AUGMENTING PHASE SPACE QUANTIZATION TO INTRODUCE ADDITIONAL PHYSICAL EFFECTS

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Abstract

Quantum mechanics can be done using classical phase space functions and a star product. The state of the system is described by a quasi-probability distribution.

A classical system can be quantized in phase space in different ways with different quasi-probability distributions and star products. A transition differential operator relates different phase space quantizations.

The objective of this thesis is to introduce additional physical effects into the process of quantization by using the transition operator. As prototypical examples, we first look at the coarse-graining of the Wigner function and the damped simple harmonic oscillator. By generalizing the transition operator and star product to also be functions of the position and momentum, we show that additional physical features beyond damping and coarse-graining can be introduced into a quantum system, including the generalized uncertainty principle of quantum gravity phenomenology, driving forces, and decoherence.

Acknowledgments

Writing a graduate thesis is a difficult undertaking; it requires perseverance, dedication, and the ability to withstand periods of self-doubt. Completing this thesis feels gratifying for it demonstrates that over the two years of work, I managed to make an original contribution to the study of phase space quantum mechanics and quantization. However, it was also sometimes immensely frustrating when I pursued paths that turned out to be dead ends or pored over my work trying to find missing minus signs.

To make it through a graduate program where a single question consumes most of your attention, there is a need for breaks, such as teaching undergraduate labs, bouldering, playing badminton, or even hanging out with friends by going to the movies or having a games night. There also must be ways to find some reason to laugh each and every day.

Early in my Master's program, my friends and I found out that every day celebrates something, such as cookies, hot air balloons, or Daleks. We began a tradition of finding out what was celebrated each day as well as the birthdays and anniversaries of scientists, inventors and new discoveries. We then combined what we found into a giant holiday, and this led to some rather interesting (absurd) days, including: Fat Squirrel Sled Race Day (February 2), Windmill Diffraction Day (May 10), Stupid Space Monkey Day (May 16), Subatomic Lunar Snack Day (July 21), and Give a Bear a Burger Day (November 16). For this, I would like to thank my penguin diagram on my whiteboard (yes, I'm thanking a Feynman diagram) and the penguin team (you know who you are). You guys tried to keep me sane and did an awful job of it!

I want to thank my supervisor, Dr. Mark Walton, for giving me suggestions as I conducted my research and for reading through my thesis at least two or three times during the editing process. I do believe that my thesis is much better as a result of your many suggestions. Listening to the comments you made will help me as I pursue a PhD program and begin an academic career.

I thank the other members of my supervisory committee (Dr. Saurya Das, Dr. Kent Peacock) for helping me through my graduate program and giving me assistance as I conducted the research. I also thank Dr. Hubert de Guise for his comments on my thesis and useful suggestions.

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Chapter 1

Introduction

Quantum mechanics can be done with several different formulations. For instance, it can be studied using operators and state vectors with wavefunctions in the position or momentum representation [1,2], path integrals [3], or quasi-probability distributions in phase space [4–7]. Studying the same system in different formulations allows for a variety of perspectives and can yield insights on a problem (for example, see [8] or [9]).

One advantage of using phase space quantum mechanics is that phase space functions are used, rather than operators. In this formulation, the probability densities for both the momentum and position are encoded in the quasi-probability distributions. Further, this formulation of quantum mechanics provides a straightforward method to study the relation between different quantizations, which describes the association between the variables of a classical system with those of a quantum system.

In this thesis, we attempt to generalize certain tools of phase space quantization to incorporate additional classical effects that were not present in the original system. We review coarse-graining and the damped quantum harmonic oscillator as prototypical examples. We then take a closer look at the damped harmonic oscillator and explore a method to introduce damping into quantization. Afterwards, we attempt to incorporate further physical effects into quantization.

Review of Quantum Mechanics in Phase Space

With the discovery of quantum mechanics, questions naturally arose over how to convert a classical system into a quantum system that would reduce to the original classical one in the classical limit. One way to to quantize a classical system is to apply a quantization map that turns classical positions and momenta into operators. This map also specifies the order of operators.

Many operator orderings have been studied, including Born-Jordan ordering [10], Weyl ordering [5,7], and symmetric ordering [11]. Refs. [12] and [13] demonstrated that different operator orderings and quantizations can give rise to physically distinct predictions.

One method to study quantization and operator ordering is to use phase space quantum mechanics (see for example, [14–16]). This formulation has emerged as a means to understand quantum systems from a different viewpoint than the position and momentum representation of operator quantum mechanics [17].

Phase space quantum mechanics was originally based on Weyl quantization. In this quantization, the Wigner transform on phase space maps a density operator to a quasi-probability distribution, known as the Wigner function. Quasi-probability distributions are similar to probability distributions, but are able to have negative values. The negative values of the Wigner function can be used to describe how nonclassical a system is. Further, a point (q, p) in phase space quantum mechanics does not have the same meaning as in classical phase space. In classical mechanics, it is possible to measure both the position and momentum simultaneously with infinite precision while Heisenberg's precludes this possibility. Therefore, it is necessary to integrate the Wigner function over the position and momentum to yield physical results; integration over the position yields a non-negative probability density for the momentum while integration over the momentum gives the probability density for the position [6, 18].

The Wigner transform also converts any operator to its Weyl-quantized phase space counterpart. It should be noted that the structure of the operator algebra is preserved in the algebra of phase space functions (formally known as a homomorphism). However, the trade-off for removing operators from this formulation of quantum mechanics is the introduction of a non-commutative binary operation known as the Moyal product, which is the exponentiation of the Poisson bracket multiplied by $\frac{i\hbar}{2}$ [6]. As a result of the presence of the Poisson bracket within the Moyal product, the intrinsic connection between classical mechanics and phase space quantum mechanics is demonstrated.

Application of the Wigner transform to the Liouville-von Neumann equation, which governs the dynamics of the density operator, yields the equation of motion of the Wigner function in phase space quantum mechanics. The time-derivative of the Wigner function is proportional to the Moyal bracket of the Hamiltonian and Wigner function. It consists of, in general, an infinite sum of derivatives of the Hamiltonian and the Wigner function with respect to the position and momentum [6, 19].

For stationary states, the Moyal bracket of the Hamiltonian and Wigner function is replaced with a similar equation, the stargenvalue equation. The stargenvalue equation is the Wigner transformation of the time-independent Schrödinger equation, $\hat{H}\hat{\rho} = E\hat{\rho}$, where $\hat{H}$ is the Hamiltonian, $\hat{\rho}$ is the density operator, and E is the energy [6, 19].

To determine the time-independent quasi-probability distributions, the stargenvalue equations can be used. Time-dependency can then be introduced by propagators, which are the Wigner transforms of the propagators in operator quantum mechanics. The time-dependent Wigner function is the Wigner transform of the time-dependent density operator [6].

Just as in operator quantum mechanics, the simple harmonic oscillator has been thoroughly analyzed in phase space quantum mechanics. This Wigner function can be derived both with the stargenvalue equation and with the Wigner transforms of creation and annihilation operators [6]. Other systems that have been studied in phase space quantum mechanics include the linear potential [19, 20], hydrogen atom [21–23], and the Morse potential [9, 24, 25].

It is possible to describe other quantizations with phase space quantum mechanics. Each quantization corresponds to a distinct quasi-probability distribution and binary operation (more generally known as a star product). Quasi-probability distributions of different quantizations can be related to the Wigner function by applying a transition operator (a differential operator), which was investigated by Bayen *et al.* in [26, 27]. In their seminal work, the mathematical study of deformation theory was applied to phase space quantum mechanics. Hence, sometimes phase space quantum mechanics is called deformation quantization. This technique has been used to examine known problems, such as the hydrogen atom, from a different perspective [28].

Further, [27] demonstrated that the transition operator can be used to transform a star product from one quantization to another. For example, the transition operator of $T_S = e^{i\hbar\partial_q\partial_p/2}$ converts the Moyal product of Weyl quantization into the standard star product $\star_S = e^{i\hbar\overleftarrow{\partial}_q\overrightarrow{\partial}_p}$ corresponding to standard ordering. Hence, it is straightforward to analyze quantization in phase space quantum mechanics; all that is necessary is to apply a transition operator.

As mentioned, the transition operator applied to the Wigner function will yield the quasi-probability distribution of a different quantization. However, these quasi-probability distributions can be obtained with another method. The Wigner transform can be modified such that it also includes a weight function that is determined by the quantization under consideration. The resultant quasi-probability distributions can then be found [29]. Weight functions have been used to study the resultant physical implications of different orderings and quantizations [18, 29]. It should be mentioned that this method of using the weight function is equivalent to applying a transition operator to the Wigner function [5].

Applications of Phase Space Quantum Mechanics

Phase space quantum mechanics has been used to introduce physical features into a quantum system that were not part of the original system. For example, the Husimi distribution studied by [30] and [31] is a Gaussian smoothed (coarse-grained) Wigner function. The Husimi distribution can be found by applying a transition operator to the Wigner function. This indicates that the transition operators can go beyond quantization and bring about additional classical effects.

Usually, the star product and transition operator depend only on the derivatives of the position and the momentum. However, they have been generalized to also be functions of the position and momentum coordinates themselves. Using weight functions, [13] showed that there is then greater freedom in quantizing monomials. Ref. [32] tried to develop a gauge theory with a position and momentum-dependent star product.

Ref. [33] suggested a method of quantizating dissipative systems by using the transition operator to incorporate damping into an initially undamped system in phase space quantum mechanics. Similar to the Husimi distribution, additional physics were brought into the system by the transition operator. With [30] and [31], this was coarse-graining, but with [33], it was damping because these features were not part of the original system. To differentiate between mapping a classical system to a quantum system (quantization) and introducing extra physics during the mapping that was not present in the original system, we refer to the latter case as augmented quantization.

Using this terminology, [33] desired to augment the quantization of a classical harmonic oscillator in such a way that it could be mapped to a linearly damped quantum harmonic oscillator. The result was the introduction of the damping parameter within the transition operator itself, which also gave a damped star product.

Phase space quantum mechanics and its methods have also been applied to many different subjects, such as condensed matter physics [34–36], quantum chaos [37, 38], the classical and semiclassical limit of quantum mechanics [39–41], spin [42–44], quantum dynamics [45,46] field theory [32,47,48], and M-theory [49–51]. In particular, in quantum optics, quantum mechanics in phase space has been used to study quantum interference as well as coherent and squeezed states [4, 52, 53]. In fact, there are techniques to experimentally determine the Wigner function [54–57]. Further, the theory of environmental decoherence (the emergence of classical mechanics from quantum mechanics) has been investigated with phase space quantum mechanics and the Wigner function [58–61].

Outline of the Thesis

In Chapter 2, we present the fundamentals of phase space quantization, first focusing on generalities, then specifically considering Weyl quantization, the Wigner function, and the Moyal product. Next, the existence of other star products and distribution functions are demonstrated using the transition operator.

Then, in Chapter 3, we concentrate on the Husimi distribution and coarse graining. We review this phase space distribution and its relationship to the minimum uncertainty wave packet. We then briefly consider a generalization of the Husimi distribution. Smoothing in the $n \to \infty$ limit of the Wigner function is also investigated, where n is the energy level.

In Chapter 4, we first review the results of [33], then consider an alternate method of augmented quantization to introduce damping into a quantum system. Modifying the technique in [33], we show that this requires the generalization of the transition operator and star product so that they also depend upon the position and momentum. It is this position and momentum dependence which forms the basis for many of our results.

We then demonstrate that augmented quantization is not limited to dissipation and coarse-graining, but can also be used to incorporate other physical features into a system. To illustrate this point, we suggest a transition operator and star product that incorporates the generalized uncertainty principle of quantum gravity phenomenology.

In Chapter 5, time-dependent transition operators are considered. We propose an augmented quantization mapping the simple harmonic oscillator to a quantum mechanical driven harmonic oscillator. This requires a time-dependent version of the local transition operator. It is then shown that decoherence can also be introduced into quantum systems with a time-dependent transition operator.

A summary of our results and possible future directions of our work are provided in Chapter 6, our Conclusion. We also comment upon the significance of our research in the overall understanding of quantum mechanics.

Chapter 2

Phase Space Quantization

2.1 Motivation

Consider a classical system that uses the function qp, where q is the position and p is the momentum. Mapping this function to a quantum mechanical system with operators is a non-trivial problem. For example, qp can be mapped to, $\hat{q}\hat{p}$, $\hat{p}\hat{q}$, or $\frac{1}{2}\left(\hat{q}\hat{p}+\hat{p}\hat{q}\right)$. Throughout this thesis, carets will refer to operators while phase space functions will not have a caret.[1] For instance, the Hamiltonian operator is $\hat{H}(\hat{q},\hat{p})$ and its phase space counterpart is $H(q,p)$.

As another example, q^3p can become $\hat{q}^3\hat{p}$, $\hat{p}\hat{q}^3$, or $\frac{1}{4}\left(\hat{q}^3\hat{p}+\hat{q}^2\hat{p}\hat{q}+\hat{q}\hat{p}\hat{q}^2+\hat{p}\hat{q}^3\right)$, among many other possibilities. The procedure of associating a classical system to a quantum mechanical one is known as quantization. In the case of operator quantum mechanics, this association also specifies an operator ordering.

One reason quantization is difficult is that imposing seemingly realistic properties can lead to contradictory results. Let $\mathcal{Q}$ be a map taking a function f to its operator counterpart, $\hat{f}$. As discussed in [62–64], the conditions

1. $\mathcal{Q}(1)=\hat{I}$, where I is the identity operator
2. $\mathcal{Q}(\{f,g\})=\frac{1}{i\hbar}[\mathcal{Q}(f),\mathcal{Q}(g)]$, where $\{\cdot,\cdot\}$ is the Poisson bracket and $[\cdot,\cdot]$ is a commutator (2.1)
3. $\mathcal{Q}(f(x))=f(\mathcal{Q}(x))$, where x is q or p

[1] The terms symbol, classical function, and c-function can also refer to a phase space function [7].

give rise to contractions. For instance, $\{q^3,p^3\}+\frac{1}{12}\{\{p^2,q^3\},\{q^3,p^3\}\}=0$. However, quantizing this expression using equation (2.1) gives a left-hand side that evaluates to a non-zero result, which is incorrect because $\mathcal{Q}(0)=0$ [65].

Several methods of quantizing a classical system exist (see, for instance, [7, 10]). In Sections 2.2-2.4, we consider operator quantization. The remainder of the thesis will be devoted to phase space quantization.

2.2 Properties of the Operator Quantization Map

There are several variations of operator ordering, including:

- Weyl ordering [66]

$$q^r p^s \to \frac{1}{2^r}\sum_{\ell=0}^{r}\binom{r}{\ell}\hat{q}^{r-\ell}\hat{p}^s\hat{q}^{\ell} = \frac{1}{2^s}\sum_{\ell=0}^{s}\binom{s}{\ell}\hat{p}^{s-\ell}\hat{q}^r\hat{p}^{\ell}\,. \tag{2.2}$$

  Weyl ordering is found by permuting $\hat{q}^r\hat{p}^s$ in all possible ways and averaging over the $(r+s)!$ permutations.

- Born-Jordan ordering [10]

$$q^r p^s \to \frac{1}{s+1}\sum_{k=0}^{s}\hat{p}^{s-k}\hat{q}^r\hat{p}^k = \frac{1}{r+1}\sum_{k=0}^{r}\hat{q}^{r-k}\hat{p}^s\hat{q}^k\,. \tag{2.3}$$

We show the ordering rules[2] of Weyl and Born-Jordan in Appendix A.

It is possible to unify these orderings within a single scheme, such that [10, 67]

$$q^r p^s \to \int_0^1 d\tau\, f(\tau)\sum_{k=0}^{s}\binom{s}{k}(1-\tau)^k\tau^{s-k}\hat{p}^{s-k}\hat{q}^r\hat{p}^k\,. \tag{2.4}$$

For $f(\tau)=\delta(\tau-\frac{1}{2})$, Weyl ordering is recovered. In the case of Born-Jordan ordering,

[2]Sometimes, the terminology of correspondence rule is also used in place of ordering rule. Application of a quantization map to a monomial gives the correspondence rule, which then defines the operator ordering [7].

$f(\tau) = 1$, so that

$$\int_0^1 d\tau \sum_{k=0}^{s} \binom{s}{k} (1-\tau)^k \tau^{s-k} \hat{p}^{s-k} \hat{q}^r \hat{p}^k = \frac{1}{s+1} \sum_{k=0}^{s} \hat{p}^{s-k} \hat{q}^r \hat{p}^k \,. \tag{2.5}$$

by using the fact that [10, 68]

$$\int_0^1 d\tau \, (1-\tau)^k \tau^{s-k} = \frac{(s-k)!k!}{(s+1)!} \,. \tag{2.6}$$

Operator quantization, as discussed in [10] and [69], is defined as having several properties. Let $z(q,p)$ be an arbitrary classical phase space function and $\mathcal{Q}$ be a quantization map taking classical functions to operators, such that $\mathcal{Q} : z(q,p) \to \hat{z}(\hat{q},\hat{p})$. Then,

1. $\mathcal{Q}(1) = \hat{I}$, where I is the identity operator
2. $\mathcal{Q}(q) = \hat{q}$
3. $\mathcal{Q}(p) = \hat{p}$
4. $\mathcal{Q}(\bar{z}(q,p)) = \hat{z}^\dagger(\hat{q},\hat{p})$, where $\bar{z}$ denotes the complex conjugate of z
5. $i\hbar\mathcal{Q}(\{q, z(q,p)\}) = [\hat{q}, \hat{z}(\hat{q},\hat{p})]$
6. $i\hbar\mathcal{Q}(\{p, z(q,p)\}) = [\hat{p}, \hat{z}(\hat{q},\hat{p})]$

(2.7)

A proof that the quantization map obeys these six axioms is given in [69]. As an example, in Section 2.3 we will show that Weyl quantization satisfies the above six properties.

2.3 Weyl Quantization

It is possible to show that Weyl quantization satisfies the six properties by using equation (2.2). However, it is easier to use the exponential form of Weyl quantization, which we illustrate below. To convert a phase space function f to an operator $\hat{f}$ under Weyl quantization, it is necessary to use the Weyl map, $\mathcal{Q}_W$ [6,70]:

$$\begin{aligned} \hat{f}(\hat{q},\hat{p}) &= \mathcal{Q}_W f(q,p)\,, \\ &= \frac{1}{(2\pi)^2}\int db\,da\,dq\,dp\,f(q,p)\,e^{ib(\hat{p}-p)+ia(\hat{q}-q)}\,, \end{aligned} \tag{2.8}$$

where $a,b \in \mathbb{R}$ and we use the notation throughout the thesis that integrals without limits implies integration between $-\infty$ and $+\infty$. With $\theta,\tau \in \mathbb{R}$, we see

$$\begin{aligned} \mathcal{Q}_W e^{i(\theta q+\tau p)} &= \frac{1}{(2\pi)^2}\int db\,da\,dq\,dp\,e^{i(\theta q+\tau p)}\,e^{ib(\hat{p}-p)+ia(\hat{q}-q)}\,, \\ &= \int db\,da\,\delta(\theta-a)\delta(\tau-b)\,e^{ib\hat{p}+ia\hat{q}}\,, \\ &= e^{i(\theta\hat{q}+\tau\hat{p})}\,. \end{aligned} \tag{2.9}$$

We also made use of the property [71],

$$\delta(x-y) = \frac{1}{2\pi}\int dw\,e^{iw(x-y)}\,. \tag{2.10}$$

Hence, the Weyl quantization map is $\mathcal{Q}_W : e^{i\theta q+i\tau p} \to e^{i\theta\hat{q}+i\tau\hat{p}}$ (i.e. it is only necessary to let $q \to \hat{q}$ and $p \to \hat{p}$ within the exponential). By expanding $e^{i(\theta q+\tau p)}$ and $e^{i(\theta\hat{q}+\tau\hat{p})}$ in powers of the position and momentum, then relating like powers of θ and τ, it is possible to recover equation (2.2), which we show in Appendix A.

Let $z(q,p) = e^{i(\theta q+\tau p)}$. Using the exponential form of Weyl quantization, the first four properties of equation (2.7) are thus automatically satisfied. To demonstrate that the fifth and sixth properties are also valid, we note that the Weyl map gives

$$\begin{aligned} i\hbar\mathcal{Q}_W(\{q,z(q,p)\}) &= i\hbar\mathcal{Q}_W(\tau\,e^{i(\theta q+\tau p)})\,, \\ &= i\hbar\tau\,e^{i(\theta\hat{q}+\tau\hat{p})}\,, \end{aligned} \tag{2.11}$$

and

$$\begin{aligned}
[\hat{q},\hat{z}(\hat{q},\hat{p})] &= e^{-i\hbar\theta\tau/2}[\hat{q},e^{\theta\hat{q}}e^{\tau\hat{p}}]\,, \\
&= e^{-i\hbar\theta\tau/2}e^{\theta\hat{q}}i\hbar\tau\sum_{n=0}^{\infty}\frac{\tau^n}{n!}\hat{p}^n\,, \\
&= i\hbar\tau\, e^{i(\theta\hat{q}+\tau\hat{p})}\,,
\end{aligned} \tag{2.12}$$

where we have applied the Zassenhaus formula, $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} e^{\frac{1}{6}[A,[A,B]]+\frac{1}{3}[B,[A,B]]}\ldots$ and $[\hat{q},\hat{p}^n] = in\hat{p}^{n-1}$ [7]. Equations (2.11) and (2.12) are equal, hence verifying Property 5 of equation (2.7). A similar procedure could be done to demonstrate Property 6.

In general, quantization maps can be represented using exponentials, in a similar form to the Weyl quantization map. This form is often easier to manipulate, which we will use in Section 2.10.

2.4 Physical Implications of Different Quantizations

The order of operators does not necessarily imply that the physical system is different. Consider $qp \to \hat{q}\hat{p}$. Using the fact that $[\hat{q},\hat{p}] = i\hbar$, we can rewrite $\hat{q}\hat{p}$ as $\hat{p}\hat{q}+i\hbar$. However, if we let $qp \to \hat{p}\hat{q}$ instead, that change in quantization may yield different physics.

In this thesis, we will regard two quantizations to be physically distinct if they yield two different values of measurable quantities. As an example, consider the classical Hamiltonian ($m = \omega = 1$),

$$H = \frac{p^2}{2} + \frac{q^2}{2} + \lambda\left(p^2+q^2\right)^3\,, \tag{2.13}$$

where λ is a constant. As the first two terms each have only a single phase space coordinate, quantization yields $p^2 \to \hat{p}^2$ and $q^2 \to \hat{q}^2$. It was shown in [13] that the most general quantization of $(p^2+q^2)^3$ is

$$(p^2+q^2)^3 \rightarrow \hat{p}^6+\hat{q}^6+\frac{3}{2}\left(\hat{p}^4\hat{q}^2+\hat{q}^2\hat{p}^4\right)+3\hbar^2\alpha\hat{p}^2+\frac{3}{2}\left(\hat{p}^2\hat{q}^4+\hat{q}^4\hat{p}^2\right)+3\hbar^2\beta\hat{q}^2\,, \quad (2.14)$$

where α and β are parameters of the quantization scheme. If we impose that $q \rightarrow -p$ and $p \rightarrow q$ yield the same quantized Hamiltonian, we find that $\alpha = \beta$. Ref. [13] demonstrated that the general quantization of the classical Hamiltonian in equation (2.13) is

$$\hat{H} = \frac{1}{2}\hat{p}^2+\frac{1}{2}\hat{q}^2+\lambda\left(\hat{p}^2+\hat{q}^2\right)^3+\lambda\left(3\hbar^2\alpha-4\right)\left(\hat{p}^2+\hat{q}^2\right)\,. \quad (2.15)$$

To solve the time-independent Schrödinger equation, let $H' = \frac{\hat{p}^2}{2}+\frac{\hat{q}^2}{2}$, which is the Hamiltonian for the simple harmonic oscillator. Then,

$$H = H' + 8\lambda H'^3 + 2\lambda\left(3\hbar^2\alpha-4\right)H'\,. \quad (2.16)$$

We see that the state vector for the simple harmonic oscillator satisfies the Schrödinger equation for this Hamiltonian with the energy levels [13],

$$E_n = \frac{1}{2}\hbar(2n+1)+\lambda\hbar\,(2n+1)^3+\lambda\hbar\left(3\hbar^2\alpha-4\right)(2n+1)\,. \quad (2.17)$$

To summarize, the quantization rule used can affect observable results (in the example above, it was the energy of a system). Hence, quantizations with different values of α may be physically distinct. This result was based upon the fact that $p^2 \rightarrow \hat{p}^2$ and $q^2 \rightarrow \hat{q}^2$. In Chapter 4, we generalize quantization such that $p^2 \not\rightarrow \hat{p}^2$ and $q^2 \not\rightarrow \hat{q}^2$.

2.5 Wigner Transform

Rather than doing quantum mechanics with operators, one alternative is to do quantum mechanics in phase space. In this description of quantum mechanics, classical phase space functions are used. To convert an operator back to a phase space function, we apply the inverse of the Weyl map of Section 2.3. This operation is called the Wigner transform (or sometimes the Weyl transform as in [72]).

The Wigner transform is [72, 73]

$$f(q,p) = \mathcal{W}\left(\hat{f}\right) = \hbar \int dy\, e^{-ipy} \left\langle q + \frac{\hbar y}{2} \middle| \hat{f} \middle| q - \frac{\hbar y}{2} \right\rangle , \tag{2.18}$$

where $\hat{f} = \hat{f}(\hat{q}, \hat{p})$ is an arbitrary operator. We label the corresponding phase space function of $\hat{f}$ as $f(q,p)$. We can show $\mathcal{W} = \mathcal{Q}_W^{-1}$ by using Fourier transform of $f(q,p)$, $\tilde{f}(a,b)$, such that

$$f(q,p) = \int da\, db\, \tilde{f}(a,b)\, e^{i(aq+bp)} . \tag{2.19}$$

With equation (2.9), we have

$$\mathcal{Q}_W f(q,p) = \int da\, db\, \tilde{f}(a,b)\, e^{i(a\hat{q}+b\hat{p})} . \tag{2.20}$$

We now need to apply $\mathcal{W}$. Note that

$$\begin{aligned} \mathcal{W}\left(e^{i(a\hat{q}+b\hat{p})}\right) &= \hbar \int dy\, e^{-ipy} \left\langle q + \frac{\hbar y}{2} \middle| e^{i(a\hat{q}+b\hat{p})} \middle| q - \frac{\hbar y}{2} \right\rangle , \\ &= \hbar e^{i\hbar ab/2} \int dy\, e^{-ipy} \left\langle q + \frac{\hbar y}{2} \middle| e^{ia\hat{q}} e^{ib\hat{p}} \middle| q - \frac{\hbar y}{2} \right\rangle . \end{aligned}$$

With

$$\begin{aligned} \hat{q}\,|q\rangle &= q\,|q\rangle , \\ e^{ib\hat{p}}\,|q\rangle &= |q - b\hbar\rangle , \end{aligned}$$

we find

$$\mathcal{W}\left(e^{i(a\hat{q}+b\hat{p})}\right) = \hbar e^{i\hbar ab/2}\int dy\, e^{-ipy} e^{ia\left(q-\frac{\hbar y}{2}-b\hbar\right)}\left\langle q+\frac{\hbar y}{2}\middle| q-\frac{\hbar y}{2}-b\hbar\right\rangle\,,$$
$$= e^{i(aq+bp)}\,.$$

Therefore,

$$\mathcal{W}\Big(\mathcal{Q}_W f(q,p)\Big) = \int da\,db\,\tilde{f}(a,b)e^{i(aq+bp)} = f(q,p)\,. \tag{2.21}$$

Hence, $\mathcal{W} = \mathcal{Q}_W^{-1}$.

2.6 Moyal Product

The tradeoff for using classical phase space functions rather than operators is that it is necessary to introduce a noncommutative binary operation known as the Moyal star product, which is denoted by $\star_M$. The transform of $\mathcal{W}$ preserves the structure of the algebra of operators, such that the Moyal product algebra is homomorphic[3] to the operator algebra. Specifically, with $\hat{f}$ and $\hat{g}$ operators [72],

$$\mathcal{W}\left(\hat{f}\hat{g}\right) = \mathcal{W}\left(\hat{f}\right)\star_M \mathcal{W}\left(\hat{g}\right) = f\star_M g\,. \tag{2.22}$$

It is possible to transform any observable from Weyl-quantized operator quantum mechanics to Weyl quantization in phase space by applying the Wigner transform.

To derive the Moyal product, consider $\hat{f}$ and $\hat{g}$ in terms of their Fourier components,

[3]Let ϕ be a map between two sets A, B and let $a_1, a_2 \in A$. We say that ϕ is homomorphic if and only if $\phi(a_1 \cdot a_2) = \phi(a_1) * \phi(a_2)$, where $\cdot$ is the operation in A and $*$ is the operation in B [74, 75].

$e^{i(\theta_1\hat{q}+\tau_1\hat{p})}$ and $e^{i(\theta_2\hat{q}+\tau_2\hat{p})}$. Then,

$$\mathcal{W}\left(e^{i(\theta_1\hat{q}+\tau_1\hat{p})}e^{i(\theta_2\hat{q}+\tau_2\hat{p})}\right) = \mathcal{W}\left(e^{i(\theta_1+\theta_2)\hat{q}+i(\tau_1+\tau_2)\hat{p}}\right)e^{-\frac{i\hbar}{2}(\theta_1\tau_2-\theta_2\tau_1)}, \tag{2.23}$$

$$= \mathcal{W}\left(e^{i(\theta_1\hat{q}+\tau_1\hat{p})}\right)\star_M \mathcal{W}\left(e^{i(\theta_2\hat{q}+\tau_2\hat{p})}\right). \tag{2.24}$$

by using equation (2.22) and the Zassenhaus formula. Using the fact that $\mathcal{W}\left(e^{i(\theta\hat{q}+\tau\hat{p})}\right) = e^{i(\theta q+\tau p)}$, equation (2.23) becomes

$$\mathcal{W}\left(e^{i(\theta_1\hat{q}+\tau_1\hat{p})}e^{i(\theta_2\hat{q}+\tau_2\hat{p})}\right) = \left(e^{i(\theta_1+\theta_2)q+i(\tau_1+\tau_2)p}\right)e^{-\frac{i\hbar}{2}(\theta_1\tau_2-\theta_2\tau_1)}, \tag{2.25}$$

and equation (2.24) is

$$\mathcal{W}\left(e^{i(\theta_1\hat{q}+\tau_1\hat{p})}e^{i(\theta_2\hat{q}+\tau_2\hat{p})}\right) = \left(e^{i(\theta_1 q+\tau_1 p)}\right)\star_M \left(e^{i(\theta_2 q+\tau_2 p)}\right). \tag{2.26}$$

Note that

$$\begin{aligned}\mathcal{I}(1,2)&e^{\frac{i\hbar}{2}\left(\partial_{q_1}\partial_{p_2}-\partial_{p_1}\partial_{q_2}\right)}e^{i(\theta_1 q_1+\tau_1 p_1)}e^{i(\theta_2 q_2+\tau_2 p_2)}\\ &= e^{i(\theta_1+\theta_2)q+i(\tau_1+\tau_2)p}e^{-\frac{i\hbar}{2}(\theta_1\tau_2-\theta_2\tau_1)},\end{aligned} \tag{2.27}$$

where $\mathcal{I}(1,2)$ means to set $q_1 = q_2 = q$ and $p_1 = p_2 = p$ at the end of the calculation.

Let $F = e^{i(\theta_1 q_1+\tau_1 p_1)}$ and $G = e^{i(\theta_2 q_2+\tau_2 p_2)}$. We can rewrite the top line of equation (2.27) as

$$e^{\frac{i\hbar}{2}\left(\partial_{q_1}\partial_{p_2}-\partial_{p_1}\partial_{q_2}\right)}FG = \sum_{n=0}^{\infty}\left(\frac{i\hbar}{2}\right)^n\frac{1}{n!}\sum_{m=0}^{n}\binom{n}{m}(-1)^m\frac{\partial^n F}{\partial q_1^{n-m}\partial p_1^m}\frac{\partial^n G}{\partial q_2^m\partial p_2^{n-m}}. \tag{2.28}$$

Applying $\mathcal{I}(1,2)$ to both sides, this equation can be written in a more concise form by introducing the differential operators, $\overleftarrow{\partial}$ (left derivative) and $\overrightarrow{\partial}$ (right derivative), which are defined as

$$f\overleftarrow{\partial}_q\, g := \frac{\partial f}{\partial q}g,$$

and similarly for the right derivative. As an example, the Leibniz rule can be written as

$$\frac{\partial}{\partial q}(fg) = \frac{\partial f}{\partial q}g + f\frac{\partial g}{\partial q} = f\left(\overleftarrow{\partial}_q + \overrightarrow{\partial}_q\right)g\,. \tag{2.29}$$

By using left and right derivatives, equation (2.28) becomes

$$\begin{aligned} &\mathcal{I}(1,2)e^{\frac{i\hbar}{2}\left(\partial_{q_1}\partial_{p_2}-\partial_{p_1}\partial_{q_2}\right)}e^{i(\theta_1 q_1+\tau_1 p_1)}e^{i(\theta_2 q_2+\tau_2 p_2)} \\ &= e^{i(\theta_1 q+\tau_1 p)}\sum_{n=0}^{\infty}\left(\frac{i\hbar}{2}\right)^n\frac{1}{n!}\sum_{m=0}^{n}\binom{n}{m}(-1)^m\left(\overleftarrow{\partial}_q\overrightarrow{\partial}_p\right)^{n-m}\left(\overleftarrow{\partial}_p\overrightarrow{\partial}_q\right)^m e^{i(\theta_2 q+\tau_2 p)}\,. \end{aligned} \tag{2.30}$$

Therefore, with equations (2.25)-(2.27), we see that the Moyal product is

$$\star_M = e^{\frac{i\hbar}{2}\left(\overleftarrow{\partial}_q\overrightarrow{\partial}_p-\overleftarrow{\partial}_p\overrightarrow{\partial}_q\right)}\,, \tag{2.31}$$

$$= \sum_{n=0}^{\infty}\left(\frac{i\hbar}{2}\right)^n\frac{1}{n!}\sum_{m=0}^{n}\binom{n}{m}(-1)^m\left(\overleftarrow{\partial}_q\overrightarrow{\partial}_p\right)^{n-m}\left(\overleftarrow{\partial}_p\overrightarrow{\partial}_q\right)^m\,. \tag{2.32}$$

We see that the Moyal product is the exponentiation of the Poisson bracket. Therefore, the Moyal product is sometimes written as $\star_M = e^{\frac{i\hbar}{2}\overleftrightarrow{P}}$, where $\overleftrightarrow{P} := \overleftarrow{\partial}_q\overrightarrow{\partial}_p - \overleftarrow{\partial}_p\overrightarrow{\partial}_q$. It is also possible to write $f(q,p)\star_M g(q,p)$ as

$$\begin{aligned} f(q,p)\star_M g(q,p) &= f\left(q+\frac{i\hbar}{2}\partial_p, p-\frac{i\hbar}{2}\partial_q\right)g(q,p)\,, \\ &= f(q,p)g\left(q-\frac{i\hbar}{2}\overleftarrow{\partial}_p, p+\frac{i\hbar}{2}\overleftarrow{\partial}_q\right)\,, \\ &= f\left(q+\frac{i\hbar}{2}\partial_p, p\right)g\left(q-\frac{i\hbar}{2}\overleftarrow{\partial}_p, p\right)\,, \\ &= f\left(q, p-\frac{i\hbar}{2}\partial_q\right)g\left(q, p+\frac{i\hbar}{2}\overleftarrow{\partial}_q\right)\,, \end{aligned} \tag{2.33}$$

where we have used the translation operator,

$$e^{a\partial_q}f(q) = f(q+a)\,. \tag{2.34}$$

In general, a star product $\star$ is defined as a bilinear map acting on smooth functions f

and g, with [26]

$$f \star g = \sum_{r=0}^{\infty} \nu^r C_r(f,g)\,, \tag{2.35}$$

where ν is a complex parameter and C_r is a bidifferential operator. The quantities C_r and $\star$ have the properties [12, 26],

1. $C_0(f,g) = fg$
2. $C_1(f,g) = \{f,g\}$
3. $C_r(g,f) = (-1)^r C_r(f,g)$ (2.36)
4. For $a \in \mathbb{R}$, $C_r(f,a) = 0$.
5. $(f \star g) \star h = f \star (g \star h)$

We will now demonstrate the associativity of the Moyal product. Let f, g, and h be functions of the position and momentum. We have

$$(f \star_M g) \star_M h = \left(\mathcal{I}(1,2) e^{\frac{i\hbar}{2}\left(\partial_{q_1}\partial_{p_2} - \partial_{p_1}\partial_{q_2}\right)} f(q_1,p_1) g(q_2,p_2) \right) \star_M h\,.$$

To condense this notation, define

$$\begin{aligned}
[\star_M(1,2)] &:= e^{\frac{i\hbar}{2}\left(\partial_{q_1}\partial_{p_2} - \partial_{p_1}\partial_{q_2}\right)}\,, \\
[\star_M(1+2,3)] &:= e^{\frac{i\hbar}{2}\left[(\partial_{q_1}+\partial_{q_2})\partial_{p_3} - (\partial_{p_1}+\partial_{p_2})\partial_{q_3}\right]}
\end{aligned}$$

so that

$$(f \star_M g) \star_M h = \mathcal{I}(1,2,3)\,[\star_M(1+2,3)]\,[\star_M(1,2)]\, f(1) g(2) h(3)$$

where $f(1) = f(q_1,p_1)$ and similarly for $g(2)$ and $h(3)$. The symbol of $\mathcal{I}(1,2,3)$ indicates that we set $q_1 = q_2 = q_3 = q$ and $p_1 = p_2 = p_3 = p$ at the end of the calculation.

