

Cayley graphs on groups with commutator subgroup of order $2p$ are hamiltonian

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Received 19 March 2017, accepted 21 December 2017, published online 17 January 2018

Abstract

We show that if G is a finite group whose commutator subgroup $[G, G]$ has order $2p$, where p is an odd prime, then every connected Cayley graph on G has a hamiltonian cycle.

Keywords: Cayley graph, hamiltonian cycle, commutator subgroup.

Math. Subj. Class.: 05C25, 05C45

1 Introduction

Let G be a finite group. It is easy to show that if G is abelian (and $|G| > 2$), then every connected Cayley graph on G has a hamiltonian cycle. (See Definition 2.1 for the definition of the term *Cayley graph*.) To generalize this observation, one can try to prove the same conclusion for groups that are close to being abelian. Since a group is abelian precisely when its commutator subgroup is trivial, it is therefore natural to try to find a hamiltonian cycle when the commutator subgroup of G is close to being trivial. The following theorem, which was proved in a series of papers, is a well-known result along these lines.

Theorem 1.1 (Marušič [13], Durnberger [4, 5], 1983–1985). *If the commutator subgroup $[G, G]$ of G has prime order, then every connected Cayley graph on G has a hamiltonian cycle.*

D. Marušič (personal communication) suggested more than thirty years ago that it should be possible to replace the prime with a product pq of two distinct primes:

Problem 1.2 (D. Marušič, personal communication, 1985). Show that if the commutator subgroup of G has order pq , where p and q are two distinct primes, then every connected Cayley graph on G has a hamiltonian cycle.

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This has recently been accomplished when G is either nilpotent [8] or of odd order [16]. As another step toward the solution of this problem, we establish the special case where $q = 2$:

Theorem 1.3. *If the commutator subgroup of G has order $2p$, where p is an odd prime, then every connected Cayley graph on G has a hamiltonian cycle.*

See the bibliography of [12] for references to other results on hamiltonian cycles in Cayley graphs.

The proof of Theorem 1.3 is a lengthy case-by-case analysis, based on the choice of certain elements a and b of the Cayley graph's connection set (see Notation 3.3). Here is an outline of the paper:

1	Introduction	1
2	Some known results	2
3	Assumptions, group theory, and connected sums	6
4	Case with $\bar{s} = \bar{t}$	11
5	Cases with $ \bar{a} > 2$ and $\bar{b} \notin \langle \bar{a} \rangle$	12
6	Cases with $\bar{b} \in \langle \bar{a} \rangle$	15
7	Cases with $ \bar{a} = 2$ and $\#S = 2$	18
8	Cases with $ \bar{a} = 2$ and $\#S = 3$	19
9	Cases with $ \bar{a} = 2$ and $\#S \geq 4$	23
	References	30

2 Some known results

We recall a few results that provide hamiltonian cycles in various Cayley graphs.

Definition 2.1 (cf. [9, p. 34]). For any subset S of a finite group G , $\text{Cay}(G; S)$ is the graph whose vertex set is G , with an edge joining g to gs , for each $g \in G$ and $s \in S$. This is called the *Cayley graph* of the connection set S on the group G .

Remark 2.2. Unlike most authors (including [9]), we do not require the connection set S to be symmetric in the definition of a Cayley graph; that is, we do not assume S is closed under inverses. This does not change the set of graphs that are considered to be Cayley graphs, because, in our notation, $\text{Cay}(G; S) = \text{Cay}(G; S \cup S^{-1})$, where $S^{-1} = \{s^{-1} \mid s \in S\}$.

Theorem 2.3 ([3, 6, 7, 12]). *Every connected Cayley graph on G has a hamiltonian cycle if $|G| = kp$ for some prime p and some $k \in \mathbb{N}$ with $1 \leq k < 32$ and $k \neq 24$.*

Notation 2.4.

- The symbol G always represents a finite group.
- For $g \in G$ and $s_1, \dots, s_n \in S \cup S^{-1}$, we use $[g](s_1, \dots, s_n)$ to denote the walk in $\text{Cay}(G; S)$ that visits (in order), the vertices

$$g, gs_1, gs_1s_2, gs_1s_2s_3, \dots, gs_1s_2 \cdots s_n.$$

We may write (s_1, \dots, s_n) for $[e](s_1, \dots, s_n)$.

- We use $(s_1, \dots, s_n)^k$ to denote the concatenation of k copies of the sequence $(s_i)_{i=1}^n$.

- Appending $\#$ to a sequence deletes the last term; that is, $(s_i)_{i=1}^n \# = (s_i)_{i=1}^{n-1}$.
- If $W = [g](s_1, \dots, s_n)$ is a walk in $\text{Cay}(G; S)$, and $h \in G$, we use hW to denote the translate $[hg](s_1, \dots, s_n)$.
- When C is an oriented cycle, we use $-C$ to denote the same cycle as C , but with the opposite orientation.
- For $g, h \in G$:

$$[g, h] = g^{-1}h^{-1}gh, \quad g^h = h^{-1}gh, \quad \text{and} \quad {}^hg = hgh^{-1} (= g^{h^{-1}}).$$

- We use G' to denote the commutator subgroup $[G, G]$ of G .
- For convenience, we let $\bar{G} = G/G'$.
- For $g \in G$, we let $\bar{g} = gG'$ be the image of g in \bar{G} .
- We use $Z(G)$ to denote the center of G .

Definition 2.5 (cf. [10, §2.1.3, p. 61]). Suppose

- N is an abelian, normal subgroup of G , and
- $C = [Nv](s_i)_{i=1}^n$ is an (oriented) cycle in $\text{Cay}(G/N; S)$.

The *voltage* of C is $v(\prod_{i=1}^n s_i)$. This is an element of N , and it may be denoted ΠC .

We have the following straightforward observations:

Lemma 2.6. Assume the notation of Definition 2.5. Then:

1. ΠC is determined by the oriented cycle C : it is independent of the choice of the vertex Nv of C , and of the choice of the representative v of Nv .
2. $\Pi gC = {}^g(\Pi C)$ for all $g \in G$.
3. $\Pi(-C) = (\Pi C)^{-1}$.

Definition 2.7. A subset S of G is an *irredundant* generating set of G if S generates G , but no proper subset of S generates G .

Lemma 2.8 (“Factor Group Lemma” [15, §2.2]). Suppose

- N is a cyclic, normal subgroup of G ,
- $(s_i)_{i=1}^m$ is a hamiltonian cycle in $\text{Cay}(G/N; S)$, and
- the voltage $\Pi(s_i)_{i=1}^m$ generates N .

Then $(s_1, s_2, \dots, s_m)^{|N|}$ is a hamiltonian cycle in $\text{Cay}(G; S)$.

Corollary 2.9 ([12, Cor. 2.11]). Suppose

- N is a normal subgroup of G , such that $|N|$ is prime,
- the image of S in G/N is an irredundant generating set of G/N ,
- there is a hamiltonian cycle in $\text{Cay}(G/N; S)$, and
- $s \equiv t \pmod{N}$ for some $s, t \in S \cup S^{-1}$ with $s \neq t$.

Then there is a hamiltonian cycle in $\text{Cay}(G; S)$.

Lemma 2.10 ([2, Lem. 1 on p. 24]). *Let $P_k \square P_\ell$ be the Cartesian product of a path of length k with a path of length ℓ . If $k\ell$ is even, and $k, \ell \geq 2$, then $P_k \square P_\ell$ has a hamiltonian path from any corner vertex v to any vertex that is at odd distance from v .*

Corollary 2.11. *Suppose N is a subgroup of an abelian group H , and $\{x, y\} \cup S_0$ is a subset of H that generates H/N . Let $k = |\langle x, N \rangle : N|$ and $\ell = |\langle x, y, N \rangle : \langle x, N \rangle|$. If $k\ell$ is even, $k, \ell \geq 2$, $0 \leq p < k$, $0 \leq q < \ell$, and $p + q$ is odd, then $\text{Cay}(H/N; \{x, y\} \cup S_0)$ has a hamiltonian path $(s_i)_{i=1}^r$, such that $s_1 s_2 \cdots s_r = x^p y^q$.*

Proof. If we identify the vertices of $P_k \square P_\ell$ with $\{(i, j) \mid 0 \leq i < k, 0 \leq j < \ell\}$ in the natural way, then the map $(i, j) \mapsto x^i y^j$ is an isomorphism from $P_k \square P_\ell$ to a subgraph X of $\text{Cay}(\langle x, y \rangle; x, y)$. So Lemma 2.10 provides a hamiltonian path $(t_i)_{i=1}^{k\ell-1}$ in X from e to $x^p y^q$. So $t_1 t_2 \cdots t_{k\ell-1} = x^p y^q$.

Let $L = (u_j)_{j=1}^n$ be a hamiltonian path in $\text{Cay}(H/\langle x, y, N \rangle)$, and let

$$(s_i)_{i=1}^r = (L, t_{2i-1}, L^{-1}, t_{2i})_{i=1}^{k\ell/2} \#.$$

From the definition of k and ℓ , we see that the natural map from X to the Cayley graph $\text{Cay}(\langle x, y, N \rangle/N; x, y)$ is an isomorphism onto a spanning subgraph. Therefore, $(s_i)_{i=1}^r$ is a hamiltonian path in $\text{Cay}(H/N; S)$. Since H is abelian, it is easy to see that $s_1 s_2 \cdots s_r = x^p y^q$. \square

Given a hamiltonian cycle C_0 in $\text{Cay}(\overline{G}; S)$, the following result often provides a second hamiltonian cycle C_1 , such that the voltage of at least one of these two cycles generates G' . (Then the Factor Group Lemma (2.8) provides a hamiltonian cycle in $\text{Cay}(G; S)$.)

Lemma 2.12 (cf. Marušič [13] and Durnberger [4], or see [16, Lem. 3.1]). *Assume:*

- N is an abelian normal subgroup of G , such that G/N is abelian,
- C_0 is an oriented hamiltonian cycle in $\text{Cay}(G/N; S)$,
- $s, t, u \in S^{\pm 1}$ and $h \in G$,
- C_0 contains:
 - the oriented path $[\overline{hs^{-1}u^{-1}}](s, t, s^{-1})$, and
 - either the oriented edge $[\overline{h}](t)$ or the oriented edge $[\overline{ht}](t^{-1})$.

Then there is a hamiltonian cycle C_1 in $\text{Cay}(G/N; S)$, such that

$$\left((\Pi C_0)^{-1} (\Pi C_1) \right)^h = \begin{cases} [u, t^{-1}][s, t^{-1}]^u & \text{if } C_0 \text{ contains } [\overline{h}](t), \\ [t^{-1}, u][s, t^{-1}]^u & \text{if } C_0 \text{ contains } [\overline{ht}](t^{-1}). \end{cases}$$

Furthermore, C_0 and C_1 have exactly the same oriented edges, except for some of the edges in the subgraph induced by $\{\overline{h}, \overline{hu^{-1}}, \overline{hs^{-1}u^{-1}}, \overline{ht}, \overline{htu^{-1}}, \overline{hts^{-1}u^{-1}}\}$.

Lemma 2.13 ([4, Lem. 2.8]). *Assume*

- S is an irredundant generating set of G ,
- $s, t \in S$, with $s \neq t$,

- s commutes with t ,
- $\langle S \setminus \{s\} \rangle \triangleleft G$, and
- there is a hamiltonian cycle in $\text{Cay}(\langle S \setminus \{s\} \rangle; S \setminus \{s\})$.

Then there is a hamiltonian cycle in $\text{Cay}(G; S)$.

We do not need the general theory of nilpotent groups, but we will make use of the following two facts. (The first is essentially the definition of a nilpotent group, which can be found in any graduate-level textbook on group theory.)

Lemma 2.14 ([14, (iii) on p. 175 and Prop. VI.1.h on page 176]).

1. Every abelian group is nilpotent.
2. If $G/Z(G)$ is nilpotent, then G is nilpotent.

Therefore, if $G' \subseteq Z(G)$ (in other words, if $G/Z(G)$ is abelian), then G is nilpotent.

Theorem 2.15 ([8]). If G is a nontrivial, nilpotent, finite group, and the commutator subgroup of G is cyclic, then every connected Cayley graph on G has a hamiltonian cycle.

The following observation is well known (and easy to prove).

Lemma 2.16 ([12, Lem. 2.27]). Let S generate a finite group G and let $s \in S$, such that $\langle s \rangle \triangleleft G$. If

- $\text{Cay}(G/\langle s \rangle; S)$ has a hamiltonian cycle, and
- either
 1. $s \in Z(G)$, or
 2. $Z(G) \cap \langle s \rangle = \{e\}$, or
 3. $|s|$ is prime,

then $\text{Cay}(G; S)$ has a hamiltonian cycle.

Corollary 2.17. Suppose

- G' is cyclic of order pq , where p and q are distinct primes,
- S is an irredundant generating set of G , and
- some nontrivial element s of S is in G' .

Then $\text{Cay}(G; S)$ has a hamiltonian cycle.

Proof. We may assume $G' = \mathbb{Z}_p \times \mathbb{Z}_q$. Since every subgroup of a cyclic, normal subgroup is also normal, we know that $\langle s \rangle \triangleleft G$. Also, there are hamiltonian cycles in $\text{Cay}(G/\mathbb{Z}_p; S)$, $\text{Cay}(G/\mathbb{Z}_q; S)$, and $\text{Cay}(G/G'; S)$ (by Theorem 1.1 and the elementary fact that Cayley graphs on abelian groups have hamiltonian cycles). Hence, we may assume $\langle s \rangle = G'$ and $G' \cap Z(G) = \mathbb{Z}_q$ (perhaps after interchanging p and q), for otherwise Lemma 2.16 applies.

Let $\widehat{G} = G/\mathbb{Z}_p$. We may assume $|\widehat{G}| \neq 27$, for otherwise $|G| = 27p$ so Theorem 2.3 applies. Then, since \widehat{G} is nilpotent (see Lemma 2.14) and its commutator subgroup is \mathbb{Z}_q , the proof in [11, §4] implies there is a hamiltonian cycle $(t_i)_{i=1}^n$ in $\text{Cay}(\widehat{G}/\widehat{G}'; S')$ whose

voltage generates \widehat{G}' . Then, since $\mathbb{Z}_p \cap Z(G) = \{e\}$, the proof of Lemma 2.16(2) in [12, Lem. 2.27(2)] tells us that $(t_i, s^{p-1})_{i=1}^n$ is a hamiltonian cycle in $\text{Cay}(G/\mathbb{Z}_q; S)$.

Note that, since \widehat{G} is a nilpotent group whose commutator subgroup is in the center and has prime order q , the order of $|\widehat{G}/\widehat{G}'|$ must be a multiple of q ; that is, n is a multiple of q (cf. Lemma 3.6 below). Calculating modulo \mathbb{Z}_p , we have

$$\begin{aligned} \Pi(t_i, s^{p-1})_{i=1}^n &\equiv s^{(p-1)n} \Pi(t_i)_{i=1}^n && (\widehat{s} \in \widehat{G}' = \widehat{\mathbb{Z}}_q \subseteq Z(\widehat{G})) \\ &\equiv \Pi(t_i)_{i=1}^n && (n \text{ is a multiple of } q) \\ &\neq e && (\Pi(t_i)_{i=1}^n \text{ generates } \widehat{G}'). \end{aligned}$$

Therefore $\Pi(t_i, s^{p-1})_{i=1}^n$ generates \mathbb{Z}_q . So the Factor Group Lemma (2.8) tells us that $((t_i, s^{p-1})_{i=1}^n)^q$ is a hamiltonian cycle in $\text{Cay}(G; S)$. \square

3 Assumptions, group theory, and connected sums

Assumptions 3.1. The remainder of this paper provides a proof of Theorem 1.3, so

- p is an odd prime,
- G is a finite group whose commutator subgroup has order $2p$, and
- S is an irredundant generating set of G .

