DESCRIBING MATHEMATICS WITHOUT REVISION: WITTGENSTEIN'S RADICAL CONSTRUCTIVISM

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ABSTRACT

In this thesis, I present a critical exposition of Wittgenstein's philosophy of mathematics, and attempt to show that Wittgenstein's philosophy of mathematics constitutes a viable alternative to the Standard View of mathematics. To this end, I show that there are two important interpretations that support the Standard View: a) Platonism; and b) Modalism. I argue that neither Platonism nor Modalism can provide a satisfactory account of mathematics. In explicating Wittgenstein's Finitistic Constructivist interpretation of mathematics, I endeavour to emphasize the main reasons why Wittgenstein's descriptive account of mathematics is better that either variant of the Standard View. Wittgenstein's formalistic view better captures what mathematicians do, and it better evaluates and rejects some verbal interpretations that mathematicians attach to their work.

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LIST OF ABREVIATIONS

FLT	Fermat's Last Theorem
GC	Goldbach's Conjecture
PA	Peano Arithmetic
PG	*Philosophical Grammar*
PI	*Philosophical Investigations*
PM	*Principia Mathematica*
PP	*Philosophy of Psychology – A Fragment*
PR	*Philosophical Remarks*
LFM	*Wittgenstein's Lectures on the Foundations of Mathematics*
RFM	*Remarks on the Foundations of Mathematics*
WVC	*Wittgenstein and the Vienna Circle*
ZFC	Zermelo-Fraenkel Set Theoretical, with the Axiom of Choice

INTRODUCTION

In this thesis I shall endeavour to demonstrate 1) that Ludwig Wittgenstein presents an interesting and viable constructivist alternative to the *standard view* of mathematics; 2) that Wittgenstein's Constructivism results from his philosophical description of what mathematics is and what mathematicians do; and 3) that the tension between Wittgenstein's avowed descriptivism and his claims, criticisms and arguments, are unavoidable consequences of untangling conceptual knots and confusions *about* and *within* mathematics.

To these ends, I first attempt to discuss some important views on the philosophy of mathematics. This initial evaluation will highlight some of the core ideas that are shared by mathematicians and philosophers about what mathematics is. I use the term "the Standard View" of mathematics to refer to the core elements of this general agreement, not just in the philosophy of mathematics, but also in the writings of mathematicians.

I will show that there are two main variants of the standard view in the philosophy of mathematics: a) Platonism; and b) Modalism. In short, both defend the view that mathematics is *objective*, and that mathematical truth is *language and mind independent*. However, the paths that enable (a) and (b) to arrive at this conclusion are very different. The former, Platonism, argues that mathematics is about abstract objects (e.g., the real numbers) that dwell in a different reality. According to Platonism, mathematical propositions are true because they *correspond* to facts in this reality. According to the second view, Modalism, mathematics is *about relations in a system*. In other words, Modalism attempts to avoid the problems related to abstract objects altogether, by claiming that mathematics is not about objects, but about *possible relations in a possible structure*. This view also relies on the fact that mathematics is essential to physics and the success and progress of science (*i.e.* the Indispensability Argument). According to Putnam (1979), Modalism can give us a realistic interpretation of mathematical truth by showing that the relations that it describes are true in contingent propositions.

The claim that we must choose between Platonism and Modalism I called "Quine's Dilemma", following (Rodych, 2005, p. 79). This dilemma, I shall argue, must be rejected.

My criticism of Platonism focuses on two well-known problems: a) the ontological problem; and b) the epistemological problem. The first questions the idea that abstract objects would be able to causally affect or causally interact with concrete *physical* objects; while the second shows the problems related to our ability to acquire knowledge of non-physical mathematical facts. In short, the first asks the question "How can abstract things, that are acausal, atemporal and aspacial, interact with *physical things*?", and the second asks "How can we, physical beings, acquire knowledge of non-physical facts in a different reality (i.e., a non-physical reality)?". Although Platonism cannot be refuted, for we cannot show that it is logically impossible for such a non-physical realm to exist, I attempt to show that its main claims generate more problems than answers.

In relation to the second horn of the dilemma, I argue that the problems related to Platonism also apply to Modalism, despite its attempt to avoid being committed to abstract objects. I argue that Modalism does not avoid being committed to abstract objects (and the problems that are associated with them), it merely hides that fact. The idea of mathematics being a language about what is possible in a possible structure (system) does not provide a clear answer to what those structures are. For if we assume that a mathematical proof is provable two days before it was actually done (*i.e.* if we assume that ‘derivability’ is the same as ‘provability’), then the *fact* that this proof was provable before it was physically executed must have existed *before* the proof was constructed. Thus, Modalism is committed to the existence of proofs and theorems as possible proofs and the terminal propositions of possible proofs. However, this sense of the words ‘exists as a possibility’ leaves unclear *where those proofs exist,* or what *‘exists’ means*. It leaves them as things that dwell in some sort of limbo: they ‘exist’ as a *shadow of reality*.

In the second part of this thesis, I will present Wittgenstein’s Constructivist philosophy of mathematics. I will show that, for Wittgenstein, mathematical propositions are not genuine propositions at all. In other words, mathematical “propositions” are not true or false, for they do not refer to anything (*i.e.* they are not about anything). Mathematics has only *intrasystemic* meaning, that is, a mathematical proposition has meaning only within a system/calculus. In different terms, the *sense* (*i.e. the meaning*) of a mathematical proposition *is* the position that it occupies in the system, including the syntactical connections that it has *in the calculus*. A mathematical proposition does not show us anything *about reality* (or about a different reality), it only shows *how we can use* that concatenations of symbols in that particular calculus.

One of the problems for the standard view is that it focuses on the image that mathematicians have about their work, not what they actually do. In other words, philosophers should focus on what mathematicians do, not what they say about (and to some extent within) their work. This constitutes Wittgenstein *descriptivism*: namely, that in philosophy, we should only describe mathematics, and never prescribe how mathematics *should* be done.

Wittgenstein says that he will only *describe* mathematics, and that his description "leaves mathematics as it is" (PI, §124). His main goal is to describe a) what mathematics is (e.g., symbols, systems, rules, etc.) and b) what mathematicians actually do.

Wittgenstein's descriptivism seems to clash immediately with what he says and does, for Wittgenstein's description of mathematics often seems to *criticize* mathematics (e.g., transfinite set theory; the assumption that mathematics contains infinite extensions; the claim that the Cantorian Diagonal Proof proves that there are *more* real numbers than natural numbers). These apparent criticisms seem to be contrary to what Wittgenstein claims the job of a philosopher to be, for he repeatedly says that we should only describe in philosophy, not argue or theorize. More than this, Wittgenstein's writings often seem suggest, or propose, or prescribe *changes* in mathematics (or at least set theory).

I believe, however, that this tension can be resolved if we better understand that a comprehensive description of mathematics is unavoidably analytic, interpretative and argumentative. Just as there is no naked, uninterpreted (empirical) observation, there is similarly no naked, uninterpreted description of a complex enterprise such as mathematics. As Wittgenstein says, the job of untangling conceptual and linguistic knots and confusions, is "just as complicated as the knots" (Wittgenstein, PR, §2). This untangling is intrinsic to philosophy, and it necessarily involves arguments for what is the case and arguments against what is not the case. For example, Wittgenstein argues that Cantor has not proved that there are more real numbers than natural numbers. On his view, what Cantor has really proved is that we cannot enumerate the reals in a way that includes *all* of what mathematicians (i.e., the Standard View) regard as real numbers.[1]

This is a core example of Wittgenstein seeming to be very critical of mathematics, mathematicians, and a core claim within pure mathematics. His argument says that *because* we cannot enumerate *all* the real numbers, there is no 'system' or *set* of all of the real numbers. In untangling this conceptual-linguistic confusion, Wittgenstein *shows* why what we say about Cantor's proof does not correspond to what the *proof proves*. If we attempt to describe only the proof, there is nothing in the proof that is a greater infinity.[2]

In the final chapter of my thesis, I attempt briefly investigate different interpretations of Wittgenstein's philosophy of mathematics. I will discuss Floyd's and Fogelin's different attempts to interpret Wittgenstein as a pure descriptivist, without any opinions on, or views about, mathematics.

In relation to this question, I believe that Wittgenstein presents a rich, interesting and defensible descriptive account of mathematics. I hope that this thesis will show that Wittgenstein is doing more than just juggling different interpretations and showing that they are all equivalent or equally arbitrary.[3]

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[1] Wittgenstein also tries to show that if we view Cantor's proof as only a proof about the recursive reals, then Cantor has proved that we cannot enumerate the recursive reals. This is going to be better explained in chapter 6.

[2] Within the scope of my thesis, it is only possible briefly to note and highlight the points at which Wittgenstein seems to criticize and revise mathematics, when he is only describing the reality of what is mathematics.

[3] See (LFM, pp. 13-14) quoted in chapter 2.

CHAPTER 1: Language and Reality

1.1. Ordinary Language, Contingent Propositions and Reality

Before discussing mathematics and its complexity, it will be helpful to examine how ordinary language works. This will enable us to work with common, basic and clear examples, which can facilitate our analysis of more complex examples in mathematics. Furthermore, by starting with ordinary language we can first clarify the use of mathematical propositions in ordinary language, for the most basic use of mathematical language is found in contingent statements, *e.g.* “Two apples plus two apples equal four apples”.

This starting point with ordinary language will not be, and does not need to be, very complex. I will assume we have a general understanding of how language works. In addition to this, I will restrict this first analysis of ordinary language to very basic linguistic moves. I do not intend to clarify how language works as a whole. For the purpose of this thesis, I will focus only on sentences that serve to describe state of affairs.

I will restrict this analysis to how we use propositions. By propositions, I mean declarative sentences in the past or present tense. This is not the same as affirming that ordinary language is, or is essentially, composed of simple and clear propositions in present or past tense. It is well known that there are different contexts that show the limitations of reducing descriptive language to simple propositions. However, the discussion of those kinds of contexts are well beyond the scope of my thesis. What I will focus on, in these first sections, is the relation between *reality and language*.

At the outset, it appears that a contingent proposition depicts or pictures or asserts how things are in reality. The proposition: “There are exactly two people at the University of Lethbridge right now” claims or asserts how things stand in reality.

1.1.1. The Ordinary Notion of Meaning in Ordinary Language

The first aspect that I want to evaluate, in ordinary language, is the *meaning of words and sentences*. By stating something, constructing a sentence, we use language for a particular purpose. The meaning of a declarative sentence, what it *says*, is related to what the sentence is about.

For example, the proposition ‘Philip is eating an apple in his house’ tells me that: a) there is a person who is referred to by the sign ‘Philip’[4]; b) he is behaving in a particular way (*i.e.* consuming something); c) this thing is an object, commonly referred to as an “apple”; and d) he is doing this in the comfort of his house.

This analysis gives us a picture. In other words, the elements of the proposition, working together, say how things stand in reality. In this sense, we can say that this piece of language refers to, or says something about, something outside of itself: the proposition is as a picture of a particular state of affairs.

In relation to this, the meaning of this proposition is (some kind of) a picture. The words of the proposition compose part of this picture. In other words, the proposition

[4] As I said in the previous section, I will not take into consideration unclear or blurred contexts. It is possible that Philip is not a person, but a dog; or that it is a fictional character of a book, or the product of the imagination of a child. For now, if not stated otherwise, I will assume that there is no ambiguous interpretation of a proposition, just by assuming the clearest and simplest context or scenario.

speaks about the things that some of its words refer to, *e.g.* ‘walking’ refers to an action of walking, ‘home’ refers to a place to which someone has an emotional attachment. How the words are related in the proposition shows how things should stand in reality.

1.1.2. Reference, Truth, and Use in Ordinary Language

The proposition gives us a picture of how things stand in reality. However, that does not mean that they necessarily do. For example, the proposition “A Boeing 747 landed in Central Park on the 6th of October 2020” purports to represent something about reality. Nonetheless, this proposition fails to represent reality in a completely accurate way. It is a picture that does not correctly describe a piece of reality. In other words, the proposition is false because it is not the case that there was a Boeing 747 in Central Park that day.

This gives us a simple idea[5] of what a true proposition is and what a false proposition is: the former gives us a picture that *corresponds* to reality, while the latter fails to correspond to reality. As I stated, a proposition has an aboutness or referential relation to reality. A proposition purports to how certain objects are related to each other in a physical state of affairs. If the objects are not configured in the same way as the picture showed us, the proposition is false.

This also shows us how we use a proposition. We use it to assert or claim that something is or is not the case. We use a proposition to say something about reality. We use it to describe reality, to say that things are configured in a particular way. For example, when I say that “There are nine people in the next room”, I am purporting to describe

[5] I am referring to this as “a simple idea” for it gives us an interesting view on propositions in a narrow context.

reality. If one were to go to the room to which I refer, he would see that there are nine people. If there are not nine people, that means that my description or assertion is incorrect, *i.e*. that proposition is false.

An important aspect revealed by this example is that a proposition gives us a picture of *existent things*. In other words, we talk *about* objects that exist. My example above is only true if *there are* nine people in the room, and if there is a room. They must exist there, in that particular room, in order for my statement to be true. This idea is not as obvious as it seems, for some contexts reveal an ambiguity in the concept of existence. For example, if someone says that "A unicorn is a horse that has a horn on his head", would we say that that proposition is true? Such a thing as a unicorn does not exist, therefore in agreement with the criterion raised above, this proposition is false.

We want to say that that proposition is true, despite the fact that there is nothing to which it refers. I ignore this kind of problem by considering only propositions that are true because they refer to something in reality. In doing so, I want to avoid any confusion that may occur by using the words 'exists' or 'refers' in alternative ways.

To similarly keep matters simple and clear, I will restrict my analysis to contingent propositions in the present and past tense. A future-tensed contingent proposition does not have a correspondence relation to reality. It tells us how things *are going to be*. In other words, a future-tensed contingent proposition does not yet correspond or fail to correspond; it cannot correspond because it is not about an existent or previously existent state of affairs. The idea of correspondence in this context is blurred because future-tensed contingent propositions are not true, and they are not false. For that reason, we cannot determine whether a future tensed proposition is true or false, for such propositions do not have a truth-value.

In summary, I am restricting the concept of truth to propositions that refer to physical things.[6] As I suggested, fictional characters could lead to an opaque context of reference, which would result in different notions of the term ‘reference’. In relation to the goal of this initial explanation, I believe that this restriction is not arbitrary. It will enable us to work with a more precise concept of reference.

By understanding this simple notion of propositions, we can now turn to a more abstract type of proposition.

1.2. Mathematical Meaning and Truth

Given the concept of ‘proposition’ presented in the last section, we can now turn to mathematical propositions. Mathematics is a very important subject. Its application to contingent propositions is indispensable (*e.g.* to count apples, to calculate the velocity of a vehicle, to construct bridges); we would not be able to describe reality the way that we do without mathematics.

At first blush, it seems obvious that mathematical propositions mean something, for they seem to tell me something. Furthermore, most of us also want to say that “2 + 2 = 4” is *true*, for that we learned long ago how to do this calculation, and we know “by heart” that this proposition is true (and not false). Indeed, we think of mathematical propositions

[6] By this I am not saying that only terms that refer are meaningful. My point is to restrict the concept of truth to propositions composed of that kind of terms.

in a stronger sense, not just as correct or incorrect, or as true or false, but as *necessarily* true or *necessarily* false.

If we pursue this matter, as philosophers usually do, we may further ask, “What makes that statement true?”. In other words, what makes that “2 + 2 = 4” true, and “2 + 2 = 7” false? I argued in the last section that what makes a contingent proposition true is its connection to reality. In other words, we say that the proposition “Mark Hamill played the character Lucas Skywalker in Star Wars Episode 4” is true, for this proposition corresponds to a state of affairs. Can we say the same thing about mathematical propositions? Does a true mathematical proposition correspond, and a false mathematical proposition fail to correspond, to something in reality?

Our instinctive answer is: Yes! Of course, it does. We can put on a table two apples and add two more apples, and this will give us four apples. This gives us an example of a true mathematical proposition, one that is true by corresponding to those objects in front of us.

However, that is not what the mathematical proposition actually says. It says that “2 + 2 = 4”; it does not say that two apples plus two apples give us four apples. The point here is that mathematical terms do not refer to particular physical things. We could have imagined anything, not just apples or oranges, but also aliens, stars, and unicorns. When we apply numerals and true mathematical equations to reality, the actual specific empirical content of the resultant contingent proposition is irrelevant.

As we know, natural numbers (positive integers) are just one type of number. Despite the use of numerals in contingent propositions, one will be inclined to say that all mathematical statements are true in the same way. They all are true by virtue of a correspondence to something. It will not take us long to realize that more complicated numbers will not give us a straightforward answer to that question. For example, we can still say that two quarters of a pizza plus two quarters of that same pizza, give us a whole pizza (*i.e.* 2/4 + 2/4 = 4/4 or 1). However, if we apply the same interpretation to a number such as π, then what are we saying that it refers to? Or, to what existent thing does $\sqrt{2}$ refer? Do we have a clear object in mind when we execute operations with such terms, or when we say that the area of a circle is $\pi \times r^2$?

Despite our inclination, or impulse, to say that a mathematical proposition is true, or that it tells us how things stand in reality, it seems that we do not know how to clearly show this. It seems that we are missing an important aspect of mathematical propositions, *i.e.*, they are *propositions about abstract things*.

On the standard interpretation, mathematical propositions do not refer to physical objects or concrete situations. That seems to be the principal reason that we differentiate mathematical propositions from contingent propositions. A pure mathematical proposition does not have any physical content, nor is it restricted to present or past tense. We say that "2 + 2 = 4" is true now, that it was true yesterday, and that it will always be true. On the other hand, a contingent proposition makes claims about past or present situations. It pictures how things stand in reality right now, or how things stood in relation to each other years or minutes earlier.

Despite apparent differences, we are still inclined to say that a contingent proposition and a mathematical proposition function in the same way. We want to hold that mathematics refers to reality in a similar way to the way that contingent propositions refer to facts; and those facts are what make a mathematical proposition and a contingent proposition true. It seems that it does not matter what type of mathematical language we use; mathematical propositions all refer to the same kind of mathematical fact. We can use a base-ten number system and construct '2 + 2 = 4', or binary notation and have '10 + 10 = 100', etc.; they are true because they all refer to the very same fact.[7] In other words, we can use different notations and/or different bases and have different mathematical propositions all be true by virtue of the fact that they all describe *one and the same mathematical fact.*

This comparison is fundamental to understanding many arguments and positions in the philosophy of mathematics. Most of the arguments that I will examine argue that mathematics is a language in the same way as our ordinary language. An initial take on this analogy constitutes the subject of the next subsection.

1.3. Comparing Mathematical Propositions with Contingent Propositions

A contingent proposition purports to picture or represent objects and their relations. If it is true, it corresponds to an *existent* state of affairs, and if it is false it does not correspond to an *existent* state of affairs. This simple conception of truth by correspondence suggests that perhaps mathematical propositions are very similar.

In Section 1.1.1, we saw that the truth of 'Philip is eating an apple in his house' requires that some of the words of this contingent proposition refer to objects or other

[7] I borrowed this example from a question that Dr. Peacock asked me during a presentation that I did.

existent things or events: “Philip” refers to Philip, “apple” refers to a piece of fruit, “eating” refers to an action of chewing and swallowing (typical an edible) thing, and “in his house” locates Philip’s eating an apple in a particular physical space. The reference of referring words and terms makes it possible for a contingent proposition to make an assertion, or it makes it possible for us to *use* a contingent proposition to make an assertion.

This kind of reference seems to be operative in mathematical propositions also. In the same way that “Philip” refers to a person in my example above, mathematical signs and terms also seem to refer. For example, we are inclined to think that the Roman numeral ‘II,’ the Portuguese term ‘dois’, and the numeral ‘2’ all refer to one and the same thing.[8] We were taught in school that they all refer to the number Two. In other words, the signs refer to something, and different signs may refer to one and the same thing.

When we think about mathematical truth, and about the truth, e.g., of “13 + 12 = 25”, it is very natural to think that this mathematical proposition is *true* because “13 + 12” and “25” refer to the same thing, namely Twenty-Five. This proposition is true, we think, because 13 and 12 and 25 are related to each other in the way described by “13 + 12 = 25”. Furthermore, just as ‘Philip is eating an apple in his house’ is false if Philip is eating a pear, of if Philip is eating an apple at work and not in his house, etc., so too “13 + 12 = 27” is false because “13 + 12” and “27” do *not* refer to the same thing (*i.e.* to the same *number*).

For these reasons, it seems natural to think that a true mathematical proposition correctly describes the relations between mathematical objects, just as a true contingent proposition correctly describes the relations between physical objects. This prompts the

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[8] Just as, in Frege’s famous example, ‘Hesperus’, ‘Phosphorus’ and ‘Venus’ all refer to one and the same planet in our solar system.

conclusion that mathematical truth is the same as contingent truth. Since contingent truth is truth-by-correspondence, mathematical truth must be truth-by-correspondence.

This uniform conception of truth also seems correct when we see, again and again, the truth of propositions such as "13 apples plus 12 apples makes 25 apples". We use the truth of arithmetic propositions to make true contingent propositions, which, again, seems to suggest that mathematical truth and contingent truth are both principally *referential matters*. Just as the truth of a contingent proposition is dependent upon something outside the proposition itself, so too the truth of a mathematical proposition also seems to be dependent upon something outside itself. The conception of the uniformity of contingent and mathematical truth seems corroborated by the fact that scientific propositions (including scientific theories) reaches outside the formalism of physics or chemistry and the truth of a mathematical proposition seems to similarly reach outside a formal system consisting of signs (or symbols) and rules.

However, the idea that mathematical signs and terms refer to mathematical things seems to generate problems. For example, we want to state that mathematical propositions are true, but on the referential account of mathematics just sketched, that implies that there is an object Two and an object Four, and that relations between these objects make "2 + 2 = 4" true. The problem here is that, when I say that "This chair is blue", we have a clear idea in our minds what a chair is, and what blue is. But the same clarity does not come nearly as naturally, or as clearly, for Two or Four.

As I said above, we might think of things that count as two, such as two bananas, two houses, or two people. However, that is not what a mathematical proposition says. A mathematical proposition such as “2 + 2 = 4” does not make reference, directly or indirectly, to any *physical* thing.

In addition to this, unlike a contingent or empirical proposition, a mathematical proposition does not require that we check reality to ascertain its truth-value. For example, if I state, “It is raining right here right now”, an immediate response to my statement is to look outside the window and see if water is falling from the skies. If it is not the case that it is raining, then my statement is false.[9] But the same does not happen for mathematical propositions. We don’t determine the truth or falsity of “987 × 789 = 778,943” by looking outside a window.

If someone says that “5 × 5 = 27”, we will quickly say that that is false, without having to make a reference to something in reality. We just know that that is false, we do not need to say or show anything else. If someone says that “987 × 789 = 778,943”, we will probably need to calculate using pen and paper, but it won’t take us long to similarly say that this mathematical proposition is also false—and if someone insists that we need to do an experiment to be sure, we will probably laugh. We may recalculate the product of 987 and 789, but we won’t think of looking out the window or going to the lab.

We can illustrate our answer with a physical example, like using lines and strokes (*i.e.* we can put five strokes in five lines, giving twenty-five total strokes)[10]. However, this is merely a mechanism to clarify our answer. We do not need empirical evidence to show

[9] In this paragraph, I am using the idea of statement and proposition interchangeable. I believe that this does not generates any problems, for the simple idea of this paragraph is just to give clear examples to this analogy (mathematics and ordinary language).
[10] Despite the simplicity of this example, Fogelin (2009, pp. 98-99) shows how such concrete examples can fail in providing the explanation that we intended.

that "$13 \times 17 = 221$". In fact, many people will insist that for memorized equations, such as "$7 \times 8 = 56$," and even for relatively simple propositions such as "$13 \times 17 = 221$", we can confidently determine the truth-value of such propositions in our minds. Whether we do a calculation in our minds, or more typically on paper with a pen, we do something *with signs in accordance with rules,* and this determines whether the arithmetic proposition is true or false.

Thus, the answer to the question "what is a number?" seems extremely important, if we want to pursue the analogy between contingent proposition reference and mathematical proposition reference. But before I evaluate some theories in the philosophy of mathematics that attempt to answer that question, there is another aspect of a proposition that I want to investigate.

1.4. Epistemology vs. Ontology: Truth and Falsity vs. Knowledge and Ignorance

So far, I have talked about how we use a proposition to describe reality and to assert that a particular state of affairs exists or obtains in reality. Before moving forward, I will first state a distinction between the epistemological aspect of a proposition, and the ontological aspect.

In reality, a state of affairs exists, whether we know it or are completely ignorant of it. The existence of something in reality is not affected by our knowledge or ignorance of it. For example, Niagara Falls existed before Father Louis Hennepin discovered it. It would be strange, for most physical facts, to assume that the existence of something in reality is dependent upon our knowledge of it. Most, if not all, physical states of affairs (and most physical objects) do not exist only if I am aware of their existence—their existence is not dependent upon human awareness or knowledge.[11]

This gives us a clear idea of the difference between the truth-value of a proposition and our ability to verify it. The former is the objective fact that makes that the proposition true or false; while the latter is our ability to *know* whether it is it is true or false. For example, we do not know (so far) whether there is intelligent life beyond Earth. Nonetheless, the proposition "There is intelligent life beyond Earth" is true or false right now. More precisely, it *must* be true or false, for it makes clear assertion, in the present tense, about the existence of something in the physical universe. The proposition is true if intelligent life exists beyond Earth, and it is false if intelligent life exists only here on Earth. Our knowledge or ignorance of its truth-value has no relevance to, or bearing on, the truth or falsity of this proposition.

If we push forward the analogy that I discussed last section, we would be inclined to say the same of mathematical propositions. That is, the seeming referentiality of mathematical terms and signs seems to commit us to the existence of mathematical facts which are independent of our knowledge of them. For example, in the same way that the contingent proposition "There is intelligent life beyond Earth right now" is determinately true or determinately false right now, even though we do not know which it is; the mathematical proposition "Every even number greater than 2 is the sum of two primes" (Goldbach's Conjecture) is true or false, whether or not we happen to know its actual truth-

[11] I am not saying that some philosophers have not tried. Famously, the Idealism program argued that reality was dependent upon our experience of it. Since it appears that no serious discussion is being developed under that program, I will not investigate this theory.

value.[12] The work of a mathematician consists in discovering mathematical facts, in the same way that Father Hennepin discovered Niagara Falls.

The picture that is here emerging is that every mathematical proposition that we have so far considered or constructed *is true or false*, despite the fact that we may or may not have a proof or refutation of each one. This is shown by the fact that some open mathematical problems, which were centuries old, have been solved. The most famous example is Fermat's Last Theorem (henceforth, FLT). For more than 300 years this problem did not have a proof, but in 1995 Andrew Wiles *found* a proof of it. If, however, Wiles discovered a proof of FLT, that proof must have existed prior to Wiles' discover of it.. Furthermore, and this gets to the heart of the matter, given the Wiles discovered a truth, that truth reflects a pre-existent fact, and when Wiles discovered the truth, he discovered a mathematical fact that existed long before his discovery. Mathematical discovery itself seems to prove the existence of mathematical facts.

For now, this notion of mathematical facts is vague. We might add that a mathematical fact is composed of abstract mathematical objects, but such an addition only postpones the problem, for it merely transfers our focus from facts to objects.

Nonetheless, it seems that this notion constitutes our first answer to such problems. We tend to believe that some propositions of mathematics are true or false, despite our lack of knowledge of them. In other words, it seems that there is a general agreement, not just for philosophers or mathematicians, that mathematics and mathematical propositions are like science and scientific proposition: both seem true or false *in relation to* independent

[12] Goldbach's Conjecture is a perfect example for the mathematical side. It makes a clear assertion about *all* even numbers greater than 2, and so far, we do not have a proof of it.

objects and facts. Be that as it may, we still do not have a clear idea of what this analogy implies. In particular, we still do not know what a number is, what constitutes a mathematical fact, and what mathematical meaning is.

In the following sections, I will evaluate some of the answers to these problems, according to different theories in the philosophy of mathematics. I will explain what the standard view of mathematics, and I will argue that a superficial answer (*i.e.* the one that I outlined in this section), as well as programs that are based on the analogy between contingent and mathematical propositions, are not satisfactory.

CHAPTER 2: Wittgenstein's Philosophical Descriptivism

2.1. Wittgenstein's Philosophical Project: Describing Without Intervening

Before I evaluate some answers to the questions that I posed last section, I will present Wittgenstein's conception philosophy and how this conception affects his philosophy of mathematics. This will give us a better understanding of Wittgenstein's philosophical position, which will help us evaluate his description of mathematics in later chapters.

Wittgenstein first radical idea is that we should not argue in philosophy, but only describe how things are.[13] As he says "[p]hilosophy must not interfere in any way with the actual use of language, so it can in the end only describe it" (PI, §124; WVC, p. 121). A successful argument would imply that a particular proposition or view is correct. The structure of a sound argument consists of premises that establish the truth of a conclusion. The conclusion, if true, pictures how things stand in reality. In other words, by giving an argument, philosophers arguing for a particular proposition, theory or position, or they are defending a particular view or interpretation by arguing against a position or argument.

For example, Wittgenstein criticizes how some programs in philosophy (*e.g.* Russell's Logicism, as well as Wittgenstein's own claims in the *Tractatus Logico-Philosophicus*) argue that a particular way of analyzing some concepts and sentences *constitutes* the essential aspect of our language (*e.g.* on Russell's account, by analyzing the

[13] Floyd (1991, p. 144) has defended an extreme form of this view, as she argued "Wittgenstein is out to undercut the notion that *particular claims must be made from within an overarching general philosophical conception or systematic logical structure*". There is no doubt about this aspect in Wittgenstein's interpretation of philosophy, that is, philosophy should only describe. However, to how far this idea stretches in Wittgenstein's work is not so clear. I will evaluate this kind of therapeutic view in later sections.

logical structure of a proposition we reveal the real nature of that proposition). The same can be said about all other sub-disciplines within philosophy, from ethics to metaphysics.

The philosophical method, despite its usefulness, cannot give us truths about reality. For we know what kind of method can give us a better understanding of reality, and that is the scientific method. Science was developed to better describe and explain how the universe works, what causes certain physical events, how our bodies function, etc. In other words, to assume that metaphysics gives us the essences of things and the essence of reality would be to claim that the philosophical method is better at describing and explaining reality that the scientific method.

The philosophical method is completely different from any method in science. Philosophers do not investigate physical events in a systematic way, based on empirical observations. The only way that philosophy comes close to reality is when philosophical conjectures find support in science, *e.g.* in scientific discoveries (as the naturalists have argued). To conceive of philosophy differently would be to assume that there is a *first philosophy*, that is, to assume that there is something prior to what can be described by science. However, Wittgenstein argues that this is merely a linguistic confusion. This idea of metaphysical programs and philosophical problems only results in a dogmatic view of philosophy. In opposition to this, Wittgenstein argues that "there is no such thing as *the* problem of philosophy, but only problems of philosophy, *i.e.* linguistic confusions which I can clear up" (WVC, p. 121). The task of philosophy, according to Wittgenstein, is to unravel linguistic knots and conceptual confusions.

2.1.1. Unraveling Linguistic Knots and Conceptual Difficulties

The work of a philosopher is to unravel conceptual-linguistic knots. This simple idea has interesting consequences. It shows us the importance of language in any human activity: how we use language reveals more than just a grammatical structure. It can show us how we represent reality.

For example, we use the word ‘is’ in very distinctive sentences, such as ‘Scott *is* the author of *Waverly*’, or ‘Trump *is* the current president of United States’. In those sentences we state that the first term refers to the same things as the definite description that follows that term. The object denoted by the name is identical to the thing referred to by the definite description. However, we also say that ‘two plus two *is* four’, or that ‘this house is white’. Should we assume that the term ‘is’ represents the same relation in all four sentences? And if we do, what implications does this assumption have? The work of a philosopher is to understand how we *use* language, and what implications our linguistic assumptions have.