By the same process,

$$f \star_M (g \star_M h) = \mathcal{I}(1,2,3)\,[\star_M(1,2+3)]\,[\star_M(2,3)]\, f(1)g(2)h(3)\,.$$

Therefore, to demonstrate associativity, it suffices to show that

$$[\star_M(1+2,3)]\,[\star_M(1,2)] = [\star_M(1,2+3)]\,[\star_M(2,3)]\;.$$

The left-hand side is equal to

$$[\star_M(1+2,3)]\,[\star_M(1,2)] = \exp\left\{\frac{i\hbar}{2}\Big[\partial_{q_3}\left(\partial_{p_1}+\partial_{p_2}\right) - \partial_{p_3}\left(\partial_{q_1}+\partial_{q_2}\right) + \partial_{p_2}\partial_{q_1} - \partial_{p_1}\partial_{q_2}\Big]\right\},$$

and the right-hand side is

$$[\star_M(1,2+3)]\,[\star_M(2,3)] = \exp\left\{\frac{i\hbar}{2}\Big[-\partial_{q_1}\left(\partial_{p_2}+\partial_{p_3}\right) + \partial_{p_1}\left(\partial_{q_2}+\partial_{q_3}\right) + \partial_{p_3}\partial_{q_2} - \partial_{p_2}\partial_{q_3}\Big]\right\}.$$

Rearranging, we find that $[\star_M(1+2,3)]\,[\star_M(1,2)] = [\star_M(1,2+3)]\,[\star_M(2,3)]$ is indeed true, thus verifying that the Moyal product is associative.

2.7 Wigner Functions

As quantum mechanics can be done in phase space, that means that there must be a phase space counterpart to the Liouville-von Neumann equation for the density operator, $\hat{\rho}$. The Liouville-von Neumann equation is [76]

$$i\hbar\frac{\partial\hat{\rho}}{\partial t} = [\hat{H},\hat{\rho}]\,, \tag{2.37}$$

where we take $\hat{H}$ as a time-independent Hamiltonian throughout Chapters 2-4. If we want to work in phase space, then the Wigner transform of equation (2.37) gives

$$i\hbar\mathcal{W}\left(\frac{\partial\hat{\rho}}{\partial t}\right) = \mathcal{W}\left(\hat{H}\hat{\rho} - \hat{\rho}\hat{H}\right), \tag{2.38}$$

$$\implies i\hbar\frac{\partial\mathcal{W}(\hat{\rho})}{\partial t} = \mathcal{W}\left(\hat{H}\right)\star_M\mathcal{W}(\hat{\rho}) - \mathcal{W}(\hat{\rho})\star_M\mathcal{W}\left(\hat{H}\right) =: [H, \mathcal{W}(\hat{\rho})]_{\star_M}, \tag{2.39}$$

where we have applied equation (2.22).

With a Hamiltonian operator of the form, $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q})$, the phase space version is $H = \frac{p^2}{2m} + V(q)$. Using equation (2.18), the analogue to the density operator is known as the Wigner function [72],

$$W(q,p) = \frac{\mathcal{W}(\hat{\rho})}{2\pi\hbar} = \frac{1}{2\pi}\int dy\, e^{-ipy}\left\langle q + \frac{\hbar y}{2}\middle|\hat{\rho}\middle|q - \frac{\hbar y}{2}\right\rangle, \tag{2.40}$$

where the $2\pi\hbar$ ensures that the Wigner function is normalized. If $\hat{\rho} = |\psi\rangle\langle\psi|$ is a pure state, then the Wigner function reduces to

$$W(q,p) = \frac{1}{2\pi}\int dy\, e^{-ipy}\,\psi^*\left(q - \frac{\hbar y}{2}\right)\psi\left(q + \frac{\hbar y}{2}\right), \tag{2.41}$$

where $\psi(q) = \langle q|\psi\rangle$ is the position space wavefunction.

It should be noted that the Wigner function is dependent on the position and the momentum, but the individual point (q,p) has no meaning in the sense of being able to assign an exact position and momentum to a quantum system, as a result of Heisenberg's uncertainty principle. It is only when integrating over the position or momentum that physical predictions are made.

The Wigner function has several properties [4, 6, 7]:

1. The marginal probabilities of the position and momentum are determined with

$$P(q) = \int dp\, W(q,p) = \langle q|\hat{\rho}|q\rangle \tag{2.42}$$

and

$$P(p) = \int dq\, W(q,p) = \langle p|\hat{\rho}|p\rangle \ . \tag{2.43}$$

If $\hat{\rho}$ is a pure state, these marginal probabilities reduce to

$$P(q) = |\psi(q)|^2 \,, \tag{2.44}$$

$$P(p) = |\phi(p)|^2 \,, \tag{2.45}$$

with $\phi(p) = \langle p|\psi\rangle$ being the wavefunction in momentum space.

2. The Wigner function is normalized:

$$\int dq\, dp\, W(q,p) = 1 \tag{2.46}$$

3. The Wigner function is real.

4. The Wigner function is an example of a quasi-probability distribution. This means that the Wigner function can take negative values. Physically, these negative values can arise as a result of interference and can be interpreted as a measure of the non-classicality present in a quantum system [18, 77, 78].

It is also possible to write the pure state Wigner function in terms of momentum space wavefunctions, $\phi(p)$ [4]:

$$W(q,p) = \frac{1}{2\pi}\int dy\, e^{-ipy}\phi^*\left(p + \frac{\hbar y}{2}\right)\phi\left(p - \frac{\hbar y}{2}\right) \,, \tag{2.47}$$

where

$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}}\int dq\, \psi(q) e^{-ipq/\hbar} \tag{2.48}$$

is the Fourier transform of the position space wavefunction.

To determine the expectation value of $\hat{A}$ using phase space quantum mechanics, we first note that $\mathcal{W}(\hat{A}\hat{B}) = A \star_M B$ from equation (2.22). Therefore, $\hat{A}\hat{B} = \mathcal{W}^{-1}(A \star_M B)$. Taking the trace, we have ($\hbar = 1$)

$$\mathrm{Tr}\left(\hat{A}\hat{B}\right) = \frac{1}{(2\pi)^2}\mathrm{Tr}\left[\int dq\,dp\,da\,db(A \star_M B)e^{ia(\hat{q}-q)+ib(\hat{p}-p)}\right],$$

where we have used the fact that $\mathcal{W}^{-1} = \mathcal{Q}_W$. Summing over position eigenstates, we find

$$\begin{aligned}\mathrm{Tr}\left(\hat{A}\hat{B}\right) &= \frac{1}{(2\pi)^2}\int dq\,dp\,da\,db\,dq'(A \star_M B)\left\langle q'\left|e^{ia(\hat{q}-q)+ib(\hat{p}-p)}\right|q'\right\rangle, \\ &= \frac{1}{(2\pi)^2}\int dq\,dp\,da\,db\,dq'(A \star_M B)\left\langle q'|q'-b\right\rangle e^{ia(q'-b)}e^{iab/2}e^{-i(aq+bp)}, \\ &= \frac{1}{2\pi}\int dq\,dp\,A \star_M B.\end{aligned}$$

If $\hat{B} = \hat{\rho}$, then we need to replace B with $\mathcal{W}(\hat{\rho})$. As $\mathcal{W}(\hat{\rho}) = 2\pi W$,

$$\mathrm{Tr}\left(\hat{A}\hat{\rho}\right) = \int dq\,dp\,A \star_M W. \tag{2.49}$$

Similarly, by using the cyclic property of the trace,

$$\mathrm{Tr}\left(\hat{\rho}\hat{A}\right) = \int dq\,dp\,W \star_M A. \tag{2.50}$$

Note that equations (2.49) and (2.50) describe the expectation value of $\hat{A}$ as $\langle\hat{A}\rangle = \mathrm{Tr}\left(\hat{A}\hat{\rho}\right) = \mathrm{Tr}\left(\hat{\rho}\hat{A}\right)$. Also note that we can integrate out the Moyal product as the boundary terms of successive partial integrations vanish. Therefore, [6]

$$\begin{aligned}\langle A(q,p)\rangle := \langle\hat{A}\rangle &= \int dq\,dp\,A(q,p) \star_M W(q,p) = \int dq\,dp\,W(q,p) \star_M A(q,p) \\ &= \int dq\,dp\,A(q,p)W(q,p),\end{aligned} \tag{2.51}$$

Using the Wigner function, we can write the Wigner transform of the Liouville-von

Neumann equation of (2.37) as

$$i\hbar\frac{\partial W}{\partial t} = [H, W]_{\star_M}\,, \tag{2.52}$$

where

$$[f, g]_{\star_M} := f \star_M g - g \star_M f \tag{2.53}$$

and $\frac{1}{i\hbar}[f,g]_{\star_M}$ is known as the Moyal bracket of $f(q,p)$ and $g(q,p)$. In the case of a stationary state, $H \star_M W = W \star_M H$.

Consider a pure state described by Schrödinger's equation, $\hat{H}\,|\psi\rangle = E\,|\psi\rangle$. Multiplying by $\langle\psi|$ on the right and taking the Wigner transform, we get the stargenvalue equation [6],

$$H \star_M W = E\,W\,, \tag{2.54}$$

Using the Hermitian-conjugate of the Schrödinger equation and assuming $\hat{H}$ is Hermitian, we similarly have the stargenvalue equation,

$$W \star_M H = E\,W\,. \tag{2.55}$$

2.8 Example: Simple Harmonic Oscillator

As an example of an application of the Wigner function and the stargenvalue equations, consider the simple harmonic oscillator Hamiltonian, $\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{q}^2$, as discussed in, for example, [4], [6], and [72]. Throughout this thesis, when considering the simple harmonic oscillator, we set $m = \omega = 1$ for the purposes of calculation.

In phase space, the stargenvalue equation is

$$\left(\frac{1}{2}p^2 + \frac{1}{2}q^2\right) \star_M W = E\,W\,. \tag{2.56}$$

Using equation (2.32), we have

$$\left(\frac{p^2}{2}+\frac{q^2}{2}\right)W+\frac{i\hbar}{2}\left(q\frac{\partial W}{\partial p}-p\frac{\partial W}{\partial q}\right)-\frac{\hbar^2}{8}\left(\frac{\partial^2 W}{\partial p^2}+\frac{\partial^2 W}{\partial q^2}\right)=EW \tag{2.57}$$

As the Wigner function is real, this implies that

$$q\frac{\partial W}{\partial p}-p\frac{\partial W}{\partial q}=0\,, \tag{2.58}$$

so that $W=W\left(\frac{p^2}{2}+\frac{q^2}{2}\right)$. We note that we have used equation (2.54) to derive the differential equation for the simple harmonic oscillator. Equivalently, equation (2.55) could have been used to obtain the same result.

Let $u=2q^2+2p^2$. Note that, for a function $f(u(y))$,

$$\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}\right)=\frac{\partial^2 u}{\partial y^2}\frac{\partial f}{\partial u}+\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 f}{\partial u^2}\,, \tag{2.59}$$

we have

$$\frac{\partial^2 W}{\partial q^2}=4\frac{\partial W}{\partial u}+16q^2\,\frac{\partial^2 W}{\partial u^2}\,, \tag{2.60}$$

$$\frac{\partial^2 W}{\partial p^2}=4\frac{\partial W}{\partial u}+16p^2\,\frac{\partial^2 W}{\partial u^2}\,. \tag{2.61}$$

Upon substitution of equations (2.60) and (2.61) into equation (2.57), we obtain

$$\frac{uW}{4}-\hbar^2\,\frac{\partial W}{\partial u}-u\hbar^2\,\frac{\partial^2 W}{\partial u^2}=EW\,. \tag{2.62}$$

As we want the solution to be normalizable, then we require $W\to 0$ as $u\to\infty$. From equation (2.62), the limit as $u\to\infty$ gives the partial differential equation,

$$\frac{1}{4}W\sim\hbar^2\,\frac{\partial^2 W}{\partial u^2}\,, \tag{2.63}$$

which has a solution $W \sim e^{-u/2\hbar}$. This indicates that we can solve equation (2.62) with an ansatz of the form, $W(u) = e^{-u/2\hbar} g(u)$. Then,

$$\hbar^2 u \frac{d^2 g}{du^2} + \hbar(\hbar - u)\frac{dg}{du} + \left(E - \frac{\hbar}{2}\right) g = 0 \tag{2.64}$$

Note that this is in the form of Kummer's differential equation, which has a solution of $g(u) = c_1 M\left(\frac{1}{2} - \frac{E}{\hbar}, 1, \frac{u}{\hbar}\right) + c_2 U\left(\frac{1}{2} - \frac{E}{\hbar}, 1, \frac{u}{\hbar}\right)$ where [79],

$$M(a,b,z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!}, \tag{2.65}$$

$$U(a,b,z) = \frac{\pi}{\sin(\pi b)} \left[\frac{M(a,b,z)}{\Gamma(1+a-b)\Gamma(b)} - z^{1-b} \frac{M(1+a-b, 2-b, z)}{\Gamma(a)\Gamma(2-b)}\right], \tag{2.66}$$

and

$$(a)_n = \begin{cases} a(a+1)\cdots(a+n-1) & n > 0 \\ 1 & n = 0 \end{cases} \tag{2.67}$$

is the Pochhammer symbol. For $u \to \infty$, $U\left(\frac{1}{2} - \frac{E}{\hbar}, 1, \frac{u}{\hbar}\right) \to \infty$ as well. Therefore, $c_2 = 0$.

From the Schrödinger equation, we know that $E = \left(n + \frac{1}{2}\right)\hbar$. We can then write $g(u) = c_1 M\left(-n, 1, \frac{u}{\hbar}\right) = c_1 L_n\left(\frac{u}{\hbar}\right)$, where L_n is the n-th Laguerre polynomial [71, 79],

$$L_n(x) = \frac{1}{n!} e^x \left(\frac{d}{dx}\right)^n \left[e^{-x} x^n\right] = \sum_{m=0}^{n} \frac{(-1)^m}{m!} \binom{n}{m} x^m. \tag{2.68}$$

Therefore, the Wigner function for the simple harmonic oscillator in the n^{th} energy level is

$$\begin{aligned} W_n(q,p) &= \frac{(-1)^n}{\pi\hbar} \exp\left[-\frac{2H}{\hbar}\right] L_n\left(\frac{4H}{\hbar}\right), \\ &= \frac{(-1)^n}{\pi\hbar} \exp\left[-\frac{\frac{p^2}{m} + m\omega^2 q^2}{\hbar\omega}\right] L_n\left(\frac{\frac{2p^2}{m} + 2m\omega^2 q^2}{\hbar\omega}\right) \end{aligned} \tag{2.69}$$

where $c_1 = \dfrac{(-1)^n}{\pi\hbar}$ is the normalization constant.

Figure 2.1 illustrates the simple harmonic oscillator Wigner function for the first four

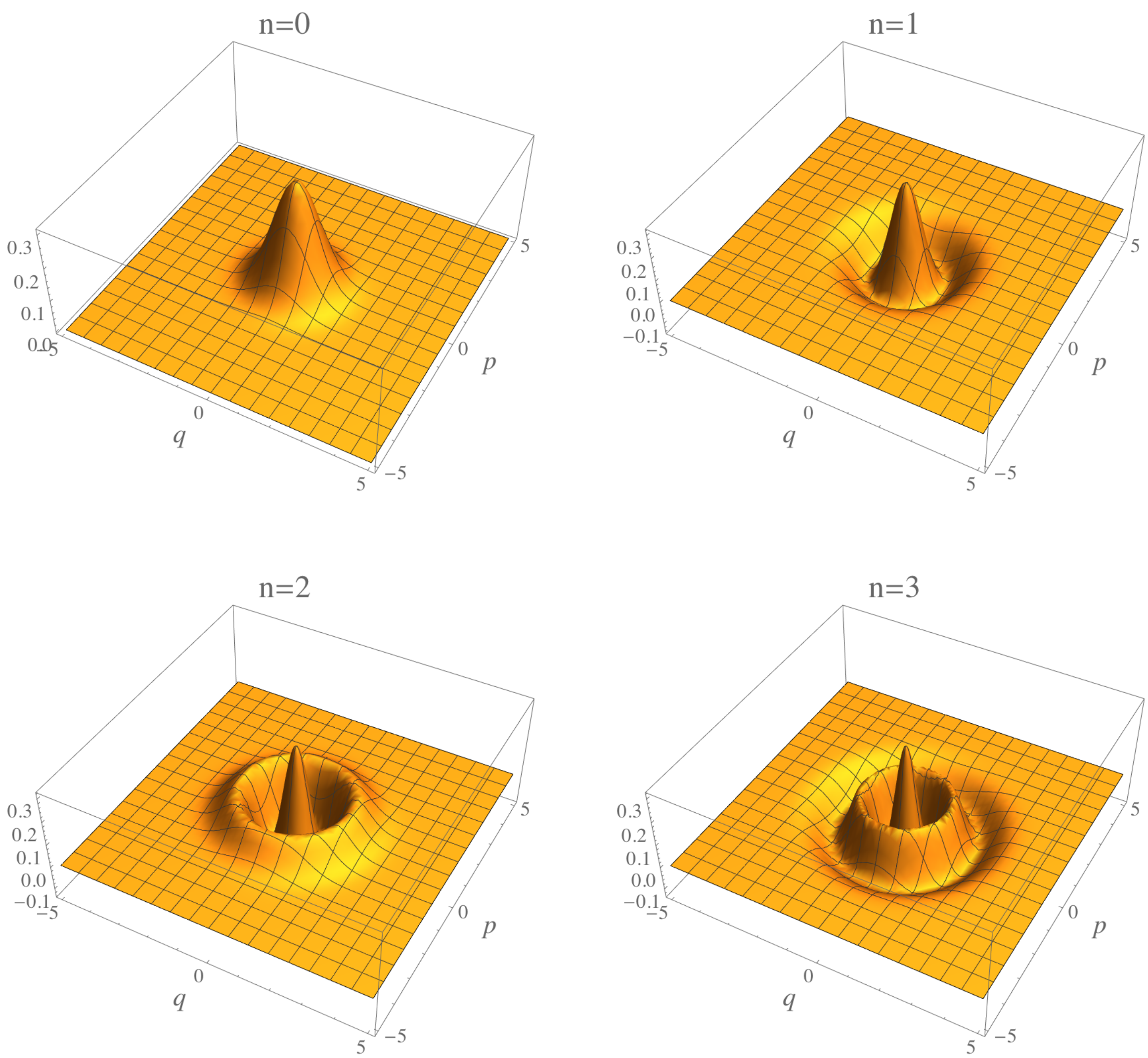

Figure 2.1: The simple harmonic oscillator Wigner function for the first four energy levels. Note that, for $n > 0$, the Wigner function has negative values.

energy levels. This simple system exemplifies the quasi-probabilistic nature of the Wigner function.

2.9 Time-Dependence of the Wigner Function

We have demonstrated that the time-independent Wigner function can be found with the stargenvalue equations of (2.54) and (2.55). However, as shown with equation (2.52), the Wigner function is generally dependent on time. To determine its time-dependence, we first note that the solution to the Liouville von-Neumann equation of (2.37) is [76]

$$\hat{\rho}(t) = e^{-i\hat{H}t/\hbar}\,\hat{\rho}(0)\,e^{i\hat{H}t/\hbar}, \tag{2.70}$$

where $\hat{U}(t) = e^{i\hat{H}t/\hbar}$ is the propagator. By applying the Wigner transform to $\hat{\rho}(t)$, we find

$$W(q,p,t) = \mathcal{W}\left(e^{-i\hat{H}t/\hbar}\right) \star_M W(q,p,0) \star_M \mathcal{W}\left(e^{i\hat{H}t/\hbar}\right), \tag{2.71}$$

and $W(q,p,0) := W(q,p)$ satisfies equations (2.54) and (2.55). The application of the Wigner transform to $\hat{U}$ yields [6]

$$\mathcal{W}(\hat{U}) =: U_{\star_M} = \mathrm{Exp}_{\star_M}\left(-\frac{itH}{\hbar}\right), \tag{2.72}$$

where [26,27]

$$\mathrm{Exp}_{\star_M}(a) =: \sum_{n=0}^{\infty} \frac{a^{\star_M^n}}{n!}, \tag{2.73}$$

is the star-exponential and $a^{\star_M^n} := \underbrace{a \star_M \cdots \star_M a}_{n \text{ times}}$. Therefore, the solution to equation (2.52) is [6]

$$W(q,p,t) = U_{\star_M} \star_M W(q,p,0) \star_M U_{\star_M}^{-1}. \tag{2.74}$$

As $\hat{U}\hat{U}^\dagger = 1$, the Wigner transform yields $U_{\star_M} \star_M U_{\star_M}^{-1} = 1$ and $U_{\star_M}^{-1} = \overline{U}_{\star_M}$. Hence,

$$W(q,p,t) = U_{\star_M} \star_M W(q,p,0) \star_M \overline{U}_{\star_M}. \tag{2.75}$$

Substitution of equation (2.75) into Wigner transform of equation (2.37) gives

$$i\hbar\frac{\partial U_{\star_M}}{\partial t} = H \star_M U_{\star_M}\,, \tag{2.76}$$

$$-i\hbar\frac{\partial \overline{U}_{\star_M}}{\partial t} = \overline{U}_{\star_M} \star_M H\,, \tag{2.77}$$

where equation (2.77) was found by taking the complex conjugate of equation (2.76), assuming that the Hamiltonian H is real.

From [26, 27], we can write the star-exponential of the Hamiltonian as the series,

$$U_{\star_M} = \sum_n W_n(q,p)e^{-iE_nt/\hbar}\,. \tag{2.78}$$

Therefore, in conjunction with equation (2.76),

$$\sum_n W_n(q,p)E_n e^{-iE_nt/\hbar} = \sum_n H \star_M W_n(q,p)e^{-iE_nt/\hbar}\,. \tag{2.79}$$

Hence, we recover equation (2.54). Similarly, we find equation (2.55) by using equation (2.77).

2.10 Transition Operators and Weight Functions

As demonstrated at the beginning of this Chapter, there are different ways of quantizing a system. Associated with those methods are different operator orderings, such as standard ordering (all positions are to the left of the momentum) and normal ordering (all creation operators are to the left of the annihilation operators). It is possible to determine star products for these other quantizations.

In Weyl quantization, application of the Wigner transform on operators $\hat{f}$ and $\hat{g}$ gives $\mathcal{W}(\hat{f}\hat{g}) = \mathcal{W}(\hat{f}) \star_M \mathcal{W}(\hat{g})$ because of the homomorphism between the algebra of operator multiplication and the Moyal product. The quantities of $\mathcal{W}(f)$ and $\mathcal{W}(g)$ are then classical phase space functions as the Wigner transform is the inverse of the Weyl quantization map.

It should be noted that this homomorphism is true for other phase space quantizations, though the transform and star product will be different. We can convert from Weyl-ordered operator quantization to a different phase space quantization with $T\mathcal{W}$, where $T = T(\partial_q, \partial_p)$ such that [26, 27]

$$T = 1 + \sum_{r=1}^{\infty} \nu^r T_r\,, \tag{2.80}$$

and T_r is a differential operator with ν being a parameter. Operation of $T\mathcal{W}$ on $\hat{f}\hat{g}$ yields

$$T\mathcal{W}(\hat{f}\hat{g}) = (T\mathcal{W}\hat{f}) \star_T (T\mathcal{W}\hat{g}) \tag{2.81}$$

where $\star_T$ is the star product corresponding to the quantization of $T\mathcal{W}$. Therefore,

$$T\,(f \star_M g) = (T f) \star_T (T g)\,. \tag{2.82}$$

If equation (2.82) is true, we would then say that $\star_M$ and $\star_T$ are cohomologically equivalent (c-equivalent). This refers to two quantizations (each with their own star product) as being mathematically equivalent, but not necessarily physically equivalent [26, 27, 80].[4]

It is also possible to consider a transition operator, T, such that $T\,(f \star_1 g) = (T f) \star_2 (T g)$, where $\star_1$ is not the Moyal product. We would then call $\star_1$ and $\star_2$ c-equivalent [82]. In this thesis, when referring to c-equivalence, we will always be referring to an operator T taking us from the Moyal product to $\star_T$.

The operator T is sometimes termed the transition operator. The purpose of the transition operator is to convert between different quantizations. As shown later in this Section, different quantizations also possess a different distribution function, which can be found by using a transition operator.

Figure 2.2 illustrates the process of converting from one quantization to another with the

[4]There are many terms meaning c-equivalence, including gauge equivalence, g-equivalence, and occasionally just equivalence [80, 81].

transition operator. Let $\hat{\mathcal{A}}_W$ be the operator algebra of observables in Weyl-ordered operator quantization and $\mathcal{A}_W$ be the star algebra of observables in phase space Weyl quantization. The observables of phase space Weyl quantization are transformed to a different star algebra ($\mathcal{A}$) and operator algebra ($\hat{\mathcal{A}}$) with the quantization map $\mathcal{Q}_T$ and transition operator T.

$$\begin{tikzcd}
\hat{\mathcal{A}}_W \arrow[d, shift right, "\mathcal{W}"'] & & \hat{\mathcal{A}} \arrow[d, shift right, "\mathcal{W}_T"'] \\
\mathcal{A}_W \arrow[u, shift right, "\mathcal{Q}_W"'] \arrow[rr, shift right, "T"'] & & \mathcal{A} \arrow[u, shift right, "\mathcal{Q}_T"'] \arrow[ll, shift right, "T^{-1}"']
\end{tikzcd}$$

Figure 2.2: There are many quantization maps. This figure illustrates one map relating Weyl operator quantization ($\hat{\mathcal{A}}_W$) to a different operator quantization ($\hat{\mathcal{A}}$) by way of two phase space quantizations ($\mathcal{A}_W$ and $\mathcal{A}$).

To determine the form of $\star_T$, let us assume that T is invertible, so that

$$f \star_M g = T^{-1} \left(\tilde{f} \star_T \tilde{g} \right) , \tag{2.83}$$

where $\tilde{f} = T f$ and $\tilde{g} = T g$. Assume that the inverse of the transition operator can be written as

$$T^{-1} = \sum_{m,n} a_{mn} \partial_p^m \partial_q^n , \tag{2.84}$$

where a_{mn} is constant, and note that

$$\partial_q^n (\tilde{f} \star_T \tilde{g}) = \sum_{k=0}^{n} \binom{n}{k} \left(\partial_q^k \tilde{f} \right) \star_T \left(\partial_q^{n-k} \tilde{g} \right) = \tilde{f} \left(\overleftarrow{\partial}_q + \overrightarrow{\partial}_q \right)^n \star_T \tilde{g} , \tag{2.85}$$

by using the Leibniz product rule in equation (2.29). Then, combining equations (2.84) and

(2.85) with equation (2.83) gives

$$f \star_M g = \sum_{m,n} a_{mn} \tilde{f} \left(\overleftarrow{\partial}_p + \overrightarrow{\partial}_p \right)^m \left(\overleftarrow{\partial}_q + \overrightarrow{\partial}_q \right)^n \star_T \tilde{g} \tag{2.86}$$

$$= f\, T\left[\overleftarrow{\partial}\right] T^{-1}\left[\overleftarrow{\partial} + \overrightarrow{\partial}\right] T\left[\overrightarrow{\partial}\right] \star_T g \tag{2.87}$$

where we define $T\left[\overleftarrow{\partial}\right]$ as replacing ∂_p with $\overleftarrow{\partial}_p$ and ∂_q with $\overleftarrow{\partial}_q$ in the transition operator. Similar definitions follow for $T\left[\overleftarrow{\partial} + \overrightarrow{\partial}\right]$ and $T\left[\overrightarrow{\partial}\right]$. We then have [83]

$$\star_T = \star_M T^{-1}\left[\overleftarrow{\partial}\right] T\left[\overleftarrow{\partial} + \overrightarrow{\partial}\right] T^{-1}\left[\overrightarrow{\partial}\right]. \tag{2.88}$$

Application of the transition operator to the stargenvalue equations of (2.54) and (2.55) yields

$$TH \star_T TW = E\, TW\,, \tag{2.89}$$

$$TW \star_T TH = E\, TW\,. \tag{2.90}$$

and the transition operator, which we assume to be independent of time, can be applied to the Moyal bracket to give

$$i\hbar \frac{\partial TW}{\partial t} = [TH, TW]_{\star_T}\,, \tag{2.91}$$

where, similar to equation (2.53),

$$[TH, TW]_{\star_T} := TH \star_T TW - TW \star_T TH\,, \tag{2.92}$$

and TW is the distribution function in the new ordering scheme. In the rest of this thesis we will call $\frac{!}{i\hbar}[\cdot,\cdot]_{\star_T}$ the $\star_T$-bracket.

Given $e^{i(\xi\hat{q}+\eta\hat{p})}$, we can can define a weight function (sometimes called a kernel) $\Phi(\xi,\eta)$, such that [5, 7]

$$e^{i(\xi\hat{q}+\eta\hat{p})}\Phi(\xi,\eta) = C(\hat{q},\hat{p}), \tag{2.93}$$

where $C(\hat{q},\hat{p}) = \mathcal{Q}\left(e^{i(\xi q+\eta p)}\right)$ is the quantization of the exponential $e^{i(\xi q+\eta p)}$, found by applying a quantization map, $\mathcal{Q}$. For example, $\Phi(\xi,\eta) = e^{-i\hbar\xi\eta/2}$ yields $C(\hat{q},\hat{p}) = e^{i\xi\hat{q}}e^{i\eta\hat{p}}$, while $\Phi(\xi,\eta) = e^{i\hbar\xi\eta/2}$ gives $C(\hat{q},\hat{p}) = e^{i\eta\hat{p}}e^{i\xi\hat{q}}$.

We can derive a general distribution function that incorporates weight functions. To do this, we write the Wigner function as

$$W(q,p) = \frac{1}{4\pi^2}\int d\xi\, d\eta\, dq'\psi^*(q'-\eta\hbar/2)\psi(q'+\eta\hbar/2)e^{i\xi(q'-q)}e^{-i\eta p}, \tag{2.94}$$

and the transition operator in the form,

$$T\left(\partial_q,\partial_p\right) = \sum_{mn} a_{mn}\partial_q^m\partial_p^n, \tag{2.95}$$

where a_{mn} is a constant. Applying the transition operator to $W(q,p)$, we note that

$$\partial_q^m\partial_p^n W = \frac{1}{4\pi^2}\int d\xi\, d\eta\, dq'\psi^*(q'-\eta\hbar/2)\psi(q'+\eta\hbar/2)(-i\xi)^m(-i\eta)^n e^{i\xi(q'-q)-i\eta p}. \tag{2.96}$$

Therefore,

$$\begin{aligned} TW = \frac{1}{4\pi^2}\int d\xi\, d\eta\, dq' \Bigg(& \psi^*(q'-\eta\hbar/2)\psi(q'+\eta\hbar/2) \\ & \times \sum_{mn} a_{mn}(-i\xi)^m(-i\eta)^n e^{i\xi(q'-q)}e^{-i\eta p}\Bigg). \end{aligned} \tag{2.97}$$

Hence,

$$\begin{aligned} TW &= \frac{1}{4\pi^2}\int d\xi\, d\eta\, dq'\psi^*(q'-\eta\hbar/2)\psi(q'+\eta\hbar/2)\,T\,(-i\xi,-i\eta)\,e^{i\xi(q'-q)}e^{-i\eta p} \\ &=: F^{\Phi}(q,p) \end{aligned} \tag{2.98}$$

is the distribution function written in terms of the weight function. Further, the weight function can be related to the transition operator with [5,7]

$$\Phi(\xi,\eta) = T(-i\xi,-i\eta) \implies \Phi(\partial_p,\partial_q) = T\left(-i\partial_p,-i\partial_q\right). \tag{2.99}$$

Using the weight function, we can then write a general distribution function for a given quantization. For an arbitrary density operator [5,7,29],

$$F^{\Phi}(q,p) = \frac{1}{4\pi^2}\int d\xi\, d\eta\, dq'\,\langle q'+\eta\hbar/2|\hat{\rho}|q'+\eta\hbar/2\rangle\,\Phi(\xi,\eta)\,e^{i\xi(q'-q)}e^{-i\eta p}. \tag{2.100}$$

As an example of the relationship between the transition operator and the weight function, consider the ground state of the simple harmonic oscillator with Wigner function $W(q,p) = \frac{1}{\pi\hbar}e^{-\frac{2H}{\hbar}}$ and wavefunction $\psi(q) = \left(\frac{1}{\pi\hbar}\right)^{1/4}e^{-\frac{q^2}{2\hbar}}$. We will determine the distribution function for standard ordering with both the weight function and the transition operator. In Table 2.1, we present the weight function for standard ordering as $\Phi(\xi,\eta) = e^{-i\hbar\xi\eta/2}$, which is derived in Appendix A.

