be derived from the metric and its second derivative is the scalar curvature R, making it the most direct gravitational action imaginable. The square root of the determinant g is from the measure in the action integral. The Einstein Lagrangian density is:

$$L \;=\; R\,\sqrt{-g}. \tag{2.8}$$

The coefficient of the action has $1/16\pi$, which we have set to 1 in the following derivation. We will include the matter field in the action denoted as ϕ.

$$S_M \ = \ \int_V L(\phi, \partial_a \phi; g_{\mu\nu}) \ \sqrt{-g} \ d^4x. \tag{2.9}$$

The S_M defined above is called the matter action. To simplify things, S_M depends only on the metric coefficients $g_{\mu\nu}$, the field (ϕ), and its first derivatives. The sum of the Einstein-Hilbert and the matter terms yields the total action functional:

$$S \ = S_{E-H} \ + \ S_M. \tag{2.10}$$

Using the above, we can demonstrate how the stationarity of S under arbitrary variations of $g_{\mu\nu}$ is the source of the Einstein field equation (Eq.(2.1)).

2.1 Arbitrary variation of the action

Let us begin with the Einstein-Hilbert term. A variation of the inverse metric $\delta g^{\mu\nu}$ will be more convenient by focusing on the integrand, *i.e.* the Einstein-Hilbert Lagrangian density (L), we can execute variation to the Einstein-Hilbert action (S_{E-H}) (by definition (2.5)) since changes can be made under the integral sign:

$$\delta \, L \ = \ \delta(g^{\mu\nu} R_{\mu\nu} \sqrt{-g}) \ = \ \frac{\delta g}{2\sqrt{-g}} g^{\mu\nu} R_{\mu\nu} \ + \ (g^{\mu\nu} \delta R_{\mu\nu}) \sqrt{-g}. \tag{2.11}$$

Jacobi's formula yields the variation of the metric determinant δg:

$$\delta g \ = \ g \ g^{\mu\nu} \ \delta g_{\mu\nu} \ = \ -g \ g_{\mu\nu} \ \delta g^{\mu\nu}. \tag{2.12}$$

We can substitute δg in Eq.(2.11) by using the second form of this identity and remembering that $g \ < 0$:

$$\delta\, L \;=\; \left[\left(R_{\mu\nu} \;-\; \frac{1}{2}\, g_{\mu\nu}\, R\right)\; \delta g^{\mu\nu} \;+\; g^{\mu\nu}\; \delta R_{\mu\nu}\right]. \tag{2.13}$$

Now it is clear that if $\delta R_{\mu\nu}$ disappears, we will recover the gravitational component of the Einstein field equation. The initial derivative of $\delta g^{\mu\nu}$ goes into the variation $\delta R_{\mu\nu}$, creating additional boundary terms. Therefore, this assumption does not necessarily hold in general cases. If we turn to the Palatini identity, we find that

$$\delta R_{\mu\nu} \;=\; \nabla_{\rho}\; \left(\delta\; \Gamma^{\rho}_{\mu\nu}\right) \;-\; \nabla_{\mu}\left(\delta\; \Gamma^{\rho}_{\rho\nu}\right). \tag{2.14}$$

Introducing the contravariant term $V^{\rho} \;=\; g^{\mu\nu}\; \delta\; \Gamma^{\rho}_{\mu\nu}$ and utilizing the Levi-Civita connection property $\nabla_{\rho}\; g_{\mu\nu} \;=\; 0$, Eq.(2.13) can be written as:

$$\sqrt{-g}\; g^{\mu\nu}\; \delta R_{\mu\nu} \;=\; \sqrt{-g}\; g^{\mu\nu}\left[\nabla_{\rho}\; \delta\; \Gamma^{\rho}_{\mu\nu} \;-\; \nabla_{\mu}\; \delta \Gamma^{\rho}_{\rho\nu}\right]$$
$$=\; \sqrt{-g}\; \nabla_{\rho}\left[g^{\mu\nu}\delta\; \Gamma^{\rho}_{\mu\nu} \;-\; g^{\rho\nu}\; \delta\; \Gamma^{\mu}_{\mu\nu}\right] \;=\; \partial_{\rho}\left(\sqrt{-g}\; V^{\rho}\right).$$

The variation δS_{E-H} separates into volume and boundary components by substituting the results into the action integral and using Stokes' theorem,

$$\delta\; S_{E-H} \;=\; \int_{\mathrm{v}} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\; R\right) \sqrt{-g}\; \delta\; g^{\mu\nu} d^4x \;+\; \oint V^{\mu} d\sigma_{\mu}\; d^3x, \tag{2.15}$$

where $d\sigma_{\mu}$ denotes the hypersurface ∂V oriented volume element, we proceed as though the surface terms can be safely ignored while ignoring the second integral. Now let us look at the variation of the matter action Eq.(2.9), which is completely generic in its dependence on the matter fields and the metric $g_{\mu\nu}$:

$$\delta S_M \;=\; \int_{\mathrm{v}} \left[\frac{\delta L_M}{\delta g^{\mu\nu}}\; \delta g^{\mu\nu} \sqrt{-g} \;+\; L_M \delta \sqrt{-g}\right] d^4x \;=\; \int_{\mathrm{v}} \left[\frac{\delta L_M}{\delta g^{\mu\nu}} \;-\; \frac{1}{2}\; L_M\; g_{\mu\nu}\right]\; \sqrt{-g}\; \delta g^{\mu\nu}\; d^4x. \tag{2.16}$$

Defining the stress-energy tensor $T_{\mu\nu}$ as:

$$-2\,\frac{\delta L_M}{\delta g^{\mu\nu}} \;+\; L_M g_{\mu\nu}, \tag{2.17}$$

it turns out that the variation of the total action is

$$\delta S \;=\int_{\mathrm{v}}\left[\left(R_{\mu\nu} \;-\; \frac{1}{2}g_{\mu\nu}R\right) \;+\frac{\delta L_M}{\delta g^{\mu\nu}} \;-\; \frac{1}{2}L_M\; g_{\mu\nu}\right]\sqrt{-g}\;\delta g^{\mu\nu}d^4x. \tag{2.18}$$

We re-introduce the factor 8π as a relative factor between the gravitational and matter actions. $S=\frac{1}{16\pi}\int_{\mathrm{v}}\left[R_{\mu\nu}\;-\;\frac{1}{2}g_{\mu\nu}\;R\;-\;8\pi\;T_{\mu\nu}\;\right]\;\sqrt{-g}\;\delta g^{\mu\nu}\;d^4x.$

The stationarity of S requires the integrand to be zero due to the arbitrary nature of $\delta g^{\mu\nu}$, ultimately resulting in the Einstein field equations

$$R_{\mu\nu} \;-\; \frac{1}{2}\;g_{\mu\nu}\;R \;=\; 8\pi\;T_{\mu\nu}. \tag{2.19}$$

By using the Einstein tensor $G_{\mu\nu}$, we can rewrite this equation as follows:

$$G_{\mu\nu} \;=\; 8\pi\;T_{\mu\nu}. \tag{2.20}$$

Using the four-divergence, we can express the desired conservation of the stress-energy tensor $T_{\mu\nu}$ as follows:

$$\nabla_\mu\;T^{\mu\nu} \;=\; 0. \tag{2.21}$$

This is possible since the Bianchi identities $\nabla_\mu\;G^{\mu\nu}\;=\;0$ result from the Riemann curvature tensor $R^{\rho}_{\sigma\mu\nu}$ symmetries. The action's invariance under an infinitesimal translation of coordinates can also be used to demonstrate this result [45].

2.2 Friedmann–Lemaître–Robertson–Walker metric

The curvature of spacetime through which light travels on its way to Earth affects the appearance of objects at cosmological distances. Einstein's general theory of relativity provides the most comprehensive description of the geometrical properties of the universe. In general relativity, the metric tensor is the fundamental quantity which describes spacetime geometry.

A metric is defined as follows: We calculate the separation between two points along a curved path $\boldsymbol{P}$ in three Euclidean space using the line element:

$$(dl)^2 \;=\; (dx)^2 \;+\; (dy)^2 \;+\; (dz)^2. \tag{2.22}$$

For a curved path $\boldsymbol{P}$, we calculate the separation between two points as follows:

$$\Delta l \;=\; \int_1^2 \sqrt{(dl)^2} \;=\; \int_1^2 \sqrt{(dx)^2 \;+\; (dy)^2 \;+\; (dz)^2}. \tag{2.23}$$

The metric for flat Lorentzian spacetime is used to measure the interval along a curved worldline, $\boldsymbol{W}$, connecting two events in spacetime without mass:

$$(ds)^2 \;=-\; (cdt)^2 \;+\; (dx)^2 + (dy)^2 + (dz)^2. \tag{2.24}$$

The total interval along the worldline W is obtained as follows:

$$\Delta s \;=\; \int_A^B \sqrt{(ds)^2} \;=\; \int_A^B \sqrt{(cdt)^2 \;-\; (dx)^2 \;-\; (dy)^2 \;-(dz)^2}. \tag{2.25}$$

The proper distance is the distance between two events, A and B, that occur simultaneously ($t_A \;=\; t_B$) in a reference frame.

$$\Delta L \;=\; \sqrt{-(\Delta s)^2}. \tag{2.26}$$

The cosmological principle simplifies finding a metric to describe a universe with matter. Space curvature is subject to change over time depending on the amount and distribution of matter and energy in the universe in an isotropic and homogeneous universe; however, on a large scale, the average curvature of space has remained relatively constant since the Big Bang.

The curvature of a sphere is obtained from $K = \frac{1}{R^2}$. 2-D curvature can, however, also be described as

$$K = \frac{3}{\pi} \lim_{B \to 0} \frac{2\pi B - C}{B^3}, \tag{2.27}$$

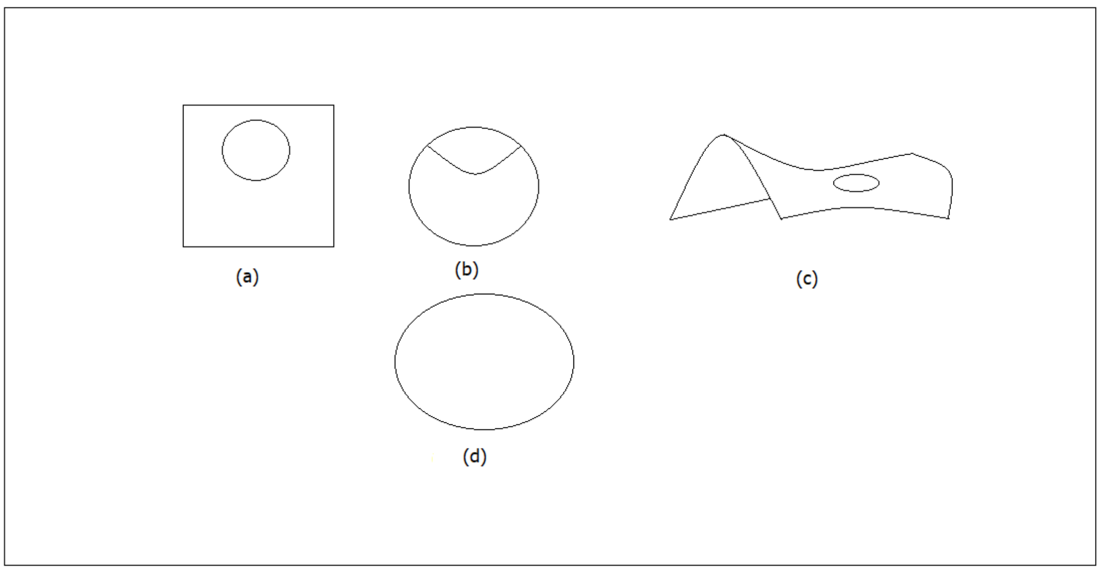

Figure 2.1: These diagrams show shapes in 2-D space with different curvatures. (a) C = $2\pi B$, (b) C < $2\pi B$, (c) $C > 2\pi B$, (d) $K = 0$ [5].

.

Where C is the circumference, and B is the radius. We can see from the diagrams above that each shape has a different circumference and, therefore, is curved differently.

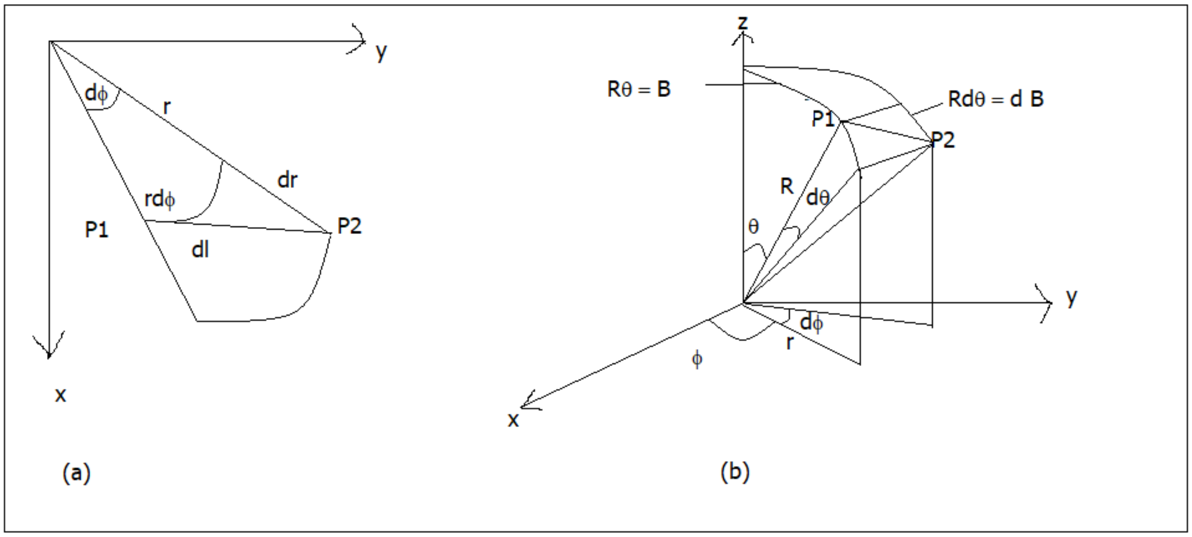

Figure 2.2: Diagrams showing the distance between two points on the surface of a sphere in (a) polar and (b) spherical coordinates [4] [5].

