Physics Letters B 755 (2016) 17–23

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Discreteness of space from GUP in a weak gravitational field

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ARTICLE INFO

Article history:
Received 8 December 2015
Received in revised form 21 January 2016
Accepted 28 January 2016
Available online 1 February 2016
Editor: M. Cvetič

ABSTRACT

Quantum gravity effects modify the Heisenberg's uncertainty principle to a generalized uncertainty principle (GUP). Earlier work showed that the GUP-induced corrections to the Schrödinger equation, when applied to a non-relativistic particle in a one-dimensional box, led to the quantization of length. Similarly, corrections to the Klein–Gordon and the Dirac equations, gave rise to length, area and volume quantizations. These results suggest a fundamental granular structure of space. In this work, it is investigated how spacetime curvature and gravity might influence this discreteness of space. In particular, by adding a weak gravitational background field to the above three quantum equations, it is shown that quantization of lengths, areas and volumes continue to hold. However, it should be noted that the nature of this new quantization is quite complex and under proper limits, it reduces to cases without gravity. These results suggest that quantum gravity effects are universal.

1. Introduction

During the last 70 years, much effort has been devoted towards the construction of a consistent theory of Quantum Gravity (QG). All approaches to QG start with an assumption about the structure of spacetime at scales that are extremely small, way beyond the current experimental advancement.

However, even if not direct, experimental evidence, e.g. analogous gravity experiments [1], suggests that gravity can show quantum effects. Therefore, since there is no direct experimental guidance, it is quite natural to try to develop a correct theory based on conceptual restrictions. Like any other active research field, what Quantum Gravity Phenomenology (QGP) ideally needs is a combination of theory and doable experiments [2].

At the moment, QGP can be thought of as a combination of all studies that might contribute to direct or indirect observable predictions [3,4] and analogous models [1]. These studies support the small and the large scale structure of spacetime consistent with String Theory, or any other approaches to QG.

The first step to identifying the relevant doable experiments for QGP research would be the identification of the working scale of this new field. This, known as the Planck scale, is first estimated from dimensional arguments. The Planck scale is uniquely defined by the fundamental constants, namely the speed of light c, the gravitational constant G, and the Planck constant $\hbar$, to provide the units of length, mass, and time

$$\ell_{Pl} = \sqrt{\frac{\hbar G}{c^3}} \sim 10^{-35}\ \text{m}\,,$$

$$m_{Pl} = \sqrt{\frac{\hbar c}{G}} \sim 10^{-8}\ \text{kg}\,,$$

$$t_{Pl} = \sqrt{\frac{\hbar G}{c^5}} \sim 10^{-44}\ \text{s}\,.$$

The smallness of this scale makes QG phenomenologists' job difficult, which is to test the Planck scale effects and extract useful information for further theoretical studies.

Among the many mathematical results of String Theory there is one which is of particular interest and relevant to QGP. This is the modification of the Heisenberg uncertainty principle (HUP), which is well known as generalized uncertainty principle (GUP). In the context, mainly but not only, of String Theory, the suggested version of GUP is [5–13]

$$\Delta x \geq \frac{\hbar}{\Delta p} + \alpha' \frac{\Delta p}{\hbar} \tag{1}$$

where $\sqrt{\alpha'} \approx 10^{-32}$ cm [14].

Recently, the theories of Doubly Special Relativity (DSRs) were introduced principally to give a physical interpretation of the Planck length, i.e., ℓ_{Pl}, in the structure of spacetime [15]. In

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http://dx.doi.org/10.1016/j.physletb.2016.01.059

particular, different values could be attributed to the Planck length by different observers. Thus, DSRs avoid these violations of Lorentz invariance by considering the Planck length as an observer-independent scale. One of the consequences of DSRs was a similar modification of the position–momentum commutation relation [16,17] which leads to a modification of the HUP as well. In this case, the suggested form of the commutator is given by [3]

$$[x_i, p_j] = i\hbar\left(\delta_{ij} - \alpha\left(p\delta_{ij} + \frac{p_i p_j}{p}\right) + \alpha^2(p^2\delta_{ij} + 3p_i p_j)\right) \tag{2}$$

where p can be interpreted as the magnitude of $\vec{p}$ since $p^2 = \sum_{i=1}^{3} p_i p_i$ and $\alpha = \frac{\alpha_0}{m_{Pl}c} = \frac{\alpha_0 \ell_{Pl}}{\hbar}$.