We wish to show that the Cayley graph $\text{Cay}(G; S)$ has a hamiltonian cycle.

3A Basic group theory

Assumption 3.2. Because of Corollary 2.17, we may assume $S \cap G' = \emptyset$.

Notation 3.3. The assumption that the commutator subgroup has order $2p$ implies that G' is cyclic (cf. [16, §2E, proof of Cor. 1.4]), so we may write

$$G' = \mathbb{Z}_2 \times \mathbb{Z}_p.$$

From Theorem 2.15, we may assume that G is not nilpotent, so $G' \not\subseteq Z(G)$ (see Lemma 2.14). This implies $\mathbb{Z}_p \cap Z(G) = \{e\}$. Hence there exists $a \in S$, such that

$$a \text{ does not centralize } \mathbb{Z}_p. \tag{3.3A}$$

Then there exists $b \in S$, such that

$$\mathbb{Z}_p \subseteq \langle [a, b] \rangle. \tag{3.3B}$$

The assumptions (3.3A) and (3.3B) are the basis of most of the arguments in the later sections of the paper.

For ease of reference, we now collect a few well-known facts from group theory (specialized to our setting).

Lemma 3.4. *If $S_0 \subseteq G$, such that $\langle S_0, \mathbb{Z}_2 \rangle = G$, then $\langle S_0 \rangle = G$.*

Proof. Since $\mathbb{Z}_2 \subseteq Z(G)$, we have

$$\langle S_0 \rangle' = \langle S_0, Z(G) \rangle' \supseteq \langle S_0, \mathbb{Z}_2 \rangle' = G'.$$

Therefore

$$\langle S_0 \rangle = \langle S_0, \langle S_0 \rangle' \rangle = \langle S_0, G' \rangle \supseteq \langle S_0, \mathbb{Z}_2 \rangle = G. \quad \square$$

Corollary 3.5. *Suppose S_0 is a proper subset of S , such that $\mathbb{Z}_p \subseteq \langle S_0 \rangle$. (In particular, this will be the case if $\{a, b\} \subseteq S_0$.) Then $\langle \overline{S_0} \rangle \neq \overline{G}$.*

Proof. Suppose $\langle \overline{S_0} \rangle = \overline{G}$. This means $\langle S_0, G' \rangle = G$. Since $G' = \mathbb{Z}_2 \times \mathbb{Z}_p$ and $\mathbb{Z}_p \subseteq \langle S_0 \rangle$, this implies $\langle S_0, \mathbb{Z}_2 \rangle = G$. So Lemma 3.4 tells us that $\langle S_0 \rangle = G$. This contradicts the fact that the generating set S is irredundant. \square

Lemma 3.6. *Let H be a group. If $x, y, z \in H$, and y centralizes H' , then $[xy, z] = [x, z][y, z]$. Therefore $[y^k, z] = [y, z]^k$ for all $k \in \mathbb{Z}$.*

Corollary 3.7. *If $x, y \in G$, such that y centralizes G' , and $\mathbb{Z}_p \subseteq \langle [x, y] \rangle$, then $|y|$ is divisible by p .*

Corollary 3.8. *Let $S_0 \subseteq G$, such that $\mathbb{Z}_2 \not\subseteq \langle S_0 \rangle'$. If $g \in G$, such that $\mathbb{Z}_2 \subseteq \langle g, S_0 \rangle'$, then $|\langle \overline{g}, \overline{S_0} \rangle : \langle \overline{S_0} \rangle|$ is even.*

In particular, if $\mathbb{Z}_2 \subseteq \langle [g, h] \rangle$, then, by taking $S_0 = \{h\}$, we see that $|\langle \overline{g}, \overline{h} \rangle : \langle \overline{h} \rangle|$ is even, so $|\overline{g}|$ is even (and, similarly, $|\overline{h}|$ must also be even).

Corollary 3.9. $|\overline{G}|$ is divisible by 4.

3B Connected sums

Definition 3.10 ([8, Defn. 5.1]). Assume C_1 and C_2 are two vertex-disjoint oriented cycles in $\text{Cay}(\overline{G}; S)$, and let $g \in G$, and $s, t \in S \cup S^{-1}$. If

- C_1 contains the oriented edge $[\overline{g}](t)$, and
- C_2 contains the oriented edge $[\overline{gst}](t^{-1})$,

then we use $C_1 \#_t^s C_2$ to denote the oriented cycle obtained from $C_1 \cup C_2$ by

- removing the oriented edges $[\overline{g}](t)$ and $[\overline{gst}](t^{-1})$, and
- inserting the oriented edges $[\overline{g}](s)$ and $[\overline{gst}](s^{-1})$.

This is called the *connected sum* of C_1 and C_2 .

If $[g](t)$ is any oriented edge of an oriented cycle C , and $s \in S$, such that sC is vertex disjoint from C , then we can form the connected sum $C \#_t^s -sC$. This construction can be iterated:

Definition 3.11. Suppose

- $[g_1](t_1), \dots, [g_k](t_k)$ are oriented edges of an oriented cycle C in $\text{Cay}(\overline{G}; S)$, such that $g_i \neq g_{i+1}$ for all i , and

- $s_1, s_2, \dots, s_k \in S \cup S^{-1}$, such that the cycles $C, s_1C, s_2s_1C, \dots, s_k s_{k-1} \cdots s_1C$ are pairwise vertex-disjoint.

Then we can form the connected sum

$$C \#_{t_1}^{s_1} -s_1C \#_{t_2}^{s_2} s_2s_1C \#_{t_3}^{s_3} \cdots \#_{t_k}^{s_k} \pm s_k s_{k-1} \cdots s_1C.$$

We call this a *connected sum of signed translates of C* .

Lemma 3.12 (cf. [8, Lem. 5.2]). *If C_1, C_2, g, s , and t are as in Definition 3.10, then*

$$\Pi(C_1 \#_t^s C_2) = \Pi C_1 \cdot {}^g[s^{-1}, t^{-1}] \cdot \Pi C_2.$$

Proof. We may assume $g = t^{-1}$ (or, in other words, $gt = e$), after translating the cycles by $(gt)^{-1}$ (cf. Lemma 2.6(2)). Write $C_1 = (s_i)_{i=1}^m$ and $C_2 = [st^{-1}](t_j)_{j=1}^n$, so

$$(C_1 \#_t^s C_2) = ((s_i)_{i=1}^{m-1}, s, (t_j)_{j=1}^{n-1}, s^{-1}).$$

By assumption, C_1 contains the edge $\overline{t^{-1}} \rightarrow \bar{e}$ and C_2 contains the edge $\bar{s} \rightarrow \overline{st^{-1}}$, so $s_m = t$ and $t_n = t^{-1}$. Therefore

$$\begin{aligned} \Pi(C_1 \#_t^s C_2) &= \prod_{i=1}^{m-1} (s_i) \cdot s \cdot \prod_{j=1}^{n-1} (t_j) \cdot s^{-1} \\ &= \prod_{i=1}^m (s_i) \cdot t^{-1} s \cdot \prod_{j=1}^n (t_j) \cdot ts^{-1} \\ &= \Pi C_1 \cdot t^{-1} s \cdot (\Pi C_2)^{st^{-1}} \cdot ts^{-1} \\ &= \Pi C_1 \cdot t^{-1} st s^{-1} \cdot \Pi C_2 \\ &= \Pi C_1 \cdot t^{-1} [s^{-1}, t^{-1}] \cdot \Pi C_2 \\ &= \Pi C_1 \cdot {}^g[s^{-1}, t^{-1}] \cdot \Pi C_2. \quad \square \end{aligned}$$

Corollary 3.13. *Assume that C_1, C_2, g, s , and t are as in Definition 3.10. If C_0 is another oriented cycle that is vertex-disjoint from C_2 and contains the oriented edge $\bar{g}(t)$, then*

$$(\Pi(C_0 \#_t^s C_2))(\Pi(C_1 \#_t^s C_2))^{-1} = (\Pi C_0)(\Pi C_1)^{-1}.$$

Corollary 3.14 ([8, Lem. 5.2]). *If C_1, C_2, g, s , and t are as in Definition 3.10, then*

$$\Pi(C_1 \#_t^s C_2) \equiv \Pi C_1 \cdot \Pi C_2 \cdot [s, t] \pmod{\mathbb{Z}_p}.$$

The following result describes a fairly common situation in which the connected sum provides hamiltonian cycles in $\text{Cay}(G; S)$:

Lemma 3.15. *Let S_0 be a nonempty subset of S , $g \in G$, $c \in S \setminus S_0$, and $s, t \in S \setminus \{c\}$. Assume C_0 and C_1 are oriented hamiltonian cycles in $\text{Cay}(\langle \overline{S_0} \rangle; S_0)$, such that*

- $(\Pi C_0)^{-1}(\Pi C_1)$ is a nontrivial element of \mathbb{Z}_p ,
- C_0 and C_1 both contain the oriented edge $[\bar{g}](s)$,

- for every $x \in S_0$, C_0 contains at least two edges that are labelled either x or x^{-1} ,
- $\mathbb{Z}_2 \subseteq \langle [c, t] \rangle$, and
- either $|\overline{G} : \langle \overline{S_0} \rangle| > 2$ or $s = t$.

If either

1. there exists $u \in S \setminus \{c\}$, such that $\mathbb{Z}_2 \not\subseteq \langle [u, c] \rangle$, or
2. $|\overline{G} : \langle \overline{S_0}, \bar{t} \rangle|$ is even,

then there is a hamiltonian cycle C in $\text{Cay}(\overline{G}; S)$, such that $\langle \Pi C \rangle = G'$, so the Factor Group Lemma (2.8) yields a hamiltonian cycle in $\text{Cay}(G; S)$.

Proof. Let $r = |\overline{G} : \langle \overline{S_0} \rangle|$. We have $\mathbb{Z}_p \subseteq \langle (\Pi C_0)^{-1}(\Pi C_1) \rangle \subseteq \langle S_0 \rangle$, so Corollary 3.5 implies $r \neq 1$.

Suppose $r = 2$. By assumption, this implies $s = t$, which means that C_0 and C_1 both contain the oriented edge $[\bar{g}](t)$. Then the translate cC_0 contains the oriented edge $[\bar{gc}](t)$. The connected sums $C = C_0 \#_t^c -cC_0$ and $C' = C_1 \#_t^c -cC_0$ are hamiltonian cycles in $\text{Cay}(\overline{G}; S)$. From Corollary 3.14, we have

$$\Pi C \equiv \Pi C_0 \cdot \Pi C_0 \cdot [c, t] \equiv [c, t] \not\equiv 0 \pmod{\mathbb{Z}_p},$$

so ΠC projects nontrivially to \mathbb{Z}_2 . Corollary 3.13 says $(\Pi C)(\Pi C')^{-1} = (\Pi C_0)(\Pi C_1)^{-1}$, which generates \mathbb{Z}_p (because it is conjugate to the inverse of $(\Pi C_0)^{-1}(\Pi C_1)$, which is assumed to be a nontrivial element of \mathbb{Z}_p). Therefore, we see that either ΠC or $\Pi C'$ generates G' , as desired. So we may assume henceforth that $r > 2$.

We now show that we may assume $t \in S_0$. To this end, suppose it is not the case that $t \in S_0$. Let $n = |\langle \overline{S_0}, \bar{t} \rangle : \langle \overline{S_0} \rangle|$. Then, by choosing a sequence $\{[g_i](s_i)\}_{i=1}^{n-1}$ of oriented edges of C_0 , we can form a connected sum C'_0 of signed translates of C_0 :

$$C'_0 = C_0 \#_{s_1}^t -tC_0 \#_{s_2}^t \cdots \#_{s_{n-1}}^t \pm t^{n-1}C_0.$$

This is a hamiltonian cycle in $\text{Cay}(\langle \overline{S_0}, \bar{t} \rangle; S_0 \cup \{t\})$. We may assume $s_1 = s$. Then another hamiltonian cycle C'_1 can be constructed by replacing the leftmost occurrence of C_0 with C_1 , and Lemma 3.12 tells us that $(\Pi C'_0)(\Pi C'_1)^{-1} = (\Pi C_0)(\Pi C_1)^{-1}$, which is a nontrivial element of \mathbb{Z}_p (and $(\Pi C_0)^{-1}(\Pi C_1)$ is conjugate to the inverse of this). From the definition of connected sum, it is obvious that C'_0 contains at least two edges labelled $t^{\pm 1}$. So the hamiltonian cycles C'_0 and C'_1 satisfy the hypotheses of the lemma with $S_0 \cup \{t\}$ in the role of S_0 and with t in the role of s .

Case 1. Assume there exists $u \in S \setminus \{c\}$, such that $\mathbb{Z}_2 \not\subseteq \langle [u, c] \rangle$.

Subcase 1.1. Assume $u \in S_0$. Fix a hamiltonian path $(s_i)_{i=1}^n$ in $\text{Cay}(\overline{G}/\langle \overline{S_0} \rangle; S \setminus S_0)$ with $s_1 = c$, and let $\pi_i = \prod_{j=1}^i s_j$. Any connected sum $C_0 \#_{t_1}^{s_1} (-\pi_1 C_0) \#_{t_2}^{s_2} \cdots \#_{t_n}^{s_n} (\pm \pi_n C_0)$ is a hamiltonian cycle C in $\text{Cay}(\overline{G}; S)$.

Since $[t, c]$ and $[u, c]$ do not have the same projection to \mathbb{Z}_2 , the voltages of $C_0 \#_t^c -\pi_1 C_0$ and $C_0 \#_u^c -\pi_1 C_0$ do not have the same projection to \mathbb{Z}_2 . Therefore, by choosing t_1 to be the appropriate element of $\{t, u\}$, we may assume the projection of ΠC to \mathbb{Z}_2 is nontrivial (see Corollary 3.14). Note also that if $|\overline{G} : \langle \overline{S_0} \rangle| = 2$, then we must have $t_1 = t$.

We may assume that $t_n = s$, and that the connected sum $(-1)^{n-1} \pi_{n-1} C_0 \#_s^{s_n} (-1)^n \pi_n C_0$ is relative to the oriented edge $[\overline{\pi_n g}](s)$ of $\pi_n C_0$ that is also in $\pi_n C_1$. Therefore, another hamiltonian cycle C' can be constructed by replacing $\pi_n C_0$ with $\pi_n C_1$ in the connected sum. Then Lemma 3.12 (together with Lemma 2.6(2)) implies that $(\Pi C)^{-1}(\Pi C')$ is conjugate to $(\Pi C_0)^{-1}(\Pi C_1)$, which is a generator of \mathbb{Z}_p . Therefore, either ΠC or $\Pi C'$ generates G' , as desired.

