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Cyclic $m$-cycle systems of complete graphs minus a 1-factor

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In honour of Dan Archdeacon.

Abstract

In this paper, we provide necessary and sufficient conditions for the existence of a cyclic $m$-cycle system of $K_n - I$ when $m$ and $n$ are even and $m \mid n$.

1 Introduction

Throughout this paper, $K_n$ will denote the complete graph on $n$ vertices, $K_n - I$ will denote the complete graph on $n$ vertices with a 1-factor $I$ removed (a 1-factor is a 1-regular spanning subgraph), and $C_m$ will denote the $m$-cycle $(v_1, v_2, \ldots, v_m)$. An $m$-cycle system of a graph $G$ is a set $C$ of $m$-cycles in $G$ whose edges partition the edge set of $G$. An $m$-cycle system is called hamiltonian if $m = |V(G)|$.

Several obvious necessary conditions for an $m$-cycle system $C$ of a graph $G$ to exist are immediate: $m \leq |V(G)|$, the degrees of the vertices of $G$ must be even, and $m$ must divide the number of edges in $G$. A survey on cycle systems is given in [4] and necessary and sufficient conditions for the existence of an $m$-cycle system of $K_n$ and $K_n - I$ were given in [1, 16] where it was shown that an $m$-cycle system of $K_n$ or $K_n - I$ exists if and only if $n \geq m$, every vertex of $K_n$ or $K_n - I$ has even degree, and $m$ divides the number of edges in $K_n$ or $K_n - I$, respectively.
Throughout this paper, \( \rho \) will denote the permutation \((0 \ 1 \ \ldots \ n-1)\), so \( \langle \rho \rangle = \mathbb{Z}_n \). An \( m \)-cycle system \( \mathcal{C} \) of a graph \( G \) with vertex set \( V(G) = \mathbb{Z}_n \) is cyclic if, for every \( m \)-cycle \( C = (v_1, v_2, \ldots, v_m) \) in \( \mathcal{C} \), the \( m \)-cycle \( \rho(C) = (\rho(v_1), \rho(v_2), \ldots, \rho(v_m)) \) is also in \( \mathcal{C} \). A cyclic \( n \)-cycle system \( \mathcal{C} \) of a graph \( G \) with vertex set \( \mathbb{Z}_n \) is called a cyclic hamiltonian cycle system. Finding necessary and sufficient conditions for cyclic \( m \)-cycle systems of \( K_n \) is an interesting problem and has attracted much attention (see, for example, \([2, 3, 6, 7, 10, 11, 13, 15]\)). The obvious necessary conditions for a cyclic \( m \)-cycle system of \( K_n \) are the same as for an \( m \)-cycle system of \( K_n \); that is, \( n \geq m \geq 3 \), \( n \) is odd (so that the degree of every vertex is even), and \( m \) must divide the number of edges in \( K_n \). However, these conditions are no longer necessarily sufficient. For example, it is not difficult to see that there is no cyclic decomposition of \( K_{15} \) into 15-cycles. Also, if \( p \) is an odd prime and \( \alpha \geq 2 \), then \( K_{p^\alpha} \) cannot be decomposed cyclically into \( p^\alpha \)-cycles \([7]\).

The existence question for cyclic \( m \)-cycle systems of \( K_n \) has been completely settled in a few small cases, namely \( m = 3 \) \([14]\), 5 and 7 \([15]\). For even \( m \) and \( n \equiv 1 \pmod{2m} \), cyclic \( m \)-cycle systems of \( K_n \) are constructed for \( m \equiv 0 \pmod{4} \) \([13]\) and for \( m \equiv 2 \pmod{4} \) \([15]\). Both of these cases are handled simultaneously in \([10]\). For odd \( m \) and \( n \equiv 1 \pmod{2m} \), cyclic \( m \)-cycle systems of \( K_n \) are found using different methods in \([2, 6, 11]\). In \([3]\), as a consequence of a more general result, cyclic \( m \)-cycle systems of \( K_n \) for all positive integers \( m \) and \( n \equiv 1 \pmod{2m} \) with \( n \geq m \geq 3 \) are given using similar methods. In \([7]\), it is shown that a cyclic hamiltonian cycle system of \( K_n \) exists if and only if \( n \neq 15 \) and \( n \not\in \{p^\alpha \mid p \text{ is an odd prime and } \alpha \geq 2\} \). Thus, as a consequence of a result in \([6]\), cyclic \( m \)-cycle systems of \( K_{2mk+m} \) exist for all \( m \neq 15 \) and \( m \not\in \{p^\alpha \mid p \text{ is an odd prime and } \alpha \geq 2\} \). In \([17]\), the last remaining cases for cyclic \( m \)-cycle systems of \( K_{2mk+m} \) are settled, i.e., it is shown that, for \( k \geq 1 \), cyclic \( km \)-cycle systems of \( K_{2km+m} \) exist if \( m = 15 \) or \( m \in \{p^\alpha \mid p \text{ is an odd prime and } \alpha \geq 2\} \). In \([19]\), necessary and sufficient conditions for the existence of cyclic \( 2q \)-cycle and \( m \)-cycle systems of the complete graph are given when \( q \) is an odd prime power and \( 3 \leq m \leq 32 \). In \([5]\), cycle systems with a sharply vertex-transitive automorphism group that is not necessarily cyclic are investigated. As a result, it is shown in \([5]\) that no cyclic \( m \)-cycle system of \( K_n \) exists if \( m < n < 2m \) with \( n \) odd and \( \gcd(m,n) \) a prime power. In \([18]\), it is shown that if \( m \) is even and \( n > 2m \), then there exists a cyclic \( m \)-cycle system of \( K_n \) if and only if the obvious necessary conditions that \( n \) is odd and that \( n(n-1) \equiv 0 \pmod{2m} \) hold.

These questions can be extended to the case when \( n \) is even by considering the graph \( K_n - I \). In \([3]\), it is shown that for all integers \( m \geq 3 \) and \( k \geq 1 \), there exists a cyclic \( m \)-cycle system of \( K_{2mk+2} - I \) if and only if \( mk \equiv 0, 3 \pmod{4} \). In \([12]\), it is shown that for an even integer \( n \geq 4 \), there exists a cyclic hamiltonian cycle system of \( K_n - I \) if and only if \( n \equiv 2, 4 \pmod{8} \) and \( n \neq 2p^\alpha \) where \( p \) is an odd prime and \( \alpha \geq 1 \). In \([8]\), it was shown that in every cyclic cycle decomposition of \( K_{2n} - I \), the number of cycle orbits of odd length must have the same parity as \( n(n-1)/2 \). As a consequence of this result, in \([8]\), it is shown that a cyclic \( m \)-cycle system of \( K_{2n} - I \) can not exist if \( n \equiv 2, 3 \pmod{4} \) and \( m \neq 0 \pmod{4} \) or \( n \equiv 0, 1 \pmod{4} \) and \( m \) does not divide \( n(n-1) \). In this paper we are interested in cyclic \( m \)-cycle systems of \( K_n - I \) when \( m \) and \( n \) are even and \( m \mid n \). The main result of this paper is the
Theorem 1.1 For an even integer $m$ and integer $t$, there exists a cyclic $m$-cycle system of $K_{mt} - I$ if and only if

1. $t \equiv 0, 2 \pmod{4}$ when $m \equiv 0 \pmod{8}$,
2. $t \equiv 0, 1 \pmod{4}$ when $m \equiv 2 \pmod{8}$ with $t > 1$ if $m = 2p^\alpha$ for some prime $p$ and integer $\alpha \geq 1$,
3. $t \geq 1$ when $m \equiv 4 \pmod{8}$, and
4. $t \equiv 0, 3 \pmod{4}$ when $m \equiv 6 \pmod{8}$.

Our methods involve circulant graphs and difference constructions. In Section 2, we give some basic definitions and lemmas while the proof of Theorem 1.1 is given in Sections 3, 4 and 5. In Section 3, we handle the case when $m \equiv 0 \pmod{8}$ and show that there is a cyclic $m$-cycle system of $K_{mt} - I$ if and only if $t \geq 2$ is even. In Section 4, we handle the case when $m \equiv 4 \pmod{8}$ and show that there is a cyclic $m$-cycle system of $K_{mt} - I$ if and only if $t \geq 1$. In Section 5, we handle the case when $m \equiv 2 \pmod{4}$. When $m \equiv 2 \pmod{8}$, we show that there is a cyclic $m$-cycle system of $K_{mt} - I$ if and only if $t \equiv 0, 1 \pmod{4}$. When $m \equiv 6 \pmod{8}$, we show that there is a cyclic $m$-cycle system of $K_{mt} - I$ if and only if $t \equiv 0, 3 \pmod{4}$. Our main theorem then follows.

2 Preliminaries

The notation $[1, n]$ denotes the set $\{1, 2, \ldots, n\}$. The proof of Theorem 1.1 uses circulant graphs, which we now define. For $x \not\equiv 0 \pmod{n}$, the modulo $n$ length of an integer $x$, denoted $|x|_n$, is defined to be the smallest positive integer $y$ such that $x \equiv y \pmod{n}$ or $x \equiv -y \pmod{n}$. Note that for any integer $x \not\equiv 0 \pmod{n}$, it follows that $|x|_n \in [1, \lceil \frac{n}{2} \rceil]$. If $L$ is a set of modulo $n$ lengths, we define the circulant graph $\langle L \rangle_n$ to be the graph with vertex set $\mathbb{Z}_n$ and edge set $\left\{\{i, j\} \mid |i - j|_n \in L\right\}$. Notice that in order for a graph $G$ to admit a cyclic $m$-cycle decomposition, $G$ must be a circulant graph, so circulant graphs provide a natural setting in which to construct cyclic $m$-cycle decompositions.

The graph $K_n$ is a circulant graph, since $K_n = \langle \{1, 2, \ldots, |n/2|\}\rangle_n$. For $n$ even, $K_n - I$ is also a circulant graph, since $K_n - I = \langle \{1, 2, \ldots, (n - 2)/2\}\rangle_n$ (so the edges of the 1-factor $I$ are of the form $\{i, i + n/2\}$ for $i = 0, 1, \ldots, (n - 2)/2$).