The second aspect is that philosophy does not have a subject matter of its on. Philosophers work with arguments, and arguments are premises connected to conclusions. In this sense, a philosopher investigates an argument by investigating what it *says*, how its reasoning goes, and how it represents the things that it refers to. The way we use language shows the assumptions of our claims and conjectures.

Nonetheless, this use of language is not usually clear or straightforward. As Wittgenstein suggests, “[p]hilosophy unties the knots in our thinking, which we have tangled up in an absurd way; but to do that, it must make movements which are just as complicated as the knots" (PR, §2). How we use language may lead to implications that are far more complex than our original assumptions.

On Wittgenstein's account, philosophers should only describe how we use language and the implications of that use. "We can only make clear the rules according to which we want to speak. An explanation cannot satisfy us because it would actually have to come to an end" (WVC, p. 123). An explanation will require another explanation, which will imply another explanation. This goes to the point that a philosopher will have to take an assumption for granted, and then we arrive at the dogmatic view that Wittgenstein wants to avoid.

2.2. Defending a Coherence Between What Wittgenstein Says He Should Do and What He Actually Does

It is evident, across Wittgenstein's work, that he is concerned not to argue for a theory about mathematics. As he regards his work and writings, he is only describing what mathematicians actually do. As Wittgenstein said, "I may occasionally produce new interpretations, not in order to suggest they are right, but in order to show that the old interpretation and the new are equally *arbitrary*" (LFM, p. 14 [italics mine]).

We can see quite clearly that Wittgenstein is (at least) trying to only describe mathematics. He consistently calls his own argument *dogmatic* when he recognizes that he is defending a particular view on mathematics, or perhaps when he is too persistent in arguing for one view at the expense of not adequately considering alternative views. For example, when he says,

> It is often *entirely* sufficient for us to show that one does not *have* to call something *this*; that one can call it *this*. For *that* in itself changes our view of objects. *In this sense my dogmatic pronouncements were incorrect*. But they could be rectified if, where I said "one must view that this way," one says: "one can also view that this way." And it would be incorrect to now believe that the sentence has thereby been deprived of its actual point. (MS, 163, III, §§ 161-162; [italics mine])

Wittgenstein argues that it is dangerous and misleading to pay too much attention to what mathematicians *say* about their work, instead of concentrating on what they actually do. On Wittgenstein's own view of his own work, he is analyzing and unravelling language and concepts to determine what mathematics really is and what mathematicians really do.

However, despite Wittgenstein's caution and advice about only describing mathematics, it seems that he makes some strong, critical claims about set theory. Wittgenstein (PR, §§145, 171; RFM, II - §§14, 15) argues that the idea of mathematical infinity as a proper extension seems "utter nonsense", and that Cantor's alleged proof that the real numbers are greater in cardinality than the natural numbers is a conceptual-linguistic confusion. As Wittgenstein says,

> [t]he [diagonal proof] procedure exhibits something – which can in a very vague way be called the demonstration that these methods of calculation cannot be ordered in a series. And here the meaning of 'these' is just kept vague. *A clever man got caught in this net of language! So, it must be an interesting net* (RFM, II - §§14-15).

I will critically evaluate this and other passages in later sections. However, it is worth noticing here that Wittgenstein suggests that the standard interpretation of Cantor's proof, that there are *more* real numbers than natural numbers, is a product of linguistic and conceptual confusions.

This seems to show us that Wittgenstein's criticism of set theory *prescribes* some changes to set theory, or our outright abandonment of it. It appears that Wittgenstein is telling us that we should abandon the notion of a) infinity as a kind of number; and also the idea that there are b) different sizes of infinite mathematical sets.

In light of this, Floyd's (1991, pp. 144-145) idea that Wittgenstein's dialectical style of writing shows that every particular theory in philosophy is as viable as any other, seems to not consider his very harsh and critical attack on set theory. It seems that Wittgenstein distinguishes between *good* and *bad* descriptions of mathematics, and that Floyd and others are unwilling to see or discuss this because it seemingly violates Wittgenstein's own strictures.

Wittgenstein's conception of philosophy indicates his take on the philosophy of mathematics. It shows why he uses the words that he uses, and why he is so concerned to only describe mathematics, and not suggest or prescribe how mathematics *should* be done. In fact, for Wittgenstein, some of the problems that we have in philosophy of mathematics derive from our confused use of certain concepts, and especially from our tendency to take a word or concept from one area of application and assume that it will apply, in the same way, in a different context. In Wittgenstein's view, philosophy oversteps itself when it tries put forward a view on mathematics that does not adequately look at and describe the actual activity of mathematicians.

Despite Wittgenstein's reluctance to go beyond description, it seems that in relation to some issues, Wittgenstein provides a strong criticism of a more standard interpretation. This criticism can be taken as an argument against one view and in favour of another. In many instances, such criticisms seem to make it very clear which description of mathematics Wittgenstein thinks is better. This does not invalidate Wittgenstein's philosophy of mathematics; rather, it serves to highlight an important aspect of his view.

I will return to Wittgenstein's philosophy of mathematics later. For now, I will investigate some of the questions raised in the previous sections.

CHAPTER 3: The Standard View on Mathematical Propositions

3.1. Mathematical Propositions and Mathematical Objects

Prior to presenting and explaining Wittgenstein's interpretation of mathematics, I will introduce what philosophers sometimes call *the standard view* of mathematics. An explanation of the standard view will serve as an interesting opposition to Wittgenstein's idea. In addition, it will highlight why Wittgenstein's arguments are considered so unorthodox.

The standard view of mathematics should not be understood *only* as a general philosophical interpretation of mathematics. The term 'standard view of mathematics' represents a *general* view of mathematics. It represents how mathematicians, logicians, and many philosophers conceive of and talk about mathematics[14]; what they say mathematics is, what they say mathematics is *about*.

The standard view represents a general consensus on some important concepts in mathematics, such as (but not restricted to) what a mathematical proposition is, what mathematical meaning and mathematical truth is, what constitutes a mathematical proof, and how to interpret a mathematical proof. This consensus may not be explicitly stated in every philosophical argument. However, an investigation of how we talk about mathematics reveals this general agreement.

Although there exist numerous the different philosophies of mathematics, it seems that there is a general agreement on some "fundamental" ideas. By talking about a general

[14] For example, see (Gödel, 1983 [1944]; Quine, 1982, pp. 148-155; Colyvan, 2001).

view, or standard view in mathematics, I aim to highlight this way of describing mathematics and mathematical activity.

First and foremost, the standard view of mathematics assumes that mathematics "is about numbers, sets, functions, etc., and that the way these things are is what makes mathematical statements true or false" (Maddy, 1990, p. 15). In other words, mathematics is about those objects. The signs and operations stand for or refer to some thing in (some) reality. They represent a mathematical fact and the objects that constitute that fact.

By *discovering* new proofs and demonstrations, mathematicians acquire *knowledge of* those mathematical objects. As argued by Maddy (1990, p.21), a common analogy is to explain mathematical propositions in comparison with empirical (contingent) propositions: in a similar way to how physics, chemistry, and biology acquire knowledge of physical objects, so too mathematicians acquire knowledge of mathematical objects and facts.

For the standard view, mathematics is similar to any scientific field: mathematical propositions are about mathematical objects, *e.g.* numbers, sets, functions*;* and mathematical propositions must be true or false, because mathematical claims must either be correct or incorrect about mathematical objects. For example, the proposition '2 + 2 = 4' is a proposition *about* the object 2 (two, *dois*, II, ||, .., etc.), the relations of addition (+) and identity (=), and the object 4 (four, *quatro*, IV, ||||, …., etc.).[15] [16] This proposition '2 + 2 = 4' is true iff it is the case that 2 + 2 = 4. Put differently, this proposition is true because

[15] I will refer to this aspect of a mathematical proposition as "the aboutness of mathematical propositions". The idea that mathematics refers to something in (some) reality can be seen in (Maddy, 1998, pp. 20-21; Putnam, 1975, p. 61; Shapiro, 1997, pp. 36-37).

[16] Here I want to show the difference between the number, or the abstract object that is referred by the symbol (*e.g.* the 4), and the numeral, or the linguistic sign that represents it in a sentence, *e.g.* 4, four, *quatro*, IV, ||||.

it *corresponds* to that fact.[17] Thus, the standard view claims that mathematical truth and empirical or contingent truth are *both* truth-by-correspondence.

However, a mathematical object or fact seems to be very different from a physical state of affairs. A scientist uses observations and empirical experiments to test a particular hypothesis.[18] Empirical (contingent) propositions describe or talk about reality; we use empirical propositions to make testable assertions about the existence of physical objects (things) exist and about how physical objects causally interact with one another.

In opposition, mathematical objects are not (and cannot) be empirical objects (*observable* physical things). The first reason for this is that a) mathematical proofs are not susceptible to change. We do not expect that, if done correctly, a mathematical operation suddenly becomes false. If a proof is true, it is *necessary true*. That implies that it was true, that it is true, and that it always will be true that 2 + 2 = 4.

Second, b) even if we concluded that a mathematical proof is incorrect, it will not be because of anything physical. It seems unreasonable to assume that after some empirical observation, we will refute[19] a branch of mathematics on the basis of that observation. In other words, it seems that no kind or amount of physical evidence can validate or invalidate

[17] I do not want to discuss in this paragraph if the proposition refers to a fact (a solid, unitary thing), or if the components of the proposition refer to objects. To avoid this, I will refer to 'facts' as a set of states of affairs. This tells us that a proposition corresponds to a state of affairs, or that the proposition picture how things stand in reality (it points to objects in a particular relation).
[18] An evaluation of what science is would require a long discussion. That would escape the scope of my work. By stating that science use observations and experiment, I merely want to highlight the connection between the empirical propositions of a scientific theory with reality.
[19] I think that there is an argument to be made about branches of mathematics that suffered some modifications (changed), based on scientific discoveries. However, I would argue that, if any, those cases would merely show that scientific discoveries served as a *stimulus* to the construction of different mathematical systems. I expect that, most philosophical interpretations of mathematics would consider the idea of a scientific discover providing evidence to refute a mathematical system to be absurd.

a mathematical proof. A proof is a precise application of syntactical rules of a calculus. It does not consider any empirical evidence, or physical content.

Even though it is hard to give a precise definition of what science is, and what constitutes a set of evidence for a particular scientific theory, it seems clear that there is a direct relation between a set of observations and some hypothesis in a particular field. This cannot be said between mathematical propositions and any empirical observation.

This leads us to conclude that such mathematical objects are not physical objects. They are not bound or affected by any physical object or event. In other words, mathematical objects are “outside of physical space, eternal and unchanging” (Maddy, 1990, p. 21). That is to say, they cannot causally interact with anything physical, for they are not physical, and must dwell in a different reality than ours.

Based on what was said, we can conclude that, for the standard view, the meaning of a mathematical proposition is something about those mathematical objects. The mathematical proposition ‘2 is a prime number’ tells me something *about 2*, in the same sense that the contingent proposition ‘Scott is the author of *Waverly*’ tells me something *about Scott*. The difference is that the objects of science can be considered physical things, in opposition to mathematical objects.

On this view, mathematics serves to create a picture of how mathematical objects stand in this distinct reality. The fact that they do, shows that a proposition is true, and false otherwise. More precisely, the work of a mathematician consists in discovering facts of that reality, and formulating proofs that pictures how things stand in this reality.

3.1.1. Discovering Mathematical Truths: Mathematical Objects as Mind-Language Independent Things

As stated above, the work of a mathematician consists in *discovering* mathematical truths. This idea relies on the premises that **a)** mathematics is *about something*, or it talks about something; **b)** what makes these propositions true are something *objective*, external to mathematics; and that **c)** mathematical objects and facts exist. Those three points lead to the view that mathematical propositions have an independent truth-value.[20] For example, in the same way that the Amazon forest exists, it took a long time for the Europeans to know of its existence. In the same way, so too mathematical objects exist, despite our *knowledge or ignorance* of them. There is no epistemological condition for a mathematical proposition to be true. What makes the mathematical proposition '2 + 2 = 4' true is something objective, independent of our knowledge or language, and necessary.

We do not know the future of mathematics, how it is going to be developed, what proofs are going to be discovered, and what branches are going to be developed. Nonetheless, according to the standard view, mathematical facts already exist, despite our ignorance of them.

By doing mathematical operations and proofs, we acquire knowledge of those abstract objects (numbers, sets, etc.). Indeed, as Maddy (1990, p. 21) states, "mathematics is the scientific study of objectively existing mathematical entities just as physics is the study of physical entities. The statements of mathematics are true or false depending on

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[20] I am not saying that an interpretation of mathematical proposition as mind-language independent is necessarily committed to the existence of abstract objects. As I am going to show in the next sections, there are some attempts to justify a realistic interpretation of mathematical propositions, without being committed to abstract objects. For now, I will not explain this difference.

the properties of those entities, independent of our ability, or lack thereof, to determine which".[21]

For example, Goldbach's conjecture ("every even number greater than two is the sum of two primes) so far does not have a proof or a refutation. We have not derived Goldbach's conjecture or its negation in any calculus. The standard interpretation argues that, so far, we do not know if Goldbach's conjecture is true or false, for we have neither a proof nor a refutation of it. However, it *is* true or false; it must be one or the other.

3.2. The Grammatical Similarity Between Mathematical Propositions and Contingent Propositions

So far, I have shown that some philosophers argue that a mathematical proposition is very similar to a contingent one. This view consists not in showing that they talk about the same things (objects), which they do not (abstract vs physical objects). Nonetheless, it proceeds by showing that their grammatical structure is the same.

The standard view argues that the structure of a contingent proposition is the same as a mathematical proposition. As shown repeatedly in the previous sections, mathematical propositions would talk about abstract objects in the same way that scientific theories talk about physical objects. This is shown by their linguistic similarities. For example, 'two *is* a prime number' has the same grammatical structure as 'apple *is* red': it associates a predicate, 'prime number' and 'red', with the subject two and apple, by using the

[21] This consistent analogy between mathematical propositions and empirical propositions can also be seen in (Gödel, 1983 [1944], pp. 455-456). It is implied that those similarities constitute a defense of referentialism in mathematics (the idea that mathematical propositions are about/refer to something in a reality). I will better show this argument in the next sections.

connective ‘is’. In the same sense, by showing that ‘5 is bigger than 2’, we show that the objects 5 and 2 have this kind of relation (bigger than); the same applies to the contingent statement ‘Philip is taller than James’.

In this sense, the main argument of the standard view is that if we evaluate mathematical propositions differently from contingent propositions, we will hold a double standard, without any justification. If their linguistic structure is similar, why is it not the case that they work in the same way? In the same way, why do we need to interpret mathematical propositions differently from contingent ones, if we use the same structure for both? This linguistic similarity would not justify treating mathematical propositions as similar to contingent propositions, but also would explain why mathematics is so useful.

This argument in favour of a referential interpretation[22] of mathematical propositions, is also based on the fact that mathematics is used, and has an important role, in contingent propositions.[23] By showing that contingent and mathematical propositions have the same linguistic structure, and that we use the former extensively as part of the latter, therefore they must be interpreted in the same way.

[22] So far, I had not used the term referentialism, or a referential interpretation in mathematics. Nonetheless, I have already stated several aspects of this interpretation. It is sufficient to say that this interpretation is argued by the standard view, and it assumes that mathematical propositions refer to things in a reality. Something that I have already talk about in this and previous sections.

[23] For my thesis, I decided not to include any form of the indispensability argument. I believed that it would consume a considerable part of this paper, taking way the focus from the main subject of the thesis – Wittgenstein’s radical Constructivism of mathematics. I decided just to make a quickly mention here, without developing the argument further.

3.2.1. The Need for a Homogeneous Semantics in Mathematics and Contingent Propositions: Benacerraf's Semantical Criterion

According to the standard view, this linguistic-structural similarity would show that mathematical and contingent propositions have *the same truth conditions*. In other words, we would have a general theory of truth for both mathematical and contingent propositions. This would imply that we would have the same criterion to evaluate what makes a mathematical and a contingent proposition true.

As Kitcher (1983, p. 105) states, "[o]ne of the primary motivations for treating mathematical statements as having a truth-values is that, by doing so, we can account for the *role* which these statements play in our commonsense and scientific investigations". More precisely, by having the truth conditions, we could explain how useful mathematics is in contingent propositions.

In this sense, one of the essential aspects of a shared concept of truth is that it enables us to explain the utility of mathematics in contingent propositions. In other words, the standard view gives us a simple account of the effectiveness or usefulness of mathematics when applied to scientific propositions. To assume otherwise would be to hold a double standard: despite the linguistic-structure similarities, and that mathematical propositions have an important role in contingent propositions, we would interpret them in very distinct ways.

This led Benacerraf (1973) to argue that one condition for any philosophy of mathematics that attempts to describe what mathematics is, is to account for mathematical truth as *truth*.[24] It is not enough to show that a mathematical proposition is a theorem of the calculus, “[t]he account should imply truth conditions for mathematical propositions that are evidently conditions of their truth” (Benacerraf, 1973, p. 666).

More precisely, Benacerraf’s argument relies on the fact that, because mathematics plays an important role in contingent propositions, any concept of truth should be the same for both propositions. This means that the mathematical proposition ‘2 + 2 = 4’ has the same type of truth conditions as the contingent proposition ‘Brasilia is the capital city of Brazil’.[25] As he points out, the conditions require a homogenous semantics for mathematical and contingent propositions.

This condition is similar to the argument that I raised in the above section: mathematical propositions and contingent propositions share the same *logical and* grammatical structure.[26] For example, the contingent proposition “(1) [t]here are at least three large cities older than New York” and the mathematical proposition, “(2) [t]here are at least three perfect numbers greater than 17”, have the same logical and grammatical structure “(3) [t]here are at least three FG's that bear ***R*** to ***a***” (Benacerraf, 1973, p. 663).[27]

[24] As Benacerraf (1973, p. 666) points out, “[f]or present purposes we can state it as the requirement that there be an over-all theory of truth in terms of which it can be certified that the account of mathematical truth is indeed an account of mathematical *truth*”.

[25] “I suggest that, if we are to meet this requirement, we shouldn't be satisfied with an account that fails to treat [a contingent proposition] and [a mathematical proposition] in parallel fashion, on the model of [the same syntactical-semantical structure]. There may well be differences, but I expect these to emerge at the level of the analysis of the reference of the singular terms and predicates.” (Benacerraf, 1973, pp. 666-667)

[26] I would argue that Benacerraf’s argument is clearer and more precise, for it defines the structural similarities of a mathematical and contingent propositions as having the same logical structure. It is not just the grammar, or how we talk about those two types of propositions.

[27] “[W]here 'There are at least three' is a numerical quantifier eliminable in the usual way in favor of existential quantifiers, variables, and identity; and 'G' are to be replaced by one-place predicates, 'R' by a two-place predicate, and 'a' by the name of an element of the universe of discourse of the quantifiers” (Benacerraf, 1973, p. 663).

Benacerraf's argument relies on Tarski's definition of truth.[28] This definition states that truth is defined in terms of satisfaction, based on the syntactical-semantical analysis of the proposition. The proposition (1) is true if there exist at least three *large cities* that are older than New York. As expressed by its logical form in (3), there is a x, y and z, that have the predicate F (city) and G (large), and those things bare the relation *r (*α is older than β) to *a* (New York). In other words, if there are three things that satisfy the conditions expressed in the proposition's logical form, then the proposition is true.

Since the logical form (3) applies exactly to the mathematical proposition (2), then (1) and (2) have the same type of truth conditions. We expect that in order for the proposition "there are at least three perfect numbers greater than 17" to be true, there must exist three things that satisfy the conditions showed by the logical form of the proposition.

To suppose otherwise would be to separate mathematical *truths* from contingent truths. For this, one would have to explain how those different "kinds" of truths relates to each other and have the same logical structure. This leave us without an explanation to how can we combine those different statements and make a true proposition, such as 'two apples plus two apples are four apples.[29] For Benacerraf, this shows that only a referential theory can adequately satisfy the semantic conditions, that is, an interpretation of mathematical

[28] "I take it that we have only one such account: Tarski's, and that its essential feature is to define truth in terms of reference (or satisfaction) on the basis of a particular kind of syntactical-semantical analysis of the language, and thus that any putative analysis of mathematical truth must be an analysis of a concept which is a truth concept at least in Tarski's sense" (Benacerraf, 1973, p. 667).

[29] "Perhaps the applicability of this requirement to the present case amounts only to a plea that the semantical apparatus of mathematics be seen as part and parcel of that of the natural language in which it is done, and thus that whatever *semantical* account we are inclined to give of names or, more generally, of singular terms, predicates, and quantifiers in the mother tongue include those parts of the mother tongue which we classify as mathematese" (Benacerraf, 1973, p. 666).

propositions that can explain the role of mathematical language in scientific (contingent) propositions.[30]

3.3. Mathematical Realism

The ideas expressed in the above section, that I classified as the *standard view of mathematics*, all converge to the same interpretation, *mathematical realism*. They all argue in favour of the view that mathematical propositions are not just concatenations of symbols. "[T]hat is, for the idea that mathematical truth must not be identified with provability" (Putnam, 2012 [2006], p. 183).

The term *realism* in mathematics is not very different from the realistic interpretation of unobservable entities in science – *e.g.* scientific realism. Both views argue that the terms of a proposition are about something, or they refer to something external. For example, scientific realism is the view that terms, such as a *electrons*, are not merely tools or fictional characters.[31] Rather, they *refer* to these things that makes the statement "there is an *x* that is a subatomic particle and it has a negative electric charge" true.

As I showed in section 3.2, the same could be said about mathematical propositions – we need mathematical objects for mathematical propositions to be true. In fact, some

[30] In his paper, "Mathematical Truth", Benacerraf (1973, p. 661) argues that any account of mathematical truth should satisfy two conditions: a) *the semantical conditions* (that I have explained in this section); and b) *the epistemological condition*. The former is usually used as a criticism to any formalistic interpretation, while the latter constitutes a criticism of any referentialistic view. Several philosophers have raised objections to Benacerraf's conditions such as (Steiner, 1973; Maddy, 1990, pp. 36-45; Balaguer, 2001, pp. 21-25). For this paper, I will present the *epistemological condition* only briefly, without dedicating a section to it.

[31] Some arguments in philosophy of science attempts to show that, terms that refer to unobservable entities, in fact do not refer at all. For example, the problem with an *electron* is that we do not have direct observation of it. We use the term to describe and predict several events, nonetheless, we cannot refer to something when we say "electron".

philosophers argue that we cannot commit to scientific realism without committing to mathematical realism.[32]

Nonetheless, mathematical realism has different consequences than scientific realism. The most relevant, I believe, are the problems raised by the fact that mathematical objects: **a)** do not exists physically; **b)** cannot causally affect physical events; and **c)** cannot be observed or otherwise accessed. Even though an electron is an unobservable entity, it is not assumed that it is something *completely* different from anything physical.

In this regard, some philosophers suggest a different kind of realistic approach. As Shapiro (1997, p. 37) states, there are two main types of mathematical realism: **a)** *ontological realism* and **b)** the *realism in truth-value*. The first (a) consists in the interpretation that I have been showing in the last sections: mathematical (abstract) objects exist, and they are what makes a mathematical proposition true.

The second type of mathematical realism is (b) r*ealism in truth-value*, which consists in the view that mathematical propositions have an objective truth-value, without assuming any ontological interpretation to mathematics. [33] This view assumes a particular interpretation of the logical structure (the logical operators) of mathematical propositions in a way that does not have *quantification over a set of abstract entities*.

Both interpretations assume that mathematical truths are mind-language independent. In other words, a mathematical proposition refers to mathematical fact. It does not depend on

[32] I will better elaborate this point in the next two sections. For a glimpse of this argument, see (Quine, 1980, pp. 45) and (Putnam, 2006, p. 183).

[33] A way to do this is to take mathematical propositions as being about possible *relations in a structure*. We can interpret the logical structure of mathematical propositions as being primarily modal propositions (*it is possible*, that there is an *x* such that x is the relation f to z…). I will better elaborate this argument in section 3.3.2.

our knowledge of it, neither in which system we have proved it. Mathematical facts are *objective*.[34]

Mathematical Platonism and mathematics as modal logic, in some sense, both satisfy Benacerraf's semantical condition: that mathematical truth should agree with a general theory of truth (*e.g.* truth by correspondence). However, those similar interpretations about mathematical truth have different premises and consequences.

3.3.1 Platonism in Mathematics: Abstract Mathematical Objects

The first of the realistic approaches to mathematics that I am going to evaluate is *ontological* realism. The best known program to defend this view in mathematics is *Platonism*. As the name suggests, Platonism originated with the philosophy of Plato. He argued that there exist two distinct realms (or realities): the first realm is our physical world, which is imperfect, contingent, mutable (changeable), etc.; and the second is a perfect reality, of immutable ideal objects, perfect and universal.[35] In short, Plato's idea was that our physical reality imitates (or participates in) this perfect realm, for example, a wooden triangle is an imitation of the perfect ideal triangle.

In relation to modern interpretations of mathematics, Platonism is associated with the idea that mathematical propositions are about abstract (platonic) objects. As Field

[34] Objectivity here refers to the assumption that a mathematical fact does not depend on our knowledge of it, neither on the calculus that it was proved. A mathematical fact is a fact, independent of how we represent it.
[35] Maddy (1990, p.21) gives an interesting summary of what Platonism in mathematics would be: "[t]raditionally, Platonism in the philosophy of mathematics has been taken to involve somewhat more than this. Following some of what Plato had to say about his Forms, many thinkers have characterized mathematical entities as abstract—outside of physical space, eternal and unchanging—and as existing necessarily— regardless of the details of the contingent make-up of the physical world. Knowledge of such entities is often thought to be a priori—sense experience can tell us how things are, not how they must be—and certain—as distinguished from fallible scientific knowledge."

(1988, p. 1) argues, mathematical Platonism assumes that **a)** mathematics is about abstract objects and **b)** those objects are mind and language independent. The former asserts that mathematics *describe*s this reality of abstract objects, that is, objects that lack some necessary properties of concrete objects[36]; as the latter assumes that this distinct reality is not dependent on our language, neither on our knowledge (or any mental state).

The peculiar aspect of mathematical objects is that they are complete distinct from physical objects. In other words, on the standard interpretation, abstract mathematical objects are acausal, atemporal and aspacial. In this sense, how we *perceive* those things must be different from physical objects. In addition, this relation does not give a precise definition of what 2 would be. The numeral '2' corresponds to this object, but there is nothing that the sign points to. As well, it is not clear if the sign '2' refers to an object, or to all objects that has the property o being 2 (*i.e.* does the sign '2' works as a name or as a property?).

A more precise approach that attempts to avoid this vagueness, is to reduce the mathematical language to set theoretical terms. This implies that mathematics would be a branch of set theory, where mathematical entities could be referred as *sets* or *collections*. For example, operations in mathematics could be translated to relations of sets, *e.g.* "2 + 2 = 4" could be treated as A: {a, b} ∪ B: {c, d} = C: {a, b, c, d}.[37] This set theoretical

[36] The most common definition of abstract objects are those kinds of objects that are not spatial, temporal, and causal. This definition classifies abstract objects by *not having* some properties that are well established for physical objects.
[37] This is a very naïve and informal description of the set theoretical approach to mathematical operations. For a clear and formal demonstration of the axiomatic system that enabled such reduction (of mathematics to a set theoretical system), see (Zermelo, 1967 [1908], pp. 183-186)

approach enables us to substitute the unclear concepts of numerals and numbers for a better defined concept, *i.e.* sets and collections.

Despite the fact that the set theoretical language provides a clear and more precise description of mathematical objects as sets or collections, it does not get rid of Platonism. We just switch the question, '*How can we perceive mathematical objects*?' to '*How can we perceive sets and collections?*'. For sets and mathematical objects enjoy the same level of abstraction.[38]

As Gödel (1983b [1947], pp. 483-484 [italics mine]) argues, following the same analogy that I raised in the sections above, "despite their remoteness from sense experience, we do have something like a perception also of the objects of set theory, as is seen from the fact that the axioms *force themselves upon us as being true*". In the same way that our sense perception "induces" us to build scientific theory that we expect to be corroborate (by our sense perception) in the future, so too our "mathematical intuition" must be regarded in the same way.

In summary, the idea is that there is something that the mathematical sign refers to. Those "intangible objects" (Quine, 1982, p. 149), *e.g.* the 4 or the 2 (or the sets A: {a, b, c, d} and B: {e, f} for a set theoretical approach), are the referent of the sign '4' and '2' respectively; and the property of "being a prime number", or "being even", are properties of those things. In addition, we know that a proof is true for intuitively it must be true (mathematical truths *force themselves upon us*).

[38] As Maddy (1990, p. 27) states, "[Zermelo] produced an axiom system that showed how mathematics could be reduced to set theory, but again, no one supposed that set theory enjoys the epistemological transparency of pure logic".

One important argument that is taken in favour of Platonism is that this referential interpretation satisfies Benacerraf's semantical condition. A mathematical proposition, such as "There is an *x* that is the sum of two and three" is true, iff there is a thing that has the property of being the sum of two and three. The quantifier ranges over a set of objects that satisfies the value of *x*, for *x* being the sum of two and three. This shows us that the values of *x* are those things that make that proposition true.

Here we see that Benacerraf's condition agrees with Quine's criterion for ontological commitment. As Quine (1948, p. 33) argues, "a theory is committed to those and only those entities to which the bound variables of the theory must be capable of referring in order that the affirmations made in the theory be true". The logical analysis shows which entities *must exis*t in order for the proposition to be true.

This notion of truth would enable us to explain the usefulness of mathematics. It is well known that mathematics is very useful in contingent propositions. In this sense, platonists argue that a formalist (fictionalist) interpretation of mathematics would leave as a mystery why it is the case that mathematics is so useful. As Kitcher (1984, p. 105) points out, a formalistic approach "can be countered by pointing to the value of mathematics in advancing our understanding of the world".

In other words, the usefulness of mathematics is justified by its being true. Otherwise, we are left with a mystery: if mathematics is a mere sign-game, how can it be so useful? Platonists criticize any formalistic interpretation by saying that it leaves mathematics as a creative activity without explaining its success. In opposition, by assuming a platonist interpretation of mathematics we are able to give a solid ground to the usefulness of mathematics, for it can explain why mathematical propositions are true.

Nonetheless, the epistemological problem of abstract entities is a barrier that Platonism cannot pass through. In face of this problem, some philosophers argue that the discussion of mathematical propositions is not about objects, "but over the *objectivity* of mathematical discourse" (Shapiro, 1997, p. 37)

3.3.2. Modalism in Mathematics: The Provability as a Mathematical Fact

As I argued in the previous section, mathematical Platonism cannot avoid the epistemological problem (even by using a set theoretical approach). For this reason, some philosophers have tried to change the focus of the discussion from mathematical objects to the objectivity of mathematical propositions.

The theory of mathematics as modal logic consists of interpreting mathematics as the study of (abstract) structures and what is possible in such structures. What is argued is that mathematics as modal logic succeeds in preserving a realistic interpretation of mathematical propositions, without being committed to abstract objects. The argument relies on two concepts: **a)** structures; and **b)** possibility.

The important aspect of structures is that it does not force us to be committed (at least not initially) to abstract objects, for it assumes that mathematics is the language about "a *pattern*, or an 'objectless template'" (Balaguer, 2001, p. 8). Mathematics, in this sense, does not talk about or refer to abstract objects in a different reality. Rather, structuralism assumes that mathematics is a language about the ***relations*** between objects, where the nature of objects is left unspecified. Or as Balaguer (2001, p. 8) argues, "the claim is that any sequence of objects with the right structure (that is, any *ω-sequence*) would suit the needs of arithmetic as well as any other". Thus, according to Balaguer, the structure that *is* number theory is an ω-sequence.

By assuming that mathematics talks about abstract structures and what is *possible* in such *structure*, it seems that we avoid any (initial) commitment to abstract objects. Mathematicians would investigate *possible relations*—they would endeavour to determine what follows from what in particular structures.. By introducing a modal interpretation, we do not have an extensional interpretation of the logical structure of mathematical propositions. This enables Modalism to support the claim that "to be, is to be the value of a bound variable", without being committed to abstract objects in mathematics.