Subcase 1.2. Assume $u \notin S_0$. Let $S_u = \{u\} \cup S_0$, let $n = |\langle \overline{S_u} \rangle : \langle \overline{S_0} \rangle| - 1$, let $(s_i)_{i=1}^m$ be a hamiltonian path in $\text{Cay}(\overline{G}/\langle \overline{S_u} \rangle; S \setminus S_u)$ with $s_1 = c$, and let $\pi_i = \prod_{j=1}^i s_j$. (Since $S \setminus S_0$ is an irredundant generating set for $\overline{G}/\langle \overline{S_0} \rangle$, we have $m, n \geq 1$.) Any connected sum

$$C_u = C_0 \#_{t_1}^u - u C_0 \#_{t_2}^u \cdots \#_{t_n}^u \pm u^n C_0$$

is a hamiltonian cycle in $\text{Cay}(\overline{G}/\langle \overline{S_u} \rangle; S_u)$, so any connected sum

$$C = C_u \#_{t'_1}^{s_1} - \pi_1 C_u \#_{t'_2}^{s_2} \cdots \#_{t'_m}^{s_m} \pm \pi_m C_u$$

is a hamiltonian cycle in $\text{Cay}(\overline{G}; S)$.

Since $t \in S_0$, we know that C_0 contains more than one edge labeled $t^{\pm 1}$, so $-u C_0$ has an edge labeled $t^{\pm 1}$ that was not removed in the construction of the connected sum $C_0 \#_{t_1}^u - \pi_1 C_0$. Furthermore, the definition of the connected sum implies that $C_0 \#_{t_1}^u - \pi_1 C_0$ also contains an edge labeled u . Therefore, we may form connected sums

$$C_u \#_{t^{\pm 1}}^c - \pi_1 C_u \text{ and } C_u \#_u^c - \pi_1 C_u$$

without removing any of the edges of C_u . Since $[c, t]$ and $[c, u]$ do not have the same projection to \mathbb{Z}_2 , the voltages of these two connected sums do not have the same projection to \mathbb{Z}_2 (see Corollary 3.14). Therefore, by choosing t'_1 to be the appropriate element of $\{t^{\pm 1}, u\}$, we may assume the projection of ΠC to \mathbb{Z}_2 is nontrivial.

We have

$$C = C_u \#_{t'_1}^{s_1} - \pi_1 C_u \#_{t'_2}^{s_2} \cdots \#_{t'_{m-1}}^{s_{m-1}} \pm \pi_{m-1} C_u \#_{t'_m}^{s_m} \\ (\pm \pi_m C_0 \#_{t_1}^u \pm \pi_m u C_0 \#_{t_2}^u \cdots \#_{t_n}^u \pm \pi_m u^n C_0),$$

so the proof can be completed almost exactly as in the final paragraph of Subcase 1.1 (by constructing another connected sum in which $\pi_m u^n C_0$ is replaced with $\pi_m u^n C_1$).

Case 2. Assume $[u, c]$ projects nontrivially to \mathbb{Z}_2 , for every $u \in S \setminus \{c\}$. In particular, $[d, c]$ projects nontrivially to \mathbb{Z}_2 , for every $d \in S \setminus (S_0 \cup \{c\})$. Since we may assume that Case 1 does not apply with d in the place of c , we conclude that we may assume

$$[u, d] \text{ projects nontrivially to } \mathbb{Z}_2, \text{ for all } d \in S \setminus S_0 \text{ and } u \in S \setminus \{d\}. \quad (3.15A)$$

Choose a hamiltonian path $(s_i)_{i=1}^n$ in $\text{Cay}(\overline{G}/\langle \overline{S_0} \rangle; S \setminus S_0)$. Any connected sum

$$C = C_0 \#_{t_1}^{s_1} - \pi_1 C_0 \#_{t_2}^{s_2} \cdots \#_{t_n}^{s_n} \pm \pi_n C_0$$

is a hamiltonian cycle in $\text{Cay}(\overline{G}; S)$. Calculating modulo \mathbb{Z}_p , and letting z be the nontrivial element of \mathbb{Z}_2 , we have

$$\begin{aligned} \Pi C &\equiv \Pi C_0 \cdot [s_1, t_1] \cdot \Pi(-\pi_1 C_0) \cdots [s_n, t_n] \cdot \Pi(\pm\pi_n C_0) && \text{(Corollary 3.14)} \\ &\equiv \Pi C_0 \cdot z \cdot \Pi C_0 \cdots z \cdot \Pi C_0 && \text{(Lemma 2.6(2) \& (3.15A))} \\ &= (\Pi C_0)^{n+1} \cdot z^n \\ &\equiv z && (n \text{ is odd}). \end{aligned}$$

The proof is now completed exactly as in the final paragraph of Subcase 1.1. \square

Corollary 3.16. *Let $S_0 \subseteq S$, $g \in G$, and $s \in S_0$. Assume C_0 and C_1 are oriented hamiltonian cycles in $\text{Cay}(\langle S_0 \rangle; S_0)$, such that*

- $(\Pi C_0)^{-1}(\Pi C_1)$ is a nontrivial element of \mathbb{Z}_p ,
- C_0 and C_1 both contain the oriented edge $[\overline{g}](s)$,
- for every $x \in S_0$, C_0 contains at least two edges that are labelled either x or x^{-1} , and
- $\mathbb{Z}_2 \not\subseteq \langle S_0 \rangle'$.

Then there is a hamiltonian cycle C in $\text{Cay}(\overline{G}; S)$, such that $\langle \Pi C \rangle = G'$, so the Factor Group Lemma (2.8) yields a hamiltonian cycle in $\text{Cay}(G; S)$.

Proof. We may assume $[c, t] \in \mathbb{Z}_p$, for all $c \in S$ and $t \in S_0$. (Otherwise, we see from Corollary 3.8 that Lemma 3.15(2) applies.) Choose $c, d \in S$, such that $[c, d] \notin \mathbb{Z}_p$, let $S_0^+ = S_0 \cup \{d\}$, and let $r = |\langle S_0^+ \rangle : \langle S_0 \rangle|$. Any connected sum of the following form is a hamiltonian cycle in $\text{Cay}(\langle S_0^+ \rangle; S_0^+)$:

$$C = C_0 \#_{s_1}^d -dC_0 \#_{s_2}^d \cdots \#_{s_{r-1}}^d \pm d^{r-1} C_0.$$

We may assume $s_1 = s$, and that the connected sum $C_0 \#_{s_1}^d -dC_0$ is formed by using the oriented edge $[\overline{g}](s)$ that is also in C_1 . Therefore, a second hamiltonian cycle C' can be constructed by replacing the leftmost occurrence of C_0 with C_1 . Then Corollary 3.8 implies that Lemma 3.15(2) applies (with S_0^+ , d , d , C , and C' in the roles of S_0 , s , t , C_0 , and C_1 , respectively). \square

4 Case with $\overline{s} = \overline{t}$

Case 4.1. *Assume there exist $s, t \in S \cup S^{-1}$ with $\overline{s} = \overline{t}$ and $s \neq t$.*

Proof. Write $t = s\gamma$ with $\gamma \in G'$. We may assume $\langle \gamma \rangle = G'$, for otherwise $|\gamma|$ is prime, so Corollary 2.9 applies with $N = \langle \gamma \rangle$. Note that the irredundance of S implies $\langle S \setminus \{s\} \rangle$ and $\langle S \setminus \{t\} \rangle$ do not contain \mathbb{Z}_p . This implies that every element of $S \setminus \{s, t\}$ centralizes \mathbb{Z}_p . So s and t do not centralize \mathbb{Z}_p .

Let $m = |\overline{t}|$ and $n = |\overline{G}|/m$.

Subcase 4.1.1. *Assume $|\overline{t}| > 2$. Since \overline{G} is abelian, it is easy to find a hamiltonian cycle $C = (t_i)_{i=1}^{mn}$ in $\text{Cay}(\overline{G}; S \setminus \{s\})$, such that $t_1 = t_2 = \cdots = t_{m-1} = t$. Since $\langle \Pi C \rangle \subseteq \langle S \setminus \{s\} \rangle$, and $\mathbb{Z}_p \not\subseteq \langle S \setminus \{s\} \rangle$, we must have $\Pi C \in \mathbb{Z}_2$.*

For each subset I of $\{1, \dots, m-1\}$, we define C_I to be the hamiltonian cycle constructed from C by changing t_i to s for all $i \in I$. The proof is completed by noting that I may be chosen such that ΠC_I generates G' , so the Factor Group Lemma (2.8) applies:

- If $\Pi C = e$, let $I = \{1\}$.
- If ΠC is the nontrivial element of \mathbb{Z}_2 , and t does not invert \mathbb{Z}_p , then we may let $I = \{1, 2\}$.
- If ΠC is the nontrivial element of \mathbb{Z}_2 , and t inverts \mathbb{Z}_p , then $|\bar{t}|$ is even, so we must have $|\bar{t}| \geq 4$. We may let $I = \{1, 3\}$.

Subcase 4.1.2. Assume $|\bar{t}| = 2$. (Since t does not centralize \mathbb{Z}_p , this implies that t inverts \mathbb{Z}_p .) Choose a hamiltonian cycle $(s_i)_{i=1}^n$ in $\text{Cay}(\overline{G}/\langle \bar{t} \rangle; S \setminus \{s, t\})$, and let

$$C_0 = (t, s_i)_{i=1}^n = (t_j)_{j=1}^{2n}.$$

Since $n = |\overline{G}|/2$ is even (see Corollary 3.9) and $S \setminus \{s\}$ is an irredundant generating set of \overline{G} , it is easy to see that C_0 is a hamiltonian cycle in $\text{Cay}(\overline{G}; S \setminus \{s\})$. Note that $t_i = t$ whenever i is odd, and that $\Pi C_0 \in \mathbb{Z}_2$ (because $\mathbb{Z}_p \not\subseteq \langle S \setminus \{s\} \rangle$).

We may assume $n \geq 6$ (for otherwise $|G| = 4np \leq 20p$, so Theorem 2.3 applies). We construct a hamiltonian cycle C_1 from C_0 :

- If $\Pi C_0 = e$, construct C_1 by changing t_1 to s .
- If $\Pi C_0 \neq e$, construct C_1 by changing both t_1 and t_5 to s .

In each case, ΠC_1 generates G' . (To see this in the second case, note that $t_2 t_3 t_4 t_5 = s_1 t s_2 t$ centralizes G' , because t inverts G' , and each s_i centralizes G' .) Therefore, the Factor Group Lemma (2.8) applies. \square

5 Cases with $|\bar{a}| > 2$ and $\bar{b} \notin \langle \bar{a} \rangle$

Recall that the elements a and b of S satisfy (3.3A) and (3.3B).

Case 5.1. Assume $|\bar{a}| > 2$, $\bar{b} \notin \langle \bar{a} \rangle$, and there exists $c \in S$, such that $\mathbb{Z}_2 \subseteq \langle [a, c] \rangle$. (It may be the case that $b = c$.)

Proof. Let $m = |\bar{a}|$ and $n = |\overline{G} : \langle \bar{a} \rangle|$. Since $\bar{b}, \bar{c} \notin \langle \bar{a} \rangle$ (and $\overline{G}/\langle \bar{a} \rangle$ is abelian), it is easy to find a hamiltonian cycle $(s_i)_{i=1}^n$ in $\text{Cay}(\overline{G}/\langle \bar{a} \rangle; S \setminus \{a\})$, such that $s_n \in \{c^{\pm 1}\}$, and $s_k = b$ for some $k < n$. Since $\mathbb{Z}_2 \subseteq \langle [a, c] \rangle$, we know m and n are both even (see Corollary 3.8). Since n is even, we have the following (well-known) hamiltonian cycle C_0 in $\text{Cay}(\overline{G}; \overline{S})$:

$$C_0 = (a, (a^{m-2}, s_{2i-1}, a^{-(m-2)}, s_{2i})_{i=1}^{n/2} \#, a^{-1}, (s_{n-j}^{-1})_{j=1}^{n-1}). \tag{5.1A}$$

Letting $\widehat{G} = G/\mathbb{Z}_p$, we have $\widehat{G}' = \mathbb{Z}_2$, so $\widehat{a}^{m-2} \in Z(\widehat{G})$ (because m is even). Therefore

$$a^{m-2} s_{2i-1} a^{-(m-2)} \equiv s_{2i-1} \pmod{\mathbb{Z}_p},$$

so, calculating modulo \mathbb{Z}_p , we have

$$\Pi C_0 \equiv a \cdot \left(\prod_{i=1}^{n-1} s_j \right) \cdot a^{-1} \cdot \left(\prod_{i=1}^{n-1} s_j \right)^{-1} \equiv a \cdot s_n^{-1} \cdot a^{-1} \cdot s_n = [a^{-1}, s_n] = [a^{-1}, c^{\pm 1}],$$

which is nontrivial $(\text{mod } \mathbb{Z}_p)$.

Recall that $s_k = b$. Let $g = \prod_{i=1}^{k-1} s_i$ and $\delta = (-1)^{k+1}$. Then C_0 contains both the oriented edge $[\overline{gb}](b^{-1})$ and the oriented path $[\overline{ga^{-2\delta}}](a^\delta, b, a^{-\delta})$. So Lemma 2.12 (with $s = a^\delta$, $t = b$, $u = a^\delta$ and $h = g$) provides a hamiltonian cycle C_1 , such that $(\Pi C_0)^{-1}(\Pi C_1)$ is conjugate to $[b^{-1}, a^\delta][a^\delta, b^{-1}]^{a^\delta}$. Since a centralizes \mathbb{Z}_2 , but not \mathbb{Z}_p , this voltage is a generator of \mathbb{Z}_p .

Thus, either ΠC_0 or ΠC_1 generates $\mathbb{Z}_2 \times \mathbb{Z}_p = G'$, so the Factor Group Lemma (2.8) provides a hamiltonian cycle in $\text{Cay}(G; S)$. \square

Case 5.2. Assume $|\overline{a}| > 2$, $\overline{b} \notin \langle \overline{a} \rangle$, and there does not exist $c \in S$, such that $\mathbb{Z}_2 \subseteq \langle [a, c] \rangle$.

Proof. Choose $c, d \in S$ with $\mathbb{Z}_2 \subseteq \langle [c, d] \rangle$. Let

$$m = |\overline{a}|, \quad n = |\langle \overline{S} \setminus \{\overline{d}\} \rangle|/m, \quad \text{and} \quad r = |\overline{G}|/(mn).$$

By assumption, we know $a \notin \{c, d\}$. Also, we may assume $d \neq b$ (after interchanging c and d if necessary). Then Corollary 3.5 tells us $r > 1$. Furthermore, we see from Corollary 3.8 that the image of c in $\overline{G}/\langle \overline{a} \rangle$ has even order, so n is even.

Subcase 5.2.1. Assume $n > 2$. It is not difficult to construct a hamiltonian cycle $(s_i)_{i=1}^n$ in $\text{Cay}(\langle \overline{S} \setminus \{\overline{d}\} \rangle / \langle \overline{a} \rangle; \overline{S} \setminus \{\overline{a}, \overline{d}\})$, such that $s_1 = b$ and $s_k = c^{\pm 1}$ for some $k \notin \{1, n\}$. Then, since n is even, we may define C_0 as in (5.1A), so C_0 is a hamiltonian cycle in $\text{Cay}(\langle \overline{S} \setminus \{\overline{d}\} \rangle; S \setminus \{d\})$.