Let $H$ be a subgraph of a circulant graph $\langle L \rangle_n$. The notation $\ell(H)$ will denote the set of modulo $n$ edge lengths belonging to $H$, that is,

$$\ell(H) = \{\ell \in L \mid \{g, g + \ell\} \in E(H) \text{ for some } g \in \mathbb{Z}_n\}.$$ 

Many properties of $\ell(H)$ are independent of the choice of $L$; in particular, the next lemma in this section does not depend on the choice of $L$. 
Let $C$ be an $m$-cycle in circulant graph $(L)_n$ and recall that the permutation
\[ \rho = (0 \ 1 \ \ldots \ n-1), \]
which generates $\mathbb{Z}_n$, has the property that $\rho(C) \in \mathcal{C}$ whenever $C \in \mathcal{C}$. We can therefore consider the action of $\mathbb{Z}_n$ as a permutation group acting on the elements of $\mathcal{C}$. Viewing matters this way, the length of the orbit of $C$ (under the action of $\mathbb{Z}_n$) can be defined as the least positive integer $k$ such that $\rho^k(C) = C$. Observe that such a $k$ exists since $\rho$ has finite order; furthermore, the well-known orbit-stabilizer theorem (see, for example [9, Theorem 1.4A(iii)]) tells us that $k$ divides $n$. Thus, if $G$ is a graph with a cyclic $m$-cycle system $\mathcal{C}$ with $C \in \mathcal{C}$ in an orbit of length $k$, then it must be that $k$ divides $n = |V(G)|$ and that $\rho(C), \rho^2(C), \ldots, \rho^{k-1}(C)$ are distinct $m$-cycles in $\mathcal{C}$.

The next lemma gives many useful properties of an $m$-cycle $C$ in a cyclic $m$-cycle system $\mathcal{C}$ of a graph $G$ with $V(G) = \mathbb{Z}_n$ where $C$ is in an orbit of length $k$. Many of these properties are also given in [7] in the case that $m = n$. The proofs of the following statements follow directly from the previous definitions and are therefore omitted.

**Lemma 2.1** Let $\mathcal{C}$ be a cyclic $m$-cycle system of a graph $G$ of order $n$ and let $C \in \mathcal{C}$ be in an orbit of length $k$. Then

1. $|\ell(C)| = mk/n$;
2. $C$ has $n/k$ edges of length $\ell$ for each $\ell \in \ell(C)$;
3. $(n/k) \mid \gcd(m, n)$;

Let $k > 1$ and let $P : v_0 = 0, v_1, \ldots v_{mk/n}$ be a subpath of $C$ of length $mk/n$. Then

4. if there exists $\ell \in \ell(C)$ with $k \mid \ell$, then $m = n/\gcd(\ell, n)$,
5. $v_{mk/n} = kx$ for some integer $x$ with $\gcd(x, n/k) = 1$,
6. $v_1, v_2, \ldots, v_{mk/n}$ are distinct modulo $k$,
7. $\ell(P) = \ell(C)$, and
8. $P, \rho^k(P), \rho^{2k}(P), \ldots, \rho^{n-k}(P)$ are pairwise edge-disjoint subpaths of $C$.

Let $X$ be a set of $m$-cycles in a graph $G$ with vertex set $\mathbb{Z}_n$ such that $\mathcal{C} = \{\rho^i(C) \mid C \in X, i = 0, 1, \ldots, n-1\}$ is an $m$-cycle system of $G$. Then $X$ is called a generating set for $\mathcal{C}$. Clearly, every cyclic $m$-cycle system $\mathcal{C}$ of a graph $G$ has a generating set $X$ as we may always let $X = \mathcal{C}$. A generating set $X$ is called a minimum generating set if $C \in X$ implies $\rho^i(C) \notin X$ for $1 \leq i \leq n$ unless $\rho^i(C) = C$.

Let $\mathcal{C}$ be a cyclic $m$-cycle system of a graph $G$ with $V(G) = \mathbb{Z}_n$. To find a minimum generating set $X$ for $\mathcal{C}$, we start by adding $C_1$ to $X$ if the length of the orbit of $C_1$ is maximum among the cycles in $\mathcal{C}$. Next, we add $C_2$ to $X$ if the length of the orbit of $C_2$ is maximum among the cycles in $\mathcal{C} \setminus \{\rho^i(C_1) \mid 0 \leq i \leq n-1\}$. Continuing in this manner, we add $C_3$ to $X$ if the length of the orbit of $C_3$ is maximum among the cycles in $\mathcal{C} \setminus \{\rho^i(C_1), \rho^i(C_2) \mid 0 \leq i \leq n-1\}$. We continue in this manner
Proof: Let \( \rho(C) \) be a cycle in an orbit of length \( k \). Suppose first that \( k \) is odd. Then, \( \ell(C) \) has an odd number of odd integers. Hence, \( \ell(C) \) contains an odd number of odd integers and, since \( |\ell(C)| \) is odd, an even number of even integers, contradicting the choice of \( \rho(C) \). Thus, \( k \) is even.

Since \( k \) is even, \( jk \) is even. Thus, \( \ell(C) \) contains an even number of even integers. If \( k \) is even, then \( \ell(C) \) also contains an odd number of odd integers, contradicting the choice of \( C \). Thus, \( \ell \) is odd.

Now suppose \( \{1, 2, \ldots, (mt - 2)/2\} \) has an odd number of odd integers. Hence there are an odd number of cycles \( C \) in \( X \) with \( \ell(C) \) containing an odd number of odd integers. Again, let \( C \in X \) be such a cycle with \( |\ell(C)| = \ell \), in an orbit of length \( k = \ell t \). Let the subpath of \( C \) starting at vertex 0 of length \( \ell \) end at vertex \( jk \) with \( \gcd(j, m/\ell) = 1 \). Now, if \( k \) is even, then \( jk \) is even so that \( \ell(C) \) contains an even number of odd integers, contradicting the choice of \( C \). Thus \( k \) is odd. Since \( k = \ell t \), we have that \( t \) is odd.

The following corollary is an immediate consequence of Lemma 2.2 and [12].

**Corollary 2.3** For an even integer \( m \) and a positive integer \( t \), if there exists a cyclic \( m \)-cycle system of \( K_{mt} - I \), then
(1) \( t \equiv 0, 2 \pmod{4} \) when \( m \equiv 0 \pmod{8} \),

(2) \( t \equiv 0, 1 \pmod{4} \) when \( m \equiv 2 \pmod{8} \) with \( t > 1 \) if \( m = 2p^\alpha \) for some prime \( p \) and integer \( \alpha \geq 1 \),

(3) \( t \equiv 0, 3 \pmod{4} \) when \( m \equiv 6 \pmod{8} \), and

(4) \( t \geq 1 \) when \( m \equiv 4 \pmod{8} \).

Let \( n > 0 \) be an integer and suppose there exists an ordered \( m \)-tuple \((d_1, d_2, \ldots, d_m)\) satisfying each of the following:

(i) \( d_i \) is an integer for \( i = 1, 2, \ldots, m \);

(ii) \( |d_i| \neq |d_j| \) for \( 1 \leq i < j \leq m \);

(iii) \( d_1 + d_2 + \cdots + d_m \equiv 0 \pmod{n} \); and

(iv) \( d_1 + d_2 + \cdots + d_r \not\equiv d_1 + d_2 + \cdots + d_s \pmod{n} \) for \( 1 \leq r < s \leq m \).

Then an \( m \)-cycle \( C \) can be constructed from this \( m \)-tuple, that is, let \( C = (0, d_1, d_1 + d_2, \ldots, d_1 + d_2 + \cdots + d_{m-1}) \), and \( \{C\} \) is a minimum generating set for a cyclic \( m \)-cycle system of \( \{d_1, d_2, \ldots, d_m\}_n \). Thus, in what follows, to find cyclic \( m \)-cycle systems of \( \langle L \rangle_n \), it suffices to partition \( L \) into \( m \)-tuples satisfying the above conditions. Hence, an \( m \)-tuple satisfying (i)-(iv) above is called a difference \( m \)-tuple and it corresponds to the \( m \)-cycle \( C = (0, d_1, d_1 + d_2, \ldots, d_1 + d_2 + \cdots + d_{m-1}) \) in \( \langle L \rangle_n \).

3 The Case when \( m \equiv 0 \pmod{8} \)

In this section, we consider the case when \( m \equiv 0 \pmod{8} \) and show that there exists a cyclic \( m \)-cycle system of \( K_{mt} - I \) for each even positive integer \( t \). We begin with the case \( t = 2 \).

**Lemma 3.1** For each positive integer \( m \equiv 0 \pmod{8} \), there exists a cyclic \( m \)-cycle system of \( K_{2m} - I \).

**Proof:** Let \( m \) be a positive integer such that \( m \equiv 0 \pmod{8} \), say \( m = 8r \) for some positive integer \( r \). Then \( K_{2m} - I = \langle S' \rangle_{2m} \) where \( S' = \{1, 2, \ldots, m-1\} = \{1, 2, \ldots, 8r-1\} \). The proof proceeds as follows. We begin by finding a path \( P \) of length \( m/2 = 4r \) ending at vertex \( m \), so that \( C = P \cup \rho^m(P) \) is an \( m \)-cycle. Note that \( \{2\}_m \) consists of two vertex disjoint \( m \)-cycles. For the remaining \( 4r - 2 \) edge lengths in \( S' \setminus (\ell(P) \cup \{2\}) \), we find \( 2r - 1 \) paths \( P_i \) of length 2, ending at vertex 4 or \(-4\), so that \( C_i = P_i \cup \rho^4(P_i) \cup \rho^8(P_i) \cup \cdots \cup \rho^{2m-4}(P_i) \) is an \( m \)-cycle. Then this collection of cycles will give a minimum generating set for a cyclic \( m \)-cycle system of \( K_{2m} - I \).

Suppose first that \( r \) is odd. For \( r = 1 \), let \( P : 0, -3, 3, 7, 8 \) and note that the edge lengths of \( P \) in the order encountered are 3, 6, 4, 1. For \( r = 3 \), let

\[
P : 0, -3, 3, -7, 7, -11, 11, 23, 19, 20, -20, -4, 24
\]
and note that edge lengths of $P$ in the order encountered are 3, 6, 10, 14, 18, 22, 12, 4, 1, 8, 16, 20. For $r \geq 5$, let
\[
P : 0, -3, 3, -7, 7, \ldots, -(4r - 1), 4r - 1, -1, -5, -4, 4, -8, 8, \ldots, -(2r - 4),
2r - 4, 2r + 8, 2r + 12, \ldots, -(4r - 8), 4r, -(4r - 4), 8r
\]
be a path of length $m/2$ whose edge lengths in the order encountered are 3, 6, 10, 14, \ldots, $8r - 6$, $8r - 2$, 4, 1, 8, 12, 16, \ldots, $4r - 4$, 4, $4r - 8$, 4, $4r + 8$, 4, $4r + 12$, \ldots, $8r - 8$, $8r - 4$, $4r + 4$.