Using equation (2.98) with $\hbar = 1$,

$$\begin{aligned} F^S(q,p) &= \frac{1}{4\pi^2\sqrt{\pi}}\int d\xi\, d\eta\, dq' e^{-(q'-\eta/2)^2/2-(q'+\eta/2)^2/2}e^{-i\xi\eta/2}e^{i\xi(q'-q)}e^{-i\eta p}, \\ &= \frac{1}{2\pi\sqrt{\pi}}\int d\eta\, e^{-q^2/2-(q+\eta)^2/2}e^{-i\eta p}, \\ &= \frac{1}{\sqrt{2}\pi}e^{-\frac{1}{2}q^2-\frac{1}{2}p^2+ipq}. \end{aligned} \tag{2.101}$$

For standard ordering, $T = e^{i\partial_p\partial_q/2}$ by equation (2.99), hence

$$Te^{-q^2-p^2} = \sum_{n=0}^{\infty} \frac{i^n}{2^n n!} \partial_p^n \partial_q^n e^{-q^2-p^2} . \tag{2.102}$$

Noting that

$$\partial_p^n e^{-p^2} = H_n(-p)\, e^{-p^2} , \tag{2.103}$$

we get

$$Te^{-q^2-p^2} = \sum_{n=0}^{\infty} \frac{i^n}{2^n n!} H_n(-q) H_n(-p) e^{-q^2-p^2} . \tag{2.104}$$

We have the identity [84],

$$\sum_{n=0}^{\infty} \frac{w^n H_n(z) H_n(z_1)}{n!} = \frac{\exp\left(\frac{2w\left(2w\left(z^2+z_1^2\right)-2zz_1\right)}{4w^2-1}\right)}{\sqrt{1-4w^2}} , \tag{2.105}$$

for $|w| < 1$. Hence,

$$TW = \frac{1}{\sqrt{2}\pi} e^{-\frac{1}{2}q^2 - \frac{1}{2}p^2 + ipq} , \tag{2.106}$$

which is the same as equation (2.101)

Let us now determine the marginal probability distributions of the distribution function $F^{\Phi}(q,p)$, in the same manner as we did for the Wigner function. Integrating over the momentum, we have

$$P^{\Phi}(q) = \int dp\, F^{\Phi} = \frac{1}{2\pi} \int d\xi\, dq' \psi^*(q) \psi(q)\, \Phi(\xi,0)\, e^{i\xi(q'-q)} , \tag{2.107}$$

We see that if $\Phi(\xi,0)=1$, then

$$P^{\Phi}(q)=\frac{1}{2\pi}\int d\xi\, dq'\psi^*(q)\psi(q)\,\delta(q'-q)=|\psi(q)|^2=P(q)\,, \tag{2.108}$$

thus we recover the same probability distribution for the position as the Wigner function in equation (2.44).

Integrating over the position,

$$P^{\Phi}(p)=\frac{1}{2\pi}\int d\eta\, dq\,\psi^*(q-\eta\hbar/2)\psi(q+\eta\hbar/2)\,\Phi(0,\eta)\,e^{-i\eta p}\,. \tag{2.109}$$

If $\Phi(0,\eta)=1$ and writing $\psi(q)$ in terms of the Fourier transform of the momentum space wave function, $\phi(p)$, then

$$P^{\Phi}(p)=\frac{1}{4\pi^2\hbar}\int d\eta\, dq\, dz\, dz'\phi^*(z')\phi(z)\,e^{-\frac{iz'q}{\hbar}+\frac{izq}{\hbar}+\frac{iz'\eta}{2}+\frac{iz\eta}{2}}e^{-i\eta p}\,. \tag{2.110}$$

Letting $q\to\hbar q$, we have

$$\begin{aligned}P^{\Phi}(p)&=\frac{1}{4\pi^2}\int d\eta\, dq\, dz\, dz'\phi^*(z')\phi(z)e^{iq(z-z')}\,e^{\frac{i\eta}{2}(z+z')}e^{-i\eta p}\,.\\&=\frac{1}{2\pi}\int d\eta\, dz|\phi(z)|^2e^{-i\eta(z-p)}\,.\\&=|\psi(p)|^2=P(p)\,,\end{aligned} \tag{2.111}$$

indicating that we have the same probability distribution for the momentum as described by the Wigner function in equation (2.45).

Several quantizations have been extensively studied, some of which are listed in Table 2.1. The weight functions used to convert Weyl quantization to other quantizations are also

Table 2.1: Different quantizations and their corresponding weight functions [5].

Quantization	Quantization Map	Weight Function $\Phi(\theta,\tau)$
Weyl	$e^{i\theta q+i\tau p}\rightarrow e^{i\theta\hat{q}+i\tau\hat{p}}$	1
Standard	$e^{i\theta q+i\tau p}\rightarrow e^{i\theta\hat{q}}e^{i\tau\hat{p}}$	$e^{-i\hbar\theta\tau/2}$
Antistandard	$e^{i\theta q+i\tau p}\rightarrow e^{i\tau\hat{p}}e^{i\theta\hat{q}}$	$e^{i\hbar\theta\tau/2}$
Normal	$e^{i\theta q+i\tau p}=e^{z\overline{\alpha}-\overline{z}\alpha}\rightarrow e^{z\hat{a}^{\dagger}}e^{-\overline{z}\hat{a}}$	$e^{\hbar\theta^2/4m\omega+\hbar m\omega\tau^2/4}$
Antinormal	$e^{i\theta q+i\tau p}=e^{z\overline{\alpha}-\overline{z}\alpha}\rightarrow e^{-\overline{z}\hat{a}}e^{z\hat{a}^{\dagger}}$	$e^{-\hbar\theta^2/4m\omega-\hbar m\omega\tau^2/4}$
Born-Jordan	$q^n p^m\rightarrow\frac{1}{n+1}\sum_{k=0}^{n}\hat{q}^{n-k}\hat{p}^m\hat{q}^k$	$\operatorname{sinc}\left(\frac{1}{2}\theta\tau\hbar\right)$
Symmetric	$e^{i\theta q+i\tau p}\rightarrow\frac{1}{2}\left(e^{i\theta\hat{q}}e^{i\tau\hat{p}}+e^{i\tau\hat{p}}e^{i\theta\hat{q}}\right)$	$\cos\left(\frac{1}{2}\theta\tau\hbar\right)$

indicated. We use the notation,

$$\begin{aligned}
\alpha &= \frac{1}{\sqrt{2\hbar m\omega}}\left(m\omega q+ip\right), \\
\hat{a} &= \frac{1}{\sqrt{2\hbar m\omega}}\left(m\omega\hat{q}+i\hat{p}\right), \\
z &= i\theta\sqrt{\frac{\hbar}{2m\omega}}-\tau\sqrt{\frac{\hbar m\omega}{2}},
\end{aligned}$$

We will denote $\overline{\alpha}$, $\overline{z}$ as complex conjugates of α, z and $\hat{a}^{\dagger}$ as the Hermitian conjugate of $\hat{a}$. In Table 2.2, we show the transition operator necessary to convert from Weyl quantization to another quantization.

To briefly summarize, in this Chapter we introduced phase space quantum mechanics and used the transition operator to convert one phase space quantization to another. This technique of using the transition operator to transform between quantizations will form the basis for introducing extra physical features into a quantum system.

Table 2.2: Mapping from Weyl quantization to another quantization in phase space. This map takes the form $e^{i(\theta q+\tau p)} \to Te^{i(\theta q+\tau p)}$, where T is the transition operator [5].

Quantization	Transition Operator	$Te^{i(\theta q+\tau p)}$
Weyl	1	$e^{i(\theta q+\tau p)}$
Standard	$T_S = e^{i\hbar\partial_p\partial_q/2}$	$e^{i(\theta q+\tau p)-i\hbar\theta\tau/2}$
Antistandard	$T_{AS} = e^{-i\hbar\partial_p\partial_q/2}$	$e^{i(\theta q+\tau p)+i\hbar\theta\tau/2}$
Normal	$T_N = e^{-\hbar\partial_q^2/4m\omega-\hbar m\omega\partial_p^2/4}$	$e^{i(\theta q+\tau p)+\hbar\theta^2/4m\omega+\hbar m\omega\tau^2/4}$
Antinormal	$T_{AN} = e^{\hbar\partial_q^2/4m\omega+\hbar m\omega\partial_p^2/4}$	$e^{i(\theta q+\tau p)-\hbar\theta^2/4m\omega-\hbar m\omega\tau^2/4}$
Born-Jordan	$T_{BJ} = \operatorname{sinc}\left(\frac{1}{2}\hbar\partial_p\partial_q\right)$	$e^{i(\theta q+\tau p)}\operatorname{sinc}\left(\frac{1}{2}\hbar\theta\tau\right)$
Symmetric	$T_{sym} = \cos\left(\frac{1}{2}\hbar\partial_p\partial_q\right)$	$e^{i(\theta q+\tau p)}\cos\left(\frac{1}{2}\hbar\theta\tau\right)$

Table 2.3: Properties of star products. Each of these quantizations obey $[q,p]_{\star T} = i\hbar$, which is the phase space analogue of Heisenberg's commutation relation. The bar over $f \star_T g$ signifies the complex conjugate. The transpose refers to $\overleftarrow{\partial} \leftrightarrow \overrightarrow{\partial}$, while the Hermitian conjugate is the complex conjugate of the transpose.

Quantization	$\star_T$	Transpose	$\overline{f\star_T g}$	Hermitian Conjugate
Weyl	$\star_M = e^{\frac{i\hbar}{2}\overleftrightarrow{P}}$	$\bar{\star}_M$	$\bar{g}\star_M\bar{f}$	$\star_M$
Standard	$\star_S = e^{i\hbar\overleftarrow{\partial}_q\overrightarrow{\partial}_p}$	$\bar{\star}_{AS}$	$\bar{g}\star_{AS}\bar{f}$	$\star_{AS}$
Antistandard	$\star_{AS} = e^{-i\hbar\overleftarrow{\partial}_p\overrightarrow{\partial}_q}$	$\bar{\star}_S$	$\bar{g}\star_S\bar{f}$	$\star_S$
Normal	$\star_N = e^{-\frac{\hbar}{2m\omega}\left(\overleftarrow{\partial}_q+im\omega\overleftarrow{\partial}_p\right)\left(\overrightarrow{\partial}_q-im\omega\overrightarrow{\partial}_p\right)}$	$\bar{\star}_N$	$\bar{g}\star_N\bar{f}$	$\star_N$
Antinormal	$\star_{AN} = e^{\frac{\hbar}{2m\omega}\left(\overleftarrow{\partial}_q-im\omega\overleftarrow{\partial}_p\right)\left(\overrightarrow{\partial}_q+im\omega\overrightarrow{\partial}_p\right)}$	$\bar{\star}_{AN}$	$\bar{g}\star_{AN}\bar{f}$	$\star_{AN}$
Born-Jordan	$\star_{BJ} = \star_M\dfrac{\operatorname{sinc}\left[\frac{1}{2}\left(\overleftarrow{\partial}_p+\overrightarrow{\partial}_p\right)\left(\overleftarrow{\partial}_q+\overrightarrow{\partial}_q\right)\right]}{\operatorname{sinc}\left(\frac{1}{2}\overleftarrow{\partial}_p\overleftarrow{\partial}_q\right)\operatorname{sinc}\left(\frac{1}{2}\overrightarrow{\partial}_p\overrightarrow{\partial}_q\right)}$	$\bar{\star}_{BJ}$	$\bar{g}\star_{BJ}\bar{f}$	$\star_{BJ}$
Symmetric	$\star_{sym} = \star_M\dfrac{\cos\left[\frac{1}{2}\left(\overleftarrow{\partial}_p+\overrightarrow{\partial}_p\right)\left(\overleftarrow{\partial}_q+\overrightarrow{\partial}_q\right)\right]}{\cos\left(\frac{1}{2}\overleftarrow{\partial}_p\overleftarrow{\partial}_q\right)\cos\left(\frac{1}{2}\overrightarrow{\partial}_p\overrightarrow{\partial}_q\right)}$	$\bar{\star}_{sym}$	$\bar{g}\star_{sym}\bar{f}$	$\star_{sym}$

Chapter 3

Coarse-Graining

3.1 Motivation

In Chapter 2, we introduced the concept of quantization and demonstrated that many different quantizations and operator orderings exist. We then showed that quantum mechanics can be done in phase space, in which distribution functions and star products are used, rather than density operators and operator multiplication.

One advantage of phase space quantum mechanics is that it is straightforward to study quantization. For the same physical system, each possible distribution function and associated star product corresponds to a distinct quantization. To convert between different quantizations, a transition operator is used.

One of the properties of the Wigner function is that it can take on negative values, hence the Wigner function cannot be interpreted as a probability distribution. The question regarding when the Wigner function is non-negative has been previously studied in, for example, [85] and [86]. Ref. [85] showed that only wavefunctions exponentiating quadratic polynomials correspond to a non-negative Wigner function. In [86], it was demonstrated that only the Wigner distributions of Gaussian wavefunctions remain non-negative under time-evolution.

In this Chapter, we will review the Husimi distribution, which is always non-negative. It will be shown that this non-negativity is the result of Gaussian smoothing the Wigner function. As an example of an application of Gaussian smoothing, we look at the $n \to \infty$ limit of the Wigner function.

3.2 The Husimi Distribution

Consider the transition operator of $T_H = e^{\hbar\partial_q^2/4m\kappa+\hbar m\kappa\partial_p^2/4}$ (or weight function $\Phi^H(\xi,\eta) = e^{-\hbar\xi^2/4m\kappa-\hbar m\kappa\eta^2/4}$), where m is the mass and κ is the smoothing parameter. By applying the transition operator to the Wigner function or using equation (2.100), we find [5],

$$F^H(q,p,t) = \frac{1}{\pi\hbar}\int dq'dp'e^{-m\kappa(q'-q)^2/\hbar-(p'-p)^2/\hbar m\kappa}W(q',p',t), \tag{3.1}$$

which is known as the Husimi distribution. This result is derived in Appendix B by way of the transition operator.

We should note two important features of the Husimi distribution. First, F^H includes an integral over the product of the Wigner function and a Gaussian; thus, the Husimi distribution describes a Gaussian-smoothed (coarse-grained) Wigner function, in which the smoothing parameter is $\hbar$. By writing $W(q',p',t)$ in the form of equation (2.41), integrating equation (3.1) over p', then writing the result as a series in $x = q - \frac{1}{2}y\hbar$ and $z = q + \frac{1}{2}y\hbar$, it is possible to show that $F_H(q,p,t)$ must be non-negative [87].

Second, the antinormal distribution function, obtained from the Wigner function with transition operator $T_{AN} = e^{\hbar\partial_q^2/4m\omega+\hbar m\omega\partial_p^2/4}$ (Table 2.2), is a special case of the Husimi distribution with $\kappa = \omega$. Here, ω is a frequency that is used to define the creation and annihilation operators using the same procedure that one does for the simple harmonic oscillator. Physically, ω may be the frequency of an external field.

One way to see why ω is required in the transition operator, rather than just $\hbar$ (in the case of T_S, for example) comes from a dimensional argument. If only $\hbar^n$ is present within the transition operator, the argument of the transition operators needs to be $\partial_q^n\partial_p^n$ as the dimensions of $\hbar$ are equal to the dimensions of the product of momentum and position. If we want a transition operator that separates ∂_q from ∂_p, it is necessary to introduce an additional parameter such that the argument in the transition operator is still dimensionless. Hence, both m and ω are required to be part of the transition operator. A similar argument follows for why κ is required in T_H.

To understand the relationship between T_H and T_{AN}, note that equation (2.100) can be written as [88]

$$F^{\Phi} = \frac{1}{4\pi^2} \int d\xi \, d\eta \, \mathrm{Tr} \left[\hat{\rho} \, e^{i(\xi \hat{q} + \eta \hat{p})} \Phi(\xi, \eta) \right] e^{-i(\xi q + \eta p)} \, . \tag{3.2}$$

With $\Phi^H(\xi, \eta) = e^{-\hbar \xi^2/4m\kappa - \hbar m \kappa \eta^2/4}$, we can then rewrite F^H as [88]

$$F^{H} = \frac{1}{4\pi^2} \int d\xi \, d\eta \, \mathrm{Tr} \left[\hat{\rho} \, e^{-\bar{z}\hat{b}} e^{z\hat{b}^\dagger} \right] e^{-i(\xi q + \eta p)} \, , \tag{3.3}$$

where

$$z = i\xi \sqrt{\frac{\hbar}{2m\kappa}} - \eta \sqrt{\frac{\hbar m \kappa}{2}} \, , \tag{3.4}$$

$$\hat{b} = \frac{1}{\sqrt{2\hbar m \kappa}} \left(m\kappa \hat{q} + i\hat{p} \right) . \tag{3.5}$$

It is possible to write $\hat{b}$ in terms of the annihilation operator $\hat{a}$ and creation operator $\hat{a}^\dagger$, such that [88]

$$\hat{b} = \mu \hat{a} + \nu \hat{a}^\dagger \, , \tag{3.6}$$

where

$$\mu = \frac{1}{2} \left(\sqrt{\frac{\kappa}{\omega}} + \sqrt{\frac{\omega}{\kappa}} \right) , \tag{3.7}$$

$$\nu = \frac{1}{2} \left(\sqrt{\frac{\kappa}{\omega}} - \sqrt{\frac{\omega}{\kappa}} \right) . \tag{3.8}$$

Using the property that $[\hat{a}, \hat{a}^\dagger] = 1$, we also have $[\hat{b}, \hat{b}^\dagger] = 1$. Therefore, as discussed in [89], equation (3.6) corresponds to the annihlation operator of a squeezed state while $\hat{b}^\dagger$ is the creation operator of a squeezed state. When $\omega = \kappa$, we see that $\mu = 1$ and $\nu = 0$, so $\hat{b} = \hat{a}$ and

$\hat{b}^\dagger = \hat{a}^\dagger$. In this case, the distribution function is related to a coherent state wave packet [90],

$$|\alpha\rangle = e^{\alpha\hat{a}^\dagger - \overline{\alpha}\hat{a}}|0\rangle\,, \tag{3.9}$$

where α is the eigenvalue for the equation $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$. It is possible to consider the coherent state as an example of a squeezed state without squeezing. Therefore, physically the antinormal distribution function is the smoothing of the Wigner function by a coherent state wave packet, while the Husimi distribution is the smoothing by a squeezed state wave packet [88].

Even though the Husimi distribution is related to the antinormal distribution function, it would be improper to say that the Husimi distribution is a quantization or ordering as an additional physical effect (coarse-graining) is introduced with the transition operator. Rather, we shall adopt the terminology of augmented quantization to describe the introduction of extra physical features into the distribution function by means of the transition operator. In other words, T_H converts Weyl quantization to an augmented quantization.

Let us rewrite F^H as [1,5]

$$F^H(q,p,t) = \frac{1}{\pi\hbar}\int dq'dp' e^{-\frac{1}{2}(q'-q)^2/(\delta q)^2 - \frac{1}{2}(p'-p)^2/(\delta p)^2}W(q',p',t)\,, \tag{3.10}$$

where $\delta q = \sqrt{\frac{\hbar}{2}}s$, $\delta p = \sqrt{\frac{\hbar}{2}}\frac{1}{s}$, and $s = \frac{1}{m\kappa}$. As $\delta q\delta p = \frac{\hbar}{2}$, we might guess that the Husimi distribution is related to the minimum uncertainty wave packet in the position representation [1],

$$\langle x|q,p\rangle = \frac{1}{(2\pi s^2)^{1/4}}e^{-(x-q)^2/4s^2}e^{ipx/\hbar}\,, \tag{3.11}$$

where the wave packet is centred at position q and momentum p.

To demonstrate how the minimum uncertainty wave packet is related to the Husimi distribution, define a phase space distribution, $\rho(q,p)$, for a system described with the density operator, $\hat{\rho}$ [1]:

$$\rho(q,p) := \frac{1}{2\pi\hbar} \langle q,p|\hat{\rho}|q,p\rangle = \frac{1}{2\pi\hbar} \mathrm{Tr}\left(|q,p\rangle\langle q,p|\hat{\rho}\right) , \tag{3.12}$$

where we have used the cyclic property of the trace to write $\rho(q,p)$ as the trace over the product of $\hat{\rho}$ and the pure state density operator, $|q,p\rangle\langle q,p|$. We will now show that $\rho(q,p) = F^H(q,p)$.

From equation (2.51),

$$\mathrm{Tr}(\hat{\rho}\hat{A}) = \int dq'dp' W(q',p')A(q',p') . \tag{3.13}$$

Thus, the trace of two density operators, $\hat{\rho}_1$, $\hat{\rho}_2$, can be written as the integral over their Wigner functions [1],

$$\mathrm{Tr}\left(\hat{\rho}_1\hat{\rho}_2\right) = 2\pi\hbar \int dq'dp' W_1(q',p')W_2(q',p') , \tag{3.14}$$

where the 2π appears because $W = \frac{1}{2\pi\hbar}\mathcal{W}(\hat{\rho})$. Applying the Wigner transform of equation (2.18) to the density operator of minimum uncertainty wave packet, the resultant Wigner function is [1]

$$W_1(q',p') = \frac{1}{\pi\hbar} e^{-\frac{1}{s^2}(q'-q)^2/\hbar - s^2(p'-p)/\hbar} . \tag{3.15}$$

If $\hat{\rho}_2$ corresponds to the Wigner function $W_2(q',p') = W(q',p')$ for an arbitrary system, we then have $\rho(q,p) = F^H(q,p)$, illustrating the intrinsic relationship between the Husimi distribution and the minimum uncertainty wave packet.

For a Wigner function describing a pure state $|\psi\rangle$, it is straightforward to use equation (3.12) to demonstrate the non-negativity of the Husimi distribution [1]. With $\hat{\rho} = |\psi\rangle\langle\psi|$ and with the fact that $\rho(q,p) = F^H(q,p)$, equation (3.12) gives

$$F^H(q,p) = \frac{1}{2\pi\hbar} \langle q,p|\psi\rangle\langle\psi|q,p\rangle = \frac{1}{2\pi\hbar} |\langle q,p|\psi\rangle|^2 \geq 0 . \tag{3.16}$$

From the weight function $\Phi^H(\xi,\eta) = e^{-\hbar\xi^2/4m\kappa - \hbar m\kappa\eta^2/4}$ of the Husimi distribution, we see that $\Phi(\xi,0) \neq 1$ and $\Phi(0,\eta) \neq 1$. This implies that the Husimi distribution does not yield the same marginal probability distributions as the Wigner function. Rather [1],

$$\begin{aligned}
P^H(q) &= \int dp\, F^H(q,p)\,, \\
&= \frac{1}{2\pi\hbar}\int dp\, dx\, dx'\; \langle q,p|x\rangle \langle x|\hat{\rho}|x'\rangle \langle x'|q,p\rangle\;, \\
&= \frac{1}{2\pi\hbar}\int dp\, dx\, dx'\; \frac{1}{\sqrt{2\pi s^2}} e^{-(x-q)^2/4s^2-(x'-q)^2/4s^2} e^{ip(x'-x)/\hbar} \langle x|\hat{\rho}|x'\rangle\;, \\
&= \int dx\, \frac{1}{\sqrt{2\pi s^2}} e^{-(x-q)^2/2s^2} \langle x|\hat{\rho}|x\rangle\;,
\end{aligned} \tag{3.17}$$

where we have applied the resolution of the identity,

$$1 = \int dx\, |x\rangle \langle x|\;. \tag{3.18}$$

By a similar procedure in which the momentum representation of the minimum uncertainty wave packet is used, we find [91],

$$\begin{aligned}
P^H(p) &= \int dq\, F^H(q,p)\,, \\
&= \int dk \sqrt{\frac{2s^2}{\pi\hbar^2}} e^{-2s^2(k-p)^2/\hbar^2} \langle k|\hat{\rho}|k\rangle\;.
\end{aligned} \tag{3.19}$$

where we have used the momentum representation of the wave packet,

$$\langle k|q,p\rangle = \left(\frac{2s^2}{\pi\hbar^2}\right)^{1/4} e^{-s^2(k-p)^2/\hbar^2} e^{-i(k-p)q/\hbar} \tag{3.20}$$

and the resolution of the identity,

$$1 = \int dk\, |k\rangle \langle k|\;. \tag{3.21}$$

Therefore, from equations (3.17) and (3.19), we find

$$P^H(q) = \int dx \frac{1}{\sqrt{2\pi s^2}} e^{-(x-q)^2/2s^2} P(x)\,, \tag{3.22}$$

$$P^H(p) = \int dk \sqrt{\frac{2s^2}{\pi\hbar^2}} e^{-2s^2(k-p)^2/\hbar^2} P(k)\,, \tag{3.23}$$

where $P(x)$ and $P(k)$ are the position and momentum marginal probability distributions calculated from the Wigner function in equations (2.44) and (2.45). Equations (3.22) and (3.23) represent the joint (simultaneous) measurements of the position and momentum of a system [92, 93] (and references therein).

Using equation (2.88), the Husimi star product is

$$\star_H = \star_M e^{\frac{m\kappa\hbar}{2}\overleftarrow{\partial}_p\overrightarrow{\partial}_p + \frac{\hbar}{2m\kappa}\overleftarrow{\partial}_q\overrightarrow{\partial}_q} = \exp\left[\frac{\hbar}{2m\kappa}\left(\overleftarrow{\partial}_q - im\kappa\overleftarrow{\partial}_p\right)\left(\overrightarrow{\partial}_q + im\kappa\overrightarrow{\partial}_p\right)\right]. \tag{3.24}$$

The antinormal star product in Table 2.3 can then be recovered by letting $\kappa \to \omega$.

Let $a = \frac{\hbar}{4m\kappa}$ and $b = \frac{1}{4}m\kappa\hbar$. As an example of time-evolution of the Husimi distribution, we will apply $T_H = e^{a\partial_q^2 + b\partial_p^2}$ to the Moyal bracket for the simple harmonic oscillator. This yields

$$\begin{aligned}
i\hbar\frac{\partial T_H W}{\partial t} &= T_H H \star_H T_H W - T_H W \star_H T_H H\,,\\
&= (H+a+b) \star_M e^{2b\overleftarrow{\partial}_p\overrightarrow{\partial}_p + 2a\overleftarrow{\partial}_q\overrightarrow{\partial}_q} T_H W\\
&\quad - T_H W \star_M e^{2b\overleftarrow{\partial}_p\overrightarrow{\partial}_p + 2a\overleftarrow{\partial}_q\overrightarrow{\partial}_q} (H+a+b)\,,\\
&= H \star_M \left(1 + 2b\overleftarrow{\partial}_p\overrightarrow{\partial}_p + 2a\overleftarrow{\partial}_q\overrightarrow{\partial}_q + 2b^2\overleftarrow{\partial}_p^2\overrightarrow{\partial}_p^2 + 2a^2\overleftarrow{\partial}_q^2\overrightarrow{\partial}_q^2\right) T_H W\\
&\quad - T_H W \star_M \left(1 + 2b\overleftarrow{\partial}_p\overrightarrow{\partial}_p + 2a\overleftarrow{\partial}_q\overrightarrow{\partial}_q + 2b^2\overleftarrow{\partial}_p^2\overrightarrow{\partial}_p^2 + 2a^2\overleftarrow{\partial}_q^2\overrightarrow{\partial}_q^2\right) H\,.
\end{aligned}$$

Simplifying, we find

$$
\begin{aligned}
i\hbar\frac{\partial T_H W}{\partial t} &= [H, T_H]_{\star_M} + 2b\, p \star_M \partial_p T_H W + 2a\, q \star_M \partial_q T_H W - 2b\, \partial_p T_H W \star_M p \\
&\quad - 2a\, \partial_q T_H W \star_M q\,, \\
&= [H, T_H]_{\star_M} - 2i\hbar\,(b-a)\,\partial_p \partial_q T_H W\,.
\end{aligned}
$$

Therefore,

$$
i\hbar\frac{\partial T_H W}{\partial t} = [H, T_H W]_{\star_M} - \frac{i\hbar}{2}\left(m\kappa\hbar - \frac{\hbar}{m\kappa}\right)\partial_p \partial_q T_H W\,. \tag{3.25}
$$

This demonstrates that the equation of motion of the simple harmonic oscillator Husimi distribution function is similar to the equation of motion of the simple harmonic oscillator Wigner function. A difference is that an extra mixed partial derivative is also present. The original equation of motion described is modified by $\mathcal{O}(\hbar^2)$ with the Husimi transition operator.

The time-independent Husimi function for the simple harmonic oscillator can be calculated with either the transition operator or the weight function, such that, with $s = 1$ [31],

$$
F^H(q,p) = \frac{1}{2\pi\hbar n!}\left(\frac{q^2+p^2}{2\hbar}\right)^n \exp\left[\frac{-q^2-p^2}{2\hbar}\right], \tag{3.26}
$$

which is plotted in Figure 3.1. The energy levels for the simple harmonic oscillator Husimi distribution are $E = \left(n + \frac{1}{2}\right)\hbar$ because the transition operator acting on the stargenvalue equation, $H \star_M W = EW$, does not affect the energy.

An advantage with using the Husimi distribution is that it is easier than the Wigner function to associate with coarse-grained classical mechanics. Consider the time-evolution of the Husimi distribution and a corresponding classical coarse-grained distribution. Initially, they evolve in a similar manner, but as time development continues, differences between the evolutions arise. The classical-course grained solutions tends to approach a smooth dis-

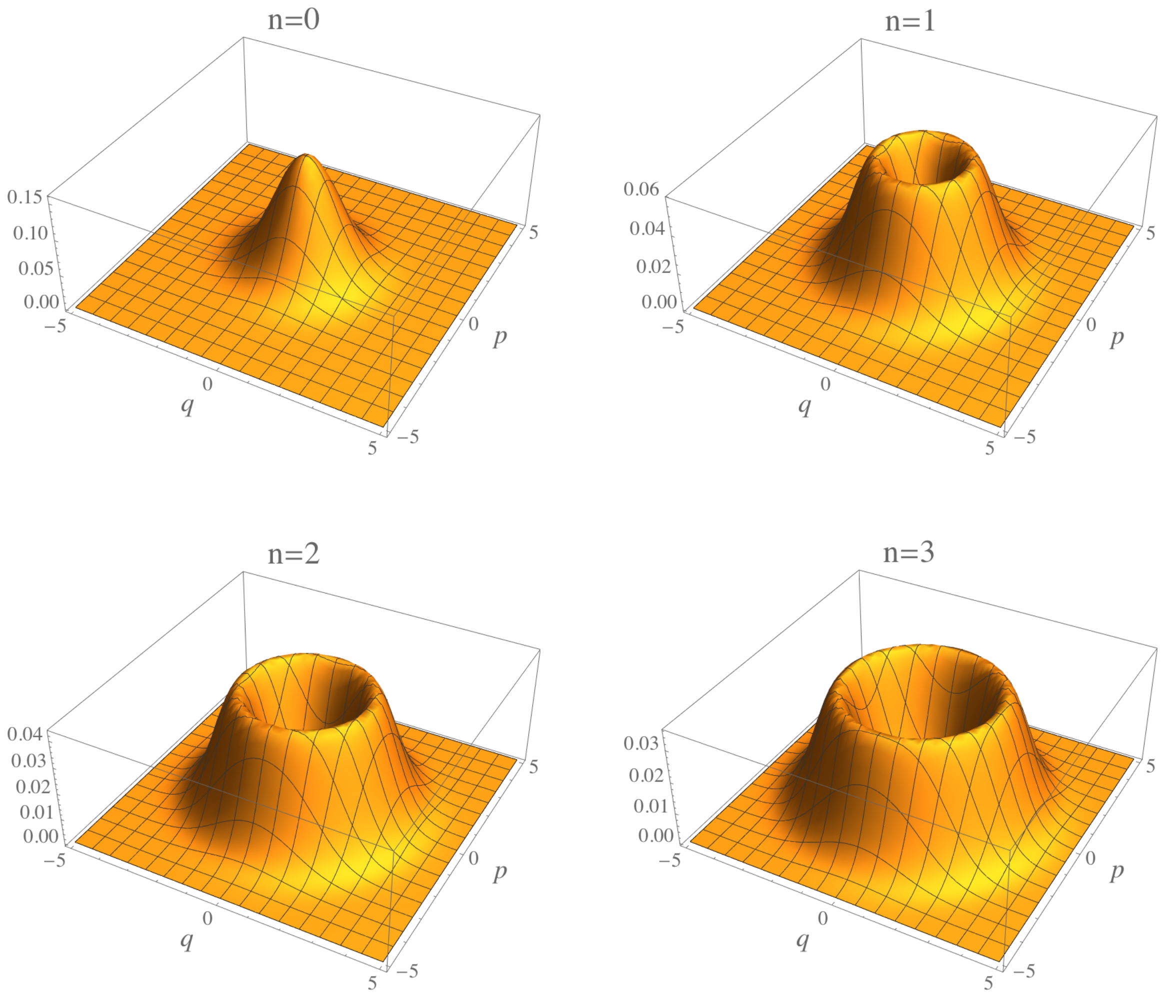

Figure 3.1: The simple harmonic oscillator Husimi distribution for the first four energy levels. Notice that the distribution is nonnegative, in contrast to Figure 2.1

tribution whereas the Husimi distribution breaks up into localized sections in phase space, which is the result of an interference process. Ref. [31] numerically quantified these effects by comparing the entropies of the Husimi and classical coarse-grained distributions during their time-evolution. This indicated that using the Husimi distribution to analyze the correspondence between quantum mechanics and classical mechanics became invalid after a certain period of time.

To conclude, the Husimi distribution is an example of augmented quantization as it incorporates additional physics (coarse-graining) into a distribution function, for coarse-graining was not present in the original system. In the next Section, we will consider a generalization of the Husimi distribution in which the Wigner function is smoothed by a different quantity, which is another example of augmented quantization.

3.3 A Generalization of the Husimi Distribution

The Husimi distribution smooths the features of the Wigner function by $\mathcal{O}(\hbar)$. If we wish to smooth by a different quantity, η, then the distribution function is,

$$F^{\eta}(q,p,t) = \frac{1}{\pi\eta}\int dq'dp' e^{-\frac{1}{2}(q'-q)^2/(\delta q)^2 - \frac{1}{2}(p'-p)^2/(\delta p)^2} W(q',p',t)\,, \tag{3.27}$$

where now

$$\delta q = \sqrt{\frac{\eta}{2}}s\,, \tag{3.28}$$

$$\delta p = \sqrt{\frac{\eta}{2}}\frac{1}{s}\,. \tag{3.29}$$

It is seen that $\delta q \delta p = \frac{\eta}{2}$, which no longer describes a minimum uncertainty wavepacket. For $\eta < \hbar$, F^{η} can still have negative values. Non-negativity only holds for $\eta \geq \hbar$ [87].

We see that $F^{\eta} \to F^{H}$ in the limit of $\eta \to \hbar$. When smoothing by η rather than $\hbar$, the Wigner function can be converted to F^{η} with the transition operator of

$$T_{\eta} = e^{\eta\left(\frac{s}{4}\partial_q^2 + \frac{1}{4s}\partial_p^2\right)}\,, \tag{3.30}$$

giving the star product of

$$\star_{\eta} = \star_M \exp\left[\frac{\eta}{2}\left(\frac{1}{s}\overleftarrow{\partial}_p\overrightarrow{\partial}_p + s\overleftarrow{\partial}_q\overrightarrow{\partial}_q\right)\right]\,, \tag{3.31}$$

which we calculated with equation (2.88). In Appendix B, we show that F^{H} and F^{η} are Weierstrass transforms of the Wigner function.

To compare with the time-evolution in equation (3.25) for the simple harmonic oscillator Husimi distribution, the equation of motion for the generalized Husimi distribution of the simple harmonic oscillator is

$$\frac{\partial T_\eta W}{\partial t} = \frac{1}{i\hbar}[H, T_\eta W]_{\star_M} - \frac{1}{2}\left(\frac{\eta}{s} - s\eta\right)\partial_p\partial_q T_\eta W \,. \tag{3.32}$$

We see that the right-hand side of equation (3.32) is the Moyal bracket between H and $T_\eta W$ plus an additional term of $\mathcal{O}(\hbar\eta)$, rather than $\mathcal{O}(\hbar^2)$. The time-independent generalized Husimi distribution for the simple harmonic oscillator are plotted in Figure 3.2 and 3.3.