Let $g = s_1 s_2 \cdots s_k$, and note that C_0 contains the oriented edges $[\overline{e}](a)$ and $[\overline{g}](c^{\mp 1})$. Since $\mathbb{Z}_2 \subseteq \langle [c, d] \rangle$, but $\mathbb{Z}_2 \not\subseteq \langle [a, d] \rangle$, we see from Lemma 3.12 that there is a connected sum

$$C = C_0 \#_{t_1}^d -dC_0 \#_{t_2}^d \cdots \#_{t_{r-1}}^d \pm d^{r-1}C_0,$$

with $t_1 \in \{a, c^{\pm 1}\}$, such that $\mathbb{Z}_2 \subseteq \langle \Pi C \rangle$. Note that C is a hamiltonian cycle in $\text{Cay}(\overline{G}; S)$.

The cycle C_0 contains both $[\overline{b}](b^{-1})$ and $[\overline{a^{-2}}](a, b, a^{-1})$, and neither of these paths contains either the edge $[\overline{e}](a)$ or the edge $[\overline{g}](c^{\mp 1})$. Therefore, C also contains both of these paths, so Lemma 2.12 (with $s = a$, $t = b$, $u = a$, and $h = e$) provides a hamiltonian cycle C' in $\text{Cay}(\overline{G}; S)$, such that $(\Pi C)^{-1}(\Pi C')$ is a conjugate of $[b^{-1}, a][a, b^{-1}]^a$, which is a generator of \mathbb{Z}_p (since a centralizes \mathbb{Z}_2 , but not \mathbb{Z}_p). Then either ΠC or $\Pi C'$ generates G' , so the Factor Group Lemma (2.8) applies.

Subcase 5.2.2. Assume $n = 2$ and $r > 2$. Since $n = 2$ (and $\overline{b} \notin \langle \overline{a} \rangle$), we have $\langle \overline{a}, \overline{b}, \overline{d} \rangle = \overline{G}$, so Corollary 3.5 implies $S = \{a, b, d\}$. (Therefore $b = c$, which means $\mathbb{Z}_2 \subseteq \langle [b, d] \rangle$.) We have the following hamiltonian cycle in $\text{Cay}(\langle \overline{a}, \overline{b} \rangle; \overline{a}, \overline{b})$:

$$C_0 = [\overline{e}](a^{m-1}, b, a^{-(m-1)}, b^{-1}).$$

Using the oriented edge $[\overline{e}](a)$, we can form the connected sum $C_0 \#_a^d -dC_0$. Then, since dC_0 contains both $[\overline{db}](b^{-1})$ and $[\overline{dab}](a^{-1})$, we can extend this to a connected sum

$$C = C_0 \#_a^d -dC_0 \#_{t_2}^d \cdots \#_{t_{r-1}}^d \pm d^{r-1}C_0,$$

with $t_2 \in \{a, b\}$, such that $\mathbb{Z}_2 \subseteq \langle \Pi C \rangle$ (see Corollary 3.14). Since C contains both $[\overline{b}](b^{-1})$ and $[\overline{a^{-2}}](a, b, a^{-1})$, we may argue as in the last paragraph of Subcase 5.2.1. Namely, Lemma 2.12 (with $s = a$, $t = b$, $u = a$, and $h = e$) provides a hamiltonian cycle C' in $\text{Cay}(\overline{G}; S)$, such that $(\Pi C)^{-1}(\Pi C')$ is a conjugate of $[b^{-1}, a][a, b^{-1}]^a$, which is

a generator of \mathbb{Z}_p . Then either ΠC or $\Pi C'$ generates G' , so the Factor Group Lemma (2.8) applies.

Subcase 5.2.3. Assume $n = r = 2$. As in Subcase 5.2.2, we must have $S = \{a, b, d\}$ and $b = c$ (so $\mathbb{Z}_2 \subseteq \langle [b, d] \rangle$).

Subsubcase 5.2.3.1. Assume $m \neq 3$. Since $m = |\bar{a}| > 2$ (by an assumption of this case), we have $m \geq 4$. We have the following hamiltonian cycle in $\text{Cay}(\bar{G}; S)$:

$$C_0 = (d, b, a, b^{-1}, d^{-1}, a^{m-2}, d, a^{-(m-3)}, b, a^{m-3}, d^{-1}, a^{-(m-1)}, b^{-1}).$$

Since a is central in G/\mathbb{Z}_p (by an assumption of this case), we know that

$$\Pi C_0 \equiv dbb^{-1}d^{-1}dbd^{-1}b^{-1} = dbd^{-1}b^{-1} = [d^{-1}, b^{-1}] \equiv [d, b] = [d, c] \pmod{\mathbb{Z}_p},$$

so $\mathbb{Z}_2 \subseteq \langle \Pi C_0 \rangle$.

Note that C_0 contains both $[\overline{dab}](b^{-1})$ and $[\overline{da^3}](a^{-1}, b, a)$ (because $m \geq 4$), so applying Lemma 2.12 (with $s = a^{-1}$, $t = b$, $u = a^{-1}$ and $h = da$) yields a hamiltonian cycle C_1 in $\text{Cay}(G; S)$, such that $(\Pi C_0)^{-1}(\Pi C_1)$ is a conjugate of $[b^{-1}, a^{-1}][a^{-1}, b^{-1}]^{a^{-1}}$, which is a generator of \mathbb{Z}_p . Then either ΠC or $\Pi C'$ generates G' , so the Factor Group Lemma (2.8) applies.

Subsubcase 5.2.3.2. Assume $m = 3$ and d does not centralize G' . Since the walk (a^{-2}, b^{-1}, a^2) is a hamiltonian path in $\text{Cay}(\langle \bar{a}, \bar{b} \rangle; a, b)$, we have the following hamiltonian cycle in $\text{Cay}(\bar{G}; S)$:

$$C = (a^{-2}, b^{-1}, a^2, d^{-1}, a^{-2}, b, a^2, d).$$

Note that

$$\Pi C = (a^{-2}b^{-1}a^2)d^{-1}(a^{-2}ba^2)d = (b^{a^2})^{-1}d^{-1}(b^{a^2})d = [b^{a^2}, d].$$

Since a^2 does not invert G' , we know that $b^{a^2} \not\equiv b^{a^{-2}} \pmod{\mathbb{Z}_2}$. Therefore, since d does not centralize G' , we may assume $[b^{a^2}, d] \not\equiv e \pmod{\mathbb{Z}_2}$ (by replacing a with its inverse if necessary). Also, since G' is central modulo \mathbb{Z}_p , we have $[b^{a^2}, d] \equiv [b, d] \not\equiv e \pmod{\mathbb{Z}_p}$. Therefore, ΠC generates G' , so the Factor Group Lemma (2.8) applies.

Subsubcase 5.2.3.3. Assume $m = 3$ and d centralizes G' . Suppose $[b, d] \in \mathbb{Z}_2$. Let $\widehat{G} = G/\mathbb{Z}_2$ and $\widehat{H} = \langle \widehat{a}, \widehat{b} \rangle$. From Theorem 1.1, we know there is a hamiltonian cycle in $\text{Cay}(\widehat{H}; a, b)$. Deleting an edge labeled $b^{\pm 1}$ (and passing to the reverse and/or a translate if necessary) yields a hamiltonian path $L = (t^i)_{i=1}^{2mp-1}$ in $\text{Cay}(\widehat{H}; a, b)$ from \widehat{e} to \widehat{b} . Let

$$C = (L^{-1}, d^{-1}, L, d).$$

Then

$$\Pi C = [\prod_{i=1}^{2mp-1} t_i, d] \in [b\mathbb{Z}_2, d] = \{[b, d]\},$$

because \mathbb{Z}_2 is in the center of G . Since $[b, d] \in \mathbb{Z}_2$, this calculation implies that C is a closed walk in $G/\mathbb{Z}_2 = \widehat{G}$. So C is a hamiltonian cycle in $\text{Cay}(\widehat{G}; S)$. The calculation also implies that the Factor Group Lemma (2.8) applies, because $\langle [b, d] \rangle = \mathbb{Z}_2$.

We may now assume $[b, d] \notin \mathbb{Z}_2$. Therefore, since d centralizes G' , and $p^2 \nmid 12 = |\overline{G}|$, we see from Lemma 3.6 that b does not centralize G' . Also, we may assume $[a, d] \neq e$, for otherwise Lemma 2.13 applies with $s = d$ and $t = a$. However, we know $\mathbb{Z}_2 \not\subseteq \langle [a, d] \rangle$ (by an assumption of this case). Therefore $\langle [a, d] \rangle = \mathbb{Z}_p$. So Subsubcase 5.2.3.2 applies after interchanging b and d . \square

6 Cases with $\bar{b} \in \langle \bar{a} \rangle$

Case 6.1. Assume $\bar{b} \in \langle \bar{a} \rangle$ and a does not invert G' .

Proof. Let $m = |\bar{a}|$. We may assume (perhaps after replacing b with its inverse) that we may write $b = a^k \gamma$ with $1 \leq k \leq m/2$ and $\gamma \in G'$. Assume $k \geq 2$, for otherwise Case 4.1 applies. This implies $m - 1 \geq k + 1$ (since $m = |\bar{a}| \geq 2k \geq k + 2$).

Subcase 6.1.1. Assume there exists $c \in S$, such that $\mathbb{Z}_2 \subseteq \langle [a, c] \rangle$. Let $n = |\bar{G} : \langle \bar{a} \rangle|$. Note that Corollary 3.8 implies m and n are even, and $c \notin \langle \bar{a} \rangle$ (so $c \neq b$).

Choose a hamiltonian cycle $(s_i)_{i=1}^n$ in $\text{Cay}(\bar{G}/\langle \bar{a} \rangle; S \setminus \{a, b\})$, such that $s_n = c$, and define C_0 as in (5.1A). Then $\langle \Pi C_0 \rangle$ contains \mathbb{Z}_2 by the same calculation as in Case 5.1.

Since $m - 1 \geq k + 1$, we may construct a hamiltonian cycle C_1 in $\text{Cay}(\bar{G}; S)$ by replacing the path (a^{k+1}) at the start of C_0 with $(b, a^{-(k-1)}, b)$. Then

$$(\Pi C_1)(\Pi C_0)^{-1} = ba^{-(k-1)}ba^{-(k+1)} = (a^k \gamma)a^{-(k-1)}(a^k \gamma)a^{-(k+1)} = a^{k+1} \gamma^a \gamma a^{-(k+1)}.$$

This is a generator of \mathbb{Z}_p , since a inverts \mathbb{Z}_2 , but not \mathbb{Z}_p . Hence, either ΠC_0 or ΠC_1 generates G' , so the Factor Group Lemma (2.8) provides a hamiltonian cycle in $\text{Cay}(G; S)$.

Subcase 6.1.2. Assume there does not exist $c \in S$, such that $\mathbb{Z}_2 \subseteq \langle [a, c] \rangle$. Choose $c, d \in S$, such that $\mathbb{Z}_2 \subseteq \langle [c, d] \rangle$. (It is possible that $b \in \{c, d\}$, but we know, by the assumption of this subcase, that $a \notin \{c, d\}$.) Let $n = |\langle \bar{a}, \bar{d} \rangle : \langle \bar{a} \rangle|$ and $r = |\bar{G}|/(mn)$. From Corollary 3.8 (and the assumption of this subcase), we know n and r are even.

We have the following hamiltonian cycle in $\text{Cay}(\langle \bar{a}, \bar{d} \rangle; a, d)$:

$$C_0 = ((a, (a^{m-2}, d, a^{-(m-2)}, d)^{n/2} \#, a^{-1}, d^{-(n-1)}).$$

As in the final paragraph of Subcase 6.1.1, another hamiltonian cycle C_1 can be constructed by replacing the path (a^{k+1}) at the start of C_0 with $(b, a^{-(k-1)}, b)$, and the calculation in Subcase 6.1.1 shows that $(\Pi C_1)(\Pi C_0)^{-1}$ generates \mathbb{Z}_p . Therefore, since $[c, d] \notin \mathbb{Z}_p$, but $[c, a] \in \mathbb{Z}_p$, we see that Lemma 3.15(1) applies (with $S_0 = \{a, b, d\}$, $g = a^{-1}$, $s = t = d$, and $u = a$). \square

Case 6.2. Assume $\bar{b} \in \langle \bar{a} \rangle$ and a inverts G' .

Proof. As in Case 6.1, we let $m = |\bar{a}|$ and write $b = a^k \gamma$ with $2 \leq k \leq m/2$ and $\gamma \in G'$. We now consider the same five subcases as in [4, pp. 60–62].

Subcase 6.2.1. Assume $2 < k < m/2$ and k is even. Let $C_1 = (a^m)$. The proof in the last paragraph of [4, p. 60] provides a hamiltonian cycle

$$C_0 = (b, a^{-(k-4)}, b, a^{m-2k-2}, b, a^{-1}, b, a^2, b^{-2}, a^{k-3})$$

in $\text{Cay}(\langle \bar{a} \rangle; a, b)$, such that $(\Pi C_0)^{-1}(\Pi C_1)$ is a generator of \mathbb{Z}_p . Therefore, Corollary 3.16 applies (with $S_0 = \{a, b\}$), because C_0 and C_1 both contain the oriented edge $[\bar{a}^{-1}](a)$.

Subcase 6.2.2. Assume $2 < k < m/2$ and k is odd. Let

$$C_0 = ((b, a, b^{-1}, a)^{(k-1)/2}, b, a^{m-2k+1})$$

and

$$C_1 = ((b, a^{-1}, b^{-1}, a^{-1})^{(k-1)/2}, b^2, a^{m-2k-1}, b).$$

Calculations in [4, p. 61] show that $(\Pi C_0)^{-1}(\Pi C_1)$ is a generator of \mathbb{Z}_p . Therefore, Corollary 3.16 applies (with $S_0 = \{a, b\}$), because C_0 and C_1 both contain the oriented edge $[\bar{e}](b)$.

Subcase 6.2.3. Assume $k = m/2$ and k is even. We follow the argument of [11, Subcase iii, p. 97]. Since k is even, we know a^k centralizes G' , so

$$b^2 = (a^k \gamma)^2 = a^{2k} \gamma^2 = a^m \gamma^2 \in \mathbb{Z}_2 \cdot \gamma^2 \not\cong e.$$

Therefore Corollary 2.9 applies (with $s = b$ and $t = b^{-1}$).

Subcase 6.2.4. Assume $k = m/2$ and k is odd. Choose $c \in S$ so that $\mathbb{Z}_2 \subseteq \langle [a, c] \rangle$, if such c exists. Otherwise, choose c so that there exists $d \in S$, such that $\mathbb{Z}_2 \subseteq \langle [c, d] \rangle$. In either case, Corollary 3.8 implies $c \in S \setminus \{a, b\}$, and $|\langle \bar{a}, \bar{c} \rangle : \langle \bar{a} \rangle|$ is even.