The distance between two points on a sphere is given by:

$$(dl)^2 = (dB)^2 + (rd\phi)^2 = (Rd\theta)^2 + (rd\phi)^2, \tag{2.28}$$

where

$$Rd\theta = \frac{dr}{\cos(\theta)} = \frac{Rdr}{\sqrt{R^2 - r^2}} = \frac{dr}{\sqrt{1 - \frac{r^2}{R^2}}}.$$

Substituting the results into Eq.(2.28), we obtain

$$(dl)^2 = \left(\frac{dr}{\sqrt{1 - \frac{r^2}{R^2}}}\right)^2 + (rd\phi)^2. \tag{2.29}$$

We could express the two-dimensional surface curvature (K) of the plane using polar coordinates as

$$(dl)^2 = \left(\frac{dr}{\sqrt{1-K\,r^2}}\right)^2 + (rd\phi)^2, \tag{2.30}$$

Extending this equation to $3-D$, we have

$$(dl)^2 = \left(\frac{dr}{1\ -K\,r^2}\right)^2 + (rd\theta)^2 + (r\sin(\theta)d\phi)^2. \tag{2.31}$$

We obtained this result by changing polar coordinates to spherical coordinates; r in the spherical coordinate is the radial coordinate. In Eq.(2.31), we can see how the curvature of our $-D$ universe influences spatial distances.

To get the metric of spacetime, we have to include time. In an expanding universe, two points should be recorded simultaneously if their separation is to be meaningful. Since an isotropic, homogeneous universe cannot have different time-passing rates for different locations, cdt should be used as the temporal term. Therefore, we have:

$$(ds)^2 = -(cdt)^2 + \left(\frac{dr}{\sqrt{1\ -K\,r^2}}\right)^2 + (r\,d\theta)^2 + (r\sin(\theta)d\phi)^2. \tag{2.32}$$

Differential proper distance is equal to

$$\Delta L = \left(\lim_{dt\ \to\ 0}\right)\sqrt{-\ (\Delta s)^2}\ .$$

Changing the radial coordinate to co-moving coordinates, we have:

$$r(t) = a(t)\ x. \tag{2.33}$$

The expansion of the 3-space impacts the geometrical and curvilinear aspects of the universe. The time-independent curvature of the universe can be expressed using the scale factor and the constant (κ):

$$K(t) \;=\; \frac{\kappa}{a(t)^2} \tag{2.34}$$

Substituting this for r and k we get the Robertson-Walker metric [40] [53]:

$$(ds)^2 \;=\; -(cdt)^2 \;+\; a(t)^2 \;\left[\left(\frac{dx}{\sqrt{1 \;-\; \kappa\, x^2}}\right)^2 \;+\; (xd\theta)^2 \;+\; (x\; \sin\theta\; d\theta)^2\right]. \tag{2.35}$$

In the usual notation we again write $x = r$ in the above. In the 1930s, Robertson and Walker proved that this metric is the most general way to describe an expanding, homogeneous, and isotropic universe.

2.3 Friedmann Equations

Einstein's field equations provide a way to determine how the metric evolves from a particular distribution of mass and energy:

$$G_{\mu\nu} \;=\; 8\;\pi\; G\; T_{\mu\nu}. \tag{2.36}$$

Where $c = 1$.

$T_{\mu\nu}$ stands for the stress-energy tensor, which explains how energy and mass influence spacetime curvature.

Using the tensor notation, the line element can be written as follows:

$$ds^2 \;=\; g_{\mu\nu}\; dx^\mu\; dx^\nu, \tag{2.37}$$

where $g_{\mu\nu}$ is the metric tensor, and the Einstein convention was utilized. To understand the intrinsic geometry of spacetime, we only need to consider the diagonal metric tensors of orthogonal coordinate systems. Using a non-vanishing diagonal matrix, the

Robertson-Walker metric components are as follows:

$$g_{00} = -1 \ , \ g_{11} = \frac{a^2}{1 - \kappa r^2}$$

,

$$g_{22} = a^2 \ r^2 \ , \ g_{33} = a^2 r^2 \sin^2 \theta$$

For Eq.(2.37), the time component T_{00} for the stress-energy tensor and the space component T_{11} for a comoving observer could be expressed as:

$$T_{00} = \rho \ , \ T_{11} = \frac{p \ a^2}{1 - \kappa r^2}, \tag{2.38}$$

where ρ is the mass-density and p is the pressure.

From Eq.(2.37), we equate G_{00} and G_{11} to T_{00} and T_{11}. The results are:

$$G_{00} = 3 \ (a)^{-2} \ (\dot{a} + \kappa \ ), \tag{2.39}$$

$$G_{11} = -(2 \ a\ddot{a} + \dot{a}^2 + \kappa) \ (1 - \kappa r^2)^{-1}. \tag{2.40}$$

Substituting Eq. (2.38), Eq. (2.39) and Eq. (2.40) into Eq. (2.36), we get:

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa}{a^2} = \frac{8\pi}{3} G \ \rho, \tag{2.41}$$

$$2\left(\frac{\ddot{a}}{a}\right) + \left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa}{a^2} = - \ 8\pi G p. \tag{2.42}$$

These two equations are known as the Friedmann equations.

Alexandr Friedmann, a Russian physicist and mathematician, calculated these equations in 1922, seven years before Hubble found that the universe is expanding when

Einstein refused to accept his equations since they did not allow the cosmos to remain static. After Friedmann died in 1927, Belgian clergyman Georges Lemaitre re-confirmed the formulas by an independent derivation [25].

Eq.(2.42) shows that the rate of the expansion of the Universe ($\dot{a}$) depends on the mass density of the universe.

Subtracting Eq.(2.42) from Eq.(2.41) we obtained the acceleration equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}G(\rho + 3\ p). \tag{2.43}$$

This equation tells us that the acceleration of the universe decreases as pressure and energy density increase, even in the case of matter or radiation energy.

Let us define $\Omega_0 = \frac{\rho_0}{\rho}$ ($\rho_0 = 3H_0^2/8\pi G$, H_0 being Hubble's constant) and make use of t_0, Eq.(2.41) takes the form:

$$\dot{a}^2 = \frac{8\pi}{3}\ G\ a_0^2\ \rho - \kappa = H_0^2 a_0^2 \Omega_0 - \kappa. \tag{2.44}$$

We can further progress to get

$$\kappa = H_0^2 a_0^2\ (\Omega_0 - 1). \tag{2.45}$$

This equation is the Newtonian relation.

Eq.(2.45) shows that the curvature parameter κ is directly proportional to the density parameter Ω_0. κ takes values of +1, 0, -1, which corresponds to overcritical density $\Omega_0 > 1$, critical density $\Omega_0 = 1$ and the under-critical density $0 < \Omega_0 < 1$, respectively.

Whether one derives it from Newtonian mechanics or General Relativity, the Friedmann equation maintains the same form. The only difference is the interpretation of the curvature constant κ. κ represents the mechanical energy of an expanding mass shell in a Newtonian universe, whereas in Einstein's universe, κ is the present value of the curvature of the universe.

2.4 Cosmological Constant

It was believed the universe was static until Hubble discovered cosmic expansion in 1929. In this case, the scale factor should not be time-dependent; instead, a_0 should be constant, resulting in $\dot{a} = \ddot{a} = 0$ (leading to the universe having an infinite age).

In this case, the Friedmann equations become:

$$\frac{\kappa}{a^2} \;=\; \frac{8\pi}{3}\, G\rho_0 \;=\; -8\pi G p_0, \tag{2.46}$$

where, ρ_0 and κ are positive but the pressure term p_0 is negative. In 1917, Einstein introduced a constant Lorentz-invariant term ($\Lambda g_{\mu\nu}$) to the Einstein field equation to correct this resulting in:

$$G_{\mu\nu} \;-\Lambda g_{\mu\nu} \;=\; 8\pi G\; T_{\mu\nu}. \tag{2.47}$$

The Lorentz-invariant term does not vanish even if we set the limit to flat space.

Including the Lorentz-invariant term into the Friedmann equations, we get:

$$\left(\frac{\dot{a}}{a}\right)^2 \;+\; \frac{\kappa}{a^2} \;=\; \frac{8\pi}{3} G\rho \;+\; \frac{\Lambda}{3}, \tag{2.48}$$

$$2\left(\frac{\ddot{a}}{a}\right) \;+\; \left(\frac{\dot{a}}{a}\right)^2 \;+\; \frac{\kappa}{a^2} \;=\; -8\pi G p \;+\; \Lambda. \tag{2.49}$$

Therefore, the vacuum energy density can be written as:

$$\Lambda = \rho \, 8 \, \pi \, G. \tag{2.50}$$

2.5 Solutions to the Friedmann Equations

The solutions to the Friedmann equations for a universe dominated by different components, such as matter, radiation, and dark energy, depend on the equation of state for each component. Here, we will provide the solutions for these three dominant components separately [21].

Matter-Dominated Universe:

In a matter-dominated universe, the equation of state for matter is $p = 0$ (pressure is negligible compared to energy density) [28]. The First Friedmann Equation is

$$H^2 = \frac{8\pi G}{3}\rho_m. \tag{2.51}$$

The Second Friedmann Equation is

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho_m, \tag{2.52}$$

where ρ_m is the energy density of matter; for a matter-dominated universe, the scale factor evolves as $a(t) \propto t^{2/3}$, where t is the cosmic time [27].

Radiation-Dominated Universe:

In a radiation-dominated universe, the equation of state for radiation is $p = \frac{1}{3}\rho_r$ where p is pressure, and ρ_r is energy density. The First Friedmann Equation is:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho_r. \tag{2.53}$$

For a radiation-dominated universe, the scale factor evolves as $a(t) \propto t^{\frac{1}{2}}$.

Dark Energy-Dominated Universe (Cosmological Constant):

In a dark energy-dominated universe with a cosmological constant (Λ), the equation of state is $p = -\rho_\Lambda$ (negative pressure due to dark energy). The Friedmann equations are:

$$H^2 = -\frac{4\pi G}{3}(\rho_\Lambda + 3p_\Lambda), \tag{2.54}$$

and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho_\Lambda + 3p_\Lambda). \tag{2.55}$$

For this case, the scale factor evolves exponentially: $a(t) \propto e^{H_\Lambda t}$ where H_Λ is the Hubble parameter associated with dark energy and it is related to the cosmological constant through the first Friedmann equation. These solutions describe the evolution of the scale factor for different dominant components in the universe. In reality, the universe's composition changes over time, and these components can coexist, leading to more complex cosmological models that require numerical simulations for accurate predictions.

2.6 Conclusion

In this Chapter, we discussed the essentials of the Lagrangian formulation of field theory. We introduced the gravitational field action as well as the matter field action. It was shown that the Einstein field equation can be found by performing an arbitrary variation over the Lagrangian term in the action and using Jacobi's identity. After we discussed the Lagrangian formulation of field theory and its application to Einstein's field equation, we discussed how to derive the FLRW metric. The discussion continued with the Friedmann equations, and the cosmological constant and the Friedmann equation solutions were introduced.

Chapter 3

Canonical Gravity

3.1 The ADM Formalism

In our previous Chapter, we discussed obtaining the field equations of general relativity using the Einstein-Hilbert action. In this Chapter, we introduce the ADM (Arnowitt, Deser, Misner) formulation of canonical gravity and study Einstein's gravity as split into a D=3 space and 1D time system. This facilitates a Hamiltonian formulation of the theory. Let us consider the Einstein-Hilbert action for the metric tensor field $g_{\mu\nu}$ with Lorentzian signature propagating on a (D + 1)-dimensional manifold M,

$$S = \frac{1}{\kappa} \int_M d^{D+1}X \sqrt{|\det(g)|}\, R^{(D+1)}, \tag{3.1}$$

where $\kappa = 16\pi G$, with G being the Newton's gravitational constant, $R^{(D+1)}$ is the Ricci curvature scalar and $|\det(g)|$ is the determinant of the metric tensor and D = 3. ADM is a Hamiltonian formulation of General Relativity. We want to obtain the Hamiltonian from the action, but the Hamiltonian is not directly contained in the action. Generally, an action is defined as the integral over the Lagrangian:

$$S = \int L \, d^4V, \tag{3.2}$$

where d^4V is the volume form and corresponds to $\sqrt{|\det(g)|}\, d^{D+1}x$ in the Einstein-Hilbert action and L is the Lagrangian function and corresponds to the Ricci curvature scalar ($R^{(D+1)}$).

Our goal is to construct a quantum theory of General Relativity. We encounter the problem that time and space are merged in General Relativity, while in Quantum Mechanics, space and time are distinct. Our mathematical solution involved foliating the Einstein-Hilbert action from a 4-D manifold to a 3 + 1 manifold (separate space and time).

In terms of one-forms, the Riemann curvature tensor is defined as follows:

$$\left[\nabla_{\mu},\nabla_{\nu}\right]\; u_{\rho} \;=\; {}^{(D\,+\,1)}R_{\mu\nu\rho}{}^{\;\sigma}\; u_{\sigma}, \tag{3.3}$$

where ∇ is the covariant derivative associated with the metric tensor $g_{\mu\nu}$.

The action principle (Eq.(3.1)) must have boundary terms to make it well-defined. The boundary term assumed for the manifold (M) is an asymptotically flat boundary. To put Eq (3.1) into canonical form, we assumed that the 4-D manifold (M) has a topology ($M = R \times \sigma$), with σ being a 3-D manifold with an asymptotic flat boundary. The Geroch theorem asserts that this type of topology is necessary for spacetime to be globally hyperbolic (with Cauchy surfaces consistent with classical physics). In the Lorentzian signature, the topology we have assumed for M has no restriction for classical physics. Unlike classical gravity, quantum gravity allows various topologies, especially topological changes. In our approach, we will use classical assumption ($M = R \times \sigma$) to develop the quantum theory of gravitational field and lift the restriction in quantum theory.