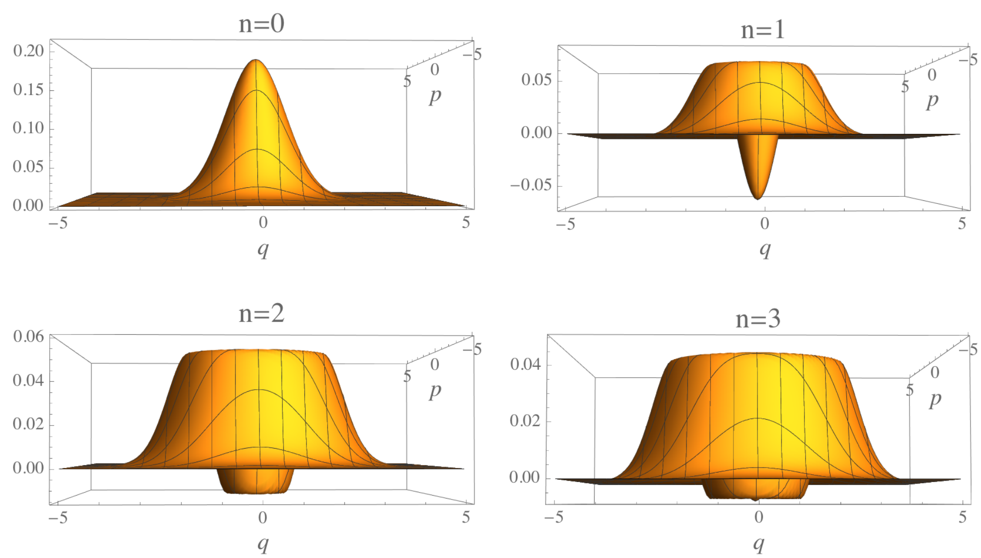

Figure 3.2: The generalized Husimi distribution for the first four energy levels of the simple harmonic oscillator. We have set $\hbar = 1$ and $\eta = 0.5$. Notice the distribution still has negative values even though smoothing was done.

3.4 Smoothing in the $n \to \infty$ Limit of the Wigner Function

As an example of coarse-graining, let us smooth the Wigner function for large n. We will focus on the simple harmonic oscillator and demonstrate that the result of this smoothing is a delta function-like distribution that describes a classical system. As argued in [58], coarse-graining may be required to recover classical mechanics from quantum mechanics.

To investigate the emergence of classical mechanics from quantum mechanics, fix the classical energy level, $E_c = r_c^2/2$. For the purposes of this calculation, we set $m = \omega = 1$ and

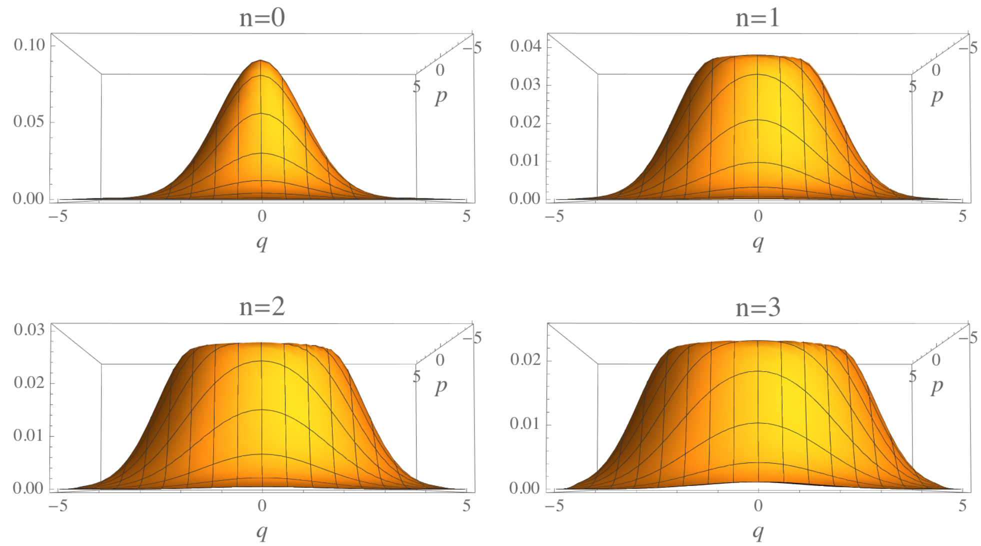

Figure 3.3: The generalized Husimi distribution for the first four energy levels of the simple harmonic oscillator. We have set $\hbar = 1$ and $\eta = 2$. Unlike Figure 3.2, the distribution is non-negative because $\eta > \hbar$.

$q^2 + p^2 = r^2$. With $E_n = \left(n + \frac{1}{2}\right)\hbar$ and $E_c = E_n$, we find that $r_c^2 \sim 2n\hbar$ for large n. Hence, as r_c is constant, so must $n\hbar$. From Figure 3.4, we see that the Wigner function features rapid oscillations at large n. Therefore, coarse-graining is needed to convert the Wigner function into a distribution for a classical system.

We want to focus on the height and radial width of the resultant distribution in phase space. However, if n increases, so does E_n. As a result, r_c also increases. We will therefore focus on the constant energy of $E_n = E_c = 1$. The effect of this is to scale r_c during Gaussian smoothing, because otherwise $r_c \to \infty$. From equation (2.69), the Wigner function of the simple harmonic oscillator at large n then behaves as

$$W_n(r) \sim (-1)^n n e^{-nr^2} L_n(2nr^2)\,. \tag{3.33}$$

In Figure 3.4, we plot, as an illustrative example, equation (3.33) for $n = 50$ and $n = 200$. At $r = 0$, the Wigner function is maximized, then rapidly decreases after the first oscillation. The inter-node distances of the oscillations slowly increase as the radial distance, r, increases. Furthermore, the amplitude of the oscillations first decreases before increasing

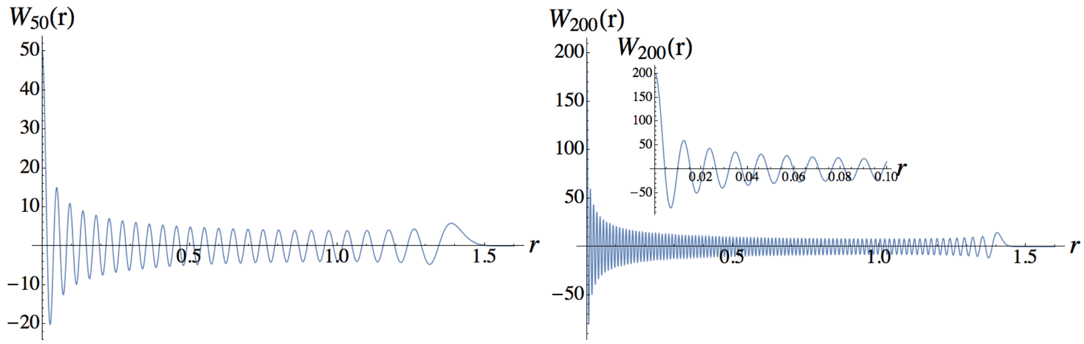

Figure 3.4: The Wigner function for $n = 50$ (left) and $n = 200$ (right). The inset plot depicts $n = 200$ for $0 \le r \le 0.1$

again.

We will smooth equation (3.33) with the Gaussian,

$$f(r) = \exp\left[-\frac{r^2}{\sigma}\right], \tag{3.34}$$

where σ is related to the inter-node separation of the oscillations and describes the coarse-graining width of the Gaussian. For simplicity, we will consider σ equal to the distance between the first two nodes of the Wigner function and the distance between the final two nodes of the Wigner function.

Combining equations (3.33) and (3.34),

$$g_n(r) = \mathcal{N} \int_0^\infty dy\, W_n(y) f(r-y) y^2 \tag{3.35}$$

is then numerically evaluated, where $\mathcal{N}$ is the normalization constant. We note that integration is done over the interval $[0, \infty)$ as $r \ge 0$ by definition and the y^2 appears because we are using spherical coordinates. In addition, we are not claiming that the two chosen values of σ are the physically correct ones that should be used to Gaussian-smooth the Wigner function. Rather, the σ's that we use are merely a demonstration of applying Gaussian smoothing to recover the classical phase space distribution of the simple harmonic oscillator with a constant energy.

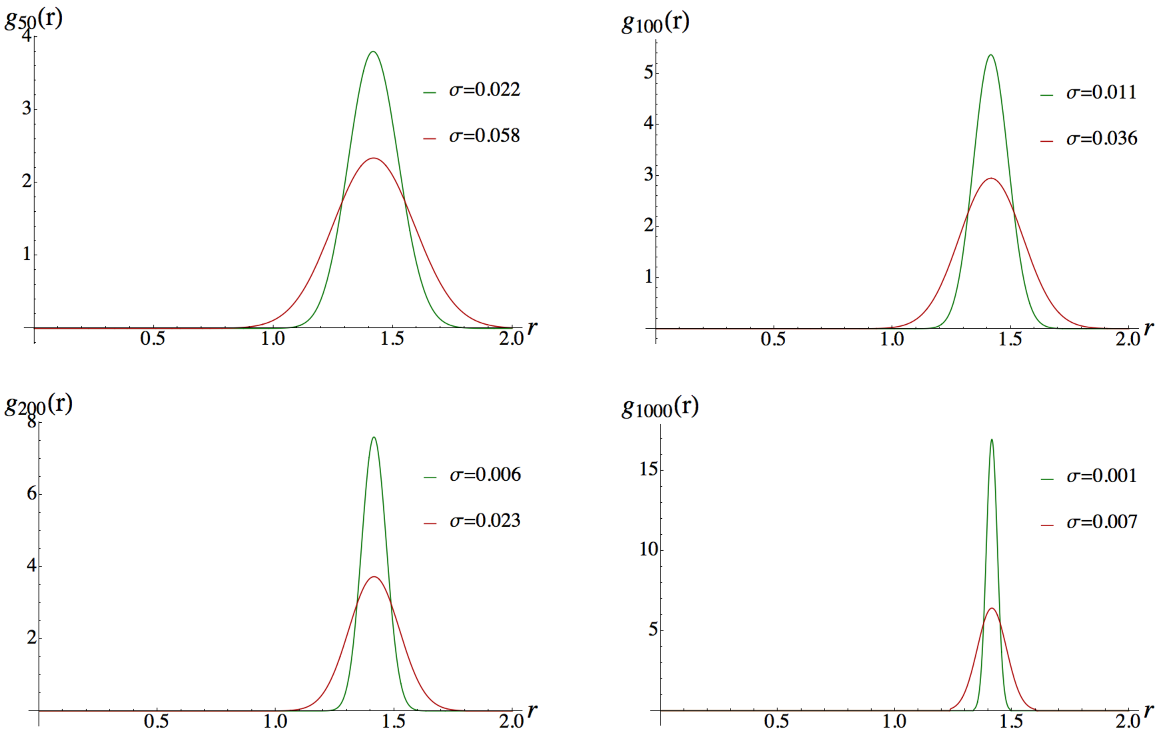

Figure 3.5: Normalized convolutions of Wigner functions smoothed with the maximum (red) and minimum (green) inter-nodal distances. We see that larger n corresponds to a decrease in width and increase in height. This seems to indicates that $n \to \infty$ implies that a delta function-like distribution will be found.

In Figure 3.5, we consider four Wigner functions convolved with equation (3.34), For $n = 50,\ 100,\ 200,\ 1000$, we look at the cases in which σ is the minimum and maximum inter-nodal distances. As n increases, σ becomes smaller because the inter-nodal distances decrease. Furthermore, with an increase in n, the resultant Gaussian becomes narrower and shifts towards $r_c = \sqrt{2}$. Therefore, as $n \to \infty$, we would expect the width of the Gaussian to go to zero, thereby becoming a delta function-like distribution. Physically, this corresponds to a distribution in a classical phase space that describes a simple harmonic oscillator of energy $E_c = 1$.

In this Chapter, we have introduced the Husimi function and demonstrated that it describes the Gaussian smoothing of the Wigner function. As this Gaussian smoothing is introduced using the transition operator, the Husimi distribution is an example of augmented quantization. A generalization of the Husimi distribution was also briefly examined so that the Wigner function is smoothed by a different parameter. We then used the technique of coarse-graining to Gaussian smooth the $n \to \infty$ limit of the Wigner function.

Coarse-graining is only one possible physical effect that can be introduced to a quantum system with the transition operator. In the next Chapter, we shall consider further physical effects in our exploration of augmented quantization.

Chapter 4

Local Transition Operators

4.1 Motivation

Dissipative forces appear in many situations, such as when investigating the interaction between multiple systems or a system and its environment [94]. One of the simplest examples of a dissipative system is the damped harmonic oscillator. Classically, the damped harmonic oscillator is well-understood. However, quantization of the damped harmonic oscillator is much more difficult than the quantization of the simple harmonic oscillator [95–97].

Our goal in this Chapter is to explore augmented quantization. Using the damped harmonic oscillator as motivation, we show that a transition operator with position and momentum dependence gives the correct equations of motion in the $\hbar \to 0$ limit of the $\star_T$-bracket. Hence, such a generalized transition operator can convert a Weyl quantized system to a system with augmented quantization. We then derive and discuss the resultant star product. We believe these results are original and of interest to the larger field of phase space quantum mechanics.

To conclude this Chapter, we apply our results to the generalized uncertainty principle of quantum gravity phenomenology and show that augmented quantization can yield the correct modification of Heisenberg's commutation relation. This will demonstrate that augmented quantization is applicable beyond damping.

4.2 The Damped Harmonic Oscillator

4.2.1 Quantization of Dissipative Systems

Consider a classical system that includes a dissipative force. This dissipative force can be introduced to a quantum system using canonical quantization, Heisenberg's equations of motion, or path integrals. However, problems arise in each of these methods, such as the lack of existence of a stable ground state [95].

To illustrate this difficulty, we will look at the Schrödinger equation for damping, summarizing the calculations of [95, 98]. Consider the classical equation of motion

$$m\ddot{q} + \frac{dV(q)}{dq} + 2\gamma\dot{q}^{\nu} = 0\,, \tag{4.1}$$

where γ is the damping constant. Multiplying this equation by $\dot{q}$, we have

$$\dot{q}\left(m\ddot{q} + \frac{dV(q)}{dq} + 2\gamma\dot{q}^{\nu}\right) = 0\,, \tag{4.2}$$

The dissipative force is equal to $F_d = -2\gamma\dot{q}^{\nu}$. Hence, the amount of energy dissipation is

$$\Delta E = 2\gamma \int dq \frac{dt}{dt}\dot{q}^{\nu} = 2\gamma \int dt\dot{q}^{\nu+1}\,.$$

From equation (4.2), we then have

$$\frac{d}{dt}\left(\frac{m}{2}\dot{q}^2 + \Delta E + V(q)\right) = 0\,, \tag{4.3}$$

because $\frac{d\Delta E}{dt} = 2\gamma\dot{q}^{\nu+1}$. Equation (4.3) shows that the sum of the kinetic energy, potential energy, and energy dissipated from the system is conserved.

We can take the canonical momentum of $p = m\dot{q}$, so the Hamiltonian is

$$H = \frac{p^2}{2m} + V(q) + \frac{2\gamma}{m^{\nu}} \int dq\, p^{\nu}\,. \tag{4.4}$$

This Hamiltonian can be quantized by using $q \to \hat{q}$ and $p \to \hat{p}$. In the position representation, the time-independent Schroödinger equation is

$$i\hbar\frac{\partial\psi(q,t)}{\partial t} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2} + V(q) + 2\gamma\left(\frac{-i\hbar}{m}\right)^{\nu}\frac{\partial^{\nu-1}}{\partial q^{\nu-1}}\right]\psi(q,t)\,. \tag{4.5}$$

With $\nu = 1$, the frictional force is proportional to velocity and describes linear damping. The equations for the wavefunction and its complex conjugate are

$$i\hbar\frac{\partial\psi(q,t)}{\partial t} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2} + V(q) + 2\gamma\left(\frac{-i\hbar}{m}\right)\right]\psi(q,t)\,, \tag{4.6}$$

$$-i\hbar\frac{\partial\overline{\psi}(q,t)}{\partial t} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2} + V(q) + 2\gamma\left(\frac{i\hbar}{m}\right)\right]\overline{\psi}(q,t)\,. \tag{4.7}$$

Thus, the continuity-like equation is

$$\frac{\partial|\psi(q,t)|^2}{\partial t} + \frac{\partial}{\partial q}j(q,t) = -\frac{4\gamma}{m}|\psi(q,t)|^2\,, \tag{4.8}$$

where

$$j(q,t) = -\frac{i\hbar}{2m}\left(\overline{\psi}\frac{\partial\psi}{\partial q} - \psi\frac{\partial\overline{\psi}}{\partial q}\right) \tag{4.9}$$

is the current density. With the assumption that the current density goes to zero at the boundaries,

$$\frac{\partial}{\partial t}\int dq|\psi(q,t)|^2 = -\frac{4\gamma}{m}\int dq|\psi(q,t)|^2\,. \tag{4.10}$$

Therefore,

$$\int dq|\psi(q,t)|^2 = e^{-4\gamma t/m}\int dq|\psi(q,0)|^2\,. \tag{4.11}$$

Some systems dissipate until they reach a ground state, but as no stable ground state is present, the above method of quantizing a damped classical system is not general and should be modified.

4.2.2 Augmented quantization of the Simple Harmonic Oscillator

By focussing on the linearly damped harmonic oscillator, let us now look at the quantization of dissipative systems from a different perspective. This will be done by using phase space quantum mechanics, the classical equations of motion for the damped harmonic oscillator, and the simple harmonic oscillator Hamiltonian.

The Poisson bracket is intimately connected to Hamilton's equations of motion [99]:

$$\dot{q} = \{q,H\} = \frac{\partial H}{\partial p}\,, \tag{4.12}$$

$$\dot{p} = \{p,H\} = -\frac{\partial H}{\partial q}\,, \tag{4.13}$$

where $H = H(q,p)$ is the Hamiltonian and $\{f,g\} = f\left(\overleftarrow{\partial}_q\overrightarrow{\partial}_p - \overleftarrow{\partial}_p\overrightarrow{\partial}_q\right)g$ is the Poisson bracket. By exponentiating the Poisson bracket (multiplied by $i\hbar/2$), the Moyal product is then formed, as was discussed in Section 2.6.

Consider the linearly damped simple harmonic oscillator, described by $\ddot{q} = -2\gamma\dot{q} - q$, where $\gamma > 0$ is a damping constant. Its equations of motion are [33]

$$\dot{q} = p\,, \tag{4.14}$$

$$\dot{p} = -q - 2\gamma p\,. \tag{4.15}$$

We will now summarize the proposal of [33], in which augmented quantization is applied to the classical simple harmonic oscillator to describe the damped quantum harmonic oscillator. Using the equations of motion of the classical damped harmonic oscillator,[5] and

[5]Note that that the equations of motion,

$$\dot{q} = p - 2\gamma q\,,$$
$$\dot{p} = -q\,,$$

the Hamiltonian of the undamped simple harmonic oscillator, we will derive a modified Poisson bracket, $M(f,g)$, that will include damping. From this new bracket, we will then find the star product and the transition operator.

Consider the new bracket M, such that

$$\dot{q} = M(q,H_0)\,, \tag{4.16}$$

$$\dot{p} = M(p,H_0)\,, \tag{4.17}$$

where $H_0 = \frac{p^2}{2} + \frac{q^2}{2}$ is the undamped simple harmonic oscillator. Thus,

$$M(q,H_0) = p\,, \tag{4.18}$$

$$M(p,H_0) = -q - 2\gamma p\,. \tag{4.19}$$

Take $M(f,g) = \{f,g\} + A(f,g)$ for arbitrary functions f and g. Then

$$\dot{q} = p + A(q,H_0)\,, \tag{4.20}$$

$$\dot{p} = -q + A(p,H_0)\,. \tag{4.21}$$

Therefore, comparing with equations (4.14) and (4.15),

$$A(q,H_0) = 0\,, \tag{4.22}$$

$$A(p,H_0) = 2\gamma p\,. \tag{4.23}$$

One possible form of A is $A(f,g) = -2\gamma \frac{\partial f}{\partial p}\frac{\partial g}{\partial p}$, yielding the modified Poisson bracket, [33]

$$M(f,g) = \{f,g\} - 2\gamma \frac{\partial f}{\partial p}\frac{\partial g}{\partial p}\,. \tag{4.24}$$

also describe a damped harmonic oscillator. The analysis in the subsequent sections would be slightly modified if these equations were used, but the general conclusions are similar.

Consider $f = f\big(q(t), p(t)\big)$. Differentiating with respect to time, we find $\frac{df}{dt} = M(f,H)$. (Here we relabel H_0 as H in order to avoid confusion with the Hermite polynomials which we use later.) Therefore, $\dot{H} = -2\gamma p^2$, implying that M describes energy dissipation as $\gamma > 0$.

To compute the new star product, we exponentiate M, such that [33]

$$\star_\gamma = e^{\frac{i\hbar}{2}M} = \exp\left[\frac{i\hbar}{2}\left(\overleftarrow{\partial}_q \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_q - 2\gamma m \overleftarrow{\partial}_p \overrightarrow{\partial}_p\right)\right]. \tag{4.25}$$

Using equation (2.88), the transition operator is [33]

$$T_\gamma = \exp\left(-\frac{i\hbar m\gamma}{2}\partial_p^2\right). \tag{4.26}$$

We have summarized the results of [33] in which they mapped an undamped simple harmonic oscillator to a damped quantum harmonic oscillator. The transition operator to convert between Weyl quantization and this damped quantization was also shown. This is an example of augmented quantization because the physical feature of damping was not present in the original system.

Consider T_γ applied to the ground state simple harmonic oscillator Wigner function. From Section 2.8, the ground state Wigner function is $W_0(q,p) = \frac{1}{\pi\hbar}e^{-\frac{1}{\hbar}(q^2+p^2)}$. Note that $\partial_p^{2n}e^{-p^2} = H_{2n}(p)$ and [33]

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}H_{2n}(p) = \frac{1}{\sqrt{1+4t}}e^{-\frac{4t}{1+4t}p^2}, \tag{4.27}$$

for small t. Therefore, the augmented quantization of the simple harmonic oscillator ground state is [33]

$$T_\gamma W_0 = \frac{1}{\pi\hbar}\frac{1}{\sqrt{1-2i\gamma/\omega}}e^{-\frac{1}{\hbar\omega}\left(m\omega^2 q^2 + \frac{p^2}{m(1-2i\gamma/\omega)}\right)}, \tag{4.28}$$

which we plot in Figure 4.1. As a result of the non-Hermitian Hamiltonian, we see that $T_\gamma W_0(q,p)$ is complex. Integration over the position will yield complex probabilities for the momentum, which is difficult to interpret.

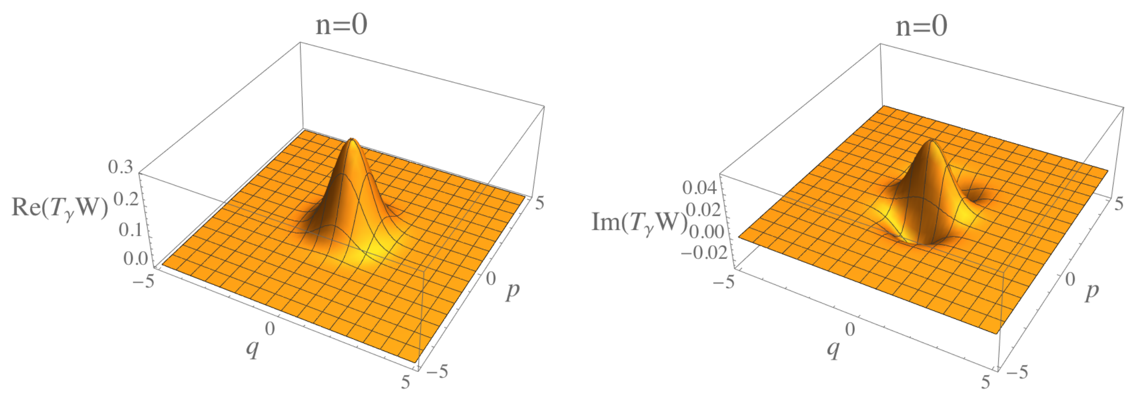

Figure 4.1: The transition operator T_γ operating upon the simple harmonic oscillator Wigner function for the first energy level. We have set $\gamma = 0.2$. Note that there is both a real and imaginary part to the distribution function. Integration over the position would then give a complex marginal probability distribution for the momentum, which is undesirable.

4.2.3 Eigenvalue Spectrum

Consider the stargenvalue equation of (2.89). We have

$$T_\gamma H \star_\gamma W_\gamma = E\, W_\gamma , \tag{4.29}$$

where $E = \left(n + \frac{1}{2}\right)\hbar\omega$ and $W_\gamma = T_\gamma W$. Thus, with $T_\gamma H = H - \dfrac{i\hbar m\gamma}{2}$, we can write

$$H \star_\gamma W_\gamma = E_\gamma W_\gamma , \tag{4.30}$$

where $E_\gamma = E + \dfrac{im\hbar\gamma}{2}$. Therefore, E_γ is the eigenvalue of the Hamiltonian H when using the star product, $\star_\gamma$.

Even though E was the eigenvalue of $T_\gamma H$, it is not clear if E_γ can be interpreted as an energy due to its complex nature. For this reason, we hesitate to call E_γ an energy, though complex energies have been examined in non-Hermitian quantum mechanics [100].

4.3 Effects of the Complex Transition Operator, T_γ

The work of Ref. [33] was a step towards using transition operators and star products to describe augmented quantization. In this Section, we will note the implications of their formulation, summarized from [83].

As T_γ is complex, this automatically implies that $\star_\gamma$ is non-Hermitian:

$$\begin{aligned}\overline{f\star_\gamma g} &= \overline{f}\, e^{-\frac{i\hbar}{2}\left(\overleftarrow{\partial}_q\overrightarrow{\partial}_p - \overleftarrow{\partial}_p\overrightarrow{\partial}_q - 2\gamma m\overleftarrow{\partial}_p\overrightarrow{\partial}_p\right)}\overline{g} = \overline{g}\, e^{\frac{i\hbar}{2}\left(\overleftarrow{\partial}_q\overrightarrow{\partial}_p - \overleftarrow{\partial}_p\overrightarrow{\partial}_q + 2\gamma m\overleftarrow{\partial}_p\overrightarrow{\partial}_p\right)}\overline{f} \\ &= \overline{g}\star_{-\gamma}\overline{f} \neq \overline{g}\star_\gamma\overline{f}\,.\end{aligned} \tag{4.31}$$

Therefore, $\star_\gamma^\dagger \neq \star_\gamma$. Further, if it is assumed that $i\hbar\frac{\partial T_\gamma W}{\partial t} = [T_\gamma H, T_\gamma W]_{\star_\gamma}$ is the correct equation of motion for $T_\gamma W$, then

$$0 = \frac{\partial W_\gamma}{\partial t} + \frac{1}{i\hbar}[H, W_\gamma]_{\star_\gamma} + i\gamma\hbar m\frac{\partial^2 W_\gamma}{\partial p\partial q}\,. \tag{4.32}$$

Hence, the evolution of $T_\gamma W$ is complex, rather than real. This means that initially real distribution functions become complex during time evolution. However, such evolution is required to be real. Therefore, the star bracket method may be invalid for non-Hermitian transition operators.

Consider now the $\hbar\to 0$ limit of the $\star_\gamma$-bracket, $\lim_{\hbar\to 0}\dfrac{[f,g]_{\star_\gamma}}{i\hbar}$. If this were the correct method of determining the equations of motion involving a complex transition operator, we should recover the equations of motion for the classical linear damped harmonic oscillator, with $f = q, p$ and $g = H$. Instead,

$$\dot{p} = \lim_{\hbar\to 0}\frac{1}{i\hbar}[p, H]_{\star_\gamma} = \{p, H\} = -q\,, \tag{4.33}$$

$$\dot{q} = \lim_{\hbar\to 0}\frac{1}{i\hbar}[q, H]_{\star_\gamma} = \{q, H\} = p\,. \tag{4.34}$$

This demonstrates that we recover the equations of motion of the simple harmonic oscillator, rather than the damped harmonic oscillator. Even though [33] used the classical equations of motion for the damped harmonic oscillator as a starting point, the $\hbar\to 0$ limit of the $\star_\gamma$-bracket does not incorporate any dependence on the damping constant.

In Section 4.4, the $\hbar\to 0$ limit of the $\star_T$-bracket is determined for any $\star_T$ found using equation (2.88). We will show that the result is always equal to the Poisson bracket, just as in equations (4.33) and (4.34).

4.4 Calculating the $\hbar \to 0$ Limit with Star Products

Let $H = \dfrac{p^2}{2m} + V(q)$. We will demonstrate that

$$\begin{aligned} \dot{p} &= \lim_{\hbar\to 0} \frac{[p,H]_{\star_T}}{i\hbar} \to -\frac{\partial V}{\partial q}\,, \\ \dot{q} &= \lim_{\hbar\to 0} \frac{[q,H]_{\star_T}}{i\hbar} \to p\,, \end{aligned} \tag{4.35}$$

holds for any star product $\star_T$ calculated with equation (2.88), regardless of the form of the (global) transition operator, $T = T(\partial_q, \partial_p)$, assuming T is real. From equation (2.88),

$$\star_T = \star_M T^{-1}\left(\overleftarrow{\partial}_q, \overleftarrow{\partial}_p\right) T\left(\overleftarrow{\partial}_q + \overrightarrow{\partial}_q, \overleftarrow{\partial}_p + \overrightarrow{\partial}_p\right) T^{-1}\left(\overrightarrow{\partial}_q, \overrightarrow{\partial}_p\right). \tag{4.36}$$

Let

$$\odot_T := T^{-1}\left(\overleftarrow{\partial}_q, \overleftarrow{\partial}_p\right) T\left(\overleftarrow{\partial}_q + \overrightarrow{\partial}_q, \overleftarrow{\partial}_p + \overrightarrow{\partial}_p\right) T^{-1}\left(\overrightarrow{\partial}_q, \overrightarrow{\partial}_p\right). \tag{4.37}$$

For arbitrary functions f and g,

$$\begin{aligned} \lim_{\hbar\to 0} \frac{[f,g]_{\star_T}}{i\hbar} &= \lim_{\hbar\to 0} \frac{f \star_M \odot_T g - g \star_M \odot_T f}{i\hbar}\,, \\ &= \lim_{\hbar\to 0} \frac{f \star_M \odot_T g - f \star^t \odot_T^t g}{i\hbar}\,, \end{aligned}$$

where we are using t to represent the transpose. As $\star_M^t = \overline{\star}_M$ and $\odot_T^t = \odot_T$,

$$\begin{aligned} \lim_{\hbar\to 0} \frac{[f,g]_{\star_T}}{i\hbar} &= \lim_{\hbar\to 0} \frac{f(\star_M - \overline{\star}_M) \odot_T g}{i\hbar}\,, \\ &= \lim_{\hbar\to 0} \frac{2f \sin\left(\frac{\hbar}{2}\overleftrightarrow{P}\right) \odot_T g}{\hbar}\,. \end{aligned}$$

If T is also a function of $\hbar$, such that $T = 1 + \sum_{r=1}^{\infty} \hbar^r T_r$ from equation (2.80), then

$$\lim_{\hbar \to 0} \frac{[f,g]_{\star T}}{i\hbar} \to \{f,g\}\,. \tag{4.38}$$

With $g = H$ and f as q or p, then we find equation (4.35).

If T does not depend on $\hbar$, however,

$$\lim_{\hbar \to 0} \frac{[f,g]_{\star T}}{i\hbar} \to f \overleftrightarrow{P} \odot_T g\,. \tag{4.39}$$

Let $f = q$ and $g = H$. Therefore,

$$\dot{q} = \lim_{\hbar \to 0} \frac{[q,H]_{\star_T}}{i\hbar} \to q \left(\overleftarrow{\partial}_q \overrightarrow{\partial}_p \odot_T - \overleftarrow{\partial}_p \overrightarrow{\partial}_q \odot_T \right) H \tag{4.40}$$

$$= \odot_T p\,, \tag{4.41}$$

as the derivatives in the Poisson bracket commute with $\odot_T$. When expanding the transition operator, all terms will be of the form $\partial_p^m \partial_q^n$, for positive integers m and n. However, the expansion of $\odot_T$ as a series will not contain the terms of $\overleftarrow{\partial}_p^m$, $\overrightarrow{\partial}_p^m$, $\overleftarrow{\partial}_q^n$, $\overrightarrow{\partial}_q^n$. Hence,

$$\dot{q} = \lim_{\hbar \to 0} \frac{[q,H]_{\star T}}{i\hbar} \to p = \{q,H\}\,. \tag{4.42}$$

Similarly, with $f = p$,

$$\dot{p} = \lim_{\hbar \to 0} \frac{[p,H]_{\star T}}{i\hbar} \to -\frac{\partial V}{\partial q} = \{p,H\}\,. \tag{4.43}$$

4.5 Transition Operators Involving Position and Momentum

4.5.1 Motivation for Generalizing the Transition Operator

We desire a $\star_T$ such that $\lim_{\hbar \to 0} \frac{[q,H]_{\star T}}{i\hbar}$ and $\lim_{\hbar \to 0} \frac{[p,H]_{\star T}}{i\hbar}$ gives the classical equations of motion for the damped harmonic oscillator. Hence, we want a transition operator and star product such that the $\hbar \to 0$ limit of the $\star_T$-bracket yields something of the form,

$$\lim_{\hbar \to 0} \frac{[f,g]_{\star_T}}{i\hbar} \to f\left(\overleftrightarrow{P} + \mathbf{physics}\right) g\,, \tag{4.44}$$

or

$$\lim_{\hbar \to 0} \frac{[f,g]_{\star_T}}{i\hbar} \to f\left(\overleftrightarrow{P} \times \mathbf{physics}\right) g\,, \tag{4.45}$$

where **physics** represents any extra physical term that should be present in the classical equations of motion for a given system.

In Section 4.4, we showed that the usual transition operator, $T = T\left(\partial_q, \partial_p\right)$, and star product of equation (2.88) always give the classical equations of motion for the classical system $m\ddot{q} + V(q) = 0$ in the $\hbar \to 0$ limit. Thus, with $T = T\left(\partial_q, \partial_p\right)$, extra physical features such as damping cannot be described. This motivates generalizing the transition operator in order to recover the classical equations of motion for a damped system in the $\hbar \to 0$ limit of the $\star_T$-bracket.

4.5.2 Transition Operator for Damping

Ref. [33] attempted to augment the quantization of the simple harmonic oscillator such that damping was incorporated into the quantum system. They did this by deriving a modified Poisson bracket from the classical equations of motion of the damped harmonic oscillator. By exponentiating this modified Poisson bracket, the star product was found, which immediately implied the transition operator. However, as [83] demonstrated, the star product and transition operator suggested by [33] does not yield the classical damped harmonic oscillator equations of motion in the $\hbar \to 0$ limit of $\frac{1}{i\hbar}[q,H]_{\star_\gamma}$ and $\frac{1}{i\hbar}[p,H]_{\star_\gamma}$.

Our objective is to find a star bracket that gives the damped harmonic oscillator equations of motion in the $\hbar \to 0$ limit. To do this, we will generalize the transition operator to include position and momentum dependence, $T = T(q, p, \partial_q, \partial_p)$. We will show that such a transition operator can convert Weyl quantization into an augmented quantization that includes damping.