One of the arguments in favour of this interpretation is Putnam's view of mathematics as modal logic. In short, Putnam argues that by taking modal operators as primitives, we construct mathematics as being about models that "[are] true under some interpretation" (Putnam, 1979, p. 73).

In this respect, Putnam's argument relies on two premises: **1)** a mathematical proof should be interpreted as a proof in modal logic; and **2)** mathematics is true under a realistic interpretation of *contingent* propositions. More precisely, Putnam argues that proof and truth are different concepts. In other words, a proof shows possible relations in a structure (which theorems follow from a set of axioms), while the successful application of mathematics to science (contingent propositions) shows that mathematical truth is mind-language independent.

In relation to (1), Putnam (1979b [1967], p. 45) argues that the "chief characteristic of mathematical propositions is the very wide variety of equivalent formulations that they possess”. According to him, mathematics talks about the same mathematical fact (*i.e.* parts of the structure, or some relations known to be derivable in the system) without using the same (set of) objects. This shows that we can have equivalent descriptions that do not exclude one another, and they can derive the *same mathematical facts*.[39]

As Putnam argues, “mathematics has no special objects of its own, but simply tells us what follows from what” (Putnam, 1979b [1967], p. 48).[40] In logical terms, a modal mathematical proposition ‘P’ is “true in all models” if it is necessary that “if Ax then P” (where ‘Ax’ stands for the conjunction of the axioms and theorems of arithmetic).[41] In other words, the modal interpretation, argued by Putnam, avoids reference to abstract object by taking a proof as the possibility of deriving a particular mathematical proposition from a set of axioms and theorems.

However, this only gives us an idea of what a proof in mathematics would be. A proof only tells us that it is necessary that the model derives the function Y. In other words, we only know that a model is true under *some* interpretation. This view, by itself, does not rule out any formalistic or intuitionistic program, for as Putnam (1979, pp. 73-74) states,

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[39] It is not clear to me what Putnam means by “mathematical fact”. I believe that he does not mean *a fact in a platonic reality*, for, as he states (Putnam, 1979 p. 72), “one does not have to ' buy' Platonist epistemology to be a realist in the philosophy of mathematics. The modal logical picture shows that one doesn't have to 'buy' Platonist ontology either”. The other possibility would be to consider a mathematical fact as the *true* mathematical propositions in contingent propositions. For example, we know that certain equations give precise and correct results that can be empirically observed and make correct predictions. I am more inclined to assume the second interpretation, rather than the first..

[40] This quotation shows similarities between Putnam’s philosophy of mathematics and the if-thenism view. In fact, I believe that, for pure mathematics, Putnam’s view can be stated as a form of If-Theism. The differences, I think, would be restricted to Putnam’s evaluation of applied mathematics (*i.e.* what he considered as a *true mathematical proposition*).

[41] Putnam (1979b [1967], pp. 47-48) gives this example in a more precise and formal way. Nonetheless, the main idea is that a mathematical proposition is a theorem in *e.g.* arithmetic if the set of axioms of that language derives the proposition.

“Bishop might say, 'indeed, most of classical mathematics is true under some interpretation; it is true under an intuitionist *re*interpretation!'”.

In order to avoid this problem, Putnam argues that the other fundamental aspect of mathematics is that the interpretation that it is true must “square” with our “physical experience”. In other words, Putnam draws attention to the fact that applied mathematics is a very important aspect of mathematics itself: mathematical propositions are true when applied (under the interpretation) to contingent propositions.[42] It seems that Putnam’s argument relies on the fact that mathematics is fundamental to scientific contingent propositions, and, therefore, it must be true in the same sense as contingent propositions (*i.e.* mind-language independent).

This kind of argument is known as the *indispensability* argument. We have several distinct indispensability arguments, which in fact have different premises and consequences.[43] In short, the indispensability arguments consist in showing that because mathematics is indispensable to scientific theories, mathematical propositions must be true; and because mathematical propositions must be true, non-physical mathematical objects must exist.

Without getting into the details, the idea of indispensability consists in the fact that science as we know it could not be done without mathematics. This leads to the conclusion

[42] Putnam (1979, p.74) says that “[t]he interpretation under which mathematics is true has to square with the application of mathematics outside of mathematics”. However, he does not clearly state what is the meaning of “true under a particular interpretation”. Based on the context of his paper, I conjecture that Putnam means that by applying mathematics in contingent propositions we have true mathematical propositions.

[43] Different indispensability arguments have different premises and conclusions. Some arguments consist in showing that because mathematical explanation is indispensable to scientific theory, therefore we must be committed to mathematical abstract objects. Others rely on some sort of holistic interpretation, where only applied mathematical propositions are true, leaving pure mathematics as merely “recreation”. For a more extensive explanation of the indispensability argument, see (Colyvan, 2001, pp. 6-17)

that “mathematics and physics are integrated in such a way that it is not possible to be a realist with respect to physical theory and a nominalist with respect to mathematical theory” (Putnam, 1979, p. 74). Put summarily, Putnam’s argument attempts to show that, because of this integrated state between mathematical propositions and scientific theories, mathematical truth is mind-language independent, for we regard scientific theories (contingent statements) as mind-language independent. To affirm otherwise, that is, to be a realistic about scientific theories and a nominalist about mathematics, would be to hold a double-standard.

In this sense, Putnam’s modal mathematics relies on two premises: **a)** mathematics is indispensable to scientific theory; **b)** a scientific theory is a set of contingent propositions about reality (scientific realism). Those premises lead to the conclusion that mathematical truth is mind-language independent, for mathematical explanation is extremely important to science and are used to make true descriptions and predictions in contingent propositions (*i.e.* mathematical propositions are useful to make true predictions, explain events, etc.).

The difference that Putnam claims here, and what makes this argument distinct from Platonism, is that we are committed to interpret the quantification operators, of mathematical propositions, in an *intensional context*. That means that the operators do not range over an infinite domain of *objects*. Instead, an intensional context suggests that the operators refer to *relations,* not to *things*. In this sense, we are able to keep Quine’s criterion for ontological commitment[44], without assuming that mathematics talks about abstract objects.

[44] In summary, Quine’s criterion for ontological commitment says that a theory is committed to those entities that must exist, in order for its propositions to be true. I will better evaluate Quine’s criterion in later sections.

On the other hand, the fact that we apply mathematics in scientific theories *guarantees* that mathematics is not just a language that evaluates what can be drawn from a set of axioms, but that it is true (mind and language independent).

The idea is that "mathematical objects" in contingent propositions are conditional upon physical objects. "[T]hey are, in a sense, merely abstract possibilities" (Putnam, 1979, p. 60) of physical things. In other words, mathematical truth is grounded in contingent propositions in the same sense that a mathematical structure consists in the abstraction of the (possible) relations of physical objects.

3.4. Quine's Dilemma: Platonism vs. Modalism

The ideas expressed above represent two distinct programs in the philosophy of mathematics. Both argue in favour of a referential interpretation of mathematical propositions, that is, mathematical propositions are about something. As a consequence of this, they share the view that mathematical truths are mind-language independent, meaning that what makes a mathematical proposition true is something external to mathematics.

In a sense, both views are classified as the standard view. As I said, the standard view on mathematics constitutes a consensus on some major ideas on what mathematics is. The differences between modal mathematics and mathematical Platonism (or mathematical set theory) arise at a fine-grained level.[45] The fundamental ideas that show how mathematicians (and philosopher alike) talk about mathematics, as I highlighted in the

[45] What I meant here is that the differences between Platonism and Modalism in mathematics will only be evident in a more detailed analysis, such as how do they interpret the logical operators.

paragraph above, are still supported by both interpretations. The differences here are only distinguishable in a more detailed analysis.

In addition, both constitute an opposition to a nominalistic view of mathematics. According to Quine (1998 [1986], p. 397) a nominalistic system cannot provide the same "explanatory power" that a system that accepts abstract entities can provide. For example, a nominalistic system cannot assume that mathematical propositions refer or talk about abstract objects, such as numbers. In this sense, sentences such as 'there exist a number x, such that x is the sum of two primes' would pose a problem to the nominalist interpretation. Nominalists would have to provide a different way to evaluate such sentences, one that would not allow quantification over abstract objects.

In the same way, Goodman and Quine (1947, p. 105) argue that the same can be seen in contingent propositions. A scientific proposition, such as "'x is a zoological species' calls for abstract objects as values".[46] [47]

For them, a nominalized logical system would not be able to represent some *true* propositions, mathematical and contingent alike. In order to avoid quantification over a set of abstract entities, we would be making mathematical systems, and scientific theories, weaker. In this sense, Quine presents a dilemma in philosophy of mathematics: **a)** we

[46] Hartry Field (1980) took the task of nominalize a part of physics, *e.g.* Newtonian gravitation theory. I will not evaluate the merits and criticism of his strategy, for that would take a considerable part of this thesis. I merely want to point that there were attempts to nominalize some branches of physics.

[47] I will not evaluate if Goodman and Quine's examples show that scientific theory necessarily quantify over a set of abstract objects. The argument in favour of a nominalistic interpretation of scientific theories would require a solid notion on some important branches of science, such as physics, chemistry, quantum mechanics, Newtonian physics, etc. For now, I will only present their example without further developing it.

accept abstract objects as the domain of our calculus; or **b)** we use modal theories to avoid referring to those things.

3.4.1 Quine's Way Out the Dilemma: Reluctant Platonism

It is important to notice that both Putnam's mathematics as modal logic and Quine's *reluctant* Platonism start from the same premise: *scientific realism*. In other words, they share the idea that scientific theories describe reality. The terms of a true scientific proposition must refer to things (objects) in reality, in order to picture reality.

In addition, it appears that they both share a similar criterion for ontological commitment: a theory is committed to the things that are taken as the value of the bound variables of its true propositions. In other words, the logical analysis of scientific theories prescribes what must exist for the propositions of the theory to be true.

The reasons that seem to motivate Putnam's Modalism is that we would be able to reject the platonist ontology (that there exist abstract objects in an ideal realm) and epistemology (the use of intuition as a way of knowing abstract things) alike. This led to the idea that mathematical truth is mind-language independent, without succumbing to the epistemological problem (of trying to explain how we acquire knowledge of acausal things).

However, for Quine the idea of resorting to modal logic, in order to avoid the epistemological problem, generates more problems than solutions. As he states,

> I recognize that we are spared such recourse to modality only by recourse to an extensional Platonism of sets. This is involved in my appeal to the successor function and again in my appeal to recursiveness. Putnam argued, conversely, that

> modality could be useful in avoiding or minimizing such recourse to mathematical objects; and Parsons has been critically scanning this prospect. My own scale of values, however, is the reverse; **my extensionalist scruples decidedly outweigh my nominalistic ones. Avoidance of modalities is as strong a reason for an abstract ontology as I can well imagine**. (Quine, 1998 [1986], p. 397 [bold mine])

Quine's disproval of modal logic is well known. In short, he argues that modality (taken strictly) generates an opaque context of reference.[48] That is to say, modality generates problems for the principle of substitutivity of equality (or substitution *salva veritate*):[49] As shown in the quotation above, modal logic presents an intensional context of reference. For example, the context of the sentence "it is possible that Thiago is wearing a white shirt today" does not enable us to substitute 'Thiago' for 'the Brazilian student of the M.A. program in philosophy at UofL', for it could be the case that in a possible world I would not have done an M.A. in philosophy. In other words, the meaning of the terms of that sentence matters for its truth-value.

Quine rejects mathematical Modalism in the same way that he rejects nominalism. This gives us a negative argument, that is, why we should *not* adopt Modalism or nominalism. On the positive side, Quine's argument in favour of Platonism relies on four premises: **a)** scientific realism; **b)** the analysis of the logical structure of propositions as a criterion for ontological commitment; **c)** the indispensable role that mathematical

[48] An opaque context of reference is such context that does not allow the principle of substitutivity. In other words, even if we substitute different terms that refer to the same object, it still does not preserve the truth-value of the proposition. For example, 'I know that Quentin Tarantino is the director of Pulp Fiction' is true only if I know that that is the case. If I do not know that Quentin Tarantino was also the director of Django, the sentence 'I know that the director of Django was the director of Pulp Fiction' is false, despite that the terms (Quentin Tarantino and 'the director of Django) refer to the same person.

[49] The principle of substitutivity of equality, in an informal sense, is the principle that enables us to substitute different terms that refer to the same objects, without changing the truth-value of the proposition. For example, the term 'evening star' can be substituted by the term 'morning star' in the sentence "the evening star is the second closest planet of the Sun", without changing the truth-value of the sentence. For they refer to the same object (Venus).

explanations have in scientific theories (Quine’s indispensability argument); and **d)** pragmatic a condition for choosing a theory (a pragmatic approach to selecting a theory).

The first three criteria, (a), (b) and (c), have been covered extensively earlier in this chapter. The first consists in arguing that, to deny that there exist objects in mathematics would be to hold a double ontological standard. On his view, there is no significant difference between physical objects and mathematical objects. The only thing that changes is the degree to which “they expedite our dealings with sense experience” (Quine, 1961 [1953], p. 45).

The second (b) is very well known: as Quine (1948, p. 33) stated it, “a theory is committed to those and only those entities to which the bound variables of the theory must be capable of referring in order that the affirmations made in the theory be true”. I do not think that I should add more to this than what I have already done. However, the next (c) requires some explanation.

Quine’s indispensability argument (c) is very different from Putnam’s indispensability argument. [50] For Quine, mathematical terms “contribute just as genuinely to physical theory as do hypothetical particles” (Quine, 1982, p. 150). Here, we see the same argument based on a realistic view of scientific theories (as well as for Putnam). Nonetheless, Quine’s realism leads to the conclusion that mathematical entities exist. For

[50] This is a matter of extensive discussion. Some authors (Rodych, 2003; Melia, 2000; Colyvan, 2001; Sober, 1993; to say a few) referred to the indispensability argument as Quine and Putnam’s Indispensability Argument. However, as I tried to show, the difference between Quine’s indispensability and Putnam’s indispensability are significative. As Putnam (2012 [2006], pp. 182-183) himself argued, as well as Pincock (2004), Putnam’s indispensability argument does not conclude that mathematical entities exist. He only supports a modal interpretation of mathematics, as I have shown in section 3.3.2. The only thing that both arguments have in common is that both rely on scientific realism.

him, if one holds that terms about non-observable in science refer to something in reality, then we should apply the same criterion for mathematical entities.

However, there is an important aspect of Quine's view on mathematics that further separates his argument from Putnam's indispensability argument. On Quine's view, only *applied* mathematics has ontological status.[51] Branches of pure mathematics that do not have any real application to scientific (contingent) propositions, are considered "mathematical recreation and without ontological rights" (Quine, 1998 [1986], p. 400).

This idea derives from Quine's representation of knowledge as a web of beliefs, known as *holism*. Holism consists in the notion that we should evaluate our system of beliefs as a whole, not as individual beliefs or propositions. For example, something that is outside of our system of beliefs (something that is not part of our best scientific theories about reality), is not known to exist.

The last premise (d) is essentially pragmatic. In the title of this section, I referred to Quine's Platonism as a 'reluctant Platonism', for it seems that the defense of Platonism comes as pragmatic in essence.[52] As Rodych (2003, p. 80) argues, Quine's Platonism relies on the "claim that we should posit entities to explain our phenomenal experiences". In order to explain and predict events in reality we must posit the existence of such abstract

[51] "Pure mathematics, in my view, is firmly imbedded as an integral part of our system of the world. Thus, my view of pure mathematics is oriented strictly to application in empirical science" (Quine, 1998 [1986], p. 400)

[52] I took this tern 'reluctant Platonism' from Putnam (2012 [2006], p. 183). However, Putnam does not further develop this notion. The explanation provided in this paragraph comes from my understanding of Quine's view on mathematics, and from my reading of Rodych's (2003, p. 80) analysis on Quine's indispensability argument.

objects. In other words, Platonism is a consequence of our criterion for selecting our best scientific theories.

CHPATER 4: Criticisms of the Standard View

4.1. Problems Concerning the Existence of Mathematical Entities and Abstract Structures

I have already touched on some difficulties that the standard view of mathematics faces, especially when a program is committed to the existence of abstract objects. I briefly talked about the problems related to abstract entities in mathematics, and how they have motivated some philosophers to switch the focus of the discussion from mathematical objects to mathematical truth or objectivity.

In this section, I am going to explain in more detail some of the problems related to the standard view of mathematics. In addition, I will argue that, so far, those problems do not have a clear answer.

I will focus on three main subjects: **a)** the ontology and epistemology of abstract objects; **b)** Modalism and possibility as a shadow of reality; and **c)** the concept of and conditions for mathematical truth. These three topics directly refer to the points that I raised in the previous sections.

First (a), the idea of existent abstract objects is, by itself, taken as problematic in the context of this discussion. That is clear for, it is well known in philosophy that the terms "Platonism", or "platonic", are used in a pejorative sense. In this regard, I evaluate two problems for abstract objects, that can be represented by the questions "How do abstract objects interact with physical objects?" and "How do we acquire knowledge of abstract entities?" On the second topic (b), I believe that the notion of possible relations in a structure cannot help us to solve the problems related to abstract entities. I will argue that Modalism "solves" the problem by merely avoiding or postponing it.

The last subject (c) relates to the notion of proof and truth in mathematics. I intend to show that the famous analogy between mathematical propositions and contingent propositions does not hold. I will argue that the differences between mathematical and contingent propositions outweigh their similarities, showing that this analogy cannot hold. In other words, I intend to show that Benacerraf's semantic argument begs the question, by assuming that mathematical proof must be interpreted as truth in contingent propositions.

In regard to these topics, I will argue that the standard view of mathematics creates an image of mathematics, without describing it. In other words, what is said about mathematics does not correspond to how mathematicians do mathematics.

4.1.1. An Epistemological Problem: How Can We Know Anything About Mathematical Entities?

In section 3.2.1, I evaluated Benacerraf's semantic condition and showed how his argument is presented as a criticism to any formalistic (nominalistic) interpretation of mathematics. However, in his paper "Mathematical Truth", Benacerraf argues that not just Formalism, but also Platonism, is not a satisfactory interpretation of mathematics.

Benacerraf's argument against Platonism is known as the epistemological condition. The argument consists in showing that it is not enough to state that a mathematical proposition is true. We should be able to show *how can we know* that that proposition is true. As he argues, "[i]t must be possible to establish an appropriate sort of connection between the *truth conditions* of p and *the grounds on* which p is said to be known" (Benacerraf, 1973, p. 672).

For example, for a contingent proposition, such as 'Trump is the president of United States of America,' the **a)** *truth condition* is satisfied by the fact that Donald Trump won the democratic process known as the presidential election; whereas **b)** *the grounds on which I know* this are that I saw the voting process, or that I saw him a fair amount of time inside The White House (via television, or radio, or the internet, etc.). Therefore, I know this proposition to be true.

However, if it is assumed that a mathematical proposition is refers to something in reality, then it is not possible to account for (b) the grounds on which we know the proposition to be true. In other words, platonists can give the truth conditions for a mathematical proposition to be true (*i.e.* it correspond to the mathematical fact/objects, or to a particular set relation); but, they cannot show what are the grounds on which we *know* that to be true. For example, if I know that "2 + 2 = 4" is true, then I should know its truth condition and how we know that. For the first aspect, a platonist claims that the truth conditions (*i.e.* what makes this proposition true) is that the proposition corresponds to a mathematical fact, or that it corresponds to a particular relation of sets (*e.g.* A: {a, b} ∩ B: {c, d,} = C {a, b, c, d}).

Nonetheless, it seems that the platonist view cannot account for that kind of correspondence. Platonists cannot show how can I know that fact, for it is *by hypothesis* that something outside of space, time, and causally inert, cannot relate in any sense to a physical object. In other words, Platonism cannot explain how it is possible that we have knowledge of facts about the platonic inaccessible realm.

My example, “Trump is the president of the United States”, shows that this picture corresponds to something in reality. In addition, I showed how I knew that to be the case. In a more precise way, the ground on which I know that a proposition to be true *causally affected me*. In other words, I could cognitively access how things stands in reality, and evaluate if it is the case that the propositional picture corresponds.

As Benacerraf’s argues, mathematical knowledge should not be significantly different from our knowledge of the physical world. If knowledge of mathematics means knowledge of properties of numbers (as the platonists claim), then we should be able to explain how we are able to know those properties.

As I said in above sections, Gödel (1983b [1947], p. 484) argues that mathematical intuition constitutes our perception of mathematical facts, for “the axioms force themselves upon us as being true”. This is now referred as platonistic epistemology, that is, a commitment to the notion of intuition as our capacity to access abstract objects.

Nonetheless, Gödel does not explain what this kind of intuition would be. Would it be a psychological state? Would it be some sort of comfort after concluding a proof? Or would it be a different aspect of our cognition? In fact, it must be a cognitive ability that is something entirely different from our “regular” perception. For, by hypothesis, a platonic object is physically inaccessible. It cannot causally interact with our sense-perception, for it cannot *causally interact with anything physical*.

In this regard, it appears that the notion that set theory or mathematical axioms “force themselves” to our cognition is too obscure. It does not constitute a good or clear indication of how we acquire knowledge of mathematical/set theoretical facts. We are left without an answer to how can we acquire knowledge of mathematical objects. In a stronger sense, one can even ask if an answer to that would be possible.

4.1.2. An Ontological Problem: How Can Mathematical Entities Causally Interact with Physical Objects?

The argument that we cannot know mathematical objects does not exclude the possibility that they might exist. In a sense, we could think of mathematical objects as theoretical entities in science: we cannot access them cognitively (directly), but they are necessary in order to explain and predict some major events.

We might even assume that our cognitive abilities are so limited that we might never be able to directly access abstract and theoretical entities. Despite our assumptions, these entities are fundamental to explain some physical phenomena, as Quine (1982, pp. 148-155) argues. In other words, a rejection of epistemological Platonism does not entail a rejection of ontological Platonism. Some would argue that the reasons for being committed to those platonic entities are not related to our ability to know them, but are pragmatic in principle.

In a pragmatic sense, as Quine suggested, we are committed to abstract entities, for those entities helps us explain why mathematics works so well in contingent propositions. In a negative sense, platonists argue that a formalistic program would leave mathematical application as a mystery. For this view, if one assumes that mathematics has no objects, that it is merely a sign-game, we would not be able to explain why it is so important in scientific theories. Mathematics would not be true or false, but merely an arbitrary set of symbols. In other words, Platonists claim that Formalism cannot explain how an meaningless arbitrary set of symbols are indispensable to explain the physical world.

Nonetheless, the apparent answer of Platonism only gives us an illusion, for the existence of mathematical objects is still questionable. As Kitcher (1983, p. 105) states, “the reasons which incline us to take the first step with the Platonist should also make us suspicious of the thesis that mathematics describes a realm of abstract objects”. We are seduced by Platonism for it gives us a straightforward answer to why mathematics is so important to science. However, this very reason should make us suspicious. In other words, the idea that mathematical entities show why mathematical propositions are true or false, should be the very idea that also raises objections to the platonistic interpretation.

According to Platonism, mathematical facts or objects help us explain why mathematics is so useful. However, that could not be further from the truth. In fact, mathematical objects do not help us explain physical events, for they *cannot help us explain physical events*. By hypothesis, mathematical objects cannot causally interact with physical objects. In this sense, how could they explain facts or events in reality? An electron helps us explain physical events for it shows how an atom is organized. But how can numbers do the same?

The mistake in the platonists’ argument is to assume that the indispensability of mathematics entails the indispensability of mathematical objects. The idea that mathematical objects helps us as much as theoretical entities in scientific theories, ignore the fact that theoretical entities are posited as *causal things*: they help us explain the connection between a physical event and the reasons or causes of that event.

The indispensability argument gives us an elusive picture of the fact that mathematics is indispensable to science. It takes the indispensability of mathematical explanations to science, and then it assumes that mathematical *entities* are indispensable. However, we do not have a clear explanation of why mathematical *entities* are indispensable to science. In other words, it does not answer the question, how a causally inert thing helps us describe a physical reality? In this sense, Platonism seems to posit abstract entities only to provide a false picture of the indispensability of mathematics to scientific theories.[53]

However, despite all that was said, we cannot affirm that Platonism has been refuted. As Rodych argues, the platonic interpretation of mathematics is not logically impossible:

> If the Platonist, Pragmatic or other, boldly asserts the existence of causally inert mathematical objects, objects in another realm entirely detached from our physical universe, there is certainly no way to refute this claim. We simply must admit that such a universe *could* exist, for there is nothing logically contradictory about the existence of two realms, one physical and one mathematical. (Rodych, 2003, p. 4)

As Rodych argues, Platonism cannot be refuted. No set of evidence could refute Platonism, for every evidence would be a physical evidence. Since Platonism claims that mathematics is about acausal, atemporal and aspatial, therefore there can be nothing in our experience of the *physical reality*, that constitute evidence that this other realm exists. It is possible that something like a platonic realm could exist, and mathematics propositions could be true by referring to the facts of this platonic realm. But as Wittgenstein (LFM, p.

[53] My argument here is very restricted. I am aware that the indispensability argument is better formulated in scientific theories that largely use theoretical terms and mathematical terms in order to explain its predicted events, *e.g.* quantum physics. For the fact that I have a very limited knowledge in quantum physics, I will not evaluate this approach. A good example of this argument in relation to quantum physics is Colyvan (2001, pp. 81-86) and Lyon and Colyvan (2008).

239) said, “it would come to saying nothing at all, a mere truism: if we leave out the question of how it corresponds, or in what sense it corresponds”.

However, I believe that by raising some questions, we are able to better evaluate the consequences entailed by a platonistic interpretation of mathematics. The idea is not to refute Platonism, but to demonstrate its limitations as a satisfactory interpretation of mathematics.

4.2. Possibility Construed as Actuality: What is the Existence of the Possibility of a Proof?

The modal account of mathematics argues that by interpreting modal operators as primitive, we avoid the problem related to the existence of abstract objects. In summary, if we assume that mathematical propositions are about possible relations, we are not assuming that they quantify over a set of abstract objects. This enables us to assume Quine’s ontological criterion for scientific theories, without being committed to abstract objects.

However, this claim comes at a cost. A modal interpretation of logical operators of mathematical propositions gives away the extensionality of the system (as I said above). Or as Quine argued, the gains of a modal interpretation do not compensate for the problems that it has.

Nonetheless, even if we do not agree with Quine’s extensionalism of first-order logic, it seems that Modalism does not solve the problem related to abstract entities in mathematics, it only postpones it. By stating that mathematics is the study of possible relations in a system, we are switching the question from "What is an abstract mathematical object?" to "What is possible in a (abstract?) modal structure/system?". In other words, the concept of "provability" does not seem to help us in relation to the epistemological and ontological problems of abstract entities.

For example, Wiles thought that he had discovered a proof of Fermat's Last theorem in 1993, but in fact the proof had an error. Despite this setback, Wiles managed to finish his proof and correct his mistake in 1994. In 1995, he published two papers in which he proved Fermat's Last Theorem. Even if Wiles failed to construct a correct proof in 1993, a modal interpretation of mathematics argues that it *was possible to prove Fermat's Last Theorem* in 1993.

This shows a connection between the idea of discoverability and provability: if Wiles could have discovered the correct proof in 1993, then Fermat's Last Theorem was provable in 1993. More precisely, mathematical Modalism is committed to the idea that it was possible to prove Fermat's Last Theorem in 1993. As Wittgenstein (LFM, p. 144) points out, "although it is said in Euclid that a straight line can be drawn between any two points, in fact the line *already exists even if no one has drawn it*."

This notion implies that possibility stands as a shadow of reality, or as Wittgenstein states, "[t]his is a most important idea: the idea of possibility as a different kind of reality; and we might call it a shadow of reality" (LFM, p. 145). In other words, every possibility already exists, parallel to what actually exists. Nonetheless, is this "realm" any different from a platonic realm? By assuming the possibility of proving something, we are implying that there *exists* a possibility of that. However, what is meant by the sentence "There exists the possibility of proving Goldbach's Conjecture" is still not clear.

By assuming that possibility exists as a shadow of reality, Modalism does not give us a better answer than Platonism. It only postpones the epistemological and ontological problems related to abstract objects, by using a blurred concept (of, possibly, abstract possibilities or abstract possible structures). It affirms that mathematics is the language that deals with possible relations in a structure. However, what is this (abstract?) structure? For it seems that the argument tries to imply that this structure is, in some sense, part of reality, and not just a language-game.

As Putnam argues, mathematics is the language that talks about what follows from what. In other words, by assuming some axioms, we determine which propositions can be derived in this language. However, this language is taken to be about to reality, in some sense. In this regard, a structure would be a set of axioms and theorems that enables us *to map reality*. His idea was that what makes a mathematical structure *true*, and therefore independent of our knowledge, is the fact that it is used in scientific theories.

However, this interpretation entails that pure mathematics is not part of this structure (as Quine said, it is only recreation with signs) and, therefore, cannot be regarded as true. But in addition, it does not tell us why mathematical propositions are necessarily true, and scientific propositions are only contingently true. In other words, if both propositions are true by corresponding to something in reality, then why is mathematical truth stronger than contingent truth? If both propositions are true by corresponding to something in reality, then why is mathematical truth necessary and contingent truth is only contingent?

In this sense, the idea of "possible relations in a structure" is still blurred: If we assume that those relations already exist, they exist as a shadow of reality (*i.e.* they do not exist in our *physical* reality now, but only as a possibility), then it seems that we are using blurred concepts and vague gestures in order to avoid the question. We do not have a concrete way of talking about possibility and structures in relation to our physical reality. In this sense, the idea of a structures only raises more questions, and those questions, I believe, will inevitably arrive at a form of Platonism.

4.3. Limitations of a General Theory of Truth: A Circularity in Benacerraf's Semantical Condition

As shown above, Benacerraf's semantic condition relies on a general theory of truth. A general theory of truth would enable us to analyze contingent and mathematical propositions in the same way, despite their differences (*e.g.* abstract objects vs physical objects, *a priori* vs *a posteriori* truth, etc.). This enables a uniform treatment of truth in any language: mathematical truth is no lesser *truth* than truth in ordinary language. This led to the conclusion that the same logical apparatus enables us to treat mathematical and contingent truth alike.

This condition reveals a criticism of formalistic interpretations of mathematics. Formalism, by taking mathematics to be just symbols and rules, would not be able to treat truth in mathematics as being equal to truth in science/ordinary language, since, from the point of view of Formalism, mathematics has no objects. More precisely, by stating that there is nothing more to mathematics than symbols and rules, there is nothing that mathematical propositions correspond to.[54] The argument attempts to show that the notion

[54] A better concept to use here is the notion of satisfaction, or reference to a ω-sequence. There is a difference between a correspondence theory of truth and Tarski's theory of truth. The former assumes a correspondence

of theoremhood cannot explain mathematics supposed[55] relation to reality.[56] In other words, merely being part of an axiomatic system does not explain why mathematics is true (or why it is useful in making predictions about and descriptions of reality).

Benacerraf's argument seems to imply that, if we understand mathematics as a system of symbols and rules without reference, we are left with no explanation of what mathematical truth is. For him, a formalistic interpretation cannot account for truth in mathematics in the same way that truth is explained for contingent propositions.

However, Benacerraf's semantical condition appears to be circular. If we require that truth in mathematics must agree with a general theory of truth, then we *assume* that truth in mathematics means truth *by correspondence*. In other words, by assuming that mathematical propositions should agree with a correspondence theory of truth, we are rejecting any formalistic account, without justifying that rejection by a cogent argument for why we need *truth in mathematics*.

The conclusion of Benacerraf's semantic condition says that truth in mathematics must be equal to truth in contingent propositions. However, in order to arrive at this conclusion, Benacerraf must first assume that *mathematical propositions are true*. In other words, Benacerraf does not consider the possibility that mathematical propositions are not true or false.

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relation between true propositions and reality, while the latter treats truth as a predicate of the metalanguage. Tarski' theory of truth by itself does not entail a referential interpretation of true propositions. It needs to be supplemented by an extensional interpretation of quantificational operators in first order logic. I will not discuss if Tarski' theory of truth needs a correspondence interpretation.