The suggested form of the commutator given in Eq. (2) is satisfied by the modified operators

$$x_i = x_{0i},$$
$$p_i = p_{0i}(1 - \alpha p_0 + 2\alpha^2 p_0^2), \qquad i = 1, 2, 3\,. \tag{3}$$

Here, x_{0i}, p_{0i} satisfy the canonical commutation relations $[x_{0i}, p_{0i}] = i\hbar\delta_{ij}$, implying that $p_{0i} = -i\hbar\frac{\partial}{\partial x_{0i}}$ is the standard momentum (operator) at low energies and p_i the modified momentum at higher energies. Note that $p_0^2 = \sum_{i=1}^{3} p_{0i}p_{0i}$ [18].

The specific modification of the commutator (see Eq. (2)), with the modified operators as given in Eq. (3), leads to a version of GUP which reads [19–21]

$$\begin{aligned}\Delta x \Delta p &\geq \frac{\hbar}{2}\left[1 - 2\alpha < p > + 4\alpha^2 < p^2 >\right] \\ &\geq \frac{\hbar}{2}\left[1 + \left(\frac{\alpha}{\sqrt{< p^2 >}} + 4\alpha^2\right)\Delta p^2 + 4\alpha^2 < p >^2 - 2\alpha\sqrt{< p >^2}\right]\end{aligned} \tag{4}$$

with the dimensionless parameter α_0 generally considered to be of order of unity.

It is evident that QGP indicates an irremovable uncertainty in distance measurements [2]. In the framework of String Theory, the modified commutation relations of position and momentum operators result in a version of GUP. A similar, but subtler, consequence of this version is that the apparently continuous-looking space on a very fine scale is actually grainy. One can ask whether this is a sole influence of gravity or a fundamental structure of the spacetime. Now, if one admits the fact that classical gravity is a derived effect of curvature of spacetime caused by mass, then one can expect to find this discontinuity even in the regions of the universe far from a massive object.

The nature of this discreteness may or may not change when the spacetime is no more flat, namely it is a curved spacetime due to the presence of a gravitational field. In order to investigate this, we trap a particle in a box with a gravitational potential inside the box and see if gravity influences the discreteness shown in [18,22].

The outline of this work is as follows. In the next Section, we briefly review the problem of a particle moving in a one-dimensional potential. Spacetime is flat but due to GUP-effects, it effectively shows a discrete structure. In Sec. 3, we investigate the discreteness of spacetime in the problem of a non-relativistic particle moving in a one-dimensional potential when gravity is present. Furthermore, we explore the discreteness of spacetime for the case of relativistic 0-spin and 1/2-spin particles moving again in a one-dimensional potential when gravity is present. Finally, in Sec. 4, we briefly present our results.

2. Discreteness of space in flat spacetime

In this section, we briefly review the non-relativistic situation where a particle is trapped in a one-dimensional box and one finds the GUP-corrected Schrödinger equation [18]. In particular, we consider one of the standard examples in quantum mechanics, namely the problem of a particle moving in a one-dimensional infinite potential well. The well or the one dimensional box of length L is defined by the potential $V(x) = 0$ for $0 \leq x \leq L$ and ∞ outside this box. The quantum mechanical equation governing such a particle is the Schrödinger equation

$$H\psi = E\psi$$

except for the fact that the position and momentum operators are now modified due to the GUP-effects.

Incorporating the GUP corrections, one can write the modified Schrödinger equation as

$$\frac{d^2}{dx^2}\psi + k_0^2\psi + 2i\alpha\hbar\frac{d^3}{dx^3}\psi = 0 \tag{5}$$

where $k_0 = \sqrt{2mE/\hbar^2}$.

At this point, it should be stressed that the α-dependent term in the above equation is only important when energies are comparable to Planck energy and lengths are comparable to Planck length. The general solution of this equation is

$$\psi = Ae^{ik_0'x} + Be^{-ik_0''x} + Ce^{ix/2\alpha\hbar}\,.$$

The first two terms along with the boundary conditions $V(x=0) = 0 = V(x=L)$ lead to the standard energy quantization. It is the new third α-dependent term that gives rise to a new condition [18]

$$\cos\left(\frac{L}{2\alpha\hbar} - \theta_C\right) = \cos(k_0 L + \theta_C) = \cos(n\pi + \theta_C + \delta_0)$$

which in turn implies that[1]

$$\frac{L}{2\alpha\hbar} = \frac{L}{2\alpha_0\ell_{Pl}} = -n\pi + 2q\pi \equiv p\pi \tag{6}$$

where $p \equiv 2q \pm n$ is a natural number. The above expression shows that the length L is quantized. This result can be interpreted as the fact that, like the energy of the particle inside the box, the length of the box can assume only certain values. In particular, L has to be in units of $\alpha_0\ell_{Pl}$. This indicates that the space, at least in a confined region and without the influence of gravity, is likely to be discrete.