We may assume $b^2 = e$, for otherwise Corollary 2.9 applies (with $s = b$ and $t = b^{-1}$). Therefore, noting that a^k inverts G' (since k is odd), we have

$$e = b^2 = (a^k \gamma)(a^k \gamma) = a^{2k} \cdot \gamma^{-1} \gamma = a^m.$$

Subsubcase 6.2.4.1. Assume $|\bar{G} : \langle \bar{a} \rangle| > 2$. It suffices to find a hamiltonian cycle C_* in $\text{Cay}(\bar{G}; S)$, such that ΠC_* projects nontrivially to \mathbb{Z}_2 , and C_* contains the paths $[a^{k-3}](a, b, a^{-1})$ and $[a^{k-1}b](b^{-1})$. For then Lemma 2.12 (with $s = a, t = b, u = a,$ and $h = a^{k-1}$) provides a hamiltonian cycle C'_* , such that $\langle (\Pi C_*)^{-1}(\Pi C'_*) \rangle = \mathbb{Z}_p$. Therefore, either ΠC_* or $\Pi C'_*$ generates G' , so the Factor Group Lemma (2.8) applies.

Note that

$$C = (a^{k-2}, b, a^{-(k-2)}, c, a^{k-1}, c^{-1}, b^{-1}, c, a^{-(k-1)}, c^{-1})$$

is a cycle through the vertices of $\text{Cay}(\bar{G}; \{a, b, c\})$ in $\langle \bar{a} \rangle \cup c\langle \bar{a} \rangle$. A connected sum of translates of C yields a hamiltonian cycle C_0 in $\text{Cay}(\bar{G}; S)$.

If $\mathbb{Z}_2 \not\subseteq \langle [a, c] \rangle$, then the connected sum defining C_0 can be chosen so that $\mathbb{Z}_2 \subseteq \langle \Pi C_0 \rangle$ (see the proof of Lemma 3.15). So we may let $C_* = C_0$.

We may now assume $\mathbb{Z}_2 \subseteq \langle [a, c] \rangle$. Construct a hamiltonian cycle C_1 in $\text{Cay}(\bar{G}; S)$ by replacing the rightmost translate of C in the connected sum with

$$C' = (a^{k-1}, b, a^{-(k-1)}, c, a^{k-1}, b^{-1}, a^{-(k-1)}, c^{-1}).$$

A straightforward calculation shows that $(\Pi C)^{-1}(\Pi C') \notin \mathbb{Z}_p$, so we have $\mathbb{Z}_2 \subseteq \langle \Pi C_i \rangle$ for some $i \in \{0, 1\}$. Let $C_* = C_i$.

Assumptions 6.2.4.2. We may now assume $|\bar{G} : \langle \bar{a} \rangle| = 2$, so the irredundance of S implies $S = \{a, b, c\}$. Since $\bar{b} \in \langle \bar{a} \rangle$, the irredundance of S also implies $\langle [a, c] \rangle = \mathbb{Z}_2$. Furthermore, we may also assume that c either centralizes G' or inverts G' . (Otherwise, a preceding case applies after interchanging a with c .)

Subsubcase 6.2.4.3. Assume c inverts G' . Let

$$L = \begin{cases} (a, b)^k \# & \text{if } p \mid k \\ (b, a)^k \# & \text{if } p \nmid k \end{cases} \quad \text{and} \quad C = (L^{-1}, c^{-1}, L, c).$$

Then L is a hamiltonian path in $\text{Cay}(\langle \bar{a} \rangle; a, b)$, so C is a hamiltonian cycle in $\text{Cay}(\bar{G}; S)$. Since $(ab)^k = \gamma^k$, we have

$$\Pi L = \begin{cases} \gamma^k b^{-1} = \gamma^{k-1} a^{-k} & \text{if } p \mid k, \\ \gamma^{-k} a^{-1} & \text{if } p \nmid k. \end{cases}$$

Thus, in either case, we have $\Pi L = \gamma^y a^z$, where $p \nmid y$ and z is odd, so

$$\begin{aligned} \Pi C &= (\Pi L)^{-1} c^{-1} (\Pi L) c = [\Pi L, c] = [\gamma^y a^z, c] \\ &= [\gamma^y, c]^{a^z} \cdot [a^z, c] = (\gamma^{-2y})^{a^z} \cdot [a, c]^z = \gamma^{2y} \cdot [a, c]. \end{aligned}$$

This generates G' , so the Factor Group Lemma (2.8) applies.

Subsubcase 6.2.4.4. Assume c centralizes G' and $k \geq 5$. Let

$$C_0 = (L, c, L^{-1}, c^{-1}),$$

where $L = (b, a)^k \#$. Since C_0 contains both $[\bar{e}](b, a, b)$ and $[\bar{a}\bar{b}\bar{c}](a^{-1})$, and also contains both $[\bar{a}^2](b, a, b)$ and $[\bar{a}^3\bar{b}\bar{c}](a^{-1})$ we can apply Lemma 2.12 twice (first with $s = b, t = a, u = c$, and $h = bc$, and then with $s = b, t = a, u = c$, and $h = a^2bc$), to obtain a hamiltonian cycle C_2 , such that

$$(\Pi C_0)^{-1} (\Pi C_2) = [a^{-1}, b]^2,$$

which generates \mathbb{Z}_p . Then, since

$$\Pi C_0 = ((ba)^k a^{-1}) c ((ba)^k a^{-1})^{-1} c^{-1} = [a, c]$$

is a generator of \mathbb{Z}_2 , we conclude that ΠC_2 generates G' , so the Factor Group Lemma (2.8) applies.

Subsubcase 6.2.4.5. Assume c centralizes G' and $k = 3$. Assume, for the moment, that $\gamma \notin \mathbb{Z}_p$. Let

$$C = (c, b, c^{-1}, a, b^{-1}, c, b, a, b^{-1}, c^{-1}, b, a).$$

Then C is a hamiltonian cycle in $\text{Cay}(\bar{G}; S)$, and a straightforward calculation shows that $\Pi C = ba^3 = \gamma^{-1}$ generates G' , so the Factor Group Lemma (2.8) applies.

Now, suppose that $p \geq 5$, and, because of the preceding paragraph, that $\gamma \in \mathbb{Z}_p$. Let

$$C = (b, a, b^{-1}, a, b, c, a^{-5}, c^{-1}).$$

Then C is a hamiltonian cycle in $\text{Cay}(\bar{G}; S)$ and

$$\Pi C = bab^{-1}abcac^{-1} = bab^{-1}aba[a, c^{-1}] = \gamma^{-3}[a, c].$$

Therefore $\langle \Pi C \rangle = G'$ (since $p \neq 3$ and γ projects trivially to \mathbb{Z}_2), so the Factor Group Lemma (2.8) applies.

We may now assume $p = 3$ (so $|G| = 72$), and that $\gamma \in \mathbb{Z}_p$. Let $\hat{G} = G/\mathbb{Z}_p$. We have the following hamiltonian cycle in $\text{Cay}(\hat{G}; S)$:

$$C = (a^2, c, a^5, c^{-1}, a^{-2}, b, a^2, c, a^{-5}, c^{-1}, a^{-2}, b).$$

Calculating modulo \mathbb{Z}_2 (so c is in the center), we have

$$\Pi C = a^2ca^5c^{-1}a^{-2}ba^2ca^{-5}c^{-1}a^{-2}b \equiv a^2a^5a^{-2}ba^2a^{-5}a^{-2}b = a^{-1}bab = [a, b] = \gamma^2.$$

This is nontrivial (mod \mathbb{Z}_2), so ΠC must be nontrivial. Therefore ΠC generates \mathbb{Z}_p , so the Factor Group Lemma (2.8) applies.

Subcase 6.2.5. Assume $k = 2 < m/2$.

Subsubcase 6.2.5.1. Assume $|\bar{G} : \langle \bar{a} \rangle| > 2$. Note that

$$C = (b, a, b^{-1}, c, b, a^{-1}, b, c^{-1}, (a, c, a, c^{-1})^{(m-4)/2})$$

is a cycle through the vertices of $\text{Cay}(\bar{G}; \{a, b, c\})$ in $\langle \bar{a} \rangle \cup c\langle \bar{a} \rangle$. A connected sum of translates of C yields a hamiltonian cycle C_0 in $\text{Cay}(\bar{G}; S)$. Since k is even, we know that $\mathbb{Z}_2 \not\subseteq \langle [b, c] \rangle$, so it is easy to choose the connected sum in such a way that $\mathbb{Z}_2 \subseteq \langle \Pi C_0 \rangle$ (see the proof of Lemma 3.15).

The cycle C contains the paths $[\bar{e}](b, a, b^{-1})$ and $[\bar{b}^2](a)$. By taking just a bit of care in the creation of C_0 (namely, not using any of these edges for the first connected sum), we may assume that C_0 also contains these paths. Then Lemma 2.12 (with $s = b$, $t = a$, $u = b$, and $h = b^2$) provides a hamiltonian cycle C_1 , such that $(\Pi C_0)^{-1}(\Pi C_1) = [a, b]^2$ (because b centralizes G'). This is a generator of \mathbb{Z}_p , so either ΠC_0 or ΠC_1 generates G' . Therefore, the Factor Group Lemma (2.8) applies.

Subsubcase 6.2.5.2. Assume $|\bar{G} : \langle \bar{a} \rangle| = 2$. The irredundance of S implies that $S = \{a, b, c\}$ (see Corollary 3.5). We have the following hamiltonian cycle in $\text{Cay}(\bar{G}; S)$:

$$C = (b^2, a^{m-5}, c, a^{-(m-4)}, c^{-1}, b^{-1}, c, a, b^{-1}, c^{-1}).$$

Since $\bar{b} \in \langle \bar{a} \rangle$, the irredundance of S implies $\langle [a, c] \rangle = \mathbb{Z}_2$. So m is even (see Corollary 3.8). However, $\mathbb{Z}_2 \not\subseteq \langle [b, c] \rangle$, because $k = 2$ is even. So

$$\begin{aligned} \Pi C &= b^2(a^{m-5}ca^{-(m-4)}c^{-1})(b^{-1}cab^{-1}c^{-1}) \\ &\equiv b^2(a^{-1})(b^{-2}a[a, c]) \equiv [a, c] \pmod{\mathbb{Z}_p}, \end{aligned}$$

which generates \mathbb{Z}_2 . We may also assume that c either centralizes G' or inverts G' (for otherwise a preceding case applies after interchanging a with c). Therefore

$$\begin{aligned} \Pi C &= b^2(a^{m-5}ca^{-(m-4)}c^{-1})(b^{-1}cab^{-1}c^{-1}) \equiv a^4\gamma^2(a^{-1})(\gamma^{-1}a^{-2}ca\gamma^{-1}a^{-2}c^{-1}) \\ &= \gamma^3 \cdot (\gamma^{-1})^c = \gamma^3 \cdot \gamma^{\pm 1} \in \{\gamma^2, \gamma^4\} \pmod{\mathbb{Z}_2}, \end{aligned}$$

which generates \mathbb{Z}_p . We now know that ΠC projects nontrivially to both \mathbb{Z}_2 and \mathbb{Z}_p , so it generates G' . Therefore, the Factor Group Lemma (2.8) applies. \square

7 Cases with $|\bar{a}| = 2$ and $\#S = 2$

Assumption 7.1. In this section, we assume

- $|\bar{a}| = 2$, for all $a \in S$, such that a does not centralize G' , and
- $\#S = 2$.

We may assume $|a| = 2$, for otherwise Case 4.1 applies with $s = a$ and $t = a^{-1}$.

We may also assume that b centralizes G' , for otherwise we must have $|\bar{b}| = 2$, so $|G| = 8p$, so Theorem 2.3 applies. Since a does not centralize G' , this implies $\bar{a} \notin \langle \bar{b} \rangle$. Let

$$n = |\overline{G} : \langle \bar{a} \rangle| = |\overline{G}|/2 = |\bar{b}|.$$

Case 7.2. Assume $n \not\equiv 1 \pmod{p}$.

Proof. Let $C = (a^{-1}, b^{-(n-1)}, a, b^{n-1})$, so C is a hamiltonian cycle in $\text{Cay}(\overline{G}; S)$ with $\Pi C = [a, b^{n-1}] = [a, b]^{n-1}$, since b centralizes G' . Note that n is even (see Corollary 3.8), and, by assumption, $n \not\equiv 1 \pmod{p}$. Therefore, $n - 1$ is relatively prime to $2p$, so ΠC generates G' , so the Factor Group Lemma (2.8) applies. \square

Case 7.3. Assume $n \equiv 1 \pmod{p}$.

Proof. We claim that $\mathbb{Z}_p \subseteq \langle b \rangle$. Suppose not. Then $|\langle b, \mathbb{Z}_2 \rangle| = 2n$. Since $\gcd(2n, p) = 1$, the abelian group $\langle b, G' \rangle$ has a unique subgroup of order $2n$, so we conclude that $\langle b, \mathbb{Z}_2 \rangle$ is normal in G . This implies that

$$\langle a \rangle \langle b, \mathbb{Z}_2 \rangle = \langle a, b, \mathbb{Z}_2 \rangle \supseteq \langle a, b \rangle = G,$$

so

$$|G| \leq |a| \cdot |\langle b, \mathbb{Z}_2 \rangle| = 2 \cdot 2n = 4n.$$

This contradicts the fact that $|G| = 4np$.

Subcase 7.3.1. Assume $\mathbb{Z}_2 \subseteq \langle b \rangle$. Combining this assumption with the above claim, we see that $G' \subseteq \langle b \rangle$. This implies $\langle b \rangle \triangleleft G$, so $G = \langle a \rangle \rtimes \langle b \rangle$. Since $|a| = 2$, this implies that $\text{Cay}(G; a, b)$ is a generalized Petersen graph. Then the main result of [1] tells us that $\text{Cay}(G; a, b)$ has a hamiltonian cycle.

Subcase 7.3.2. Assume $\mathbb{Z}_2 \not\subseteq \langle b \rangle$. Since $\langle b, G' \rangle$ is abelian, $\gcd(n, p) = 1$, and $\mathbb{Z}_2 \not\subseteq \langle b \rangle$, we may write

$$\langle b, G' \rangle = \mathbb{Z}_2 \times \mathbb{Z}_p \times \mathbb{Z}_n.$$

Then $G = \langle a \rangle \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_p \times \mathbb{Z}_n)$, and we may assume $b = (0, 1, 1)$ and $[a, b] = (1, 2, 0)$. For $\overline{G} = G/\langle b^2 \rangle = G/(\mathbb{Z}_p \times 2\mathbb{Z}_n)$, it is straightforward to check that $((a, b)^4 \#, b^{-1})$ is a hamiltonian cycle in $\text{Cay}(\overline{G}; a, b)$ whose voltage is $(0, -2, 2)$. (This hamiltonian cycle is taken from the final paragraph of Case 1 of the proof of [3, Prop. 6.1].) This voltage generates $\mathbb{Z}_p \times 2\mathbb{Z}_n$ (since $\gcd(p, n) = 1$), so the Factor Group Lemma (2.8) applies. \square

8 Cases with $|\bar{a}| = 2$ and $\#S = 3$

Assumption 8.1. In this section, we assume

$$S = \{a, b, c\},$$

and

$$|\bar{s}| = 2, \text{ for all } s \in S, \text{ such that } s \text{ does not centralize } G'.$$

We also assume Case 4.1 does not apply. (So $|s| = 2$.) In particular, we have $|a| = 2$.

Note that $\bar{a} \notin \langle \bar{b} \rangle$. (If $\bar{a} \in \langle \bar{b} \rangle$, then b , like a , does not centralize G' , so our assumption implies $|\bar{b}| = 2$. Then $\bar{a} = \bar{b}$, contradicting the fact that Case 4.1 does not apply.)