Considering the assumption we made for the topology of M, it is evident that it foliates into hypersurfaces ($\sum_t = X_t\,(\sigma)$). For each fixed $t \in R$, an embedding exists:

$$X_t \;:\; \sigma \rightarrow\; M.$$

This is given by the definition $X_t(x) = X(x,t)$, where the local coordinates of the 3-D compact manifold(σ) is given by $x^a\,(a,b,c,..... = 1,2,3,....)$. We also have a diffeomorphism

$$X \ : R \times \sigma \mapsto M \ ; (t,x) \ \mapsto \ X(t,x) \ = \ X_t(x).$$

These special diffeomorphisms enable the decomposition of action (Eq.(3.1)) into

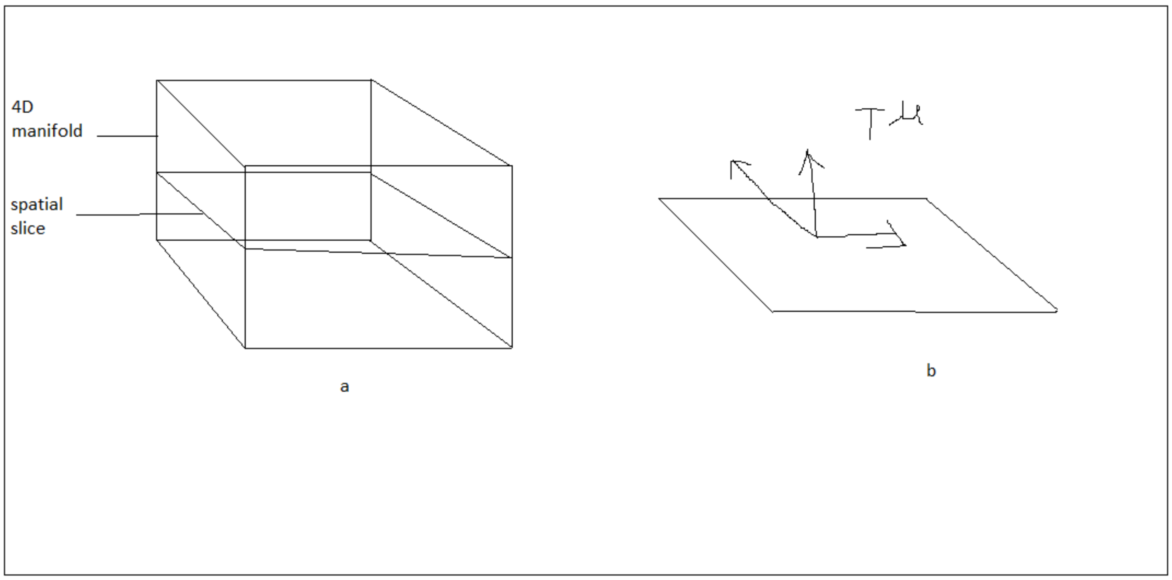

Figure 3.1: The diagrams for the 4-D manifold, the spatial slice and the decomposition of a vector field on the spatial slice.

$D \ + \ 1$ (space and time). Taking into account the fact that the action is invariant under all diffeomorphisms, we allow the embeddings to be arbitrary, and it is expressed using the vector field:

$$T^\mu(X) \ = \ \left(\frac{\partial X^\mu(t,x)}{\partial t}\right)_{| \ X \ = \ X(x,t)} \ = \ N(X)n^\mu(X) \ + \ N^\mu(X), \tag{3.4}$$

where T^μ is the time evolution vector field, n^μ is the unit normal vector to $\sum_t$ which means $g_{\mu\nu} \ n^\mu n^\nu \ = \ s$, N $(N : \sum_t \ \to \ R)$ is the lapse function, which measures clock speed according to the local observer. N^μ is the shift vector field, which specifies the local velocities of observers along the hypersurface. Consider a hypersurface σ that is embedded into the $4-D$ manifold M through X. We can take n as its unit normal vector field and $\Sigma = X(\sigma)$ as its image. When developing the tensor calculus of spatial tensor fields, we now have the option of working with either Σ or σ; working on Σ gives the advantage of comparing spatial tensor fields with arbitrary tensor fields restricted to Σ since both are tensor fields on a subset of M. In addition, after developing tensor calculus on the hypersurface (Σ), we pull back the tensor fields through the embedding to obtain the tensor calculus on σ. $\mu,\nu,\rho,....$ are spacetime indices (0,1,2,3) and $a,b,c...$ are spatial indices (1,2,3).

3.1.1 The Gauss-Codazzi Equations

The Gauss-Codazzi equations relate the $4-D$ Riemann tensor (Eq. (3.3)) to the 3-D Riemann tensor and the extrinsic curvature tensor when there is a 3-D hypersurface Σ embedded in the 4-D manifold M.

Our next step is to derive the Gauss-Codazzi equation using the induced coordinates on σ (a,b,c = 1,2,3.). Let the induced metric on the hypersurface Σ be (First fundamental form): $q_{\mu\nu} = g_{\mu\nu} - s\, n_\mu\, n_\nu$ and the derivative operator on σ be D_μ. Hence, D_μ could be mathematically expressed as follows:

$$D_\mu\phi = q_\mu{}^\nu \nabla_\nu \phi. \tag{3.5}$$

Note that the induced metric ($q_{\mu\nu} = g_{\mu\nu} - s\, n_\mu n_\nu$) is the projector onto the hypersurface (Σ).

The second fundamental form of the hypersurface Σ is the extrinsic curvature tensor, given by:

$$K_{\mu\nu} = q_\mu^\rho\, q_\nu^\sigma\, (2\nabla_{(\rho}n_{\sigma)}). \tag{3.6}$$

In addition, one important property of the second fundamental form is its symmetry:

$$K_{\mu\nu} = K_{\nu\mu}; \tag{3.7}$$

this implies that

$$K_{\mu\nu} + K_{\nu\mu} = 2K_{\mu\nu}. \tag{3.8}$$

In terms of covariant derivative, Eq.(3.8) can be expressed as

$$2K_{\mu\nu} = q_\mu^\rho \, q_\nu^\sigma \, (2\nabla_{(\rho} n_{\sigma)}). \tag{3.9}$$

In differential geometry, a useful identity exists: the relation between the Lie and covariant derivatives. mathematically, we can express that as [22]:

$$(L_n \, g)_{\mu\nu} = 2\nabla_{(\mu} \, n_{\nu)}. \tag{3.10}$$

Using Eq.(3.5) and Eq.(3.10) we can express Eq.(3.9) as

$$2K_{\mu\nu} = q_\mu^\rho q_\mu^\sigma (L_n g)_{\rho\sigma} = q_\mu^\rho q_\nu^\sigma (L_n q + s \, L_n n \otimes n)_{\rho\sigma}. \tag{3.11}$$

The unit normal can be expressed as:

$$n^\mu = \frac{T^\mu - N^\mu}{N}. \tag{3.12}$$

Using Eq.(3.12) we get

$$2K_{\mu\nu} = \frac{1}{N}(L_T \, q_{\mu\nu} - D_\mu N_\nu + D_\nu N_\mu). \tag{3.13}$$

The first form and second fundamental form of tensors are spatial, so they vanish when contracted with the timelike unit normal n^μ:

$$n^\mu q_{\mu\nu} \;=\; 0. \tag{3.14}$$

Our definition of the $4-D$ Riemannian tensor in Eq.(3.3) was expressed using the $4-D$ covariant derivative, but if we want to obtain the $3-D$ Riemannian tensor, we must define it using the $3-D$ covariant derivative. The $3-D$ covariant derivative can also be obtained as a projection of the $4-D$ covariant derivative [52]:

$$D_\mu\, u_\nu \;=\; q^\rho_\mu\, q^\sigma_\nu\, \nabla_\rho \tilde{u}_\sigma, \tag{3.15}$$

where $\tilde{u}_\mu(u)$ is the induced vector in $\sum$ and $n^\mu u_\mu = 0$.

Using Eq.(3.3), we define the $3-D$ Riemann curvature as:

$$\left[D_\mu, D_\nu\right]\; u_\rho \;=\; (D_\mu\, D_\nu - D_\nu\, D_\mu)\; u_\rho \;=\; {}^3R_{\mu\nu\rho}{}^{\sigma} u_\sigma. \tag{3.16}$$

Using Eq.(3.15) and cyclic chain rule:

$$\begin{aligned} \left[D_\mu, D_\nu\right] u_\rho \;&=\; (D_\mu D_\nu - D_\nu D_\mu)\; u_\rho \\ &=\; q^{\mu'}_\mu q^{\nu'}_\nu q^{\rho'}_\rho\; \nabla_{\mu'}(\; q^{\nu''}_{\nu'} q^{\rho''}_{\rho'}\; \nabla_{\nu''} u_{\rho''}) \;-\; q^{\mu'}_\nu q^{\nu'}_\mu q^{\rho'}_\rho\; \nabla_{\nu'}(\; q^{\mu''}_{\mu'} q^{\rho''}_{\rho'}\; \nabla_{\mu''} u_{\rho''}). \end{aligned} \tag{3.17}$$

Let us focus on the first part of Eq.(3.17):

$$D_\mu D_\nu u_\rho \;=\; q^{\mu'}_\mu q^{\nu'}_\nu q^{\rho'}_\rho q^{\rho''}_{\rho'} (\nabla_{\mu'} q^{\nu''}_{\nu'}) \nabla_{\nu''} u_{\rho''} \;+\; q^{\mu'}_\mu q^{\nu'}_\nu q^{\nu''}_{\nu'} q^{\rho'}_\rho (\nabla_{\mu'} q^{\rho''}_{\rho'}) \nabla_{\nu''} u_{\rho''}$$

$$+ q_\mu^{\mu'} q_\nu^{\nu'} q_{\nu'}^{\nu''} q_\rho^{\rho'} q_{\rho'}^{\rho''} \nabla_{\mu'} \nabla_{\nu''} u_{\rho''} + q_\mu^{\mu'} q_\nu^{\nu'} q_\rho^{\rho''} \nabla_{\mu'} (n^{\nu''} n_{\nu'}) \nabla_{\nu''} u_{\rho''} + q_\mu^{\mu'} q_\nu^{\nu''} q_\rho^{\sigma'} \nabla_{\mu'} (n_{\sigma'} n^{\sigma''}) \nabla_{\nu''} u_{\sigma''}$$

$$+$$

$$q_\mu^{\mu'} q_\nu^{\nu''} q_\rho^{\sigma''} \nabla_{\mu'} \nabla_{\nu''} u_{\sigma''} + q_\mu^{\mu'} q_\nu^{\nu'} q_\rho^{\sigma''} n^{\nu''} \nabla_{\mu'} n_{\nu'} \nabla_{\nu''} u_{\sigma''} - q_\mu^{\mu'} q_\nu^{\nu''} q_\rho^{\sigma'} \nabla_{\mu'} n_{\sigma'} (\nabla_{\nu''} n^{\sigma''}) u_{\rho''}$$

$$+ q_\mu^{\mu'} q_\nu^{\nu''} q_\rho^{\rho''} \nabla_{\mu'} \nabla_{\nu''} u_{\rho''}$$

$$= K_{\mu\nu} q_\rho^{\rho''} n^{\nu''} \nabla_{\nu''} u_{\rho''} - K_{\mu\nu} q_\rho^{\nu''} \nabla_{\nu''} n^{\sigma''} u_{\sigma''} + q_\mu^{\mu'} q_\nu^{\nu''} q_\rho^{\rho''} \nabla_{\mu'} \nabla_{\nu''} u_{\rho''} \tag{3.18}$$

as $q_\mu^{\mu'} n_{\mu'} = 0$.

Going through the steps for the antisymmetrization part in Eq.(3.17) we get:

$$D_\nu D_\mu u_\rho = q_\nu^{\nu'} q_\mu^{\mu'} q_\rho^{\sigma'} q_{\sigma'}^{\rho''} (\nabla_{\nu'} q_{\mu'}^{\mu''}) \nabla_{\mu''} u_{\rho''} + q_\nu^{\mu'} q_{\mu'}^{\mu''} q_{\mu'}^{\nu'} q_\rho^{\sigma'} (\nabla_{\nu'} q_{\sigma'}^{\rho''}) \nabla_{\mu''} u_{\rho''} + q_\nu^{\mu'} q_{\mu'}^{\mu''} q_\mu^{\nu'} q_\rho^{\sigma'} q_{\sigma'}^{\rho''} \nabla_{\nu'} \nabla_{\mu''} u_{\rho''} \tag{3.19}$$

$$+ q_\nu^{\mu'} q_\mu^{\nu'} q_\rho^{\rho''} \nabla_{\nu'} (n^{\mu''} n_{\mu'}) \nabla_{\mu''} u_{\rho''} + q_\nu^{\mu''} q_{\mu'}^{\nu'} q_\rho^{\sigma'} \nabla_{\nu'} (n_{\sigma'} n^{\rho''}) \nabla_{\mu''} u_{\rho''} + q_\nu^{\mu''} q_\mu^{\nu'} q_\rho^{\rho''} \nabla_{\nu'} \nabla_{\mu''} u_{\rho''}$$

$$+ q_\nu^{\mu'} q_\mu^{\nu'} q_\rho^{\rho''} n^{\mu''} \nabla_{\nu'} n_{\mu'} \nabla_{\mu''} u_{\rho''} - q_\nu^{\mu''} q_\mu^{\nu'} q_\rho^{\sigma'} \nabla_{\nu'} n_{\sigma'} (\nabla_{\mu''} n^{\rho''}) u_{\rho''} + q_\nu^{\nu'} q_\mu^{\mu''} q_\rho^{\sigma''} \nabla_{\nu'} \nabla_{\mu''} u_{\rho''}$$

$$= K_{\nu\mu} q_\rho^{\rho''} n^{\mu''} \nabla_{\mu''} u_{\rho''} - K_{\nu\mu} q_\rho^{\rho''} \nabla_{\mu''} n^{\rho''} u_{\rho''} + q_\nu^{\nu'} q_\mu^{\mu''} q_\rho^{\rho''} \nabla_{\nu'} \nabla_{\mu''} u_{\rho''}$$

using the fact that

$$q_\nu{}^\mu n^\sigma \nabla_\mu u_\sigma = q_\nu^\mu \nabla_\mu (n^\sigma u_\sigma) - q_\nu{}^\mu u_\sigma \nabla_\mu n^\sigma = -K_\nu{}^\sigma u_\sigma, \tag{3.20}$$

Thus we get:

$$\begin{aligned} &(D_\mu D_\nu - D_\nu D_\mu) u_\sigma \\ &= K_{\nu\mu} K_\sigma{}^{\sigma''} u_{\sigma''} - K_{\nu\sigma} K_\mu{}^{\sigma''} u_{\sigma''} + q_\mu^{\mu'} q_\nu^{\nu''} q_\sigma^{\sigma''} (\nabla_{\mu'} \nabla_{\nu''} - \nabla_{\nu''} \nabla_{\mu'}) u_{\sigma''}. \end{aligned} \tag{3.21}$$

Combining Eq(3.18), Eq.(3.20) and Eq.(??), we get the Gauss-Codazzi equation:

$${}^3R_{\mu\nu\sigma}{}^\rho = K_{\nu\sigma} K_\mu{}^\rho - K_{\mu\sigma} K_\nu{}^\rho + q_\mu^{\mu'} q_\nu^{\nu'} q_\sigma^{\sigma'} q_{\rho'}^\rho \, {}^4R_{\mu'\nu'\sigma'}{}^{\rho'}. \tag{3.22}$$

Next, we take the contracted version of the above indices and work in three dimensions with indices labelled by $a, b, c, d, ...$