When considering transition operators of the form, $T = T(q, p, \partial_q, \partial_p)$, the resultant distribution function will incorporate a quantization that is related to the values of q and p. Hence, by incorporating explicit dependence on q and p, it is possible to interpret the transition operator as describing a form of "local" operator ordering.

When using local transition operators, we still enforce the defining relation of $\star_T$: $T(f \star_M g) = Tf \star_T Tg$, implying that $T\left(T^{-1}f \star_M T^{-1}g\right) = f \star_T g$. However, it is important to note that the star product can no longer be written in the same manner as equation (2.88).

As a simple example, consider $T_p = e^{p\partial_p}$ giving rise the star product, $\star_p$. Expanding T_p, we get

$$T_p = 1 + p\partial_p + \frac{1}{2}\left(p^2\partial_p^2 + p\partial_p\right) + \frac{1}{6}\left(p^3\partial_p^3 + 3p^2\partial_p^2 + p\partial_p\right) + \cdots , \tag{4.46}$$

so that

$$\begin{aligned} T_p(f \star_M g) = e^{p\partial_p}(f \star_M g) &= f \star_M g + p(f \star_M (\overleftarrow{\partial}_p + \overrightarrow{\partial}_p)g) \\ &\quad + \frac{1}{2}\left[p^2(f \star_M (\overleftarrow{\partial}_p + \overrightarrow{\partial}_p)^2 g) + p(f \star_M (\overleftarrow{\partial}_p + \overrightarrow{\partial}_p)g)\right] + \cdots , \\ &\neq f \star_M e^{p(\overleftarrow{\partial}_p + \overrightarrow{\partial}_p)} g = Tf \star_p Tg \,. \end{aligned}$$

Hence, we find

$$\star_p \neq \star_M e^{-p\overleftarrow{\partial}_p} e^{p(\overleftarrow{\partial}_p + \overrightarrow{\partial}_p)} e^{-p\overrightarrow{\partial}_p} = \star_M \,, \tag{4.47}$$

in conflict with equation (2.88).

Using $T = T(q, p, \partial_q, \partial_p)$ and $T\left(T^{-1}f \star_M T^{-1}g\right) = f \star_T g$, the goal is to recover Hamilton's equations of motion for the damped harmonic oscillator with the undamped simple harmonic oscillator Hamiltonian, $H = \dfrac{p^2}{2} + \dfrac{q^2}{2}$ so that

$$\begin{aligned} \dot{p} &= -q - 2\gamma p\,, \\ \dot{q} &= p\,, \end{aligned} \tag{4.48}$$

where $\gamma > 0$ is a damping constant.

As shown in Section 4.4, the $\hbar \to 0$ limit of the $\star_T$-bracket of two phase space functions, f and g, is

$$\lim_{\hbar\to 0} \frac{[f,g]_{\star_T}}{i\hbar} = \lim_{\hbar\to 0} \frac{T[T^{-1}f, T^{-1}g]_{\star_M}}{i\hbar}\,, \tag{4.49}$$

so that

$$\lim_{\hbar\to 0} \frac{[p,H]_{\star_T}}{i\hbar} = \lim_{\hbar\to 0} \frac{T[T^{-1}p, T^{-1}H]_{\star_M}}{i\hbar}\,. \tag{4.50}$$

Motivated by the simple form $H = \frac{p^2}{2} + \frac{q^2}{2}$, consider the transition operator

$$T_\gamma = \exp\left[\gamma\Big(a\partial_q + b\partial_p + c\partial_q^2 + d\partial_p^2\Big)\right]\,, \tag{4.51}$$

where a, b, c, d are functions of q and p.

If $a = c = d = 0$ and $b = p$, then $T_\gamma = e^{\gamma p\partial_p}$. As shown in equation (4.46), the exponentiation of $p\partial_p$ results in a non-terminating series when acting on the Hamiltonian. To avoid this, we will focus upon small damping, such that

$$T_\gamma \approx 1 + \gamma\Big(a\partial_q + b\partial_p + c\partial_q^2 + d\partial_p^2\Big)\,, \tag{4.52}$$

and

$$T_\gamma^{-1} \approx 1 - \gamma\Big(a\partial_q + b\partial_p + v\partial_q^2 + d\partial_p^2\Big)\,. \tag{4.53}$$

Calculations yield

$$
\begin{aligned}
T_\gamma(T_\gamma^{-1} p \star_M T_\gamma^{-1} H) = p \star_M H - \gamma b \star_M H - \gamma p \star_M \Big[aq + bp + c + d\Big] \\
+ \gamma\Big(a\partial_q + b\partial_p + c\partial_q^2 + d\partial_p^2\Big)\Big(p \star_M H\Big) + \mathcal{O}(\gamma^2)\,.
\end{aligned}
\tag{4.54}
$$

Hence,

$$
\dot{p} = \lim_{\hbar\to 0} \frac{[p,H]_{\star_T}}{i\hbar} \approx \{p,H\} + \gamma\Big(\partial_q\Big[aq + bp + c + d\Big]\Big) - \gamma\{b,H\} - \gamma a\,. \tag{4.55}
$$

Similarly, we find

$$
\dot{q} = \lim_{\hbar\to 0} \frac{[q,H]_{\star_T}}{i\hbar} \approx \{q,H\} - \gamma\Big(\partial_p\Big[aq + bp + c + d\Big]\Big) - \gamma\{a,H\} + \gamma b\,. \tag{4.56}
$$

We therefore need to solve

$$
\partial_q\Big[aq + bp + c + d\Big] - \{b,H\} - a = -2p\,, \tag{4.57}
$$

$$
-\partial_p\Big[aq + bp + c + d\Big] - \{a,H\} + b = 0\,, \tag{4.58}
$$

to determine the functions a, b, c, and d; the solutions are shown in Appendix C.

Therefore, the transition operator in equation (4.52) gives the damped harmonic oscillator equations of motion in the $\hbar \to 0$ limit of the $\star_T$-brackets. Hence, with this transition operator, we have mapped the Weyl-quantized simple harmonic oscillator to a system that also includes damping.

This method of incorporating additional physical effects into a quantum system with a transition operator of the form $T = T(q, p, \partial_q, \partial_p)$ can be extended to other potentials and systems. For instance, with an arbitrary potential of $V(q)$, Hamilton's equations are

$$
\begin{aligned}
\dot{p} &= -\frac{dV}{dq} - 2\gamma p\,, \\
\dot{q} &= p\,.
\end{aligned}
\tag{4.59}
$$

The differential equations that result (for small damping) are:

$$\partial_q \left[a\frac{dV}{dq} + bp + c\frac{d^2V}{dq^2} + d \right] - \{b, H\} - \left(a\frac{d^2V}{dq^2} + c\frac{d^3V}{dq^3} \right) = -2p\,, \tag{4.60}$$

$$-\partial_p \left[a\frac{dV}{dq} + bp + c\frac{d^2V}{dq^2} + d \right] - \{a, H\} + b = 0\,. \tag{4.61}$$

This indicates that the general form of T_γ in equation (4.52) is valid beyond harmonic oscillators, but the coefficients of a, b, c, and d are dependent on the potential.

With a non-linearly damped equation of motion of the form,

$$\ddot{q} + 2\gamma f(\dot{q}) + q = 0\,, \tag{4.62}$$

the Hamiltonian equations of motion are

$$\begin{aligned} \dot{p} &= -q - 2\gamma f(p)\,, \\ \dot{q} &= p\,. \end{aligned} \tag{4.63}$$

By the same process used for linear damping, the functions of a, b, c, and d in equation (4.52) can be found. The coupled partial differential equations are

$$\partial_q \left[aq + bp + c + d \right] - \{b, H\} - a = -2f(p)\,, \tag{4.64}$$

$$-\partial_p \left[aq + bp + c + d \right] - \{a, H\} + b = 0\,. \tag{4.65}$$

In Appendix C, we show the solutions to equations (4.64) and (4.65) for quadratic damping.

To summarize, we have generalized the transition operator so that it is now also a function of the position and momentum coordinates. To illustrate its usefulness, we showed that such a transition operator can yield the the damped harmonic oscillator equations of motion in the $\hbar \to 0$ limit of the $\star_T$-bracket for weak damping. We say that this transition operator was able to convert a Weyl-quantized simple harmonic oscillator to an augmented quantized simple harmonic oscillator.

It was also shown that this method of using $T = T(q, p, \partial_q, \partial_p)$ to convert to an augmented quantized system has applications beyond the harmonic oscillators. We illustrated this by briefly considering linear damping for a classical system described by an arbitrary potential and by also examining the non-linearly damped harmonic oscillator.

We note that we have only considered small damping, hence we were able to expand the transition operator as a series. It would be difficult to treat equation (4.51) otherwise, as no closed form may exist for arbitrary a, b, c, d. By considering forms besides exponential for the transition operator, it may be possible to describe the augmented quantization of an even larger class of systems.

In Section 4.8, we will use a local transition operator to realize the commutation relation of the generalized uncertainty principle of quantum gravity phenomenology. Then, in Chapter 5, we will consider a time-dependent local transition operator.

4.5.3 Eigenvalue Spectrum

In this Section, we will illustrate the difficulty of determining the eigenvalue of H when using the star product, $\star_T$. The following argument holds for both global and local transition operators.

In equation (2.89), we presented the stargenvalue equation, $TH \star_T TW = E\,TW$. Thus, the eigenvalue of TH in the $\star_T$ formulation is E. If $H = \frac{p^2}{2m} + V(q)$ and $T = e^{\varepsilon \partial_p^2}$, where $\varepsilon \in \mathbb{C}$, then $TH = H + \varepsilon$. Thus,

$$H \star_T TW = (E - \varepsilon) TW\,, \tag{4.66}$$

so the eigenvalue of H in the $\star_T$ formulation is $E - \varepsilon$. This is similar to what was shown in Section 4.2.3, in which the eigenvalue of $H = \frac{p^2}{2} + \frac{q^2}{2}$ was found for the damped augmented quantization in [33].

Consider the application of an arbitrary transition operator to the stargenvalue equation, then rearranging so that we have an equation of the form $H \star_T TW = E_T(q,p)\,TW$. If we want to convert phase space quantum mechanics to operator quantum mechanics, then operation of a quantization map $\widetilde{\mathcal{Q}}_T$ yields

$$\widetilde{\mathcal{Q}}_T(H)\,\widetilde{\mathcal{Q}}_T(TW) = \widetilde{\mathcal{Q}}_T(E_T(q,p)TW)\,. \tag{4.67}$$

We can write $\widetilde{\mathcal{Q}}_T(H \star_T TW) = \widetilde{\mathcal{Q}}_T(H)\,\widetilde{\mathcal{Q}}_T(TW)$ because the operator algebra and $\star_T$ algebra are homomorphic. However, as $E_T(q,p)$ is not necessarily constant, we are unable to determine $\widetilde{\mathcal{Q}}_T(E_T(q,p)TW)$. Hence, it is not possible to treat equation (4.67) as an eigenvalue equation. It might only make sense to find the eigenvalues of H in the $\star_T$ formulation in some situations.

4.5.4 Relation of the Local Transition Operator to the Weight Function

In Section (2.10), it was demonstrated that a transition operator, $T(\partial_q,\partial_p)$, yielded the weight function $\Phi(\partial_p,\partial_q) = T\left(-i\partial_p,-i\partial_q\right)$. Let us now determine the weight function corresponding to a transition operator of the form, $T = T(q,p,\partial_q,\partial_p)$. As a revealing example, consider $T = e^{p\partial_p}$. If we apply this to a Wigner function, W, we have

$$TW = \frac{1}{4\pi^2}\int d\xi\, d\eta\, dq'\,\psi^*(q'-\eta\hbar/2)\psi(q'+\eta\hbar/2)\,e^{i\xi(q'-q)}e^{p\partial_p}e^{-i\eta p}\,. \tag{4.68}$$

However, unlike in the previous case in which the transition operator was only dependent on differential operators, it is not possible to let $\partial_p \to -i\eta$ because ∂_p also acts upon the p in the exponential, such that

$$e^{p\partial_p} = 1 + p\partial_p + \frac{1}{2}\left(p^2\partial_p^2 + p\partial_p\right) + \frac{1}{6}\left(p^3\partial_p^3 + 3p^2\partial_p^2 + p\partial_p\right) + \cdots\,. \tag{4.69}$$

Instead, to determine the relationship between $T(q, p, \partial_q, \partial_p)$ and the weight function, the transition operator should be written in a similar form as equation (2.95),

$$T(q, p, \partial_q, \partial_p) = \sum_{mn} t_{mn}(q, p)\, \partial_q^m \partial_p^n, \tag{4.70}$$

by using the commutation relations of $[\partial_q, q] = 1$ and $[\partial_p, p] = 1$. This will separate the differential operators from any functions of q and p also present in the transition operator. Thus,

$$\begin{aligned} T(q, p, \partial_q, \partial_p)W = \frac{1}{4\pi^2} \int d\xi\, d\eta\, dq' \Bigg(& \psi^*(q' - \eta\hbar/2)\psi(q' + \eta\hbar/2) \\ & \times \sum_{mn} t_{mn}(q, p)\, \partial_q^m \partial_p^n\, e^{i\xi(q'-q)} e^{-i\eta p} \Bigg), \end{aligned} \tag{4.71}$$

giving

$$\begin{aligned} T(q, p, \partial_q, \partial_p)W = \frac{1}{4\pi^2} \int d\xi d\eta dq' \Bigg(& \psi^*(q' - \eta\hbar/2)\psi(q' + \eta\hbar/2) \\ & \times \sum_{mn} t_{mn}(q, p)(-i\xi)^m (-i\eta)^n\, e^{i\xi(q'-q)} e^{-i\eta p} \Bigg). \end{aligned} \tag{4.72}$$

Therefore, the weight function can be related to the transition operator with

$$\Phi(\xi, \eta) = \sum_{mn} t_{mn}(q, p)(-i\xi)^m (-i\eta)^n. \tag{4.73}$$

To illustrate this method of finding the weight function, consider $T = e^{p\partial_p}$ and note that it can be written as

$$T = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=0}^{n} \left\{ \begin{matrix} n \\ m \end{matrix} \right\} p^m \partial_p^m, \tag{4.74}$$

where $\left\{ \begin{matrix} n \\ m \end{matrix} \right\}$ is a Stirling number of the second kind, defined as [79]

$$\left\{ \begin{matrix} n \\ m \end{matrix} \right\} = \frac{1}{m!} \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} k^n . \tag{4.75}$$

Therefore, the weight function for $T = e^{p\partial_p}$ is

$$\Phi(\xi, \eta) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=0}^{n} \left\{ \begin{matrix} n \\ m \end{matrix} \right\} p^m (-i\eta)^m . \tag{4.76}$$

4.6 Star Product with Position and Momentum Dependence

4.6.1 Derivation of the Star Product

When deriving the star product for the transition operator, $T = T(q, p\partial_q, \partial_p)$, one starts with $T(f \star_M g) = Tf \star_T Tg$, then letting $F = Tf$ and $G = Tg$, we get $T(T^{-1}F \star_M T^{-1}G) = F \star_T G$. Note that

$$\begin{aligned} &\left[T^{-1}\left(q, p, \partial_q, \partial_p\right) F(q, p)\right] \star_M \left[T^{-1}\left(q, p, \partial_q, \partial_p\right) G(q, p)\right] \\ &= \mathcal{I}(1,2) \left[\star_M(1,2)\right] T^{-1}\left(q_1, p_1, \partial_{q_1}, \partial_{p_1}\right) T^{-1}\left(q_2, p_2, \partial_{q_2}, \partial_{p_2}\right) F(1) G(2) \end{aligned} \tag{4.77}$$

where $[\star_M(1,2)] := e^{\frac{i\hbar}{2}\left(\partial_{q_1}\partial_{p_2} - \partial_{p_1}\partial_{q_2}\right)}$, $F(1) := F(q_1, p_1)$, $G(2) := G(q_2, p_2)$, and $\mathcal{I}(1,2)$ sets $q_1 = q_2 = q$ and $p_1 = p_2 = p$ at the end of the calculation.

We then have

$$\begin{aligned} &T\left(\left[T^{-1}\left(q, p, \partial_q, \partial_p\right) F(q, p)\right] \star_M \left[T^{-1}\left(q, p, \partial_q, \partial_p\right) G(q, p)\right]\right) \\ &= \mathcal{I}(1,2) T(1,2) \left[\star_M(1,2)\right] T^{-1}\left(q_1, p_1, \partial_{q_1}, \partial_{p_1}\right) T^{-1}\left(q_2, p_2, \partial_{q_2}, \partial_{p_2}\right) F(1) G(2) , \end{aligned} \tag{4.78}$$

where $T(1,2)$ represents the transition operator written in terms of q_1, p_1, q_2, p_2, such that it satisfies equation (4.78). Therefore,

$$\star_T = \mathcal{I}(1,2) T(1,2) \left[\star_M(1,2)\right] T^{-1}\left(q_1, p_1, \partial_{q_1}, \partial_{p_1}\right) T^{-1}\left(q_2, p_2, \partial_{q_2}, \partial_{p_2}\right) . \tag{4.79}$$

The goal is now to determine $T(1,2)$. We conjecture that

$$T(1,2) = T\left(\frac{q_1+q_2}{2}, \frac{p_1+p_2}{2}, \partial_{q_1}+\partial_{q_2}, \partial_{p_1}+\partial_{p_2}\right), \tag{4.80}$$

so equation (4.78) becomes

$$\begin{aligned} F \star_T G =& \mathcal{I}(1,2) T\left(\frac{q_1+q_2}{2}, \frac{p_1+p_2}{2}, \partial_{q_1}+\partial_{q_2}, \partial_{p_1}+\partial_{p_2}\right) [\star_M(1,2)] \\ & \times T^{-1}\left(q_1, p_1, \partial_{q_1}, \partial_{p_1}\right) T^{-1}\left(q_2, p_2, \partial_{q_2}, \partial_{p_2}\right) F(1) G(2), \end{aligned} \tag{4.81}$$

hence the star product is

$$\begin{aligned} \star_T =& \mathcal{I}(1,2) T\left(\frac{q_1+q_2}{2}, \frac{p_1+p_2}{2}, \partial_{q_1}+\partial_{q_2}, \partial_{p_1}+\partial_{p_2}\right) [\star_M(1,2)] \\ & \times T^{-1}\left(q_1, p_1, \partial_{q_1}, \partial_{p_1}\right) T^{-1}\left(q_2, p_2, \partial_{q_2}, \partial_{p_2}\right). \end{aligned} \tag{4.82}$$

When $T = T(\partial_p, \partial_q)$, equation (4.82) reduces to

$$\begin{aligned} \star_T &= \mathcal{I}(1,2) T\left(\partial_{q_1}+\partial_{q_2}, \partial_{p_1}+\partial_{p_2}\right) [\star_M(1,2)] T^{-1}\left(\partial_{q_1}, \partial_{p_1}\right) T^{-1}\left(\partial_{q_2}, \partial_{p_2}\right), \\ &= \mathcal{I}(1,2) [\star_M(1,2)] T^{-1}\left(\partial_{q_1}, \partial_{p_1}\right) T\left(\partial_{q_1}+\partial_{q_2}, \partial_{p_1}+\partial_{p_2}\right) T^{-1}\left(\partial_{q_2}, \partial_{p_2}\right), \\ &= \star_M T^{-1}\left[\overleftarrow{\partial}\right] T\left[\overleftarrow{\partial}+\overrightarrow{\partial}\right] T^{-1}\left[\overrightarrow{\partial}\right], \end{aligned}$$

which is precisely equation (2.88), as desired.

As a first step towards confirmation of the conjecture, note that equation (4.82) can be expanded in a series so that each term in $T\left(\frac{q_1+q_2}{2}, \frac{p_1+p_2}{2}, \partial_{q_1}+\partial_{q_2}, \partial_{p_1}+\partial_{p_2}\right)$ will have the form,

$$\begin{aligned} \cdots \times A\left(\frac{q_1+q_2}{2}, \frac{p_1+p_2}{2}\right) \left(\partial_{p_1}+\partial_{p_2}\right)^{n'} \left(\partial_{q_1}+\partial_{q_2}\right)^{m'} \\ \times B\left(\frac{q_1+q_2}{2}, \frac{p_1+p_2}{2}\right) \left(\partial_{p_1}+\partial_{p_2}\right)^{n} \left(\partial_{q_1}+\partial_{q_2}\right)^{m} \times \cdots, \end{aligned} \tag{4.83}$$

where n, m, n', m' are non-negative integers and are not necessarily equal. We need only consider equation (4.83) acting on the term, $p_1^a p_2^b q_1^{a'} b_2^{b'}$ where $a+b \geq n$ and $a'+b' \geq m$ because it is possible to expand $[\star_M(1,2)]\, T^{-1}\left(q_1, p_1, \partial_{q_1}, \partial_{p_1}\right) T^{-1}\left(q_2, p_2, \partial_{q_2}, \partial_{p_2}\right) F(1)G(2)$ as a series in terms of the position and momentum coordinates.

To verify the conjecture, it is only necessary to show

$$\begin{aligned} \mathcal{I}(1,2) B\left(\frac{q_1+q_2}{2}, \frac{p_1+p_2}{2}\right) \left(\partial_{p_1}+\partial_{p_2}\right)^n \left(\partial_{q_1}+\partial_{q_2}\right)^m p_1^a\, p_2^b\, q_1^{a'}\, q_2^{b'} \\ = B(q,p)\, \partial_p^n \partial_q^n\, p^{a+b}\, q^{a'+b'}\,, \end{aligned} \tag{4.84}$$

from the second line of equation (4.83). We can then expand the right-hand side of equation (4.84) as a series so that it will be in the form of $p_1^a\, p_2^b\, q_1^{a'}\, b_2^{b'}$. Therefore, the argument that we will use to validate equation (4.84) can then be applied to the next term in equation (4.83), containing $A\left(\frac{q_1+q_2}{2}, \frac{p_1+p_2}{2}\right) \left(\partial_{p_1}+\partial_{p_2}\right)^{n'} \left(\partial_{q_1}+\partial_{q_2}\right)^{m'}$. We will use Mathematica to evaluate the resultant series.

Assume a, b, a', and b' are non-negative integers. We have,

$$\begin{aligned} \left(\partial_{p_1}+\partial_{p_2}\right)^n \left(\partial_{q_1}+\partial_{q_2}\right)^m p_1^a p_2^b q_1^{a'} q_2^{b'} = \sum_{\ell=0}^{n} \sum_{\ell'=0}^{m} \Bigg\{ \binom{n}{\ell} \binom{a}{\ell} \ell! \binom{b}{n-\ell} (n-\ell)! \binom{m}{\ell'} \\ \times \binom{a'}{\ell'} \ell'! \binom{b'}{m-\ell'} (m-\ell')!\, p_1^{a-\ell} p_2^{b+\ell-n} q_1^{a'-\ell'} q_2^{b'+\ell'-m} \Bigg\}, \\ = \binom{b}{n} n! \binom{b'}{m} m!\, {}_2F_1\left[-a, -n, 1+b-n; \frac{p_2}{p_1}\right] {}_2F_1\left[-a', -m, 1+b'-m; \frac{q_2}{q_1}\right] \\ \times p_1^a p_2^{b-n} q_1^{a'} q_2^{b'-m}\,, \end{aligned}$$

where we have expanded $\left(\partial_{p_1}+\partial_{p_2}\right)^n \left(\partial_{q_1}+\partial_{q_2}\right)^m$ with the binomial theorem and then applied the Leibniz identity

$$\frac{d^n}{dx^n}\Big[y(x)z(x)\Big] = \sum_{k=0}^{n} \binom{n}{k} \frac{d^k y(x)}{dx^k} \frac{d^{n-k} z(x)}{dx^{n-k}}$$

for any functions $y(x)$ and $z(x)$. We also used

$$\sum_{\ell=0}^{n}\binom{n}{\ell}\binom{a}{\ell}\ell!\binom{b}{n-\ell}(n-\ell)!p_1^{a-\ell}p_2^{b+\ell-n}=p_1^a p_2^{b-n}\binom{b}{n}n!{}_2F_1\left[-a,-n,1+b-n;\frac{p_2}{p_1}\right]$$

By expanding the Gauss hypergeometric functions as a series and letting $p_1=p_2=p$ and $q_1=q_2=q$,

$$\begin{aligned}&\mathcal{I}(1,2)\left(\partial_{p_1}+\partial_{p_2}\right)^n\left(\partial_{q_1}+\partial_{q_2}\right)^m p_1^a p_2^b q_1^{a'}q_2^{b'}\\&=\binom{b}{n}n!\binom{b'}{m}m!\sum_{\alpha=0}^{a}\frac{(-a)_\alpha(-n)_\alpha}{\alpha!(1+b-n)_\alpha}\sum_{\alpha'=0}^{a'}\frac{(-a')_{\alpha'}(-m)_{\alpha'}}{(\alpha')!(1+b'-m)_{\alpha'}}p^{a+b-n}q^{a'+b'-m}\end{aligned}\tag{4.85}$$

as [79]

$${}_2F_1(w,x,y;z)=\sum_{\alpha=0}^{x}\frac{(w)_\alpha(x)_\alpha}{(y)_\alpha}\frac{z^\alpha}{\alpha!}\tag{4.86}$$

if x is a negative integer. Using

$$\sum_{\alpha=0}^{a}\frac{(-a)_\alpha(-n)_\alpha}{(\alpha)!(1+b-n)_\alpha}=\frac{(a+b)!(b-n)!}{b!(a+b-n)!},\tag{4.87}$$

we find,

$$\mathcal{I}(1,2)\left(\partial_{p_1}+\partial_{p_2}\right)^n=\binom{a+b}{n}\binom{a'+b'}{m}n!m!p^{a+b-n}q^{a'+b'-m}.\tag{4.88}$$

From the right-hand side of equation (4.84)

$$\partial_p^n\partial_q^m p^{a+b}q^{a'+b'}=\binom{a+b}{n}\binom{a'+b'}{m}n!m!p^{a+b-n}q^{a'+b'-m}.\tag{4.89}$$

Hence, the conjecture is verified if a, b, a', and b' are non-negative integers. We have therefore shown that equation (4.82) is the generalization of the star product resulting from a local transition operator for the conditions on a, b, a', and b'.

If a, b, a', and b' are not non-negative integers, we expect a similar procedure as outlined above to hold, but the factorials will be replaced by the Γ function. It is important to note, though, that the sums in equation (4.85) should then range from 0 to m and 0 to n, rather than to a' and a.

4.6.2 Properties of the Generalized Star Product

The properties of equation (4.82) will now be analyzed. Note that the transpose (using the definition in Table 2.3) of $\star_T$ is itself because, instead of letting $\overleftarrow{\partial} \leftrightarrow \overrightarrow{\partial}$, we now let $1 \leftrightarrow 2$. Therefore, if $T(q,p,\partial_q,\partial_p)$ is also real, then $\star_T$ is Hermitian because $\star_M$ is also Hermitian.

Let us now consider $[q,p]_{\star_T}$. We can write this as

$$
\begin{aligned}
[q,p]_{\star_T} = \mathcal{I}(1,2)\Bigg\{ & T\left(\frac{q_1+q_2}{2}, \frac{p_1+p_2}{2}, \partial_{q_1}+\partial_{q_2}, \partial_{p_1}+\partial_{p_2}\right)[\star_M(1,2)]\,T^{-1}\left(q_1,p_1,\partial_{q_1},\partial_{p_1}\right) \\
& \times T^{-1}\left(q_2,p_2,\partial_{q_2},\partial_{p_2}\right)\Bigg\} q_1 p_2 \\
-\mathcal{I}(1,2)\Bigg\{ & T\left(\frac{q_1+q_2}{2}, \frac{p_1+p_2}{2}, \partial_{q_1}+\partial_{q_2}, \partial_{p_1}+\partial_{p_2}\right)[\star_M(1,2)]\,T^{-1}\left(q_1,p_1,\partial_{q_1},\partial_{p_1}\right) \\
& \times T^{-1}\left(q_2,p_2,\partial_{q_2},\partial_{p_2}\right)\Bigg\} q_2 p_1 \,.
\end{aligned}
$$

Expanding $T^{-1}\left(q,p,\partial_q,\partial_p\right)$ in a similar manner as equation (4.83), the only relevant derivative with respect to p is the first derivative (all other derivatives will give zero when acting on p). Similarly, only the first derivative of the position will give a non-zero result in the expansion of $T^{-1}\left(q,p,\partial_q,\partial_p\right)$. Therefore, the transition operator can be expanded as

$$
T = 1 + f_1(q,p)\partial_q + f_2(q,p)\partial_p + \cdots\,, \tag{4.90}
$$

As will be shown in Section 4.6.3, the first term must be 1 rather than a function of q and p because the transition operator acting on the identity must always give back the identity.

This holds for both global and local transition operators.

Letting

$$T^{-1} = 1 + g_1(q,p)\partial_q + g_2(q,p)\partial_p\,, \tag{4.91}$$

we find

$$\begin{aligned}&\mathcal{I}(1,2)\,[\star_M(1,2)]\,T^{-1}\left(q_1,p_1,\partial_{q_1},\partial_{p_1}\right)T^{-1}\left(q_2,p_2,\partial_{q_2},\partial_{p_2}\right)q_2p_1\\ &\quad= \Big(p+g_2(q,p)\Big)\star_M\Big(q+g_1(q,p)\Big)\,.\end{aligned} \tag{4.92}$$

As a result,

$$\begin{aligned}[q,p]_{\star_T} = &\Big(1+f_1(q,p)\partial_q\Big)\,[q+g_1(q,p),p+g_2(q,p)]_{\star_M}\\ &+f_2(q,p)\partial_p\,[q+g_1(q,p),p+g_2(q,p)]_{\star_M}\,.\end{aligned} \tag{4.93}$$

We see that in general, $[q,p]_{\star_T} \neq i\hbar$, in contrast to the global transition operators presented in Table 2.3.

Let f, g, h be functions of q and p. We can show that $\star_T$ is associative because,

$$\begin{aligned}(f\star_T g)\star_T h &= T(T^{-1}f\star_M T^{-1}g)\star_T h\,,\\ &= T((T^{-1}f\star_M T^{-1}g)\star_M T^{-1}h)\,,\\ &= T(T^{-1}f\star_M (T^{-1}g\star_M T^{-1}h))\,,\end{aligned}$$

by using the associativity of the Moyal product demonstrated in Section 2.6. Then

$$\begin{aligned}(f\star_T g)\star_T h &= T(T^{-1}f\star_M T^{-1}(T(T^{-1}g\star_M T^{-1}h)))\,,\\ &= T(T^{-1}f\star_M T^{-1}(g\star_T h))\,,\\ &= f\star_T (g\star_T h)\,,\end{aligned}$$

demonstrating the associativity of $\star_T$, We note that this derivation holds true regardless of whether $T = T(\partial_q,\partial_p)$ or $T = T(q,p,\partial_q,\partial_p)$.

With $T = T(q,p,\partial_q,\partial_p)$, we have generalized the star product so it includes local transition operators and is able to help describe augmented quantization. It is still possible to write $f \star_T g$ in a similar form as equation (2.35), such that

$$f \star_T g = \sum_{r=0}^{\infty} \nu^r C_r(q,p,f,g)\,, \tag{4.94}$$

where ν is a complex parameter and C_r now includes explicit dependence on q and p, regardless of f and g. We can write C_r as

$$C_r = b_r(q,p)\left[D_r f(q,p)\right]\left[\widetilde{D}_r g(q,p)\right]\,,$$

where $b_r(q,p)$ is a function determined by the form of $f \star_T g$, D_r and $\widetilde{D}_r$ are functions of ∂_q, ∂_p and the coordinates of q ,p.

Comparing with equation (2.36), similar properties of C_r are:

1. $C_0(q,p,f,g) = fg$
2. For $a \in \mathbb{R}$, $C_r(q,p,f,a) = C_r(q,p,a,f) = 0$

(4.95)

where, unlike in Section 2.6, $C_1(f,g)$ is not necessarily the Poisson bracket nor is $C_r(f,g)$ antisymmetric in f and g. We note that equation (4.95) is a non-exhaustive list; additional properties of $C_r(q,p,f,g)$ could be determined.

4.6.3 Converting to Quantizations with Global and Local Transition Operators

To convert from Weyl quantization to a different quantization in phase space, the transition operator is applied, such that

$$\left(e^{i(\theta q+\tau p)}\right)_W \to T e^{i(\theta q+\tau p)}\,, \tag{4.96}$$

where the subscript W indicates that Weyl quantization is our reference quantization. In Table 2.2, we showed how to convert between different operator quantizations and phase space quantizations.

With $T=1$, equation (4.96) gives Weyl quantization, so $e^{i(\theta q+\tau p)}$ is unaffected. For $T=T(\partial_q,\partial_p)$, $e^{i(\theta q+\tau p)}$ will pick up additional terms dependent on θ and τ. As an example, $T_S=e^{i\hbar\partial_p\partial_q/2}$ of Table 2.2 gives standard ordering. In this case, $T_S e^{i(\theta q+\tau p)}=e^{i(\theta q+\tau p-\hbar\theta\tau/2)}$. To then relate the function of $e^{i(\theta q+\tau p)}$ in classical mechanics to $Te^{i(\theta q+\tau p)}$ in phase space quantum mechanics, it is necessary to expand both quantities in terms of θ and τ and relate like powers.

For $T=T(q,p,\partial_q,\partial_p)$, equation (4.96) also describes augmented quantization induced by local transition operators, so a similar procedure to associate the classical function of $e^{i(\theta q+\tau p)}$ with $Te^{i(\theta q+\tau p)}$ could be done. Thus, the mapping from the classical position and momentum to their augmented quantized counterparts would look like:

$$\begin{aligned} q &\to q+c(q,p)\,, \\ p &\to q+d(q,p)\,, \end{aligned} \tag{4.97}$$

where $c(q,p)$ and $d(q,p)$ are functions determined by the transition operator. In contrast, with the transition operators of Table 2.2, we have $c(q,p)=d(q,p)=0$.

In general, augmented quantization of a classical function $z(q,p)$ obeys

$$\begin{aligned} &1.\ \ 1\to 1 \\ &2.\ \ q\to Tq \\ &3.\ \ p\to Tp \\ &4.\ \ z^*(q,p)\to Tz^*(q,p) \end{aligned} \tag{4.98}$$

These rules are similar to the first four properties of quantization from Section 2.2.

To conclude this Section, we recall that with $T=T(\partial_q,\partial_p)$, the star products of $\star_M$ and $\star_T$ are c-equivalent, as discussed in Section 2.10. This form of mathematical equivalence holds because $T(f\star_M g)=Tf\star_T Tg$. With $T=T(q,p,\partial_q,\partial_p)$, we still have $T(f\star_M g)=$

$Tf \star_T Tg$, hence $\star_M$ and the resultant $\star_T$ are also mathematically equivalent. However, as the $\star_T$ will also depend on position and momentum, rather than just derivatives, $\star_T$ may not have the same form of mathematical equivalence as for global star products.

4.7 Star Product for the Damped Harmonic Oscillator

In Section 4.5, we generalized the transition operator to investigate its ability to augment the quantization of a system. We did this by using $T = T(q,p,\partial_q,\partial_p)$ and showing that the $\hbar \to 0$ limit of $[q,H]_{\star_\gamma}$ and $[p,H]_{\star_\gamma}$ recovered the classical equations of motion for the damped harmonic oscillator. It was then demonstrated that the star product was also generalized. As a result, additional physical features were incorporated into both $T = T(q,p,\partial_q,\partial_p)$ and the associated star product, which is the hallmark of augmented quantization.