Notation 8.2. Let

$$n = |\bar{b}| = |\langle \bar{a}, \bar{b} \rangle : \langle \bar{a} \rangle| \geq 2 \quad \text{and} \quad \ell = |\bar{G} : \langle \bar{a}, \bar{b} \rangle| = |\bar{G}|/(2n) \geq 2.$$

The last inequality is because the irredundance of S implies $\bar{c} \notin \langle \bar{a}, \bar{b} \rangle$ (see Corollary 3.5).

Case 8.3. Assume $|\bar{b}| = 3$.

Proof. Since $|\bar{b}| \neq 2$, Assumption 8.1 implies that b centralizes G' . Also, since $|\bar{b}|$ is odd, Corollary 3.8 implies that $[a, b]$ and $[b, c]$ project trivially to \mathbb{Z}_2 , so $[a, c]$ must project nontrivially (and ℓ must be even). We have the following hamiltonian path in $\text{Cay}(\bar{G}/\langle \bar{a} \rangle; S)$:

$$L = (c^{\ell-1}, b, c^{-(\ell-1)}, b, c^{\ell-1}).$$

Then $C = (L, a, L^{-1}, a)$ is a hamiltonian cycle in $\text{Cay}(\bar{G}; S)$. Since $\ell - 1$ is odd, it is easy to see that $\mathbb{Z}_2 \subseteq \langle \Pi C \rangle$.

Since C contains both $[\bar{c}^{\ell-2}](c, b, c^{-1})$ and $[\bar{c}^{\ell-1}\bar{a}\bar{b}](b^{-1})$, Lemma 2.12 (with $s = c, t = b, u = a$, and $h = c^{\ell-1}a$) provides a hamiltonian cycle C' , such that $(\Pi C)^{-1}(\Pi C')$ is conjugate to $[t^{-1}, u][s, t^{-1}]^u = [b^{-1}, a][c, b^{-1}]^a = [a, b][c, b]$. This is an element of \mathbb{Z}_p . If it generates \mathbb{Z}_p , then either ΠC or $\Pi C'$ generates G' , so the Factor Group Lemma (2.8) applies.

Thus, we may assume $[a, b][c, b]$ is trivial. Since $\mathbb{Z}_p \subseteq \langle [a, b] \rangle$ (see (3.3B)), this implies that $[c, b]$ is nontrivial. So we may assume that c does not centralize \mathbb{Z}_p (for otherwise replacing c with c^{-1} would replace $[c, b]$ with $[c, b]^{-1}$, which would not cancel $[a, b]$).

Now, Assumption 8.1 implies $|\bar{c}| = 2$, so we have the hamiltonian cycle

$$C_0 = (b^2, a, b^2, c, a, b, a, b, a, c),$$

in $\text{Cay}(\bar{G}; S)$. This contains both the path $[bac](a, b, a)$ and the edge b, so applying Lemma 2.12 (with $s = a, t = b, u = c$, and $h = b$) provides a hamiltonian cycle C_1 , such that $(\Pi C_0)^{-1}(\Pi C_1)$ is conjugate to $[u, t^{-1}][s, t^{-1}]^u = [c, b^{-1}][a, b^{-1}]^c$. This is not equal to $[a, b][c, b]$ (which is trivial), because $[a, b^{-1}]^c = [a, b]$, but $[c, b^{-1}] = [c, b]^{-1} \neq [c, b]$. So $(\Pi C_0)^{-1}(\Pi C_1)$ is nontrivial, and therefore generates \mathbb{Z}_p . Since a straightforward calculation shows that \mathbb{Z}_2 is contained in $\langle \Pi C_0 \rangle$, this implies that either ΠC_0 or ΠC_1 generates G' , so the Factor Group Lemma (2.8) applies. \square

Case 8.4. Assume $\ell = 2$.

Proof. We may assume $|\bar{b}| \geq 4$, for otherwise either $|\bar{b}| = 2$, so Theorem 2.3 applies (because $|G| = 16p$), or $|\bar{b}| = 3$, so Case 8.3 applies. Let

$$L = (a, b, a, b^{n-2}, a, b^{-(n-3)}) \quad \text{and} \quad C = (L, c, L^{-1}, c^{-1}),$$

so L is a hamiltonian path in $\text{Cay}(\langle \bar{a}, \bar{b} \rangle; a, b)$ and C is a hamiltonian cycle in $\text{Cay}(\bar{G}; S)$.

Subcase 8.4.1. Assume $[a, c]$ and $[a, b][b, c]$ are not both in \mathbb{Z}_p . A straightforward calculation (using Lemma 3.6) shows that $\Pi C \equiv [a, c] \pmod{\mathbb{Z}_p}$. If this is in \mathbb{Z}_p , then, by assumption, $[a, b][b, c] \notin \mathbb{Z}_p$, so applying Lemma 2.12 to the paths $[\bar{c}](a, b, a)$ and $[\bar{a}\bar{b}\bar{c}](b^{-1})$ in C (so $s = a, t = b, u = c$, and $h = ac$) yields a hamiltonian cycle C' , such that $\Pi C'$ projects nontrivially to \mathbb{Z}_2 . Therefore, we have a hamiltonian cycle (either C or C') whose voltage is not in \mathbb{Z}_p .

Now, since $|\bar{b}| \geq 4$, we see that C (and also C') contains the path $[\overline{b^{-2}ac}](b, a, b^{-1})$ and $[\overline{ac}](a)$. Furthermore, we know that $[b, a][b, a]^b$ is a nontrivial element of \mathbb{Z}_p (because b does not invert $[a, b]$). Therefore, Lemma 2.12 (with $s = b, t = a, u = b$, and $h = ac$) yields a hamiltonian cycle C_1 (or C'_1) whose voltage generates G' , so the Factor Group Lemma (2.8) applies.

Subcase 8.4.2. Assume $[a, c]$ and $[a, b][b, c]$ are both in \mathbb{Z}_p . Since $[a, c], [a, b]$, and $[b, c]$ generate G' , they cannot all be in \mathbb{Z}_p , so this assumption implies that neither $[a, b]$ nor $[b, c]$ is in \mathbb{Z}_p . Also, we may assume $\langle [a, c] \rangle = \mathbb{Z}_p$, for otherwise $[a, c] = e$, so we could apply Lemma 2.13 with $s = c$.

We have the following hamiltonian cycle in $\text{Cay}(\bar{G}; S)$:

$$C_0 = (b^{n-1}, c, b^{-(n-2)}, a, b^{n-2}, c^{-1}, b^{-(n-1)}, c, a, c^{-1}).$$

Then

$$\begin{aligned} \Pi C_0 &= b^{n-1}c(b^{-(n-2)}ab^{n-2})c^{-1}b^{-(n-1)}cac^{-1} \\ &= b^{n-1}c(a[a, b]^{n-2})c^{-1}b^{-(n-1)}cac^{-1} \\ &= ([a, b]^{-(n-2)})^c \cdot b^{n-1}(cac^{-1})b^{-(n-1)}(cac^{-1}) \\ &= ([a, b]^{-(n-2)})^c \cdot [b, cac^{-1}]^{-(n-1)} \\ &= ([a, b]^{-(n-2)})^c \cdot [b, a]^{-(n-1)} \quad (cac^{-1} \in aG' \text{ and } G' \subseteq C_G(b)) \\ &= ([a, b]^{-(n-2)})^c \cdot [a, b]^{n-1}. \end{aligned}$$

If c centralizes \mathbb{Z}_p , then $\Pi C_0 = [a, b]$ generates G' , so the Factor Group Lemma (2.8) applies.

We may now assume c does not centralize \mathbb{Z}_p . Then Assumption 8.1 tells us that c inverts \mathbb{Z}_p , so $\Pi C_0 = [a, b]^{2n-3}$ (and $|c| = 2$). Hence, we may assume $2n \equiv 3 \pmod{p}$, for otherwise ΠC_0 generates G' , so the Factor Group Lemma (2.8) applies. We now consider the following hamiltonian cycle in $\text{Cay}(\bar{G}; S)$:

$$C_* = (b^{n-3}, c, b^{-(n-4)}, a, b^{n-4}, c^{-1}, b^{-(n-3)}, c, (b^{-1}, c)^2, a, (c, b)^2, c^{-1}).$$

We have

$$\Pi C_* = b^{n-3}c(b^{-(n-4)}ab^{n-4})c^{-1}b^{-(n-3)}c((b^{-1}c)^2a(cb)^2)c^{-1}.$$

Since cb inverts G' , we know that $(b^{-1}c)^2a(cb)^2 = a$, so ΠC_* is exactly the same as the voltage of C_0 , but with n replaced by $n - 2$; that is,

$$\Pi C_* = [a, b]^{2(n-2)-3} = [a, b]^{2n-7}.$$

Since $2n \equiv 3 \pmod{p}$, we have

$$2n - 7 \equiv 3 - 7 = -4 \not\equiv 0 \pmod{p},$$

so ΠC_* generates G' , so the Factor Group Lemma (2.8) applies. \square

Case 8.5. Assume $|\bar{b}| \neq 3$ and $\ell \neq 2$.

Proof. Since $\ell \neq 2$, we know $|\bar{c}| > 2$, so c must centralize G' (by Assumption 8.1). Also, Corollary 3.8 implies that $|\bar{b}|$ and ℓ cannot both be odd.

- If $|\bar{b}|$ is odd (so ℓ is even), let

$$L = (c^{\ell-1}, b, c^{-1}, b, c, b, (b^{n-4}, c^{-1}, b^{-(n-4)}, c^{-1})^{\ell/2}, b^{-1}, c^{\ell-3}, b^{-1}, c^{-(\ell-3)}).$$

- If $|\bar{b}|$ is even, let

$$L = (c^{\ell-1}, b^{n-1}, c^{-1}, (c^{-(\ell-2)}, b^{-1}, c^{\ell-2}, b^{-1})^{(n-2)/2}, c^{-(\ell-2)}).$$

In either case, L is a hamiltonian path in $\text{Cay}(\overline{G}/\langle \bar{a} \rangle; \{b, c\})$ from \bar{e} to \bar{b} . Now, let

$$C = (L, a, L^{-1}, a) \quad \text{and} \quad (g, \epsilon) = \begin{cases} (c^{\ell-1}, -1) & \text{if } |\bar{b}| = 2 \text{ or } |\bar{b}| \text{ is odd,} \\ (ab^2, 1) & \text{if } |\bar{b}| > 2 \text{ and } |\bar{b}| \text{ is even,} \end{cases}$$

so C is a hamiltonian cycle in $\text{Cay}(\overline{G}; S)$ that contains the paths

$$[\bar{bc}](c^{-1}, a, c), \quad [\bar{ca}](c^{-1}, a, c), \quad [\bar{g}](b), \quad \text{and} \quad [\overline{gbac^\epsilon}](c^{-\epsilon}, b^{-1}, c^\epsilon).$$

Note that $[\bar{bc}](c^{-1}, a, c)$ contains $[\bar{b}](a)$ and that $[\bar{ca}](c^{-1}, a, c)$ contains $[\bar{a}](a)$. Also note that all of these paths are vertex-disjoint (except for the vertices \bar{ac} and $\{abc\}$ when $|\bar{b}| = 2$ and $\ell = 3$). We introduce some terminology:

- Applying Lemma 2.12 to the oriented paths $[\bar{ca}](c^{-1}, a, c)$ and $[\bar{b}](a)$ (so $s = c^{-1}$, $t = a$, $u = b$, and $h = ab$) will be called the “ a -transform.” This multiplies the voltage by γ_a , where $\gamma_a = [a, b^{-1}][c, a]$.
- Applying Lemma 2.12 to the oriented paths $[\bar{g}](b)$ and $[\overline{gbac^\epsilon}](c^{-\epsilon}, b^{-1}, c^\epsilon)$ (so $s = c^{-\epsilon}$, $t = b^{-1}$, $u = a$, and $h = gb$) will be called the “ b -transform.” This multiplies the voltage by a conjugate of γ_b , where $\gamma_b = [b, a][b, c^{-\epsilon}]$.

Subcase 8.5.1. Assume precisely one of γ_a and γ_b is in \mathbb{Z}_p . Write $\{a, b\} = \{x, y\}$, such that $\gamma_x \in \mathbb{Z}_p$ and $\gamma_y \notin \mathbb{Z}_p$. We may assume $\langle \gamma_x \rangle = \mathbb{Z}_p$ (by replacing c with its inverse, if necessary). Choose C' to be either C or the y -transform of C , such that $\Pi C'$ projects nontrivially to \mathbb{Z}_2 . Then choose C'' to be either C' or the x -transform of C' , such that $\Pi C''$ generates G' , so the Factor Group Lemma (2.8) applies.

Subcase 8.5.2. Assume γ_a and γ_b are both in \mathbb{Z}_p . Since $[a, b]$, $[a, c]$, and $[b, c]$ cannot all be in \mathbb{Z}_p , this assumption implies that none of them are in \mathbb{Z}_p . Therefore, since the path L has odd length, we see that ΠC has nontrivial projection to \mathbb{Z}_2 .

We may assume (by replacing c with its inverse, if necessary), that γ_a has nontrivial projection to \mathbb{Z}_p , so $\langle \gamma_a \rangle = \mathbb{Z}_p$. Therefore, by choosing C' to be either C or the a -transform of C , such that $\Pi C'$ generates G' , we may apply the Factor Group Lemma (2.8).

Subcase 8.5.3. Assume neither γ_a nor γ_b is in \mathbb{Z}_p , and b centralizes G' . Note that the sum of the exponents of the occurrences of b in L is 1, and the sum of the exponents of the occurrences of c is 0. Therefore, since b and c centralize G' , Lemma 3.6 implies that $\Pi C = [a, b]$. Hence, we may assume $[a, b] \in \mathbb{Z}_p$ (for otherwise $\langle \Pi C \rangle = G'$, so the Factor Group Lemma (2.8) applies). Then, by the assumption of this subcase, we conclude that

$[a, c] \notin \mathbb{Z}_p$. So we may assume $\langle [a, c] \rangle = \mathbb{Z}_2$, for otherwise b and c could be interchanged, resulting in a situation in which $[a, b] \notin \mathbb{Z}_p$, and which has therefore already been covered. Also, since $[a, b] \in \mathbb{Z}_p$ and $[a, c] \notin \mathbb{Z}_p$, Corollary 3.8 tells us that ℓ is even (and recall that $\ell \neq 2$).

Since $[a, b]$ is a nontrivial element of \mathbb{Z}_p , and b centralizes G' , we see from Corollary 3.7 that $|b|$ is divisible by p . Therefore, $|b| \neq 2$, so we may assume $|\bar{b}| > 2$ (for otherwise Case 4.1 applies with $s = b$ and $t = b^{-1}$). Since $|\bar{b}| \neq 3$ (by the assumption of this case), this implies $n = |\bar{b}| \geq 4$, so we may let

$$L_0 = (c^{\ell-1}, b, c^{-(\ell-1)}, b^2, (b^{n-4}, c, b^{-(n-4)}, c)^{\ell/2} \#, b^{-1}, c^{-(\ell-2)}),$$

so L_0 is a hamiltonian path from \bar{e} to \bar{b}^2c in $\text{Cay}(\overline{G}/\langle \bar{a} \rangle; \{b, c\})$. Note that the sum of the exponents of the occurrences of b in L is 2, and the sum of the exponents of the occurrences of c is 1. Therefore, since b and c centralize G' , Lemma 3.6 implies $\Pi(L_0, a, L_0^{-1}, a) = [a, b]^2[a, c]$. This generates G' , so the Factor Group Lemma (2.8) applies.