Now suppose that $r$ is even. For $r = 2$, let $P : 0, -3, 3, -7, 7, -1, -5, -4, 16$ and note that the edge lengths of $P$ in the order encountered are 3, 6, 10, 14, 8, 4, 1, 12. For $r \geq 4$, let
\[
P : 0, -3, 3, -7, 7, \ldots, -(4r - 1), 4r - 1, -1, -5, -4, 4, -8, 8, \ldots, -(2r - 4),
2r - 4, 2r - 8, 2r + 8, 2r + 12, \ldots, -(4r - 8), 4r, -(4r - 4), 8r
\]
be a path of length $m/2$ whose edge lengths in the order encountered are 3, 6, 10, 14, \ldots, $8r - 6$, $8r - 2$, 4, 1, 8, 12, 16, \ldots, $4r - 8$, $4r - 4$, $4r + 8$, $4r + 12$, \ldots, $8r - 8$, $8r - 4$, $4r + 4$.

In each case, let $C = P \cup \rho^m(P)$ and observe that $C$ is an $m$-cycle $C$ with $\ell(C) = \{1, 3, 4, 6, 8, \ldots, 8r - 2\}$. Let $C' = (0, 2, 4, 6, \ldots, 2m - 2)$ and note that $C'$ is an $m$-cycle with $\ell(C') = \{2\}$.

For $0 \leq i \leq r - 2$, let $P_i : 0, 9 + 8i, 4$ be the path of length 2 with edge lengths 9 + 8i, 5 + 8i and let $P_i' : 0, 11 + 8i, 4$ be the path of length 2 with edge lengths 11 + 8i, 7 + 8i. Let $C_i = P_i \cup \rho^i(P_i) \cup \rho^8(P_i) \cup \cdots \cup \rho^{2m-4}(P_i)$ and $C_i' = P_i' \cup \rho^i(P_i') \cup \rho^8(P_i') \cup \cdots \cup \rho^{2m-4}(P_i')$ and note that each is an $m$-cycle with $\ell(C_i) = \{5 + 8i, 9 + 8i\}$ and $\ell(C_i') = \{7 + 8i, 11 + 8i\}$.

Finally, let $P_0 : 0, 8r - 3, -4$ be the path of length 2 with edge lengths 8r - 3 and 8r - 1. Let $C'' = P_0'' \cup \rho^i(P_0'') \cup \rho^8(P_0'') \cup \cdots \cup \rho^{2m-4}(P_0'')$ and note that $C''$ is an $m$-cycle with $\ell(C'') = \{8r - 3, 8r - 1\}$.

Then $\{C, C', C_0, \ldots, C_{r-2}, C_0', \ldots, C_{r-2}', C''\}$ is a minimum generating set for a cyclic $m$-cycle system of $K_{2m} - I$.

We now consider the case when $t$ is even and $t > 2$.

**Lemma 3.2** For each positive integer $k$ and each positive integer $m \equiv 0 \pmod{8}$, there exists a cyclic $m$-cycle system of $K_{2mk} - I$.

**Proof:** Let $m$ and $k$ be positive integers such that $m \equiv 0 \pmod{8}$. Lemma 3.1 handles the case when $k = 1$ and thus we may assume that $k \geq 2$. Then $K_{2km} - I = \langle S' \rangle_{2km}$ where $S' = \{1, 2, \ldots, km - 1\}$. Since $K_{2m} - I$ has a cyclic $m$-cycle system by Lemma 3.1 and $\langle \{k, 2k, \ldots, mk\} \rangle_{2km}$ consists of $k$ vertex-disjoint copies of $K_{2m} - I$, we need only show that $\langle S \rangle_{2km}$ has a cyclic $m$-cycle system where $S = \{1, 2, \ldots, mk\} \setminus \{k, 2k, \ldots, mk\}$.  

Let \( A = [a_{i,j}] \) be the \((k-1) \times m\) array

\[
\begin{bmatrix}
  k-1 & 2k-1 & 3k-1 & 4k-1 & (m-1)k-1 & mk-1 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  2 & k+2 & 2k+2 & 3k+2 & (m-2)k+2 & (m-1)k+2 \\
  1 & k+1 & 2k+1 & 3k+1 & (m-2)k+1 & (m-1)k+1 \\
\end{bmatrix}.
\]

It is straightforward to verify that \( A \) satisfies

\[
\sum_{j \equiv 0,1 \pmod{4}} a_{i,j} = \sum_{j \equiv 2,3 \pmod{4}} a_{i,j},
\]

and

\[
a_{i,1} < a_{i,2} < \ldots < a_{i,m}
\]

for each \( i \) with \( 1 \leq i \leq k-1 \).

For each \( i = 1, 2, \ldots, k-1 \), the \( m \)-tuple

\[
(a_{i,1}, -a_{i,3}, a_{i,5}, -a_{i,7}, \ldots, a_{i,m-3}, -a_{i,m-1}, -a_{i,m-2}, a_{i,m-4}, -a_{i,m-6}, \ldots, -a_{i,6}, a_{i,4}, -a_{i,2}, a_{i,m})
\]

is a difference \( m \)-tuple and corresponds to an \( m \)-cycle \( C_i \) with \( \ell(C_i) = \{a_{i,1}, a_{i,2}, \ldots, a_{i,m}\} \). Hence, \( X = \{C_1, C_2, \ldots, C_{k-1}\} \) is a minimum generating set for a cyclic \( m \)-cycle system of \( \langle S \rangle_{2km} \).

\[\square\]

4 The Case when \( m \equiv 4 \pmod{8} \)

In this section, we consider the case when \( m \equiv 4 \pmod{8} \) and show that there exists a cyclic \( m \)-cycle system of \( K_{mt} - I \) for each \( t \geq 1 \). We begin with the case when \( t \) is odd, say \( t = 2k+1 \) for some nonnegative integer \( k \).

**Lemma 4.1** For each nonnegative integer \( k \) and each \( m \equiv 4 \pmod{8} \), there exists a cyclic \( m \)-cycle system of \( K_{m(2k+1)} - I \).

**Proof:** Let \( m \) and \( k \) be nonnegative integers such that \( m \equiv 4 \pmod{8} \). Since \( K_m - I \) has a cyclic hamiltonian cycle system [12], we may assume that \( k \geq 1 \). Let \( m = 4r \) for some positive integer \( r \). Then \( K_{m(2k+1)} - I = \langle S' \rangle_{(2k+1)m} \) where \( S' = \{1, 2, \ldots, 4rk + 2r - 1\} \). Again, since \( K_m - I \) has a cyclic hamiltonian cycle system [12] and \( \langle \{2k+1, 4k+2, \ldots, (2r-1)(2k+1)\} \rangle_{(2k+1)m} \) consists of \( 2k+1 \) vertex-disjoint copies of \( K_m - I \), we need only show that \( \langle S' \rangle_{(2k+1)m} \) has a cyclic \( m \)-cycle system where

\[
S = \{1, 2, \ldots, 4rk + 2r - 1\} \setminus \{2k+1, 4k+2, \ldots, (2r-1)(2k+1)\}.
\]
Let \( r \) and \( k \) be positive integers. Let \( A = [a_{i,j}] \) be the \( k \times m \) array
\[
\begin{bmatrix}
  k & 2k & 3k + 1 & 4k + 1 & 5k + 2 & (4r - 2)k + 2r - 2 & (4r - 1)k + 2r - 1 & 4rk + 2r - 1 \\
  2 & k + 2 & 2k + 3 & 3k + 3 & 4k + 4 & (4r - 3)k + 2r & (4r - 2)k + 2r + 1 & (4r - 1)k + 2r + 1 \\
  1 & k + 1 & 2k + 2 & 3k + 2 & 4k + 3 & (4r - 3)k + 2r - 1 & (4r - 2)k + 2r & (4r - 1)k + 2r
\end{bmatrix}.
\]

It is straightforward to verify that \( A \) satisfies
\[
\sum_{j \equiv 0,1 \mod 4} a_{i,j} = \sum_{j \equiv 2,3 \mod 4} a_{i,j},
\]
and
\[
a_{i,1} < a_{i,2} < \ldots < a_{i,m}
\]
for each \( i \) with \( 1 \leq i \leq k \).

For each \( i = 1, 2, \ldots, k \), the \( m \)-tuple
\[
(a_{i,1}, -a_{i,3}, a_{i,5}, -a_{i,7}, \ldots, a_{i,m-3}, -a_{i,m-1}, -a_{i,m-2}, a_{i,m-4}, -a_{i,m-6}, \ldots, -a_{i,6}, a_{i,4}, -a_{i,2}, a_{i,m})
\]
is a difference \( m \)-tuple and corresponds to an \( m \)-cycle \( C_i \) with \( \ell(C_i) = \{a_{i,1}, a_{i,2}, \ldots, a_{i,m}\} \). Hence, \( X = \{C_1, C_2, \ldots, C_k\} \) is a minimum generating set for a cyclic \( m \)-cycle system of \( K_{m(2k+1)} - I \). \( \Box \)

We now handle the case when \( t \) is even, say \( t = 2k \) for some positive integer \( k \).

**Lemma 4.2** For each positive integer \( k \) and each \( m \equiv 4 \mod 8 \), there exists a cyclic \( m \)-cycle system of \( K_{2mk} - I \).

**Proof:** As before, let \( m \) and \( k \) be positive integers such that \( m \equiv 4 \mod 8 \). Thus \( m = 4r \) for some positive integer \( r \). Then \( K_{2mk} - I = \langle S' \rangle_{2km} \) where \( S' = \{1, 2, \ldots, 4rk - 1\} \). Since \( K_m - I \) has a cyclic hamiltonian cycle system \([12]\) and \( \{2k, 4k, \ldots, (2r - 1)(2k)\}_{2km} \) consists of \( 2k \) vertex-disjoint copies of \( K_m - I \), we need only show that \( \langle S \rangle_{2km} \) has a cyclic \( m \)-cycle system where
\[
S = \{1, 2, \ldots, 4rk - 1\} \setminus \{2k, 4k, \ldots, (2r - 1)(2k)\}.
\]
Since \( |S| = m(k - 1) + m/2 \), we will start by partitioning a subset \( T \subseteq S \) with \( |T| = m(k - 1) \) into \( k - 1 \) difference \( m \)-tuples.

Let \( T = \{1, 2, \ldots, 4rk - 1\} \setminus \{1, 2k, 4k - 1, 4k + 1, 6k, 8k - 1, 8k, 8k + 1, \ldots, (4r - 4)k - 1, (4r - 4)k, (4r - 4)k + 1, (4r - 2)k, 4rk - 1\} \), and observe that \( |T| = (k - 1)m \).