Contracting the 3-D Riemann curvature with q^{ab} q_d^c leads to:

$${}^3R = K^2 - K_d{}^b K_b{}^d + {}^4R, \tag{3.23}$$

making the Riemann curvature on the 4-D manifold the subject we get:

$${}^4R = {}^3R + K_d{}^b K_b{}^d - K^2. \tag{3.24}$$

Eq.(3.22), Eq.(3.23) and Eq.(3.24) are known as the Gauss-Codazzi equation, the components of the metric $g_{\mu\nu}$ are now in terms of the 3-D metric and the lapse and the shift given as:

$$g_{\mu\nu} \;=\; \begin{pmatrix} A & B_b \\ B_a & q_{ab} \end{pmatrix} \;,\; g^{\mu\nu} \;=\; \frac{1}{N^2}\begin{pmatrix} -1 & N^b \\ N^a & C^{ab} \end{pmatrix}, \tag{3.25}$$

where A, B_b and C^{ab} are unknowns. Utilizing the identity $g^{ac}g_{bc} = \delta^a_b$ allows us to solve for the unknown elements.

$$g_{a\rho}g^{\rho 0} \;=\; \delta^0_a = \frac{1}{N^2}(-B_a \;+\; q_{ab}N^b) \;=\; 0 \longrightarrow \; B_a \;=\; q_{ab}N^b, \tag{3.26}$$

$$g_{0\sigma}g^{\sigma 0} \;=\; \delta^0_0 = \frac{1}{N^2}(-A \;+\; B_aN^a) \;=\; 1 \longrightarrow \; A \;=\; q_{ab}N^aN^b \;-\; N^2, \tag{3.27}$$

$$g_{a\sigma}g^{\sigma d} \;==\; \frac{1}{N^2}q_{ad}(N^aN^d \;+\; C^{bd}) \;=\; \delta^d_a \;\longrightarrow\; C^{ab} \;=\; N^2q^{ab} \;-\; N^aN^b. \tag{3.28}$$

Replacing A, B_b and C^{ab} into the metric we get:

$$g_{\mu\nu} \;=\; \begin{pmatrix} N_aN^a \;-\; N^2 & N_b \\ N_a & q_{ab} \end{pmatrix} \;,\; g^{\mu\nu} \;=\; \frac{1}{N^2}\begin{pmatrix} -1 & N^b \\ N^a & q^{ab} \;-\; N^aN^b \end{pmatrix}. \tag{3.29}$$

We can define the determinant $g_{\mu\nu}$ and q_{ab} as follows:

$$\det(g_{\mu\nu}) \;=\; g, \;\; \det(q_{ab}) \;=\; q. \tag{3.30}$$

The indices $(\mu\nu)$ run from 0 to 3, so this is a 4x4 matrix.

Using Cramer's rule to express the inverse of the metric results in:

$$g^{00} \;=\; \frac{D_{00}}{\det(g_{\mu\nu})}. \tag{3.31}$$

D_{00} is the (0,0) element of the cofactor matrix of $g_{\mu\nu}$. This is obtained from $D_{00} = (-1)^0 E_{00} = E_{00}$ is the determinant of the 3 x 3 matrix obtained from suppressing the first line and the first column of $g_{\mu\nu}$. From Eq.(3.30):

$$g^{00} = \frac{q}{g} \tag{3.32}$$

From Eq.(3.31), $g^{00} = -\frac{1}{N^2}$, Eq.(3.32) can be expressed as:

$$-\frac{1}{N^2} = \frac{q}{g}, \quad \sqrt{g} = N\sqrt{q}. \tag{3.33}$$

Finally, we can cast the action into $D+1$ form using the Gauss-Codazzi equations and the ADM split of the determinant as

$$S = \frac{1}{\kappa}\int_R dt \int_\sigma d^D x \sqrt{\det(q)}|N|(R - s[K_a{}^b K_b{}^a - K^2]), \tag{3.34}$$

where, $K_a{}^b K_b{}^a = K_{ab}K^{ab}$.

The action can be written as:

$$S = \frac{1}{\kappa}\int_R dt \int_\sigma d^D x \sqrt{\det(q)}|N|(R - s[K_{ab}K^{ab} - K^2]). \tag{3.35}$$

In this case, $(\sqrt{\det(q)}|N|(R - s[K_{ab}K^{ab} - K^2])$ is the Lagrangian defined on a $3-D$ manifold:

$$L_G = \sqrt{\det(q)}|N|(R - s[K_{ab}K^{ab} - K^2]). \tag{3.36}$$

Now that the action has been cast into $D+1$ (foliation of the space-time) form, we want to cast it into canonical form, which means we want to execute the Legendre transform on the Lagrangian density in Eq.(3.36).

To cast the action into canonical form, we must specify the configuration variable and the corresponding conjugate momenta; the configuration variable is the metric on the $3-D$ manifold (q_{ab}). We need the velocities of q_{ab} to get the conjugate momenta. Note that the action is dependent on the velocities ($\dot{q}_{ab}$) of the configuration variable q_{ab}, but not on the time derivatives of the shift function (N^a) and the lapse function (N). From Eq.(3.13) it is easy to deduce the velocities ($\dot{q}_{ab}$) in terms of the configuration variable (q_{ab}):

$$K_{ab} \;=\; \frac{\dot{q}_{ab} \;-\; D_a n_b - D_b n_a}{2N}. \tag{3.37}$$

The conjugate momenta can be solved by using:

$$P^{ab} \;=\; \frac{\delta L_G}{\delta \dot{q}_{ab}} \;=\; \sqrt{\det(q)}[K^{ab} - \; q^{ab}K]. \tag{3.38}$$

$K \;=\; q^{ab}K_{ab}$,

$$\Pi \;=\; \frac{\delta L_G}{\delta \dot{N}} \;=\; 0, \tag{3.39}$$

$$\Pi_a \;=\; \frac{\delta L_G}{\delta N^a} \;=\; 0. \tag{3.40}$$

Eq.(3.39) and Eq.(3.40) are the primary constraints, and to do the Legendre transform, we used the Dirac method for the Hamiltonian treatment of systems with constraints, which required us to introduce Lagrange multipliers (λ, λ_a). Solving for the other velocities in the system, we get the following:

$$\dot{q}_{ab}P^{ab} \;=\; 2N\sqrt{\det(q)}[K_{ab}K^{ab} - K^2], \tag{3.41}$$

$$\begin{aligned} P_{ab} \;=\; q_{ac}q_{bd}P^{cd} \;&=\; \sqrt{\det(q)}[q_{ac}q_{bd}K^{cd} - \; q_{ac}q_{bd}q^{cd}K] \\ &= \sqrt{\det(q)}[K_{ab} - \; 3\; q_{ab}K], \end{aligned} \tag{3.42}$$

$$P^{ab}P_{ab} \;=\; \sqrt{\det(q)}[K_{ab}K^{ab} - 9K^2 \;-\; K^2 \;+\; 9K^2] = \sqrt{\det(q)}[K_{ab}K^{ab} \;-\; K^2], \quad (3.43)$$

$$P \;=\; q_{ab}P^{ab} \;=\; \sqrt{\det q}[q_{ab}K^{ab} - q_{ab}q^{ab}K] = \; -\sqrt{\det q}[2K], \quad (3.44)$$

$$(P)^2 \;=\; 4\; \det(q)\; K^2. \quad (3.45)$$

When we incorporate these formulas into the action, we obtain a canonical form and perform integration by parts on the spatial components. Considering that we are dealing with an asymptotic flat boundary, the boundary term is zero.

$$S \;=\; \frac{1}{\kappa}\int_R\int_\sigma d^3x\; (\dot{q}_{ab}P^{ab} \;+\; \dot{N}\Pi \;+\; \dot{N}^a\Pi_a \;-\; (\lambda B \;+\; \lambda^a B_a \;+ N^a\; H_a + N\; H)), \quad (3.46)$$

where H_a is the spatial constraint, H is the Hamiltonian constraint and B and B^a are the primary constraints ($B = \Pi \;= 0$ and $\;B^a = \Pi^a \;= 0$),

$$H^a \;=\; -2D_bP^{ab}, \quad (3.47)$$

$$H \;=\; (\frac{1}{\sqrt{\det q}}[q_{ac}q_{bd} \;-\; \frac{1}{2}q_{ab}q_{cd}]P^{ab}P^{cd} \;+\; \sqrt{\det q}\; R). \quad (3.48)$$

Our interest lies in Hamiltonian constraints, so other constraints are set to zero, leaving only Hamiltonian constraints. Consequently, the action becomes:

$$S \;=\; \frac{1}{\kappa}\int_R dt\int_\sigma d^3x\; [\dot{q}_{ab}P^{ab} \;-\; (N^aH_a \;+\; |N|H)]. \quad (3.49)$$

The Hamiltonian is entirely a constraint. We can still define a quantum Hilbert space using the canonical algebra of q_{ab} and P^{ab}. The Hamiltonian constraint acts on the wave function of the universe to give zero,

$$H\ \Psi = 0. \tag{3.50}$$

This equation is known as the Wheeler-DeWitt equation and it plays the role of the Schr*ö*dinger equation in canonical quantum gravity theory. The next chapter will discuss the Wheeler-Dewitt equation in more detail [13].

3.1.2 Importance of the constraints

- The constraints are automatically maintained in time, making them consistent with the equations of motion,

$$\frac{dH_a}{dt} = \{H_a(D_b), H[N, N^b]\}. \tag{3.51}$$

  The closure of the algebra causes the right-hand side of Eq.(3.51) to disappear in the phase space of the constraint.

- Specifying N and N^a, the unknown quantities, results in a well-posed Cauchy problem for the dynamical equations [11].

- The constraint equations (3.39)-(3.40) are satisfied on all spacelike hypersurfaces of the manifold (M) if the Lorentzian metric g, on the spacetime manifold M fulfills the vacuum Einstein equations $G_{\mu\nu} = 0$. (P^{ab} is calculated from the given spacetime geometry using the definition of the second fundamental form just as we did and resulted in the relationship between P^{ab} and K_{ab} Eqn. 3.37.)

- Assume that $(M, g_{\mu\nu})$ is a Lorentzian spacetime in which the constraint equations (3.39 - 3.40) have properties satisfied on any spacelike hypersurface. All ten Einstein field equations $G_{\alpha\beta}(X, g) = 0$ must be satisfied.

The last result is significant because it indicates that the dynamic aspects of Einstein's equations are already included in the constraints.

3.2 Conclusion

In conclusion, the Arnowitt-Deser-Misner (ADM) formalism stands as a pivotal achievement in the realm of general relativity, offering a profound insight into the geometric intricacies of spacetime through the elegant concept of splitting 4D spacetime into 3+1 space and time. This formulation has reshaped our understanding of the dynamics of gravity and provided a robust foundation for various applications across classical and quantum physics.

At the heart of the ADM formalism lies the concept of the first and second fundamental forms, which allow us to dissect spacetime into spatial and temporal dimensions, respectively. This spatial-temporal decomposition provides a clear and intuitive description of the evolving gravitational field. The first fundamental form, encapsulating the spatial geometry, captures how distances and angles change on spatial slices, while the second fundamental form captures the extrinsic curvature of these slices within the larger spacetime.

By splitting the Einstein-Hilbert action into its 3+1 form, the ADM formalism allows us to reframe the gravitational dynamics in terms of the spatial metric, the lapse function, and the shift vector. This separation is instrumental in analyzing the evolution of the gravitational field, enabling us to derive the Hamiltonian formulation of general relativity. Utilizing the Lagrangian derived from the 3+1 action, one can obtain the canonical momentum variables conjugate to the spatial metric. The Hamiltonian can then be expressed in terms of these momentum variables and their derivatives, providing a powerful tool for understanding the underlying dynamics of gravity.

The ADM formalism's significance extends to both theoretical and practical domains. It has paved the way for numerical simulations of gravitational phenomena, including black hole mergers and neutron star collisions. Furthermore, this formalism has played a crucial role in canonical approaches to quantizing gravity, offering insights into the quantum nature of spacetime.

In essence, the ADM formalism's success lies in its ability to unravel the intricate nature of spacetime geometry and gravitational dynamics. By embracing the 3+1 split, the formalism offers a new perspective on the interplay between space and time, providing a powerful toolkit for investigating the fundamental behaviour of gravity and its consequences on the universe at a grand stage.

Chapter 4

Quantum Cosmology

4.1 Introduction

The idea of quantum cosmology is that quantum physics should apply to the universe as a whole. According to the standard particle physics model, there are four forces in nature. They are the gravitational force, electromagnetic force, weak nuclear force, and strong nuclear force. Three forces have been unified, except gravity, which must be unified with quantum mechanics. Unifying General Relativity and Quantum Mechanics is not an easy task because, in General Relativity, both space and time are physical objects, and they obey dynamical laws and exhibit gravitational waves that interact with one another and with matter, so it is necessary to quantize space and time. Quantum Cosmology aims to quantize the universe as a whole, and it has direct connections with quantum gravity (quantum theory of gravitation). Since quantum gravity remains unfinished, the theoretical basis of quantum cosmology remains unclear. Making things worse, there are several difficult conceptual problems to be overcome.

The theory of quantum cosmology has a relatively long history, even though it remains highly speculative and controversial. Lemaître's idea of the primordial atom combines insights from General Relativity with quantum mechanics concepts (in later years to be referred to as the Big Bang singularity)[24] [26]. Tolman's efforts to solve the singularity problem of General Relativity using quantum physics were in many ways prescient, and the entropy problem of an eternal universe is still relevant today

[47] [48]. Despite this, further progress was hampered not only by the incomplete nature of quantum mechanics at the time and the lack of interest from mainstream theorists. (atomic physics, after all, presented much more urgent and practical problems).

In this chapter, we will use the results we obtained in Chapter 3 to compute the Hamiltonian for matter fields and scalar fields using the definition of the Lagrangian as it pertains to quantum field theory, as well as the method and results we obtained for canonical gravity in Chapter 3.

4.2 Cosmological metric

The FLRW metric was derived in Chapter 2; in this section, it will be expressed in conformal time and transformed using Rindler-like coordinates.