[55] By using the term "supposed' I am not questioning the application of mathematics to reality. I merely want to question the idea of *correspondence*.

[56] What Benacerraf seems to argue here is that we cannot just assume that the theorems in a calculus are true *because* we prove it in relation to a calculus. For him, there is no connection between truth and being a theorem in the calculus. One needs to show how theoremhood shows the truth of mathematical statements.

We have no reason to assume that mathematical propositions must have a truth value. Benacerraf argues that mathematical truth is no less true than truth in contingent propositions, based on the fact that mathematical and contingent propositions have the same logical structure. However, that does not tell us why mathematical propositions must be true or false. The fact that contingent and mathematical propositions have a similar logical structure does not force us to assume that mathematical propositions are true.

Thus, Benacerraf's semantic criterion seems to be an attempt at an impossible task. By requiring that anti-platonistic theories agree with a correspondence theory of truth in mathematics, Benacerraf's first criterion begs the question. For if, by definition, anti-Platonism (Formalism) assumes that mathematical symbolism does not refer to anything, it can never satisfy the semantical condition.

CHAPTER 5: Wittgenstein Philosophy of Mathematics

5.1. Wittgenstein's Radical Constructivism in Mathematics

Given the problems raised in the last sections, I believe that there are more interesting answers to the question "What is mathematics?" In particular, I will show that Wittgenstein's descriptive account of mathematics is a very interesting and viable alternative to both Platonism and Modalism.

According to Wittgenstein, a mathematical calculus (system) consists of *only* symbols and syntactical rules. This claim constitutes the core of Wittgenstein's philosophy of mathematics. If we look to mathematics, and if we especially look at what mathematicians do, we will see that there is nothing more than symbols and rules.

According to Rodych (1997, p. 196), on Wittgenstein's formalistic interpretation of mathematics, "[a] mathematical calculus is defined by its accepted or stipulated propositions and rules of operations. Mathematics is syntactical, not semantical: the meaningfulness of propositions within a calculus is an entirely intrasystemic matter".

On Wittgenstein's view, we invent mathematics "bit-by-little-bit"[57]; we construct it. There is nothing more to mathematics than the symbols that we have put on paper and the rules that we use to construct that sequence of symbols. There is nothing to be *discovered in* mathematics, in the sense that we cannot *find a proof or path that was already there*. As Wittgenstein points out "[m]athematics is always a machine. The calculus does

[57] I borrowed this interesting term from Professor Victor Rodych (2008). It gives this idea of something being *constructed* very gradually.

not describe anything” (Wittgenstein, 1979, p. 106). In other words, mathematics does not talk about anything.

However, before moving forward and explicating Wittgenstein’s philosophy of mathematics, it is important to highlight some of his main ideas. The first impression that one gets of Wittgenstein’s radical remarks is that Wittgenstein is *arguing in favour of a philosophical view*. It seems that Wittgenstein is *arguing* in defense of particular ideas and propositions.

I will endeavour to show that, for the most part, Wittgenstein *is only describing mathematics*. Wittgenstein’s argument does not prescribe any change in what constitutes a mathematical proof, or what *should be* accepted as a proof. His arguments do not demand a change to *what mathematics already is*.

As is going to be clear later on, by arguing in favour of a constructivist account of mathematics, Wittgenstein is not questioning or suggesting changes to what is *actually done in mathematics*. He is only questioning what mathematicians *say* about their work; he is only showing the linguistic confusions that arise when (some) mathematicians and philosophers attempt to describe mathematics.

In the next sections I will present and evaluate some of the main consequences of Wittgenstein’s description of mathematics. The first aspect to notice is that, for Wittgenstein, a) mathematics is *not about* anything. The symbols and rules that we use to construct a mathematical system do not *refer* to anything. In addition, b) this compels us to conclude that there is nothing else to mathematics besides what have been done (what have been constructed, or what have been actually proved). Furthermore, c) we do not *discover* mathematical *truths*, we *construct correct* mathematical propositions. Finally, d) I will argue that Wittgenstein's philosophical description succeeds in only describing mathematics.

5.2. Symbols and Rules: Non-Referential Nouns in Mathematical Propositions [58]

One *absolutely undisputed* aspect of mathematics is that mathematicians work with specific symbols under precise and well-defined rules of operation (transformation rules). It seems that there is no debate about this simple idea: in mathematics, we work with symbols and rules to construct proofs. What makes Wittgenstein's philosophy of mathematics so original (and radical) is that, for him, that is all that mathematics is.[59]

As Rodych (2008, p. 87) argues, for Wittgenstein, mathematical propositions are not genuine propositions.[60] According to Wittgenstein, "the only genuine propositions that we can use to make assertions about reality are 'empirical' (contingent) propositions, which are true if they agree with reality and false otherwise" (Rodych, 2008, p. 87). Mathematical propositions as simple as '2 + 2 = 4', or as complex as the most complex mathematical theorems, do not refer to a particular *existent* state of affairs. Put very concretely and clearly, mathematical propositions do not talk about reality.

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[58] I thank Dr. Gillman Payette for our many discussions that gave me the tittle of this section.

[59] Here we see again that Wittgenstein is making a strong claim. Nonetheless, his argument only attempts to show that that is all that mathematics is. In other words, as I am going to show in the next sections, Wittgenstein is not trying to change anything related to *how we do mathematics*: *how we construct proofs*.

[60] "Mathematics is a logical method. The propositions of mathematics are equations, *and therefore pseudo-propositions*" (Tractatus, §6.2 [italics mine]), or that "[a]n equation merely marks the point of view from which I consider the two expressions: it marks their equivalence in meaning" (Tractatus, §6.2323).

An important consequence that follows from this is that mathematical propositions do not have a truth-value, for they *cannot* have a truth-value. We use mathematical propositions, meaning that we perform operations on symbols by following syntactical rules in a particular calculus. We do not "check" reality in order to construct a mathematical proof—we do not examine reality to know if a particular mathematical proposition is true or false. We do not "go outside" of mathematics to "discover" that a particular proof is correct or not.

In simple terms, a mathematical proposition cannot have a truth-value for it does not talk about anything. For example, if the mathematical proposition "2 + 2 = 4" is true, there must be a state of affairs to which that proposition points—there must be a reality that makes that proposition true. However, if we reject the idea that there is a different (platonic) reality in which mathematical objects dwell, *e.g.* as *Two* or *Two-ness*, then what is the proposition talking about?

In fact, we cannot meaningfully ascribe a truth-value to a mathematical proposition. For example, "Tokyo is a city in Japan" is a true proposition because the content of the proposition corresponds to an actual state of affairs. We know that "Tokyo" is the name of a particular territory in Japan, which shows that Tokyo is a city in Japan.

On Wittgenstein's view, the comparison between a mathematical proposition and a contingent proposition "is a matter only of a very superficial relationship" (RFM, III - §4). The use of the term 'truth' for both contingent propositions and mathematical concatenations of signs generates a conceptual-linguistic problem – by giving us the impression that how we use the term, in both situations, is the same.

To enable us to talk, with Wittgenstein, about strings of signs (symbols), strings of signs that are wffs, strings of signs that are not wffs, and string of symbols that are genuine mathematical proposition, Rodych (2008, p. 85) coined the term 'Csign,' which he defines as follows:[61]

'**Csign' = df.** A finite concatenation of signs.

This definition requires a definition of 'sign,' which is as follows.

'Sign' = df.

A sign is a symbol in one of the following finite sets:

1. A mathematical symbol used in an article in a Mathematics Journal from 1800 through 2021. (Here "mathematical symbol" excludes letters in words of natural language prose.)
2. Any letter of the English, German, Portuguese, Italian, Spanish, or French alphabets.

With this term in hand, Rodych (2008, p. 85) uses "Csign" to a) refer to so-called "mathematical" conjectures, in order to avoid the confusion of talking about concatenation of signs that are mathematical propositions, and those that are not; b) to avoid using the term "well-formed formulas", since mathematicians and logicians regard, *e.g.*, a well-formed formula of Peano Arithmetic as a number-theoretical proposition with a truth-value; and c) it demarcates a clear difference between genuine mathematical propositions, and concatenations of symbols that are *not* mathematical propositions. To make this distinction clear, I will henceforth use the term 'mathematical proposition' for a Csign that is a part of a calculus/mathematical system, while using the term 'Csign' for a concatenation of signs which may or may not be a mathematical proposition. This enables

[61] Dr. Victor Rodych has communicated this definition to me. His (Rodych, 2008) did not provide a precise definition.

us to state, e.g., Wittgenstein's view that a mathematical conjecture, such as Goldbach's conjecture, is a Csign (in, say, the language of PA), but *not* a mathematical proposition proved in a mathematical calculus. As will be seen, it also enables us to talk about how one and the same Csign can *not* be a mathematical proposition before it is proved; and then becomes a mathematical proposition, with mathematical sense, after it is proved in a mathematical calculus.

5.2.1. Truth vs. Correctness: How to Evaluate an Incorrect Mathematical Csign

Despite the fact that mathematical conjectures do not have a truth-value, we still want to discriminate propositions such as "2 + 2 = 5" from "2 + 2 = 4". At first, we may think that "2 + 2 = 4" is true, for that is what we were taught. I learned that two apples plus two apples give four apples, and then I learned that the correct symbol to replace 'x' in the equation '2 + 2 = x' is four —— namely, '4'. It is false that that kind of sum would give us five apples, or any other number. We simply know that to be the case. So, we must ask ourselves: How do we arrive at the answer "it is four apples" or "it is four".

The first idea that may come to mind is that our answer is based on the empirical fact that two apples plus two apples gives us four apples. There is no other correct answer. We could check to determine that, no matter the object, every two things that we add to two other things give us four things.

However, it appears that this would only "explain" simple operations using the natural numbers. If we are dealing with more complex sets of numbers, *e.g.* irrationals and imaginary, we are still asking the same questions; *e.g.* What does π or i refer to?

In addition to this, an important aspect of mathematics is how we prove a particular mathematical proposition. The rules followed and the operations executed in a proof show that a mathematical proof is not about anything in reality. We do not need to *check* that two objects plus two objects give us four objects for *every instance/application* of those numbers in contingent propositions. Mathematicians do not want to say that we can check reality to determine that "2 + 2 = 4" is true. They will assert that this is *necessarily true*. No past, present or future experience can give us a different answer than '4 things'. In another sense, it appears that the empirical content[62] of the proposition does not matter for its proof. Mathematicians do not need to check reality — at any time — when they construct a proof.

Another way to evaluate our answer is that we were taught that "2 + 2 = 4". As a child, we slowly learned the mathematical symbols, how to put them in a sequence and how to do basic operations with them. At first, using objects as reference of the results of the operations (*e.g.* two bananas equal two oranges); and later, progressively using only symbols (*e.g.* $x^2 + x + cx = 0$).[63] Therefore, "2 + 2 = 4" is true for it agrees with the rules of the system.

As Wittgenstein (2009, pp. 63-64, §145, 146) reminds us, if children fail to construct the sequence of natural number from 1 to 10, we simple correct them by showing them the right way. We taught them by saying which is the correct symbol that follows the previous one, *e.g.* that 9 comes after 8, or 126 comes after 125. If a child questions why

[62] For the term 'empirical content' I am only suggesting the things that can be designated by a number and are cognitively accessible.
[63] Although there are several ways to teach mathematics to someone, all of them have the same structure, starting first by relating symbols with concrete objects and then moving to using only symbols (*i.e.*, an abstraction from physical objects to symbols).

the sequence of numbers that he constructed is not correct, we just say that that is not a move allowed in the system.

That leads us to conclude that there is no difference between the incorrect sequence that the child built and the correct one that we learned. We just learned that what we learned is a mathematical calculus, and we call any Csign that belongs in that system "true". In saying that the mathematical proposition "2 + 2 = 4" is true, we merely mean that *it is correct*,[64] for it follows the rules of the calculus and it is itself one of the rules of the calculus, a point Wittgenstein frequently emphasizes. A correct mathematical proposition dictates the correct *use* of symbols in the calculus. The notion of truth here seems out of place, for a proof shows us that a mathematical proposition is part of calculus (correct) or is not part of a calculus (incorrect)—that a mathematical proposition is a correct mathematical construction or an incorrect mathematical construction.

A child is making a mistake for he does not know *how to use* certain symbols. This idea tells us that we learn how to differentiate an incorrect mathematical proposition from a correct one: by correctly *using* them; and we can only do that by knowing the rules of the system. We are not using *true propositions* to correct the child, in the way that we would do so for any false description of reality. For example, if a child said that the sky is red, we would then correct her statement by *showing* that the sky is in fact blue, and perhaps contrasting it with a red stop sign; that is, we would correct the child by showing how what was said does not agree with reality.

[64] "And the possibility of proving the propositions of mathematics means simply that their correctness can be perceived without its being necessary that what they express should itself be compared with the facts in order to determine its correctness" (Tractatus, §6.2321).

5.2.2. Counting Apples and Oranges: Mathematical Meaning in Contingent Propositions

We saw earlier that, for the standard view, the meaning of a mathematical proposition is partly *referential*. In other words, the meaning of a mathematical proposition constitutes something *about* a mathematical reality. The main idea was that mathematics is *about* something (even if it is not about something in physical reality), and by calculating we are discovering certain properties of those things. The standard view claims that mathematical language stands in relation to *a certain* reality, or a certain *possible* reality, assuming that a mathematical proposition refers to aspects (facts) of those things (mathematical objects), which collectively constitute that reality.[65]

However, as was shown, the referential aspect of mathematical propositions raises more problems than it solves. If we assume that the meaning of a mathematical proposition is partly a reference to abstract objects, then we must further elaborate how mathematical propositions are about abstract objects. E.g.: How does a proof relate to, or refer to, a mathematical fact?

One attempt to answer these questions is John Stuart Mill's idea that mathematics is essentially justified by *inductive reasoning*. For him, despite the fact that we use

[65] It is hard to avoid a vague language when talking about the supposed objects of mathematics. Any comment or any assertion may lead to the idea that such things as mathematical objects, mathematical facts, do exist. In addition, because it is impossible to know what those objects are, we cannot meaningfully describe them using clear terms.

deductive terms to make proofs and construct new propositions, a mathematical proposition is only as strong as its inductive justification.[66]

One interesting aspect of Mill's view is that it is closely related to how we commonly think about mathematics. If questioned about why it is the case that '2 + 2 = 4', a person with no mathematical or philosophical background will probably answer: 'Well, if you have two apples and put two more apples, then you have four apples'. In other words, the person's answer justifies the truth of that mathematical proposition by appealing to a *physical and empirical fact,* or to multiple physical/empirical facts. The mathematical proposition '2 + 2 = 4' is merely a generalization of the empirical fact that two things plus two things equal four things. The proof shows that '2 + 2 = 4' holds because it represents a generalization of those empirical facts (*i.e.* the proof is justified by the inductive reasoning of putting four things together). For, as Mill (1981, VII, p. 254) says, "there is in every step of an arithmetical or algebraical calculation a real induction, a real inference of facts from facts".

To the contrary, Wittgenstein claims that both a mathematical proposition and a Csign do not refer to anything. Mathematical propositions are not *about* reality in any sense. Despite the fact that we use mathematics in contingent propositions, to make predictions and to refer to objects in reality (*e.g.* "Can I have those *two* pens on the table?"), it seems reasonably clear that a mathematical proof does not indicate a relation between a sequence of symbols and a particular state of affairs. How we *use mathematics in*

[66] For this small summary of Mill's conjectures about mathematics, I mostly use Christopher Macleod's *Stanford Encyclopedia Article*, "John Stuart Mill".

contingent statements only shows how useful it can be in our daily lives, not the *meaning* of mathematical propositions.

This idea that the sentence ‘two bananas plus two bananas equal four bananas’ gives us the meaning of the mathematical proposition ‘2 + 2 = 4’ has at least two problems: a) first, it assumes that a proof of a simple equation, *e.g*. 2 + 2 = 4, proves an empirical fact, or needs an empirical fact. This notion, however, does not consider the actual *proof* that gives (yields) the result: that is, the set of rules and mathematical propositions that position that mathematical proposition in a mathematical system.

However, as Frege (1960 [1884], p. 10) argues, “if the definition of each individual number did really assert a special physical fact, then we should never be able sufficiently to admire, for his knowledge of nature, a man who calculates with nine-figure numbers”. In other words, Frege suggests that simple arithmetic operations, such as addition of 9- or 10- or 12-digit numbers, does not have a physical event that justifies the result. The man lacks *knowledge of nature*, that is, he does not know a physical fact that describes the mathematical operation that he just executed.[67]

In addition to this, b) this approach to the question of meaning does not consider situations in which it is not the case that ‘two things plus two things equal four things’. For example, if I add two drops of water plus two drops of water in a crucible, do I have four drops inside the crucible? We would be inclined to say “yes!’, however the only thing that

[67] I am not arguing that there is *no* physical fact that corresponds to this operation. Rather, it certainly exists, but we do not proceed to *count* it. In other words, the whole operation is done without knowing any empirical/physical fact that would *justify* the result of the operation. In addition, to count such operations is not even feasible, for it would take more than a person’s entire life (much more than that).

we can *see* in the bucket is one very small puddle. This idea cannot adequately be applied to objects that cannot be clearly separated from one another.

In summary, the fact that we use mathematics to make predictions, to make requests, and to do all sorts of things, does not tell us the *meaning* of a mathematical proposition (*e.g.* it does not tell me anything about its *proof*). The result of a mathematical operation is only the last piece of an intricate sequence of constructions in accordance with rules, which cannot be excluded from the meaning of the proposition. To affirm otherwise, would be to a) assume that pure mathematics is meaningless or idle "recreation" because mathematical propositions do not have an application in contingent propositions, or b) to assume that a proof is not part of the meaning of a mathematical proposition.

These considerations show that the *use* of mathematics in contingent propositions does not give us the meaning of mathematical propositions, it only shows *how useful this system of symbols and rules can be in contingent propositions*. The limitations of a referential interpretation of mathematics seems to show that the nature of mathematics does not require something *external*. The meaning of a mathematical proposition is not something *outside of mathematics*.

5.2.3. Mathematical Meaning: Proof and Intra-Systemic Sense

At first blush, it seems that the meaning of a mathematical proposition "must" tell us what mathematics is *about*. In the case of the standard view, mathematics is about mathematical objects and/or facts. However, if we apply the same analysis to a formalistic interpretation of mathematics, it sometimes then mathematics is about the symbols and rules that we write on a piece of paper. In other words, mathematics would tell us something *about* those concatenations of signs. As Wittgenstein states:

> One asks such a thing as what mathematics is about – and someone replies that it is about numbers. Then someone comes along and says that it is not about numbers but about numerals; for numbers seem very mysterious things. *And then it seems that mathematical propositions are about scratches on the blackboard. That must seem ridiculous even to those who hold it, but they hold it because there seems to be no way out.* (LFM, p. 112)

The first horn of this dilemma, that mathematics is about numbers (objects), generates the problems and limitations that were stated in the previous sections (about platonic objects and facts). However, the second horn of this dilemma forces us to be committed to the "ridiculous" idea that mathematics is about those scratches (symbols) that we write on a piece of paper.

Wittgenstein criticizes the idea that mathematics is *about* something. It is the alleged *aboutness* of mathematics that generates this false dilemma. In his view, we should abandon the idea that a mathematical proposition refers to anything or talks about anything. Mathematics is not *about* anything. A meaningful mathematical proposition does not refer to or talk *about* anything *external to mathematics*, rather it has only *intrasystemic meaning*. More precisely, a mathematical proposition is meaningful only if we know[68] of *a decision procedure* that *positions* it in a calculus.

As was said in the sections above, we invent mathematics, and in so doing we construct calculi and formulate rules for each calculus. What a correct mathematical proposition is and what it is not is determined by the rules of the system. According to

[68] For Wittgenstein, one important aspect of a proof is that we *must know* it. We must know a decision procedure that puts the Csign in the calculus. Without this epistemological aspect, we cannot say that we have a meaningful mathematical Csign. I will better elaborate this idea in the next sections.

Wittgenstein, the first criterion for something to be a mathematical "proposition" (*i.e.* to be mathematically meaningful) is that we must have a *known* mechanical process that, by a finite number of steps, determines if a particular mathematical proposition belongs or does not belong to a calculus.

For example, imagine a computer that has the capacity to solve and prove several equations[69]. This computer operates under a set of rules and has a set of symbols as its language. If we were to give it a concatenation of signs (of the language that it has), it would be able to tell us if that Csign is derived in the system/calculus or not. How does it do that? By a certain number of operations (steps) under the rules that it has. If we were to give a Csign to the computer that it is not derivable in his system, would we consider that Csign to be a mathematical proposition (according to the computer)? It seems not: the computer will have no use for that Csign[70]. It would not enable it to construct a new sequence of signs, derive new proofs, etc. It would just be discarded.

This raises the question: what do mathematicians actually do? Do they do something different from the computer, when they are constructing the proof of a particular mathematical proposition?[71] In a more concrete picture, do we do something different when we multiply 253 × 120? Do we not calculate it by constructing a proof in a particular way (by writing the first term on top, the second term right below, multiplying every single

[69] This example was given by Professor Victor Rodych several times during our meetings. Despite that I am using his example here in a different sense, than when he explained it to me, I still thought that I could use his example to show how the process of calculating and proving something might be exemplified.
[70] The concept of "use" here is very important. The idea is that a Csign that has been positioned in a mathematical system has syntactical connections in that system. In this sense, the computer tell us *if* that Csign can be *positioned* in the calculus.
[71] I am not referring here to the process of coming up with a proof. Obviously, the way that we humans operate with symbols is different from a machine. I merely wanted to show the *mechanical process* of constructing a proof, and what those Csigns are in a system.

number separately and adding the result at the end)? And we can only do this because we know how a multiplication works (we know how to apply the rule of multiplication for natural numbers).

The central idea is that, by using the rules of the calculus, we decide whether a Csign is mathematically meaningful—i.e. whether it is a mathematical proposition. A Csign is a mathematical proposition iff we have a known decision procedure that tells us whether that Csign is correct or incorrect relative to a particular calculus.[72] If there is no known proof or known decision procedure for a Csign, that Csign is *not* a mathematical proposition.

This poses a question about the status of undecided mathematical Csigns. For example, one of the most famous *unsolved problems* in mathematics is Goldbach's Conjecture (henceforth "GC"). GC states: "Every even number greater than 2, is the sum of two primes". Despite its simple formulation, Goldbach's Conjecture has not been proved or refuted.

According, however, to Wittgenstein's criterion of mathematical meaningfulness, GC is not a mathematical proposition. To mathematicians and most philosophers, this is very counterintuitive, for it appears that GC is meaningful, and its meaning seems very clear and very precise. Mathematicians would certainly say: 'I understand what "every" means, as well as "number", "greater than", "2", "prime", etc.' GC seems to clearly have mathematical meaning, for it *talks about numbers*—it makes a very precise claim about all

[72] Here, it seems again that Wittgenstein is telling mathematicians what should be considered a mathematical proposition: a mathematical proposition (Csign) must have a known decision procedure, otherwise it would be a concatenation of symbols (*i.e.* mathematical meaningless). However, as I tried to point in my computer example, Wittgenstein is doing nothing different from what he said: he is attempting to clear language confusions.

even numbers great than 2. Furthermore, we can even check if GC holds a large number of even numbers, for it is a simple matter to verify that it holds for 4 (1 + 3), 6 (5 + 1), 8 (5 + 3), 10 (7 + 3). Indeed, in 2020, we know that GC holds for every even number less than 4×10^8.

In light of this, how is GC not mathematically meaningful? How is it possible that GC is not a mathematical proposition? The standard view would argue that we can understand GC and what it says, and we know how to determine whether it holds for a particular even number or for a particular finite range of even numbers. The fact that we do not have a proof of it, or a decision procedure for it, does not tell us that it is not a mathematical proposition.

For Wittgenstein, the central flaw of this kind of argument is how it relates mathematics to the empirical sciences. The confusion here is the assumption that mathematical meaning is, in some sense, similar to meaning in contingent propositions.

5.2.3.1. An Elusive Comparison: Mathematical Meaning and Propositional Meaning

In the previous section, I tried to show what makes a Csign a mathematical proposition (*i.e.* what a mathematically meaningful Csign is). The idea that a proof or decision procedure is necessary for a Csign to be mathematically meaningful, raises the problem that some core examples of mathematical problems would not be considered a mathematical propositions. Those Csigns, such as GC, do not have a decision procedure and, therefore, they are not mathematical propositions at all.

The central idea is that a decision procedure is a necessary condition for a Csign to be a mathematical proposition. This criterion is controversial because it seems that we can clearly understand certain non-mathematical Csigns (in symbols or in a natural language), such as “Every even number greater than two is the sum of two primes”.

Proponents of the standard (referentialistic) view on mathematics argue that GC is clearly meaningful, and that it has a clear and precise meaning. GC makes a precise claim about all even numbers greater than 2. The fact that we do not have a proof only indicates that we do not yet know whether GC is true or false. We understand GC completely right now, and a proof of GC would not change its meaning. GC has a fully determinate sense in the absence of a decision of its truth-value. [73] [74]

It is important to note that the concept of “mathematical meaningfulness” should not be confused with the ordinary use of the word “meaning”. The opposite of “mathematically meaningful” is not “meaningless”. A Csign can be constructed using a mathematical language (*i.e.* it can be a well-formed formula), but if it does not have a decision procedure or a proof, it cannot be *used* in any particular mathematical system (we cannot position it or its negation in a calculus, perform operations with it, prove other mathematical proposition *from* it, etc.). A Csign that is not a mathematical proposition cannot be used to make a move in a mathematical language-game (*i.e.* in a mathematical calculus). This means that, despite its appearance and beliefs about its mathematical

[73] I thank again Dr. Victor Rodych for this explanation. The distinction between “mathematically meaningful” and “full of meaning” was one of his ways of explaining how easy it is to misunderstand Wittgenstein’s arguments.

[74] A criticism of Wittgenstein’s notion of mathematical problem is going to be evaluate in the section “Is There Only Certainty in Mathematics? Criticism to the idea of Mathematics as a Human Invention”. For now, I will retain my analysis only to answer the problems related to mathematical meaning and contingent meaning.

meaningfulness, such a Csign is not a mathematical proposition. As Wittgenstein argues, "[i]f you want to know what 2 + 2 = 4 means, you have to ask how we work it out" (PG, p. 333). If we *cannot* use a Csign (or its negation) in a calculus, how can it be a *mathematically meaningful* proposition? For all we know, such a Csign may one day be proved to be syntactically independent of *all* existent mathematical calculi.

The claim that a Csign is not a mathematical proposition does not entail that it is not understandable, or unintelligible. We understand, e.g., that GC would be proved to be *incorrect* by a single counter-example—by a proof that one even number greater than 2 is *not* the sum of two primes. The misleading idea is to understand "mathematically meaningful" as "complete(d) meaning".[75] We confuse the use of ordinary language in Csigns, understanding similar terms as having the same meaning. As Wittgenstein argues, "[w]e are used to saying '2 times 2 is 4', and the verb 'is' makes this into a proposition, and apparently establishes a close kindship with everything that we call a 'proposition'" (RFM, I - App. III - §4). Thus, in response to the claim that GC's meaning is complete right now, we may ask: Is GC's meaning truly complete in the absence of a mathematical decision? For example, why do we presently have a far better idea of what would refute GC than of what might prove it (and in what system it might one day be proved).?[76]

What misleads us here is that we think we understand a Csign in the same way that we understand an empirical (contingent) proposition. By using the term "is", in "every even

[75] I thank Professor Victor Rodych for this explanation. The distinction between "mathematical meaning" and "complete(d) meaning" was one of his ways to explain how easy it is to misunderstand Wittgenstein's views on mathematical conjectures, such as GC.
[76] Wiles' proof of FLT is strong evidence of this. Before Wiles' proof, it would have been impossible for anyone to even sketch a proof of FLT in whatever system it has actually be proved in. Even in 2021, experts are not certain whether Wiles' proof requires a system stronger than ZFC. See (Glivický and Kala, 2016) and footnote #87.

number greater than two *is* the sum of two primes”, we are led to believe that the infinite totality of even numbers is a thing, in a drawer (like a drawer filled with socks), and that our Csigns, such as GC, make specific claims about a concrete and fully determinate thing. We say, e.g., that GC makes a precise claim about this determinate thing, despite the fact that GC cannot presently be used in any calculus.

Wittgenstein shows us that the resemblances are a matter of a “superficial relationship” (RFM, I - App. III - §4). If we analyze the proposition “Scott *is* the author of Waverly”, the term “is” shows that both the name and the description refer to the same thing. The proposition either (1) expresses the fact that two terms refer to one and the same thing or (2) states that two terms refer to one and the same thing. Furthermore, the proposition is meaningful, despite the fact that we do not know whether it is true or false. It tells me something about this person (Scott). If true, it tells us that Scott is the author of *Waverly*. If false, Scott is not the author of *Waverly*. More precisely, it gives us a picture: whether or not that the proposition agrees with reality, we still know what *would make it true or false*. We understand the proposition because we understand its truth conditions.

As I said in previous sections, according to the standard view, the meaning of a mathematical conjecture such as GC is complete. A proof or a refutation does not change its meaning, it only shows it to be true (or false), in the same way that an empirical verification shows that an empirical proposition is true without altering that proposition’s meaning. The Csign that is GC says something *about* the natural numbers and its meaning is complete before it is proved or refuted.

Some of the problems with the aboutness of mathematical propositions were already stated in previous sections. The claim that mathematics is about something commits us to the existence of abstract objects and all of the problems associated with such an existence claim.[77]

Unlike a mathematical proposition, a contingent proposition makes a claim about reality. We can know the truth conditions of a contingent proposition prior to knowing that it is true. We can picture what would make the proposition "Scott is the author of *Waverly*" true, *e.g*. there is a document that attests that Scott is the author of *Waverly*, or we have observed Scott writing the book entitled *Waverly*. However, the same cannot be said about mathematical propositions. As Wittgenstein argues: "the mathematical proof couldn't be described before it is discovered" (PG, p. 371). We only know the proof of a mathematical proposition when we have constructed it.

In this regard, an analysis of mathematical propositions similar to the analysis of contingent propositions is misleading. These two kinds of "propositions" are very different. In mathematics, the symbols do not refer to anything in reality. By saying that "two times two is four", we are given the impression that there exist two objects that are (or constitute) this other thing (object). But, as Wittgenstein says, "[i]n mathematics everything is algorithm and nothing is meaning; even when it doesn't look like that because we seem to be using words to talk about mathematical things" (PG, p. 468).

In this sense, are we doing something else in mathematics, different from the **obvious** transformation of one symbol into another (in accordance with the rules of the

[77] I am aware that some authors (such as Putnam, 1975; 2006) argue that a modal interpretation of mathematical propositions may retain the referentialistic aspect of a Csign, without committing our analysis to abstract entities. However, as already stated in above sections, I believe that they all have the same ontological problem that Platonism has.

system)? As already stated, it is a proof that enables us to conclude that “2 + 2 = 4” is *correct*, and a proof seems to be only an application of precise rules.

Wittgenstein does not say that we should refrain from using certain words and terms (*e.g.* mathematical *proposition*, *true* or *false*). It is not a matter of a natural-language grammatical mistake. Rather, the point is that the use of those terms, both in contingent propositions and in mathematical propositions (and in mathematical conjectures), give us the illusory idea that they have the same meaning. As he asks, “[s]hould we not shake our heads, though, when someone shewed us a multiplication done wrong, as we do when someone tells us it is raining, if it is not raining? — Yes; and here is a point of connexion. But we also make gestures to stop our dog, e.g. when he behaves as we do not wish” (RFM, III - §4).

By conflating mathematical meaning with the referential meaning of contingent propositions, we are led to believe that contingent propositions and so-called mathematical propositions are the same. Still, pointing out that they are different does not, in and of itself, tell us what the *sense* of a mathematical proposition is.

5.3. Mathematical Sense: The Position of a Mathematical Proposition in a Mathematical System

For Wittgenstein, a Csign is a mathematical proposition iff we have a decision procedure for it. In other words, a mathematical Csign must be algorithmically decidable in a mathematical calculus, otherwise it is just a piece of pseudo-mathematical symbolism.