Further work has shown that this consequence of the GUP effects can be extended to relativistic scenarios in one, two, and three dimensions [22]. There are several reasons why one needs to investigate the relativistic cases. High energy particles are much more likely to probe the fabric of spacetime near the Planck scale, which means that they are necessarily relativistic or ultra-relativistic particles. In addition, the fact, that most elementary particles are fermions, leads us to investigate Dirac equation instead of the Schrödinger equation.

[1] As already mentioned, for brevity the mathematical details have been omitted here. However, for the interested reader, the derivation of the quantization condition, i.e. Eq. (6), can be found in [18]. The whole analysis goes from Eq. (11) to Eq. (21) of reference [18].

3. Discreteness in curved spacetime

It has been proven that the GUP corrections imposed on a free particle lead to the discreteness of space. Although the moving particle was kept in a box, no force field inside the box was assumed, i.e., the particle was free to move in a flat spacetime. If we wish to claim that the quantum gravity effects are universal then we should expect that the length quantization will also emerge in the presence of external forces. In other words, discreteness of space must hold whether or not there is an external field present.

3.1. Non-relativistic case

The first step towards this generalization would be to consider gravity as the external force field inside the box, since it is the weakest among the four fundamental forces as well as being universal. Additionally, as we have discussed before, our goal is to find how gravity determines the nature of discreteness. With a gravitational potential present inside the box, we ignore all but the first term of the Taylor expansion of this potential, which is a linear term. This is reasonable because we are interested in the behavior of spacetime fabric near Planck scale and the gravitational potential changes very little over such small distances. Furthermore, in practice, we often use the gravitational potential energy approximated as $V(h) = mgh$ over a small vertical distance h and, thus, the field strength reads $E_h = -\frac{1}{m}\frac{\partial dV(h)}{\partial dh} = -g$. It is evident that this also justifies the previous claim of utilizing a linearized potential term.

Let us now consider a one-dimensional box of length L ($0 < x < L$) with a linear potential inside, which has the form

$$V(x) = \begin{cases} kx, & \text{if } 0 < x < L \\ \infty, & \text{otherwise} \end{cases} \tag{7}$$

with k a parameter of unit J/m and the smallness of k is assumed.

Without considering the GUP effects, the Schrödinger equation governing the motion of a particle of mass m inside this box is given by [23]

$$\frac{d^2\psi_0(x)}{dx^2} - \frac{2m}{\hbar^2}(kx - E)\psi_0(x) = 0\,, \tag{8}$$

with $\psi_0(x) = 0$ when $x \le 0$ or $x \ge L$, since the potential outside the box becomes ∞.

The above equation, namely Eq. (8), is an Airy equation whose general solution reads [24]

$$\psi_0(x) = C_1 Ai\left[\frac{\frac{2m}{\hbar^2}(kx - E)}{(\frac{2m}{\hbar^2}k)^{\frac{2}{3}}}\right] + C_2 Bi\left[\frac{\frac{2m}{\hbar^2}(kx - E)}{(\frac{2m}{\hbar^2}k)^{\frac{2}{3}}}\right] \tag{9}$$

where $Ai[u]$ and $Bi[u]$ are Airy functions of the first and second kind, respectively.

We now use this wavefunction, i.e., ψ_0, for solving the GUP-corrected Schrödinger equation. Utilizing the GUP-modified operators given in Eq. (3) in order to modify the Hamiltonian of the system under study, the GUP-corrected one-dimensional Schrödinger equation for a non-relativistic particle moving in a box of length L with a linear potential reads, cf. Eq. (5),

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E - kx)\psi + 2i\alpha\hbar\frac{d^3\psi}{dx^3} = 0\,. \tag{10}$$

It is seen that the additional third term, $2i\alpha\hbar\frac{d^3\psi}{dx^3}$, which depends on the GUP parameter, i.e., α, becomes significant at high energies (comparable to Planck energy), or, equivalently, at small lengths (comparable to Planck length). Therefore, we can consider a perturbative approach in order to solve Eq. (10). A suitable trial solution can be of the form

$$\begin{aligned}\psi_1 &= \psi_0(E + \epsilon\alpha, k, x) \\ &= \psi_0(E, k, x) + \epsilon\alpha\frac{d}{dE}\psi_0(E, k, x)\end{aligned} \tag{11}$$

where the form of ψ_0 is given by Eq. (9), and ϵ is a coefficient that will be determined later.