The flat FLRW metric is expressed as:

$$ds^2 = -dt^2 + a(t)^2 \left(dx^2 + dy^2 + dz^2\right). \tag{4.1}$$

Alternatively, Eq.(4.1) can be expressed as:

$$ds^2 = -dt^2 + a(t)^2 \left(dx^a dx_a\right). \tag{4.2}$$

Consider a light beam that travels in the x direction. With $ds = 0$, it travels on a null geodesic obeying the relation:

$$dt = \pm a(t) dx. \tag{4.3}$$

Using the relation of proper velocity:

$$\eta = \frac{dx}{d\tau}. \tag{4.4}$$

Assuming the proper velocity is 1, then Eq.(4.3) becomes:

$$d\tau = \frac{dt}{a(\tau)}. \tag{4.5}$$

This implies that

$$dt = a(\tau)\, d\tau. \tag{4.6}$$

Substituting Eq.(4.6) into Eq.(4.2), we can express the cosmological metric in terms of conformal time:

$$ds^2 = a(\tau)^2(-d\tau^2 + dx^a dx_a). \tag{4.7}$$

Transforming using Rindler-like coordinates (X,T), which represents accelerated observer in X-direction, an example of which (α is a constant):

$$\tau = \tau(X,T) \text{ and } x = x(X,T)$$

which implies that [39]:

$$\tau = X\sinh(\alpha T), \quad x = X\cosh(\alpha T). \tag{4.8}$$

Differentiating τ and x with respect to T and X, respectively, and substituting that into Eq.(4.7) results in:

$$ds^2 = a(T,X)^2(-\alpha^2 X^2 dT^2 + dX^2 + dy^2 + dz^2). \tag{4.9}$$

This implies that the metric is no longer of the conformally maximally symmetric form.

4.3 Solving the total Hamiltonian

We want to examine the solutions of the Wheeler-Dewitt equation in the accelerated frames' observer. First, we must obtain the Hamiltonian for the scalar field and the matter. We need the Lagrangian for the scalar and matter fields to construct the Hamiltonian through the Legendre transformation. The Lagrangian for the scalar field can be expressed as:

$$L = \frac{\sqrt{-g}}{2}\left[\nabla_\mu \, \phi \, g^{\mu\nu} \, \nabla_\nu \, \phi \, + \, V(\phi)\right]. \tag{4.10}$$

In terms of the Lapse and scale factor, we can express Eq.(4.10) as:

$$L = \frac{\sqrt{-g}}{2}\left[-\frac{1}{N^2}(\partial_o\phi)^2 + \frac{1}{a(\tau)^2}\,\partial_i \, \phi \, \delta^{ij} \, \partial_j\phi\right]. \tag{4.11}$$

The Lagrangian is a function of the reduced metric q_{ab} and the velocity $\dot{q}_{ab}$. Based on the results from Eq.(4.7) , we can express the reduced metric in terms of the scale factor as:

$$q_{ab} = a(\tau)^2 \, \delta_{ab} \text{ and } q^{ab} = \frac{1}{a(\tau)^2}\,\delta^{ab}, \tag{4.12}$$

$$\dot{q}_{ab} = 2 \, a(\tau)\dot{a}(\tau)\delta_{ab}, \tag{4.13}$$

the second fundamental form can be expressed as a function of the velocity factor as:

$$K_{ab}(\tau) = \frac{1}{2N}(\dot{q}_{ab})(\tau), \tag{4.14}$$

$$N = a(\tau), \tag{4.15}$$

$$K_{ab}(\tau) \;=\; \dot{a}(\tau)\delta_{ab}. \tag{4.16}$$

The generalized momentum can be expressed as follows:

$$P_a \;=\; \frac{\delta S}{\delta \dot{a}(\tau)}, \tag{4.17}$$

$$q^{ac}q^{bd}K_{ab}(\tau) \;=\; K^{cd} \;=\; \frac{\dot{a}(\tau)}{a(\tau)^4}\delta^{cd}. \tag{4.18}$$

This implies that:

$$K^{ab} \;=\; \frac{\dot{a}(\tau)}{a(\tau)^4}\delta^{ab}$$

.

Putting the result we obtained for the second fundamental form K_{ab} into we get:

$$P^{ab}(\tau) \;=\; \frac{2s\dot{a}(\tau)\delta^{ab}}{a(\tau)}. \tag{4.19}$$

We can now insert these results into the Hamiltonian relation we obtained in Chapter 3 to obtain:

$$H \;=\; \frac{6\ s^3\ \dot{a}(\tau)}{a(\tau)}. \tag{4.20}$$

With Eq.(4.17) and Eq.(4.20), we can express Hamiltonian terms in terms of generalized momentum:

$$H_a \;=\; \frac{-P_a^2}{24a(\tau)} \tag{4.21}$$

The Lagrangian for the scalar field ϕ could be expressed as:

$$L = \frac{\sqrt{-g}}{2}[\partial_\mu \phi \; g^{\mu\nu}\partial_\nu\phi - V(\phi)]. \tag{4.22}$$

Going through the steps we used to obtain P_a, the momentum term corresponding to the scalar field came out to be:

$$\frac{P_\phi^2}{2a(\tau)^3}. \tag{4.23}$$

When one works in an accelerated frame (which we discuss in detail in Chapter 5), the momenta remain the same; therefore, the cosmological metric in the accelerated frame is expressed as:

$$ds^2 = a(X,T)^2 \; (-\alpha^2 \; X^2 dT^2 + dX^2 + dy^2 + dz^2). \tag{4.24}$$

We added matter to this system in the form of a scalar field ϕ to describe matter-driven cosmology. The Hamiltonian takes the form:

$$H_{\text{RW}} = -\frac{p_a^2}{24a} + \frac{p_\phi^2}{2a^3} + a^3 V(\phi). \tag{4.25}$$

4.3.1 Solutions to the Wheeler-DeWitt Equation

In Eq.(4.25), p_ϕ and p_a are the momentum corresponding to the scalar field and the scale factor. We calculated them by considering a Universe filled with a scalar field ϕ.

The classical observables are promoted to operators and Eq.(4.25) becomes:

$$\hat{H}_{\text{RW}}\psi(a,\phi) = \left(-\frac{\hat{p}_a^2}{24a} + \frac{\hat{p}_\phi^2}{2a^3}\right)\psi(a,\phi) = 0. \tag{4.26}$$

To solve the WDW, we need to deal with the configuration space of the entire universe. The configuration space is a mathematical space representing the possible states or wave functions of the entire universe as a whole. This gives all possible ways that the geometry of the universe can evolve. However, solving the full equations of general relativity for the entire configuration space can be extremely challenging, if not impossible, due to its high dimensionality and complexity.

To make the problem more manageable, we often consider minisuperspace models, which restrict the range of possible geometries or fields to a smaller, finite-dimensional subset of the full configuration space. These restrictions typically involve simplifications such as assuming spatial homogeneity and isotropy, reducing the number of dimensions of the configuration space, and often imposing certain symmetries. By doing so, physicists can simplify the equations of motion and analyze the behaviour of the system.

In the minisuperspace model, the cosmological wavefunctions have been calculated using two degrees of freedom (a, ϕ). We have the tunnelling solution and the Hartle-Hawking boundary conditions, and both wavefunctions predict the Universe is nucleated by quantum fields in de Sitter invariant vacuum fields [23]. We will not go into the detailed derivation of the wavefunction; we will give a piece of highlight information about the two wavefunctions. Detailed derivation could be in Vilenken's paper [50]. From a mathematical standpoint, the tunneling solution is analogous to quantum tunneling through a potential barrier, sometimes called

quantum tunneling from nothing or the "creation of the universe from nothing" [50]. The nucleation of the Universe is a non-singular event according to the quantum tunneling approach. In the semiclassical model, the underbarrier propagation is described by an imaginary time evolution; the tunneling process is also described by ordinary Euclidean field equations, which are mapped to the Lorentzian solution at the point of nucleation. Although the Universe started in a non-singular way, it will eventually evolve into a singularity known as the Big Crunch. The general solution for the wavefunction is:

$$\psi = \sum_n c_n \, e^{is_n}. \tag{4.27}$$

Where s_n is the Euclidean action associated with the metric. In the Hartle-Hawking proposal, the universe is considered a closed, compact, and smooth manifold without any boundaries in spacetime. The no-boundary condition implies that the universe has no initial singularity; instead, it has a quantum state described by a wave function that depends on certain geometric properties of the three-dimensional spatial slices.

The wave function proposed by Hartle and Hawking is a function of the three-metric (spatial geometry) of the universe. It is given by a path integral over all possible compact, smooth Euclidean 4-geometries (spacetimes with imaginary time). The integral includes contributions from different topologies and geometries, effectively summing over all possible ways the universe could be.

The tunnelling solution predicts the initial state of the universe and leads to inflation, but the Hartle-Hawking solution does not.

A detailed derivation of each solution can be found here [9], [14], [15], [49].

Using the WKB approximation of Vilenkin [50], a semi-classical solution is of the form

$$\psi(\xi', \phi) = e^{ik(\xi' \pm \phi)}. \tag{4.28}$$

Where ξ' is $\sqrt{12}\,\ln a$, we have derived Eq. (4.28) in detail in Chapter 6

4.4 Conclusion

Conformal and accelerated frames have produced intriguing and consistent results for cosmological metrics. First, we considered the Lagrangian formulation of the system, which encompassed both matter and scalar fields. We discovered that the Hamiltonian obtained in both frames were identical, suggesting a fundamental equivalence between these two perspectives.

The Hamiltonian of the system was derived using the Lagrange transform. Our calculations revealed that the Hamiltonian is fundamentally a function of both the momentum associated with the scalar field and the matter field. The implications of this observation for understanding the dynamics of the universe are profound.
We found that it is important to consider multiple frames and perspectives when investigating complex cosmological systems. The equivalence of solutions across frames underlines the robustness and symmetry of our approach. Furthermore, the dependence of the Hamiltonian on both scalar field momentum and matter field parameters offers valuable insights into the interplay between these components in shaping the evolution of the cosmos. It implies that the scalar field dynamics and the behaviour of matter fields are intimately connected.

We discussed the two solutions for the cosmological wavefunction in the minisuperspace model, which are the tunnelling solution and the Hartle-Hawking boundary condition; the tunnelling solution has an advantage over the Hartle-Hawking boundary solution by giving a prediction of the initial state of the universe and also lead to inflation, as the universe tunnels into existence, it can enter a phase of rapid exponential expansion known as inflation. We will use the tunnelling solution in our calculations for determining the thermalization in the universe filled with a scalar field. Overall, this provides a deeper understanding of the dynamics of cosmological metrics and their connection between different frames. The consistency of our solutions between the conformal and accelerated frames underscores the robustness of our mathematical approach and the fundamental principles underlying cosmological models.

Chapter 5

Thermalization in curved space-time

5.1 Introduction

Thermalization in quantum field theory has been subject to intense research and discussion for decades. Thermalization in curved space-time has been explored recently by advances in theoretical physics. Many of these studies have concerned flat space-time, but advances have also led to exploring thermalization in curved space-time [3]. Both cosmologists and physicists are fascinated by the mysterious realm of curved space-time described by general relativity. This warped spacetime produces profound phenomena that fundamentally challenge our understanding of the universe. One of these phenomena is thermalization, a process at the intersection of quantum field theory and gravitational physics. This chapter examines the intriguing interplay between thermalization, the Bogoliubov transformation, the Unruh effect, and Hawking radiation.

Thermalization, or the tendency for a physical system to reach thermal equilibrium, has fascinated researchers across many disciplines. Thermalization takes on a unique and profound meaning in curved space-time. The geometric structure of the universe, influenced by massive celestial bodies and intense gravitational fields, affects the dynamics of quantum fields in such a way that thermal behaviour arises. The emergence of these thermalization processes is governed by the principles of quantum field theory in curved spacetime, set amid some of the most mysterious and awe-inspiring aspects of our universe. We study thermalization in curved space-time through the Bogoliubov transformation, a mathematical tool that helps bridge the gap between early- and late-time quantum fields. Spacetime curvature transforms the quantum vacuum to mimic the temperature-like aspects of thermal equilibrium, revealing the subtle yet transformative effects of spacetime curvature. In investigating the Bogoliubov transformation, we gain insight into how the fabric of space-time itself can induce thermal behaviour in quantum fields.

In curved space-time, there is a tangible manifestation of thermalization called the Unruh effect, named after physicist William G. Unruh. According to this theory, an observer in uniform acceleration perceives the vacuum state of a quantum field as a thermal bath, complete with particles that appear to emerge from it. This effect provides a striking link between gravitational acceleration and thermal phenomena, highlighting the profound connection between the geometry of space-time and the behaviour of quantum fields.

Near black hole event horizons, we encounter the renowned Hawking radiation, a phenomenon that emerges from gravitational physics. Stephen Hawking predicted this radiation due to the quantum mechanical interaction between black holes and vacuum fluctuations within quantum fields. Hawking radiation introduces the concept that even black holes, known for their insatiable appetite for matter and energy, emit particles and eventually evaporate.

Thermalization in curved space-time, the Bogoliubov transformation, the Unruh effect, and Hawking radiation are explored in this chapter. We aim to unravel the intricate web that binds these concepts together to deepen our understanding of how curved space-time affects quantum fields and our understanding of the universe. In this Chapter we review some concepts of quantum field theory (QFT) and thermalization in curved space-time.

5.2 Thermalization in Quantum Field theory

5.2.1 Quantum Fields and Thermalization

Quantum Field Theory (QFT) unifies two of the most successful theories of physics: quantum mechanics and special relativity. We understand the fundamental forces and particles of the universe through particle physics, which is the foundation of modern physics. QFT consists of the following components:

- Quantum Fields: Quantum Field Theory describes physical observables in terms of a field that exists throughout spacetime. The fields are associated with electrons, photons, and quarks, among other fundamental particles. Quantum fields are the building blocks of the theory.

- Quantization: QFT applies the principles of quantum mechanics to these fields. By quantizing the fields, the theory treats them as operators capable of creating and annihilating particles. Excitations of fields are interpreted as particles.

- Interactions: The theory allows for the description of particle interactions through interaction terms in the field equations. These interactions are mediated by particles known as gauge bosons, such as photons for electromagnetism or gluons for the strong nuclear force.

- Lagrangian Formulation: QFT is typically formulated using a Lagrangian density, which encapsulates the dynamics of the fields and their interactions. The equations of motion for the fields are derived from this Lagrangian through the principle of least action.

- Renormalization: In practical applications of QFT, infinities can arise in perturbation theory. Renormalization techniques are employed to remove these divergences and obtain physically meaningful results.

5.2.2 The Concept of Thermalization

Thermalization is a concept that arises in the context of statistical mechanics and quantum systems, including those described by QFT. It refers to the process by which a system reaches a state of thermal equilibrium characterized by energy distribution among its constituents. Here are the key aspects of thermalization:

- Equilibration: Thermalization refers to the process by which a quantum system, initially in a non-equilibrium state, evolves to reach a state of thermal equilibrium. During equilibration, energy is redistributed among the system's degrees of freedom until a balance is achieved.

- Relaxation Time: The time it takes for a quantum system to reach thermal equilibrium is characterized by a relaxation time. This time can vary significantly depending on the system's properties, interactions, and initial conditions.

- Quantum vs. Classical Thermalization:

  Quantum systems may differ from classical thermalization due to quantum interference and entanglement effects. Understanding quantum aspects of thermalization is crucial in various contexts, such as in condensed matter systems and high-energy particle collisions.