For the damped harmonic oscillator transition operator of equation (4.51), using equation (4.82), the star product is

$$
\begin{aligned}
\star_\gamma = \exp\Bigg\{\gamma\Bigg[& a\left(\frac{q_1+q_2}{2},\frac{p_1+p_2}{2}\right)(\partial_{q_1}+\partial_{q_2}) + b\left(\frac{q_1+q_2}{2},\frac{p_1+p_2}{2}\right)(\partial_{p_1}+\partial_{p_2}) \\
&+c\left(\frac{q_1+q_2}{2},\frac{p_1+p_2}{2}\right)(\partial_{q_1}+\partial_{q_2})^2 + d\left(\frac{q_1+q_2}{2},\frac{p_1+p_2}{2}\right)(\partial_{p_1}+\partial_{p_2})^2\Bigg]\Bigg\} \\
&\times[\star_M(1,2)]\exp\Big\{-\gamma\Big[a(1)\partial_{q_1}+b(1)\partial_{p_1}+dc(1)\partial_{q_1}^2+d(1)\partial_{p_1}^2\Big]\Big\} \\
&\times\exp\Big\{-\gamma\Big[a(2)\partial_{q_2}+b(2)\partial_{p_2}+c(2)\partial_{q_2}^2+d(2)\partial_{p_2}^2\Big]\Big\}
\end{aligned}
$$

where we have adopted the notation that $a(1) := a(q_1,p_1)$, $a(2) := a(q_2,p_2)$, and similarly for b, c, and d. Letting $\widetilde{T}_\gamma(q,p,\partial_q,\partial_p) = a\partial_q + b\partial_p + c\partial_q^2 + d\partial_p^2$, up to $\mathcal{O}(\gamma)$, the damped harmonic oscillator star product is

$$
\begin{aligned}
\star_\gamma \approx \mathcal{I}(1,2)\Bigg\{&[\star_M(1,2)] + \gamma\Bigg[\widetilde{T}_\gamma\left(\frac{q_1+q_2}{2},\frac{p_1+p_2}{2},\partial_{q_1}+\partial_{q_2},\partial_{p_1}+\partial_{p_2}\right)[\star_M(1,2)] \\
&-[\star_M(1,2)]\,\widetilde{T}_\gamma(q_1,p_1,\partial_{q_1},\partial_{p_1}) - [\star_M(1,2)]\,\widetilde{T}_\gamma(q_2,p_2,\partial_{q_2},\partial_{p_2})\Bigg]\Bigg\}.
\end{aligned}
\tag{4.99}
$$

In Section 4.5, we calculated the $\hbar \to 0$ limit of the $\star_\gamma-$product between position/momentum and the Hamiltonian by using the transition operator and the Moyal product. We can find the $\hbar \to 0$ limit of $[p,H]_{\star_\gamma}$ directly using equation (4.99) by noting that

$$\begin{aligned}
&\mathcal{I}(1,2)T_\gamma(1,2)\,[\star_M(1,2)]\,p_1H(q_2,p_2) = \mathcal{I}(1,2)\Bigg\{\,[\star_M(1,2)]\,p_1H(q_2,p_2)\\
&-\gamma\Big(\,[\star_M(1,2)]\,[b(1)H(q_2,p_2)+a(2)p_1q_2+b(2)p_1p_2+c(2)p_1+d(2)p_1]\Big)\\
&+\gamma\Bigg[a\left(\frac{q_1+q_2}{2},\frac{p_1+p_2}{2}\right)(\partial_{q_1}+\partial_{q_2})+b\left(\frac{q_1+q_2}{2},\frac{p_1+p_2}{2}\right)(\partial_{p_1}+\partial_{p_2})\\
&+c\left(\frac{q_1+q_2}{2},\frac{p_1+p_2}{2}\right)\left(\partial_{q_1}+\partial_{q_2}\right)^2+d\left(\frac{q_1+q_2}{2},\frac{p_1+p_2}{2}\right)(\partial_{p_1}+\partial_{p_2})^2\Bigg]\\
&\times[\star_M(1,2)]\left(p_1H(q_2,p_2)\right)\Bigg\},\\
&= p\star_M H-\gamma b\star_M H-\gamma p\star_M\left[aq+bp+c+d\right]+\gamma\Big(a\partial_q+b\partial_p+c\partial_q^2+d\partial_p^2\Big)\Big(p\star_M H\Big),
\end{aligned}$$

which is equation (4.54). Evaluating $\mathcal{I}(1,2)T_\gamma(1,2)_\gamma[\star_M(1,2)]\,p_2H(q_1,p_1)$, we see that

$$\mathcal{I}(1,2)\lim_{\hbar\to 0}\frac{1}{i\hbar}\Big(T_\gamma(1,2)\,[\star_M(1,2)]\;p_1H(q_2,p_2)-T_\gamma(1,2)\,[\star_M(1,2)]\,p_2\,H(q_1,p_1)\Big)=-2p$$

if

$$\partial_q\Big[aq+bp+c+d\Big]-\{b,H\}-a=-2p$$

holds. This corresponds to equation (4.57). Similarly, by using equation (4.99), the $\hbar\to 0$ limit of $[p,H]_{\star_\gamma}$ gives

$$\mathcal{I}(1,2)\lim_{\hbar\to 0}\frac{1}{i\hbar}\Big(T_\gamma(1,2)\,[\star_M(1,2)]\;q_1H(q_2,p_2)-T_\gamma(1,2)\,[\star_M(1,2)]\,q_2\,H(q_1,p_1)\Big)=0$$

if equation (4.58) is satisfied. We have therefore shown that we recover the same classical equations of motion for the damped harmonic oscillator if we write the $\star_T$-bracket using the damped star product, $\star_\gamma$, or using both the Moyal product and damped transition operator, T_γ, of Section 4.5.

In Section 4.5, we also considered linear damping for an arbitrary potential and the non-linearly damped harmonic oscillator. Similar calculations as those shown in this Section recover the differential equations in (4.60), (4.61) and (4.64), (4.65).

4.8 Generalized Uncertainty Principle

In Chapter 3, we showed that the Gaussian smoothing of the Wigner function was an example of augmented quantization as additional physics were introduced by the transition operator. Earlier in this Chapter, we also used augmented quantization to map the classical simple harmonic oscillator to its quantum mechanical damped counterpart.

We will now show that augmented quantization and the transition operator have applications beyond incorporating coarse graining or damping into a quantum system. Our objective is to apply augmented quantization to a classical system such that its quantum mechanical counterpart includes perturbations resulting from quantum gravity effects. We will derive a transition operator that will introduce these effects into a system. This transition operator will then be applied to the Wigner function of the simple harmonic oscillator. Ref. [101] has studied the quantum gravity-modified simple harmonic oscillator by determining its wavefunction in momentum space, then finding the resultant phase space distribution function.

The largest particle accelerator in present use is the Large Hadron Collider with energies of $\sim$10 TeV. A new particle accelerator is currently under consideration with energies of $\sim$ 100 TeV [102]. However, this collider energy is many, many orders of magnitude below the natural energy scale of quantum gravity, $\sim 10^{16}$ TeV (Planck energy). Consequently, to study the effects of quantum gravity, it is necessary to consider low-energy corrections.

Several theories of quantum gravity, including loop quantum gravity and string theory, predict the existence of a minimum length scale that is believed to be similar to a minimum uncertainty in the position (on the order of the Planck length). This leads to a modification of Heisenberg's uncertainty principle, in which the quantum gravity corrections are truncated to be either quadratic (no linear term) or linear + quadratic in a small parameter dependent upon the Planck length [103, 104].

This modification of Heisenberg's uncertainty principle is known as the generalized uncertainty principle (GUP), which is valid at very low energies. The commutation relation of GUP is [104],

$$[\hat{q}_i, \hat{p}_j] = i\hbar \left(\delta_{ij} - \alpha \left(\hat{p}\delta_{ij} + \frac{\hat{p}_i \hat{p}_j}{\hat{p}} \right) + \alpha^2 (\hat{p}^2 \delta_{ij} + 3\hat{p}_i \hat{p}_j) \right), \tag{4.100}$$

where $\alpha \sim \alpha_0 \ell_{Pl}/\hbar$ is a small parameter, α_0 is a constant, $\hat{p}^2 = \sum_{j=1}^{3} \hat{p}_j^2$ and ℓ_{Pl} is the Planck length. As shown in [104], one method to find equation (4.100) from $[\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij}$ is to make the transformation, $\hat{p}_i \to \hat{p}_i \left(1 - \alpha\hat{p} + 2\alpha^2\hat{p}^2\right)$. In one dimension, this mapping modifies the Hamiltonian of $H = \frac{\hat{p}^2}{2m} + V(\hat{q})$, such that the GUP-modified Hamiltonian is,

$$H \to \frac{\hat{p}^2}{2m} + V(\hat{q}) - \frac{\alpha}{m}\hat{p}^3 + \frac{5\alpha^2}{2m}\hat{p}^4. \tag{4.101}$$

As a proof of concept of using a transition operator to introduce quantum gravity effects, we will consider only the quadratic form of GUP, such that $[\hat{q}, \hat{p}] = i\hbar(1 + 4\alpha^2\hat{p}^2)$. It is possible to obtain this modified commutation relationship from $[\hat{q}, \hat{p}] = i\hbar$ with $\hat{p} \to \hat{p}\left(1 + \frac{4}{3}\alpha^2\hat{p}^2\right)$ [103], so that the Hamiltonian of $H = \frac{\hat{p}^2}{2m} + V(\hat{q})$ becomes

$$H \to \frac{\hat{p}^2}{2m} + V(\hat{q}) + \frac{4\alpha^2}{3m}\hat{p}^4 + \mathcal{O}(\alpha^4). \tag{4.102}$$

Therefore our goal is to find the transition operator and star product incorporating GUP, such that

$$[q, p]_{\star_\alpha} = i\hbar(1 + 4\alpha^2 p^2). \tag{4.103}$$

As an ansatz, take $T_\alpha = e^{b\partial_q + c\partial_p}$, where $b = b(q,p)$, $c = c(q,p)$ so that

$$T_\alpha^{-1} q = q - b\,, \tag{4.104}$$

$$T_\alpha^{-1} p = p - c\,. \tag{4.105}$$

Recalling that $T(T^{-1}f \star_M T^{-1}g) = f \star_T g$ for any transition operator, we have

$$T_\alpha^{-1} q \star_M T_\alpha^{-1} p = q \star_M p - q \star_M c - b \star_M p + b \star_M c\,, \tag{4.106}$$

$$T_\alpha^{-1} p \star_M T_\alpha^{-1} q = p \star_M q - p \star_M b - c \star_M q + c \star_M b\,, \tag{4.107}$$

The generalized uncertainty principle is a perturbative concept, so we need only keep terms up to $\mathcal{O}(\alpha^2)$. As $T_\alpha \to 1$ in the limit of $\alpha \to 0$, this implies that b and c are both of $\mathcal{O}(\alpha^2)$. Hence,

$$\begin{aligned} T_\alpha[T_\alpha^{-1} q, T_\alpha^{-1} p]_{\star_M} &= [q,p]_{\star_M} + (b\partial_q + c\partial_p)[q,p]_{\star_M} - [q,c]_{\star_M} - [b,p]_{\star_M} \\ &= i\hbar\left(1 - \partial_p c - \partial_q b\right)\,, \end{aligned} \tag{4.108}$$

implying that

$$\partial_p c + \partial_q b = -4\alpha^2 p^2\,, \tag{4.109}$$

from equation (4.103).

We want the transition operator to operate on the Hamiltonian so that the GUP-modified Hamiltonian is $T_\alpha H = H + \dfrac{4\alpha^2}{3m} p^4$. Let us expand the potential as the series, $V(q) = a_1 + a_2 q + a_3 q^2 + V_3(q)$, where $V_3(q)$ contains powers of q greater than or equal to 3. Applying T_α to $H = \frac{p^2}{2m} + V(q)$ gives

$$T_\alpha H = \frac{p^2}{2m} + V(q) + b\left[a_2 + 2a_3 q + \frac{\partial V_3(q)}{\partial q}\right] + \frac{c}{m} p\,. \tag{4.110}$$

Therefore, $b = 0$ as the GUP-modified Hamiltonian does not contain extra position dependence beyond $V(q)$. Hence, equation (4.109) indicates that $c = -\frac{4\alpha^2}{3}p^3$. As a result,

$$T_\alpha H = \frac{p^2}{2m} + V(q) - \frac{4\alpha^2}{3m}p^4\,, \tag{4.111}$$

but this is not equal to $\frac{p^2}{2m} + V(q) + \frac{4\alpha^2}{3m}p^4$.

To circumvent this difficulty, note that the ansatz of $T_\alpha = e^{b\partial_q + c\partial_p}$ only included derivatives of ∂_q and ∂_p because q and p are both linear functions; it was not necessary to include higher order derivatives in T_α. As H is quadratic in p, consider the revised ansatz of

$$T_\alpha = e^{-\frac{4}{3}\alpha^2 p^3 \partial_p + d\partial_p^2}\,, \tag{4.112}$$

where $d = d(q,p)$. We now have

$$T_\alpha H = \frac{p^2}{2m} + V(q) - \frac{4\alpha^2}{3m}p^4 + \frac{d}{m}\,, \tag{4.113}$$

implying that $d = \frac{8}{3}\alpha^2 p^4$. Therefore, the transition operator introducing GUP into a system is

$$T_\alpha = e^{-\frac{4}{3}\alpha^2 p^3 \partial_p + \frac{8}{3}\alpha^2 p^4 \partial_p^2}\,, \tag{4.114}$$

Using equation (4.82), the star product, $\star_\alpha$, is

$$\begin{aligned} \star_\alpha =& \mathcal{I}(1,2)\left[e^{-\frac{4\alpha^2}{3}\left(\frac{p_1+p_2}{2}\right)^3(\partial_{p_1}+\partial_{p_2}) + \frac{8\alpha^2}{3}\left(\frac{p_1+p_2}{2}\right)^4(\partial_{p_1}+\partial_{p_2})^2} e^{\frac{i\hbar}{2}(\partial_{q_1}\partial_{p_2} - \partial_{p_1}\partial_{q_2})} \right. \\ & \left. \times\, e^{\frac{4\alpha^2}{3}p_1^3\partial_{p_1} - \frac{8\alpha^2}{3}p_1^4\partial_{p_1}^2} e^{\frac{4\alpha^2}{3}p_2^3\partial_{p_2} - \frac{8\alpha^2}{3}p_2^4\partial_{p_2}^2} \right]. \end{aligned} \tag{4.115}$$

We note that we have only considered quadratic effects in GUP to find the transition operator and star product. However, a similar procedure could be done to determine the transition operator and star product if the linear+quadratic form of GUP is used or if higher order quantum gravity corrections are desired.

One advantage in knowing the transition operator for GUP is that it is straightforward to then determine the equation of motion for the distribution function $T_\alpha W$,

$$\begin{aligned} i\hbar\frac{\partial T_\alpha W}{\partial t} &= [T_\alpha H, T_\alpha W]_{\star\alpha} \\ &= [H,W]_{\star M} - \left(\frac{4}{3}\alpha^2 p^3\partial_p - \frac{8}{3}\alpha^2 p^4 \partial_p^2\right)[H,W]_{\star M} \end{aligned}$$

As an example, consider the simple harmonic oscillator with $W = \frac{(-1)^n}{\pi\hbar}\exp\left[-\frac{2H}{\hbar}\right] L_n\left(\frac{4H}{\hbar}\right)$. We have

$$\begin{aligned} T_\alpha W = \frac{(-1)^n e^{-\frac{q^2+p^2}{\hbar}}}{3\pi\hbar^3} \Bigg\{ 16\alpha^2 p^4 \Bigg[& 8p^2 L_{n-2}^2\left(\frac{2\left(q^2+p^2\right)}{\hbar}\right) \\ & + \left(8p^2-\hbar\right) L_{n-1}^1\left(\frac{2\left(q^2+p^2\right)}{\hbar}\right)\Bigg] \\ & \left(3\hbar^2 - 8\alpha^2 p^4\hbar + 32\alpha^2 p^6\right) L_n\left(\frac{2\left(q^2+p^2\right)}{\hbar}\right)\Bigg\} \end{aligned} \tag{4.116}$$

where $L_n^m(x) = (-1)^m \frac{d^m}{dx^m} L_{n+m}(x)$ is an associated (generalized) Laguerre polynomial [71]. We illustrate the GUP transition operator applied to the simple harmonic oscillator Wigner function in Figure 4.2. In Figure 4.3, we plot the difference between $T_\alpha W$ and W, setting $\alpha = 0.02$.

We also consider in Figures 4.4 and 4.5 the effect of GUP by plotting the probability in position, $P_n(q) = \int dp\, T_\alpha W$, and the probability for the momentum, $P_n(p) = \int dq\, T_\alpha W$, for the n^{th} energy level with $\alpha = 0.05$. We see that the greatest effects appear at the locations of maximum probability for the simple harmonic oscillator ($\alpha = 0$). The perturbations become more noticeable as n increases.

In this Chapter, we have investigated local transition operators as a means to understand augmented quantization. Focussing on the damped harmonic oscillator and the generalized uncertainty principle of quantum gravity phenomenology, our results show that local transition operators are able to incorporate more physical features than a global transition

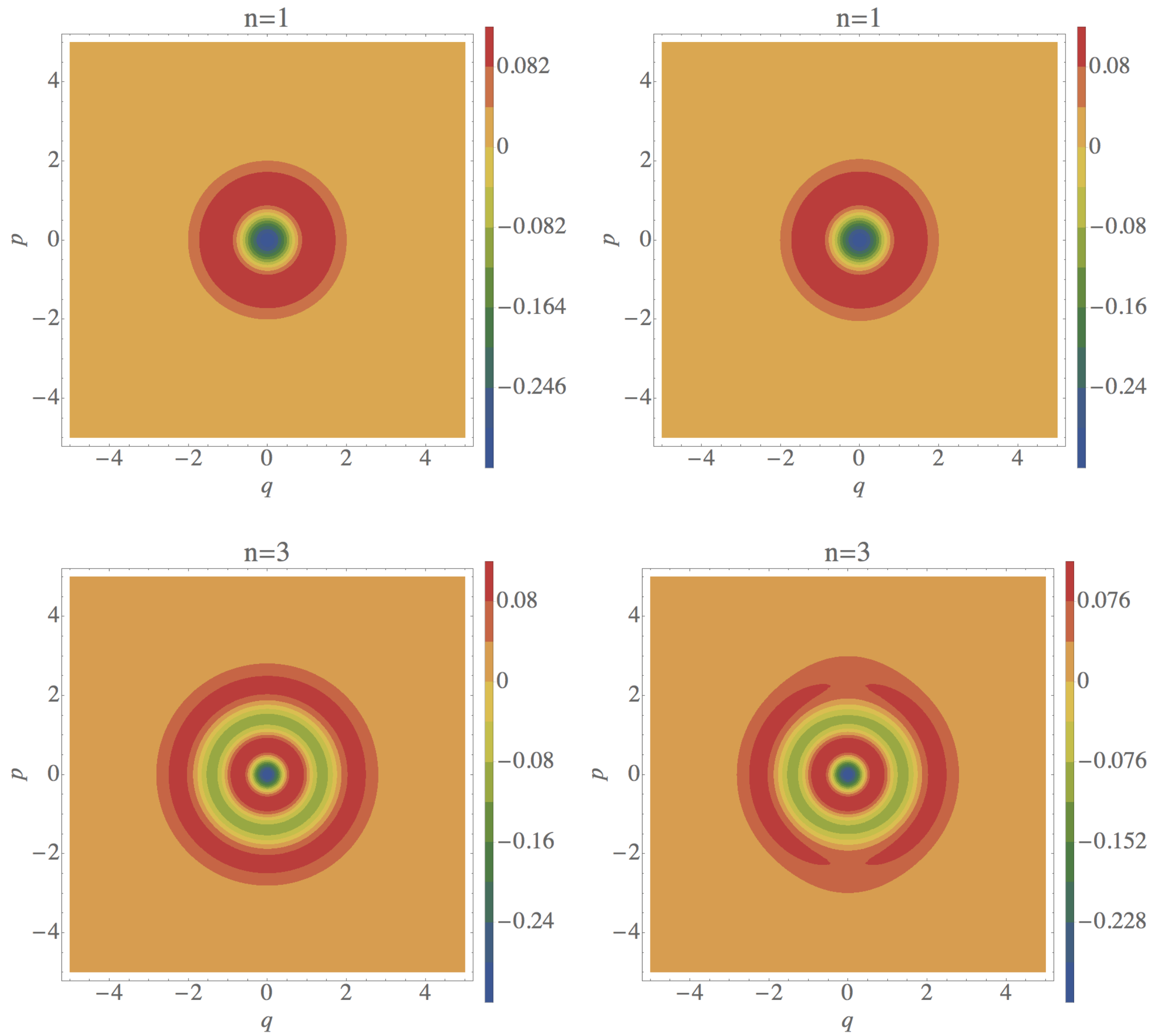

Figure 4.2: Comparing the contour plot of the simple harmonic oscillator Wigner function (left) with the contour plot of $T_\alpha W$ (right) for the $n=1$ and $n=3$ energy levels. GUP does not have a large effect on the $n=1$ energy level, but the perturbations from GUP are more prominent at $n=3$.

operator. In the next Chapter, we will consider augmented quantization from the perspective of time-dependent local and global transition operators.

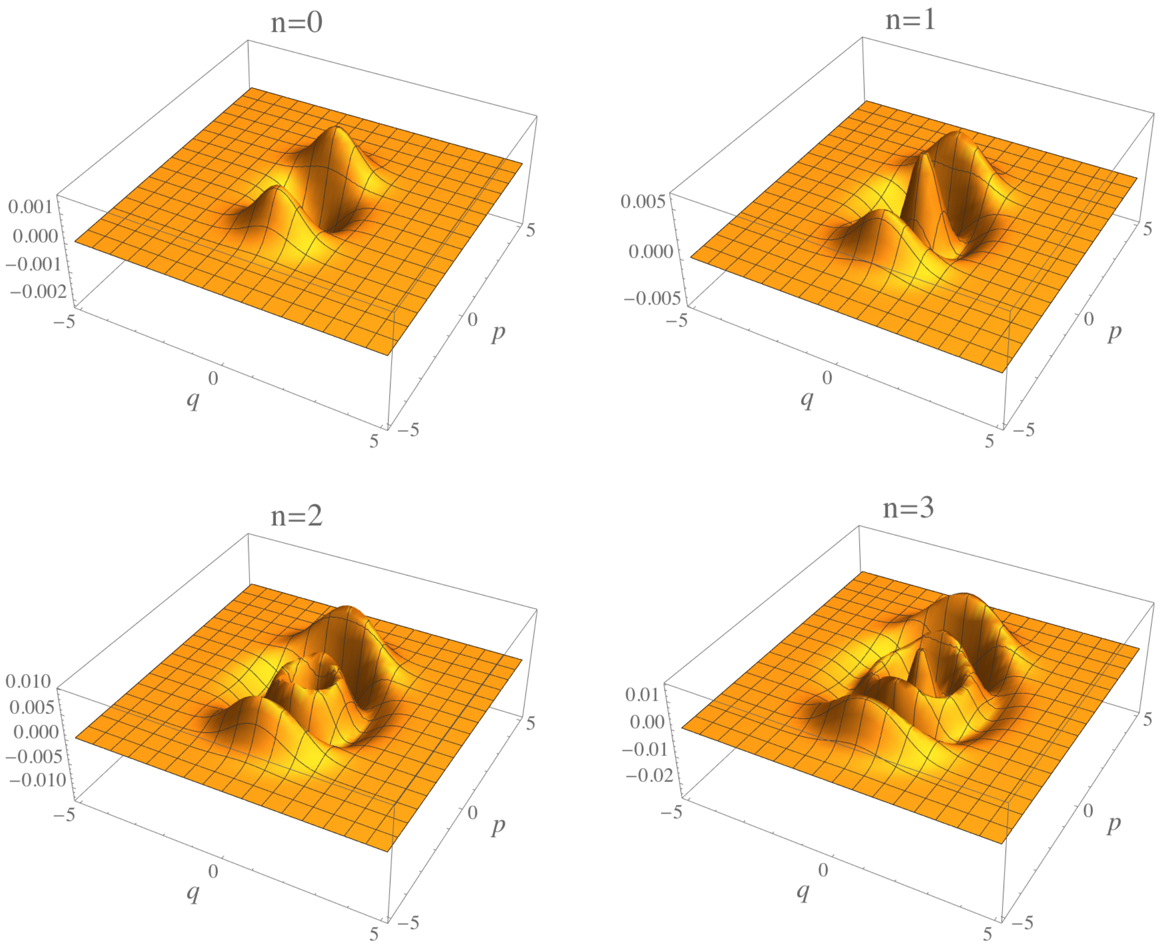

Figure 4.3: The difference $T_\alpha W - W$ for the first four energy levels of the simple harmonic oscillator. As n increases, so does the effect of the quantum gravity corrections.

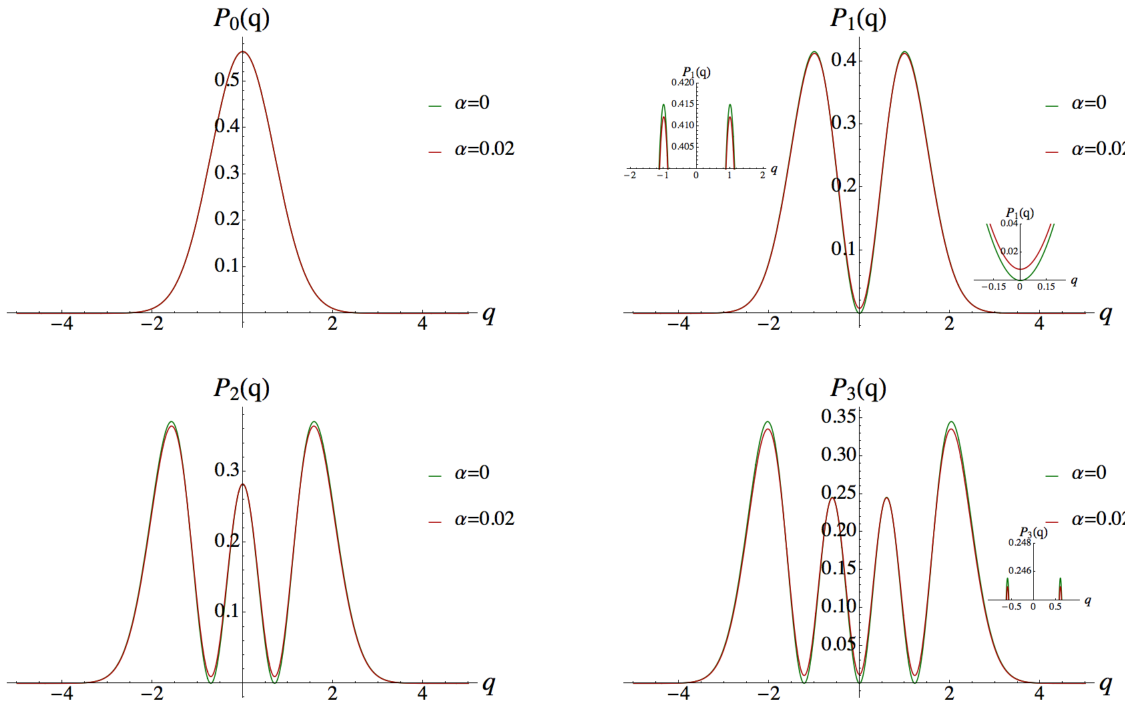

Figure 4.4: The marginal probability distribution of the position for the first four energy levels of the GUP-modified simple harmonic oscillator. The marginal probability density of the position for $n = 0$ is the same for both the simple harmonic oscillator and the GUP-modified simple harmonic oscillator.

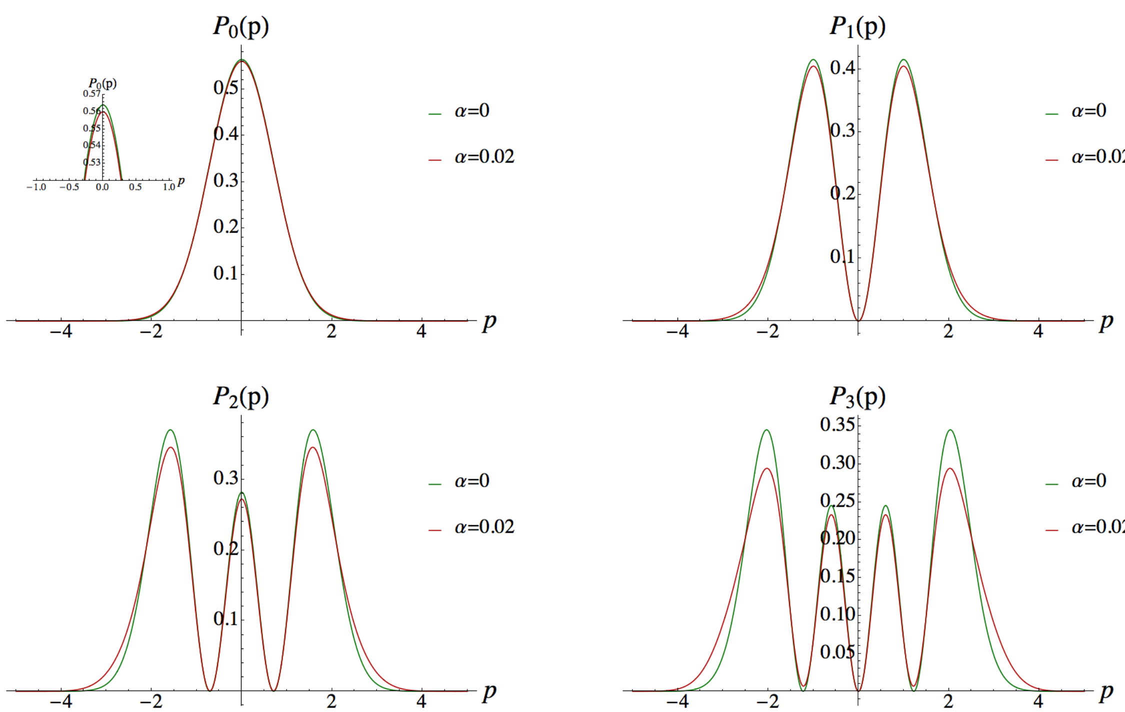

Figure 4.5: The marginal probability distribution of the momentum for the first four energy levels of the GUP-modified simple harmonic oscillator.

Chapter 5

Time Dependent Transition Operators

5.1 Motivation

In the previous chapters, we have considered time-independent Hamiltonians. For a time-dependent Hamiltonian, the density operator is [76]

$$\hat{\rho}(t) = \hat{U}(t)\hat{\rho}(0)\hat{U}^{\dagger}(t)\,, \tag{5.1}$$

where (for $t \geq t_0$).

$$\begin{aligned}
\hat{U}(t) &= \sum_{n=0}^{\infty} U_n;\;\; U_n = \frac{(-i)^n}{\hbar^n n!}\int_{t_0}^{t} d\tau_1 \int_{t_0}^{t} d\tau_2 \cdots \int_{t_0}^{t} d\tau_n \mathcal{T}\left\{\hat{H}(\tau_1)\hat{H}(\tau_2)\cdots\hat{H}(\tau_n)\right\}, \\
&= \sum_{n=0}^{\infty} \frac{(-i)^n}{\hbar^n n!}\int_{t_0}^{t} \cdots \int_{t_0}^{t} \mathcal{T}\left\{\hat{H}(\tau_1)\cdots\hat{H}(\tau_n)\right\} d\tau_1 \cdots d\tau_n\,, \\
&= \mathcal{T}\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t}\hat{H}(\tau)d\tau\right],
\end{aligned} \tag{5.2}$$

is the propagator and $\mathcal{T}$ is the time-ordering operator defined as [105]

$$\mathcal{T}\left\{\hat{H}(\tau_1)\hat{H}(\tau_2)\right\} = \begin{cases} \hat{H}(\tau_1)\hat{H}(\tau_2) & \text{if } \tau_1 \geq \tau_2\,, \\ \hat{H}(\tau_2)\hat{H}(\tau_1) & \text{if } \tau_2 \geq \tau_1\,, \end{cases} \tag{5.3}$$

and

$$\mathcal{T}\left\{\hat{H}(\tau_1)\hat{H}(\tau_2)\cdots\hat{H}(\tau_n)\right\} = \hat{H}(\tau_1')\hat{H}(\tau_2')\cdots\hat{H}(\tau_n')\,. \tag{5.4}$$

Here, $\tau_1', \tau_2', \ldots, \tau_n'$ is a permutation of $\tau_1, \tau_2, \ldots, \tau_n$ with $\tau_1 \geq \tau_2 \geq \cdots \geq \tau_n$. The purpose of the time-ordering operator is to guarantee that the correct Hamiltonian is applied at the correct time.

By differentiating equations (5.1) and (5.2) and making use of the Leibniz integral rule (differentiation under the integral sign), the resultant equation is still the Liouville-von Neumann equation,

$$i\hbar \frac{\partial \hat{\rho}}{\partial t} = [\hat{H}(t), \hat{\rho}], \tag{5.5}$$

so that the application of the Wigner transform from equation (2.18) yields

$$i\hbar \frac{\partial W}{\partial t} = [H(t), W]_{\star M}. \tag{5.6}$$

In this Chapter, we will assume the Hamiltonian to be time-independent, but the transition operator to be time-dependent. As a result, TH will be a function of time.

The properties of time-dependent transition operators and star products will be briefly analyzed. Such time-dependent products have recently been used to investigate dissipation in quantum systems (for example, see [106] and [107]). We will then propose a transition operator for a driven harmonic oscillator quantum system. To conclude this Chapter, we investigate the potential of the transition operator to describe decoherence.

5.2 Time-dependent Transition Operators and Star Products

Let us first consider the transition operator as a function of time, position and momentum, which we denote as $T = T(t, q, p, \partial_q, \partial_p)$. Applying the transition operator to the

Moyal bracket,

$$i\hbar T\left(\frac{\partial W}{\partial t}\right) = [TH, TW]_{\star_T}\,, \tag{5.7}$$

$$\implies i\hbar\frac{\partial TW}{\partial t} = [TH, TW]_{\star_T} - i\hbar\frac{\partial T}{\partial t}W\,, \tag{5.8}$$

by applying the product rule. We see that there is an additional term present in the equation of motion, unlike the case of equation (2.91), in which the transition operator is time-independent. This new term can be interpreted as being responsible for incorporating additional physical effects. For instance, if $T = e^{f(t,q,p,\partial_q,\partial_p)}$, then $\frac{\partial T}{\partial t}W = \dot{f}(t,q,p,\partial_q,\partial_p)TW$, where $\dot{f}$ is the partial derivative of f with respect to time. In general, $\dot{f}(t,q,p,\partial_q,\partial_p)$ seeks to scale derivatives of TW both as a function of time and its location, similar to the case of a coupling a harmonic oscillator to a thermal reservoir of non-interacting harmonic oscillators, as discussed in [108, 109].

Further, the star product itself will depend on time. It was demonstrated in [110] and [111] that the Moyal product can be related to the area of a triangle in phase space by using the representation of the star product,

$$f \star_M g = \frac{1}{\pi^2\hbar^2}\int dp'dp''dq'dq'' f(q',p')g(q'',p'')e^{-\frac{2i}{\hbar}[p(q'-q'')+p'(q''-q)+p''(q-q'')]}\,. \tag{5.9}$$

The exponent is proportional to the area of a triangle whose vertices have the coordinates of (q,p), (q',p'), (q'',p''). In general a polygon of coordinates $(q_1,p_1),\ldots,(q_n,p_n)$ has an area of [112]

$$A = \frac{1}{2}\left(\begin{vmatrix} q_1 & q_2 \\ p_1 & p_2 \end{vmatrix} + \begin{vmatrix} q_2 & q_3 \\ p_2 & p_3 \end{vmatrix} + \cdots + \begin{vmatrix} q_n & q_1 \\ p_n & p_1 \end{vmatrix}\right), \tag{5.10}$$

where we use vertical lines to represent the determinant. This equation is sometimes called the shoelace formula.