Skipping intermediate mathematical steps, the general solution of the GUP-corrected Schrödinger equation is given by

$$\begin{aligned}\psi(x) = \frac{A}{\sqrt{\pi}}&\left[\xi^{-1/4}\sin\left(\frac{2}{3}\xi^{\frac{3}{2}} + \frac{\pi}{4}\right) + \right. \\ &\left(\frac{2m}{\hbar^2}\right)^{1/3} k^{-2/3}\epsilon\alpha\left(-\frac{1}{4}\xi^{-5/4}\sin\left(\frac{2}{3}\xi^{3/2} + \frac{\pi}{4}\right)\right. \\ &\left.\left. + \xi^{1/4}\cos\left(\frac{2}{3}\xi^{3/2} + \frac{\pi}{4}\right)\right)\right] + \\ &\frac{B}{\sqrt{\pi}}\left[\xi^{-1/4}\cos\left(\frac{2}{3}\xi^{\frac{3}{2}} + \frac{\pi}{4}\right) + \right. \\ &\left(\frac{2m}{\hbar^2}\right)^{1/3} k^{-2/3}\epsilon\alpha\left(-\xi^{1/4}\sin\left(\frac{2}{3}\xi^{3/2} + \frac{\pi}{4}\right)\right. \\ &\left.\left. - \frac{1}{4}\xi^{-5/4}\cos\left(\frac{2}{3}\xi^{3/2} + \frac{\pi}{4}\right)\right)\right] + Ce^{ix/2\hbar\alpha}\end{aligned} \tag{12}$$

with

$$\xi = \left(\frac{2m}{\hbar^2}\right)^{\frac{1}{3}} k^{-\frac{2}{3}}(E - kx)$$

$$\begin{aligned}\epsilon = &\left[(2i\hbar)\frac{3}{4}\left(\frac{2m}{\hbar^2}\right)^{\frac{11}{12}} k^{\frac{7}{6}} E^{-\frac{1}{4}}\right. \\ &\times\left(C_1\sin\left(\xi_0 + \frac{\pi}{4}\right) - C_2\cos\left(\xi_0 + \frac{\pi}{4}\right)\right) \\ &+ \alpha\,(2i\hbar)\left(\frac{2m}{\hbar^2}\right)^{\frac{17}{12}} k^{\frac{1}{6}} E^{\frac{5}{4}} \\ &\left.\times\left(C_2\sin\left(\xi_0 + \frac{\pi}{4}\right) - C_1\cos\left(\xi_0 + \frac{\pi}{4}\right)\right)\right] \\ &\div\left[\left(\frac{2m}{\hbar^2}\right)^{\frac{11}{12}} k^{\frac{1}{6}} E^{-\frac{1}{4}}\right. \\ &\left.\times\left(C_1\sin\left(\xi_0 + \frac{\pi}{4}\right) - C_2\cos\left(\xi_0 + \frac{\pi}{4}\right)\right)\right]\end{aligned}$$

$$\xi_0 = \frac{2}{3}\left(\left(\frac{2m}{\hbar^2}\right)^{\frac{1}{3}} k^{-\frac{2}{3}} E\right)^{\frac{3}{2}}$$

and A, B, C are constants. In particular, we can absorb the phase of A in ψ, such that A can be treated as a real constant while B can be written as $B = |B|e^{i\theta_B}$. Furthermore, C is such a constant that its magnitude $|C|$ becomes zero in the limit $\alpha \to 0$, since the last term must vanish in this limit.

Next, by imposing the boundary conditions $\psi(x = 0) = 0$ and $\psi(x = L) = 0$, we arrive at the following condition on the length of the box

$$\cos(L/2\hbar\alpha) = \left(1 - \frac{kL}{E}\right)^{-1/4} \times \left[A_* \sin\left(\frac{2}{3}\sqrt{\frac{2m}{\hbar^2}}\frac{(E-kL)^{3/2}}{k} + \frac{\pi}{4}\right) + B_* \cos\left(\frac{2}{3}\sqrt{\frac{2m}{\hbar^2}}\frac{(E-kL)^{3/2}}{k} + \frac{\pi}{4}\right)\right] \quad (13)$$

where A_* and B_* are constants that depend on A, B, k, and E.