In addition to that, there is an epistemological condition: we must *know* that we have a decision procedure for that Csign. We must be aware of the use of the rules and aware of the results derived. As Rodych (2008, p. 90) argues "[a] Csign is a mathematical proposition *of* calculus Γ *iff* it is algorithmically decidable *in* calculus Γ *and we know this to be the case*".

However, these two criteria do not tell us what the *sense* of a mathematical Csign is. For now, we can only determine whether a particular Csign is mathematically meaningful or not (i.e., whether or not it is a mathematical proposition). Nonetheless, the fact that a particular Csign has a known decision procedure does not tell us how to use a Csign in a particular calculus, or whether a Csign is incorrect and has no use in a particular calculus. More precisely, if a Csign is correct, that fact does not tell us how to *use* that mathematical Csign in the calculus.

In the previous section I stated that a mathematical proposition (i.e., a Csign that is a mathematical proposition) does not have *sense* in the same way that a contingent proposition has meaning. Mathematical propositions are semantically empty. However, we know that each mathematical proposition is *used* in a unique way.

When I use a particular mathematical Csign to perform certain operations, what am I doing? How do I know how to use a particular Csign? For example, what is this operation that, in order to solve the equation "2/3 + 5/6 = x", we multiply the first term by 2/2 (2/3 × 2/2 = 4/6), and then add the new first term to the second term (4/6 + 5/6), and arrive at the result '9/6'? How do I know that these are correct symbolic transformations of this symbolic equation?

The immediate answer is that the rules of the system (*e.g.* Peano Arithmetic) enable us to transform the Csign in the way that I transformed above. We know that, to solve equations that have a sum of two distinct fractions, with different denominators, we must find or calculate a common denominator for both. The transformations and calculations given above constitute a proof of the mathematical proposition "2/3 + 5/6 = 9/6", with each step being the application of a rule. The rules *yielded* the final line of the proof.

In summary, the sense of a mathematical proposition is the *position* that it occupies in a particular mathematical system or calculus (as Wittgenstein tends to call it). We can better understand Wittgenstein's conception of a mathematical calculus (system) if we use the analogy of as a complex web. Every node in the web is a proved proposition. All the nodes are connected by syntactical derivations or calculations; each proposition is derived from one or more other propositions.[78] By algorithmically deciding a mathematical proposition we position it or its syntactical negation in a particular mathematical system.

That is what was done in the operation above. We positioned the proposition "2/3 + 5/6 = 9/6" in the calculus. The operation by which I did that, the operations performed in accordance with rules, is the "path" that goes from the axioms (*e.g.* the axioms of Peano Arithmetic) to that sequence of signs. (i.e., to that Csign, which is a mathematical proposition in a mathematical calculus). That is the sense of a mathematical proposition: the position of that mathematical proposition in a calculus, including the derivation of that

[78] "The method by which mathematics arrives at its equations is the method of substitution. For equations express the substitutability of two expressions and, starting from a number of equations, we advance to new equations by substituting different expressions in accordance with the equations" (Tractatus, §6.24).

proposition (and *all other syntactical connections* that mathematical proposition has to other propositions in the calculus).

However, it is important to understand that the concept of "*proved*" is that we know the syntactical rules that allow us, or license us, to *transform* one Csign into another, by a *finite number of steps*. For 'proved' and 'provable', according to Wittgenstein, can only mean *syntactically derived or derivable*: the symbolic or syntactical steps (i.e., operations) that connect every node create paths that show us the unique sense (position) of a particular node (derived mathematical proposition).

In addition to that, it is not enough that we know that a mathematical Csign is *syntactically derivable*.[79] In order for a mathematical Csign to have sense, in a particular calculus, we must execute the requisite operations of the derivation. In other words, the proof does not exist if we do not actually construct it. For example, the Csign '4,789 × 8,762 = 41,961,218' is mathematically meaningful (i.e., it is a mathematical proposition), for we have a known decision procedure for it. However, the mathematical ***proposition*** '4,789 × 8,762 = 41,961,218' only has sense when we actually do the operations and have its results in hand.

After calculating, the mathematical proposition, '4,789 × 8,762 = 41,961,218' has a position in our system, which shows us the rules that enabled us to put it in our system. So, it has a decision procedure, *and* it has a position (sense) in a calculus. However, we might have a mathematically meaningful Csign that does not have sense. The *incorrect*

[79] Provability ('derivability') is a blurred concept to use here, for it may entail some sort of modality. In order to avoid that kind of modal interpretation, what I mean by *provable* here is *only* that we know the syntactical rules that enable us to position the Csign in the system. By stating that a mathematical Csign is *provable*, I mean that we know how to prove it, despite the fact that we have not written down the calculation yet.

Csign “8 + 15 = 24” does not have mathematical sense, simply because it does not have a position in a mathematical calculus, and it cannot be used *within* a mathematical calculus.

As I said, the proof only exists if we have constructed it. Let us imagine that, by Monday (12nd October 2020) no one has ever done the calculation of “2,987,354 × 3,466,877”. We know that that is a mathematical proposition, for it has a known decision procedure (the rule of multiplication of natural numbers). Nonetheless, it does not have sense. If by Wednesday someone actually does the calculation and writes “2,987,354 × 3,466,877 = 10,356,788,873,458”, then it has sense. The mathematical proposition did not have a sense on Tuesday, or on Monday, or any other day. As Rodych argues:

> Wittgenstein’s reasoning about mathematical sense is best understood in connection with the rest of his radical *constructivist philosophy* of mathematics. According to this view, human beings invent mathematics *bit-by-little-bit*, which means, *in part*, *that we don’t discover pre-existing proofs – they exist only when we have constructed them*. (Rodych, 2008, p. 86)

If we assume that the mathematical proposition was provable on Monday, then we are committed to the idea of *discovering* a mathematical proposition. This would mean that, somewhere, a proof already exists. This pictures mathematics as a constellation, where each star constitutes a mathematical proof that exists in a cosmic system, independent of our knowledge, just waiting to be discovered.

However, I have already showed the problems with this kind of interpretation. The way that Wittgenstein avoids this kind of criticism is by understanding a mathematical proposition, and a decision of a mathematical proposition, as intrinsically a human invention.

5.4. Wittgenstein's Constructivism and the Epistemological Aspect of a Proof

One important aspect of Wittgenstein's Constructivism is our knowledge of mathematics. Our knowledge of mathematics is what enables us to create new proofs in a calculus. This simple idea has a stronger sense in Wittgenstein's Constructivism. As I said above, only a Csign that has a *known* decision procedure is a mathematical proposition.

Wittgenstein's argument avoids any commitment to the notion of modality by excluding derivability or possibility as an aspect of a mathematical proposition. We must have a known decision procedure for the particular Csign, otherwise it will only be a concatenation of symbols, and nothing more.

Our knowledge of a decision procedure enables us to position a mathematical proposition in a mathematical calculus. In addition to that, a mathematical proposition only has sense when we actually do the calculation and construct the proof. Prior to that, a proposition can be mathematically meaningful, but it does not have *sense*.

This argument is based on Wittgenstein's epistemological criteria, in agreement with his Constructivism: A proof does not exist before we have written it down. As I said above, despite the fact that a proposition has a known decision procedure, if there does not exist a constructed prove of it in a system, then "the path" that connects that proposition to other propositions in the calculus does not exist.

All processes related to executing a proof are related to our ability to construct it. There is nothing in the calculus that we have not put in there.[80] In this sense, we cannot understand the sense of an unproved mathematical Csign for there is nothing to understand.

5.4.1 Mathematical Knowledge: Understanding and Accepting a Proof of a Mathematical Proposition

So far, I have highlighted the importance of a mathematical proof for Wittgenstein's philosophy of mathematics, and I have given some examples of proofs. However, I have not given a precise definition of what a proof might be. At this juncture, it is important to clarify: "what constitutes a proof in mathematics?".

A mathematical proof is a sequence of mathematical propositions, beginning with one or more axioms or stipulated propositions. Each mathematical proposition is derived from one or more previous propositions (except for the axioms), in accordance with the rules of the calculus. According to Wittgenstein, the last link in a proof is a mathematical *proposition*; it is sometimes called "the theorem" proved by the proof. For example, when the mathematical proposition '24 × 32 = 768' is proved or derived in a calculus, it is positioned in that calculus by the construction of a syntactical derivation.

What counts as proof of '24 × 32 = 768' is a written construction in accordance with a rule-governed procedure, whereby the first term (24) is on top, the sign of multiplication (×) is on the left one line below, and the second term (32) is below the first. The complete proof is executed from this initial construction: We first multiply the top

[80] "What a mathematical proposition says is always what its proof proves. That is to say, it never says more than its proof proves" (PR, §154)

number (24) by each singular number of the below term (2 and then 30). After we have multiplied 24 by 2 and then 24 by 30, we then add the two products (48 and 720), and write the sum of these products at the bottom of the multiplication. This multiplication, in accordance with rules of our calculus, constitutes a proof of the correct mathematical proposition ‘24 × 32 = 768’.

This simple multiplication gives us an example of a proof—of a proof of ‘24 × 32 = 768’. This example shows that a proof must be a finite number of calculations or transformations or “inferences”, each performed in accordance with a rule of the calculus, which results in a proved proposition on the last line of the derivation/calculation.

About a mathematical proof, Wittgenstein emphasizes:

> ‘A mathematical proof must be perspicuous’. Only a structure whose reproduction is an easy task is called a ‘proof.’ It must be possible to decide with certainty whether we actually have here before us the same proof twice, or not. A proof must be a picture, a drawing that can, with certainty, be exactly reproduced. Or again: it must be possible to reproduce exactly, with certainty, what is essential to a proof. (MS 122, §13, WML, Vol. 3; RFM, III - §1, 1)

In order for a sign-construction to be a proof, that sign-construction must be perspicuous: we must be able to differentiate each unique symbol as a different symbol from all of the other symbols of the calculus. A sign-construction that includes two or more symbols that cannot easily be discriminated is not a proof. We must be able to “reproduce” the proof, and, in doing so, we *must be able to see and discriminate* the different symbols and the symbolic-syntactical rules correctly.

According to Marion (2011), this aspect of a proof adds a visual element as a condition for something to be a mathematical proof. As he argues, “Wittgenstein’s ‘surveyability argument’[81] is best understood not as an argument concerning the length of proofs but as involving the recognition that *formal proofs possess a non-eliminable visual element*” (Marion, 2011, p. 141). This non-eliminable visual element seems also to have a concomitant non-eliminable cognitive element, since we must be able to visually discriminate two distinct symbols and we must cognitively discriminate them as well.

Marion does not disregard the length of a proof as an important aspect of a proof. For a finite number of steps can still yield a “proof” that we cannot construct (*e.g.* a proof that would have one billion steps). Indeed, a proof must be feasible, in the sense that we must be able to construct it and survey it both visually and cognitively. Marion stresses that, for Wittgenstein, there is an important visual element that is ignored. These visual and cognitive elements add to Wittgenstein’s Constructivism, where a proof does not exist prior to its known and understood construction.

If it is the case that a series of calculations or transformations leads to (yields) the result, we must *know that they do, and we must be able to reproduce that series of transformations*. If we do not have knowledge that the sign-constructions of a proof constitute a sufficient condition to derive the sign-construction (what we believe/know to have be proved), then we do not have a proof at all. In addition, if we cannot reproduce a “proof”, by showing that the construction of each sign, Csign, or sign-construction leads (yields) to the same result, it is not a mathematical proof.

[81] Marion uses the term “surveyability” instead of “perspicuous”. He argues that the term “surveyability” better express Wittgenstein’s epistemological condition for something to be a mathematical proof. I will not for or against Marion’s approach. I will simply consider his argument and use it to explain this aspect of Wittgenstein’s philosophy of mathematics. I believe that using Marion’s argument of the surveyability of proofs, without engaging in a discussion about the correct term that better express Wittgenstein’s epistemological criterion, will not generate any problems for the argument that I want to present.

We do not reproduce a mathematical proof expecting that it may give us a different result. If the reproduction is done correctly, we know that it cannot give a different result. In other words, if we know a proof of a mathematical proposition, we know how to use it, for a proof must allow us to put that mathematical proposition *inside a particular calculus*.

This shows the important aspect that I already highlighted in previous sections: by understanding a proof, we understand the sense of a mathematical proposition (we understand the meaning of a mathematical proposition). In coming to understand a proof, we come to understand *everything that there is to know about that proof*, and the proposition proved, and the sense of the proposition (and *the way in which* the proved proposition is 'correct'). Or as Wittgenstein argues, "a child has got to the bottom of arithmetic in knowing how to apply numbers, and that's all that there is to it" (LFM, p. 271).

I believe that, complementary to Marion's interpretation of Wittgenstein on surveyability, it seems that Wittgenstein's epistemological condition also highlights the fact that, not just the proof, but also *the symbols* used in a proof, must be distinguishable in *shape*. We must be able to look at the shape of a symbol and discriminate it from other different symbols.

Wittgenstein (RFM, III - §§10-11) gives us two sequence of symbols: a) "||||||||||||||||||||||||||| ||||||||||||||||" as a proof for "27 + 16 = 43"; and b) "251… 1; 252… 2; 253… 3; etc. 3470…3220"[82] as a proof of the mathematical proposition "250 + 3220 = 3470".

[82] I must say that it is very difficulty to reproduce those sequences of symbols, in Microsoft Word, as precisely as Wittgenstein's draws then in his book. This is a significant limitation, for in this very section I argued that *the visual element* is very important for a proof to be taken as a proof. The way that Wittgenstein draws the proof of "250 + 3220 = 3470" is different from what I manage to do. He drew them in vertical lines, where

Those examples show us the important aspect of the shape (the visual element) of a proof. While (b) has precise and clearly distinguishable symbols, (a) does not. We have *to count* the strokes in order to conclude that "27 + 16 = 43". In other words, we need something else beside the proof in order to conclude that that mathematical proposition is correct. As Wittgenstein argues, "[w]ell, can't one say *after all* that he proves the proposition with the aid of the pattern [the strokes]? Yes; but the pattern is not the proof" (RFM, III - §11).

A proof yields the result. As in (b), we *see* that those steps yield "3470…3220" on the last line of the proof. It is not required that we count or do anything else. Therefore, (b) is a proof of the mathematical proposition "'250 + 3220 + 3470", while (a) is merely a pattern.

5.5. Are There No Real Open Problems in Mathematics?

A Csign is a mathematical proposition if and only if we have a known decision procedure for deciding it. This claim of Wittgenstein's seems to have the immediate consequence that there are no genuine open problems in mathematics, for if we lack a decision procedure for a Csign, that Csign is not a meaningful mathematical proposition.

In the previous section, I use only simple examples using the natural numbers (and some fractions). Despite the fact that the naturals are the most commonly known numbers, the equations that I used (with numbers up to 7 digits) did not have straightforward answers,

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we can easily *see* the 1-1 correspondence that *show* us the derivation. I only hope that I have provided a faithful approximation.

i.e. we had to actually calculate the results. It would be different if I used as example "7 × 5", that has a straightforward answer, "35".

This, I believe, exemplifies the distinction between an open mathematical problem and a mathematical question: in relation to a mathematical question, we already *have* the means to decide it, *e.g.* 2,987,354 × 3,466,877. Even though it requires that we perform some mathematical operations, we *already know how to work it out*.

In opposition to this, a Csign (e.g., a mathematical conjecture) is an open mathematical problem if and only if we do not posses a known decision procedure for deciding it. For example, FLT was a famous open problem. Until 1994, we did not have a decision procedure for it. After Andrew Wiles' proof we now know that there is a way to show that there do not exist three natural numbers that satisfy "$a^n \times b^n = c^n$", for $n > 2$.

However, as I have argued, in order for a Csign to be a mathematical proposition, it must have a known decision procedure. This leads us to conclude that FLT was *not* a mathematical proposition, for we did not have an applicable decision for deciding it prior to Wiles' proof. In addition to this, because FLT did not have a constructed proof, FLT did not have sense.

Such Csigns, according to Wittgenstein, are not mathematical propositions. Undecided 'mathematical' Csigns are not meaningful mathematical propositions within a particular calculus. This kind of mathematical conjecture would be considered a Csign, but not a mathematical proposition.

For this reason, it certainly seems that Wittgenstein claims that mathematics does not have open mathematical problems that require *unsystematic* decisions.[83] For Wittgenstein, there seem to be only algorithmically decidable questions. A mathematical conjecture poses a problem that does not have a method for solving it (i.e. a decision procedure).[84]

5.5.1. Mathematical Problems: Open Problems vs. Questions

Wittgenstein's investigation of the sense of mathematical Csigns raises some questions about mathematical conjectures and *open problems* in mathematics. These problems refer to questions in mathematics for which we do not have, right now, a decision procedure. Csigns that constitute open problems "are not known to be provable or disprovable within currently accepted systems of mathematics" (Säätelä, 2011, p. 162). In contrast to open mathematical problems, we have "elementary mathematical problems": algorithmically decidable mathematical propositions, such as "$234 \times 456 = 106{,}704$", for which we can algorithmically construct a proof.

Elementary mathematical problems can be related to simple *tasks*, where one follows the rules, stated by the mathematical system, in order to achieve the correct result. A task gives us the idea of a mechanical procedure to construct a proof. We simply execute operations in accordance with the rules of the system in the lines of our proof in a very *perspicuous way*.

[83] "Where you can't look for an answer, you can't ask either, and that means: *Where there's no logical method for finding a solution, the question doesn't make sense either*" (PR, §149 [italics mine]).
[84] As Wittgenstein claimed, "[o]nly where there's a method of solution is there a problem". (PR, §149)

The main difference between a question and an open problem in mathematics is that for the former we have a known decision procedure, while for the latter we do not. For example, for a simple sentence '2 + 2 = 4' we have a decision procedure. However, the same cannot be said of Goldbach's Conjecture (hereafter 'GC'). We do not know of a decision procedure for algorithmically deciding GC.

A question, which is posed by an elementary mathematical proposition, can be algorithmically decided. If, e.g., "13 × 12 = 156" is correct, it will be positioned within a particular mathematical calculus (system); if, e.g., "13 x 12 = 166" is incorrect it will be refuted and the correct proposition (i.e., "13 × 12 = 156") will be positioned within a particular calculus. In this manner, Wittgenstein attempts to show that we have *two* importantly different kinds of "mathematical problems": mathematical propositions that are easily decidable by known decision procedure, and Csigns for which we do not know of a way or method by which to decide them.

However, the vast majority of interesting and difficult open problems are Csigns for which there is no known decision procedure. Thus, it seems that Wittgenstein's account of "meaningful mathematical propositions" is entirely at odds with how mathematicians view the most challenging mathematical Csigns.

According to the Standard View, all mathematical propositions are on a par: they are all either true or false, and they are all either decidable or not, relative to a particular mathematical system, regardless of whether we know whether or not they are decidable. The Standard View does, obviously, grant that most of the difficult open problems are mathematical propositions that we do not know how to decide; it also grants that we do not even know whether they are decidable in this or that system.

The important aspect that I want to highlight is that the standard view's interpretation assumes that open problems in mathematics entail that a mathematical proposition *is true or false*. If we are committed to the idea that mathematical conjectures are mathematically meaningful, then they must be true or false. We investigate a mathematical conjecture such as GC in order to determine its truth-value: we want to determine *that* it is true if it is true, and we want to determine *that* it is false if it is false. Mathematical conjectures are mathematically meaningful propositions that are open problems for mathematicians until they are solved, standardly by either by a proof or a refutation. It assumes that we now have mathematically meaningful Csigns for which we do not have decision procedures, but which, one day, we may decide by *finding a proof of the Csign or its syntactical negation*.

This conceptual difference shows one of the main criticisms of Wittgenstein's view in relation to the use of the word 'problem' in mathematics. The misuse of the term "mathematical problem" gives the illusion that two very different kinds of mathematical question are the same. It assumes that questions that we know how to answer, and Csigns that we do not have a clue (so far) how to proof or refute, are essentially the same. As Wittgenstein says, "[a] mathematical proposition, if it is not proved, is — one might say — the expression of a problem" (MS 123, II, §158).

It is not that Wittgenstein "wants to *prohibit* the use of the word [problem], but he wants to point out that we must realize that the word is used in fundamentally different senses within mathematics" (Säätelä, 2011, p. 164). In other words, it is not the case that there are no problems in mathematics, and Wittgenstein does not deny the existence of open problems (e.g., mathematical conjectures such as GC) in mathematics. However, what we consider a problem, and how we view and talk about certain Csigns (e.g., GC), might reveal a commitment to some sort of Platonism.

5.5.2 The Paradox of the Triviality of Mathematical Problems

On Wittgenstein's view, a Csign is not necessarily a mathematical proposition, for, as he argues, "[a] mathematical question must be no less exact than a mathematical proposition" (PG, p. 375).

However, this leads to the view that every mathematically meaningful *question* must already have a *known decision procedure*. Therefore, in order for something to be a genuine mathematical question, we must already have a decision procedure for deciding it. Since we do not possess decision procedures for most interesting and challenging mathematical conjectures, such as GC, it seems, on the surface at least, that GC cannot pose a question or a problem for mathematicians.

It seems that there is a conflict here, between what mathematicians claim to be a *genuine mathematical problem*, and what Wittgenstein's says a mathematical problem is. For, if Wittgenstein claims that mathematical conjectures (that are well-formed formulas in mathematics), which constitute open problems for mathematicians, are not mathematically meaningful, then is he not suggesting or demanding modifications to how mathematicians do their work? Is he not, here, *meddling in mathematicians' affairs*?

At first, Wittgenstein's argument appears to dictate what mathematicians should work on: they should only work on *genuine mathematical problems* (*i.e.* as I called them, following Wittgenstein, *questions*). But that could not be further from the truth. So far, what I have argued is that what mathematicians claim to be a "mathematical problem" does not adequately distinguish between Csigns and algorithmically decidable mathematical proposition; they do not adequately distinguish between a mathematical question that is answered by an algorithmic task and mathematical conjectures that cannot be solved or decided algorithmically.

I believe that Wittgenstein is, again, *only untangling conceptual-linguistic knots*. He does not argue that mathematicians should not attempt to prove FLT or GC. Rather, he is arguing that, if, in 1980, we claim that these two Csigns—FLT and GC—are genuine mathematical propositions (and, therefore, mathematically meaningful Csigns), then we are presupposing some form or other of Platonism. As I stated, the claim that those undecided Csigns are genuine mathematical propositions (constituting *definite and decidable* mathematical problems), entails that they *must be true or false* – or at least that they are decidable in some existent mathematical system.[85] The principal problem for the claim that undecided mathematical conjectures are *genuine* mathematical problems is that it assumes that certain Csigns, *e.g.* GC, are *decidable* in a system (*e.g.* Peano Arithmetic). But do we actually know that GC is decidable in PA? Do we actually know that GC is decidable in ZFC? In the year 2020, do we actually know that FLT is decidable in PA? Do we even know that FLT is decidable in ZFC? At present, the mathematical community

[85] And, as I showed in Section ___?, the reasons that lead some authors to claim that undecided mathematical propositions must be true or false, are supported by a form of Platonism. They must be committed to the idea that what makes these Csigns true or false is something *outside of mathematics*, namely some kind of non-physical fact, which is mind- and language-independent.

is very uncertain what system, and what strength of system, is required for Wiles' 1995 proof.[86] Without a decision procedure, we have no argument to support this claim that GC or FLT is decidable in PA and/or ZFC.

Wittgenstein's view does not claim that there exist only trivial problems in mathematics. By constructing new proofs, we are continually constructing new calculi by extending our constructed calculus. We are constructing new paths in our syntactical system. These new paths will enable us to use different mathematical propositions in different ways.

By constructing a proof, we construct a new mathematical system that enables us to work with different questions in mathematics. For example, Wiles' proof was only possible because of several theorems that had been proved in the recent past. In other words, Wiles was only able to prove FLT because mathematicians constructed a *new calculus* (or new calculi).

5.5.3. Open Mathematical Problems as Stimuli for Mathematicians' Creative Activity

According to the foregoing, a Csign that is a mathematical conjecture is not a mathematical question, *i.e.* a Csign that does not have a method for solving is not a meaningful mathematical proposition. However, that does not entail that Csigns, such as GC, are not important in mathematics. As Wittgenstein himself said, "[m]y explanation mustn't wipe out the existence of mathematical problems" (PR, §148).

[86] See (Glivický and Kala, 2016)

As we know, centuries after Fermat's conjecture, Wiles managed to construct a proof for that Csign. It seems that Fermat's conjecture was fundamental to start the *search* for its proof. Even though we did not have a proof of the Csign that was FLT, this conjecture stimulated mathematicians to *try* to proof it.

This seems to contradict Wittgenstein's argument, that a mathematical conjecture *qua* open problem is senseless. However, Wittgenstein does not claim that an open problem, such as FLT was in 1980, is meaningless, in the same way I which the term "carro" is meaningless in English. The idea here is that such a Csign does not have mathematical meaning (*i.e.* mathematical *sense*, *i.e.* a position in a constructed, existent calculus), and it is not a mathematical proposition (i.e. because it does not have an applicable decision procedure). On Wittgenstein's view, this fact *does not mean* that we cannot ask "Can FLT be proved in system PA?" or "Can FLT be proved in some existent mathematical calculus?".

This distinction enables us to say in 1990, FLT was indeed senseless (i.e., it had not been *positioned*, by a proof, within a particular calculus). However, this does mean that it could not *affect us*. On Wittgenstein's view, a Csign such as FLT can be a *stimulus* to mathematical activity.[87]

For example, imagine that an engineer had to solve a problem related to the stability of a concrete bridge. After some unsuccessful attempts, he was finally able to work out an

[87] "I do not claim that it is wrong or illegitimate if anyone concerns himself with FLT. Not at all! If e.g. I have a method for looking integers [sic] that satisfy the equation $x^2 + y^2 = z^2$, then the formula $x^n + y^n = z^n$ *may stimulate me*. *I may let a formula stimulate me*. *Thus I shall say, Here there is a stimulus – but not a question*. Mathematical 'problems' are always such stimuli" (WVC, p. 144 [italics mine]). See this also in (PG, 371 [italics mine]), "Unproved mathematical propositions – signposts for mathematical investigation, *stimuli to mathematical constructions*".

answer to his bridge-problem. That happened because one of his friends told him how he managed to fix a problem related to some ground squirrels in his garden. His **friend's** solution gave him an idea on how to fix the concrete bridge.

His friend's anecdote definitely helped the engineer to solve the problem; however, I believe that no one would ever say that a gardening problem is a bridge-engineering problem. Nonetheless, the anecdote helped the engineer by causing him to think in a certain way. He was guided by an analogy.

An analogy can affect us in a way that enables us to try to expand a particular calculus, to construct new paths for other Csigns. As Wiles himself said, the proof of FLT was only possible because new branches and systems were developed.[88]

One of the issues with mathematical conjectures and open problems in mathematics is that this notion entails that there are undecided *mathematical* Csigns: a Csign that so far is undecided, but for which we will one day discover a proof or refutation, "for in mathematics there is no ignorabimus" (Hilbert, 1902, p. 445). As I already said, this idea has as consequence that *there exists a proof or refutation already*, we just have not found it yet. We are led to believe that we discover proofs in mathematics, and with assumption we bring with us all of problems of Platonism and/or Modalism.

It is worth noting in this connection that, in his Ph.D. dissertation, L.E.J. Brouwer states (1975 [1907], p. 79, footnote #3): "[a] A fortiori it is not certain that any

[88] As Wiles said in *FLT*, "[i]t was one evening at the end of the summer of 1986 when I was sipping iced tea at the house of a friend. Casually in the middle of a conversation he told me that Ken Ribet had proved the link between Taniyama–Shimura and Fermat's Last Theorem. I was electrified. I knew that moment that the course of my life was changing because this meant that to prove Fermat's Last Theorem all I had to do was to prove the Taniyama–Shimura conjecture."

mathematical problem can either be solved or proved to be unsolvable, though HILBERT, in 'Mathematische Probleme', believes that every mathematician is deeply convinced of it". In other words, we do not have a justification for Hilbert's claim.

Since Wittgenstein rejects the idea that Csigns for which we do not have decision procedures are mathematical propositions, we must now determine what the status of undecided Csign is. In other words, if undecided Csigns are not mathematically meaningful, what *are* these well-formed formulas?

According to Säätelä (2011, pp. 162), Wittgenstein faces a dilemma, "[on one hand] that it is nonsensical to treat such 'propositions' [as GC or FLT prior to 1994] as genuine mathematical propositions. On the other hand, he does not want to claim that it is illegitimate for mathematicians to concern themselves with problems such as [GC or FLT]".

This dilemma leaves undecided Csigns in a limbo, because, although they do not have sense, they can be useful various ways.

5.6. The Status of Mathematical Conjectures: Undecided Csigns

Wittgenstein argues in favour of a very radical view on Csigns for which we do not have a decision procedure: a Csign is mathematically meaningful (is a mathematical proposition) iff we have a decision procedure that demonstrates that it is correct and positions it in a calculus or shows that it is incorrect (*e.g.* $2 + 2 = 5$). This raises the question: What is the status of Csigns for which we do not have a decision procedure?

As I said, we do not have a decision procedure for GC, therefore it is not a mathematical proposition. Nevertheless, we are inclined to argue that it is meaningful, because we believe we can *understand* what it talks about —because we believe that it talks about *all even numbers greater than 2*. We can argue, e.g., that GC can clearly be checked for some even numbers. For example, we can determine that it holds for 4 (1+3), 6 (5+1), 8 (5+3) … So far, it has been checked for 400 trillion even numbers; all of them are the sum of two primes.

This way of thinking exemplifies the standard view argument, where "[w]e understand [Goldbach's Conjecture] insofar as we are able to understand this meaning – insofar as we are able to understand what it is, or what it would be like, for every even number greater than 2 to be the sum of two primes" (Rodych, 2008: 86). On this view, GC is a meaningful mathematical proposition because we can understand *what it refers to or talks about* (*i.e.* we can understand the specific claim it makes: we can understand what makes it false, and we can also understand — and, in a certain way, even visualize — what makes it true).

For example, the contingent proposition "there are intelligent beings elsewhere in the universe" is meaningful, for we can clearly understand what it *talks about* and what specific claims it makes. We know what physical fact would make this proposition true, namely, the physical existence of intelligent beings somewhere other than Earth.

The standard view assumes that mathematical propositions work in a very similar way. A Csign is a meaningful mathematical proposition if we know the conditions for it to be true. This gives us a picture of what happened to FLT. We know what would take for it to be true, that there do not exist three natural numbers that satisfy the equation "$a^n + b^n =$

c^n" for $n > 2$. It turns out that, according to Wiles' proof, that three such numbers do not exist!

What the standard view seems to show is that nothing changed after Wiles' proof. In particular, and most importantly, the *meaning* of FLT did not change. It still says the very same thing about the natural numbers. Its meaning was complete before Wiles' proof, and since Wiles' proof did not alter its meaning, its meaning remains complete and unchanged.[89] Similarly, if we construct a proof **of** GC tomorrow, it will not have a different sense (meaning) tomorrow. It will still say that "every even number greater than two is the sum of two primes". A proof would not modify that—a proof of GC would not alter GC's meaning.

At first sight, it appears that we understand Goldbach's Conjecture because it seems that it *talks about* something, and makes a specific claim *about something*. "'[I]t says' that every even number greater than 2 is the sum of two primes and we know what the words 'every,' 'even' and 'prime' mean in mathematics" (Rodych, 2008: 96). In other words, by understanding the meaning of the terms that constitute the proposition we can fully grasp the meaning of the proposition.

Wittgenstein argues, however, that in order to determine the status of a given Csign, **we** must ask the question: Can we prove it in a particular calculus?[90] As he (PG: 370) argues, "[w]e must first ask ourselves: is the mathematical proposition proved? If so, how?

[89] I thank Professor Victor Rodych for this insight. The idea of "complete meaning" gives us a good picture of the opposition between the standard view and Wittgenstein's radical Constructivism.
[90] "What is it that goes on when, while we've as yet no idea how a certain proposition is to be proved, we still ask 'Can it be proved or not?' and proceed to look for a proof? If we 'try to prove it', what do we do? Is this a search which is essentially unsystematic, and therefore strictly speaking not a search at all, or can there be some plan involved? How we answer this question is a pointer as to whether the as yet unproved—or as yet unprovable—proposition is senseless or not" (PR, §148).