Different star products correspond to different areas and geometries (right, isosceles, etc). of triangles. Therefore, a time-dependent star product can be interpreted as giving rise to a time-dependent area and geometry of a triangle.

As no derivatives of time are present in $TH \star_T TW = ETW$, all properties of the transition operator and star product will be the same as when $T = T(\partial_q, \partial_p)$ or $T = T(q, p, \partial_q, \partial_p)$. It is only when applying the transition operator to the Moyal bracket that the presence of t will introduce additional features, as illustrated in equation (5.8).

We will now briefly consider the time-dependent transition operator, $T = T(\partial_t)$, such that the transition operator does not dependent on t, p, q, ∂_p, or ∂_q. Such a transition operator will not have an effect upon stargenvalue equations of (2.54) and (2.55). Further, we still have $i\hbar\frac{\partial TW}{\partial t} = [TH, TW]_{\star_T}$, without the additional term of (5.8). The star product, however, will now include left and right derivatives of time, so that, by the same argument of Section 2.10, $\star_T = \star_M T^{-1}\left[\overleftarrow{\partial}_t\right] T\left[\overleftarrow{\partial}_t + \overrightarrow{\partial}_t\right] T^{-1}\left[\overrightarrow{\partial}_t\right]$. Mathematically, $T = T(\partial_t)$ acts to incorporate higher-order time derivatives into the equation of motion.

With $T = T(\partial_t)$, it is not possible to define a weight function as $\Phi(\xi, \eta)$ is defined for only derivatives of p and q. For $T = T(t, \partial_q, \partial_p)$, however, the weight function of equation (2.99) is then $\Phi(\xi, \eta) = T(t, -i\xi, -i\eta)$. Therefore, a time-dependent transition operator automatically implies a time-dependent weight function. Such time-dependencies physically mean the existence of a time-dependent augmented quantization. Theoretically, it is possible to use a time-dependent transition operator to describe a system that is initially normal ordered at an initial time, $t = t_0$, but antinormal ordered at a later time, $t = t'$. This can be demonstrated with $T = e^{f(t)\left(\hbar\partial_q^2/4m\omega + \hbar m\omega\partial_p^2/4\right)}$, where $f(t) = 1 - 2t$, for example.

5.3 The Driven Harmonic Oscillator

In Section 4.5, we used the simple harmonic oscillator Hamiltonian to find the transition operator to convert to a damped system described by augmented qantization. We did this by imposing that the $\hbar \to 0$ limit of the star brackets, $\frac{1}{i\hbar}[p, H]_{\star_\gamma}$ and $\frac{1}{i\hbar}[q, H]_{\star_\gamma}$, must give the classical equations of motion for the damped harmonic oscillator. The same procedure can be used to determine a transition operator that maps the classical simple harmonic oscillator to a quantum driven harmonic oscillator, which has the classical equations of motion,

$$\begin{aligned} \dot{p} &= -q + F(t)\,, \\ \dot{q} &= p\,. \end{aligned} \tag{5.11}$$

Our objective is to demonstrate the proof of concept that we can recover equation (5.11) in the $\hbar \to 0$ limit of the star brackets. Therefore, consider $T_F = e^{a(q,t)\partial_p^2}$. We choose a transition operator of this form because additional derivatives and functions of q and p result in non-commuting individual factors when expanding the transition operator as a series, as demonstrated in equation (4.46). Such a transition operator containing additional derivatives and functions of q or p can cause successive powers of $F(t)$ to be present, and hence cannot give equation (5.11) in the $\hbar \to 0$ limit of the $\star_T$-bracket.

Recalling that $f \star_T g = T(T^{-1} f \star_M T^{-1} g)$, we have

$$\begin{aligned} \lim_{\hbar \to 0} \frac{1}{i\hbar} T_F \left[T_F^{-1} p, T_F^{-1} H \right]_{\star_M} &= \lim_{\hbar \to 0} \frac{1}{i\hbar} T_F \left[p, H - a(q,t) \right]_{\star_M}\,, \\ &= -q + \frac{\partial a(q,t)}{\partial q}\,, \end{aligned}$$

and

$$\begin{aligned} \lim_{\hbar \to 0} \frac{1}{i\hbar} T_F \left[T_F^{-1} q, T_F^{-1} H \right]_{\star_M} &= \lim_{\hbar \to 0} \frac{1}{i\hbar} T_F \left[q, H - a(q,t) \right]_{\star_M}\,, \\ &= p\,. \end{aligned}$$

Therefore, $\dfrac{\partial a(q,t)}{\partial q} = F(t)$, implying $a(q,t) = F(t)q + C(t)$, where $C(t)$ is an arbitrary function of time. We note that $C(t)$ may have physical consequences, but we are concerned with a proof of concept demonstrating that a transition operator can be found for the driven harmonic oscillator, so we take $C(t) = 0$ for simplicity. Therefore,

$$T_F = e^{qF(t)\partial_p^2}\,, \tag{5.12}$$

with the weight function of $\Phi(\xi,\eta) = e^{-qF(t)\eta^2}$. We further note that this transition operator holds for an arbitrary potential, $V(q)$, such that

$$\dot{p} = -\frac{\partial V}{\partial q} + F(t)\,, \tag{5.13}$$

$$\dot{q} = p\,. \tag{5.14}$$

The transition operator of $T_F = e^{qF(t)\partial_p^2}$ is equivalent to the Weierstrass transform (Appendix B) [113] (for $qF(t) > 0$), hence

$$T_F W(q,p,0) = \frac{1}{\sqrt{4\pi qF(t)}} \int dp' W(q,p',0)\, e^{-\frac{(p-p')^2}{4qF(t)}}\,. \tag{5.15}$$

For illustrative purposes, let us consider a time-independent driving force, such that $F(t) = F_0\,[\theta(q) - \theta(-q)]$, where $\theta(q)$ is the Heaviside step function defined as [71]

$$\theta(q) = \begin{cases} 0 & q < 0 \\ 1 & q \geq 0 \end{cases}\,, \tag{5.16}$$

and F_0 is a constant. We will apply T_F with this driving force to the first four energy levels of the simple harmonic oscillator. We present this simple example to demonstrate the effect that the driving force can have upon the probability densities for the momentum. By varying F_0, it will be possible to determine how more complicated driving forces will affect the probability density.

The probability densities, $P_n(p)$, for the momentum are shown in Figures 5.1 and 5.2. The position probability density will not be affected because expansion of the transition operator and integration with respect to p yields terms of the form

$$\int dp\, \frac{d^{2n}W(q,p)}{dp^{2n}} = 0 \tag{5.17}$$

for the simple harmonic oscillator Wigner function. This is the result of our assumption that the transition operator only depends on $a(q,t)$ and momentum derivatives.

Figure 5.1 illustrates small F_0, while Figure 5.2 shows the effects of larger forces. We see that as time progresses, the marginal probabilities spread in width and damp out oscillations. Hence, if time-dependence is periodic, we would expect the marginal probabilities in the momentum to also be periodic.

Using equation (4.82), the star product is

$$\star_F = \mathcal{I}(1,2)e^{\left(\frac{q_1+q_2}{2}\right)F(t)\left(\partial_{p_1}+\partial_{p_2}\right)^2} e^{\frac{i\hbar}{2}\left(\partial_{q_1}\partial_{p_2}-\partial_{p_1}\partial_{q_2}\right)}\, e^{-q_2F(t)\partial_{p_2}^2}\, e^{-q_1F(t)\partial_{p_1}^2}\,. \tag{5.18}$$

With equation (5.18), the equation of motion for the simple harmonic oscillator is

$$\begin{aligned} \frac{\partial T_F W}{\partial t} &= \frac{1}{i\hbar}[H+qF(t),T_FW]_{\star_F} - q\dot{F}(t)\,\partial_p^2 T_F W\,, \\ &= \frac{1}{i\hbar}[H,T_FW]_{\star_F} + F(t)\,\partial_p T_F W - q\dot{F}(t)\,\partial_p^2 T_F W \end{aligned} \tag{5.19}$$

The right hand side of this equation of motion consists of three terms: the first is analogous to the Moyal bracket between H and T_FW, the second term illustrates that $\frac{\partial T_F W}{\partial t}$ is dependent upon the change in momentum of the original system (noting that ∂_p and T_F commute), while the third shows that the derivative of the force also dictates the time-dependence of T_FW. Equation (5.19) is similar in structure to the Fokker-Planck equation [114].

5.4 Environmental Decoherence

When a quantum system is open, it may couple with its surroundings. As a result of this interaction, it could lose its ability to form a coherent superposition; this process is known

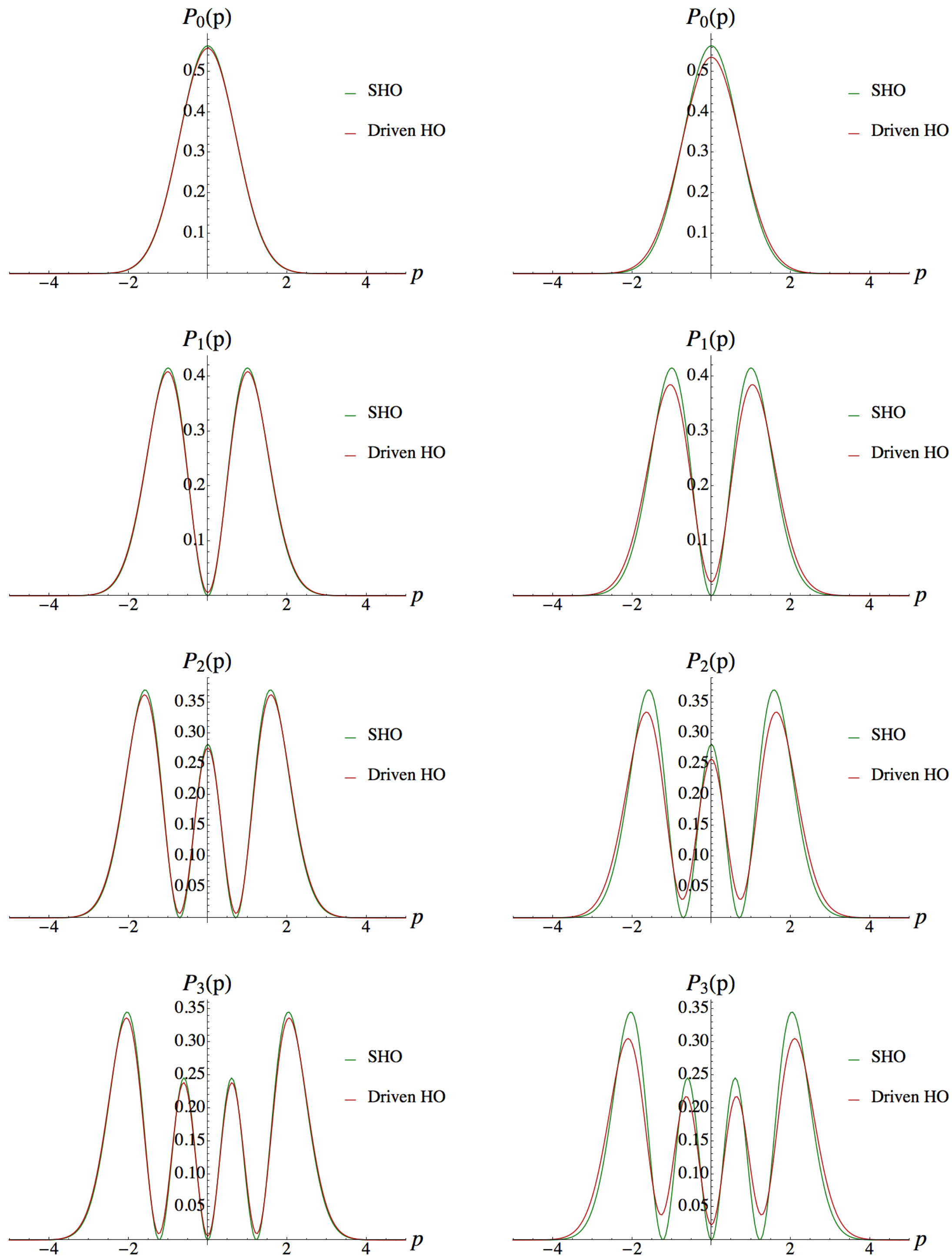

Figure 5.1: Small effects in the marginal probability density of momentum for an augmented quantization of the simple harmonic oscillator. The augmented quantization maps the simple harmonic oscillator to a quantum driven harmonic oscillator. The left column of plots use $F_0 = 0.1$ while the right column has $F_0 = 0.5$, where $F(t) = F_0\,[\theta(q) - \theta(-q)]$.

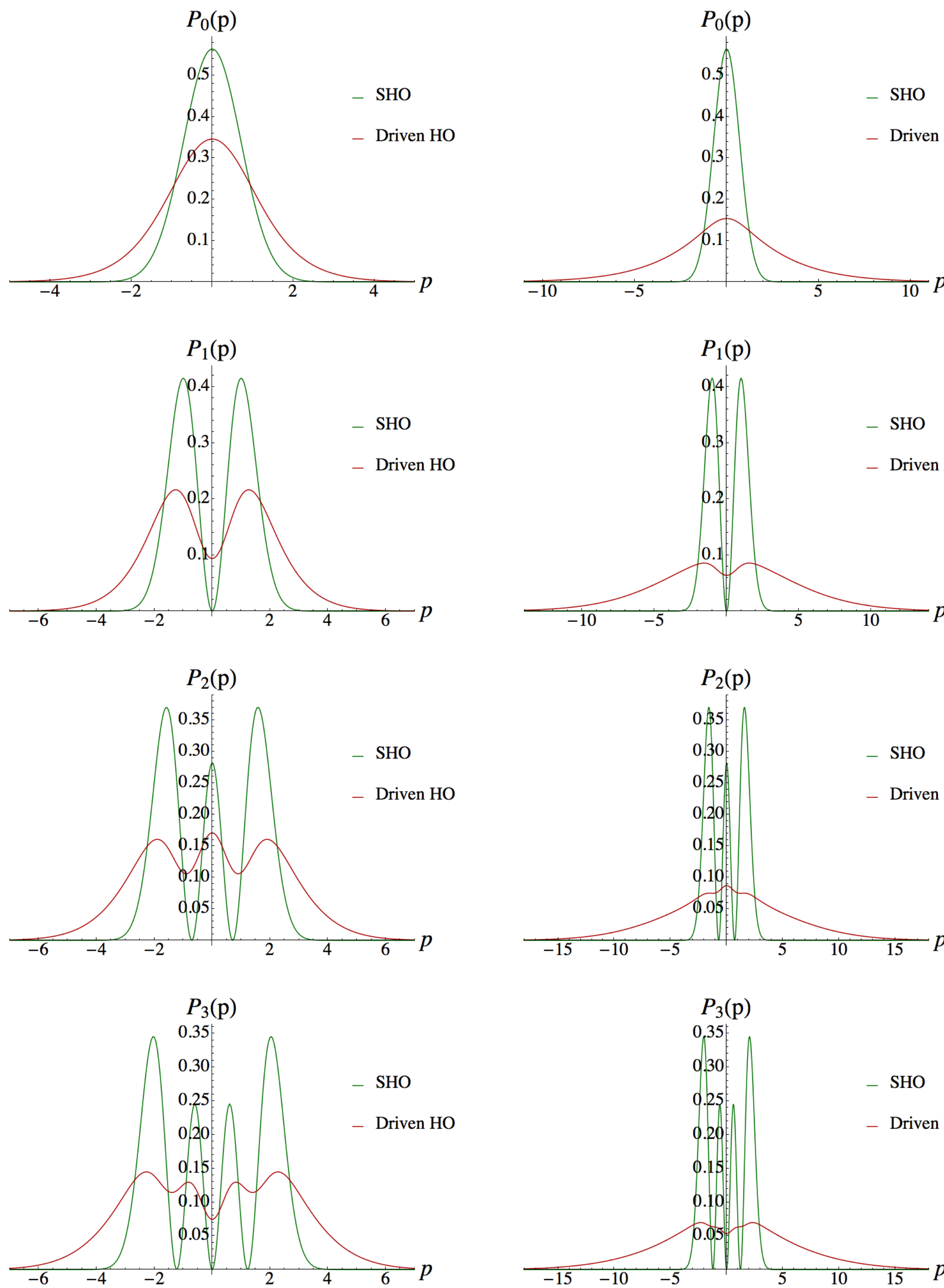

Figure 5.2: Large effects in the marginal probability density for an augmented quantization of the simple harmonic oscillator. The augmented quantization maps the simple harmonic oscillator to a quantum driven harmonic oscillator. The left column of plots use $F_0 = 1$ while the right column has $F_0 = 10$, where $F(t) = F_0\,[\theta(q) - \theta(-q)]$.

as environmental decoherence, which we will refer to as decoherence.

Initially, before the decohering process occurs, the system possesses a set of states that can be measured. After decoherence, many of the states will be unobservable, leaving a smaller subset of states that are able to be observed. This process is known as environmentally-induced superselection. For systems of macroscopic size, it is believed that decoherence brings about the emergence of classical mechanics from quantum mechanics [59, 108, 115].

Open systems must be described by a density operator, rather than a state vector because interaction between the system and the environment often results in an initially pure state becoming a mixed state during time evolution. In the position representation, decoherence manifests itself by the decay of off-diagonal elements of the density operator [108, 115]. As an example to see this, consider the system, S, entangled with its environment, E, such that the composite system-environment is initially in the state [115],

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|S_1\rangle|E_1\rangle + |S_2\rangle|E_2\rangle\right), \tag{5.20}$$

where $|S_i\rangle$ and $|E_i\rangle$ are states of the system and environment, respectively. Then, the density operator is [115],

$$\hat{\rho} = |\psi\rangle\langle\psi| = \frac{1}{2}\sum_{i,j=1}^{2} |S_i\rangle\langle S_j| \otimes |E_i\rangle\langle E_j| . \tag{5.21}$$

When analyzing decoherence, it is often desirable to remove the states of the environment from the density operator $\hat{\rho}$, such that the dynamics of the density operator corresponding to the system can be directly studied. This is achieved using the mathematical technique of the partial trace. It is valid because the partial trace accounts for Born's rule and the projection postulate [116, 117].

Let the density operator of two systems A, B be $\hat{\rho}^{AB}$. Also let $|a_1\rangle, |a_2\rangle \in A$, $|b_1\rangle, |b_2\rangle \in B$ be normalized but initially non-orthogonal. Denoting the partial trace over system B as

Tr_B, the partial trace of $|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|$ is [116]

$$\mathrm{Tr}_B(|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|) = |a_1\rangle\langle a_2| \,\mathrm{Tr}(|b_1\rangle\langle b_2|) \tag{5.22}$$

Therefore, with our example of S and E,

$$\hat{\rho}_S = \frac{1}{2}\Big(|S_1\rangle\langle S_1| + |S_2\rangle\langle S_2| + |S_1\rangle\langle S_2| \langle E_2|E_1\rangle + |S_2\rangle\langle S_1| \langle E_1|E_2\rangle \Big). \tag{5.23}$$

Often, $\langle E_2|E_1\rangle \sim e^{-t/\tau_d}$, where t is the time and τ_d is the time required for decoherence to occur. As $\hat{\rho}_S$ evolves, $\langle E_2|E_1\rangle \to 0$, so that after a sufficient period of time,

$$\hat{\rho}_S \approx \frac{1}{2}\Big(|S_1\rangle\langle S_1| + |S_2\rangle\langle S_2| \Big). \tag{5.24}$$

This indicates that the measurement of the density operator will show the system to be in $|S_1\rangle\langle S_1|$ or $|S_2\rangle\langle S_2|$, rather than, for example, $|S_1\rangle\langle S_2|$.

Decoherence can also be analyzed in phase space by taking the Wigner transform of the density operator. Within phase space, decoherence is exhibited through the decay of interference terms of the Wigner function. Depending on the system modelled, decoherence can also be represented by the diffusion in the position or the momentum [61,108,115,118–121].

The length of time necessary for the off-diagonal elements to be suppressed is determined by the system, environment, and method of interaction. For instance, if the system is a free electron being scattered by an environment of solar neutrinos, the decoherence time could be approximately 32 million years, while a bowling ball being scattered by sunlight could have a decoherence time around 10^{-28} s. In these cases, it is believed that decoherence will result in the electron being unable to form a superposition, while decoherence will cause the bowling ball to not just lose the ability to form a superposition, but also to behave in a classical manner [108].

5.4.1 Scattering Decoherence

To demonstrate decoherence in terms of the vanishing of interference and the effect in phase space, we will consider scattering, which has been extensively treated in, for example, [108, 115, 122–124]. Here, we briefly summarize the results presented in [108] and [115], in which density operator of the system was determined. We will then show that, in this situation, decoherence can be applied to augmented quantization with the transition operator to produce an effective description of decoherence.

For simplicity, the scattered particle will be taken to be more massive than the scattering particles, thus the system will have negligible recoil. It is also assumed that the system and environment are initially separable, so that the total density operator can initially be written as $\hat{\rho}_{SE} = \hat{\rho}_S \otimes \hat{\rho}_E$. If the scattering is isotropic, then, using the properties of the S-matrix and scattering amplitudes, it is possible to find the equation of motion for the density operator. Accounting for the number density $\mu(p')$ of incoming particles whose magnitude of the momentum is p' and the speed of those particles $v(p')$,

$$\frac{\partial \rho(q,q',t)}{\partial t} = -F(q-q')\rho(q,q',0)\,, \tag{5.25}$$

where $\rho(q,q',t) = \langle q|\hat{\rho}(t)|q'\rangle$ and $F(q-q')$ describes the rate (sometimes called localization rate or decoherence rate) at which the spatial coherence between q and q' vanishes, which is related to the scattering amplitude.

5.4.2 Long Wavelengh Limit

If the wavelength of the incoming particles is much larger than $q-q'$, equation (5.25) reduces to [125]

$$\frac{\partial}{\partial t}\rho(q,q',t) = -\Lambda(q-q')^2\rho\,, \tag{5.26}$$

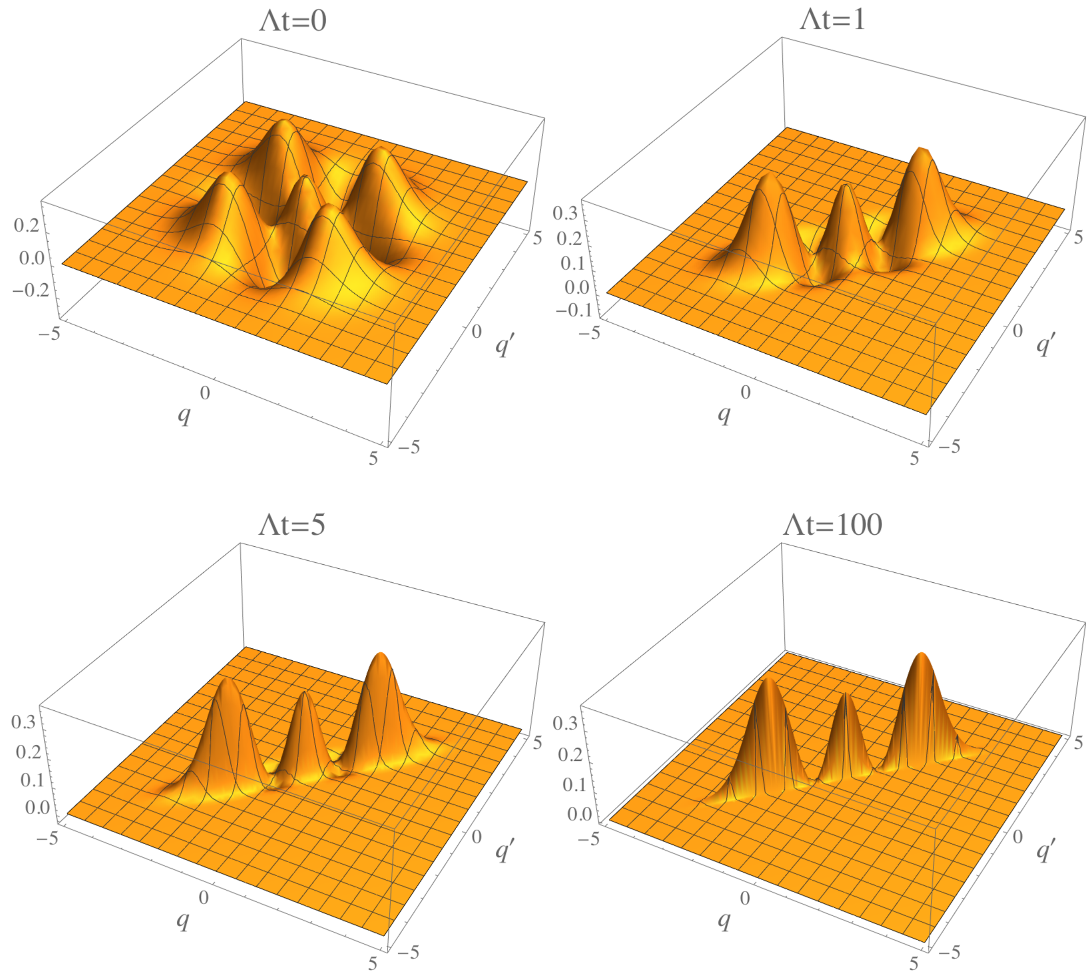

Figure 5.3: Decoherence in the long wavelength limit applied to the simple harmonic oscillator wavefunction, $\psi(q)$, so that $\rho(q,q') = \psi(q)\psi(q')$. Here, we plot the $n = 2$ energy level.

where Λ describes the decay of the coherence length $q - q'$ (decoherence rate). Therefore, the density operator is [125]

$$\rho(q,q',t) = \rho(q,q',0)e^{-\Lambda t(q-q')^2}, \tag{5.27}$$

which we plot in Figure 5.3 for the simple harmonic oscillator. We see that time-evolution will cause the off-diagonal terms of $q \neq q'$ to vanish.

By applying the Wigner transform of equation (2.18) to equation (5.26), we find [108]

$$\frac{\partial}{\partial t}\widetilde{W}(q,p,t) = \Lambda \frac{\partial^2 \widetilde{W}}{\partial p^2}\,, \tag{5.28}$$

which is solved by the Wigner function ($\Lambda t > 0$) [108],

$$\widetilde{W}(q,p,t) = \frac{1}{\sqrt{4\pi\Lambda t}} \int dp' \widetilde{W}(q,p',0)\, e^{-\frac{(p-p')^2}{4\Lambda t}}\,, \tag{5.29}$$

and

$$\widetilde{W}(q,p',0) = \frac{1}{2\pi} \int dy\, e^{-ip'y} \rho\left(q+\frac{\hbar y}{2}, q-\frac{\hbar y}{2}, 0\right). \tag{5.30}$$

is the Wigner transform of $\rho(q,q',0)$ and is found by using the resolution of the identity twice in equation (2.40).

Equation (5.29) describes time-dependent coarse-graining and is similar in form to the Husimi distribution of equation (3.1). We can also understand equation (5.29) as converting a Weyl-quantized system to an augmented quantization of the system as a result of the introduction of decoherence through Gaussian-smoothing the Wigner function.

Given an arbitrary system initially described by $W(q,p,0)$, we want to write a transition operator, T_Λ such that $T_\Lambda W(q,p,0)$ solves equation (5.28). If we let

$$T_\Lambda = e^{\Lambda t \partial_p^2}\,, \tag{5.31}$$

then,

$$\frac{\partial}{\partial t}\Big(T_\Lambda W(q,p,0)\Big) = \Lambda \partial_p^2\, T_\Lambda W(q,p,0)\,, \tag{5.32}$$

showing that $T_\Lambda W(q,p,0) = \widetilde{W}(q,p,t)$. As in Section 5.3, $T_\Lambda = e^{\Lambda t \partial_p^2}$ is equivalent to the Weierstrass transform. Figures 5.4 and 5.5 illustrate the Wigner function for the simple harmonic oscillator undergoing decoherence and its resultant momentum probability density.

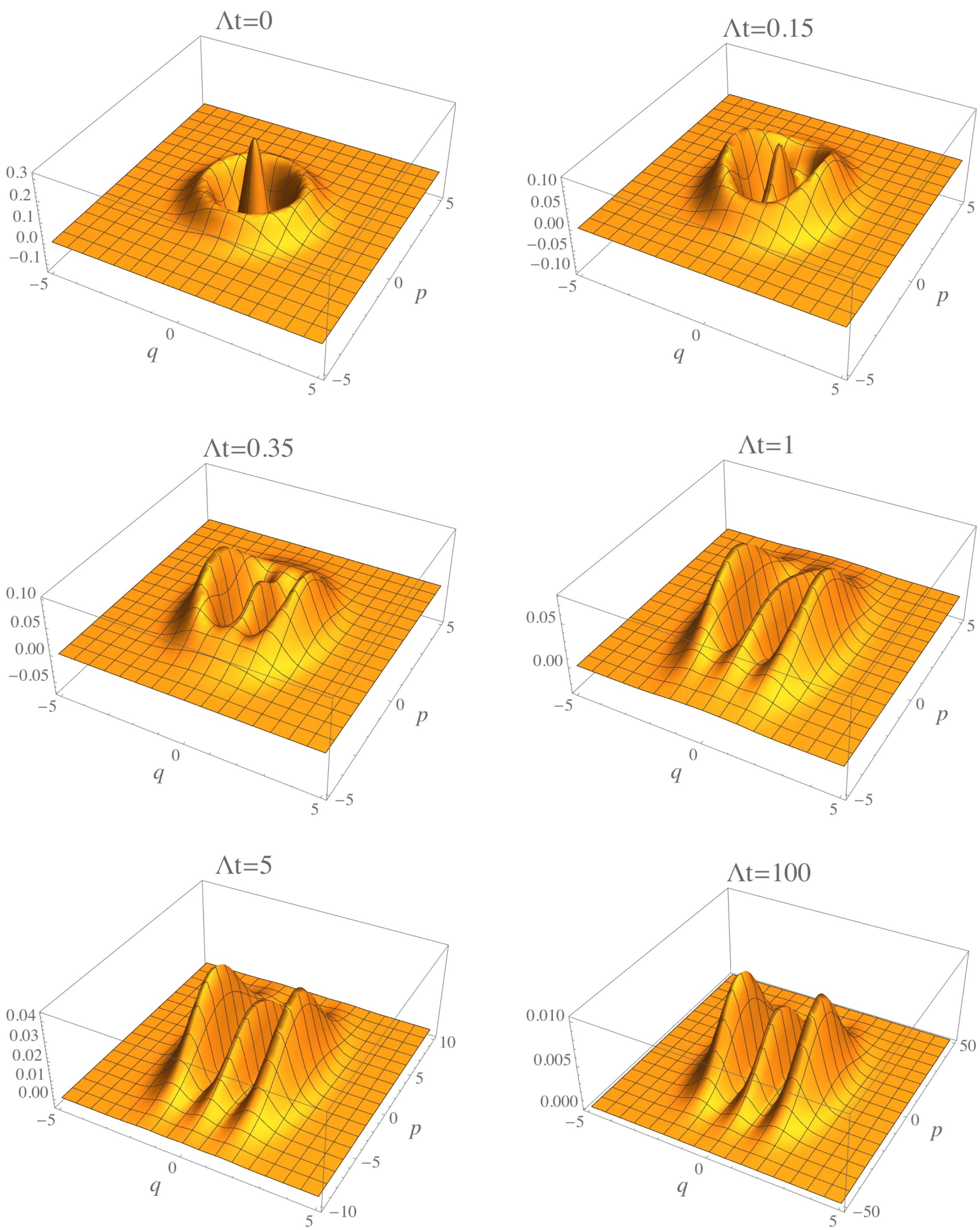

Figure 5.4: The transformed Wigner function of the $n = 2$ simple harmonic oscillator, such that it incorporates decoherence. As shown, there is diffusion in the momentum.

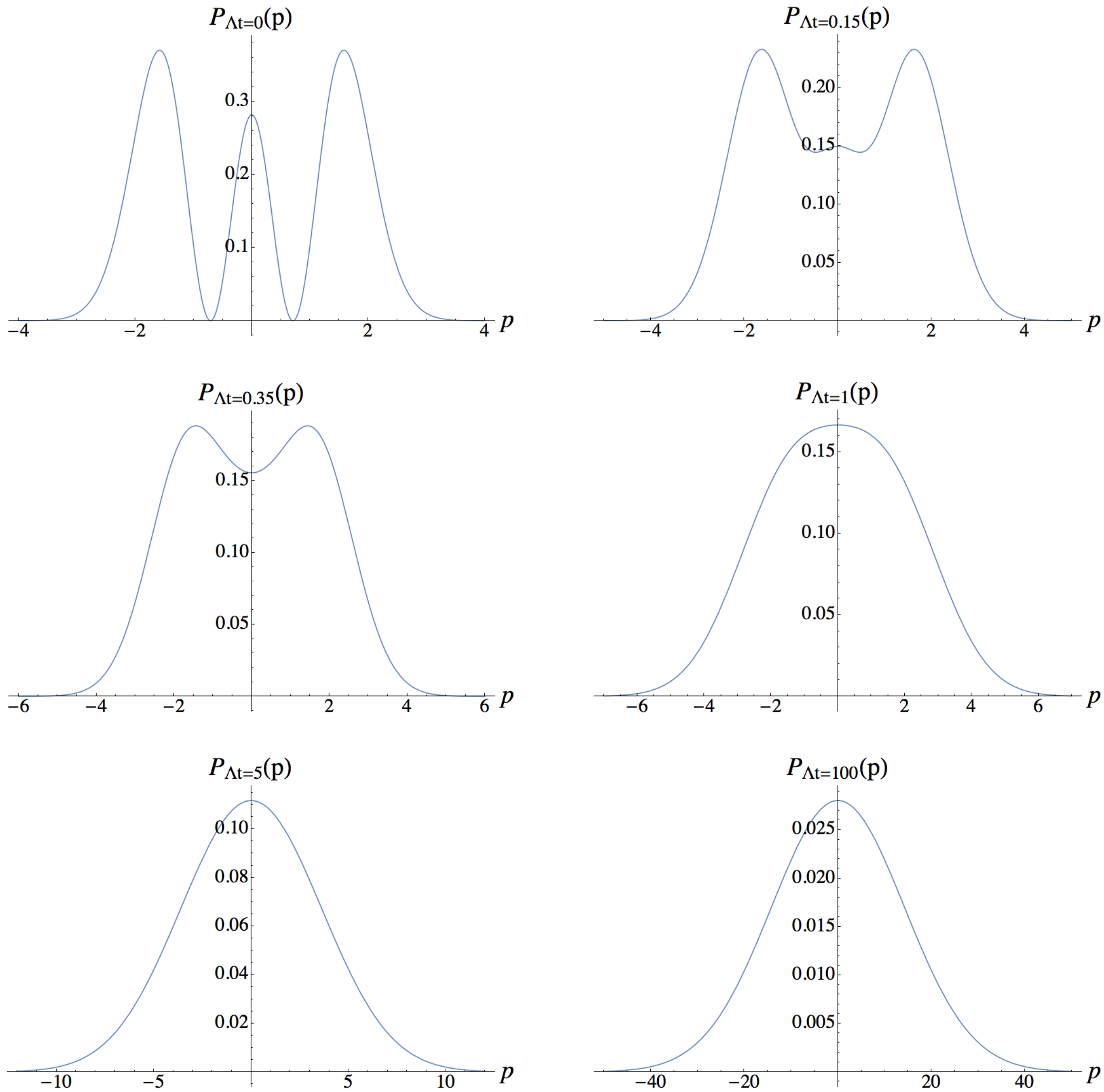

Figure 5.5: The probability of the momentum of the $n = 2$ augmented quantized simple harmonic oscillator so that it includes decoherence. As the transition operator is only dependent on derivatives of the momentum, the probability of position does not change, by equation (5.17).