Let \( A = [a_{i,j}] \), with entries from the set \( T \), be the \((k - 1) \times m\) array
\[
\begin{bmatrix}
  k & 2k - 1 & 3k - 1 & 4k - 2 & 5k & 6k - 1 & 7k - 1 & 8k - 2 & 9k \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  3 & k + 2 & 2k + 2 & 3k + 1 & 4k + 3 & 5k + 2 & 6k + 2 & 7k + 1 & 8k + 3 \\
  2 & k + 1 & 2k + 1 & 3k & 4k + 2 & 5k + 1 & 6k + 1 & 7k & 8k + 2
\end{bmatrix}.
\]
It is straightforward to verify that the array $A$ satisfies

$$\sum_{j \equiv 0, 1 (\text{mod } 4)} a_{i,j} = \sum_{j \equiv 2, 3 (\text{mod } 4)} a_{i,j},$$

and

$$a_{i,1} < a_{i,2} < \ldots < a_{i,m}$$

for each $i$ with $1 \leq i \leq k - 1$.

For each $i = 1, 2, \ldots, k - 1$, the $m$-tuple

$$(a_{i,1}, -a_{i,3}, a_{i,5}, -a_{i,7}, \ldots, a_{i,m-3}, -a_{i,m-1}, -a_{i,m-2}, a_{i,m-4}, -a_{i,m-6}, \ldots, -a_{i,6}, a_{i,4}, -a_{i,2}, a_{i,m})$$

is a difference $m$-tuple and corresponds to an $m$-cycle $C_i$ with $\ell(C_i) = \{a_{i,1}, a_{i,2}, \ldots, a_{i,m}\}$. Hence, $X = \{C_1, C_2, \ldots, C_{k-1}\}$ is a minimum generating set for a cyclic $m$-cycle system of $\langle B \rangle_{2km}$.

It now remains to find a minimum generating set for a cyclic $m$-cycle system of $\langle B \rangle_{2km}$ where $B = \{1, 4k - 1, 4k + 1, 8k - 1, 8k + 1, \ldots, (4r - 4)k - 1, (4r - 4)k + 1, 4rk - 1\}$. For $i = 1, 2, \ldots, r$, define $d_{2i-1} = 4(i - 1)k + 1$ and $d_{2i} = 4ik - 1$. Observe that $B = \{d_1, d_2, \ldots, d_{2r}\}$ and $d_{j+2} - d_j = 4k$ for $j = 1, 2, \ldots, 2r - 2$. Since $m \equiv 4 \pmod{8}$, it follows that $r$ is odd. Let $P_1 : 0, 1, 4k$, and let $P_i : 0, d_{2i+1}, 4k$ if $i$ is even and let $P_i : 0, d_{2i}, 4k$ if $i$ is odd. Let $C_i' = P_1 \cup \rho^{4k} (P_1) \cup \rho^{8k} (P_1) \cup \cdots \cup \rho^{(2m-4)k} (P_1)$, and note that $C_i'$ is an $m$-cycle with $\ell(C_i') = \{1, 4k - 1\}$, $\ell(C_i') = \{d_{2i-1}, d_{2i+1}\}$ if $i$ is even, and $\ell(C_i') = \{d_{2i-2}, d_{2i}\}$ if $i$ is odd. Then $\ell(C_i) = \ell(C_i') \cup \cdots \cup \ell(C_r') = B$ so that $\{C_1', C_2', \ldots, C_r'\}$ is a minimum generating set for $\langle B \rangle_{2km}$.

5 The Case when $m \equiv 2 \pmod{4}$

In this section, we consider the case when $m \equiv 2 \pmod{4}$ and prove parts (2) and (4) of Theorem 1.1. We divide this proof into three parts, each dealt with in its own subsection. First we consider the case $t \equiv 0 \pmod{4}$. Then we consider the case $m \equiv 2 \pmod{8}$ and $t \equiv 1 \pmod{4}$. Finally we consider the case $m \equiv 6 \pmod{8}$ and $t \equiv 3 \pmod{4}$.

5.1 The case when $t \equiv 0 \pmod{4}$.

We consider the case $t \equiv 0 \pmod{4}$, starting with the special case $t = 4$.

Lemma 5.1 For each positive integer $m \geq 6$ with $m \equiv 2 \pmod{4}$, there exists a cyclic $m$-cycle system of $K_{4m} - I$. 
Proof: Let $m \geq 6$ be a positive integer with $m \equiv 2 \pmod{4}$. Then $K_{4m} - I = \langle S' \rangle_{4m}$ where $S' = \{1, 2, \ldots, 2m - 1\}$. The proof proceeds as follows. We begin by finding one difference $m$-tuple which corresponds to an $m$-cycle $C$ with $|\ell(C)| = m$. Note that $\langle \{4\} \rangle_{4m}$ consists of four vertex-disjoint $m$-cycles. For the remaining $m - 2$ edge lengths in $S' \setminus (\ell(C) \cup \{4\})$, we find $(m - 2)/2$ paths $P_i$ of length 2, ending at vertex 8 or $-8$, so that $C_i = P_i \cup \rho^8(P_i) \cup \rho^{16}(P_i) \cup \cdots \cup \rho^{4m-8}(P_i)$ is an $m$-cycle. Then this collection of cycles will give a minimum generating set for a cyclic $m$-cycle system of $K_{4m} - I$.

Consider the difference $m$-tuple

$$(1, -2, 6, -10, \ldots, 2m - 6, -(2m - 2), -3, 8, -12, \ldots, 2m - 12, -(2m - 8), 2m - 4)$$

and the corresponding $m$-cycle $C$ with $\ell(C) = \{1, 2, 3, 6, 8, \ldots, 2m - 2\}$. It is straightforward to verify that the odd vertices visited all lie between $-m + 1$ and $m - 1$ with no duplication. Similarly, the even vertices visited all lie between $-2m + 4$ and $-4$, and have no duplication.

Let $C' = (0, 4, 8, \ldots, 4m - 4)$ and note that $C'$ is an $m$-cycle with $\ell(C') = \{4\}$.

Let $m = 8k + m'$, so $m'$ is either 2 or 6. If $k = 0$, then $m' = 6$ and let $P : 0, 13, 8$ be the path of length 2 with edge lengths 11, 5. Then, $C'' = P \cup \rho^8(P) \cup \rho^{16}(P)$ is a 6-cycle with $\ell(C'') = \{1, 5\}$. Then $\{C, C', C''\}$ is a minimum generating set for cyclic 6-cycle system of $K_{24} - I$. Now suppose that $k \geq 1$. For $0 \leq i \leq k - 1$, let $P_i : 0, 13 + 16i, 8$ be the path of length 2 with edge lengths $13 + 16i, 5 + 16i$; let $P'_i : 0, 15 + 16i, 8$ be the path of length 2 with edge lengths $15 + 16i, 7 + 16i$; and let $P''_i : 0, 17 + 16i, 8$ be the path of length 2 with edge lengths $17 + 16i, 9 + 16i$; and let $P'''_i : 0, 19 + 16i, 8$ with edge lengths $19 + 16i, 11 + 16i$. Let $C_i = P_i \cup \rho^8(P_i) \cup \rho^{16}(P_i) \cup \cdots \cup \rho^{4m-8}(P_i)$, $C'_i = P'_i \cup \rho^8(P'_i) \cup \rho^{16}(P'_i) \cup \cdots \cup \rho^{4m-8}(P'_i)$, $C''_i = P''_i \cup \rho^8(P''_i) \cup \rho^{16}(P''_i) \cup \cdots \cup \rho^{4m-8}(P''_i)$, and $C'''_i = P'''_i \cup \rho^8(P'''_i) \cup \rho^{16}(P'''_i) \cup \cdots \cup \rho^{4m-8}(P'''_i)$ and note that each is an $m$-cycle with $\ell(C_i) = \{5 + 16i, 13 + 16i\}$, $\ell(C'_i) = \{7 + 16i, 15 + 16i\}$, $\ell(C''_i) = \{9 + 16i, 17 + 16i\}$, and $\ell(C'''_i) = \{11 + 16i, 19 + 16i\}$.

If $m' = 2$, then $\{C, C', C_0, C'_0, C''_0, \ldots, C_{k-1}, C'_{k-1}, C''_{k-1}\}$ is a minimum generating set for a cyclic $m$-cycle system of $K_{4m} - I$. If $m' = 6$, then let $P_k : 0, 2m - 1, -8$ and $P'_k : 0, 2m - 3, -8$ be paths of length 2 with $\ell(P_k) = \{2m - 1, 2m - 7\}$ and $\ell(P'_k) = \{2m - 3, 2m - 5\}$. Let $C_k = P_k \cup \rho^8(P_k) \cup \rho^{16}(P_k) \cup \cdots \cup \rho^{4m-8}(P_k)$ and $C'_k = P'_k \cup \rho^8(P'_k) \cup \rho^{16}(P'_k) \cup \cdots \cup \rho^{4m-8}(P'_k)$ and observe that each is an $m$-cycle with $\ell(C_k) = \{2m - 1, 2m - 7\}$ and $\ell(C'_k) = \{2m - 3, 2m - 5\}$. Thus, $\{C, C', C_0, C'_0, C''_0, \ldots, C_{k-1}, C'_{k-1}, C''_{k-1}, C_k, C'_k\}$ is a minimum generating set for a cyclic $m$-cycle system of $K_{4m} - I$. □

We now consider the case when $t \equiv 0 \pmod{4}$ with $t > 4$.

**Lemma 5.2** For each positive integer $k$ and each positive integer $m \equiv 2 \pmod{4}$ with $m \geq 6$, there exists a cyclic $m$-cycle system of $K_{4mk} - I$.

**Proof:** Let $m \geq 6$ and $k$ be positive integers such that $m \equiv 2 \pmod{4}$. Lemma 5.1 handles the case when $k = 1$ and thus we may assume that $k \geq 2$. Then
$K_{4km} - I = \langle S'\rangle_{4km}$ where $S' = \{1, 2, \ldots, 2km - 1\}$. Since $K_{4m} - I$ has a cyclic $m$-cycle system by Lemma 5.1 and $\langle \{k, 2k, \ldots, 2km\}\rangle_{4km}$ consists of $k$ vertex-disjoint copies of $K_{4m} - I$, we need only show that $\langle S\rangle_{2km}$ has a cyclic $m$-cycle system where $S = \{1, 2, \ldots, 2km\} \setminus \{k, 2k, \ldots, 2km\}$.

Let $A = [a_{i,j}]$ be the $2k \times m$ array

$$
\begin{bmatrix}
2k & 4k & 6k & 8k & (m-1)2k & 2km \\
2k-1 & 2k+1 & 6k-1 & 8k-1 & (m-1)2k-1 & 2km-1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
2 & 4k-2 & 4k+2 & 6k+2 & (m-2)2k+2 & (m-1)2k+2 \\
1 & 4k-1 & 4k+1 & 6k+1 & (m-2)2k+1 & (m-1)2k+1 \\
\end{bmatrix}.
$$

(Observe that the second column does not follow the same pattern as the others.)