(…) The fact that this is so often not understood arises from our thinking once again along the lines of a misleading analogy”. This “misleading analogy” is the analogy that I mentioned several times, namely, that a Csign is no different from a contingent proposition.[91]

The idea that an undecided Csign is mathematically meaningful is based partially on the fact that some Csigns, such as GC, are *well-formed formulae*. However, the claim or intimation that if GC is a well-formed formula, then it must be decidable in a calculus, is elusive. It *assumes* that because a Csign is a well-formed formula it must be decidable: it does not show us why that would be the case. As Brouwer wrote in 1907-08, we do not have a *proof* that *all* Csigns that are regarded as meaningful mathematical propositions are mathematically decidable or decidable in existent mathematical calculi. It seems that the rules governing well-formedness of, say, well-formed formula of PA are *based* on our English understanding of, *e.g.* what GC *says* in English.

In addition, if we analyze several proofs in mathematics, we will conclude that they do not proceed in the way that we think. For example, Wiles’ proof used several different branches in mathematics, being supported by different lemmas that were demonstrated over the previous 2-3 decades. This indicates that FLT, despite appearing to say something very elementary about the natural numbers, is far more complex than is indicated by the appearance of the Csign itself and by the natural language interpretation for it in 1980 or 1990.

[91] “What seduces us into thinking that GC is a mathematical proposition with a fully determinate sense is a ‘faulty analogy’ between mathematical and contingent propositions” (Rodych, 2008, pp. 95)

This shows why this analogy between mathematical Csigns and contingent propositions does not hold. If the sense of a mathematical proposition is its position in a particular calculus, “the proof is part of the grammar of the proposition” (Wittgenstein 1974: 370). Much to the contrary, the meaning of a contingent proposition can be fully understood if we understand the meaning of its terms, for it is possible to imagine (picture) the state of affairs, even if we do not know if the proposition is true or false. But we cannot imagine a proof of GC, today, when we do not have a proof of GC today.

For example, “this man died two hours ago”[92] is a proposition that we can picture, even though we do not know what establishes the time of the man’s death. It is meaningful because the proposition tells us what makes it true, that is, this man died two hours ago. Furthermore, even if we *discover* new methods for proving that the man died two hours ago, the meaning of the proposition “this man died two hours ago” would not change. It would still be the same picture; it would still be the same proposition that pictures a possible state of affairs in physical reality. Any method for justifying the picture of the proposition does not change the picture that it shows us[93].

To the contrary, “a mathematical proof incorporates the mathematical proposition into a new calculus, and alters its position in mathematics” (Wittgenstein, 1974: 371). The syntactical derivation that we construct for a specific mathematical proposition in a calculus shows us the sense of that mathematical proposition (it shows where to position

[92] Wittgenstein (PG, p. 370)

[93] As Rodych (2008:96) points out: “One can understand “the man died two hours ago” completely without knowing whether it is true because, given our linguistic conventions, this sentence has a fully determinate sense which we can understand, picture, etc.

that proposition in the system). In this sense, a proof has a direct relation to the proved proposition, for it alters (or establishes) how we can use that proposition inside the calculus.

Goldbach's Conjecture is not be positioned in any calculus because it does not have a derivation. Did mathematicians know what exactly was required of a proof of FLT before Wiles proved it? This shows us that "[w]e cannot understand [Goldbach's Conjecture] until it has a sense, and it does not have a sense until we have given it a sense by proving it in a calculus" (Rodych, 2008: 97).[94] Analogously, can we say that we understood FLT prior to Wiles' proof? Could we have known that it would require a set of complex lemmas and formulas in different branches of mathematics? Wiles did not prove FLT by showing that there do not exist three natural numbers that satisfy …; his proof is vastly more complex than one would think, to the point that, so far, we *do not have a proof* of FLT in either Peano Arithmetic or in ZFC.[95] As Wittgenstein suggests, a proof "does not merely shew *that* it is like this, but: *how* it is like this" (RFM, III - §22).

Goldbach's Conjecture is not a mathematical proposition because we do not possess a decision procedure by means of which we can decide it. If we assume that GC is a mathematical proposition, we must then say that it "hangs in the air". It is a well-formed formula, but it is not presently a proposition positioned within Peano Arithmetic, or any other mathematical system. As Wittgenstein states, "[an undecided][96] proposition makes

[94] To understand Goldbach's Conjecture as a genuine mathematical proposition would be to accept that some "propositions" in mathematics are neither correct nor incorrect. In other words, one would have to accept a third possible scenario. To hold that, would be to give way the law of the excluded middle in mathematics. Furthermore, if mathematics is a human invention that we construct, how would be possible to have problems that we cannot find the solution?
[95] FLT is a very complicated example. The proof itself is very complex and as Säätelä states (2011, pp. 177-178), "not 'surveyable' or 'perspicuous' in the same sense as elementary mathematical proofs".
[96] In this section, Wittgenstein is referring to FLT that, when he wrote the book, did not have a proof. I made those insertions to better state his argument.

no *sense* until I can *search* for a solution to the equation… And 'search' must always mean: search systematically. Meandering about in infinite space on the look-out for a gold ring is no kind of search" (PR, §150).

5.6.1. The Problem of Mathematical Sense: The Impossibility of Proving a Csign

In addition to the problem of undecided Csigns, which seems to constitute the main problems in mathematics, Wittgenstein's notion of mathematical sense also raises problems related to how we prove a mathematical proposition.

As Wittgenstein himself recognizes, the idea of the sense of a mathematical proposition being its proof would entail that by proving a proposition and by proving it with a different proof, we seemingly alter the sense of a mathematical proposition. When we first prove a mathematical proposition, we give it a sense it previously did not have. Secondly, if we construct a new (e.g., second) proof of a proved mathematical proposition, its position in a calculus is altered by our giving it new syntactical connections. Furthermore, if every new proof establishes a new path that modifies the calculus (*i.e.* makes new syntactical connections), then we would do not learn anything new about the old system that we were working with yesterday when we execute a new proof today. [97]

This problem can be understood in relation to two aspects: a) the sense of a mathematical proposition; and b) our knowledge of its sense. As I stated in the sections above, we must construct a proof in order for it to be *an actual proof*. Therefore, the concept

[97] "Wouldn't this imply that we can't learn anything new about an object in mathematics, since, if we do, it is a new object?" (PR, §155)

of proof and our knowledge of a proof is necessarily connected (*i.e.* everything that is related to a proof is also related to how we constructed that proof).

This shows that, if the sense of a mathematical proposition is its position in a particular calculus, and there is nothing else to it, then by unsystematically proving a Csign we prove a mathematical proposition with a new sense, although the Csign—*qua* concatenation of signs—remains the same. The Csign that is proved has different syntactical connections than it than it had before the proof; therefore, it is a new proposition. What we learn from this new proof is something new: we learn something new about a new mathematical proposition; we do learn something about the (same) Csign in the old calculus. For Wittgenstein, construction, sense, and calculus all go together: a new construction (proof) gives or changes sense and the calculus in which the newly proved proposition resides. We now know something new about this new proposition and this new calculus. In other words, we can never learn something new about a Csign.

As I already said before, it seems that we cannot have more than one proof of the same proposition. In fact, we cannot even prove the non-mathematical Csign that we started to work with. The moment that I finish a proof, I have a new proposition, and what I was working with simply disappears, for "when I learn the proof, I learn something *completely new*, and not just the way leading to a goal with which I'm already familiar" (PR, §155).

In summary, the problem shows that we cannot prove a Csign, for when I finish a proof, that Csign, which is now a mathematical proposition, has a very different sense than when I set out to proof it. More precisely, a Csign is equal to another Csign if it has the same *meaning* (*sense*). So, if I am comparing one and the same Csign before and after it has been proved, we must conclude that it is a new mathematical proposition with a new sense after it is proved, where before it was not a mathematical proposition and did not have a sense. And if, after first proving a Csign (i.e., a mathematical proposition) we give it a second proof, we alter the sense of a mathematical proposition, by giving a proved proposition a new syntactical connection in its calculus.

Wittgenstein himself struggled with this problem.[98] It seems doubtful that I or any other commentator can give a satisfactory answer to this problem, on Wittgenstein's behalf, which will satisfy a proponent of the standard view. This is partly because Wittgenstein's view is very different as regards the meaning (sense) of a mathematical proposition. However, I think that a reasonable explanation lies in the fact that, on Wittgenstein's account, we are not just comparing the same concatenation of symbols that different senses, *we are comparing two different calculi*.[99] A new proof does not alter only the sense of a Csign, it alters the calculus (by constructing new syntactical connections). When we compare one and the same Csign before and after it is first proved, we are comparing a Csign that has no sense with a Csign, that is in a different calculus, which has a sense (and which, therefore, is a mathematical proposition). But what is the relevance of comparing two different calculi?

This is merely a sketch of an answer. It only indicates a possible answer. For as Wittgenstein suggested, "[i]t all depends *what* settles the sense of a proposition, what we choose to say settles its sense" (RFM, VII - §10).

[98] Wittgenstein states this problem at various places in both the middle and later periods. See (RFM, III - §59-60; RFM, VI - §13; RFM, VII - §§10-11; PR, §155; PG, pp. 367-369).
[99] "[A] mathematical proof incorporates the mathematical proposition into a new calculus, and alters its position in mathematics. The proposition with its proof doesn't belong to the same category as the proposition without a proof" (PG, p. 371).

5.7. The Later Wittgenstein's Extrasystemic Application Criterion: Distinguishing Mathematical Calculi and Sign-Games

So far, I have only presented Wittgenstein's formalist account of mathematics. The reason for that is that his formalist conception does not change between his earlier work and the later period: the nature of a mathematical proposition, intra-systemic meaning (i.e., sense), and the relationship between proof and sense *do not change* in the later period.

However, we have strong evidence that the *strength* of his formalistic interpretation varies significantly throughout his work. We can see that Wittgenstein changes his views from the *Tractatus* to his later work. I will analyze how his views change between his earlier writings (*Tractatus*) and his later writings (*Remarks on the Foundations of Mathematics*). I will here focus on the differences between the middle period (*Philosophical Remarks* and *Philosophical Grammar*) and the later period (*Remarks on the Foundations of Mathematics*) in order to evaluate a crucial difference between the middle and the later period, namely Wittgenstein's *the extrasystemic-application criterion*.

As Rodych (1997, pp. 196-203) argues, Wittgenstein's formalistic interpretation varies from *strong Formalism* in the middle period, to *weak Formalism* in the later period. As I showed, in the middle period Wittgenstein argues that mathematics is syntactical. Additionally, in the middle period, Wittgenstein (PR, §109, [italics mine]) argues that "[o]ne always has an aversion to giving arithmetic a foundation by saying something about its application. *It appears firmly enough grounded in itself*. And that of course derives from the fact that *arithmetic is its own application*". In other words, a mathematical calculus does not require any form of extra-mathematical (extrasystemic) application, for “arithmetic isn’t at all concerned about this application [to count strokes]. Its application takes care of itself” (PG, p. 308).

At this period, Wittgenstein’s argues in defense of *strong Formalism* (Rodych, 1997, p. 196).[100] The middle Wittgenstein seems to claim that it is irrelevant for a mathematical Csign that it, or it terms, has meaning in contingent propositions. The sense of a mathematical proposition is entirely an intrasystemic matter.

The difference in the later period is that Wittgenstein now argues for a weaker variant of Formalism. The main aspect of this weaker form of Formalism is that, in addition to the decision-procedure criterion, a system is only a mathematical system if its concepts *have meaning in contingent propositions*. As Frascolla argues, “in order that the meaning of the symbols for finite cardinals can be understood, […] one must consider their use in the empirical statements describing the results of processes of counting objects.” (Frascolla, 1994, p. 163)[101]. The later Wittgenstein draws attention to this important (and necessary) aspect of mathematical words: some concepts and words of a mathematical calculus must have meaning in contingent propositions.

[100] It is important to highlight that Wittgenstein did not call his view Formalism. In fact, he criticizes the formalistic approach to mathematics, in particular, what Wittgenstein assumed to be Hilbert’s view. However, the concept that Wittgenstein has of Formalism is very different than the one that I have attributed to him (or in fact what we now call “Formalism”). In fact, it is very different from what Hilbert himself defended (see Rodych, 1997, pp. 198-199). For Wittgenstein (LFM, p. 112), a formalistic approach assumes that mathematics is *about* the signs on a piece of paper, meaning that mathematics is a concatenation of meaningless signs.

[101] Frascolla interprets the passages of *RFM V-§2* and *RFM V-§41* as Wittgenstein’s “quasi-revisionary” attitude. I do not agree with Frascolla’s view that Wittgenstein’s philosophy of mathematics is semi-revisionary, in order to preserve the interpretation that Wittgenstein is only describing mathematics. Nonetheless, it seems that Rodych (1997) and Frascolla (1994) share a similar interpretation of these passages: that the later Wittgenstein draws attention to another condition for a calculus to be mathematically meaningful (that a Csign must also be meaningful in contingent propositions in order to be mathematically meaningful). Rodych also argues (Rodych, 1997, pp. 217-218) that this new criterion serves to “dissolve a tension” between Wittgenstein’s criticism of set theory and the idea that mathematical systems are mere sign-games.

However, one important aspect of this new criterion is that it does not alter the sense of a mathematical proposition. Wittgenstein is not arguing that the sense of a mathematical proposition is related to something external to the calculus. "The difference in [*Remarks on the Foundation of Mathematics*], […] is that Wittgenstein now requires that a sign-game must have a real world application to constitute a mathematical 'language-game'" (Rodych, 1997, p. 217). Mathematics still has only intrasystemic meaning. This extrasystemic criterion does not alter the sense of a mathematical proposition, it only establishes that its terms *must have meaning in contingent propositions*: "it is essential to mathematics that its signs are also employed in *mufti*" (RFM, V - §2). This change raises the question: Why does Wittgenstein change his mind? What reasons does Wittgenstein give in support of this new criterion?

I believe, with (Rodych, 1997), that this change to a form of "weak Formalism" serves to solve a "tension" in Wittgenstein's philosophy of mathematics. As I stated in chapter 2, despite Wittgenstein's best efforts to only describe mathematics, he seems to make some very strong criticisms of set theory — in particular, of *transfinite set theory*. Wittgenstein says, for example, that he believes and hopes "that a future generation will laugh at this hocus pocus" (RFM, II - §22). However, his own view on the philosophy mathematics, assumes that mathematics is only a sign-game, no different than other sign-games, such as transfinite set theory.[102] As Rodych (1997, p. 218) argues, "[o]n Wittgenstein's intermediate terms, [transfinite set theory] *should* qualify as a meaningful mathematical calculus, but it does not".

[102] "The difference between my point of view and that of contemporary writers on the foundations of arithmetic is that I am not obliged to despise particular calculi like the decimal system. *For me one calculus is as good as another*" (PG, p. 334 [italics mine]).

Thus, on Wittgenstein's intermediate account, elementary number theory is merely a sign-game in the same way that transfinite set theory is a sign-game. By Wittgenstein's own criteria, transfinite set theory should be as mathematical as any core mathematical system, for they are just sign-games..

This is the "tension" that Wittgenstein attempts to resolve by highlighting an important aspect of mathematical systems: "[c]oncepts which occur in 'necessary'[103] propositions must also occur and have meaning in non-necessary ones" (RFM, V - §41). This new necessary condition of a mathematical calculus enables the later Wittgenstein to argue, without reservation, that transfinite set theory "is for the time being a piece of mathematical architecture which hangs in the air, […] not supported by anything and supporting nothing" (RFM, II - §35). Transfinite set theory is merely a sign-game, it is not (yet) a mathematical calculus. It "hangs in the air" because it is not "anchored in any real world language-game" (Rodych, 1997, p. 219). The fact is that some mathematical calculi have extra-mathematical applications outside of pure mathematics. Indeed, this is the very reason that Indispensability Argument seems so appealing. The extra-mathematical application of mathematical calculi in contingent propositions shows a fundamental difference between mathematics and any other pure sign-game, such as transfinite set theory, chess, etc.[104]

[103] Here it is important to highlight that Wittgenstein writes the word *necessary* in between quotation marks. That is the case for he is not claiming that mathematical propositions are *necessarily true*. Rather, he is just using a more familiar way to refer to mathematical propositions, *i.e.* a way that is common to the standard view.

[104] We can see a similar explanation in (Frascolla 1994, p. 163 [italics mine]): "*There is a further principle which underlies his attitude to the theory of transfinite cardinals* and which is again linked to the question of the meaning of mathematical expressions". Despite the fact that Frascolla does not call this new criterion as "the application condition/criterion", his explanation of why Wittgenstein adopts this new criterion is similar way that I have stated in this section.

This new criterion leads us to further evaluate what is and is not mathematics. As we have just seen, Wittgenstein's later writing shows that he is as worried about applied mathematics as Quine or Putnam. The main difference here is that, despite the fact that Wittgenstein may be granting that mathematics *is indispensable* to contemporary physics or chemistry, he would never grant or say that this *proves* or *indicates* that mathematical propositions are true. And Wittgenstein would certainly deny that extra-mathematical application of mathematical calculi shows that we need to posit special mathematical entities to account for the success of mathematical physics.

With the conditions/criterion that I have raised in these last sections, we can now focus on the concept of mathematical infinity, and what role it has, according to Wittgenstein, in mathematical systems.

CHAPTER 6: Wittgenstein's Finitistic Constructivism

6.1. The Finite and the Infinite in Mathematics

We have seen that the standard view maintains referentialism for mathematics and Wittgenstein denies that mathematical propositions refer to objects or facts. Nowhere is this more evident than in these two opposing accounts of mathematical infinity. On the standard view, infinite extensions are essential to mathematics, and a principal desideratum of an adequate philosophical account of mathematics. Wittgenstein, to the contrary, denies the existence of infinite mathematical extensions, and with that denial he offers an alternative description of "infinite sets", irrational numbers, quantification over an infinite domain, and proof by mathematical induction.

The first common notion that we have about mathematics is that it talks about an infinite domain. In other words, if we think about mathematics it seems that it talks about an *infinite number of things*. For example, the set of all natural numbers is represented by N = {0,1,2,3,4,5,6 …}, where the three dots "…" tell us that there is no end to that sequence. We all agree that it would be absurd if, one day, someone says that we have finally arrived at the last natural number.

However, I have being arguing that, for Wittgenstein, mathematics is a human invention. We construct a calculus bit-by-little-bit. Wittgenstein maintains that there are no infinite extensions in mathematics (e.g., there are no infinite extensions in mathematical calculi) and that what people take to be "mathematical infinity" is always an unlimited rule for constructing finite extensions.

Thus, there is a conflict between our idea of an infinite sequence of numbers and Wittgenstein's argument that, in mathematics, there are no infinite extensions. Most pure mathematicians claim or assume that infinite sets are fundamental for mathematics, and that the three dots of any infinite set, or the decimal expansion of an irrational (also 'ending' with three dots), *represents* something infinitely long—an infinite extension. According to the standard view, in the same way that π represents an infinitely long number, where we can discover any particular digit in its decimal expansion, we similarly represent, in a variety of ways, the actual infinity of every infinite set of numbers.

This tells us that there are many forms that the concept of infinity takes in mathematics: we can talk about infinite sets, such as the natural numbers, or the even numbers; as well as about the infinite extension of π, or any other irrational number. However, on the standard view, this is just the beginning, for since Cantor's late 19^{th} Century proofs, modern mathematics has *also* maintained that some infinite sets are larger that other infinite sets—that the cardinality of the former is greater than the cardinality of the latter.

In order to understand mathematical infinity, we must first explain the role of infinity in mathematics. To do so, it is important to start with different sets of numbers, and then examine putative proofs of infinity in mathematics. Our goal here is to understand what mathematical infinity is.

6.1.1. Understanding the Concept of Infinity in Mathematics: Rational and Irrational Numbers

The rational numbers, as the name suggests (ratio), are fractions. For example, the fraction 1/7 has the symbolic structure ‘n/m.’ ‘n’ is called the numerator and ‘m’ the denominator. It can also be written in decimal notation: 0.142857142857142…, where the three dots stand for the unending repetition of ‘142857’. In this example, the dots stand for the *periodicity* of ‘142857’.

In addition to this, most logicians and mathematicians claim, the set of all rationals numbers is denumerable. By this they mean that we can show the rational numbers can be put in a 1-1 correspondence with the sequence of the natural numbers. According to set theory and modern mathematics, the set of rationals numbers and the set of natural numbers have the same *cardinality* because they can be brought into a 1-1 correspondence.

On the other side, we have the irrational numbers. Those are numbers that are not equal to any fraction (*i.e.* we cannot write an irrational as a fraction of naturals). A very famous example is π. The symbol of π stands for the number 3.1425926…, however, unlike the decimal expansion of 1/7, namely 0.142857…, π’s three dots do not stand for a periodicity. There is no periodicity in the decimal expansion of π , for if there were, it would be a rational number. Thus, when we expand π, we cannot know what the next number is until we calculate it.

On this basis, and with the help of Cantor’s Diagonal, mathematicians and logicians claim that the set of all irrationals numbers is nondenumerable. We cannot construct a sequence that has *all* irrational numbers, which means that there is no method for putting the irrationals in a 1-1 correspondence with the natural numbers.

One important aspect of the rationals is that we do not need to operate with the decimal expansion at all. We can ignore the infinitely long decimal expansion, *e.g.* 0.142857…. This is not to say that the decimal expansion is useless, but only that we do not need to rely on it to use the mathematical Csign 1/7 in a meaningful way. We can work only with fractions.

However, in the case of the irrational numbers, the three dots do not stand for a periodicity. This leads us to think that we need infinity and actual infinite extensions in mathematics, otherwise we would miss or omit infinitely long irrational numbers.

6.2. The Standard View on Irrational Numbers as Infinite Extensions

The standard view claims that real numbers, such as π, are infinitely long extensions. On this view, when we expand π we discover each next digit of π. This entails that there *exist* such infinitely long numbers, as mathematical facts, in a platonic realm. In other words, in the same way that a biologist discovers new properties of a species of cicada, so too a mathematician discovers the 369^{th} decimal place of π, or of √2. More precisely, humans discovered that the digit 0 *was always at* 369^{th} decimal place of π.

The older 18^{th} and 19^{th} Century views of infinite extensions in mathematics was extended in the late 19^{th} Century by Georg Cantor's proof that there are more real numbers than rational numbers or natural numbers. On the standard view, by showing that the set of all real numbers is nondenumerable, Cantor proved that the set of all real numbers is larger than the set of the natural numbers. We cannot put the set of all real numbers in a 1-1 correspondence with the sequence of the natural numbers; therefore, the cardinality of the set of real numbers is greater than the cardinality of the natural numbers, *i.e.* there are *more* real numbers than natural numbers.

The proof can be easily executed by *reductio ad absurdum*. Let us assume that we can put the real numbers between 0 and 1 and the natural numbers into a 1-1 correspondence. The first real number, matched with the natural number 1, would have the digits in the following structure: 0. a_{11} a_{12} a_{13} a_{14} $a_{ij\ldots}$. where *a* is a positive integer and the subscript *i* and *j* tells us the position that the term *a* occupies in the table, *i.e.* the row and the column, respectively. The structure of the first four real numbers in the list is given in Figure 1 below:[105]

0.	$\mathbf{a_{11}}$	a_{12}	a_{13}	a_{14}	a_{15}…	→	1
0.	a_{21}	$\mathbf{a_{22}}$	a_{23}	a_{24}	a_{25}…	→	2
0.	a_{31}	a_{32}	$\mathbf{a_{33}}$	a_{34}	a_{35}…	→	3
0.	a_{41}	a_{42}	a_{43}	$\mathbf{a_{44}}$	a_{45}…	→	4
0.	a_{51}	a_{52}	a_{53}	a_{54}	$\mathbf{a_{55}}$**…**	→	**5…**

Figure 1: Cantor's Diagonal Proof

Given this assumption, we can construct a real number that is different in every a_{ij} where i = j. To do this, we can take, e.g., the first digit of the first real number, a_{11}, and add one (+1). So, if this digit is '1,' we begin constructing our *different* real number by making its first digit '2.' For the next real number, we take a_{22} and add 1; so if a_{22} is '2,' our new number will have an '3' at the 2nd decimal place. And so on. For every a_{ij} in the array where i = j (the red bold terms in each line, illustrated above), our newly constructed

[105] Figure 1 owes much to Professor Kent Peacock's (unpublished) logic book.

number will have a digit at the i^{th} decimal place that differs from the $a_{ij}{}^{th}$ digit by one. This can be shown more concretely, as in Figure 2:

0.	**1**	1	2	2	2…	→	1
0.	1	**3**	3	7	4…	→	2
0.	1	4	**8**	5	6…	→	3
0.	1	3	3	**6**	9…	→	4
0.	2	1	2	3	**4…**	→	**5**

Figure 2: An Example of Cantor's Diagonal Proof

In this more concretely example of the proof, the newly constructed number differs from each real number in the list at the n^{th} decimal place. Since our newly constructed number differs from each real number at the nth decimal place, it follows that this sequence of real numbers does not contain *all* real numbers, for our newly constructed number, *e.g.* **0.24975**…, is not in the sequence.

This contradicts our assumption that we had listed all of the real numbers. It follows, according to set theory and modern mathematics, that the initial assumption that *all* the reals between 0 and 1 can be brought into a 1-1 correspondence with the naturals is false. Therefore, the set of all real numbers are nondenumerable in the sense that the set of real numbers cannot be brought into a 1-1 correspondence with the naturals.

Put differently, we cannot put the set of all real numbers, or even the reals between 0 and 1, in a 1-1 correspondence with the natural numbers. This leads most mathematicians and logicians to conclude that Cantor's diagonal proof *shows* that the set of all real numbers are greater in size than the set of the natural numbers. They draw this conclusion because 1-1 correspondence is the criterion of equinumerosity.

6.2.1. A Critique of the Standard Conception of Mathematical Infinity: Infinity is not a Number

Wittgenstein gives us a completely different understanding of the notion of infinite in mathematics. The first aspect of his argument is to show that "infinite" is not a number. "We mistakenly treat the word 'infinite' as if it were a number word, because in everyday speech both are given as answers to the question 'how many?'" (PG, p. 463). If, e.g., someone asks us "How many natural (rational, irrational) numbers are there?", we are first inclined to say (i.e., before leaning of Cantor's Diagonal proof), "Well, infinitely many". The word 'infinite', in this sense, refers to the cardinality of sets, *i.e.* the *size* of non-finite sets. In other words, infinite works as a number that represents something *endless*.

Secondly, we behave as if we understand infinite mathematical extensions when, in fact, mathematical infinity is not something that we can imagine in its (alleged) totality. For example, people say that they can imagine a line of apples that is infinitely long. We can just imagine that no matter how long we walk, we cannot see the end of the line, that is, there will be no last apple. In reality, however, we cannot actually imagine an *infinite row apples*. We can only imagine a non-ending row of apples, where every time that we walk further along the row, we see a new apple. However, I do not believe that anyone can imagine *an infinite set of apples*. This raises the question: Can we actually imagine the infinite extension of π? In other words, can we imagine *all of the* numbers in a line that is infinitely long given that there is no last or terminal digit in decimal expansion of π? To borrow from Wittgenstein, we seem to confuse our ability to imagine an enormously long (finite) decimal expansion (e.g., with $100{,}000^{100{,}000}$ digits) with our inability to imagine an infinite long row of digits or apples.

We are drawn to this way of thinking of mathematical infinity because we wrongly associate infinity with something very large. However, this view is just a "figure of speech". As Hilbert famously said in 1925:

> "Just as in limit processes of the infinitesimal calculus, the infinite in the sense of the infinitely large and the infinitely small proved to be merely a figure of speech, so too we must realize that the infinite in the sense of an infinite totality, where we still find it used in deductive methods, is an illusion" (Hilbert, 1983 [1925], p. 184)

The actual infinite is nowhere to be found in reality. It is merely a figure of speech, that gives us the incorrect idea that infinity is a number. "The infinite divisibility of a continuum is an operation which exists only in thought. It is merely an idea which is in fact impugned by the results of our observation of nature of our physical and chemical experiments" (Hilbert, 1983 [1925], p. 186). In other words, the idea of space that is infinitely divisible is only a result of our imagination. Infinity, as Hilbert (1983 [1925] suggest, is nowhere to be found in reality. And yet, very interestingly, we *can* construct a rational number between *any* two rational numbers, i.e., the rational numbers are dense in a constructive sense.

As Rodych (1999a, p. 281) argues, "[a]n irrational number is, therefore, both a sign, such as '√2', and the unique rule for constructing finite, rational 'approximations' in some base or other". An irrational number stands for a rule that enables us to construct fractions. For example, π stands for the rule that enables us to construct and use the rational number 3.14, or the rational number 314/100, etc. In other words, the three dots of 3.14… only indicate that there are no in principle limitation on the length of the fractions for π that we construct, namely whether we stop at 3.1425… or at 31425/10000. In this sense, an irrational number is a law or rule that yields finite extensions.

In summary, an irrational number is only a number so far as we have a way to compare it with a rational number. In other words, we have a procedure that enables us to work with finite expansions of an irrational number, for the rule that is π gives us a fractions. "[N]o matter how the rule is formulated, in every case I still arrive at nothing else but an endless series of rational numbers" (PR, §180). An irrational is not an "infinite extension, but once again an infinite rule, with which an extension can be formed" (PR, §183).

6.2.2. Wittgenstein's Constructivist Interpretation of Genuine Irrationals and Cantor's Diagonal Proof

As I already highlighted in previous sections, Wittgenstein argues that an irrational is not an infinitely long extension, but it is an unlimited rule that enables us to construct fractions. For example, the √2 gives us "longer rational expansions (rationals 'approximations') whose squares converge to 2" (Rodych, 1999a, p. 281). The irrational √2 enables us to construct the fractions 141/100, 1414/1000,14142/10000, etc.; where the square of each number approximates ever more closely the number two (*e.g.* in decimal base: 1.9881, 1.999396, 1.99996164).

In this sense, mathematical infinity is not an *extension*. We cannot use the word 'infinite' to answer the question 'How many stars are there, in the universe?". As

Wittgenstein says, "[w]here the nonsense starts is with our habit of thinking of a large number as closer to infinity than a small one" (PR, §138). We believe that 300 trillion are closer to infinity than 10, however this is an illusion based on our understanding of infinity (as something really big). This would be the same as to assume that if I walk 10,000 kilometers, I am closer to my infinitely distant destination than if I walked only 10 kilometers, even though I will *never* be able to reach my destination.

Thus, an "infinite set" is not an extension or a sequence. In the same sense as for the irrationals, an infinite set is only a rule. It enables us to construct an unlimited sequence of mathematical terms.

In the same way that an irrational number is a recursive rule for constructing fractions, an infinite set in mathematics is a recursive rule that enables us to construct an unlimited sequence of new finite symbolic extensions. Or as Rodych (2000a, p. 286) argues, "[t]here is no such thing as *the set* of natural numbers, or *the set* of even numbers, or *the set* of real numbers", where these are meant as a mathematical extension. We only have "the infinite possibility of finite series of numbers" (PR, §144).

Wittgenstein's interpretation of the irrationals and of infinite sets already indicates Wittgenstein's criticism of the standard interpretation of Cantor's diagonal proof. Wittgenstein does not agree with the idea that such a proof *proves that* there are different cardinalities of infinite sets or different sizes of infinite sets. For him, Cantor's proof only proves that *we cannot construct the set of all real numbers*.

The first important aspect of Wittgenstein's argument is that he reconstructs Cantor's proof without using infinitely long decimal expansions of real numbers. Instead, Wittgenstein constructs an array of square roots of natural numbers, and then proceeds to show that we cannot construct a sequence of *rules* for *all* real numbers.