5.2.3 Equilibrium and Non-Equilibrium state in Quantum Field Theory:

The concept of equilibrium and nonequilibrium states plays an important role in quantum field theory (QFT), just as it does in statistical mechanics and thermodynamics [19]. Nevertheless, in QFT, these concepts are often applied differently due to the quantum nature of fields. Our next step is to explore equilibrium and nonequilibrium states in quantum field theory:

Equilibrium State

In quantum field theory, an equilibrium state is when the properties of a system do not change on average with time. It is a state of equilibrium where macroscopic observables like particle density, energy, and momentum remain constant or fluctuate within a narrow range.

Thermal Equilibrium

In quantum field theory, equilibrium states are commonly associated with thermal equilibrium states. During thermal equilibrium, the properties of a system are described by its density matrix in a fixed temperature state.
Thermal equilibrium in quantum field theory is crucial in cosmology and studying the early universe, particularly during the epochs of cosmic inflation and the Big Bang. To model the evolution of the universe, it is crucial to understand how quantum fields behave under extreme thermal conditions. There are quite a few papers describing the origin of thermalization in QFT using the above techniques.

5.2.4 Non-Equilibrium State

In QFT, non-equilibrium states are situations where the system is not in a time-independent, stable configuration. The presence of external forces, rapid changes, or deviations from equilibrium are often the characteristics of these states. The study of non-equilibrium QFT is more challenging mathematically and requires specialized techniques.

Some examples of non-equilibrium situations in QFT include:

- Particle collisions: High-energy particle collisions create non-equilibrium states with many particles and energy density.
- Quenches: Non-equilibrium dynamics can result from sudden changes in the system parameters, like changes in coupling constants.

- Early Universe: The evolution of the universe during its early stages involves non-equilibrium QFT, including phase transitions and particle production.

Non-equilibrium QFT usually requires solving time-dependent equations of motion, such as in the Schwinger-Dyson equations or the Kadanoff-Baym equations. These equations describe the time evolution of field operators and require specialized numerical or analytical methods.

For a comprehensive understanding of the physical universe, from particle behaviour in particle accelerators to the evolution of the cosmos, it is essential to comprehend the behaviour of quantum fields in equilibrium and non-equilibrium states.

5.2.5 Pure and Mixed state

Thermalization in QFT requires the concept of pure and mixed states [44]. The fundamental entities in QFT are fields (ψ) that obey quantum principles. Normally, when we define a quantized field theory, we use free field quantization, which is the Fock space quantization, and introduce interactions using perturbative methods. The Fock space states are number operator states for the quantum field; they are pure states. In this context, a pure state is a quantum field configuration precisely defined without any statistical uncertainty. In other words, a specific quantum field operator exists at each point in space-time. Pure states are essential to understand the initial conditions of a quantum field system.

QFT, however, often deals with interacting fields in complex environments, so maintaining pure states is challenging. The presence of an external environment and interactions with other fields and particles can introduce uncertainties and cause mixed states to emerge. Mixed states occur when the quantum field system loses its coherence because of interactions or the environment, becoming a statistical ensemble of pure states. In the process of a system evolving toward thermal equilibrium, mixed states are crucial for understanding the dynamics of the system.

Thermalization occurs when a system reaches a maximum entropy state, with its particles and fields distributed thermally. Mixed states describe this distribution, which is inherently probabilistic. Quantum field systems lose information about their initial conditions and settle into states defined by temperature and chemical potentials as they thermalize.

The density matrix, or density operator, bridges pure and mixed states in quantum field theory. This operator encapsulates the statistical information of a quantum field system. For a given pure state $|\psi>$, the density operator is defined as:

$$\Lambda = |\psi><\psi|. \tag{5.1}$$

This is a pure state when $\Lambda^2 = \Lambda$

In the case of a mixed state, the density matrix is expressed as a weighted sum of pure-state density matrices:

$$\Lambda = \sum a_i|\psi_i><\psi_i|. \tag{5.2}$$

We normalize the trace of the density matrix to ensure unity probabilities for all possible states.

When thermalization takes place in QFT, the density matrix is used to track the evolution of the system as it loses coherence and transitions from pure states to a statistical ensemble. This is a powerful tool for understanding classical behaviour and calculating thermal averages.

Rindler Spacetime

Rindler spacetime is a simplified solution to Einstein's field equations in general relativity. It describes the spacetime experienced by an observer who is undergoing constant acceleration in a flat (Minkowski) spacetime. In other words, if you were in a spaceship accelerating constantly at a certain rate, the spacetime you experience would be described by Rindler coordinates [34].

The Rindler metric in Minkowski space-time is given by:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2. \tag{5.3}$$

A vector field is a Killing field if the Lie derivative of the metric along the vector vanishes. In Minkowski spacetime, a boost-killing field is given by:

$$t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t}. \tag{5.4}$$

These Killing fields have time-like integral curves in the right and left Rindler wedges.

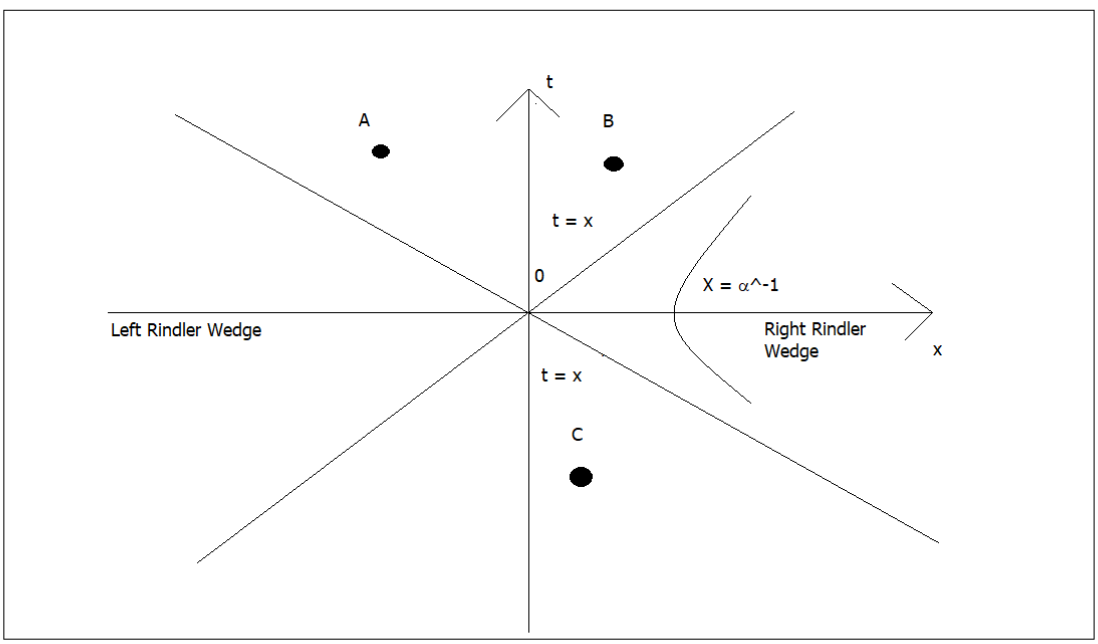

Figure 5.1: Worldlines of observers moving with uniformly constant acceleration α in Minkowski spacetime, the observers cannot receive signals from events A, B and cannot transmit signals to C [31].

When a body is moving hyperbolically at a constant proper acceleration in the x-direction, which depends on the proper time and the rapidity, the coordinates are given by [35]:

$$x = X \ \sinh \alpha T \ , \ t = X \ \cosh \alpha T, \tag{5.5}$$

α represents the acceleration of the system. The metric in the T and X is given by:

$$ds^2 = -\alpha^2 X^2 dT^2 + dX^2 + dy^2 + dz^2. \tag{5.6}$$

Rindler coordinates are often used to study the physics of accelerating observers and provide a useful framework for understanding certain aspects of relativistic physics. In the discussion of the Unruh effect, we will see that Rindler coordinates provide a natural framework for understanding and calculating the Unruh effect.

In Rindler coordinates, the space-time metric simplifies, making calculations involving accelerating observers more tractable.

Rindler space-time is the spacetime experienced by an accelerating observer, and the Unruh effect is the phenomenon of particles being detected by that observer due to their acceleration.

5.2.6 Bogoliubov Transformation

The Bogoliubov transformation is a powerful mathematical tool in quantum field theory that is used to study particle creation and annihilation in systems described by second quantization, particularly in the context of many-body physics and quantum field theory. It was first introduced by the Russian physicist Nikolay Bogoliubov in the mid-20th century and has found applications in various areas of physics, including condensed matter physics and particle physics [43].

Quantum fields and particle creation: Quantum field theory treats particles as quanta of excitations in a quantum field. Fields are continuous entities that permeate space-time, and particles correspond to a specific mode or state of these fields. Particle creation and annihilation processes involve changing the number of particles in a particular mode or state of the field.

Annihilation and Creation Operators: To mathematically describe these processes, physicists use annihilation and creation operators. Annihilation operators, denoted as "a," are responsible for removing a particle from a given mode of the field, while creation operators, denoted as "$a^\dagger$" (a-dagger), add a particle to that mode. These operators allow us to build and manipulate quantum states representing different particle configurations.
The Bogoliubov transformation is a unitary transformation that relates the annihilation and creation operators in one set of modes to those in another set of modes. It is typically used when there is a change of basis in the field's Hilbert space due to, for example, a change in the field's ground state or the introduction of external perturbations.

Mathematical Derivation

In a quantum system involving interacting particles, the Bogoliubov transformation diagonalizes the Hamiltonian. Here, we will present a simplified formulation of the Bogoliubov transformation for bosonic particles.
Consider the following Hamiltonian for a system of non-interacting Bosonic particles:

$$H = \sum_k \frac{k^2}{2m} a_k^\dagger a_k, \tag{5.7}$$

where $a_k^\dagger$ and a_k represent the creation and annihilation operators for particles with momentum k and m is the mass of the particle.
Our next step is introducing a Bogoliubov transformation that will diagonalize this Hamiltonian. Typically, the transformation is written as follows:

$$a_k \ = \ u_k \ \alpha_k \ + \ v_k \ \alpha^{\dagger}_{-k}, \tag{5.8}$$

where a_k is the original annihilation operator, α_k and $\alpha_k^{\dagger}$ are the new annihilation and creation operators after the transformation, u_k and v_k are the Bogoliubov coefficients.

The new operators satisfy the commutation relation

$$[\alpha_k, \alpha^{\dagger}_{k'}] \ = \ \delta_{kk'}. \tag{5.9}$$

We progress by finding the values of u_k and v_k, to do this we substitute the transformation into the Hamiltonian:

$$H \ = \ \sum_k \frac{k^2}{2m}(u_k\alpha_k + v_k\alpha^{\dagger}_{-k})^{\dagger}(u_k\alpha_k + v_k\alpha^{\dagger}_{-k}). \tag{5.10}$$

Expanding the equation, we get:

$$H = \ \sum_k \frac{k^2}{2m}(u_k^* u_k \alpha_k^{\dagger}\alpha_k + u_k^* v_k \alpha_k \alpha^{\dagger}_{-k} + v_k^* u_k \alpha_{-k}\alpha_k + v_k^* v_k \alpha_{-k}\alpha^{\dagger}_{-k}). \tag{5.11}$$

The vacuum state is defined as

$$\alpha_k|0> = \ 0. \tag{5.12}$$

From Eq.(5.12) the only term that will remain as non-zero in the transformed vacuum is:

$$<0|H|0> = <0|v_k^* v_k \alpha_{-k}\alpha^{\dagger}_{-k}|0> . \tag{5.13}$$

Using the identity:

$$[\alpha_k, \alpha_k^{\dagger}] = 1, \tag{5.14}$$

$$< 0|H|0 > = v_k^* v_k < 0|\alpha_{-k}\alpha_{-k}^\dagger|0 >, \tag{5.15}$$

$$< 0|H|0 > | = |v_k|^2 < 0|\alpha_{-k}\alpha_{-k}^\dagger|0 > . \tag{5.16}$$

Based on the modulus, the vacuum state of one observer contains particles in the vacuum state of the other observer. This is the Bogoliubov transformation [33].

It is important to note that each Bogoliubov coefficient has a specific form depending on the specifics of the system and Hamiltonian. For our system in which the Hamiltonian is a constraint, together with our WKB solution, I will use this procedure in the next chapter to solve Bogoliubov coefficients in quantum cosmology.

This section briefly explains how the Bogoliubov transformation and its coefficients are derived and calculated mathematically.

We must note that the Bogoliubov transformation lets us express the new creation and annihilation operators in terms of their old ones. Observing the evolution of coefficients u_k and v_k in time or space can provide insights into particle creation and annihilation. For example, non-zero coefficients v_k indicate particle creation for certain modes, while u_k coefficients indicate particle annihilation.
Bogoliubov transformations are useful tools in quantum field theory for studying particle creation and annihilation processes by relating creation and annihilation operators in different modes. It plays a crucial role in understanding quantum phenomena in different physical systems.

5.2.7 Unruh Effect

The Unruh effect is a fascinating and somewhat counterintuitive phenomenon in the field of theoretical physics, particularly in the context of Special Relativity and quantum field theory. It suggests that an accelerating observer in empty space would perceive a thermal bath of particles, even though an inertial observer at rest

or moving at a constant velocity would perceive empty space. This effect is named after Canadian physicist William Unruh, who first proposed it in 1976 [6].
To understand the Unruh effect, let's break down the key concepts.

Special Relativity: Special Relativity, developed by Albert Einstein in 1905, is a fundamental theory of physics that deals with the behaviour of objects moving at constant velocities, particularly in the absence of gravitational forces. It introduces the concept of space-time, where space and time are intertwined into a four-dimensional continuum.

Accelerating Observers: In special relativity, there is a crucial distinction between inertial observers (those in constant, straight-line motion) and accelerating observers (those undergoing non-uniform motion). This distinction is fundamental to understanding the Unruh effect.

Now, let's delve into how the Unruh effect emerges as a consequence of special relativity:

In the framework of quantum field theory, empty space is not truly empty [51]. It is filled with quantum fields, which can fluctuate, briefly producing pairs of particles and antiparticles (a phenomenon known as vacuum fluctuations). These particle-antiparticle pairs annihilate each other almost immediately, so inertial observers cannot directly detect them.

However, when an observer accelerates through space, the boundary conditions change for what qualifies as "empty space" from their perspective. This change in boundary conditions effectively modifies the vacuum state as perceived by the accelerating observer.

The Unruh effect arises when these accelerated observers detect particles that appear to be emanating from the seemingly empty space. From their point of view, these particles behave as if they are part of a thermal distribution, meaning they have a temperature associated with them. This temperature is directly proportional to the observer's acceleration.