Let $A'$ be the $(2k-2) \times m$ array obtained from $A$ by deleting rows 1 and $k+1$. Then the entries in $A'$ are precisely the elements of $S$. Also, it is straightforward to verify that $A'$ satisfies

$$a_{i,j} + a_{i,j+3} = a_{i,j+1} + a_{i,j+2}$$

for each positive integer $j \equiv 3 \pmod{4}$ with $j \leq m - 3$,

$$a_{i,1} + a_{i,2} + a_{i,m-3} + a_{i,m-1} = a_{i,m-2} + a_{i,m},$$

and

$$a_{i,1} < a_{i,2} < \ldots < a_{i,m}$$

for each $i$ with $1 \leq i \leq 2k - 2$.

For each $i = 1, 2, \ldots, 2k - 2$, the $m$-tuple

$$(a_{i,1}, a_{i,2}, -a_{i,4}, a_{i,6}, -a_{i,8}, a_{i,10}, \ldots, -a_{i,m-2}, -a_{i,m}, a_{i,m-3}, -a_{i,m-5}, a_{i,m-7}, \ldots, a_{i,3}, a_{i,m-1})$$

is a difference $m$-tuple and corresponds to an $m$-cycle $C_i$ with $\ell(C_i) = \{a_{i,1}, a_{i,2}, \ldots, a_{i,m}\}$. Hence, $X = \{C_1, C_2, \ldots, C_{2k-2}\}$ is a minimum generating set for a cyclic $m$-cycle system of $\langle S\rangle_{4km}$.

What remains is to find cyclic $m$-cycle systems of $K_{mt} - I$ for the appropriate odd values of $t$, which we do in the following subsections.

### 5.2 The case when $m \equiv 2 \pmod{8}$ and $t \equiv 1 \pmod{4}$.

In this subsection, we find a cyclic $m$-cycle system of $K_{mt} - I$ when $m \equiv 2 \pmod{8}$ and $t \equiv 1 \pmod{4}$. We begin with two special cases, namely when $m = 10$ or $t = 5$.

**Lemma 5.3** For each positive integer $t \equiv 1 \pmod{4}$ with $t > 1$, there exists a cyclic $10$-cycle system of $K_{10t} - I$. 
Proof: Let \( t \equiv 1 \pmod{4} \) with \( t > 1 \), say \( t = 4s + 1 \) where \( s \geq 1 \). Then \( K_{10t} - I = \langle S' \rangle_{10t} \) where \( S' = \{1, 2, \ldots, 20s + 4\} \). Consider the paths \( P_1 : 0, 5t - 1, 2t \) and \( P_2 : 0, 5t - 2, 2t \). Then, \( \ell(P_1) = \{3t - 1, 5t - 1\} \) and \( \ell(P_2) = \{3t - 2, 5t - 2\} \). For \( i \in \{1, 2\} \), let \( C_i = P_i \cup \rho^{2i}(P_i) \cup \rho^{4i}(P_i) \cup \cdots \cup \rho^{8i}(P_i) \). Then clearly each \( C_i \) is an \( 10 \)-cycle and \( X = \{C_1, C_2\} \) is a minimum generating set for \( \{3t - 2, 3t - 1, 5t - 2, 5t - 1\}\) of \( S \). Since \( 3t - 3 = 12s \) and \( 5t - 2 = 20s + 3 \), it remains to find a cyclic 10-cycle system of \( \langle S \rangle_{10t} \) where \( S = \{1, 2, \ldots, 12s, 12s + 3, 12s + 4, \ldots, 20s + 2\} \). Let \( A = [a_{i,j}] \) be the \( 2s \times 10 \) array

\[
\begin{bmatrix}
1 & 2 & 3 & 4 & 8s + 1 & 8s + 3 & 12s + 3 & 12s + 4 & 12s + 5 & 12s + 6 \\
5 & 6 & 7 & 8 & 8s + 2 & 8s + 4 & 12s + 7 & 12s + 8 & 12s + 9 & 12s + 10 \\
& & & & & & & & & \\
& & & & & & & & & \\
8s - 3 & 8s - 2 & 8s - 1 & 8s & 12s - 2 & 12s & 20s - 1 & 20s & 20s + 1 & 20s + 2
\end{bmatrix}
\]

Clearly, for each \( i \) with \( 1 \leq i \leq 2s \),

\[
a_{i,2} + \sum_{j \equiv 0,1 \pmod{4}} a_{i,j} = a_{i,1} + \sum_{j \equiv 2,3 \pmod{4}} a_{i,j} \quad \text{(where } 3 \leq j \leq 10)\]

and

\[
a_{i,1} < a_{i,2} < \ldots < a_{i,10}.
\]

Thus the 10-tuple

\[
(a_{i,1}, -a_{i,2}, a_{i,3}, -a_{i,4}, a_{i,5}, -a_{i,6}, a_{i,7}, -a_{i,8}, a_{i,9}, -a_{i,10})
\]

is a difference 10-tuple and corresponds to a 10-cycle \( C'_i \) with \( \ell(C'_i) = \{a_{i,1}, a_{i,2}, \ldots, a_{i,10}\} \). Hence, \( X' = \{C'_1, C'_2, \ldots, C'_{2s}\} \) is a minimum generating set for a cyclic 10-cycle system of \( \langle S \rangle_{10t} \).

We now consider the case when \( t = 5 \).

Lemma 5.4 For each positive integer \( m \equiv 2 \pmod{8} \), there exists a cyclic \( m \)-cycle system of \( K_{5m} - I \).

Proof: Let \( m \) be a positive integer such that \( m \equiv 2 \pmod{8} \), say \( m = 8r + 2 \) for some positive integer \( r \). By Lemma 5.3, we may assume \( r \geq 2 \). Then \( K_{5m} - I = \langle S' \rangle_{5m} \) where \( S' = \{1, 2, \ldots, 20r + 4\} \).

Let \( 2r = 6q + 4 + b \) for integers \( q \geq 0 \) and \( b \in \{0, 2, 4\} \). Let \( a \) be a positive integer such that \( 1 + \log_2(q + 2) \leq a \leq 1 + \log_2(5q + 2) \), and note that \( a \) exists since if \( q = 0 \) then \( \log_2(q + 2) \) is an integer, while if \( q \geq 1 \) then \( 2(q + 2) = 2q + 4 \leq 4q + 2 < 5q + 2 \).

For nonnegative integers \( i \) and \( j \), define \( d_{i,j} = 10(2r - i) + j \). Consider the path \( P_{i,j} : 0, d_{i,j}, 5 \cdot 2^a \) and observe that \( \ell(P_{i,j}) = \{10(2r - i) + j, 10(2r - i) + j - 5 \cdot 2^a\} \).

If \( 0 < j < 10 \), then \( C_{i,j} = P_{i,j} \cup \rho^{10}(P_{i,j}) \cup \rho^{20}(P_{i,j}) \cup \cdots \cup \rho^{5m-10}(P_{i,j}) \) is an \( m \)-cycle since \( m \equiv 2 \pmod{8} \) gives \( \gcd(5 \cdot 2^a, 5m) = 10 \). Thus, if \( 0 < j < 10 \), \( \ell(C_{i,j}) = \{10(2r - i) + j, 10(2r - i) + j - 5 \cdot 2^a\} \).

Let \( X = \{C_{0,j} \mid 1 \leq j \leq 4\} \cup \{C_{i,j} \mid 1 \leq i \leq q \text{ and } 1 \leq j \leq 6\} \cup \{C_{q+1,j} \mid 6 - b + 1 \leq j \leq 6\} \)
and let
\[
B = \{20r + j, 20r + j - 5 \cdot 2^a | 1 \leq j \leq 4\}
\]
\[
\cup \{10(2r - i) + j, 10(2r - i) + j - 5 \cdot 2^a | 1 \leq i \leq q \text{ and } 1 \leq j \leq 6\}
\]
\[
\cup \{10(2r - q - 1) + j, 10(2r - q - 1) + j - 5 \cdot 2^a | 6 - b + 1 \leq j \leq 6\},
\]
where if \(q = 0\) or \(b = 0\), we take the corresponding sets to be empty as necessary. Now \(B\) will consist of \(4r\) distinct lengths and \(X\) will be a minimum generating set for \(\langle B\rangle_{5m}\) if \(20r + 4 - 5 \cdot 2^a \leq 10(2r - q - 1) + 6 - b\). Note that \(1 + \log_2(q + 2) \leq a \leq 1 + \log_2(5q + 2)\) gives \(q + 2 \leq 2^{a-1} \leq 5q + 2\). So,
\[
20r + 4 - [10(2r - q - 1) + 6 - b] = 10q + 8 + b \leq 10q + 12
\]
and
\[
(10q + 12)/10 < q + 2 \leq 2^{a-1}.
\]
Thus \(20r + 4 - 5 \cdot 2^a \leq 10(2r - q - 1) + 6 - b\) so that \(B\) consists of \(4r\) distinct lengths, and \(X\) is a minimum generating set for \(\langle B\rangle_{5m}\).

It remains to find a cyclic \(m\)-cycle system of \(\langle S' \setminus B\rangle_{5m}\). The smallest length in \(B\) is \(10(2r - q - 1) + 6 - b + 1 - 5 \cdot 2^a\), and we wish to show \(10(2r - q - 1) + 6 - b - 5 \cdot 2^a \geq 12\). So,
\[
10(2r - q - 1) + 6 - b - 12 = 20r - 10q - 16 - b \geq 20r - 10q - 20
\]
and \((20r - 10q - 20)/10 \geq 2r - q - 2\). Now
\[
2r - q - 2 = 5q + 2 + b \geq 5q + 2 \geq 2^{a-1}.
\]
Hence, \(10(2r - q - 1) + 6 - b - 5 \cdot 2^a \geq 12\). Since \(|B| = 4r\), we have \(|S' \setminus B| = 20r + 4 - 4r = 2(8r + 2)\). Now
\[
S' \setminus B = \{1, 2, \ldots, 10(2r - q - 1) + 6 - b - 5 \cdot 2^a\}
\]
\[
\cup \{10(2r - i) - 5 \cdot 2^a - 3, 10(2r - i) - 5 \cdot 2^a - 2, 10(2r - i) - 5 \cdot 2^a - 1, 10(2r - i) - 5 \cdot 2^a | 0 \leq i \leq q\}
\]
\[
\cup \{10(2r) + 5 - 5 \cdot 2^a, \ldots, 10(2r - q - 1) + 6 - b\}
\]
\[
\cup \{10(2r - i) - 3, 10(2r - i) - 2, 10(2r - i) - 1, 10(2r - i) | 0 \leq i \leq q\}.
\]

Note that each the sets \(\{1, 2, \ldots, 10(2r - q - 1) + 6 - b - 5 \cdot 2^a\}, \{10(2r - i) - 5 \cdot 2^a - 3, 10(2r - i) - 5 \cdot 2^a - 2, 10(2r - i) - 5 \cdot 2^a - 1, 10(2r - i) - 5 \cdot 2^a | 0 \leq i \leq q\}, \{10(2r) + 5 - 5 \cdot 2^a, \ldots, 10(2r - q - 1) + 6 - b\}, and \{10(2r - i) - 3, 10(2r - i) - 2, 10(2r - i) - 1, 10(2r - i) | 0 \leq i \leq q\} \) has even cardinality and consists of consecutive integers. Therefore, we may partition \(S' \setminus B\) into sets \(T, S_1, S_2, \ldots, S_{sr-4}\) where \(T = \{1, 2, \ldots, 12\}\) and for \(i = 1, 2, \ldots, 8r - 4\), let \(S_i = \{b_1, b_i + 1\}\) with \(b_1 < b_2 < \cdots < b_{8r-4}\).

