

PARAMETERIZED QUERY COMPLEXITY IN QUANTUM COMPUTATION

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Dedication

To my parents and sister

Abstract

Given a function promised to be constant or balanced. Deutsch's algorithm and its extension Deutsch-Jozsa are the algorithms that can determine the property of the function in constant number of queries. The algorithm works only on the functions that are promised to be either constant or balanced. There exist functions that are neither constant nor balanced. Our proposal is to analyze the query complexity of two such functions as a function of some parameter. We apply the methodology to two different problems. We parameterize the degree of imbalance for an arbitrarily chosen function. The same parameterization is used for the functions that are not self-dual. We give global and local adiabatic algorithms for both the problems. Our adiabatic algorithms have smaller query complexity as compared to the deterministic algorithms.

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Chapter 1

Introduction

1.1 Introduction

In the classical setting, a bit has values either 0 or 1. In quantum computation, a qubit can be in a state of $|0\rangle$, $|1\rangle$ or any superposition of the two states. The superposition of the states enables a qubit to be in any state at the same time. This property distinguishes quantum computation from classical computation. When a classical computer can work with either 0 or 1, a quantum computer gets the privilege to be in a state of $|0\rangle$, $|1\rangle$ or in a superposition of $|0\rangle, |1\rangle$ at a time. For this reason, it would be possible for a quantum computer to solve problems faster and efficiently compared to classical computer. It might be possible for a quantum computer to solve the problems that are considered intractable to be solved classically.

Deutsch algorithm [7] and its extension Deutsch-Jozsa algorithm [8] are two quantum algorithms for solving a problem faster using quantum computation over classical computation. A function $f(x)$ is constant if the output is always same for any input x . A balanced function outputs 0 and 1 equal number of times. Given is a function,

$$f : \{0, 1\} \rightarrow \{0, 1\} \tag{1.1}$$

and promise that the function is either constant or balanced. Classically, two queries are required to determine the property of the function. It is necessary to check the output for the inputs 0 and 1. If the output is same no matter what is the input value then the function

is constant. Otherwise, the given function will be balanced. Deutsch gave an algorithm that will require one query. Now we consider a generalized function,

$$f : \{0, 1\}^n \rightarrow \{0, 1\} \tag{1.2}$$

To decide over n inputs whether the function is constant or balanced, the classical algorithm will require $2^{n-1} + 1$ queries. Deutsch-Jozsa quantum algorithm will take only one query and n measurements.

A function g is called self-dual if for any input x , $g(x) \neq g(\bar{x})$.

What would be the query complexity if a given function is neither constant nor balanced? We introduce a parameter for all boolean functions of n variables and observe the behavior of Deutsch-Jozsa algorithm as a function of the parameter. Our proposal is to analyze the query complexity of two problems with some parameters. The first problem is determining whether a given function is balanced or not. The second problem is to determine the self-duality of a function. We also give quantum adiabatic algorithms for the two problems with better performance/query complexity than the deterministic case.

1.2 Organization of the thesis

We discuss basic concepts and definitions regarding the quantum computation in Chapter 2. We will also describe how to analyze a quantum circuit. Gershgorin circle theorem, Deutsch and Deutsch-Jozsa quantum algorithms are also described in this Chapter.

In Chapter 3, we focus on the literature review. The motivation for our thesis is also discussed in this Chapter.

In Chapter 4, we present the idea of parameterized query complexity. We also give two theorems to establish the query complexity of the correctness of our given algorithms with a fixed probability. Algorithms are given to determine the degree of imbalance and self-duality of a function.

In Chapter 5, we describe the analysis of adiabatic computation for the ε -balanced function. It also includes some numerical results and a discussion of the results.

Chapter 6 describes the adiabatic computation for the δ -dual function.

Finally, we conclude with the future research directions and a summary of our work.

Chapter 2

Basic Concepts

2.1 Definitions

Dirac Notation

Any superposition can be represented in Dirac notation [9]. With the increase of qubits in a system, the dimension of matrices and vectors needed to represent the entire quantum system grow exponentially. A superposition of quantum state is written in matrix and Dirac notation below,

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} |0000\rangle + \frac{1}{\sqrt{2}} |1111\rangle = \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |15\rangle$$

In the above example the matrix representation uses a column vector with 14 zero and 2 nonzero entries. In Dirac notation, we represent only the quantum states with nonzero amplitudes.

Qubit

A qubit is in state $|0\rangle, |1\rangle$ or in a superposition of $|0\rangle, |1\rangle$. Classical bits have values 0 or 1. A qubit can be represented as,

$$\alpha|0\rangle + \beta|1\rangle$$

where α and β are the amplitudes of the states and α^2, β^2 are the probabilities of the qubit being in state $|0\rangle, |1\rangle$ respectively. Note that, $\alpha^2 + \beta^2 = 1$

Postulates of Quantum Computing

We will describe three important postulates of quantum computing. These postulates are state space postulate, evolution postulate and composite system postulate. Most of the description of these postulates is from [19].