Mathematical Derivation

Let us start with the Minkowski metric in natural units ($c = 1$).

In this task, we want to calculate the quantum field seen by an observer moving uniformly with constant acceleration.

To illustrate the transformation from an inertial observer to an accelerated observer, we use the simplest field, the scalar field. We use a Lorentz-invariant action to define a scalar quantum field:

$$-\frac{1}{2}\int d^4x\sqrt{-g}\;\partial_\mu\phi\; g^{\mu\nu}\;\partial_\nu\phi, \tag{5.17}$$

To quantize the field, we introduce creation and annihilation operators, a_k and $a_k^\dagger$, corresponding to plane wave modes with wavevector k. The operators satisfy the following commutation relations:

$$[a_k, a_{k'}^\dagger] \;=\; \delta(k-k'). \tag{5.18}$$

The quantum field operator $\phi(x)$ can be expanded in terms of these creation and annihilation operators:

$$\phi(x) \;=\; \frac{1}{\sqrt{(2\pi)^3}}\int \frac{d^3k}{\sqrt{2\omega_k}}[a_k e^{ikx} \;+\; a_k^\dagger e^{-ikx}], \tag{5.19}$$

where $\omega_k = |\mathrm{k}|$ is the Energy mode.

The next step is to find the mode functions for our accelerating observer. In a uniformly accelerating frame, these are obtained by solving the Klein-Gordon equation

$$\frac{1}{\sqrt{-g}}\,\partial_\mu(\sqrt{-g}g^{\mu\nu}\,\partial_\nu\,\phi) = 0, \tag{5.20}$$

where ϕ is taken as massless, but the derivation is similar for a massive field. In the Rindler coordinates the Klein-Gordon equation of motion is given as

$$\frac{-1}{\alpha^2 X^2}\partial_T^2(e^{i\omega T}\tilde{\phi}(X)) + \frac{e^{i\omega T}}{X}\partial_X\tilde{\phi}(X) + e^{i\omega T}\partial_X^2\tilde{\phi}(X) \;=\; 0, \tag{5.21}$$

where we have taken $\phi(T,X) = \exp(i\omega T)\tilde{\phi}(X)$ and from the solution of Eq. 5.21

$$\tilde{\phi}(X) \;= C_1 \sin\left(\frac{\omega}{\alpha}\ln(X)\right) + C_2\cos\left(\frac{\omega}{\alpha}\ln(X)\right). \tag{5.22}$$

If we set $C_1 = i; C_2 = 1$, one can write the solution as

$$\phi(T,X) = A\exp\left(i\omega\left(T + \frac{1}{\alpha}\ln(X)\right)\right) = A\exp(i\omega T)X^{i\omega/\alpha}. \tag{5.23}$$

We can calculate the Bogoliubov transformations

$$\alpha_{\omega\omega'} = \int \phi(T,X)\exp(-i\omega' t)dt, \tag{5.24}$$

$$\beta_{\omega\omega'} = \int \phi(T,X)\exp(+i\omega' t)dt. \tag{5.25}$$

We are not calculating the Bogoliubov coefficients explicitly, but one can see that

$$|\alpha_{\omega\omega'}| = \exp(\frac{\pi\omega'}{\alpha})|\beta_{\omega\omega'}|. \tag{5.26}$$

Now, let us calculate the expectation value of the number of particles observed by our accelerating observer. If $a_{kR}, a_{kR}^\dagger$ are the Rindler creation and annihilation operators, the number operator is given by:

$$N_{kR} \;=\; a^{\dagger}_{kR} a_{kR}. \tag{5.27}$$

The expectation value of N_{kR} in the accelerating frame is

$$< N_{kR} > = < 0|N_{kR}|0 >, \tag{5.28}$$

where $|0>$ is the vacuum state in the static frame.

Using the mode expansion, we can calculate $N_k R$ by Wick's theorem:

$$< N_{kR} > = \int \frac{d\omega}{(2\pi)} [e^{-2\pi\omega/\alpha} |\alpha_{\omega\omega'}|^2]. \tag{5.29}$$

This is the expression for the Unruh effect, describing how quantum particles are perceived by uniformly accelerating observers. It shows that the observer sees a thermal distribution of particles with a temperature proportional to their acceleration.

The exponential can be interpreted as Boltzmann's weight of blackbody radiation with temperature $T = \alpha/2\pi$.

Discussions on Unruh Temperature

The Gibbons-Hawking effect for curved space-time is related to the Unruh effect, which relates the temperature the accelerating observer perceives to their proper acceleration.

The thermal nature of vacuum states for accelerating observers is a concept in quantum field theory and relativistic physics closely related to the Unruh effect and Hawking radiation. It arises from the interplay between quantum mechanics and the effects of acceleration in the context of special relativity [2]. Let us break down the key points:

Vacuum State: In quantum field theory, the vacuum state is often referred to as the lowest energy state of a quantum field, such as the electromagnetic field. It is not necessarily "empty" in the classical sense, but it has no real particles, and its energy is considered zero-point energy.

Accelerating Observers: According to the equivalence principle of general relativity, there is no experiment an observer inside an accelerating reference frame (like an accelerating rocket) can perform to distinguish between the effects of acceleration and the presence of a gravitational field. This principle is crucial for understanding the behaviour of quantum fields in such situations.

Unruh Radiation: When an observer is constantly accelerating, they will perceive the vacuum state of a quantum field as a state with particles. This apparent existence of particles, which are not present for an inertial observer, is known as Unruh radiation. Unruh radiation is similar in concept to Hawking radiation near black holes.

Thermal Nature: The Unruh effect leads to the conclusion that the vacuum state for an accelerating observer appears to be in a thermal state, meaning it behaves as if it has a temperature. This temperature is proportional to the observer's acceleration, known as the Unruh temperature (T_U). Mathematically, the Unruh temperature (T_U) is given by as derived earlier in appropriate units:

$$T_U = \frac{a\hbar}{2\pi c k_B}. \tag{5.30}$$

Where: $\hbar$ is the reduced Planck's constant, and a is the proper acceleration of the observer, c is the speed of light, and k_B is Boltzmann's constant. Here, we have restored the constants we initially set to 1. It is important to note that this concept is still theoretical and has not been experimentally confirmed, primarily due to the immense difficulty in achieving the necessary accelerations to observe the Unruh effect directly. However, it remains an intriguing and fundamental aspect of the interface between quantum mechanics and gravity.

The Unruh effect results from the interplay between special relativity and quantum field theory. It tells us that an accelerating observer will perceive a thermal bath of particles, whereas an inertial observer sees empty space. This intriguing phenomenon highlights the deep connections between our understanding of space, time, and quantum physics in the relativistic framework.

5.2.8 Quantum Field Theory in Accelerated Frame

When observers uniformly accelerate in Minkowski space, they perceive the vacuum state differently than an inertial observer. In particular, they detect particles, or more precisely, they perceive the vacuum fluctuations as particles.

The Bogoliubov transformation connects the creation and annihilation operators that describe particles in the accelerated observer's frame to those in the inertial frame. This transformation essentially relates the particle content of the vacuum state in the two frames.

This transformation results in the accelerated observer seeing the vacuum fluctuations as a thermal distribution of particles, which implies a temperature associated with these fluctuations. This temperature is known as the Unruh temperature.

The Unruh effect can be understood using Bogoliubov transformations by mathematically connecting the quantum field description of the vacuum state as seen by an inertial observer to that seen by an accelerated observer. This transformation reveals that an accelerated observer perceives the vacuum fluctuations as thermal radiation, leading to the prediction of the Unruh effect.

5.2.9 Application of the Unruh Effect

We will discuss some of the applications of the Unruh effect in this section.

Hawking Radiation

In 1974, Stephen Hawking came up with the theory that black holes emit particles and eventually lose mass. Basically, this phenomenon results from the interplay between quantum mechanics and general relativity. Despite providing a simplified derivation, it is important to note that the full derivation requires complex calculations based on quantum field theory in curved space-time. The Schwarzschild metric is commonly used to describe black holes without rotation. This metric specifies the space-time geometry around a black hole as follows:

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) dt^2 + \frac{dr^2}{(1 - \frac{2GM}{c^2 r})} + r^2 (d\theta^2 + \sin^2\theta d\phi^2), \tag{5.31}$$

where M is the mass of the black hole, r is the radial coordinate, θ and ϕ are the angular coordinates.

Spacetime tends to curve near the event horizon of a black hole; due to this curvature, particle-antiparticle pairs are created from vacuum fluctuations. Close to the event horizon, one particle of the pair falls into the black hole (inside the event horizon) while the other escapes to infinity. Due to the uncertainty principle, short-lived particle-antiparticle pairs can spontaneously form if their total energy is conserved. The particle that escapes must have positive energy (since it escapes to infinity), while the particle falling into the black hole must have negative energy. Positive energy particles become Hawking radiation, while negative energy particles reduce the black hole's mass.

The Hawking temperature T_H is derived from the energy conservation equation, and it is given by:

$$k_B T_H = \frac{\hbar c^3}{8\pi GM}. \tag{5.32}$$

The Hawking radiation spectrum follows a blackbody distribution, resembling the Planck spectrum with a temperature T_H. This means that black holes can emit radiation in the form of particles (typically photons) with a thermal spectrum. Hawking radiation can be derived using the same techniques as the Unruh effect as mathematically the near-horizon metric is that of a Rindler space-time.

5.3 Conclusion

In conclusion, our exploration of thermalization in curved spacetime and its connection to quantum field theory has shed light on some profound phenomena in the realm of theoretical physics. Thermalization, in this context, refers to the process by which a quantum field adapts to the unique characteristics of curved spacetime, ultimately leading to the emergence of thermal effects, such as Hawking radiation.

We delved into the intricate world of quantum field theory, which provides us with a powerful framework to understand the behaviour of quantum fields in curved spacetime. Within this framework, we discussed the concept of Bogoliubov transformations, a mathematical tool that allows us to bridge the gap between the quantum field modes in different regions of spacetime. This transformation helps us uncover the fascinating phenomenon of Hawking radiation [30].

As Stephen Hawking proposed, Hawking radiation is a remarkable consequence of the interplay between quantum field theory and the gravitational effects of a black hole. It reveals that contrary to classical expectations, black holes are not entirely black but can emit thermal radiation due to quantum fluctuations near their event horizons. This groundbreaking idea has had profound implications for our understanding of the fundamental laws of physics, challenging the boundaries of classical general relativity and quantum mechanics.

In summary, our discussion of thermalization, quantum field theory, Bogoliubov transformations, and Hawking radiation has illustrated the intricate and fascinating connections between the quantum world and the curvature of spacetime. These concepts have expanded our theoretical understanding of the universe and inspired ongoing research and exploration at the forefront of modern physics. As we continue to unravel the mysteries of the cosmos, these ideas remain pivotal in our quest to unify the fundamental forces of nature and deepen our comprehension of the underlying principles that govern the universe.

Chapter 6

Thermalization in Quantum Cosmology

6.1 Introduction

Toward the end of Chapter 4, we discussed some existing solutions to the cosmological wavefunction and their conclusions about the quantum wavefunction of the universe. The Hartle-Hawking state suggests that the universe has no boundary. In this model, there is no initial singularity, and time is treated as an imaginary concept in its early stages. Quantum descriptions of the early universe were derived from path integrals of all possible compact, smooth Euclidean 4-geometries. In addition, we discussed the tunnelling solution, which implies that the Universe was created from nothing.

In this Chapter, we are not interested in the singularity of the system but rather in the thermalization of the wavefunction. We want to study the universe using thermalization because it allows the quantum state to exchange energy with the matter quanta, thereby making it an open system. This prevents cosmological singularities from forming, which enables us to connect them to the exchange of particles of the gravitational wave function with the matter fields [38].

6.1.1 Operator ordering Problem

The operator ordering problem in quantum cosmology refers to the issue of how to properly order the operators representing physical observables in the equations of quantum cosmology [15]. In quantum mechanics, the order in which operators are written in an equation can affect the physical predictions of the theory. Different orderings of operators can result in different mathematical representations of quantum states and different predictions for physical measurements. In quantum cosmology, which seeks to describe the universe as a quantum system, there is no clear and unique prescription for how to order the operators. This ambiguity is often called the "operator ordering problem" in the context of quantum cosmology. It's an issue that needs to be addressed in formulating quantum theories of the universe. Various proposals and approaches have been made to deal with the operator ordering problem in quantum cosmology. Some of these approaches are:

1. DeWitt's Superspace Formulation: This represents the quantum version of Einstein's equations. In this formulation, the ordering of operators is often chosen to be symmetric, where the momenta and coordinates are treated on equal footing. However, this is just one of many possible choices.

2. Minisuperspace Models: These are simplified models of the universe that reduce the degrees of freedom to a finite number, making the operator ordering problem more tractable. These models can help analyze the quantum behaviour of cosmological systems while avoiding some of the complexities of full quantum cosmology. The operator ordering problem in quantum cosmology is a fundamental challenge when applying quantum mechanics to the entire universe. Different approaches and formulations have been proposed to address this problem, but it remains an open question in the field of theoretical physics..

In our situation, we order the WDW equation introduced in Chapter 4 in the following form:

$$\left(-\frac{1}{12}\hat{a}\hat{p}_a\hat{a}\hat{p}_a+\hat{p}_\phi^2\right)\psi(a,\phi)=0. \tag{6.1}$$

6.1.2 Thermalization

Using the operator representation $p_a = i\frac{\delta}{\delta a}$ ($\kappa = 1$) and $p_\phi = i\frac{\delta}{\delta\phi}$, one gets

$$\left(\frac{a}{12}\frac{\delta}{\delta a}\left(a\frac{\delta\psi}{\delta a}\right) - \frac{\delta^2\psi}{\delta\phi^2}\right) = 0. \tag{6.2}$$

Using the redefinition of the coordinate in superspace (space of a, ϕ) as $\xi = \ln a$, one gets the WDW as

$$\left(\frac{1}{12}\frac{\delta^2\psi}{\delta\xi^2} - \frac{\delta^2\psi}{\delta\phi^2}\right) = 0. \tag{6.3}$$

One can absorb the constant 12 into a redefinition of ξ and eventually obtain the WDW on superspace as

$$\frac{\delta^2\psi}{\delta\xi'^2} - \frac{\delta^2\psi}{\delta\phi^2} = 0. \tag{6.4}$$

Using the WKB approximation of Vilenkin [46] [50] , a semi-classical solution is of the form

$$\psi(\xi', \phi) = e^{ik(\xi' \pm \phi)}. \tag{6.5}$$

One can draw this wave function in the $-\infty < \xi' < \infty$ and $-\infty < \phi < \infty$ 'causal diamond' with null lines extending from null past to null infinity.