Let \(A = [a_{ij}]\) be the \(2 \times m\) array
\[
\begin{bmatrix}
1 & 2 & 3 & 4 & 9 & 11 & b_1 & b_1 + 1 & b_2 & b_2 + 1 & \cdots & b_{4r-2} & b_{4r-2} + 1 \\
5 & 6 & 7 & 8 & 10 & 12 & b_{4r-1} & b_{4r-1} + 1 & b_{4r} & b_{4r} + 1 & \cdots & b_{8r-4} & b_{8r-4} + 1
\end{bmatrix}
\]
Thus, let \( B \) be positive integers such that \( \sum_{1 \leq i \leq 2} b_i \leq 2 \).

\[
a_{i,2} + \sum_{j \equiv 0,1 \pmod{4}} a_{i,j} = a_{i,1} + \sum_{j \equiv 2,3 \pmod{4}} a_{i,j} \quad \text{(where } 3 \leq j \leq m)\]

and

\[
a_{i,1} < a_{i,2} < \ldots < a_{i,m}.\]

Hence, for \( 1 \leq i \leq 2 \), the \( m \)-tuple

\[
(a_{i,1}, -a_{i,2}, a_{i,3}, -a_{i,5}, a_{i,7}, \ldots, a_{i,m-3}, -a_{i,m-1}, -a_{i,m-2}, a_{i,m-4}, -a_{i,m-6}, \ldots, a_{i,6}, -a_{i,4}, a_{i,m})
\]

is a difference \( m \)-tuple and corresponds to an \( m \)-cycle \( C_i \) with \( t(C_i) = \{a_{i,1}, a_{i,2}, \ldots, a_{i,m}\} \). Hence, \( X' = \{C_1, C_2\} \) is a minimum generating set for a cyclic \( m \)-cycle system of \( \langle S' \setminus B \rangle_{5m} \).

We are now ready to prove the main result of this subsection, namely, that \( K_{mt} - I \) has a cyclic \( m \)-cycle system for every \( t \equiv 1 \pmod{4} \) and \( m \equiv 2 \pmod{8} \) with \( t > 1 \) if \( m = 2p^\alpha \) for some prime \( p \) and integer \( \alpha \geq 1 \).

**Lemma 5.5** For each positive integer \( t \equiv 1 \pmod{4} \) and each \( m \equiv 2 \pmod{8} \) with \( t > 1 \) if \( m = 2p^\alpha \) for some prime \( p \) and integer \( \alpha \geq 1 \), there exists a cyclic \( m \)-cycle system of \( K_{mt} - I \).

**Proof:** Let \( m \) and \( t \) be positive integers such that \( m \equiv 2 \pmod{8} \) and \( t \equiv 1 \pmod{4} \). Thus \( m = 8r + 2 \) for some positive integer \( r \). Then \( K_{mt} - I = \langle S' \rangle_{mt} \) where \( S' = \{1, 2, \ldots, (mt - 2)/2\} \). Since \( K_m - I \) has a cyclic hamiltonian cycle system [12] if and only if \( m \neq 2p^\alpha \) for some prime \( p \) and integer \( \alpha \geq 1 \), we may assume that \( t > 1 \). Thus, let \( t = 4s + 1 \) for some positive integer \( s \). By Lemmas 5.3 and 5.4, we may assume that \( s \geq 2 \) and \( r \geq 2 \).

The proof proceeds as follows. We begin by finding a set \( B \subseteq S' \) such that \( |B| = 4r \) and \( \langle B \rangle_{mt} \) has a cyclic \( m \)-cycle system with a minimum generating set \( X \) consisting of cycles each with two distinct lengths and orbit \( 2t \). We then construct an \((|S' \setminus B|/m) \times m\) array \( A = [a_{i,j}] \) with the property that for each \( i \) with \( 1 \leq i \leq |S' \setminus B|/m \),

\[
a_{i,2} + \sum_{j \equiv 0,1 \pmod{4}} a_{i,j} = a_{i,1} + \sum_{j \equiv 2,3 \pmod{4}} a_{i,j} \quad \text{(where } 3 \leq j \leq m)\]

and

\[
a_{i,1} < a_{i,2} < \ldots < a_{i,m}.
\]

Thus for each \( i = 1, 2, \ldots, |S' \setminus B|/m \), the \( m \)-tuple

\[
(a_{i,1}, -a_{i,2}, a_{i,3}, -a_{i,5}, a_{i,7}, \ldots, a_{i,m-3}, -a_{i,m-1}, -a_{i,m-2}, a_{i,m-4}, -a_{i,m-6}, \ldots, a_{i,6}, -a_{i,4}, a_{i,m})
\]
is a difference $m$-tuple and corresponds to an $m$-cycle $C_i$ with $\ell(C_i) = \{a_i, a_i + 1, \ldots, a_i + m\}$. Hence, $X' = \{C_1, C_2, \ldots, C_{|S' \setminus B|/m}\}$ will be a minimum generating set for a cyclic $m$-cycle system of $(S' \setminus B)_{mt}$.

Let $w = \lfloor r/2 \rfloor$, and let $\delta_0 = 2(r/2 - w)$, so that $\delta_r = 1$ if $r$ is odd and $\delta_r = 0$ if $r$ is even. Write $w = qs + b$ where $q$ and $b$ are non-negative integers with $0 \leq b < s$ (note that it may be the case that $q = 0$). For integers $i$ and $j$, define $d_{i,j} = 4(r-2i)t+j$. Consider the path $P_{i,j} : 0, d_{i,j}, 4t$ and observe that $\ell(P_{i,j}) = \{4(r-2i)t+j, 4(r-2i-1)t+j\}$.

If $0 < j < t$, then $C_{i,j} = P_{i,j} \cup \rho^2(P_{i,j}) \cup \rho^4(P_{i,j}) \cup \cdots \cup \rho^{(m-2)t}(P_{i,j})$ is an $m$-cycle since $m \equiv 2 \pmod{8}$ gives $\gcd(4t, mt) = 2t$. Thus, if $0 < j < t$, $\ell(C_{i,j}) = \{4(r-2i)t+j, 4(r-2i-1)t+j\}$. Let

$$X = \{C_{i,j} \mid 0 \leq i < q-1 \text{ and } 1 \leq j < t-1\} \cup \{C_{q,j} \mid t-4b - 2\delta_r \leq j \leq t-1\}$$

and let

$$B = \{4(r-2i)t+j, 4(r-2i-1)t+j \mid 0 \leq i < q-1 \text{ and } 1 \leq j < t-1\} \cup \{4(r-2q)t+j, 4(r-2q-1)t+j \mid t-4b - 2\delta_r \leq j \leq t-1\},$$

where we take the appropriate sets to be empty if $q = 0$ or $b = 0$. Observe that $X$ is a minimum generating set for $(B)_{mt}$, and consider the set $S' \setminus B$. Now $|X| = 4qs + 4b$ so that $|B| = 2(4qs + 4b) = 4r$. Hence $|S' \setminus B| = (4r+1)t - 1 - 4r = 2s(8r+2)$ and

$$S' \setminus B = \{1, 2, \ldots, 4(r-2q-1)t + t - 1 - 2\delta_r - 4b\} \cup \{4(r-2q-1)t + t, 4(r-2q-1)t + t + 1, \ldots, 4(r-2q)t + t - 1 - 2\delta_r - 4b\} \cup \{4kt + t, 4kt + t + 1, \ldots, 4k + 1)t \mid r-2q \leq k \leq r-1\}.$$

Note that $S' \setminus B$ has been written as the disjoint union of sets, each of which has even cardinality and consists of consecutive integers.

The smallest length in $B$ is $4(r-2q-1)t + t - 4b - 2\delta_r$, and we wish to show this length is at least $12s + 1$. Now $r \geq 2w = 2(qs + b) > 2q + 1$ since $s \geq 2$. Next since $0 \leq b < s$ and $t = 4s + 1$, we have $t - 1 - 4b = 4s - 4b \geq 4$. Therefore, $4(r-2q-1)t \geq 4t > 16s$, and thus $4(r-2q-1)t + t - 3 - 4b > 16s + 2 > 12s$. Since the smallest length is $S' \setminus B$ is at least $12s + 1$ and since $S' \setminus B$ consists of sets of consecutive integers of even cardinality, we may partition $S' \setminus B$ into sets $T, S_1, \ldots, S_{8rs-4s}$ where $T = \{1, 2, \ldots, 12s\}$, and for $i = 1, 2, \ldots, 8rs - 4s$, $S_i = \{b_i, b_i + 1\}$ with $b_1 < b_2 < \cdots < b_{8rs-4s}$. Let $A = [a_{i,j}]$ be the $2s \times m$ array

\[
\begin{bmatrix}
1 & 2 & 3 & 4 & 8s+1 & 8s+3 & b_1 & b_1+1 \\
5 & 6 & 7 & 8 & 8s+2 & 8s+4 & b_{4r-1} & b_{4r-1}+1 \\
8s-3 & 8s-2 & 8s-1 & 8s & 12s-2 & 12s & b_{8rs-4s-4r+3} & b_{8rs-4s-4r+3}+1 \\
& b_2 & b_2+1 & \cdots & b_{4r-2} & b_{4r-2}+1 \\
& b_{4r} & b_{4r}+1 & \cdots & b_{8r-4} & b_{8r-4}+1 \\
& \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
& b_{8rs-4s-4r+4} & b_{8rs-4s-4r+4}+1 & \cdots & b_{8rs-4s} & b_{8rs-4s}+1
\end{bmatrix}.
\]
Clearly, for each $i$ with $1 \leq i \leq 2s$,

$$a_{i,2} + \sum_{j \equiv 0,1 \text{ (mod 4)}} a_{i,j} = a_{i,1} + \sum_{j \equiv 2,3 \text{ (mod 4)}} a_{i,j} \quad \text{(where 3 \leq j \leq m)}$$

and

$$a_{i,1} < a_{i,2} < \ldots < a_{i,m}.$$

Thus the $m$-tuple

$$(a_{i,1}, a_{i,2}, a_{i,3}, a_{i,7}, \ldots, a_{i,m-3}, a_{i,m-1}, a_{i,m-2}, a_{i,m-4}, a_{i,m-6}, \ldots)$$

is a difference $m$-tuple and corresponds to an $m$-cycle $C_i$ with $\ell(C_i) = \{a_{i,1}, a_{i,2}, \ldots, a_{i,m}\}$. Hence, $X' = \{C_1, C_2, \ldots, C_{2s}\}$ is a minimum generating set for a cyclic $m$-cycle system of $\langle S' \setminus B \rangle_{mt}$. $\square$

### 5.3 The Case when $m \equiv 6 \text{ (mod 8)}$ and $t \equiv 3 \text{ (mod 4)}$

In this subsection, we find a cyclic $m$-cycle system of $K_{mt} - I$ when $m \equiv 6 \text{ (mod 8)}$ and $t \equiv 3 \text{ (mod 4)}$. We begin with three special cases, namely when $m = 6$, $m = 14$, or $t = 3$. We first consider the case $m = 6$.