Wittgenstein does not use numbers in decimal notation, where the three dots at the end of every real number *represents* an infinitely long expansion. He presents the proof as constructible enumeration of rules for the square roots of the natural numbers, *e.g.* $\sqrt{1}$, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$, …, $\sqrt{n}$…; that is, he presents the proof as a proof that constructs *rules* from a rule, where each constructed rule enables us to *generate* a fraction, a finite extension, for that specific real number (e.g. for √3). The goal of Wittgenstein's proof construction is the same, *i.e.* to show that we cannot have a sequence of all real numbers, by giving a method to construct the diagonal number: a number that its n-th term is different in every n-th decimal place of the corresponding square root:

$\sqrt{1}$	1.	0	0	0	0	0…	→	1
$\sqrt{2}$	1.	4	1	4	2	1…	→	2
$\sqrt{3}$	1.	7	3	2	0	5…	→	3
$\sqrt{4}$	2.	0	0	0	0	0…	→	4

Figure 3: Wittgenstein's Finitistic View of Cantor's Diagonal Proof

Wittgenstein's construction of Cantor's proof (as well as Cantor's proof itself) does not show that *the set of all real numbers* "has a greater 'multiplicity' than others" (Rodych, 2000a, p. 293). As he argued, "[i]f we want to see what has been proved, we ought to look at nothing but the proof" (PR, §163).[106] What Wittgenstein's construction of Cantor's proof

[106] "If you want to know what the verbal expression means, look at the calculation; not the other way about" (RFM, II - §7)

shows is that the set of all recursive reals (or all computable reals) is *not enumerable by a rule*. We cannot enumerate the set of recursive reals; the so-called set of all real numbers is non-enumerable.

As we can see, Cantor's proof gives us a proof that shows that we cannot have or construct a sequence of all real numbers, meaning that we cannot have or construct such a thing as *the set of all real numbers*. "When people say 'The set of all transcendental numbers is greater that that of algebraic numbers', that's nonsense. The set is of a different kind. It isn't 'no longer' denumerable, it's simply not denumerable!" (PR, §174).

For Wittgenstein, it means nothing to say that "[*t*]*herefore* the X numbers are not denumerable" (RFM, II - §10). Cantor's proof only shows us that, "if anyone tried day-in day-out 'to put all irrationals numbers into a series' we could say: 'Leave it alone; it means nothing; don't you see, if you established a series, I should come along with the diagonal series!'" (RFM, II - §13).

6.3. Mathematical Induction

One of the main consequences of Wittgenstein rejection of infinite mathematical extensions (including infinite mathematical sets *as* extensions) is that we cannot quantify over an infinite domain, or as Rodych (2000b, p. 255) argues, "we cannot *meaningfully* quantify over an infinite mathematical domain, simply because there is no such thing as an infinite mathematical domain (i.e. totality, set)".

This leads Wittgenstein to claim that there is no such thing as a proposition about *all numbers*, "simply because there are infinitely many" (PR, §126). We cannot represent statements about all numbers by means of a proposition (WVC, p. 82). Such “propositions” are mathematically meaningless “because the Law of the Excluded Middle does not hold in the sense that we do not know of a decision procedure by means of which we can make the expression either ‘true’ [correct] or ‘false’ [incorrect]” (Rodych, 2000b, p. 264).

We can now clearly see that Wittgenstein’s view on quantification over an infinite domain derives directly from his Finitism and his conception of mathematically meaningful propositions. In addition, the notion of an infinite mathematical extension, as I have already shown, does not makes sense:

> Let’s imagine a man whose life goes back for an infinite time and who says to us: ‘I’m just writing down the last digit of π, and it’s a 2’. Every day of his life he has written down a digit, without ever having begun; he has just finished. This seems utter nonsense, and a *reductio ad absurdum* of the concept of an infinite *totality* (PR, 145).

Since there is nowhere to arrive, the man has never even begun to write a number, that is, he is not closer to finish than when he started. Nonetheless, because his life goes on to infinite, as allegedly the values of π, then he already finished it, *i.e.* he is in the same “place” as the last digit of π. This attempt to describe and explain Wittgenstein’s thought experiment already shows how misleading and incomprehensible the idea of an infinite mathematical extension is.

In opposition to Wittgenstein’s Finitism, Michael Potter argues that it is a mistake to assume that mathematics does not need quantification over an infinite domain. He argues that a) Wittgenstein’s thought experiment is the closest thing that he has to an argument against the actual infinite in mathematics; and b) that Wittgenstein’s argument does not clearly state a problem for infinite sets, but only for "our conception of a task performed in time"[107] (Potter, 2011, p. 126).

However, I believe that Potter's first argument (a) does not take into consideration Wittgenstein's philosophy of mathematics as a whole. Potter does not recognize, it seems, several of Wittgenstein's arguments against the notion of the actual infinity. It seems clear that that thought experiment only illustrates Wittgenstein's general argument against such idea. As we can see, Wittgenstein has multiple criticisms, direct and indirect, of the idea of an actual infinite set: 1) the idea that infinity is not a number (PG, p. 463; PR, §138); 2) the mistake of conflating *extension* with *intension* in relation to irrational numbers (PR, §§180-186); 3) Wittgenstein's criticism to nondenumerable sets (as I have shown in the previous sections of this chapter); and 4) Wittgenstein's view that only Csigns that have a known decision procedure are mathematically meaningful (as I have shown in chapter 5); to list only a few. I believe that those arguments constitute a reasonable argument against the standard interpretation of infinite sets in mathematics and, as far as I can tell, Potter fails to consider these arguments in his evaluation.

In relation to Potter's second argument (b), it appears that he misconstrued Wittgenstein's philosophy of mathematics. First, it seems that Potter assumes that an infinite set is not related to our capacity to construct such set. However, for Wittgenstein, if we cannot construct a proof of a Csign in a calculus, that Csign is not a mathematical proposition. Potter's idea that Wittgenstein's thought experiment only speaks to our ability or inability to construct the proof, and not to the thing that we are trying to prove,

[107] "But even if we agree with Wittgenstein that this is utter non-sense, it is far from clear that what is nonsensical about it is the fault of the infinite totality involved, rather than something inherently directional in our conception of a task performed in time" (Potter, 2011, p. 126).

completely dismisses Wittgenstein's idea of proof and mathematically meaningful propositions.

However, as Wittgenstein himself recognizes (PR, §126), his Finitism generates a conception of mathematical induction which conflicts with how mathematical induction is standardly interpreted and presented symbolically.

As Rodych (2000b, p. 258) puts it, "[g]iven that one cannot quantify over an infinite domain, the question arises: What, if anything, does any number-theoretic proof by mathematical induction actually *prove*"? As we can see a proof by mathematical induction is constructed in the following way: we prove $\varphi(1)$ (*i.e.* the *Inductive Base*) and the $\forall n(\varphi(n) \rightarrow \varphi(n + 1))$ (*i.e.* the *Inductive Step*); and then we infer or conclude *that* $\forall n\varphi(n)$. This is the standard view of mathematical induction. The important aspect of this proof is that by proving the inductive base and the inductive step we prove that *all numbers have the property φ*, because we can validly infer the universally quantified conclusion from the proved inductive base and the proved inductive step.

As Wittgenstein argues, the standard interpretation of mathematical induction claims that it *shows* a (syntactical) route or path that establishes that *all numbers* have the property: It "sounds as if here a proposition saying that such and such holds for all cardinal numbers is proved true by a particular route, and as if this route was a route through a space of conceivable routes" (PG, p. 406).

However, the alleged conclusion of an inductive proof is *not shown by the proof*. A proof by induction does not give us a *method of verifying* the conclusion ($\forall n\varphi(n)$). Rather, it proves that a base number has the property φ, and that if *any* arbitrary number has the property φ, then that number's successor also has the property φ. But "that it holds for every number" is not shown by the proof. We *jump* to that conclusion. Or we erroneously think that we must conclude something about the amorphous "all numbers". However, by concluding that the property φ holds *for all numbers* we go beyond what an inductive proof actually proves. As Wittgenstein argues:

> We are not saying that when f(1) holds and when f(c + 1) follows from f(c), the proposition f(x) is therefore true of all cardinal numbers; but: 'the proposition f(x) holds for all cardinal numbers' means 'it holds for x = 1, and f(c + 1) follows from f(c). (PG, p. 406)

Wittgenstein argues that a proof by induction actually *show us* the *infinite possibility* of the application of a recursive rule. It is not a proof in the ordinary sense, where each derived proposition connects syntactically to the alleged terminal proposition (∀nφ(n)). As Marion argues (1995, p. 151 [italics mine]), "the distinguishing feature of proofs by induction was that the proposition proved *does not* appear as the last step of the proof, so that the proof is not a proof of the proposition *per se*".

A proof by mathematical induction is a recursive rule that constitutes "a general guide to an arbitrary special proof. A signpost that shows every proposition of a particular form a particular way home" (PR, §164). The distinction between an arithmetical proof and a proof by induction is that mathematical induction does not *prove* the conclusion, (∀nφ(n)) – the last line of a standard inductive proof.[108]

[108] "The most striking thing about a recursive proof is that what it alleges to prove is not what comes out of it" (PR, §163).

As Rodych (2000b, p. 259) shows, Wittgenstein argues that a proof by mathematical induction proves, e.g., $\varphi(1)$ (*i.e.* the *Inductive Base*) and $\varphi(n) \rightarrow \varphi(n + 1)$ (*i.e.* the *Inductive Step*). *That* is what we actually prove. If someone speaks of the "conclusion" $\varphi(m)$, we can only say that there is no such conclusion, and a pseudo-proposition such as $\varphi(m)$ only "stands proxy for" the proved inductive base and inductive step. But it is really the proof of the base and step that shows us that we can reiterate *modus ponens* to arrive at a proof that any particular number has property φ. We did not *prove $\varphi(m)$*.

We are now in a position to compare Wittgenstein's description of mathematical induction and the standard view of mathematical induction:

Mathematical Induction according to the Standard View	***Mathematical Induction according to Wittgenstein***
Inductive Base: $\varphi(1)$	**Inductive Base:** $\varphi(1)$
Inductive Step: $\forall n(\varphi(n) \rightarrow \varphi(n + 1))$	**Inductive Step:** $\varphi(n) \rightarrow \varphi(n +1)$
Conclusion: $\forall n \varphi(n)$	**----**

Figure 4: Mathematical Induction – Wittgenstein vs Standard View

By avoiding quantification over an infinite domain, Wittgenstein's description of proof by mathematical induction is "not meant as a rule for algebraic calculation, but as a *device* for *explaining arithmetical expressions*. It *represents an operation,* which I *can* apply to an arbitrary pair of numbers (PR, §164 [italics mine]). An actual inductive proof *shows* that we can construct a direct proof to *any particular number*. In this sense, Wittgenstein's view of mathematical induction shows that we do not *need* quantification over an infinite domain. A proof by mathematical induction enables us to *cognitively see* that we have a direct route/path, by *modus ponens*, to prove F for *any particular number*.

Once we know a recursive rule that shows us that it holds for the base case and the inductive step, it "spares me the trouble of proving each proposition of the form [φ(m)]" (PR, §164). The "conclusion merely shows us this, and "once you've got the induction, it's all over" (PG, p. 406). That is why the conclusion is left *blank* in Figure 3 — on Wittgenstein's account there is no conclusion or third-step of induction; the so-called conclusion is nothing more than the restatement of the inductive base and the inductive step.

In this specific way, Wittgenstein is arguing that after we have proved the inductive base and the inductive step, "it is all over': the proof is complete, now we can *see* that *any number has the property* φ. Why do we need to show that *all numbers have the property* φ? The alleged conclusion, ∀nφ(n), is not part of the proof. In other words, can we *show that all numbers have the property φ*?

Interestingly, Henri Poincaré claimed that the conclusion of a proof by mathematical induction is synthetic *a priori*, rather than analytic, "since its conclusion 'goes beyond' its premises rather than being a mere restatement of them "in other words" (Detlefsen, 1992, p. 213). Thus, Poincaré argues that a proof by mathematical induction is ampliative and synthetic *a priori.* It is ampliative in that its conclusion says more than its premises, since it premises state *only* that a particular number has the property φ, and that if any number has the property φ, the successor of that number has the property φ. The conclusion of an inductive proof states that *all numbers* (from the base number onward) have the property φ, and thus, in this way, the conclusion goes beyond the premises.

On Poincaré's view, this type of proof is synthetic *a priori* because it is known to be non-analytic and to be valid and truth-preserving by the mind. Most mathematicians and philosophers agree with Poincaré: They believe that proof by mathematical induction proves that every element of an infinite set has a property, and it does this because we can see that we can repeat *modus ponens* inferences *ad infinitum*; *therefore*, all numbers have the property φ.

The most important question is: Do we need the conclusion that *all numbers have the property φ*? Everyone grants that the proved inductive base and the inductive step, together, enable us to prove of *any* number that *it* has the property φ. Wittgenstein agrees that that is what has been proved. Poincaré and Wittgenstein agree that an inductive proof licences a direct proof that any particular number has the property in question. Why do we need or why do we say that *all* numbers have the property φ? The reason seems to be that there is a tradition of thinking and speaking this way. We only ever need, however, to prove that a particular number has the property φ.

In a similar sense, Potter (2011, p. 128) argues that Wittgenstein's account on induction "does not explain why we feel entitled to infer from (x)φ(x) to φ(n) for any number *n*". Potter's main argument states that:

> "if we wish to prove (x)φ(x) by mathematical induction, we must prove two things: first we check that φ(0) holds; then we prove (x)(φ(x) → φ(x + 1)). If the meaning of (x)φ(x) is an induction, then in the same way we would expect the meaning of (x)(φ(x) → φ(x + 1)) to be an induction too. Moreover, the second of these expressions is logically *more* complex than the first. We might try to relieve this difficulty by making a distinction between generalizations which have inductive proofs and ones (such as $(x+y)^2=x^2+2xy+y^2$) which have free variable proofs; but the relief is only temporary. The problem is that in most systems these proofs depend in their turn on other inductive proofs (Potter, 2011, p. 128).

Potter argues that this leads to an infinite regress, where the meaning of each proposition appeals to the next one (Potter, 2011, p. 128). According to him, since Wittgenstein suggests that the meaning of an arithmetical generalization is its inductive proof, then every occurrence of the universal quantifier must be replaced by an inductive proof.

First, it is important to notice that Potter seems to be mistaken about Wittgenstein's view on induction. As I have shown, Wittgenstein does not claim that we "infer from $\forall n\varphi(n)$ to $\varphi(n)$"[109]. On the contrary, we do not infer φm (a proxy statement) from a proved inductive base and a proved inductive step. The proxy statement is eliminable, and ought to be eliminated if it misleads (as it seems to mislead Potter). Furthermore, we don't use $\forall n\varphi(n)$ either as a conclusion or as a premise in our mathematical proofs. What Wittgenstein claims is that mathematical induction is a recursive rule. It does not need quantification over an infinite domain. On his account, we do not infer the universally quantified $\forall n\varphi(n)$ from the inductive base and inductive step. Instead, we prove that a particular number has the property φ, and we prove that if any arbitrary number n has the property φ, then $n + 1$ has the property φ.

Although Potter raises some issues for Wittgenstein's Finitism and his view on mathematical induction, it is still not clear what those issues are. In fact, despite Potter's claims that Wittgenstein's account of induction leads to an infinite regress, he does not explicitly show how that is the case..

It also seems that Potter thinks that we need the universal quantifier both on the conclusion and on the Inductive Step. However, this seems to beg the question, for he does not give an argument for why that must be the case. It is not clear why we need the universal

[109] As we can see in the above citation, Potter uses the symbol '(x)' as the universal quantifier and the variable 'x' to show the structure of a proof by mathematical induction. In order to maintain a consistent formatting with my previous argument, I change his notion to '∀' and 'n', respectively.

quantifier in the Inductive Step, or why Wittgenstein's account of induction (as a recursive rule of reiterations of *modus ponens*) does not work. In fact, it seems that there were others attempts to show how quantifier-free induction works.[110]

6.4. Wittgenstein's Criticism of Set Theory as an Argument in Favour of his Philosophical View

Another aspect that is important to mention here is that Wittgenstein's construction of Cantor's proof *is an argument in defense of his view*. So far, I have shown that Wittgenstein's interpretation only attempts to *solve linguistic confusions*. Even though he has some strong criticisms of the standard view, Wittgenstein's philosophy of mathematics tries to *accurately describe mathematics as it is*, *i.e.* how mathematicians actually *do* mathematics.

However, as I already shown, Wittgenstein's criticisms of set theory (especially transfinite set theory) attempt to *change the ways in which logicians and mathematicians work.* Nowhere can we see this better than in relation to the concept of mathematical infinity. The stronger consequences of Wittgenstein's claims—*e.g.* that there is a system for the rationals, but no system of all irrationals—shows that he believes that there is something essentially incorrect in the workings of set theory. Moreover, Wittgenstein explicitly says that future generations will abandon and laugh about this "hocus pocus".

[110] I believe that Thoralf Skolem has attempted to demonstrate a different constructive approach to mathematical induction. However, I will only cite his name here, and not try to explain his view, since neither I, nor Dr. Victor Rodych, has adequate knowledge of Skolem's work.

We can definitely see that Wittgenstein is *trying* to only describe mathematics. Nonetheless, in relation to transfinite set theory, it seems that he considered it, as a whole, a conceptual-linguistic confusion. As he boldly says, "[s]et theory is wrong because it apparently presupposes a symbolism which doesn't exist instead of one that does exist (is alone possible). *It builds on a fictitious symbolism, therefore on nonsense*" (PR, §174 [italics mine]).

Wittgenstein's arguments, it seems, are not just attempting to better describe mathematics. They seem to revise set theory, or prescribe revisions to set theory (e.g., abandoning transfinite set theory entirely) by criticizing the concept of infinity and how logicians and mathematicians use and interpret Cantorian diagonalizations.

It seems that the justification for such reformulation of set theories is similar to his analysis of psychology:

> The confusions and barrenness of psychology is not to be explained by its being a 'young science' its state is not comparable with that of physics, for instance, in its beginning. (*Rather, with that of certain branches of mathematics. Set theory*.) For in psychology, there are experimental methods *and conceptual confusion*. (*As in other case, conceptual confusion and methods of proof.)* (PP, §371).

Here it seems that Wittgenstein is telling us that the problems in set theory (and in mathematics) are "conceptual confusions". That would show that some *language-moves in set theory*, that logicians and mathematicians make or work with, are not justified, because they are based on conceptual knots. Although Wittgenstein's argument aims only to untangle conceptual knots, it does seem to be more than just a description.

6.5. Wittgenstein on Mathematical Undecidability

In his famous "Mathematical Problems" (1900) David Hilbert declared that there is no *ignorabimus* (*i.e.*, there are no unanswerable questions) in mathematics. Hilbert reiterated this claim many times in print throughout his life, and, as far as we know, until his death in 1943. Although Hilbert's original 1900 pronouncement was probably that there are no "absolutely undecidable" mathematical propositions, later in his career he focused more on the decidability of First-Order Logic and Elementary Number Theory (e.g., PA). In 1928 in particular, he announced four questions, two of which were the decidability of First Order Logic and the completeness of elementary number theory.

In his 1907 Ph.D. Dissertation, L.E.J. Brouwer explicitly questioned Hilbert's claim about the decidability of all mathematical propositions and wrote that Hilbert has not *proved* that, in principle, all mathematical propositions are decidable. Brouwer went on to give several putative examples of undecidable mathematical propositions: e.g., "*Do there occur in the decimal expansion of π infinitely many pairs of consecutive equal digits*? (Brouwer 1908, p. 110).[111]

[111] [In (Brouwer 1907), Brouwer writes (Footnote #3): "A fortiori it is not certain that any mathematical problem can either be solved or proved to be unsolvable, though HILBERT, in "*Mathematische Probleme*", believes that every mathematician is deeply convinced of it. But for this question as well it is of course uncertain whether it will ever be possible to settle it, i.e. either to solve it or to prove that it is unsolvable (a logical question is nothing else than a mathematical problem)". In his (1908), Brouwer adds: "It follows that the question of the validity of the *principium tertii exclusi* is equivalent to the question whether unsolvable mathematical problems exist. There is not a shred of a proof for the conviction, which has sometimes been put forward, that there exist no unsolvable mathematical problems". The footnote is, again, to (Hilbert 1900, 1902).

6.5.1. The Intermediate Wittgenstein on Mathematical Undecidability

Wittgenstein's earliest writings on decidability and undecidability in mathematics can be seen immediately after his return to philosophy, in 1929, about a year after attending Brouwer's March 1928 Vienna lecture "Science, Mathematics, and Language" (Brouwer, 1929). In his earliest 1929 manuscripts, Wittgenstein re-started his work on the philosophy of mathematics by framing the notion of mathematical *propositions* as algorithmic decidability within a calculus. This novel approach to mathematics immediately raises the question, "What, if anything, does Wittgenstein say about mathematical *undecidability*?"

The first aspect of Wittgenstein's earlier work on undecidable mathematical propositions is that he rejects them. For Wittgenstein, "if there is no method provided for deciding whether the proposition is true or false, then it is pointless, *and that means senseless*" (PG, p. 452 [italics mine]). Wittgenstein has two main reasons for rejecting undecidable mathematical propositions: a) a Csign that quantifies over an infinite domain is not algorithmically decidable; and b) that if we do not have a decision procedure for a Csign, then that Csign is not a mathematical proposition (Rodych, 1999).

Wittgenstein's first reason (a) for denying undecidable mathematical *propositions* is that, given that they quantify over an infinite domain (extension), they are *not* mathematical propositions. Goldbach's Conjecture (GC) is one such example. If GC is restricted to the first 1,000,000,000 even numbers, it "quantifies over a finite domain," which means that we have a decision procedure for it (even if the length of complexity of the proof, for a larger number of even numbers, would require a computer to execute the check).

Second (b), a Csign that has an unrestricted universal quantifier (and, therefore, which allegedly makes a claim about *all numbers*) is not a mathematical proposition, and, therefore, it does not have *sense*. In 1929-30, before Gödel's 1930-31 paper, Wittgenstein asserts, "[o]nly where there's a method of solution is there a [mathematical] problem" (PR, §149). Two to three years later, he similarly claims that "[i]f there is no method provided for deciding whether the proposition is true or false, then it is... senseless" (PG, p. 451).[112] Wittgenstein rejects mathematical propositions about infinite extensions and infinite sets, and argues, as I said in section 6.3, that even in the case of a proof by mathematical induction, where people *think* that we have proved that all elements of an infinite set have a particular property, we have really only proved an inductive base and an inductive step, *not* that infinitely many numbers have a particular property.

Contrary to Brouwer, for Wittgenstein the statement that "it is not the case that there are three consecutives seven in the decimal expansion of π" is not a mathematical proposition at all. It is not the case that the Law of the Excluded Middle does not apply here (as Brouwer suggested), but that we are not dealing with a (meaningful, genuine) mathematical proposition. Or as Wittgenstein (PR, §151) states it, "where the law of the excluded middle doesn't apply, no other law of logic applies either, because in that case we aren't dealing with propositions of mathematics (Against Weyl and Brouwer)".

Later in PR, we see that Wittgenstein makes a direct reference to undecidability and undecidable propositions:

> Undecidability presupposes that there is, so to speak, a subterranean connection between the two sides; that the bridge cannot be made with symbols. A connection between symbols which exists but cannot be represented by symbolic transformations is a thought that cannot be thought. If the connection is there, then it must be possible to see it. (PR, §174)

[112] See also (PR, §§148-152), (PG, p. 366), (PG, p. 387), (PG, p. 451-52), and (PG, p. 468).

In this passage, Wittgenstein draws our attention to the fact that we *assume* that undecidable propositions *are meaningful mathematical propositions* in some mathematical language (e.g., the language of PA or of ZFC). However, we know that there is no such connection between the conjecture (e.g., GC) and the system that in which it allegedly resides.

For instance, if we assume that GC is undecidable in PA, it must be true and undecidable in PA (since, if it is false, it is decidable by enumeration to a counter-example), and "this implies that '(*x*)...' is meant extensionally and … [that] all *x* happen to have a property," whereas, "[i]n truth, … it's impossible to talk of such a case at all and the '(*x*)...' in arithmetic cannot be taken extensionally" (PR, §174). For Wittgenstein, the idea that GC is true and undecidable, presupposes that there is a structural property or reason *why* every even number greater than 2 is the sum of two primes, but this symbolic-syntactical property cannot be seen, thought, or shown by a mathematical construction (e.g., an inductive proof).

Thus, for Wittgenstein, an undecidable proposition is a "contradiction-in-terms" (Rodych, 1999b, p. 176). A genuinely undecidable Csign does not *belong to a calculus* because it is syntactically independent of that calculus. Additionally, because an undecidable Csign does not have a proof within an existent mathematical calculus, it also does not have any mathematical sense. It is, therefore, unreasonable to suggest that a Csign such as GC—a mathematical conjecture in number theory—is a mathematical proposition, since we *know* that it does not belong to any calculus.

6.5.2. Wittgenstein on Undecidable Mathematical Propositions in Relation to Gödel's First Undecidability Theorem

In 1937-38, Wittgenstein returns to mathematical undecidability, but now his focus is Gödel's First "Incompleteness Theorem", or, as it should more accurately be called, Gödel's First *Undecidability Theorem*. In (RFM, I – App. III), Wittgenstein discusses what he takes to be Gödel's First Undecidability Theorem, although, interestingly, Wittgenstein does not mention the name 'Gödel' in the appendix.

After some interesting preliminaries, Wittgenstein asks, at §5: "Are there true propositions in Russell's system, which cannot be proved in his system?" (RFM, I – App. III, §5). An affirmative answer to this question constitutes what most people take Gödel's First Undecidability Theorem to be: there are true but unprovable propositions in PM (*Principia Mathematica*). Wittgenstein quickly follows this question by asking another question: "What is called a true proposition in Russell's system, then?" (RFM, I – App. III, §5). If we assume that a particular Csign *P* is unprovable in PM, what *then* does it mean for it to be *true*? Wittgenstein is here questioning the alleged difference between "proved in PM" and "true" (or "true in PM").

Wittgenstein's answer to this question is simple and clear: "'p' is true = p. (That is the answer.)" (RFM, I – App. III, §6). On Wittgenstein's own descriptive account, if a mathematical proposition *P* is true, that means *only* that it is asserted in a mathematical calculus such as PM, and a mathematical proposition is only asserted in PM "at the end of one of his proofs, or as a 'fundamental law' (Pp.). There is no other way in this system of employing asserted propositions in Russell's symbolism" (RFM, I – App. III, §6). This is Wittgenstein's own long-standing position on so-called mathematical truth: a mathematical proposition is true in PM only in the sense that it is either an axiom (*i.e.* a primitive proposition [‘p.p.’]) of PM or a proved proposition in PM. About 8 years earlier (1930), Wittgenstein put the matter in exactly the same way, saying (PR, §202): “A mathematical *proposition* can only be either a stipulation, or a result worked out from stipulations in accordance with a definite method”.

Thus, it seems clear that, in 1938, Wittgenstein still regards an undecidable mathematical proposition as a *contradiction-in-terms*. For him, 1) a mathematical proposition is decidable by a known decision procedure, and 2) a “true mathematical proposition” is either an axiom of a calculus or a proved proposition within a calculus. If a particular Csign is actually undecidable in a calculus (i.e., syntactically independent of that calculus), it is not a mathematical proposition *of* that calculus. And if a particular Csign is actually a “true proposition of PM”, it must be an axiom or a proved proposition *in* PM.

In the remainder of (RFM I, App. III), Wittgenstein wrestles with his understanding of Gödel’s constructed proposition *P* and, what Wittgenstein takes to be, the reasoning of Gödel’s proof. It seems that Wittgenstein thinks, in 1938, that Gödel’s proof requires a self-referential, natural language interpretation of ‘*P*’ and ‘~ *P*’. The most important passage in this connection is the following:

> I imagine someone asking my advice; he says: "I have constructed a proposition (I will use '*P*' to designate it) in Russell's symbolism, and by means of certain definitions and transformations it can be so interpreted that it says: '*P* is not provable in Russell's system'. (RFM, I – App. III, §8)

This seems to be Wittgenstein’s core 1937-38 construal of the reasoning of Gödel’s proof. Wittgenstein seems to think that Gödel proves that *P* is “true but unprovable in PM” by constructing *P* in such a way that *P* says, *self-referentially*, that '*P* in not provable in Russell's system'. It certainly seems as if Wittgenstein interpreted *P* according to Gödel's informal statement in the introduction of his famous 1931 paper, where Gödel says: "the undecidable proposition [*R*(*q*);*q*] states... that [*R*(*q*);*q*] is not provable," and "[f]rom the remark that [*R*(*q*);*q*] *says about itself* that it is not provable it follows at once that [*R*(*q*);*q*] is true" (Gödel, 1992 [1931], p. 41 [italics mine]). So, speaking informally, Gödel claims that *P* actually says "*P* is not provable [in PM]." So, Wittgenstein seems to model his proposition *P* on Gödel's [*R*(*q*);*q*], and Wittgenstein seems also to follow Gödel's Introductory formulation[113] when he provides, what Wittgenstein takes to be, the Gödelian reasoning:

> Must I not say that this proposition on the one hand is true, and on the other hand is unprovable? For suppose it were false; then it is true that it is provable. And that surely cannot be! And if it is proved, then it is proved that it is not provable. Thus it can only be true, but unprovable (RFM, I – App. III, §8).

As we have just seen, this reasoning is very similar to Gödel's informal reasoning, namely that "[f]rom the remark that [*R*(*q*);*q*] *says about itself* that it is not provable it follows at once that [*R*(*q*);*q*] is true". For if [*R*(*q*);*q*] were *false*, it would be provable, and, thus, it would be provable that it is unprovable, and, hence, both provable and unprovable. And if

[113] In 1998, Georg Kreisel claimed, both in conversation with V. Rodych (Cambridge, U.K.) and in print, that he had personal conversations with Wittgenstein in which Wittgenstein said that, prior to 1942, he had not read Gödel's 1931 proof because he had been "put off by the introduction" (Kreisel, 1998, 119). Rodych points out (2003) that there is evidence in Wittgenstein's *Nachlass* indicating that Wittgenstein read more of Gödel's paper, in the period 1939-41, and also in 1942. In particular, 'Wittgenstein's (MS 163, III, §90) discussion of "I'm not K provable" certainly suggests that Wittgenstein is referring to Gödel's use of "κ-PROVABLE" and Theorem VI ("*For every ω–consistent recursive class κ of FORMULAS*…") in the actual body of his 1931 proof (van Heijenoort, 1967, 607-609), rather than "Gödel's casual [informal], opening proof-guide," where Gödel does *not* speak of "κ-PROVABLE" or of "κ-PROVABILITY," but rather defines "a class K of natural numbers" in terms of "*Bew x*," which is defined as "x is a provable formula" (1967, 598)' (Rodych 2003, p. 307).

[$R(q);q$] were provable, it would, again, be both provable and unprovable (i.e., it would be possible to prove that it is unprovable by virtue of its self-referential meaning).

By assuming that the Gödel's proof requires *P* to be interpreted as a self-referential proposition in natural language, Wittgenstein mistakenly represents Gödel's proof as *requiring* this natural-language interpretation of *P* (Rodych 1999b).[114] However, in his proof, Gödel constructs his number-theoretic *P* so that it is number-theoretically 'true' *iff* there does not exist a natural number that stands in a particular relation to another natural number—and Gödel constructs his number-theoretic Csign *P* so that there is no such natural number *iff P* is unprovable in PM.[115] As Rodych (1999b, p. 183) argues, "it is entirely unnecessary to give [this number-theoretic proposition] a natural language [self-referential] interpretation *to establish the bi-conditional relationship*".