It can be shown that in the limit $k \to 0$ the wavefunction given by Eq. (12) becomes the solution of Schrödinger equation for an infinite potential well. Thus, taking the limit $k \to 0$ the RHS of Eq. (13) reads

$$B_1 \cos(\kappa L_0) - A_1 \sin(\kappa L_0) \quad (14)$$

where L_0 is the length of the box in flat spacetime, $\kappa = \sqrt{\frac{2mE}{\hbar^2}}$, and

$$A_1 = H_1\left(A_* \cos\left(\frac{H_2}{k} + \frac{\pi}{4}\right) - B_* \sin\left(\frac{H_2}{k} + \frac{\pi}{4}\right)\right) \quad (15)$$

$$B_1 = H_1\left(A_* \sin\left(\frac{H_2}{k} + \frac{\pi}{4}\right) + B_* \cos\left(\frac{H_2}{k} + \frac{\pi}{4}\right)\right) \quad (16)$$

with

$$H_1 = \frac{1}{\sqrt{\pi}}\left[\left(\frac{2m}{\hbar^2}\right)^{\frac{1}{3}} \frac{(E-kx)}{k^{\frac{2}{3}}}\right]^{-\frac{1}{4}} \quad (17)$$

$$H_2 = \frac{2}{3}\left(\frac{2m}{\hbar^2}\right)^{\frac{1}{2}} E^{\frac{3}{2}} . \quad (18)$$

Without loss of generality, we let $A_1 = \sin\theta$ and $B_1 = \cos\theta$ for an arbitrary θ; thus, Eq. (13) becomes

$$\cos(L_0/2\hbar\alpha) = \cos\theta\cos(\kappa L_0) - \sin\theta\sin(\kappa L_0) \quad (19)$$

$$= \cos(\kappa L_0 + \theta) . \quad (20)$$

According to the analysis in [18], the above equation implies that $\frac{L_0}{2\hbar\alpha} = p\pi$, $p \in \mathbb{N}$. Since L is the perturbation of L_0, Eq. (13) yields

$$\frac{L}{2\hbar\alpha} = f(k)p_1\pi + p\pi \quad (21)$$

where $p_1 \in \mathbb{N}$ and for each p there is a finite set of values of $p_1 \in \mathbb{N}$. Moreover, since the first term on the RHS of Eq. (21) is a small perturbative term, the number of p_1 values, for each p, depends on the smallness of function $f(k)$.

As in the case of flat spacetime, we have arrived at a length quantization condition. Moreover, we have a fine structure (splitting) of the length quantization due to the presence of gravity (see Fig. 1). This is similar to the energy quantization of the hydrogen atom, in presence of an external electromagnetic field.

3.2. Relativistic case

The small-scale structure of spacetime should not depend on the use of relativistic or non-relativistic test particles. However, particles with speeds comparable to the speed of light should be treated relativistically, and the fundamental spacetime structure should be reexamined. In this subsection, we take a closer look at the relativistic equivalent of the Schrödinger equation, i.e., Klein–Gordon equation and, in particular, the modification induced by GUP. First, we will derive the GUP-version of the Klein–Gordon equation with a linear potential and then we will try to solve it in order to obtain possible length quantization. Notwithstanding its relative simplicity, Klein–Gordon equation has mathematical difficulties, especially when it comes to dimensions higher than one. For this reason, it is much easier to implement the more versatile Dirac equation. Therefore, we will also solve the Dirac equation in order to find a similar length quantization as in [22].

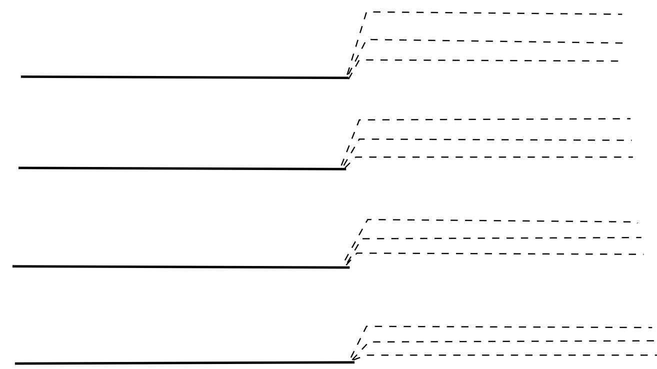

Fig. 1. Comparison between L_0 (solid lines) which is the quantized length with GUP corrections in flat spacetime and L (dotted lines) which is the quantized length with GUP corrections in curved spacetime.