**Lemma 5.6** For all positive integers $t \equiv 3 \text{ (mod 4)}$, there exists a cyclic 6-cycle system of $K_{6t} - I$.

**Proof:** Let $t$ be a positive integer such that $t \equiv 3 \text{ (mod 4)}$, say $t = 4s + 3$ for some non-negative integer $s$. Then $K_{6t} - I = \langle S' \rangle_{6t}$ where $S' = \{1, 2, \ldots, 12s + 8\}$.

Consider the paths $P_i: 0, 3t - i, 2t$, for $1 \leq i \leq 4$; then $\ell(P_i) = \{3t - i, t - i\}$. Next, let $C_i = P_i \cup P_i'^t \cup P_i'^{2t}$. Then each $C_i$ is a 6-cycle and $X = \{C_1, C_2, C_3, C_4\}$ is a minimum generating set for $\langle B \rangle_{6t}$ where $B = \{3t - i, t - i \mid 1 \leq i \leq 4\}$. Now, $t - 5 = 4s - 2$ and thus $S' \setminus B = \{1, 2, \ldots, 4s - 2, 4s + 3, 4s + 4, \ldots, 12s + 4\}$, and so we must find a cyclic 6-cycle system of $\langle S' \setminus B \rangle_{6t}$. Let $A = [a_{i,j}]$ be the $2s \times 6$ array

$$
\begin{bmatrix}
1 & 2 & 3 & 4 & 8s + 5 & 8s + 7 \\
5 & 6 & 7 & 8 & 8s + 6 & 8s + 8 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
4s - 3 & 4s - 2 & 4s + 3 & 4s + 4 & \alpha & \alpha + 2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
8s + 1 & 8s + 2 & 8s + 3 & 8s + 4 & 12s + 2 & 12s + 4 \\
\end{bmatrix}
$$

where

$$\alpha = \begin{cases} 
10s + 2 & \text{if } s \text{ is even}, \\
10s + 3 & \text{if } s \text{ is odd}.
\end{cases}$$
Clearly, for each \( i \) with \( 1 \leq i \leq 2s \),
\[
a_{i,2} + \sum_{j \equiv 0,1 \pmod{4}} a_{i,j} = a_{i,1} + \sum_{j \equiv 2,3 \pmod{4}} a_{i,j} \quad \text{(where } 3 \leq j \leq 6)\]
and
\[
a_{i,1} < a_{i,2} < \ldots < a_{i,6}.
\]
Thus the 6-tuple
\[
(a_{i,1}, -a_{i,2}, a_{i,3}, -a_{i,4}, -a_{i,5}, a_{i,6})
\]
is a difference 6-tuple and corresponds to a 6-cycle \( C'_i \) with \( \ell(C'_i) = \{a_{i,1}, a_{i,2}, \ldots, a_{i,6}\} \).
Hence, \( X' = \{C'_1, C'_2, \ldots, C'_{2s}\} \) is a minimum generating set for a cyclic 6-cycle system of \( \langle S' \setminus B \rangle_{14t} \).
\[
\]
Next we consider the case when \( m = 14 \).

**Lemma 5.7** For all positive integers \( t \equiv 3 \pmod{4} \), there exists a cyclic 14-cycle system of \( K_{14t} - I \).

*Proof:* Let \( t \) be a positive integer such that \( t \equiv 3 \pmod{4}, \) say \( t = 4s + 3 \) for some non-negative integer \( s \). Then \( K_{14t} - I = \langle S' \rangle_{14t} \) where \( S' = \{1, 2, \ldots, 28s + 20\} \).
Consider the paths \( P_i : 0, 7t - i, 2t, \) for \( 1 \leq i \leq 10 \); then \( \ell(P_i) = \{7t - i, 5t - i\} \).
Next, let \( C_i = P_i \cup \rho^{2t}(P_i) \cup \rho^{4t}(P_i) \cup \cdots \cup \rho^{12t}(P_i) \). Then each \( C_i \) is a 14-cycle and \( X = \{C_1, C_2, \ldots, C_{10}\} \) is a minimum generating set for \( \langle B \rangle_{14t} \) where \( B = \{7t - i, 5t - i \mid 1 \leq i \leq 10\} \).
Now, \( 5t - 10 = 20s + 5 \) and thus \( S' \setminus B = \{1, 2, \ldots, 20s + 4, 20s + 15, 20s + 16, \ldots, 28s + 10\} \), and so we must find a cyclic 14-cycle system of \( \langle S' \setminus B \rangle_{14t} \).
Let \( A = [a_{i,j}] \) be the \( 2s \times 14 \) array
\[
\begin{bmatrix}
1 & 2 & 3 & 4 & 8s + 1 & 8s + 3 & 12s + 1 & 12s + 2 & 12s + 3 & 12s + 4 \\
5 & 6 & 7 & 8 & 8s + 2 & 8s + 4 & 12s + 5 & 12s + 6 & 12s + 7 & 12s + 8 \\
9 & 10 & 11 & 12 & 8s + 5 & 8s + 7 & 12s + 9 & 12s + 10 & 12s + 11 & 12s + 12 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
8s - 3 & 8s - 2 & 8s - 1 & 8s & 12s - 2 & 12s & 20s - 3 & 20s - 2 & 20s - 1 & 20s \\
20s + 1 & 20s + 2 & 20s + 3 & 20s + 4 \\
20s + 15 & 20s + 16 & 20s + 17 & 20s + 18 \\
20s + 19 & 20s + 20 & 20s + 21 & 20s + 22 \\
\vdots & \vdots & \vdots & \vdots \\
28s + 7 & 28s + 8 & 28s + 9 & 28s + 10
\end{bmatrix}
\]
Clearly, for each \( i \) with \( 1 \leq i \leq 2s \),
\[
a_{i,2} + \sum_{j \equiv 1 \pmod{4}} a_{i,j} = a_{i,1} + \sum_{j \equiv 2,3 \pmod{4}} a_{i,j} \quad \text{(where } 3 \leq j \leq 14)\]
and
\[
a_{i,1} < a_{i,2} < \ldots < a_{i,14}.
\]
Thus the 14-tuple
\[(a_{1,1}, a_{1,2}, a_{1,3}, a_{1,5}, a_{1,7}, a_{1,9}, a_{1,11}, a_{1,13}, -a_{1,12}, a_{1,10}, -a_{1,8}, a_{1,6}, -a_{1,4}, a_{1,14})\]
is a difference 14-tuple and corresponds to a 14-cycle \(C'_r\) with \(\ell(C'_r) = \{a_{1,1}, a_{1,2}, \ldots, a_{1,14}\}\). Hence, \(X' = \{C'_1, C'_2, \ldots, C'_{2n}\}\) is a minimum generating set for a cyclic 14-cycle system of \(\langle S' \setminus B \rangle_{14t}\).

We now consider the case when \(t = 3\).

**Lemma 5.8** For all positive integers \(m \equiv 6 \pmod{8}\), there exists a cyclic \(m\)-cycle system of \(K_{3m} - I\).

**Proof:** Let \(m\) be a positive integer such that \(m \equiv 6 \pmod{8}\), say \(m = 8r + 6\) for some non-negative integer \(r\). By Lemmas 5.6 and 5.7, we may assume \(r \geq 2\). Then \(K_{3m} - I = \langle S' \rangle_{m} \) where \(S' = \{1, 2, \ldots, 12r + 8\}\). Write \(2r = 4q + b + 2\) for integers \(q \geq 0\) and \(b \in \{0, 2\}\), and let \(a\) be a positive integer such that \(1 + \log_2(q + 1) \leq a \leq 1 + \log_2(3q + 4/3 + 5b/6)\). For integers \(i\) and \(j\), define \(d_{i,j} = 6(2r - i) + j\). Then consider the path \(P_{i,j} : 0, d_{i,j}, 3 \cdot 2^a; \) so \(\ell(P_{i,j}) = \{6(2r - i) + j, 6(2r - i) + j - 3 \cdot 2^a\}\).

Now, let \(C_{i,j} = P_{i,j} \cup \rho^b(P_{i,j}) \cup \cdots \cup \rho^{3(m-2)}(P_{i,j})\). Then \(C_{i,j}\) is an \(m\)-cycle since \(m \equiv 6 \pmod{8}\) implies \(\gcd(3 \cdot 2^a, 3m) = 6\). Thus, \(\ell(C_{i,j}) = \ell(P_{i,j})\).

Now, let
\[
X = \{C_{0,j} \mid j = 7, 8\}
\cup \{C_{i,j} \mid 0 \leq i \leq q - 1 \text{ and } 1 \leq j \leq 4\}
\cup \{C_{q,j} \mid 5 - b \leq j \leq 4\}
\]
and let
\[
B = \{12r + 7, 12r + 7 - 3 \cdot 2^a, 12r + 8, 12r + 8 - 3 \cdot 2^a\}
\cup \{6(2r - i) + j, 6(2r - i) - 3 \cdot 2^a + j \mid 0 \leq i \leq q - 1 \text{ and } 1 \leq j \leq 4\}
\cup \{6(2r - q) + j, 6(2r - q) - 3 \cdot 2^a + j \mid 5 - b \leq j \leq 4\}
\]
where, if \(q = 0\) or \(b = 0\), we take the corresponding sets to be empty as necessary.