1. **State Space Postulate:** The state space postulate defines the qubit and its representation. We can represent the state of a quantum system by a two dimensional vector as follows,

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \alpha|0\rangle + \beta|1\rangle$$

where

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

2. **Evolution Postulate:** We can describe the evolution of a new state in a system by unitary transformations. Let, x be a vector representing a state and U a unitary operator. State x evolves to state x' under the action of U as $x' = Ux$. U is a unitary operator if $U^\dagger U = I$ where U^\dagger is the hermitian conjugate transpose of the matrix U and I is the identity matrix. Every transformation in quantum computing is unitary to preserve the norm of operators.
3. **Composite System Postulate** According to the state space postulate we can represent any quantum system by a vector. If $|v_1\rangle = [a_1 \ a_2 \ \dots \ a_n]$ and $|v_2\rangle = [b_1 \ b_2 \ \dots \ b_n]$ are two such vectors representing the superposition of two individual subsystems with n qubits then the tensor product can be defined as,

$$|v_1\rangle \otimes |v_2\rangle = \begin{bmatrix} a_1 |v_2\rangle \\ a_2 |v_2\rangle \\ \cdot \\ \cdot \\ \cdot \\ a_n |v_2\rangle \end{bmatrix}$$

Composite system postulate states that we can write the state of composite system V as follows,

$$V = |v_1\rangle \otimes |v_2\rangle$$

provided that the subsystems are independent of each other.

Quantum Entanglement

There exist such quantum states that are dependent and the qubits of the quantum system can't be separated. Quantum entanglement occurs when a quantum state can't be written as a tensor product of other states. For example, consider the following superposition. This state can not be described as a tensor product.

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle$$

For two qubits there exist four entangled states known as *bell states* [5, 22].

Quantum Circuit

Quantum circuits are different from classical circuits. In a quantum circuit, the inputs are qubits. Horizontal single line indicates qubit and parallel lines represent classical bits

after measurement. A quantum circuit is always evaluated from left to right over time. Measurements and different unitary operators are represented by symbols. Next we describe how to read a quantum circuit. Figure 2.1 is a quantum circuit with an initial state

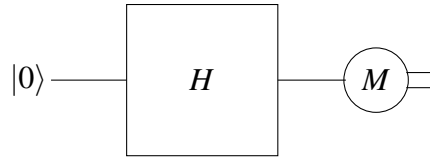
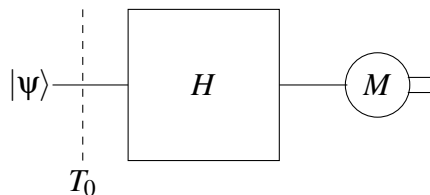


Figure 2.1: An example of Quantum Circuit

$|\psi\rangle$. H is a unitary transformation performed on the qubits. Finally, M represents a measurement procedure to obtain classical bits.

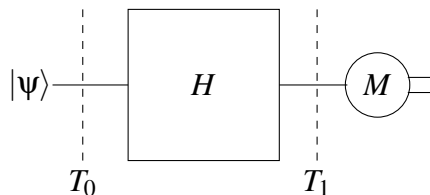
Step 1:

We have a quantum circuit with the initial state $|\psi\rangle$ that can be either $A = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ or $B = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$.



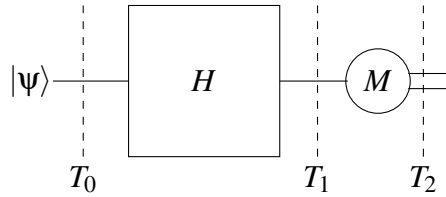
Step 2:

The unitary Hadamard transformation is defined by the matrix, $H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. After applying Hadamard transformation, the state will be $1|0\rangle$ for A or $1|1\rangle$ for B.



Step 3:

In the end, if we measure the qubit we will get 0 if our initial state $|\psi\rangle$ was A. For input B, the output will give 1 after measurement.



Constant and Balanced Function

There are four boolean functions over one variable shown in table 2.1. The output of f_1

Table 2.1: All Possible Functions

x	f_1	f_2	f_3	f_4
0	0	0	1	1
1	0	1	0	1

and f_4 are constant at 0 or 1 whereas the output of f_2 and f_3 depend on the input. f_1 and f_4 are constant. f_2 and f_3 are balanced. Now we give the definition of constant and balanced function.

Constant Function

A function $f : \{0, 1\}^n \rightarrow \{0, 1\}$ is constant if for all, $x \in \{0, 1\}^n$, $f(x) = 0$ or $f(x) = 1$. In other words, a constant function has the same output for all the inputs. The following example shows two constant functions over two variables.

Table 2.2: Constant Function

x	$f(x)$	x	$f(x)$
00	0	00	1
01	0	01	1
10	0	10	1
11	0	11	1

Balanced Function

A function $f : \{0, 1\}^n \rightarrow \{0, 1\}$ is balanced if for all inputs $x \in \{0, 1\}^n$, $f(x)$ outputs 0 and 1 equal number of times. The output of a balanced function depends on the input. An

example of balanced function is below.