We should not assume, however, that this mistake invalidates all of Wittgenstein's remarks on Gödel's proof. In fact, although Wittgenstein seems not to fully understand *how* Gödel's proof works, he seems to fully understand that a proof of the Gödelian *P* in PM would enable a proof ~ *P*, and that this would prove PM to be inconsistent. The key sections are (RFM, I App III, §§11, 17). At §17, Wittgenstein says:

> Suppose however that not-*P* is proved.—Proved *how*? Say by *P*'s being proved directly—for from that follows that it is provable, and hence not-*P*. What am I to say now, "*P*' or "not-*P*"? Why not both? If someone asks me "Which is the case, *P*, or not-*P*?" then I reply: *P* stands at the end of a Russellian proof, so you write *P* in the Russellian system; on the other hand, however, it is then provable and this is expressed by not-*P*, but this

[114] Some authors disagree that Wittgenstein mistakenly represented Gödel's proof. For example, Schoeder argues that Rodych's argument was "doubly unfair". On his view, Wittgenstein knew, and was correct in represent Gödel's proof in a self-referential way. Because this section consists of a simple explanation of Wittgenstein's idea on mathematical undecidability, I will not further evaluate Schoeder's argument. See (Schoder, 2020) for his complete argument.
[115] See (Rodych, 1999b, p. 183)

> proposition does not stand at the end of a Russellian proof, and so does not belong to the Russellian system. (RFM, I – App. III, §17)

Even though Wittgenstein's self-referential interpretation of Gödel's *P* makes him erroneously think that the proof of ~ *P* would not be in PM, he certainly thinks, *correctly*, that a direct proof of *P* would prove ~ *P*. At §11, Wittgenstein explicitly says that a proof of *both P* and ~ *P* would be a contradiction, which means, of course, that PM would be inconsistent.

> Let us suppose I prove the unprovability (in Russell's system) of *P*; then by this proof I have proved *P*. Now if this proof were one in Russell's system—I should in that case have proved at once that it belonged and did not belong to Russell's system. —That is what comes of making up such sentences.—But there is a contradiction here!—Well, then there is a contradiction here. Does it do any harm here? (RFM, I – App. III, §11)

This is Wittgenstein's *correct understanding* of what Gödel proved. Wittgenstein correctly thinks that Gödel has proved that a proof of *P* within PM would prove ~ *P*, and that, in that case, PM would contain a contradiction, and would, therefore, be inconsistent. This is *exactly* what Gödel proved: "If *P* is proved (or provable) in PM, then ~ *P* is proved (or provable) in PM, and PM is inconsistent". Gödel proved the conditional "if PM is consistent, *P* is not provable in PM" (and, with B. Rosser, more strongly: "If PM is consistent, *P* is undecidable in PM"). Wittgenstein's correct understanding of this, namely "If *P* is proved in PM, then PM is inconsistent", is the contrapositive of the first formulation, namely "If *P* is proved in PM, then PM is inconsistent".

Wittgenstein gives us the contrapositive of Gödel's First Undecidability Theorem. Gödel proved that "If *P* is proved/provable in PM, then PM is inconsistent." This proposition is logically equivalent to the conditional: "If PM is consistent, *P* is unprovable in PM." Gödel did *not* prove that PM is consistent, and, therefore, Gödel *did not* prove that *P* is unprovable in PM. Thus, Gödel *did not* prove that *P is number-theoretically true*, and he did not prove that *P* is *true* via the biconditional, because he did not prove that *P* is unprovable in PM.

6.5.3. Wittgenstein's Later Views on Gödel's First Undecidability Theorem

The most important fact about Wittgenstein's 1941 writings on Gödel's conditional undecidability theorem is that he seems to clearly understand a) how Gödel-numbering works, and b) how Gödel-numbering enables a direct proof of ~ *P* in PM from a direct proof of *P* in PM. At (RFM, VII - §22), Wittgenstein says: "Let us assume that we have an arithmetical proposition saying that a particular number... cannot be obtained from the numbers ..., ..., ..., by means of such and such operations". One section earlier, Wittgenstein states: "'The proposition says that this number cannot be got from these numbers in this way'" (RFM, VII - §21). This is exactly the form of Gödel's number-theoretic proposition *P*. Thus, in 1941, Wittgenstein seems to have an improved understand of Gödel's *P* and how Gödel's reasoning works with *P*.

However, in the same passages, Wittgenstein does say that "Gödel's proposition, which asserts something about itself, does not mention itself" (RFM, VII - §21).[116] Thus, it seems that Wittgenstein has not completely abandoned his idea that Gödel's proof is executed by interpreting '*P*' in a self-referential way.

[116] We can see that Wittgenstein insists on this self-referential approach to Gödel's proof in several passages, for example: "Do not forget that the proposition asserting of itself that it is unprovable is to be conceived as a mathematical assertion—for that is not a matter of course" (RFM, VII - §21).

I believe that if we consider especially RFM, VII - §21), Wittgenstein's RFM, VII - §21) does *not* invalidate Wittgenstein's descriptive account of what Gödel proved, and also the number-theoretic *way in which* Gödel proves that a proof of *P* in PM makes possible a proof of ~ *P* in PM. As I have briefly shown, Wittgenstein now understands how the proof is executed. It is simply unclear whether Wittgenstein still thinks that *P* must be self-referential. At §22, Wittgenstein certainly seems to recognize that Gödel proves that a proof of *P* in PM would prove, *number-theoretically,* ~ *P* in PM. Thus, it seems that Wittgenstein is suggesting that we can have Gödel's proof *both* ways: a number-theoretic version and a self-referential version.

That seems to be the case, for Wittgenstein main point about *what* Gödel's proof actually proves does not change between his middle and his later writings on undecidability. As Wittgenstein says, '[h]owever queer it sounds, my task as far as concerns Gödel's proof seems merely to consist in making clear what such a proposition as: "Suppose this could be proved" means in mathematics' (RFM, VII - §22). As we see again, Wittgenstein argues that Gödel proved a *conditional*, that is, *if* P is proved in PM, then PM is inconsistent. Whether, in 1941, Wittgenstein views Gödel's proof as requiring a self-referential proposition or as being executable using either a self-referential proposition *or* a number-theoretic proposition, does not change the fact that the proof proves the afore-mentioned conditional, and *not the consistency of PM,* and hence not the *unprovability* of *P* in PM.

6.5.4. Wittgenstein's Description of Mathematics and Undecidability

Undecidable mathematical propositions seem to pose a problem for Wittgenstein formalistic approach to mathematics. For, according to the standard interpretation of Gödel's proof, Gödel showed that there is a clear demarcation between a true mathematical proposition and a provable mathematical proposition. This led to the conclusion, in the 1930s, but especially by the 1960s, that mathematical truth is not equivalent with *provability*, as Hilbert and many other great mathematicians thought for centuries, and up to 1931. Since 1931, many mathematicians, logicians, and philosophers have thought that Gödel proved that "no axiomatic calculus can capture all number-theoretic truth" (Rodych, 1999b, p. 196) – *i.e.,* that this shows a clear demarcation between syntax and semantics in mathematics.

For these reasons, it is thought that Gödel's First Undecidability Theorem refutes Wittgenstein's claim that a genuine mathematical proposition must have a known decision procedure, and also Wittgenstein's claim that mathematics is only exclusively syntactical, and in no part semantical. Since the 1930s, it has been thought and written that we can show, using Gödel's proof, that there exist true but unprovable propositions of PA.

However, as Rodych (1999b, p. 196) argues, "[w]hat is crucial [in Wittgenstein's description of Gödel's undecidability], of course, is "if Γ is consistent". In order to show that there are true but unprovable propositions of Γ (the consequent), we must first show that it is the case that Γ is consistent (the antecedent). In the case of Peano Arithmetic this is not as straightforward and entirely unproblematic as it might seem.[117] Thus, as I have

[117] In this thesis I will not evaluate some of the proofs for the consistence of PA. I believe that they would require a careful and systematic approach, which would demand a large portion of this thesis and would deviate from its aim and scope. Gödel's Second Undecidability Theorem (co-discovered by J. von Neumann

suggested, it is a mistake to think that "Gödel's proof of true but unprovable propositions of PA refutes Wittgenstein's conception of mathematics" because Gödel did not prove that there are true but unprovable propositions of PM.

in November of 1930) proves that a consistency proof of PA cannot be represented in PA, and this refutes Hilbert's conjecture that a *finitary* consistency proof of PA (and of Analysis) was possible.

CHAPTER 7: Opposing Interpretations of Wittgenstein's Work

7.1. Difficulties Related to Wittgenstein's Work: Multiple Interpretations of Wittgenstein's Philosophy of Mathematics

To this point, I have presented Wittgenstein's philosophy of mathematics in opposition to the standard view. I have argued that Wittgenstein's view constitutes a more interesting and more viable approach to the philosophy of mathematics. In addition, I have shown that Wittgenstein tries to describe mathematics by unraveling conceptual-linguistic knots. Despite his best attempt at only describing mathematics, it seems that Wittgenstein *prescribes that we should not quantify over an infinite domain in mathematics.* This is best shown in his criticisms of infinite mathematical extensions and of to transfinite set theory.

However, some authors attempt to show that, in fact, Wittgenstein is *only describing mathematics*, in the stronger sense. For them, as I will briefly show, Wittgenstein's method only shows that every philosophical interpretations of mathematics are equally viable and equally irrelevant.

This shows us that there exists a lot of disagreement among contemporary philosophers about which interpretations constitute a more accurate description of Wittgenstein's philosophy of mathematics. Some of Wittgenstein's main concepts that I have shown here are not interpreted in the same way by some commentators. In addition, most of Wittgenstein's work was published after his death, leaving most of what he wrote (especially in the philosophy of mathematics) to be organized and translated by other people, leaving a high degree of interpretation and (sometimes) speculation as to what his view might be.

In this regard, it seems imperative to discuss and explain some interpretations of Wittgenstein's philosophical work that deviate from what I have presented. In briefly presenting other interpretations, I will continue my defense of Wittgenstein's *weaker descriptivism*. In my view, the fact that Wittgenstein aims to describe and does describe mathematics and mathematical activity does not mean that Wittgenstein has *no* particular views about mathematics, nor that he should not argue for a particular philosophical position. Rather, Wittgenstein's weaker descriptivism consists in an attempt to *explain and argue* in defense of what is being described.

7.1.1. Taking Wittgenstein's at his Word: Floyd's and Fogelin's Readings of the Later Wittgenstein

Juliet Floyd's[118]and Robert Fogelin's[119] similar interpretations of Wittgenstein's work consist in assuming that Wittgenstein is not arguing for any particular view about mathematics. As I said in section 2, Wittgenstein says that we should not argue for any particular view in mathematics. This has led Floyd, and later Fogelin, to interpret Wittgenstein's writing on mathematics by *taking Wittgenstein at his own word*. Put differently, Floyd and Fogelin endeavour to interpret Wittgenstein's words, including some of his more argumentative and more critical passages, as non-judgmental and non-assertive. They want to show that Wittgenstein has no position on mathematics, and, therefore, he is not a formalist or a finitist or a constructivist or an anti-extensionalist (i.e., an intensionalist about real numbers).

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[118] See (Floyd, 1991; 1995)
[119] See (Fogelin, 2009)

To this end, Floyd (1991) argues that Wittgenstein's dialectical method of writing (most known in his Later Period) shows that he is not arguing for any view. As she states,

> [a]s I read him, the dialectical style of his later writing – and its seemingly interminable character – reflects his conception of the problems under discussion. We are left in the end with dispensable comments, for purposes of linguistics, empirical psychology or mathematical practice, **but the point of the investigation is to show that what the essentialists says about language, thought and mathematics is equally dispensable**" (Floyd, 1991, p. 144 [bold mine])

On Floyd's account, if we take Wittgenstein at his own word, he is not presenting his ideas as true or correct. He is only showing that any particular philosophical view, or position, is equally *dispensable*. To support her view, Floyd points to a passage on *Wittgenstein's Lectures on the Foundations of Mathematics*:

> A philosopher provides gas, or decoration – like squiggles on the wall of the room. I may occasionally produce new interpretations, not in order to suggest they are right, but in order to show that the old interpretation and the new are equally arbitrary. I will only invent a new interpretation to put side by side with an old one and say, 'Here, choose, take your pick'" (LFM I, pp. 13-14).

Floyd's (1991, p. 143) claims that the first sections of *The Remarks on the Foundation of Mathematics* sets are "the beginning of a powerful and total reworking of a tradition philosophical picture of logic and mathematics". Indeed, the radical claims that Wittgenstein wrote in *RFM* as a whole are impressive. As I have shown, it is a completely different approach, one that strongly opposes the standard view of mathematics, *e.g.* to "rework a traditional philosophical picture of logic and mathematics".

Floyd (1991, p. 144) claims that "Wittgenstein is out to undercut the notion that *particular claims must be made from within an overarching general philosophical conception* or systemic logical structure". On her account, Wittgenstein is not arguing in favour of any particular view. Floyd assumes that Wittgenstein does not base his argument in an overarching general philosophical conception.

Floyd's main argument is that Wittgenstein uses a dialectical method to present his ideas, and in so doing, he is just comparing opposing views (*i.e.* thesis and anti-thesis), in an attempt to show that these different views do not matter for mathematics at all. In addition, Floyd points out that Wittgenstein claims that the work of a philosopher is only to describe, never to say what is the case or to prescribe what should be done in mathematics.

I contend that Floyd's argument does not take into consideration Wittgenstein's criticism of transfinite set theory. It seems that Wittgenstein is not just attempting to "rework a philosophical picture of logic and mathematics", but he *is reworking logic and mathematics* (as I have argued in the previous section), despite his best efforts.

7.1.1.1. Fogelin's View on Mathematical Meaning as Necessarily Connected to Contingent Propositions

In a similar way, Fogelin attempts to analyze Wittgenstein's philosophy of mathematics, by assuming that Wittgenstein is only doing a kind of philosophical therapy. As Fogelin says in the preface of his book *Taking Wittgenstein at his Word*, he wants to determine "[w]hat kind of interpretation emerges if we adhere strictly to Wittgenstein's methodological pronouncements, in particular, his claims that his aim is purely therapeutic and that he is not in the business of presenting and defending philosophical theses?" (2009, p. xi). In other words, if we read Wittgenstein's writings on mathematics as a "therapeutic" investigation, what interpretation emerges if, wherever possible, we interpret his statements and his putative pronouncements and criticisms only as philosophical therapy?

Fogelin's experiment, in my view, constitutes a praiseworthy enterprise. In several passages, we see Fogelin attempting to explain several controversial and radical passages of Wittgenstein's work, in an attempt to read Wittgenstein's words as, presumably descriptions and therapy. The most noticeable of Fogelin's attempts to defend Wittgenstein's descriptivism is Fogelin's interpretation of Wittgenstein's views on Cantor's Diagonal Proof (Fogelin, 2009, pp. 126-129).

However, by taking Wittgenstein at his word, it seems that Fogelin ignores certain passages, or forces a particular interpretation onto certain passages, in order to maintain a kind of skeptical descriptivist interpretation of Wittgenstein's work. For example, Fogelin resists calling Wittgenstein's view a formalistic or finitist approach to mathematics despite considerable evidence that Wittgenstein does argue for and defend a type of Finitism and a type of Formalism. In fact, because Wittgenstein himself criticizes those two approaches, Fogelin seems to not further pursue this subject, *i.e.* "Does Wittgenstein *defend* a formalistic or finitistic view of mathematics?". Because Fogelin does not distinguish between a type of Formalism and a type of Finitism that Wittgenstein *may* once or twice criticize, and a type of Formalism and type of Finitism that Wittgenstein seems to present, argue for, and defend, it certainly seems as if Fogelin's experiment is not unbiased both in the passages he selects to interpret and 'biased' no-thesis-thesis way he interprets them.

Take, for example, Formalism. Fogelin first quotes from the student notes that constitute LFM:

> One asks such a thing as what mathematics is about—and someone replies that it is about numbers. Then someone comes along and says that it is not about numbers but about numerals; for numbers seem very mysterious things. **And then it seems that mathematical propositions are about scratches on the blackboard**. That must seem ridiculous even to those who hold it, but hold it because there seems to be no way out—I am trying to show in a very general way how the misunderstanding of supposing a mathematical proposition to be like an experiential proposition leads to the misunderstanding of supposing that a mathematical proposition is about scratches on the blackboard. (LFM, p. 113 [bold mine])

In this well-known LFM passage, Wittgenstein seems to lay out a dilemma. As Fogelin says: "Under the influence of a misleading comparison between mathematical propositions and experiential propositions, we seem forced to choose between a theory that is mysterious (Platonism) and one that is ridiculous (Formalism)" (2009, p. 94). Not wanting to ascribe a position or thesis or ISM to Wittgenstein, Fogelin says that Wittgenstein's "way out of this impasse is utterly simple," for, on Fogelin's reading, Wittgenstein contends (asserts, argues, defends?) that "[n]umerals, even when employed in pure mathematics, owe their mathematical significance to their adjectival employment outside pure mathematics" (Fogelin 2009, p. 95). To support or defend this interpretation Fogelin offers (RFM, V - §2 & §41) as proof text, passages which I have similarly use to show that the later Wittgenstein adopts an extra-systemic, extra-mathematical application criterion (the APC) to distinguish mere sign-games from mathematical calculi.

As I have argued in section 5.7., Wittgenstein's later work shows that in order for a Csign to be a mathematical proposition, some of its concepts and terms must also have extrasystemic meaning. Here Fogelin seems to be similarly arguing that Wittgenstein avoids Platonism and one variant of Formalism by claiming that some mathematical terms have meaning in contingent propositions. What is confusing is that Fogelin seems to *minimally* claim that this is Wittgenstein's position—that Wittgenstein goes between the horns of the dilemma by adopting a third option, namely that this particular position.

This shows that Fogelin's experiment does not want to attribute any kind of "*ism*" to Wittgenstein's philosophy of mathematics, such as *Formalism, Platonism, Finitism, Constructivism*, etc. As Fogelin (2009, p. 127 [italics mine]) argues, "[w]e might say that [Wittgenstein] is a deflationist with respect to transfinite cardinals, but that label can trigger inappropriate associations as well. *We might better call him a descriptivist*".[120]

Fogelin uses the term "descriptivist" to designate Wittgenstein's philosophical view, in order to show that Wittgenstein is not against or in favour of any particular philosophical interpretation. As he argues: "But saying that Wittgenstein is *against* the inflation of these proofs [*i.e.*, the standard interpretation of Cantor's Diagonal proof] may be wrong too, for that suggests he *has an opposing opinion on this matter,* and this is something he explicitly denies" (Fogelin, 2009, p. 128 [italics mine]). For Fogelin, Wittgenstein is only describing mathematics and doing philosophical therapy.

Despite his best efforts to interpret everything that Wittgenstein says as description or therapy, here and there Fogelin can not help but attribute a particular view on mathematics to Wittgenstein. For example, it seems that Fogelin (2009, p. 94-95) is attributing a particular view to Wittgenstein when he claims that, for Wittgenstein,

[120] We can see that Floyd and Mühlhölzer have a very similar argument, avoiding attribute *any* philosophical label to Wittgenstein. As they state: "Above all we shall argue against the idea that Wittgenstein was a constructivist or Brouwerian, or even a finitist or a verificationist. [..] This, we believe, was no part of Wittgenstein's philosophical line of thought" (Floyd & Mühlhölzer, 2020, p. x-xi). However, the reasons for that are still obscure. As I have shown, in several passages Wittgenstein gives us *an argument defending a particular view*. The problem of the claim that Wittgenstein is not defending any particular view is that it leaves out a good portion of his work that strongly suggests that he is defending his description of mathematics.

“[n]umerals even when employed in pure mathematics, owe their mathematical significance to their adjectival employment outside pure mathematics” (Fogelin, 2009, p. 95).

Similarly, in discussing Wittgenstein’s remarks on transfinite set theory, Fogelin (2009, p. 127 [italics mine]) says, almost without noticing it, that Wittgenstein is a “deflationist with respect to transfinite cardinals”, but quickly catches himself, and says that the label “deflationist” might “trigger inappropriate associations” (whatever they might be), and so, it is better in some sense to “call him a descriptivist”. Whether or not a position has a known philosophical label attached to it, does not change the fact that, at various places, Wittgenstein is “presenting and defending philosophical theses”.

In addition, I believe that a) Fogelin did not pursue the matter of whether or not Wittgenstein’s extrasystemic criterion is compatible with a formalistic approach of mathematics; and that b) Fogelin’s interpretation of (LFM, p. 113) seems to not focus on the main issue raised by Wittgenstein, that is, the issue of assuming that mathematical propositions *are about something*.

As I have also argued in section 5.7., the extrasystemic criterion is not just compatible with Wittgenstein’s formalistic approach, but it also serves to solve a tension between Wittgenstein’s middle radical Formalism (that we can see in (PR, §109; PR, §152) and (PG, p. 308; PG, p.468)) and his criticism of transfinite set theory. As I argued, this new criterion does not change the fact that, for Wittgenstein, mathematical propositions have only intrasystemic meaning (*i.e. sense*); nor does it change the fact that Wittgenstein argues that the *sense* of a mathematical proposition is its position in a calculus. Its main purpose is to distinguish mere sign games from mathematical language-games (calculi).

Moreover, the LFM passage quoted by Fogelin seems not to be about Platonism and Formalism. Instead, Wittgenstein's principal point is that this is *false dilemma*, but not because of the 3rd "way out" Fogelin highlights, but because both of these accounts of mathematics assume incorrectly that mathematics is about something. As we know, Wittgenstein argues *against* the view that mathematical propositions *are about something* (based on, what Fogelin correctly calls, "a misleading comparison between mathematical and contingent propositions"), and thereby avoids the dilemma by adopting a different variant of Formalism. As we can in the continuation of the very same LFM passage (which Fogelin does not quote), Wittgenstein is "trying to show in a very general way how the *misunderstanding of supposing a mathematical proposition to be like an experiential proposition leads to the misunderstanding of supposing that a mathematical proposition is about scratches on the blackboard*" (LFM, p. 113), and how *both* misunderstandings are entirely due to erroneously assuming that mathematics and its signs are about *anything*.[121]

Indeed, when you interpret the entire passage, we see that Wittgenstein is saying that both Platonism and Formalism, *if interpreted in this referential way* (of being about something), are ridiculous. However, that is only a problem for a formalistic approach *if* we defend that idea, *i.e.* if we defend the idea that mathematics is *about something*. A formalistic approach *does not* have claim that the signs and propositions of a mathematical calculus are about anything. As I have argued, Wittgenstein rejects the idea that

[121] This seems evident, as Wittgenstein continues to engage with and against the notion of mathematical aboutness in the very following paragraphs: "One might say that it is a statement about numbers. Is it wrong to say that? Not at all; that is what we call a statement about numbers. But this gives the impression that it's not about some coarse thing like scratches, but about something very thin and gaseous. Well, what is a number, then? I can show you what a numeral is. But when I say it is a statement about numbers it seems as though we were introducing some new entity somewhere" (LFM, p. 113 [bold mine]).

mathematical propositions are true or false, by rejecting a referentialistic account of mathematics.

Despite what are perhaps unavoidable pitfalls such as the foregoing, Fogelin's experiment is laudable and very interesting. Contrary to Floyd's (1991; 1995) descriptivist analysis, Fogelin tries to explain some of the most controversial passages of RFM and LFM, such as Wittgenstein's criticism of Cantor's Diagonal Proof (RFM, II - §§21-22; LFM, p. 103) in accordance with Wittgenstein's descriptive-therapeutic approach.

However, because Fogelin does not want to ascribe a philosophical view to Wittgenstein, he dismisses some important passages that give us good reason to infer that Wittgenstein's description of mathematics constitutes a formalistic and finitistic view of mathematics.

7.2. Wittgenstein and Set Theory: Wittgenstein's Argument in the Philosophy of Mathematics.

It seems that Wittgenstein is not simply criticizing Cantor's diagonal proof on the grounds that it is just an interpretation, equally irrelevant like any other philosophical positions. I believe that what Wittgenstein *says*, and also his tone, and his finitistic approach to mathematics, imply that Cantor's diagonal proof, taken as a proof for the different cardinality of the real numbers and natural numbers, is mistaken. As he puts it, "[a] clever man [who] got caught in this net of language! So it must be an interesting net" (RFM II, §§14-15). The "clever man" is almost certainly a reference to Cantor, and the "net of language" that he "got caught in" is the linguistic-conceptual confusions that gave birth to the equinmerosity of infinite sets, and the very conflation of extensions and intensions that engenders mistaken interpretations of Cantor's Diagonal Proof and, especially, *what* it proves.

In other words, Wittgenstein is not merely comparing Finitism with an infinitistic interpretation of mathematics (and Cantor's diagonal proof), showing that they are equally arbitrary explanations. His criticisms of set theory are clear: "Set theory is wrong because it apparently presupposes a symbolism which doesn't exist instead of one that does exist (is alone possible). It builds on a fictitious symbolism, therefore on nonsense" (PR, §174).

The problem with Floyd's approach to Wittgenstein is that it ignores these (and other) important (and reoccurring) passages in which Wittgenstein is emphatically defending *his description of mathematics*. Indeed, this dialectic process of raising different views and confronting each of them is prominent in Wittgenstein's writing. However, it does not seem that it serves to show that different philosophical views are equally arbitrary or equally viable.

The main issue here is that Floyd, but also Fogelin, ignores the fact that Wittgenstein, in describing mathematics, is *arguing against certain descriptions and explanations, and in favour of other descriptions*. In other words, in attempting to describe what mathematics is, Wittgenstein is giving us the *reasons* why he says what is the case. Wittgenstein can only describe something if he describes it correctly; there are correct and incorrect descriptions, and both Floyd and Fogelin seem to think that Wittgenstein qua descriptivist cannot argue for the better or correct descriptions and against the worse and incorrect descriptions. In other words, we should not associate Wittgenstein's view with any philosophical interpretation of mathematics, such as Formalism or Finitism.

Arguments in philosophy are unavoidable, for even an attempt to describe a particular activity requires an argumentative structure, where the person doing the describing provides reasons for a particular description and also reasons against other possible descriptions. If those descriptions/arguments fall under a certain philosophical view, that does not show that the argument *prescribe something for mathematics*.

7.3. Understanding Wittgenstein's Descriptivism

Throughout the thesis I have highlighted what one might considered that Wittgenstein was contradicting his own principle, *i.e.* the principle that we must not formulate theses in philosophy, but only describe. This has led me to show that, despite the fact that several of Wittgenstein's arguments seem very radical or reversionistic, he was, for the most part, only describing mathematics and unravelling conceptual-linguistic knots. This activity of unravelling linguistic confusions and thereby clarifying our concepts constitutes Wittgenstein's descriptivist account of mathematics.

It is important to notice that Wittgenstein's descriptivist account of mathematics is not the same as the one suggested by Floyd or Fogelin. Floyd's view of Wittgenstein's philosophy of mathematics seems to say that Wittgenstein is only making a "therapeutic argument". For Floyd, Wittgenstein's descriptivism implies that he is only comparing equally arbitrary philosophical theses. In this sense, Wittgenstein is doing nothing more than a skeptical exercise – he is criticizing important views on mathematics, within the philosophy of mathematics, without being committed to any of them.

Indeed, I believe that Wittgenstein does not suggest that a philosophical view of mathematics should *prescribe* what mathematicians are entitled to do, what a mathematical proof should be, or what mathematicians should consider as a mathematical problem.

Nonetheless, even though Wittgenstein is not suggesting any changes in relation to what we do in mathematics, he *did argue* to defend *his account* of several philosophical topics in mathematics. This can be clearly shown by examining, as I have, his views on a) mathematical sense; and b) the difference between mathematical questions and ('open') mathematical problems (e.g., infinitistic mathematical conjectures, such as GC).

In relation to (a), I have argued that Wittgenstein attempts to *clarify* what a mathematically meaningful proposition is and what the *sense* of that mathematical proposition is. In fact, his argument tries to show how linguistic confusions originated from the ways in which mathematicians (and some philosophers) talk about mathematics. In other words, Wittgenstein wants to show that the *way that we talk about mathematics* does not correspond to what we do in mathematics. For example, we can check "the account books of mathematicians" (PG, p. 295) by examining articles in mathematics journal over the last 60 years. We do not find mathematicians checking a non-physical reality and reporting their intuitive or "mathematical-perceptual" observations. Instead, not surprisingly, we find pages and pages of symbols and words—journal issues filled with proofs and symbolic transformations.

It is clear that Wittgenstein shows the reasons (gives us an argument) in favour of *his view* of what (a) would be. His argument shows us the linguistic confusions that arise by our way of talking about mathematics. By arguing that the sense of a mathematical proposition *is* its syntactical position in a particular mathematical calculus, Wittgenstein is *describing* what mathematicians do. By showing that Wittgenstein's view about (a) constitutes a better *description* of how we do mathematics, we are led to his view about (b) – a mathematical question must have a known decision procedure. Mathematical conjectures—mathematical problems in the absence of a known decision procedure—can constitute open problems for centuries, as has GC. We do not know whether GC will ever be proved in PA; for all we know, we may one day prove that GC is syntactically independent of PA.

Superficially, it may seem as if Wittgenstein tells the mathematician not to try to decide mathematical conjectures, which would fly in the face of pure mathematics. But this is not what Wittgenstein says or means. The idea that a meaningful mathematical problem must have a decision procedure does not prohibit mathematicians from trying to prove an undecided Csign such as Goldbach's Conjecture, or FLT (in 1990). Wittgenstein's distinction serves to remind us of the large difference between algorithmic decidability and unsystematic searches for proofs. Wittgenstein reminds us not to confuse an "open problem" — an unsystematic search for an unknown proof — with a mathematical question – the process by which we position a mathematical proposition within a calculus by means of a known decision procedure.

In this regard, it seems that Floyd's view that Wittgenstein's philosophy of mathematics constitutes a form of skeptical descriptivism does not take into consideration all of the many passages where it is clear that Wittgenstein is arguing for his description of mathematics. Therefore, it seems incorrect to say or insist that Wittgenstein was merely comparing equally arbitrary (and useless?) philosophical views. The very idea of describing something seems to require that we provide our reasons for claiming that our description is correct, and others are incorrect. If Wittgenstein is indeed describing what mathematicians do – and, therefore, what mathematics is – he must argue that alternative descriptive accounts are incorrect. By describing mathematics, Wittgenstein defends this concrete position, *i.e.* that mathematics is a sign-game, nothing more. It is only syntactical, not semantical.

Still, this argument does not seem to hold for Wittgenstein's criticism of set theory. As I have shown, Wittgenstein's criticisms of set theory suggest changes in how we do set theory. In fact, Wittgenstein is suggesting not just changes, but he openly believes and hopes that one day we will abandon this "*hocus pocus*". It seems, therefore, that Wittgenstein's considerations of set theory go beyond the mere enterprise of only describing mathematics. It suggests that this set theoretical language should be abandoned (at least partially abandoned, *e.g.* transfinite set theory).

We have seen that Wittgenstein has some strong criticisms of the concept of infinity in mathematics. Indeed, it seems that even though Wittgenstein's view on Cantor's Diagonal Proof seems very different from the standard interpretation, he is showing what the proof actually proved. His constructivist approach to Cantor's proof does not seem to pose a contradiction to his descriptivist aims. We can see that his constructivist reconstruction of Cantor's proof does not alter what the proof proves. It *clarifies* what we *thought* that the proof showed, *i.e.* it clarifies the linguistic confusions that our interpretation of Cantor's proof generated. If Wittgenstein is right, we do not have any grounds for saying that there are more real numbers than natural numbers, but we do have good reason to say that Cantor *has proved* that we cannot enumerate all those things we wish to call "real numbers" (i.e., all decimal expansions).

However, his criticisms of transfinite set theory and its alleged implications are far more than linguistic and conceptual clarifications. This shows us that by unraveling and solving some linguistic knots/problems Wittgenstein prescribes changes to at least some parts of set theory. In other words, his descriptivist account seems to imply something more than just linguistic revisions.

This, I believe, does not pose a problem for Wittgenstein's descriptivism; it only shows us what the activity of *describing something* involves. By describing what mathematics is, Wittgenstein arrives at the conclusion that this set theoretical language was built on linguistic knots and conceptual confusions. The act of revising transfinite set theory does not only implies the clarification of the concepts that we are dealing with, but it implies a complete reformulation of the field itself.

Wittgenstein's descriptive account of mathematics shows us that by describing something we must know and show the reasons why our description is correct. This can be shown as easily as, for example, pointing to someone buying milk and relating that to the description "John is buying milk"; or it can be as complicated as showing the reasons why a mathematically meaningful proposition must have a known decision procedure.

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