Subcase 8.5.4. Assume neither γ_a nor γ_b is in \mathbb{Z}_p , and b does not centralize \mathbb{Z}_p . From Assumption 8.1, we know $\bar{b} = 2$ (so b must invert G').

We may assume $[a, c] \in \mathbb{Z}_2$, for otherwise Case 8.4 could be applied by interchanging b and c . Then we may assume $[a, c]$ is the nontrivial element of \mathbb{Z}_2 , for otherwise the assumption that $\gamma_a \notin \mathbb{Z}_p$ implies $\langle [a, b] \rangle = G'$, so $\langle a, b \rangle \triangleleft G$, and then Lemma 2.13 applies with $s = c$.

By applying the same argument, with a and b interchanged, we may assume $[b, c]$ is also the nontrivial element of \mathbb{Z}_2 . This implies $[a, b] \in \mathbb{Z}_p$, since $\gamma_b \notin \mathbb{Z}_p$.

Note that, since a and b both have order 2 (and invert G'), the image of $\langle a, b \rangle$ in G/\mathbb{Z}_2 is the dihedral group of order $2p$. Also, the preceding two paragraphs imply that c is in the center of G/\mathbb{Z}_2 . Therefore, we have the following hamiltonian cycle in $\text{Cay}(G/\mathbb{Z}_2; S)$:

$$C = (c, (c^{\ell-2}, a, c^{-(\ell-2)}, b)^p \#, c^{-1}, (a^{-1}, b^{-1})^p \#).$$

Since $[a, b]$ projects trivially to \mathbb{Z}_2 , Corollary 3.8 implies that ℓ is even, so, calculating modulo \mathbb{Z}_p , we have

$$\begin{aligned} \Pi C &= c(c^{\ell-2}ac^{-(\ell-2)}b)^pb^{-1}c^{-1}(a^{-1}b^{-1})^pb \\ &\equiv c(ab)^pb^{-1}c^{-1}(a^{-1}b^{-1})^pb && \left(\begin{array}{l} \ell - 2 \text{ is even, so } c^{\ell-2} \\ \text{is central modulo } \mathbb{Z}_p \end{array} \right) \\ &\equiv z^{2p-1}(ab)^pb^{-1}(a^{-1}b^{-1})^pb && \left(\begin{array}{l} \text{letting } z = [a, c] = [b, c] \text{ be} \\ \text{the nontrivial element of } \mathbb{Z}_2 \end{array} \right) \\ &\equiv z && (z^2 = e \text{ and } [a, b] \in \mathbb{Z}_p). \end{aligned}$$

Since this generates \mathbb{Z}_2 , the Factor Group Lemma (2.8) applies. □

9 Cases with $|\bar{a}| = 2$ and $\#S \geq 4$

Assumption 9.1. In this section, we assume

- $\#S \geq 4$, and
- $|\bar{s}| = 2$, for all $s \in S$, such that s does not centralize G' .

We also assume Case 4.1 does not apply. (So $|s| = 2$.)

Furthermore, we assume $\bar{b} \notin \langle \bar{a} \rangle$ (otherwise, Case 4.1 applies). Then it is easy to see that we also have $\bar{a} \notin \langle \bar{b} \rangle$.

Outline. This final section of the proof is longer than the others, so here is an outline of the cases and subcases that it considers.

9.4: Assume no element of S centralizes G' .

9.4.1: Assume $\#S \geq 5$.

9.4.2: Assume $\#S = 4$.

9.5: Assume there exists $s \in S$, such that $[a, s] \notin \mathbb{Z}_p$, and, in addition, either $s = b$, or b centralizes G' , or $\mathbb{Z}_p \subseteq \langle S \setminus \{a\} \rangle'$.

9.5.1: Assume $\mathbb{Z}_p \not\subseteq \langle S \setminus \{a\} \rangle'$.

9.5.2: Assume $\mathbb{Z}_p \subseteq \langle S \setminus \{a\} \rangle'$.

9.6: Assume b centralizes G' .

9.6.1: Assume there exists $c \in S$, such that $[c, b] \notin \mathbb{Z}_p$.

9.6.2: Assume $[c, b] \in \mathbb{Z}_p$ for all $c \in S$.

9.7: Assume that none of the preceding cases apply.

Since Case 9.4 does not apply, some element c of S centralizes G' .

9.7.1: Assume $\langle [s, c] \rangle \neq \mathbb{Z}_2$, for some $s \in S \setminus \{c\}$.

9.7.2: Assume $\langle [s, c] \rangle = \mathbb{Z}_2$, for all $s \in S \setminus \{c\}$.

Notation 9.2. Let $n = |\bar{b}|$ and $\ell = |\bar{G} : \langle \bar{a}, \bar{b} \rangle| = |\bar{G}|/(2n)$.

Note 9.3. The irredundance of S implies $S \setminus \{a, b\}$ is an irredundant generating set for $\bar{G}/\langle \bar{a}, \bar{b} \rangle$ (see Corollary 3.5), so $\ell \geq 4$.

Case 9.4. Assume no element of S centralizes G' .

Proof. From Assumption 9.1, we see that every element of S inverts G' (and has order 2). We may assume no two elements of S commute, for otherwise it is not difficult to see that Lemma 2.13 applies.

Let $c, d \in S \setminus \{a, b\}$, and let $\gamma = [a, b][a, c]$. We claim that we may assume $\gamma \notin \mathbb{Z}_2$, by permuting b, c, d . To this end, first note that if $\gamma \in \mathbb{Z}_2$, then $\mathbb{Z}_p \subseteq \langle [a, c] \rangle$, so there is no harm in putting c into the role of b . Now, let us suppose $[a, b][a, c]$, $[a, c][a, d]$, and $[a, d][a, b]$ are all in \mathbb{Z}_2 . Then

$$[a, b] \equiv [a, c]^{-1} \equiv [a, d] \equiv [a, b]^{-1} \pmod{\mathbb{Z}_2},$$

which contradicts the fact that $[a, b] \notin \mathbb{Z}_2$ (and p is odd).

Let

$$C = ((c, a, c, b)^2 \# d)^2,$$

so C is a hamiltonian cycle in $\text{Cay}(\langle \bar{a}, \bar{b}, \bar{c}, \bar{d} \rangle; \{a, b, c, d\})$ that contains the vertex-disjoint paths $[\bar{e}](c, a, c)$, $[\bar{abc}](a)$, $[\bar{bd}](c, a, c)$, and $[\bar{acd}](a)$. Applying Lemma 2.12 to the paths

$[\bar{e}](c, a, c)$ and $[\overline{abc}](a)$ (so $s = c, t = a, u = b$, and $h = bc$) will multiply the voltage by γ . Applying Lemma 2.12 to the other two paths $[\overline{bd}](c, a, c)$ and $[\overline{acd}](a)$ (so $s = c, t = a, u = b$, and $h = cd$) will also multiply the voltage by γ (because bc and cd both centralize G'). Therefore, applying Lemma 2.12 twice yields a hamiltonian cycle C'' , such that $(\Pi C) \cdot C'' = \gamma^2$, which is a generator of \mathbb{Z}_p .

Subcase 9.4.1. Assume $\#S \geq 5$. If there exist $s, t \in S$, such that $s \notin \{a, b, c\}$, and $[s, t] \notin \mathbb{Z}_p$, then the preceding paragraph implies that Lemma 3.15(2) applies.

Thus, we may assume that the preceding condition does not apply (for any legitimate choice of a, b , and c). Fix two elements $x, y \in S \setminus \{a, b, c\}$. The failure of the condition implies $[x, S] \subseteq \mathbb{Z}_p$. In particular, $[x, y]$ must be a generator of \mathbb{Z}_p (because no two elements of S commute), so we may let $\{x, y\}$ play the role of $\{a, b\}$. So we may let $\{x, y, b, c\}$ play the role of $\{a, b, c, d\}$. Then, since $a \notin \{x, y, b, c\}$, the failure of the condition implies $[a, S] \subseteq \mathbb{Z}_p$. Similarly, $[b, S]$ and $[c, S]$ are also in \mathbb{Z}_p . So $[s, t] \subseteq \mathbb{Z}_p$ for all $s, t \in S$. This contradicts the fact that $\langle [S, S] \rangle = G' \not\subseteq \mathbb{Z}_p$.

Subcase 9.4.2. Assume $\#S = 4$. For convenience, in this subcase (and only in this subcase), we drop our standing assumption that $\langle [a, b] \rangle$ contains \mathbb{Z}_p . Instead, choose $b, d \in S$, such that $[b, d]$ projects nontrivially to \mathbb{Z}_2 . A straightforward calculation (using the fact that a, b, c , and d all invert G') shows that

$$\Pi C = [c, d]^4 [d, a]^2 [d, b].$$

Since $[d, b]$ projects nontrivially to \mathbb{Z}_2 , but $[c, d]^4$ and $[d, a]^2$ have even exponents, so they obviously do not, we see that $\mathbb{Z}_2 \subseteq \langle \Pi C \rangle$. Therefore, we may assume $\Pi C \in \mathbb{Z}_2$, for otherwise the Factor Group Lemma (2.8) applies.

We may assume $\gamma \in \mathbb{Z}_2$, for otherwise applying Lemma 2.12 twice (as in the paragraph immediately before Subcase 9.4.1) yields a hamiltonian cycle whose voltage generates G' , so the Factor Group Lemma (2.8) applies. By the definition of γ , this means $[a, b][a, c] \in \mathbb{Z}_2$. And we may assume the same is true when b and d are interchanged, which means $[a, d][a, c] \in \mathbb{Z}_2$. So

$$[a, b] \equiv [a, c]^{-1} \equiv [a, d] \pmod{\mathbb{Z}_2}.$$

By interchanging a and c , we conclude that we may also assume

$$[c, b] \equiv [c, a]^{-1} \equiv [c, d] \pmod{\mathbb{Z}_2}.$$

So

$$[c, d] \equiv [c, a]^{-1} = [a, c] \equiv [a, d]^{-1} = [d, a] \pmod{\mathbb{Z}_2}.$$

Therefore

$$[d, a]^6 [d, b] = [d, a]^4 [d, a]^2 [d, b] \equiv [c, d]^4 [d, a]^2 [d, b] = \Pi C \equiv 0 \pmod{\mathbb{Z}_2}.$$

If $p \neq 3$, then, since we may assume the same is true when we interchange a and c , we conclude that $[d, c] \equiv [d, a] \pmod{\mathbb{Z}_2}$. Since we also have $[c, d] \equiv [d, a] \pmod{\mathbb{Z}_2}$, we conclude that $[c, d]$ and $[a, d]$ are in \mathbb{Z}_2 . This implies $[b, d] \notin \mathbb{Z}_2$ (since d does not centralize \mathbb{Z}_p , and is therefore not in the center of G/\mathbb{Z}_2), so

$$\Pi C = [c, d]^4 [d, a]^2 [d, b] \equiv e^4 e^2 [d, b] = [d, b] \neq 0 \pmod{\mathbb{Z}_2}.$$

This contradicts the fact that $\Pi C \in \mathbb{Z}_2$.

We now assume $p = 3$. Then the equation $[d, a]^6[d, b] \equiv 0 \pmod{\mathbb{Z}_2}$ implies $[d, b] \in \mathbb{Z}_2$. This conclusion came from assuming only that $[d, b] \notin \mathbb{Z}_p$. Therefore, for all $s, t \in S$, the commutator $[s, t]$ must be in either \mathbb{Z}_2 or \mathbb{Z}_p . However,

$$[a, b] \equiv [c, a] \equiv [a, d] \equiv [b, c] \equiv [d, c] \pmod{\mathbb{Z}_2},$$

and $[a, b] \notin \mathbb{Z}_2$. Therefore, we conclude all five of these other commutators are in \mathbb{Z}_p . (Therefore, the stated congruences between these commutators are actually equalities.)

Now, interchanging $a \leftrightarrow b$ and $c \leftrightarrow d$ in C yields a hamiltonian cycle C^* , such that

$$\Pi C^* = [d, c]^4[c, b]^2[c, a] = [d, c][b, c][c, a] = [c, a]^3 = e$$

(because $p = 3$). Let $\gamma^* = [b, a][b, d]$, so γ^* is obtained from $\gamma = [a, b][a, c]$ by interchanging $a \leftrightarrow b$ and $c \leftrightarrow d$. Then, since applying Lemma 2.12 to C can multiply the voltage by $\gamma = [a, b][a, c]$, we know that applying Lemma 2.12 to C^* can multiply the voltage by γ^* , which generates G' . So the Factor Group Lemma (2.8) applies. \square

Case 9.5. Assume there exists $s \in S$, such that $[a, s] \notin \mathbb{Z}_p$, and:

$$\text{either } s = b, \text{ or } b \text{ centralizes } G', \text{ or } \mathbb{Z}_p \subseteq \langle S \setminus \{a\} \rangle'.$$

Proof. Let $S_0 = S \setminus \{a\}$. Note that the irredundance of S implies $a \notin \langle S_0 \rangle \mathbb{Z}_2$ (see Lemma 3.4).

Subcase 9.5.1. Assume $\mathbb{Z}_p \not\subseteq \langle S_0 \rangle'$. If $[a, b] \notin \mathbb{Z}_p$, we assume that $s = b$. Let

$$g = \begin{cases} s & \text{if } [s, a] \notin \mathbb{Z}_2, \\ sb^2 & \text{if } [s, a] \in \mathbb{Z}_2. \end{cases}$$

Note that $\langle [g, a] \rangle = G'$.

Let $H^* = \langle S_0 \rangle \mathbb{Z}_2 / \mathbb{Z}_2$. From the assumption of this subcase, we know that H^* is abelian. Therefore, Corollary 2.11 provides a hamiltonian path $L = (s_i)_{i=1}^r$ in $\text{Cay}(\overline{H^*}; S_0)$, such that $s_1 s_2 \cdots s_r \in g \mathbb{Z}_2$. Then (L^{-1}, a, L, a) is a hamiltonian cycle in $\text{Cay}(\overline{G}; S)$, and

$$\Pi C = [s_1 s_2 \cdots s_r, a] \in [g \mathbb{Z}_2, a] = \{[g, a]\}$$

(since \mathbb{Z}_2 is in the center of G). This voltage generates G' , so the Factor Group Lemma (2.8) applies.

Subcase 9.5.2. Assume $\mathbb{Z}_p \subseteq \langle S_0 \rangle'$. Suppose $w, x, y \in S^{\pm 1} \setminus \{a\}$, such that

$$\langle \overline{w} \rangle \subsetneq \langle \overline{w}, \overline{x} \rangle \subsetneq \langle \overline{w}, \overline{x}, \overline{y} \rangle. \tag{9.5A}$$

It is easy to construct a hamiltonian cycle C_0 in $\text{Cay}(\overline{S_0}; S_0)$, such that C_0 contains the oriented paths $[\overline{hw^{-1}y^{-1}}](w, x, w^{-1})$ and $[\overline{hx}](x^{-1})$, for some $h \in G$. Furthermore, if

$$\text{either } x \notin \{s^{\pm 1}\} \text{ or } |\overline{G}| > 16, \tag{9.5B}$$

then, for some $\epsilon \in \{\pm 1\}$, it is not difficult to arrange that the hamiltonian cycle C_0 contains the oriented edge $[\overline{s^\epsilon}](s^{-\epsilon})$, and that this edge is not in either of the above-mentioned paths.