3.2.1. Klein–Gordon equation

The Klein–Gordon equation with no force field is given by [25]

$$(\hbar^2\Box + m^2c^2)\psi = 0 \quad (22)$$

where $\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$ and $\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$.

Next, we take into account a gravitational force field by utilizing a linearized potential. In this case, the GUP-corrected Klein–Gordon equation in one dimension reads

$$\frac{d^2\psi}{dx^2} + \frac{1}{\hbar^2c^2}\left(E^2 - m^2c^4 - 2Ekx\right)\psi + 2i\alpha\hbar\frac{d^3\psi}{dx^3} = 0 . \quad (23)$$

Comparing Eq. (23) with Eq. (10), i.e.,

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E-kx)\psi + 2i\alpha\hbar\frac{d^3\psi}{dx^3} = 0 ,$$

and by making the following "transformations"

$$\frac{2m}{\hbar^2}E \to \frac{1}{\hbar^2c^2}(E^2 - m^2c^4)$$

$$\frac{2Ek}{\hbar^2c^2} \to \frac{2mk}{\hbar^2}$$

we arrive at a length quantization similar to the one given by Eq. (21).

3.2.2. Dirac equation in one dimension

The three-dimensional version of Klein–Gordon equation suffers from the non-locality of the differential operators. In particular, the term p^2, when GUP is considered, becomes

$$p^2 = p_0^2 - 2\alpha p_0^3 = -\hbar^2\nabla^2 + 2i\alpha\hbar^3\nabla^3$$

and, thus, the second term reads

$$2i\alpha\hbar^3\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)^{3/2} .$$

We can deal with this term which is a non-local one using fractional calculus [26], but a much simpler approach would be to employ the Dirac equation.

The free-particle Dirac equation is given by [27]

$$i\frac{\partial\Psi}{\partial t} = \left(\beta mc^2 + c\vec{\alpha}\cdot\vec{P}\right)\Psi \quad (24)$$

with

$$\beta \equiv \gamma^0 = \begin{pmatrix} \mathbf{I}_2 & 0 \\ 0 & -\mathbf{I}_2 \end{pmatrix} \tag{25}$$

and

$$\alpha^i \equiv \gamma^0\gamma^i = \begin{pmatrix} \mathbf{I}_2 & 0 \\ 0 & -\mathbf{I}_2 \end{pmatrix} \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

where σ_i, with $i = x, y, z$ for the 3 spatial dimensions, are the Pauli spin matrices. These matrices are given by [28]

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$
$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{26}$$

Here $\beta mc^2 + c\vec{\alpha} \cdot \vec{P}$ is the Dirac Hamiltonian with no force field to be present. It should be noted that $\vec{\alpha}$ is distinct from the GUP parameter, i.e., α.

At this point, we take into account a gravitational force field by utilizing a potential term in the form $V(\vec{r})$. In this case, the GUP-corrected Dirac equation reads

$$i\frac{\partial \Psi}{\partial t} = \left(\beta mc^2 + c\vec{\alpha} \cdot \vec{P} + V(\vec{r})\mathbf{I_4}\right)\Psi \,. \tag{27}$$

Specifically, for the case of one spatial dimension, say z, the GUP-corrected Dirac equation reads

$$\left(-ic\hbar\alpha_z\frac{d}{dz} + c\alpha\hbar^2\frac{d^2}{dz^2} + \beta mc^2 + kz\mathbf{I}_4\right)\psi(z) = E\psi(z) \,.$$

This equation represents a relativistic particle in a one-dimensional box with a potential of the form kz inside.

The four linearly independent solutions to this equation are given by

$$\psi_1 = N_1\left(1 - \frac{4ik\alpha\kappa z}{c/z + 2i\alpha\kappa\left(c(1-2\alpha\kappa\hbar^2) - 2E\right)}\right) \times$$
$$e^{i\kappa z}\begin{pmatrix} \chi \\ r\sigma_z\chi \end{pmatrix}$$
$$\psi_2 = N_2 e^{iz/\alpha\hbar}\begin{pmatrix} \chi \\ \sigma_z\chi \end{pmatrix} \tag{28}$$

with χ being a normalized spinor that satisfies the relation $\chi^\dagger\chi = I$.