We note that the resultant star product for this simplification of decoherence is

$$\star_\Lambda = \star_M e^{2\Lambda t \overleftarrow{\partial}_p \overrightarrow{\partial}_p}, \tag{5.33}$$

by using equation (2.88). Applying T_Λ to equation (2.54), we have

$$H \star_\Lambda T_\Lambda W = (E - \Lambda t)\, T_\Lambda W\,, \tag{5.34}$$

because $TH = H + \Lambda t$ (similar to Section 4.2.3). As $t \to \infty$, we see that $(E - \Lambda t) \to -\infty$. This could suggest that the above method of analyzing decoherence with the Wigner function may only be valid for $\Lambda t \ll E$ if we want to use the original Hamiltonian H, rather than the transformed Hamiltonian, TH, in the $\star_\Lambda$ formulation.

To derive equation (5.32), we assumed that we initially had a time-independent Wigner function W, so that time-dependence arose through application of the transition operator $T_\Lambda = e^{\Lambda t \partial_p^2}$. Let us now consider W time-dependent initially, such that it solves the Moyal bracket. Application of $T_\Lambda = e^{\Lambda t \partial_p^2}$ on the Moyal bracket yields,

$$\begin{aligned} i\hbar T_\Lambda \frac{\partial W}{\partial t} &= [T_\Lambda H, T_\Lambda W]_{\star_\Lambda} \\ &= [H, T_\Lambda W]_{\star_M} + 2\Lambda t [p, \partial_p T_\Lambda W]_{\star_M}\,. \end{aligned} \tag{5.35}$$

Therefore

$$i\hbar \frac{\partial T_\Lambda W}{\partial t} = [H, T_\Lambda W]_{\star_M} - 2i\hbar\Lambda t \partial_q \partial_p T_\Lambda W - i\hbar\Lambda \partial_p^2 T_\Lambda W\,. \tag{5.36}$$

The term of $[H, T_\Lambda W]_{\star_M}$ is the original evolution, while the other two terms are due to the decoherence process.

5.4.3 Short Wavelength Limit

To conclude this Chapter, we will briefly consider the case that the wavelength of the incoming particles is much shorter than the coherence separation, $q - q'$. In this limit, equation (5.25) simplifies to,

$$\frac{\partial}{\partial t}\rho(q, q', t) = -\Gamma_{tot}\rho(q, q', t)\,, \tag{5.37}$$

where Γ_{tot} is the total scattering rate dependent on the momentum of the scattering particles, their speed, and their cross section. Therefore

$$\rho(q,q',t) = \rho(q,q',0)e^{-\Gamma_{tot}t}. \tag{5.38}$$

Using the Wigner transform, equations (5.37) and (5.38) become

$$\frac{\partial}{\partial t}\widetilde{W}(q,p,t) = -\Gamma_{tot}\widetilde{W}(q,p,t), \tag{5.39}$$

$$\implies \widetilde{W}(q,p,t) = \widetilde{W}(q,p,0)e^{-\Gamma_{tot}t}. \tag{5.40}$$

Similar to the long wavelength case, we seek a transition operator such that $\widetilde{W}(q,p,t) = TW(q,p,0)$. If $T = e^{-\Gamma_{tot}t}$, then

$$\frac{\partial}{\partial t}\Big(TW(q,p,0)\Big) = -\Gamma_{tot}\,TW(q,p,0) \tag{5.41}$$

showing that $\widetilde{W} = TW$. However, $T = e^{-\Gamma_{tot}t}$ is not a valid transition operator as it is not a differential operator. This demonstrates the possibility that not all decoherence phenomena may be described with the transition operator and star products.

In this Chapter, we have considered time-dependent augmented quantization by using both global and local transition operators. We have shown that it is possible to recover the classical equations of motion for the driven harmonic oscillator using a time-dependent local transition operator. Further, we have demonstrated that the results in [108, 115] can be written in terms of transition operators to describe a simplified version of decoherence.

Chapter 6

Conclusion

Quantum mechanics can be done in may ways, such as with operators or path integrals. In this thesis, we have focussed on doing quantum mechanics in phase space so that classical phase space functions can be used. As a result, operators are no longer required; the tradeoff is that it is necessary to introduce a binary non-commuting operation, known as the star product. It is straightforward to use phase space quantum mechanics to understand the physics of different quantizations and orderings; either the transition operator or the weight function can be used to give the distribution function and star product for a given quantization.

To conclude this thesis, we will first give a summary of the previous chapters and highlight the main results. Next, some possible extensions and applications of our work will be given. Then, we will consider the significance of our research in a broader context.

Summary

In Chapter 2, the operator quantization map was formally defined and it was shown that the map for Weyl quantization (which is the basis of the original version of phase space quantum mechanics) satisfies this definition. We then illustrated the main results of [13], which showed that different quantizations can give different physical results.

Next, we reviewed the fundamentals of phase space quantization by discussing the Wigner transform, Wigner functions, and the Moyal product. We showed that different quantizations are described with different distribution functions and star products. To relate the distribution functions and observables of different quantizations, the transition operator or the weight function could be used. Similarly, the transition operator can relate different star products.

In Chapter 3, the Husimi distribution was illustrated. It was shown that coarse-graining by $\hbar$ can be introduced using the transition operator. Hence, the Husimi distribution is a prototypical example of using the transition operator to incorporate additional physical effects in a quantum system.

To differentiate between quantizing a classical system and the introduction of additional physical features (such as coarse-graining or damping) during quantization, we used the term augmented quantization. Hence, when coarse-graining the Wigner function of the simple harmonic oscillator, we would say that the resultant Husimi distribution is an augmented quantization of the simple harmonic oscillator.

We then briefly considered a possible generalization of the Husimi function, such that the Wigner function was coarse-grained by a different parameter, as shown in equation (3.27). We determined its corresponding transition operator and star product in equations (3.30) and (3.31). This was a natural extension of the Husimi distribution and coarse-graining.

As an example of coarse-graining, smoothing in the classical limit of the Wigner function was analyzed. We illustrated that the coarse-grained Wigner function for $n \to \infty$ can yield a delta function-like phase space distribution.

In Chapter 4, the method of [33] to introduce damping within augmented quantization was first summarized. They proposed a method to map an undamped classical harmonic oscillator to a damped quantum harmonic oscillator. Using the classical equations of motion for the damped harmonic oscillator, [33] found the star product and transition operator to be equations (4.25) and (4.26), respectively.

We then showed that all transition operators of the form $T = T(\partial_q, \partial_p)$, including the one proposed by [33], yield the $\hbar \to 0$ limits to the star brackets:

$$\lim_{\hbar\to 0} \frac{[q,H]_{\star T}}{i\hbar} \to p = \{q,H\}\,,$$
$$\lim_{\hbar\to 0} \frac{[p,H]_{\star T}}{i\hbar} \to -\frac{\partial V}{\partial q} = \{p,H\}\,.$$

This demonstrated that transition operators $T = T(\partial_q, \partial_p)$ are not able to provide extra terms that would be present in the $\hbar \to 0$ limit, assuming the $\star_T$-bracket described the equation of motion. Hence, it was not possible to find a transition operator of the form $T = T(\partial_q, \partial_p)$, such that the $\hbar \to 0$ limit of the star bracket would give the equations of motion for damping.

Our ultimate goal was to introduce additional physics, such as damping, during quantization. We also required that, if the extra physics includes classical features, they must be present in the $\hbar \to 0$ limit. This motivated us to generalize the transition operator by giving it position and momentum dependence. With $T = T(q, p, \partial_q, \partial_p)$, the star product was also local:

$$\begin{aligned}\star_T =&\mathcal{I}(1,2)\left\{T\left(\frac{q_1+q_2}{2}, \frac{p_1+p_2}{2}, \partial_{q_1}+\partial_{q_2}, \partial_{p_1}+\partial_{p_2}\right)[\star_M(1,2)]\right.\\ &\left.\times T^{-1}\left(q_1, p_1, \partial_{q_1}, \partial_{p_1}\right) T^{-1}\left(q_2, p_2, \partial_{q_2}, \partial_{p_2}\right)\right\},\end{aligned}$$

as displayed in equation (4.82).

Using this generalized transition operator, it was possible to introduce (small) damping to the undamped harmonic oscillator with equation (4.52), if the equations of (4.57) and (4.58) were satisfied. We then found the resultant star product for small damping, given in equation (4.99). The effect of such a generalization of the transition operator and star product was to introduce new physical effects that were not present in the original system. Therefore, $T = T(q, p, \partial_q, \partial_p)$ and its associated quantities can describe augmented quantization.

Augmented quantization is not limited to coarse-graining or damping. Using the transition operator of equation (4.114) and star product of equation (4.115), we demonstrated that we could incorporate quantum gravity corrections from the generalized uncertainty principle into an augmented quantization. Considering the simple harmonic oscillator as an example, we illustrated that GUP-modified transition operator applied to the Wigner function resulted in perturbative corrections to both the overall structure of the distribution function and the momentum probability distribution.

In Chapter 5, time-dependence was introduced into the transition operator. With a local time-dependent transition operator, it was possible to describe the augmented quantization of a simple harmonic oscillator so that it was mapped to a driven harmonic oscillator. We used the transition operator of equation (5.12), which gave the distribution function in equation (5.15) and star product of equation (5.18).

Employing a simplified description of scattering decoherence, we showed that the resultant distribution function could be found with a transition operator, with the same results of [108, 115]. We presented the transition operator and star product in equations (5.31) and (5.33).

Tables 6.1 and 6.2 include all the quantizations and augmented quantizations discussed within this thesis. We also show how each maps the function $e^{i(\theta q+\tau p)}$ to a phase space (augmented) quantization.

Possible Extensions and Applications of Augmented Quantization

To determine the expectation value of an observable $A(q,p)$ in phase space quantum mechanics and Weyl-ordering, one uses [6]

$$\langle A\rangle = \int dq\,dp\,A(q,p)\star_M W(q,p) = \int dq\,dp\,A(q,p)W(q,p)\,.$$

For other ordering possibilities, the method of determining the expectation value is less clear. As an example, [31] presented the expectation value of an observable for a system

Table 6.1: Quantizations discussed in this thesis. These have previously been studied in great detail (see, for example, [5])

Quantization	Transition Operator	Operator Ordering: $e^{i(\theta q+\tau p)} \to$
Weyl	1	$e^{i(\theta q+\tau p)}$
Standard	$e^{i\hbar\partial_p\partial_q/2}$	$e^{i(\theta q+\tau p)}e^{-i\hbar\theta\tau/2}$
Antistandard	$e^{-i\hbar\partial_p\partial_q/2}$	$e^{i(\theta q+\tau p)}e^{i\hbar\theta\tau/2}$
Normal	$e^{-\hbar\partial_q^2/4m\omega-\hbar m\omega\partial_p^2/4}$	$e^{i(\theta q+\tau p)}e^{\hbar\theta^2/4m\omega+\hbar m\omega\tau^2/4}$
Antinormal	$e^{\hbar\partial_q^2/4m\omega+\hbar m\omega\partial_p^2/4}$	$e^{i(\theta q+\tau p)}e^{-\hbar\theta^2/4m\omega-\hbar m\omega\tau^2/4}$
Born-Jordan	$\operatorname{sinc}\left(\frac{1}{2}\hbar\partial_p\partial_q\right)$	$e^{i(\theta q+\tau p)}\operatorname{sinc}\left(\frac{1}{2}\hbar\theta\tau\right)$
Symmetric	$\cos\left(\frac{1}{2}\hbar\partial_p\partial_q\right)$	$e^{i(\theta q+\tau p)}\cos\left(\frac{1}{2}\hbar\theta\tau\right)$

Table 6.2: Augmented quantizations discussed in this thesis. The Husimi distribution is designed to coarse-grain the Wigner function [31]. Ref. [33] first proposed the damping augmented quantization discussed in Section 4.2.2. The remainder are original and are designed to incorporate specific physical effects during quantization. Note that a, b, c, and d are functions of the position and momentum.

Augmented Quantization	Transition Operator	Operator Ordering: $e^{i(\theta q+\tau p)} \to$
Husimi	$e^{\hbar\partial_q^2/4m\kappa+\hbar m\kappa\partial_p^2/4}$	$e^{i(\theta q+\tau p)}e^{-\hbar\theta^2/4m\kappa-\hbar m\kappa\tau^2/4}$
Generalized Husimi	$e^{\eta\partial_q^2/4m\kappa+\eta m\kappa\partial_p^2/4}$	$e^{i(\theta q+\tau p)}e^{-\eta\theta^2/4m\kappa-\eta m\kappa\tau^2/4}$
Damping up to $\mathcal{O}(\gamma)$ (Section 4.2.2)	$e^{-\frac{i\hbar m\gamma}{2}\partial_p^2}$	$e^{i(\theta q+\tau p)}e^{\frac{i\hbar m\gamma}{2}\tau^2}$
Damping up to $\mathcal{O}(\alpha^2)$ (Section 4.7)	$e^{\gamma\left(a\partial_q+b\partial_p+c\partial_q^2+d\partial_p^2\right)}$	$e^{i(\theta q+\tau p)}e^{\gamma\left(ia\theta+ib\tau-c\theta^2-d\tau^2\right)}$
GUP	$e^{-\frac{4\alpha^2}{3}p^3\partial_p+\frac{8\alpha^2}{3}p^4\partial_p^2}$	$e^{i(\theta q+\tau p)}e^{-\frac{4\alpha^2}{3}ip^3\tau-\frac{8\alpha^2}{3}p^4\tau^2}$
Driven	$e^{qF(t)\partial_p^2}$	$e^{i(\theta q+\tau p)}e^{-qF(t)\tau^2}$
Decoherence	$e^{\Lambda t\partial_p^2}$	$e^{i(\theta q+\tau p)}e^{-\Lambda t\tau^2}$

described by the Husimi distribution as

$$\langle A\rangle_H = \int dq\,dp\,T_H A \star_H T_H W\,,$$

where T_H is the Husimi transition operator and $\star_H$ is the Husimi star product. In contrast, [6] suggested that the expectation value should instead be

$$\langle A \rangle_H = \int dq\, dp\, A\, e^{-\hbar\partial_q^2/4m\kappa - \hbar m\kappa\partial_p^2/4}\, T_H W\,.$$

It is not clear if, for an arbitrary transition operator, $\langle A \rangle = \langle A \rangle_T$ when considering augmented quantization. Further, it is uncertain if

$$\langle A \rangle_T = \int dq\, dp\, TA \star_T TW$$

is correct when the transition operator is also a function of the position and momentum. This question should be studied as it will help determine if any of the augmented quantizations in this thesis suggest non-physical conclusions.

Another extension of our work is to study the theoretical ramifications of local transition operators beyond observables. Let us consider, for example, mapping the simple harmonic oscillator to a damped harmonic oscillator. As shown in [95], there are several methods to quantize a damped harmonic oscillator, but many of them yield undesirable physical implications, such as the violation of Heisenberg's uncertainty principle.

We focussed on studying the equations of motion for the damped augmented quantization of the simple harmonic oscillator. Further work is therefore required to determine if the damped local transition operator also results in non-physical implications. Similarly, the other augmented quantizations incorporating the generalized uncertainty principle, driving forces, and decoherence should be further investigated to determine if they should be modified or eliminated on non-physical grounds.

Quantum mechanics usually involves Hermitian operators to calculate physical results. However, quantum mechanics has been generalized so that Hermiticity is no longer the fundamental indication of whether a result is physical [100, 126–131]. There has also been interest in using the Moyal product and phase space quantum mechanics to describe non-

Hermitian quantum systems [65, 132, 133].

A possible application of the work in this thesis is to use complex transition operators as a possible means to transition between Hermitian quantum mechanics and non-Hermitian quantum mechanics. For instance, consider a non-Hermitian Hamiltonian $\tilde{H} = \frac{p^2}{2} + \tilde{V}(q)$, where $\tilde{V}(q)$ is complex. If $T = e^{[\tilde{V}(q) - V(q)]\partial_p^2}$ acts upon the real Hamiltonian $H = \frac{p^2}{2} + V(q)$, then $TH = \tilde{H}$.

In this thesis, we have predominantly considered phase space in terms of the position and momentum. Alternatively, phase space distributions can be written in terms of α and $\overline{\alpha}$, where [5]

$$\alpha = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega q + ip),$$

where ω has the same meaning as in Section 3.2. In this case, the measure of $dqdp$ is replaced by $d^2\alpha = d(\mathrm{Re}\,\alpha)\mathrm{d}(\mathrm{Im}\,\alpha)$, which is equal to $(1/2\hbar)dqdp$. Using α and $\overline{\alpha}$, it is possible to write the density operator in terms of coherent states $\langle\alpha|$, $|\alpha\rangle$ and the P-representation of quantum optics, such that [134–136]

$$\hat{\rho} = \int d^2\alpha P(\alpha, \overline{\alpha}, t) \left|\alpha\right\rangle \left\langle\alpha\right|,$$

where $P(\alpha, \overline{\alpha}, t)$ is the P-representation. As shown in [5], the P-representation is equivalent to the antinormal distribution in phase space quantum mechanics

In Section 5.2, we demonstrated a time-dependent transition operator that could make a system normal ordered at one time and antinormal ordered at a later time. Hence, such a time-dependent transition operator, or even a local transition operator, will introduce additional features into the P-representation. Therefore, when the P-representation is no longer valid, it may be possible to utilize a time-dependent transition operator or a local transition operator acting on the Wigner function to describe a larger range of quantum phenomena.

In Chapter 1, we identified that the techniques of phase space quantum mechanics have been used to investigated spin. To describe spin, it is necessary to analyze Wigner functions and star products on the surface of a sphere [137]. The difficulty arises in part due to the Poisson bracket in the Moyal product. In spherical coordinates, the Poisson bracket contains a $\sin\theta$ term that does not commute with the derivatives [138, 139].

A possible application of local transition operators is to directly encode the curvature of the space. As a result, a local transition operator may be able to transform a Wigner function on flat space to a distribution function on curved space. The star product would then also incorporate the curvature of space. If this is possible, such transition operators could be applicable to other curved surfaces beyond the sphere.

Significance of Results

In this thesis, we have explored augmented quantization and developed a local transition operator/star product. The work presented is one of a few proposals that we are aware of which consider the applications of position and momentum-dependent quantization.

Our original results deal with introducing a physical single feature into a local transition operator, namely damping, the generalized uncertainty principle, a driving force, and decoherence. Considering damping in particular, it is unknown if our proposed augmented quantization of the simple harmonic oscillator is a bona-fide quantization of the damped harmonic oscillator. To verify if it is, it would be necessary to compare our augmented quantization of the simple harmonic oscillator to experimental measurements of a quantized damped harmonic oscillator. Only experiments will be able to validate or reject this idea of using a local transition operator to describe some augmented quantizations.

We have considered only a small class of physical systems, though it should be possible to apply augmented quantization (or a modification thereof) to more systems, such as those discussed in [95]. Therefore, it is anticipated that augmented quantization as well as local transition operators and star products can open additional avenues of research into understanding quantization and fundamental questions in quantum mechanics.

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Appendix A

Derivation of Ordering Rules

In this Appendix, we will demonstrate the ordering rules of Weyl and of Born-Jordan quantization, based on the procedure and properties outlined in [7] and [10]. It will also be necessary to determine the weight function for standard ordering.

A.1 Weyl Ordering

Consider a classical function $f(q,p)$. We can define a quantization map $\mathcal{Q}_\Phi$, such that the quantization of $f(q,p)$ is

$$\begin{aligned}\mathcal{Q}_\Phi f(q,p) &= \frac{1}{(2\pi)^2}\int db\,da\,dq\,dp\,f(q,p)\Phi(a,b)\,e^{ib(\hat{p}-p)+ia(\hat{q}-q)}\,,\\ &= \frac{1}{(2\pi)^2}\int db\,da\,dq\,dp\,f(q,p)\Phi(a,b)\,e^{\frac{i\hbar}{2}ab}e^{ia(\hat{q}-q)}e^{ib(\hat{p}-p)}\,,\end{aligned}\tag{A.1}$$

where $\Phi(a,b)$ is the weight function of Section 2.10 and we have used the simplified Zassenhaus formula of $e^{A+B}=e^Ae^Be^{-[A,B]/2}$ if $[A,[A,B]]=[B,[A,B]]=0$.

With $\Phi(a,b)=1$, equation (A.1) reduces to the Weyl transform of equation (2.8). To derive the order of operators in Weyl quantization, we will relate it to standard ordering. This will simplify the resultant calculations.

In equation (A.1), we see that if $\Phi(a,b)=e^{-\frac{i\hbar}{2}ab}$, then the quantization map for standard ordering is found:

$$\mathcal{Q}_S f(q,p) = \frac{1}{(2\pi)^2}\int db\,da\,dq\,dp\,f(q,p)\,e^{ia(\hat{q}-q)}e^{ib(\hat{p}-p)}\,. \tag{A.2}$$

Therefore, the Weyl quantization map can be written in terms of the map for standard ordering, such that

$$\mathcal{Q}_W f(q,p) = \mathcal{Q}_S\left(e^{\frac{i\hbar}{2}ab}f(q,p)\right)\,. \tag{A.3}$$

Therefore,

$$\mathcal{Q}_W f(q,p) = \mathcal{Q}_S\left(e^{-\frac{i\hbar}{2}\partial_p\partial_q}f(q,p)\right)\,. \tag{A.4}$$

Hence, to determine the Weyl-ordered form of the operator counterpart of $f(q,p) = q^r p^s$, we only need apply a differential operator and standard-order the result.

With $f(q,p) = q^r p^s$, we have

$$e^{-\frac{1}{2}i\hbar\partial_p\partial_q} q^r p^s = \sum_{k=0}^{\min(r,s)} \left(-\frac{i\hbar}{2}\right)^k k! \binom{r}{k}\binom{s}{k} q^{r-k} p^{s-k}. \tag{A.5}$$

Equation (A.4) can be used so that the Weyl-ordered analogue of $q^r p^s$ is

$$\mathcal{Q}_W\left(q^r p^s\right) = \sum_{k=0}^{\min(r,s)} \left(-\frac{i\hbar}{2}\right)^k k! \binom{r}{k}\binom{s}{k} \hat{q}^{r-k} \hat{p}^{s-k}. \tag{A.6}$$

Rearranging operators, we find equation (2.2),

$$\mathcal{Q}_W\left(q^r p^s\right) = \frac{1}{2^s}\sum_{\ell=0}^{s}\binom{s}{\ell}\hat{p}^{s-\ell}\hat{q}^r\hat{p}^\ell = \frac{1}{2^r}\sum_{\ell=0}^{r}\binom{r}{\ell}\hat{q}^{r-\ell}\hat{p}^s\hat{q}^\ell, \tag{A.7}$$

which describes Weyl-ordered operators.

A.2 Born-Jordan ordering

For Born-Jordan ordering, we need a weight function Φ_{BJ}, such that the Born-Jordan quantization map is

$$\mathcal{Q}_{BJ} f(q,p) = \frac{1}{(2\pi)^2}\int db\,da\,dq\,dp\, f(q,p)\Phi_{BJ}(a,b)\, e^{ib(\hat{p}-p)+ia(\hat{q}-q)}, \tag{A.8}$$

This is related to the Weyl map by

$$\mathcal{Q}_{BJ}\Big(f(q,p)\Big) = \mathcal{Q}_W\Big(f(q,p)\Phi_{BJ}(a,b)\Big). \tag{A.9}$$

Let us consider a Fourier component,S $f(q,p) = e^{i(aq+bp)}$. Applying equation (A.9), we have

$$\Phi_{BJ}(a,b)e^{i(a\hat{q}+b\hat{p})} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(ia)^n(ib)^m}{n!m!}\mathcal{Q}_{BJ}(q^n p^m) \tag{A.10}$$

We will now show that the weight function that corresponds to Born-Jordan ordering is $\Phi_{BJ} = \operatorname{sinc}\left(\frac{\hbar}{2}ab\right)$. As

$$\mathcal{Q}_{BJ}(q^n p^m) = \frac{1}{n+1}\sum_{k=0}^{n}\hat{q}^{n-k}\hat{p}^m\hat{q}^k, \tag{A.11}$$

we have

$$\Phi(a,b)\hat{I} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(ia)^n (ib)^m}{n!m!} \frac{1}{n+1} \sum_{k=0}^{n} e^{-i(a\hat{q}+b\hat{p})} \hat{q}^{n-k} \hat{p}^m \hat{q}^k \,. \tag{A.12}$$

Summing over m and applying the simplified Zassenhaus formula to $e^{-i(a\hat{q}+b\hat{p})}$, we find

$$\Phi(a,b)\hat{I} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(ia)^n}{n!(n+1)} e^{\frac{i\hbar}{2}ab} e^{-ia\hat{q}} e^{-ib\hat{p}} \hat{q}^{n-k} e^{ib\hat{p}} \hat{q}^k \,. \tag{A.13}$$

We will now simplify $e^{-ib\hat{p}}\hat{q}^{n-k}e^{ib\hat{p}}\hat{q}^k$. Noting that

$$e^{-ib\hat{p}} \left|q\right\rangle = \left|q+\hbar b\right\rangle \tag{A.14}$$

we have

$$\hat{q}e^{-ib\hat{p}} \left|q\right\rangle = (q+\hbar b) \left|q+\hbar b\right\rangle \,. \tag{A.15}$$

We could write the left-hand side as

$$\hat{q}e^{-ib\hat{p}} \left|q\right\rangle = \left([\hat{q}, e^{-ib\hat{p}}] + e^{-ib\hat{p}}\hat{q}\right) \left|q\right\rangle \,. \tag{A.16}$$

Therefore,

$$[\hat{q}, e^{-ib\hat{p}}] = \hbar b\, e^{-ib\hat{p}} \,. \tag{A.17}$$

Expanding the commutator, we see that

$$\hat{q} - \hbar b\hat{I} = e^{-ib\hat{p}} \hat{q} e^{ib\hat{p}} \tag{A.18}$$

Therefore,

$$\left(\hat{q} - \hbar b\hat{I}\right)^k = e^{-ib\hat{p}} \hat{q}^k e^{ib\hat{p}} \,. \tag{A.19}$$

As a result, equation (A.13) becomes

$$\Phi_{BJ}(a,b)\hat{I} = e^{iab\hbar/2} \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(ia)^n}{n!(n+1)} e^{-ia\hat{q}} (\hat{q} - \hbar b\hat{I})^{n-k} \hat{q}^k \,. \tag{A.20}$$

Applying an inverse quantization map, we find

$$\Phi_{BJ}(a,b) = e^{iab\hbar/2} \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(ia)^n}{n!(n+1)} e^{-iaq} (q - \hbar b)^{n-k} q^k \,. \tag{A.21}$$

By Mathematica,

$$\sum_{k=0}^{n}(q-\hbar b)^{n-k}q^{k}=\frac{q^{n+1}-(q-\hbar b)^{n+1}}{\hbar b}, \tag{A.22}$$

$$\sum_{n=0}^{\infty}\frac{(ia)^{n}}{n!(n+1)}\frac{q^{n+1}-(q-\hbar b)^{n+1}}{\hbar b}=\frac{i\left(e^{ia(q-\hbar b)}-e^{iaq}\right)}{\hbar ab}, \tag{A.23}$$

we find that $\Phi_{BJ}(a,b)=\operatorname{sinc}\left(\frac{\hbar}{2}ab\right)$, as desired. The same result would have been found if we had used

$$\mathcal{Q}_{BJ}(q^{n}p^{m})=\frac{1}{m+1}\sum_{k=0}^{m}\hat{p}^{m-k}\hat{q}^{n}\hat{p}^{k} \tag{A.24}$$

instead.

Appendix B

The Weierstrass transform

In Sections 3.2, 3.3, and 5.4, we used transition operators of the form, $T = e^{b\partial_p^2}$, where $b \in \mathbb{R}$. In this Appendix, it will be shown that this transition operator acting on the Wigner function $W = W(q,p,0)$ is related to the heat equation, where $W(q,p,b) = TW$. We will then show that the $e^{b\partial_p^2}$ Gaussian smooths the Wigner function, and is thus related to the Weierstrass transform of mathematics and the Husimi distribution of physics.

Assuming $b > 0$, by differentiating TW with respect to b, we have

$$\frac{\partial}{\partial b} TW = \partial_p^2 TW \,. \tag{B.1}$$

This is the heat equation (sourceless diffusion equation). In equation (B.1), b acts as the time coordinate and p behaves as the spatial coordinate. We will now solve this equation to evaluate TW.

Taking the Fourier transform $\mathscr{F}$ of equation (B.1) with respect to the momentum and using the property [114],

$$\mathscr{F}\left\{\partial_p^2 W(q,p,b)\right\} = -\left(\frac{k}{\hbar}\right)^2 \mathscr{F}\left\{W(q,p,b)\right\}, \tag{B.2}$$

where k is the transform variable, then

$$\frac{\partial f(q,k,b)}{\partial b} = -\left(\frac{k}{\hbar}\right)^2 f(q,k,b)\,. \tag{B.3}$$

where we have let $f(q,p,b) := \mathscr{F}\left\{W(q,p,b)\right\}$. The solution to equation (B.3) is

$$f(q,k,b) = C(k) e^{-k^2 b/\hbar^2}\,, \tag{B.4}$$

and $f(q,p,0) = \mathscr{F}\left\{W(q,p,0)\right\} = C(k)$. Applying the inverse Fourier transform with respect to k, we have [114]

$$\begin{aligned} W(q,p,b) &= \frac{1}{\sqrt{2\pi\hbar}} \int dk C(k) e^{-k^2 b/\hbar^2} e^{ikp/\hbar}\,, \\ &= \frac{1}{2\pi\hbar} \int dk\, dp'\, W(q,p',0) e^{ik(p-p')/\hbar - k^2 b/\hbar^2}\,. \end{aligned} \tag{B.5}$$

Noting that [114],

$$\int dk\, e^{ik(p-p')/\hbar - k^2 b/\hbar} = \sqrt{\frac{\pi\hbar^2}{b}} e^{-\frac{(p-p')^2}{4b}}\,, \tag{B.6}$$

substitution into equation (B.5) yields

$$W(q,p,b) = \frac{1}{\sqrt{4\pi b}} \int dp'\, W(q,p',0) e^{-\frac{(p-p')^2}{4b}}\,. \tag{B.7}$$

This corresponds to the Gaussian smoothing of the momentum in the Wigner function. Therefore,

$$e^{b\partial_p^2} W(q,p,0) = \frac{1}{\sqrt{4\pi b}} \int dp'\, W(q,p',0) e^{-\frac{(p-p')^2}{4b}}\,. \tag{B.8}$$

for $b \geq 0$.

Mathematically, equation (B.8) is known the Weierstrass transform of the Wigner function. The properties of this transform are discussed in, for example, [140].

It is possible to relate the Weierstrass transform to the Husimi distribution of Section 3.2 as the transition operator for the Husimi distribution is of the form $T_H = e^{a\partial_q^2 + b\partial_p^2}$ with $a, b > 0$. Noting that T_H is can be separated into a product of momentum derivatives and position derivatives ($T_H = e^{\hbar\partial_q^2/4m\kappa} e^{\hbar m\kappa\partial_p^2/4}$), we can repeat the above steps for the position to find that the Husimi distribution is the Weierstrass transform of the Wigner function with respect to both the position and momentum. A similar argument can be made for the generalization of the Husimi distribution of Section 3.3.

Appendix C

Damping Transition Operator

C.1 Linear Damping

In Section 4.5, we presented coupled partial differential equations in (4.57) and (4.58) that would determine the form of the transition operator to convert a Weyl-quantized simple harmonic oscillator to one that included small damping. This system is underdetermined as there are four unknowns but only two equations. It is thus possible to have two of $a(q,p)$, $b(q,p)$, $c(q,p)$, $d(q,p)$ arbitrary and then solve equations (4.57) and (4.58) solely in terms of the remaining two functions. Setting two of the functions equal to zero, we will present the solution to equations (4.57) and (4.58) for each of the six cases.

With $a=b=0$ and $c=d=0$, we find that the resultant pair of differential equations are inconsistent and no solution exists. Now consider $a=c=0$. Then, the systems of differential equations reduce to

$$\frac{\partial d}{\partial q}+q\frac{\partial b}{\partial p}=-2p\,, \tag{C.1}$$

$$-p\frac{\partial b}{\partial p}-\frac{\partial d}{\partial p}=0\,, \tag{C.2}$$

yielding the solution which we determine with Maple (where we are doing indefinite integration),

$$b=g(q)+2\int dp\left(\arctan\left(\frac{q}{p}\right)-\frac{d}{du}f(u)|_{u=p^2+q^2}\right)\,, \tag{C.3}$$

$$d=-(p^2+q^2)\arctan\left(\frac{q}{p}\right)-qp+f(p^2+q^2)\,, \tag{C.4}$$

with $f(p^2+q^2)$ and $g(q)$ being arbitrary functions. For $a=d=0$, $b=c=0$, and $b=d=0$, the non-zero functions in each of those remaining cases are equations (C.3), (C.4), and

$$a=g(q)+2\int dp\left(\arctan\left(\frac{q}{p}\right)-\frac{d}{du}f(u)|_{u=p^2+q^2}\right)\,, \tag{C.5}$$

$$c=-(p^2+q^2)\arctan\left(\frac{q}{p}\right)-qp+f(p^2+q^2)\,. \tag{C.6}$$

In other words, the differential equations for the two non-zero parameters will have the same form as equations (C.1) and (C.2), whose solutions will be two of equations (C.3)-(C.6). To determine the non-zero functions and form of f and g, it may be necessary to consider additional constraints on the system.

C.2 Non-linear Damping

We will now present the solutions to equations (4.64) and (4.65) for quadratic damping. For this case, $f(p) = p^2$.

With $a = b = 0$ and $c = d = 0$, the system of differential equations is inconsistent, so consider $a = c = 0$. We have

$$\frac{\partial d}{\partial q} + q\frac{\partial b}{\partial p} = -2p^2\,, \tag{C.7}$$

$$-p\frac{\partial b}{\partial p} - \frac{\partial d}{\partial p} = 0\,. \tag{C.8}$$

The solutions are (by Maple)

$$b = 4pq + F_1(q) - 2\int dp\frac{d}{du}F_2(u)|_{u=p^2+q^2}\,, \tag{C.9}$$

$$d = -\frac{4}{3}q^3 - 2p^2q + F_1(u)|_{u=p^2+q^2}\,, \tag{C.10}$$

where F_1 and F_2 are arbitrary functions.

For the remaining cases, the differential equations for the non-zero parameters have the same form as equations (C.7) and (C.8). Thus, when b or d are non-zero, they will still be equal to equations (C.9) and (C.10). When a or c are non-zero, we have

$$a = 4pq + F_1(q) - 2\int dp\frac{d}{du}F_2(u)|_{u=p^2+q^2}\,, \tag{C.11}$$

$$c = -\frac{4}{3}q^3 - 2p^2q + F_1(u)|_{u=p^2+q^2}\,. \tag{C.12}$$