We can make Rindler transformations in this super-space and obtain new wave functions in 'accelerating coordinates':

$$\xi' = X\sinh(\alpha T); \quad \phi = X\cosh(\alpha T), \tag{6.6}$$

where we have used the same notation as coordinate transformation in real space (Eq.(6.10)), as it is our idea that one can identify these transformations in super-space with real coordinate transformations in the classical solutions for a, ϕ. However, the transformations will keep the WDE invariant, but only wedges of the super-space causal diamonds will be accessible in the X, T frame. Note these transformations are

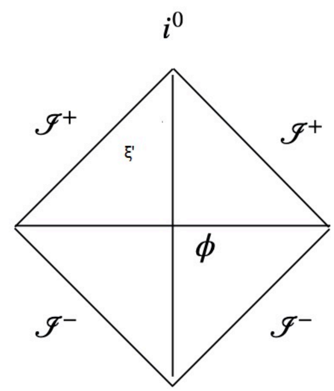

Figure 6.1: A diagram of Dewitt superspace showing $-\infty < \xi' < \infty$ and $-\infty < \phi < \infty$ with null lines extending from null past to null infinity.

similar to the Rindler transformation defined in the Unruh effect. As we are not analyzing the metric is De-Witt superspace, which is six-dimensional, this is a transformation of the scalar-gravity cosmological system. If we implement the coordinate transformation of the super-space on the wavefunction,

$$\psi(X,T) = e^{ik(Xe^{\pm\alpha T})}, \tag{6.7}$$

and if we see the scale factor for the scalar energy-momentum tensor as a function of co-moving coordinates, then

$$a(t) \propto (t - t_0)^{1/3}, \tag{6.8}$$

with the scalar field as

$$\phi(t) \propto \frac{1}{3}\ln(t - t_0). \tag{6.9}$$

We can use the real space coordinate identifications with τ, ϕ to see how the super-space coordinate transformation is identified in real space. The coordinate transformations are very complicated.

If we identify the modes of the wave function in the T coordinate as (the Hamiltonian does not depend on T except for an overall factor)

$$\psi_T = \exp(ik'T), \tag{6.10}$$

Then, a Bogoliubov transformation relates the two wave functions. We then formulate a thermalized evolution of the Universe. Questions of how a background-independent observer can exist to measure this thermal process is a work in progress. However, the result we try to show is that the wave function in Quantum Cosmology is defined in a frame, which is 'accelerating' in superspace.

The Bogoliubov coefficients can be calculated as

$$\alpha_{kk'} = \int e^{ik(Xe^{\pm\alpha T})} \exp(ik'T)dT, \tag{6.11}$$

and

$$\beta_{kk'} = \int e^{ik(Xe^{\pm\alpha T})} \exp(-ik'T)dT. \tag{6.12}$$

We get the following results by solving the integrals

$$\alpha_{kk'} = \exp\left(\pm\frac{\pi k'}{2\alpha}\right) \exp\left(\frac{i\pi}{2}\right), \left(\frac{1}{kx}\right)^{\pm\frac{ik'}{\alpha}} \Gamma\left(\pm\frac{ik'}{\alpha}\right). \tag{6.13}$$

$$\beta_{kk'} = \exp\left(\mp\frac{\pi k'}{2\alpha}\right) \exp\left(\frac{i\pi}{2}\right) \left(\frac{1}{kx}\right)^{\mp\frac{ik'}{\alpha}} \Gamma\left(\mp\frac{ik'}{\alpha}\right). \tag{6.14}$$

From the above , $\beta_{kk'}$ can be expressed in terms of $\alpha_{kk'}$:

$$\frac{\alpha_{kk'}}{\beta_{kk'}} = \exp\left(\pm\frac{\pi k'}{\alpha}\right) \frac{\Gamma\left(\pm\frac{ik'}{\alpha}\right)}{\Gamma\left(\mp\frac{ik'}{\alpha}\right)}. \tag{6.15}$$

It is noted that the exponential term in Eq.(6.15), which gives us a result that corresponds precisely to the Boltzmann distribution of blackbody radiation when we calculate the modulus of $\alpha_{kk'}$:

$$|\alpha_{kk'}|^2 = \exp\left(\pm\frac{2\pi k'}{\alpha}\right) |\beta_{kk'}|^2 \tag{6.16}$$

The exponential term in Eq.(6.15) corresponds to the Boltzmann distribution for blackbody radiation, interpreted as thermalization.

As both coefficients($\alpha_{kk'}, \beta_{kk'}$) are non-zero (Eq.(6.15), Eq.(6.16)), it shows a mixing of negative and positive frequencies. As these are semi-classical modes of the universe, we cannot interpret this as the 'creation and annihilation' of universes. In other words, the thermalization cannot be obtained as a temperature-dependent distribution of a number operator or energy, as in Hawking radiation [17].

6.2 Thermalization Interpretation

However, if we analyze the same system as the accelerated system, is the time evolution Unitary?

To answer the question, we must first find the reduced Hamiltonian in the Rindler coordinates using the super-Hamiltonian Equation (Eq.(4.26); it follows then that the intrinsic time for $\kappa = 0$ (flat universe) is $a(t) \propto (t - t_0)^{1/3}$. Based on our discussion above, the reduced Hamiltonian for this system can be found by solving the constraint.

In the ξ and ϕ frame the super-Hamiltonian takes the form:

$$H_{\text{RW}} = \frac{-p_{\xi}^{2}}{24e^{3\xi}} + \frac{p_{\phi}^{2}}{2e^{3\xi}} + e^{3\xi}V(\phi). \tag{6.17}$$

Our natural choice for the intrinsic time in this coordinate is $\xi = \ln a$. Solving for the reduced Hamiltonian in this coordinate gives:

$$h_{\xi} = \pm\sqrt{12}[p_{\phi}^{2} + 2e^{6\xi}V(\phi)]^{1/2}. \tag{6.18}$$

In the X and T frame, the super-Hamiltonian takes the form:

$$H_{\text{RW}} = \frac{1}{2e^{X\sinh(\alpha T)}}(\frac{-p_{T}^{2}}{\alpha^{2}X^{2}} + p_{X}^{2}). \tag{6.19}$$

The reduced Hamiltonian in the T frame is:

$$h_{T} = p_{X}\alpha X. \tag{6.20}$$

Following the formulations of [7], [8] and [37] we try to find if a thermalization process is involved. To study, we use the density matrix formalism.

$$\Lambda = |\psi><\psi|; \tag{6.21}$$

one then evolves the density matrix forward in time, using the Hamiltonian of the Rindler time, or h_T.Showing that this time evolution is non-unitary in thermal evolution is work in progress. However next we observe whether in the Rindler frame, we have a mixed density matrix.

6.2.1 Mixed State

To check for thermalization, we also need to see if we have a mixed state in the new basis. We know that the state ψ is a linear combination of the basis states of

$|k>, |-k>$. Is the state mixed in the new basis? To find that, we compute ρ^2 and see if that equals ρ. By definition, the static state density matrix is

$$\rho = |\psi><\psi| = \int dk' dk'' \left(\alpha_{kk'}\alpha^*_{kk''}|k'><k''| + \alpha_{kk'}\beta^*_{kk''}|k'><-k''| \right.$$
$$\left. + \beta_{kk'}\alpha^*_{kk''}|-k'><k''| + \beta_{kk'}\beta^*_{kk''}|-k'><-k''| \right). \qquad (6.22)$$

If we take the inner product of this with the states $|m>$, which are such that the $<m|n> = \delta(m-n)$, one gets the elements of the density matrix to be

$$\rho_{mn} = \alpha_{km}\alpha^*_{kn} + \alpha_{km}\beta^*_{k-n} + \beta_{k-m}\alpha^*_{kn} + \beta_{k-m}\beta^*_{k-n} . \qquad (6.23)$$

We can set $\mathrm{Tr}(\rho) = 1$ as discussed earlier in Chapter 5. To compute ρ^2, we use the integrals

$$\rho^2_{mn} = \int \rho_{ml}\rho_{ln} dl. \qquad (6.24)$$

This explicitly gives

$$\rho^2 = \alpha_{km}\left(\int (\alpha^*_{kl}\alpha_{kl} + \alpha^*_{kl}\beta_{k-l} + \beta^*_{k-l}\alpha_{kl} + \beta^*_{k-l}\beta_{k-l})dl\right)\alpha^*_{kn} + \alpha_{km}\left(\int (\alpha^*_{kl}\alpha_{kl} + \beta^*_{k-l}\alpha_{kl})dl\right)\beta^*_{k-n}$$
$$+ \beta_{k-m}\left(\int (\alpha^*_{kl}\alpha_{kl} + \alpha^*_{kl}\beta_{k-l})dl\right)\alpha^*_{kn} + \beta^*_{k-m}\left(\int (\beta_{k-l}\alpha_{kl} + \beta_{k-l}\beta^*_{k-l})dl\right)\beta_{k-n}$$
$$+ \alpha_{km}\left(\int (\alpha^*_{kl}\beta^*_{k-l} + \beta^*_{k-l}\beta^*_{k-l})dl\right)\beta_{k-n} + \beta_{k-m}(\int \alpha^*_{kl}\alpha_{kl}dl)\beta^*_{k-n}$$
$$+ \beta_{k-m}\left(\int \alpha^*_{kl}\beta^*_{k-l}dl\right)\beta_{k-n} + \beta^*_{k-m}\left(\int (\beta_{k-l}\alpha_{kl} + \beta_{k-l}\beta_{k-l})dl\right)\alpha^*_{kn} .$$

The first term has the $\mathrm{Tr}\ \rho$ in it, within the integral which can be set to 1. But the above has terms which are definitely not in ρ as $\int \beta_{k-m}(\alpha^*_{kl}\beta^*_{k-l})\beta_{k-n}dl$. As the Bogoliubov coefficients are explicitly known, one can calculate the integrals exactly. They are infinite, as expected, due to $l \subset [0, \infty]$. However, the integrals can be regularized using zeta functions. The following integrals are relevant [12]. We find that for terms of the type $\alpha^*_{kl}\alpha_{kl}$ and $\beta^*_{kl}\beta_{kl}$ the integral is

$$\int_0^\infty \frac{\exp(\pm x)dx}{x(\sinh(x))} = 2\zeta(0), \tag{6.25}$$

while integrals of the type

$$\int_1^\infty \frac{dx}{x(\sinh(x))} = -2\sum_{k=0}^{\infty} \mathrm{Ei}[-(2k+1)].$$

The other integrals involving $(\Gamma(ik))^2$ have to be computed but can be shown to be non-zero. Thus, $\rho^2 \neq \rho$ is based on the infinite regularizations. This shows that the quantum state is a mixed state, and thermalization is predicted.

6.3 Conclusion

In conclusion, this chapter has unveiled a remarkable set of discoveries and insights that contribute significantly to our understanding of the quantum fabric of the universe. By implementing coordinate transformations on the wavefunction solution to the Wheeler-DeWitt equation, we have embarked on a journey that bridges the realms of theoretical physics and cosmology.

Our primary focus centred on the transformation in superspace, a mathematical construct reflecting the consequences of an observer's acceleration within the framework of real spacetime. This transformation revealed profound consequences, specifically in the form of a Bogoliubov transformation applied to the semi-classical wave function of the Universe defined within the DeWitt superspace. Note here, our DeWitt superspace coordinates are related to 'static' coordinates T and X.

One of the most captivating outcomes of our research is the identification of entropy production in the time evolution of the density matrix within this new accelerated time frame. This observation signifies the presence of a thermal flow within the quantum vacuum of the universe. This introduces a thermodynamic perspective to the cosmos, challenging our traditional understanding and opening the door to a deeper comprehension of its dynamic evolution.

Our work holds significant implications for the broader cosmology and theoretical physics field. It prompts us to question and explore the fundamental nature of the universe, pushing the boundaries of established theories and mathematical frameworks.

Identifying entropy production in the quantum vacuum invites further investigation and serves as a stepping stone for the pursuit of answers to profound questions related to the origins of dark energy and dark matter and the very fabric of space-time.

As we progress in our exploration, we anticipate that these findings will serve as a catalyst for future research, leading to a more comprehensive and integrated understanding of the universe's structure, evolution, and underlying physical principles. This chapter represents a significant contribution to the ongoing quest to unravel the mysteries of our cosmos, and it sets the stage for exciting discoveries and breakthroughs in the realm of theoretical physics and cosmology.

Chapter 7

Conclusion

In conclusion, this thesis has embarked on a multifaceted journey through the realms of quantum cosmology, uncovering profound insights into the thermalization processes within the universe. Our exploration began by setting the stage in Chapter 1, where we elucidated the successes and limitations of General Relativity, paving the way for the necessity of Quantum General Relativity.

Chapter 2 delved into the foundations of theoretical cosmology, introducing the Friedmann equations and their various solutions, each representing different cosmic eras. These equations serve as the backbone of our understanding of the evolving universe.

In Chapter 3, we ventured into the Hamiltonian formulation of general relativity, a crucial step in bridging classical and quantum physics. We unveiled the Wheeler-DeWitt equation, a cornerstone in the quest to unite gravity with quantum mechanics.

Chapter 4 took us into the heart of Quantum Cosmology, where we transformed the cosmological metric into the language of scale factors and matter terms, a critical step towards a quantum description of the cosmos. The derivation of the total Hamiltonian and the application of the WKB solution by Vilenkin were key milestones in this chapter.

In Chapter 5, we delved into the intriguing world of thermalization in curved spacetime. The Unruh effect, Bogoliubov transformations, Hawking radiation, and the distinction between pure and mixed states enriched our understanding of quantum fields in gravitational backgrounds.

In Chapter 6 we unveiled a fascinating connection between the concept of an accelerated observer in the cosmological context and the mathematical framework of coordinate transformations within the DeWitt superspace. Through this transformation, we have derived a Bogoliubov transformation of the semi-classical wave function of the Universe, which is a significant achievement in the study of quantum gravity.

One of the most striking findings of our work is the revelation that the density matrix was a mixed state ($\rho^2 \neq \rho$), and also the time evolution of the density matrix in this new accelerated time might exhibit entropy production indicating the presence of a thermal flow within the universe. This discovery introduces a thermodynamic aspect to the study of cosmology, opening up exciting avenues for further exploration.

By bridging the gap between accelerated observers, DeWitt superspace transformations, and thermal flows, we aim to unearth new physics that may contribute to our understanding of these cosmic mysteries.

The implications of this research extend beyond the boundaries of established theories, offering a fresh perspective on the fundamental workings of the universe. As we continue to investigate the consequences of our findings, we are optimistic that they will lead to breakthroughs in our comprehension of the evolution of the Universe and the elusive dark energy and dark matter, potentially reshaping our understanding of the cosmos.

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