Applying Lemma 2.12 to the first two paths (so $s = w$, $t = x$, and $u = y$) yields a hamiltonian cycle C_1 , such that $(\Pi C_0)^{-1}(\Pi C_1)$ is conjugate to $[x^{-1}, y][w, x^{-1}]^y$. Removing the edge $[s^\epsilon](s^{-\epsilon})$ yields hamiltonian paths $C_0\#$ and $C_1\#$ from \bar{e} to \bar{s}^ϵ .

From Lemma 3.4 and the assumption of this subcase, we see that $\langle \bar{S}_0 \rangle \neq \bar{G}$. So

$$C_0^+ = (C_0\#, a, (C_0\#)^{-1}, a) \text{ and } C_1^+ = (C_1\#, a, (C_1\#)^{-1}, a)$$

are hamiltonian cycles in $\text{Cay}(\bar{G}; S)$. For $k = 0, 1$, we have

$$\Pi C_k^+ = [((\Pi C_k)s^\epsilon)^{-1}, a].$$

Since $\Pi C_k \in G'$, and G' is central modulo \mathbb{Z}_p (and from the choice of s), we have

$$\Pi C_k^+ \equiv [s^\epsilon, a] \not\equiv e \pmod{\mathbb{Z}_p}.$$

Furthermore, if $[x^{-1}, y][w, x^{-1}]^y$ projects nontrivially to \mathbb{Z}_p , then $(\Pi C_0^+)^{-1}(\Pi C_1^+)$ does not centralize a modulo \mathbb{Z}_2 , so ΠC_0^+ and ΠC_1^+ are not both in \mathbb{Z}_2 . This implies that ΠC_k^+ generates G' for some k , so the Factor Group Lemma (2.8) applies. Therefore (after replacing x^{-1} with x for simplicity), we may assume

$$[w, x]^y [x, y] \in \mathbb{Z}_2 \text{ for all } w, x, y \in S^{\pm 1} \setminus \{a\} \text{ that satisfy (9.5A) and (9.5B)}. \quad (9.5C)$$

We will show that this leads to a contradiction.

Assume, for the moment, that b centralizes G' . Then $n = |\bar{b}| > 2$ (because Corollary 3.7 implies that $|b| \neq 2$), so $|\bar{G}| = 2n\ell > 2 \cdot 2 \cdot 4 = 16$. Therefore (9.5B) is automatically satisfied. Let $x, y \in S_0 \setminus \{b\}$, such that $x \neq y$. We see from Note 9.3 that (9.5A) is satisfied for $w = b^{\pm 1}$, so (9.5C) tells us

$$[b, x]^y [x, y] \text{ and } [b^{-1}, x]^y [x, y] \text{ are both in } \mathbb{Z}_2.$$

However, we also know that $[b^{-1}, x] = [b, x]^{-1}$ (because we are assuming in this paragraph that b centralizes G'). Therefore

$$[b, x]^y \equiv [x, y]^{-1} \equiv [b^{-1}, x]^y = ([b, x]^{-1})^y \pmod{\mathbb{Z}_2},$$

so $[b, x] \in \mathbb{Z}_2$ (for all $x \in S_0$). Then, since $[b, x]^y [x, y] \in \mathbb{Z}_2$, we conclude that $[x, y] \in \mathbb{Z}_2$, for all $x, y \in S_0$. This contradicts the assumption of this subcase.

Now assume b does not centralize G' . We may assume Case 9.4 does not apply, so G' is centralized by some $t \in S$ (and $t \neq b$). Let $w, x \in S_0 \setminus \{t\}$ with $w \neq x$. Combining the irredundance of S with the fact that $t \neq b$ implies that (9.5A) is satisfied for $y = t^{\pm 1}$ (unless $\bar{w} = \bar{x}$, when Case 4.1 applies). We may assume $x \neq s$ (by interchanging w and x , if necessary), so (9.5B) is satisfied. Then (9.5C) tells us

$$[w, x]^t [x, t] \text{ and } [w, x]^{t^{-1}} [x, t^{-1}] \text{ are both in } \mathbb{Z}_2.$$

Since t centralizes G' , this implies $[x, t] \equiv [x, t^{-1}] = [x, t]^{-1} \pmod{\mathbb{Z}_2}$, so $[x, t] \in \mathbb{Z}_2$ (for all $x \in S_0$). Since $[w, x]^t [x, t] \in \mathbb{Z}_2$, this implies $[w, x] \in \mathbb{Z}_2$ (for all $w, x \in S_0$). This contradicts the assumption of this subcase. \square

Case 9.6. Assume b centralizes G' .

Proof. We consider two subcases.

Subcase 9.6.1. Assume there exists $c \in S$, such that $[c, b] \notin \mathbb{Z}_p$. We use some of the arguments of Case 8.5. We may assume $[a, s] \in \mathbb{Z}_p$ for all $s \in S$. (Otherwise, Case 9.5 applies, because b centralizes G' .) Therefore $c \neq a$. Let $L = (s_i)_{i=1}^r$ be a hamiltonian path from \bar{c} to \bar{b} in $\text{Cay}(\overline{G}/\langle \bar{a} \rangle; S \setminus \{a\})$, such that $s_1 = c = s_r^{-1}$, and L contains a path of the form $[\overline{gc^\epsilon}](c^{-\epsilon}, b^\delta, c^\epsilon)$ (for some $\delta, \epsilon \in \{\pm 1\}$) that is vertex-disjoint from $\{\bar{c}, \bar{c}, \bar{b}, \bar{bc}\}$. Now let $C = (L, a, L^{-1}, a)$. Then C contains vertex-disjoint paths of the form

$$[\bar{b}](a), \quad [\overline{ca}](c^{-1}, a, c), \quad [\overline{gc^\epsilon}](c^{-\epsilon}, b^\delta, c^\epsilon), \quad \text{and} \quad [\overline{gab^\delta}](b^{-\delta}).$$

- Applying Lemma 2.12 to $[\bar{b}](a)$ and $[\overline{ca}](c^{-1}, a, c)$ (so $s = c^{-1}, t = a, u = b$, and $h = ab$) will be called the “ a -transform.” It multiplies the voltage by

$$\gamma_a = [b, a][a, c^{-1}].$$

- Applying Lemma 2.12 to $[\overline{gc^\epsilon}](c^{-\epsilon}, b^\delta, c^\epsilon)$ and $[\overline{gab^\delta}](b^{-\delta})$ (so $s = c^{-\epsilon}, t = b^\delta, u = a$, and $h = ga$) will be called the “ b -transform.” It multiplies the voltage by a conjugate of

$$\gamma_b = [a, b][c^{-\epsilon}, b].$$

Since $[a, b], [a, c] \in \mathbb{Z}_p$ and $[b, c] \notin \mathbb{Z}_p$ we know $\gamma_a \in \mathbb{Z}_p$ and $\gamma_b \notin \mathbb{Z}_p$. Also, we may also assume γ_a is nontrivial (by replacing b with b^{-1} if necessary). Therefore, the argument of Subcase 8.5.1 applies. Namely, choose C' to be either C or the b -transform of C , such that $\Pi C'$ projects nontrivially to \mathbb{Z}_2 . Then choose C'' to be either C' or the a -transform of C' , such that $\Pi C''$ generates G' , so the Factor Group Lemma (2.8) applies.

Subcase 9.6.2. Assume $[c, b] \in \mathbb{Z}_p$ for all $c \in S$. Choose $c, d \in S$, such that $[c, d] \notin \mathbb{Z}_p$. Assuming that Case 9.5 and Subcase 9.6.1 do not apply, we have

$$[s, t] \in \mathbb{Z}_p \text{ for all } s \in \{a, b\} \text{ and } t \in S.$$

Therefore, $c, d \notin \{a, b\}$, and the element $\gamma = [a, b][d^{-1}, a]$ is in \mathbb{Z}_p , and we may assume (by replacing b with its inverse, if necessary) that γ generates \mathbb{Z}_p .

Let $S_0 = \{a, b, d\}$, and choose a hamiltonian cycle C_0 in $\text{Cay}(\overline{S_0}; S_0)$ that contains the oriented paths $[\bar{d}](d^{-1}, a, d)$ and $[\bar{ab}](a)$, and has at least two edges labelled $x^{\pm 1}$, for every $x \in S_0$. Lemma 2.12 (with $s = d^{-1}, t = a, u = b$, and $h = b$) provides a hamiltonian cycle C_1 , such that $(\Pi C_0)^{-1}(\Pi C_1)$ is conjugate to γ , and therefore generates \mathbb{Z}_p . Furthermore, C_1 contains all of the oriented edges of C_0 that are not in these two above-mentioned paths, so Lemma 3.15(2) applies (with $g = b$ and $t = d$). \square

Case 9.7. Assume that none of the preceding cases apply.

Proof. This implies that:

- #1. $[a, b] \in \mathbb{Z}_p$. (Otherwise, Case 9.5 applies.)
- #2. If $s \in S$, and there exists $t \in S$, such that t inverts G' and $\mathbb{Z}_p \subseteq \langle [s, t] \rangle$, then s inverts G' . (If s does not invert G' , then we see from Assumption 9.1 that s centralizes G' , so Case 9.6 applies with s and t in the roles of b and a , respectively.)

#3. There exists $c \in S$, such that c centralizes G' . (Otherwise, Case 9.4 applies.) From (#2), we know $[a, c] \in \mathbb{Z}_2$.

Subcase 9.7.1. Assume $\langle [s, c] \rangle \neq \mathbb{Z}_2$, for some $s \in S \setminus \{c\}$. Suppose, for the moment, that s centralizes G' . Then Lemma 3.6 implies $[a, [s, c]] = [[a, s], [a, c]] = e$ (because G' is abelian), so $[s, c]$ projects trivially to \mathbb{Z}_p . Since $\langle [s, c] \rangle \neq \mathbb{Z}_2$, we conclude from this that $[s, c] = e$, so Lemma 2.16 applies.

We may now assume s does not centralize G' , so there is no harm in assuming that $s = a$. Since (#2) implies that $[a, c] \in \mathbb{Z}_2$, we see that $[a, c]$ must be trivial. Let $H = \langle S \setminus \{c\} \rangle$. We may assume $\mathbb{Z}_2 \not\subseteq H$, for otherwise $H \triangleleft G$, so Lemma 2.13 applies with $s = c$ and $t = a$. Therefore, $[x, y] \in \mathbb{Z}_p$ for all $x, y \in S \setminus \{c\}$, but there is some $d \in S \setminus \{c\}$, such that $[c, d]$ projects nontrivially to \mathbb{Z}_2 .

Similarly, we may assume $\mathbb{Z}_p \not\subseteq \langle S \setminus \{a\} \rangle$, for otherwise we have $\langle S \setminus \{a\} \rangle \triangleleft G$, so Lemma 2.13 applies with $s = a$ and $t = c$. This means $[x, y] \in \mathbb{Z}_2$ for all $x, y \in S \setminus \{a\}$. In particular, since b and d are in both $S \setminus \{a\}$ and $S \setminus \{c\}$, we must have $[b, d] \in \mathbb{Z}_2 \cap \mathbb{Z}_p = \{e\}$.

Choose a hamiltonian cycle C_0 in $\text{Cay}(\overline{H}; S \setminus \{c\})$ that contains the oriented paths $[\overline{d}](d^{-1}, b, d)$ and $[\overline{ab}](b)$. If we apply Lemma 2.12 to these paths (so $s = d^{-1}$, $t = b$, $u = a$, and $h = a$), then the voltage is multiplied by a conjugate of $[b, a][b, d^{-1}]$, which is a generator of \mathbb{Z}_p (since $[a, b]$ generates \mathbb{Z}_p and $[b, d]$ is trivial). Therefore, Lemma 3.15(1) applies with $s = t = d$ and $u = a$.

Subcase 9.7.2. Assume $\langle [s, c] \rangle = \mathbb{Z}_2$, for all $s \in S \setminus \{c\}$. For convenience, let $\widehat{G} = G/\mathbb{Z}_2$ and $\widehat{H} = \langle \widehat{S} \setminus \{\widehat{c}\} \rangle$. Then $|\widehat{H}'| = p$ is prime, so Theorem 1.1 provides a hamiltonian path L in $\text{Cay}(\widehat{H}; S \setminus \{c\})$. Since \widehat{c} is central in \widehat{G} , there is a spanning subgraph of $\text{Cay}(\widehat{G}; S)$ that is isomorphic to the Cartesian product $L \square (\widehat{c}^{\ell-1})$, where $\ell = |\widehat{G} : \langle S \setminus \{c\} \rangle|$. Since $|\widehat{G}|$ is even, it is easy to find a hamiltonian cycle C in $L \square (\widehat{c}^{\ell-1})$ (see Lemma 2.10), and this yields a hamiltonian cycle \widehat{C} in $\text{Cay}(\widehat{G}; S)$.

To complete the proof, we carry out a straightforward (and well-known) calculation to verify that $\Pi \widehat{C}$ is nontrivial, so the Factor Group Lemma (2.8) applies.

If we view the Cartesian product $L \square (\widehat{c}^{\ell-1})$ as a grid of squares, then the interior of the hamiltonian cycle C is a union of squares of the grid. Graph theoretically, this means C is the connected sum of some number N of digons of the form $[g](t, t^{-1})$ (where $t \in S^{\pm 1}$). Note that if C is an r -cycle (with $r \geq 2$), then $C \#_t^s (t, t^{-1})$ is an $(r+2)$ -cycle. Therefore, since the length of C is $|\widehat{G}|$, we have $2N = |\widehat{G}| \equiv 0 \pmod{4}$, so N is even.

Now, each 4-cycle in $L \square (\widehat{c}^{\ell-1})$ is of the form $[\widehat{g}](s^{-1}, t^{-1}, s, t)$, where one of s and t is in $\{c^{\pm 1}\}$, and the other is in $S^{\pm 1} \setminus \{c^{\pm 1}\}$. This means that in any connected sum $C \#_t^s [g](t, t^{-1})$, one of s and t is in $\{c^{\pm 1}\}$, and the other is in $S^{\pm 1} \setminus \{c^{\pm 1}\}$. By the assumption of this subcase, we conclude that $[s, t] = z$, where z is the generator of \mathbb{Z}_2 . Therefore

$$\begin{aligned} \Pi C &= \Pi \left([\widehat{g}_1](t_1, t_1^{-1}) \#_{t_2}^{s_2} [\widehat{g}_2](t_2, t_2^{-1}) \#_{t_3}^{s_3} \cdots \#_{t_N}^{s_N} [\widehat{g}_N](t_N, t_N^{-1}) \right) \\ &\equiv \prod_{i=2}^N [s_i, t_i] && \text{(Corollary 3.14 and } \Pi(t, t^{-1}) = e) \\ &= z^{N-1} \\ &\not\equiv e \pmod{\mathbb{Z}_p} && (N-1 \text{ is odd).} \quad \square \end{aligned}$$

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