Imposing the boundary conditions directly here, we end up having the so-called Klein paradox. In order to avoid this, we will resort to the MIT bag model of quark confinement [29]. Imposing the MIT bag boundary conditions and omitting some straightforward steps, the condition on the length of the box is given by

$$\frac{L}{\alpha\hbar} = arg\left[\frac{\rho_1(ir-1)\left(e^{i(\delta-\kappa L)} - e^{i\left(\kappa L - tan^{-1}\left(\frac{2r}{r^2-1}\right)\right)}\right)}{F'}\right]$$
$$-\frac{\pi}{4} + 2n\pi\,,\; n \in \mathbb{N}, \tag{29}$$

where $\kappa = \kappa_0 + \alpha\hbar\kappa_0^2$ with κ_0 being the wavenumber that satisfies the relation $E^2 = (\hbar\kappa_0)^2 + (mc^2)^2$. Additionally, r, δ, ρ_1, and F' are defined as

$$r = \frac{\hbar\kappa_0 c}{E + mc^2}$$
$$\delta = \tan^{-1}\left(\frac{2r}{r^2-1}\right)$$
$$\rho_1 = \left(1 - \frac{4ik\alpha\kappa z}{c/z + 2i\alpha\kappa\left(c(1-2\alpha\kappa\hbar^2) - 2E\right)}\right)$$
$$F' = \sqrt{2}F$$

with $F \sim \alpha^s$ and $s > 0$.

3.2.3. Dirac equation in three dimensions

In the most general case, let us consider a box defined by $0 \leq x_i \leq L_i,\ i = 1\ldots d$, d being the dimension of the box, i.e., $d = 1, 2$, or 3. That is, this box can be one, two, or three-dimensional. The box has a linearized potential inside, as before. Without loss of generality, we orient the box such that the direction in which the potential changes is our x-direction. The Dirac Hamiltonian with the linear potential term can now be written as

$$\begin{aligned} H &= c\vec{\alpha} \cdot \vec{p} + \beta mc^2 + V(\vec{r})I \\ &= c\left(\alpha_x p_x + \alpha_y p_y + \alpha_z p_z\right) + \beta mc^2 + kxI \\ &= c\vec{\alpha} \cdot \vec{p_0} - c\alpha\left(\vec{\alpha} \cdot \vec{p_0}\right)\left(\vec{\alpha} \cdot \vec{p_0}\right) + \beta mc^2 + kxI \,. \end{aligned}$$

Note that we employed the GUP-corrected momenta, i.e., $p_i = p_{0i}(1 - \alpha p_0)$, $i = 1, .., 3$, where $p_{0i} = -i\hbar\frac{d}{dx_i}$, and followed Dirac prescription, i.e., we replaced p_0 by $\vec{\alpha} \cdot \vec{p_0}$.

The wavefunction inside the box turns out to be

$$\psi = \begin{pmatrix} \left[\prod_{i=1}^{d}\left(\rho_1^{\delta_{i1}}e^{i\kappa_i x_i} + \rho_2^{\delta_{i1}}e^{-i(\kappa_i x_i - \delta_i)}\right) + Fe^{i\frac{\hat{q}.\vec{r}}{\alpha\hbar}}\right]\chi \\ \sum_{j=1}^{d}\left[\prod_{i=1}^{d}\left(\rho_1^{\delta_{i1}}e^{i\kappa_i x_i} + (-1)^{\delta_{ij}}\rho_2^{\delta_{i1}}e^{-i(\kappa_i x_i - \delta_i)}\right)r\hat{\kappa}_j + Fe^{i\frac{\hat{q}.\vec{r}}{\alpha\hbar}}\hat{q}_j\right]\sigma_j\chi \end{pmatrix}$$

where δ_{ij} is the usual Kronecker delta, $\hat{q}$ is an arbitrary unit vector, and δ_l is given by

$$\delta_l = \kappa_l L_l = \tan^{-1}\left(\frac{2r\hat{\kappa}_l}{r^2\hat{\kappa}_l^2 - 1}\right) + \mathcal{O}(\ln\alpha)$$

with $\hat{\kappa}_l$ being the l component of the unit vector of the wave vector $\vec{\kappa}$ with components κ_l.

Moreover, ρ_1 and ρ_2 are defined as

$$\rho_1 = \left(1 - \frac{4ik\alpha\kappa_1 x}{c/x + 2i\alpha\kappa_1\left(c(1-2\alpha\kappa_1\hbar^2) - 2E\right)}\right)$$
$$\rho_2 = \left(1 + \frac{4ik\alpha\kappa_1 x}{c/x - 2i\alpha\kappa_1\left(c(1+2\alpha\kappa_1\hbar^2) - 2E\right)}\right) \,.$$

The number of terms in the first row is $2^d + 1$ and that in the second row is $(2^d + 1) \times d$.