Now \(B\) will consists of \(4r\) distinct lengths and \(X\) will be a minimum generating set for \(\langle B \rangle_{3m}\) if \(12r + 8 - 3 \cdot 2^a \leq 6(2r - q) + 5 - b - 1\). Note that \(1 + \log_2(q + 1) \leq a\) so that \(q + 1 \leq 2^{a-1}\). Next,
\[
12r + 8 - [6(2r - q) + 5 - b - 1] = 6q + 4 + b \leq 6q + 6 = 6(q + 1) \leq 6 \cdot 2^{a-1} = 3 \cdot 2^a,
\]
and hence \(12r + 8 - 3 \cdot 2^a \leq 6(2r - q) + 5 - b - 1\). Thus, \(B\) consists of \(4r\) distinct lengths, and \(X\) is a minimum generating set for \(\langle B \rangle_{3m}\). Now, the smallest length in \(B\) is \(6(2r - q) + 5 - b - 3 \cdot 2^a\) and we want this length to be greater than 8. Recall that \(a \leq 1 + \log_2(3q + 3/2 + 5b/6)\) and thus \(2^{a-1} \leq 3q + 3/2 + 5b/6\). Hence, \(3 \cdot 2^a \leq
18q + 9 + 5b = 12r - 6q - 3 - b since 2r = 4q + b + 2. Therefore, 6(2r - q) + 5 - b - 3 \cdot 2^a \geq 8. Since |B| = 4r, we have |S' \setminus B| = 8r + 8. Note that  
\[
S' \setminus B = \{1, 2, \ldots, 6(2r - q) + 5 - b - 3 \cdot 2^a - 1\} \\
\cup \{6(2r - i) - 3 \cdot 2^a + 5, 6(2r - i) - 3 \cdot 2^a + 6 \mid 0 \leq i \leq q\} \\
\cup \{12r - 3 \cdot 2^a + 9, \ldots, 6(2r - q) + 5 - b - 1\} \\
\cup \{6(2r - i) + 5, 6(2r - i) + 6 \mid 0 \leq i \leq q\}.
\]

Note that $S' \setminus B$ has been written as the disjoint union of sets, each of which has even cardinality and consists of consecutive integers. Therefore, we may partition $S' \setminus B$ into sets $T, S_1, S_2, \ldots, S_{4r}$ where $T = \{1, 2, \ldots, 8\}$ and for $i = 1, 2, \ldots, 4r$, let $S_i = \{b_i, b_i + 1\}$ with $b_1 < b_2 < \cdots < b_{4r}$. Consider the $m$-tuple  
\[(1, -3, 6, -7, b_1, -b_2, b_3, -b_4, \ldots, b_{4r-1}, -b_{4r}, -(b_{4r-1} + 1), b_{4r-2} + 1, \ldots, (b_{4r-3} + 1), b_{4r-4} + 1, \ldots, b_2 + 1, -(b_1 + 1), 8, -5, b_{4r} + 1)\]

which is a difference $m$-tuple and corresponds to an $m$-cycle $C_1$ with  
\[
\ell(C_1) = \{1, 3, 5, 6, 7, 8, b_1, b_1 + 1, b_2, b_2 + 1, \ldots, b_{4r}, b_{4r} + 1\}.
\]

Then consider the path $P : 0, 2, 6$; so $\ell(P) = \{2, 4\}$. Now, let $C_2 = P \cup \rho^5(P) \cup \cdots \cup \rho^{3(m-2)}(P)$. Then $C_2$ is an $m$-cycle since $m \equiv 6 \pmod{8}$ implies $\gcd(6, 3m) = 6$. Thus, $\ell(C_2) = \ell(P) = \{2, 4\}$. Hence, $X' = \{C_1, C_2\}$ is a minimum generating set for a cyclic $m$-cycle system of $\langle S' \setminus B \rangle_{3m}$. \hfill \Box

We now prove the main result of this subsection, namely that $K_{mt} - I$ has a cyclic $m$-cycle system for every $t \equiv 3 \pmod{4}$ and $m \equiv 6 \pmod{8}$.

**Lemma 5.9** For all positive integers $t \equiv 3 \pmod{4}$ and $m \equiv 6 \pmod{8}$, there exists a cyclic $m$-cycle system of $K_{mt} - I$.

**Proof:** Let $m$ and $t$ be positive integers such that $m \equiv 6 \pmod{8}$ and $t \equiv 3 \pmod{4}$. Then $m = 8r + 6$ and $t = 4s + 3$ for some non-negative integers $r$ and $s$. Then $K_{mt} - I = \langle S' \rangle_{mt}$ where $S' = \{1, 2, \ldots, (4r + 3)t - 1\}$.

By Lemmas 5.6, 5.7, and 5.8, we may assume $s \geq 1$ and $r \geq 2$. First, write $6r + 4 = (2t - 2)q + (t - 1)\ell + b$ for integers $q, \ell$ and $b$ with $q \geq 0, 0 \leq b < 2t = 2, \ell = 0$ if $6r + 4 < t - 1$, or $\ell = 1$ otherwise. For integers $i$ and $j$, define $d_{i,j} = 2t(2r - 2i - 1) + j$. Consider the path $P_{i,j} : 0, d_{i,j}, 2t$ and note that $\ell(P_{i,j}) = \{2t(2r - 2i - 1) + j, 2t(2r - 2i - 2) + j\}$. If $0 < j < 2t$, then $C_{i,j} = P_{i,j} \cup \rho^2(P_{i,j}) \cup \rho^{4t}(P_{i,j}) \cup \cdots \cup \rho^{(m-2)\ell}(P_{i,j})$ is an $m$-cycle since $m \equiv 6 \pmod{8}$ implies $\gcd(2t, mt) = 2t$. Thus, if $0 < j < 2t$, then $\ell(C_{i,j}) = \ell(P_{i,j})$.

Now, let  
\[
X = \{C_{-1,j} \mid 1 \leq j \leq t - 1\} \\
\cup \{C_{i,j} \mid 0 \leq i \leq q - 1 \text{ and } 1 \leq j \leq 2t - 2\} \\
\cup \{C_{q,j} \mid 2t - 1 - b \leq j \leq 2t - 2\}
\]
and let

\[ B = \{2t(2r + 1) + j, 2t(2r) + j \mid 1 \leq j \leq t - 1\} \]
\[ \cup \{2t(2r - 2i - 1) + j, 2t(2r - 2i - 2) + j \mid 1 \leq j \leq 2t - 2\text{ and } 0 \leq i \leq q - 1\} \]
\[ \cup \{2t(2r - (2q + 1)) + 2t - 1 - b + j, 2t(2r - 2q - 2) + 2t - 1 - b + j \mid 0 \leq j \leq b - 1\}. \]

where we take the first set to be empty if \( \ell = 0 \), the second to be empty if \( q = 0 \), and the third to be empty if \( b = 0 \). Then \( X \) is a minimum generating set for \( \langle B \rangle_{mt} \).

Now we must find a cyclic \( m \)-cycle system of \( \langle S' \setminus B \rangle_{mt} \). First, \( |B| = 2[(2t - 2)q + (t - 1)\ell + b] = 12r + 8 \) so that \( |S' \setminus B| = (4r + 3)t - 1 - 12r - 8 = (8r + 6)(2s) \).

Moreover,

\[ S' \setminus B = \{1, 2, \ldots, 2t(2r - 2q - 1) - b - 2\} \]
\[ \cup \{2t(2r - 2q - 1) - 1, 2t(2r - 2q - 1), \ldots, 2t(2r - 2q) - b - 2\} \]
\[ \cup \{2t(2r - i) - 1, 2t(2r - i) \mid 0 \leq i \leq 2q\} \]
\[ \cup \{4rt + t, 4rt + t + 1, \ldots, 4rt + 2t\}. \]

The smallest length in \( B \) is \( 4t(r - q - 1) + (2t - 1) - b \), and we must verify that this length is at least \( 12s + 1 \). Note that we have \( 2t - 1 - b > 1 \). Thus, it is sufficient to prove that \( 4t(r - q - 1) \geq 12s \), or \( t(r - q - 1) \geq 3s \). This inequality follows if \( r > q + 1 \). Clearly, this is true if \( q = 0 \) since \( r \geq 2 \), so assume \( q \geq 1 \). Then \( \ell = 1 \), and so \( 6r + 4 = 2q(4s + 2) + (4s + 2) + b \), or

\[ 3r + 2 = q(4s + 2) + 2s + 1 + b/2 \]
\[ = 4qs + 2q + 2s + 1 + b/2 \]
\[ \geq 6q + 3 \text{ (since } s \geq 1\text{).} \]

So, \( r \geq 2q + 1/3 > q + 1 \) since \( q \geq 1 \). Since the smallest length in \( B \) is at least \( 12s + 1 \) and \( S' \setminus B \) consists of sets of consecutive integers of even cardinality, we may partition \( S' \setminus B \) into sets \( T, S_1, \ldots, S_{8rs} \) where \( T = \{1, 2, \ldots, 12s\} \), and for \( i = 1, 2, \ldots, 8rs, S_i = \{b_i, b_i + 1\} \) with \( b_1 \leq b_2 \leq \cdots \leq b_{8rs} \). Let \( A = [a_{i,j}] \) be the \( 2s \times m \) array

\[
\begin{bmatrix}
1 & 2 & 3 & 4 & 8s + 1 & 8s + 3 & b_1 & b_1 + 1 \\
5 & 6 & 7 & 8 & 8s + 2 & 8s + 4 & b_{4r+1} & b_{4r+1} + 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
8s - 3 & 8s - 2 & 8s - 1 & 8s & 12s - 2 & 12s & b_{8rs-4r+1} & b_{8rs-4r+1} + 1
\end{bmatrix}
\]

with

\[
\begin{bmatrix}
b_2 & b_2 + 1 & \cdots & b_{4r} & b_{4r} + 1 \\
b_{4r+2} & b_{4r+2} + 1 & \cdots & b_{8r} & b_{8r} + 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
b_{8rs-4r+2} & b_{8rs-4r+2} + 1 & \cdots & b_{8rs} & b_{8rs} + 1
\end{bmatrix}.
\]
Clearly, for each $i$ with $1 \leq i \leq 2s$,
\[
a_{i,2} + \sum_{j \equiv 0,1 \pmod{4}} a_{i,j} = a_{i,1} + \sum_{j \equiv 2,3 \pmod{4}} a_{i,j} \quad \text{(where } 3 \leq j \leq m)\]
and
\[
a_{i,1} < a_{i,2} < \ldots < a_{i,m}.
\]
Thus the $m$-tuple
\[
(a_{i,1}, -a_{i,2}, a_{i,3}, -a_{i,5}, a_{i,7}, \ldots, a_{i,m-3}, -a_{i,m-1}, -a_{i,m-2}, a_{i,m-4}, -a_{i,m-6}, \ldots, a_{i,6}, -a_{i,4}, a_{i,m})
\]
is a difference $m$-tuple and corresponds to an $m$-cycle $C_i$ with $\ell(C_i) = \{a_{i,1}, a_{i,2}, \ldots, a_{i,m}\}$. Hence, $X' = \{C_1, C_2, \ldots, C_{2s}\}$ is a minimum generating set for a cyclic $m$-cycle system of $\langle S' \setminus B \rangle_{mt}$.

References


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