Using the MIT bag model again, we obtain conditions on the dimensions of the box. In this case, these conditions are not symmetrical unlike the case in flat spacetime. Along x-direction, the length quantization has the following form

$$\frac{\hat{q}_1 L_1}{\alpha\hbar} = \frac{\hat{q}_1 L_1}{\alpha_0 \ell_{Pl}} = -\theta_1 + arg\left(\frac{\rho_1(ir\hat{\kappa}_1 - 1) - \rho_2(ir\hat{\kappa}_1 + 1)e^{i\delta_1}}{F'} f_{\bar{1}}\right) + 2n_1\pi\,,\ n_1 \in \mathbb{N} \quad (30)$$

with $f_{\bar{l}}(x_i, \kappa_i, \delta_i) = \prod_{i=1(i\neq l)}^{d} \left(e^{i\kappa_i x_i} + e^{-i(\kappa_i x_i - \delta_i)}\right)$. Along y and z directions, the quantization conditions are identical

$$\frac{\hat{q}_l L_l}{\alpha\hbar} = \frac{\hat{q}_l L_l}{\alpha_0 \ell_{Pl}} = -2\theta_l + 2n_l\pi \quad (31)$$

with $n_l \in \mathbb{N}$ and $\theta_l = \tan^{-1}(\hat{q}_l)$.

This is also consistent with the fact that the potential inside the box increases linearly along x-direction and remains zero along y and z directions.

To obtain the area and volume quantizations, we simply multiply the above conditions

$$A_N = \prod_{l=1}^{N} \frac{\hat{q}_l L_l}{\alpha_0 \ell_{Pl}} = \prod_{l=2}^{N} (2n_l\pi - 2\theta_l)\left(2n_1\pi - \theta_1 + arg\left(\frac{\rho_1(ir\hat{\kappa}_1 - 1) - \rho_2(ir\hat{\kappa}_1 + 1)e^{i\delta_1}}{F'} f_{\bar{1}}\right)\right) \quad (32)$$

with $n_l \in \mathbb{N}$, and where $N = 2$ and $N = 3$ represent the area and volume quantization, respectively.

4. Conclusions

In this work, we have shown that if we trap a particle in a one-dimensional box of size L, include a gravitational potential inside the box and then try to measure the length of the box, the length L will appear as a quantized quantity in units of $\alpha_0 \ell_{Pl}$ where ℓ_{Pl} is the Planck length. This result can be interpreted as the discreteness of space near the Planck scale holding for curved spacetime as it holds for flat spacetime, as shown in previous works [18,22].

For the gravitational potential, we have used the first term of a Taylor series to describe it as a linearized potential. This is reasonable because we are interested in the behavior of spacetime fabric near Planck scale and the gravitational potential changes at a very slow rate over such small distances.

We have implemented our method for a non-relativistic particle in curved spacetime and for a relativistic one. In the latter case, the GUP-corrected Klein–Gordon equation in one dimension has been solved as well as the GUP-corrected Dirac equation in one, two and three dimensions. As already mentioned, in all cases the length of the box appears as a quantized quantity in units of $\alpha_0 \ell_{Pl}$. The presence of lengths that are proportional to the Planck length in all cases strengthens the claim of the existence of a minimum measurable length. Furthermore, in two and three dimensions, the area and volume quantizations were also obtained.

Extension of the method employed in this work for arbitrary curved spacetime would be quite interesting. In particular, it is expected that subsequent terms in the Taylor series would give rise to a more general curved spacetime. Hence, an arbitrary form of the gravitational potential could be analyzed following the same approach. This would still assume a fixed classical background. A complete theory of quantum gravity, once formulated, should be able to address the issues discussed here, with background spacetime which may be fluctuating. In this case, we hope that the results derived in this work would continue to hold, at least approximately, and almost exactly in the limit when such fluctuations can be ignored.

Finally, one may be interested in delving into the possible connection between the non-relativistic particle moving in a box inside which a linear potential is present, and the hydrogen atom. In both systems, a fine structure (splitting) shows up. In particular, for the first system it is the fine structure of the length quantization, while for second system it is the fine structure of the energy quantization. This apparent coincidence suggests further investigation of the discreteness of spacetime. In addition, although the original HUP is restricted to position–momentum commutation while the time–energy uncertainty principle has been merely thought of as a statistical measure of variance, a more generalized idea of GUP-corrected commutation relation involving 4-momentum might give rise to discontinuity of time.

Acknowledgements

We would like to thank the referee for constructive comments. This work is supported in part by the Natural Sciences and Engineering Research Council of Canada.

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