Table 2.3: Balanced Function

x	$f(x)$
00	0
01	0
10	1
11	1

Cauchy-Schwarz Inequality

For any two vectors x and y in \mathbb{R}^n , Cauchy-Schwarz inequality [18] states,

$$|\langle x, y \rangle|^2 \leq \langle x, x \rangle \cdot \langle y, y \rangle \quad (2.1)$$

or as,

$$\langle x, y \rangle \leq \|x\|_2 \cdot \|y\|_2 \quad (2.2)$$

where, $\|x\|_2$ and $\|y\|_2$ are euclidian or 2-norm of the corresponding vector x and y .

Convex Combination

A matrix A is a linear convex combination [27, 17, 1] of matrices B and C if

$$A = (1 - \alpha)B + \alpha C$$

for $0 \leq \alpha \leq 1$.

Spectrum

For a given matrix $A \in \mathbb{R}^{n \times n}$, the multiset of all the eigenvalues of the matrix is called it's spectrum [33]. If A has the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, then the spectrum of $A = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$.

Euclidean Norm of a Vector

A normalized vector \hat{x} of vector x is a unit length vector in the same direction. Given a vector,

$$v = [a_1 \quad a_2 \quad \dots \quad a_n].$$

The Euclidean norm of the vector v is defined as,

$$\|v\|_2 = \sqrt{a_1^2 + a_2^2 + \dots + a_n^2}$$

The Euclidean norm of a normalized vector is 1.

Matrix Trace

Given a square matrix $A \in \mathbb{R}^{n \times n}$. The trace of the matrix is the sum of all the diagonal entries.

Matrix Rank and singularity

Rank of a matrix can be defined as the maximum number of linearly independent rows or columns. If we consider a matrix $M \in \mathbb{R}^{m \times n}$ having linearly independent a rows and b columns then,

$$\text{Rank}(M) = \max(a, b)$$

A square matrix of the dimension $n \times n$ will be nonsingular if the rank of the matrix is equal to n .

Unitary Matrix

A matrix M with complex entries is unitary if the following property holds.

$$MM^\dagger = I$$

where, M^\dagger is the hermitian conjugate of the matrix M and I is an identity matrix. The following example shows a unitary matrix.

$$H = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

Permutation Matrix

A square matrix M with 0/1 entries is called Permutation matrix if it has exactly one 1 in each row and column and 0's in other entries.

Hermitian Matrix

A matrix M is self-adjoint having the property,

$$M = M^*$$

where, M^* is the conjugate transpose of M . A Hermitian matrix is a square matrix that is self-adjoint [36]. A Hermitian matrix has real eigenvalues. If the eigenvalues of a Hermitian matrix are distinct then it is diagonalizable.

Symmetric Matrix

A square matrix is called symmetric if it is equal to its transpose [23]. A real symmetric matrix is always Hermitian.

Multiplicities

For the spectrum of any matrix, there are two multiplicities called Algebraic and Geometric multiplicities. We describe these two types of multiplicities along with examples.

Algebraic Multiplicity

Algebraic multiplicity of an eigenvalue in a matrix is the number of repetitions of the eigenvalue. Consider a matrix, A . If the matrix has an eigenvalue λ repeating t times then we call the algebraic multiplicity of λ is t . An example is illustrated below with a matrix,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Matrix A has the eigenvalue $\lambda = 1$ repeated two times. So, the algebraic multiplicity of $\lambda = 1$ is 2.

Geometric Multiplicity

For the matrix A , null space [25] can be defined as set of all vectors u such that, $Au = 0$. Geometric multiplicity for an eigenvalue λ of a matrix A is the dimension of the null space of $A - \lambda I$. The relation between algebraic and geometric multiplicity is,

$$\text{geometric_multiplicity}(\lambda) \leq \text{algebraic_multiplicity}(\lambda)$$

So, the geometric multiplicity of a matrix is bounded by algebraic multiplicity. Consider the following example,

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$$

For the eigenvalue, $\lambda = 2$ the dimension of the null space of $A - \lambda I$ is 1.

$$A - 2I = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$$

So, the geometric multiplicity of the above matrix is 1. The rank of the matrix is also 1. The rank and nullity of a matrix A obey the following [3].

$$\text{Rank}(A) + \text{Nullity}(A) = n$$

where, n is the number of columns of the matrix A .

Taylor Theorem

Given a function, $f(x) : \mathbb{R} \rightarrow \mathbb{R}$. At a specific point x_0 , the value of the function is $f(x_0)$. To evaluate the function at any point x , we can use a polynomial $F(x)$ that gives us an approximated value $F(x_0)$ which is close to the original value $f(x_0)$. Taylor theorem [12] gives us such polynomial for which we can approximate a function at a point without

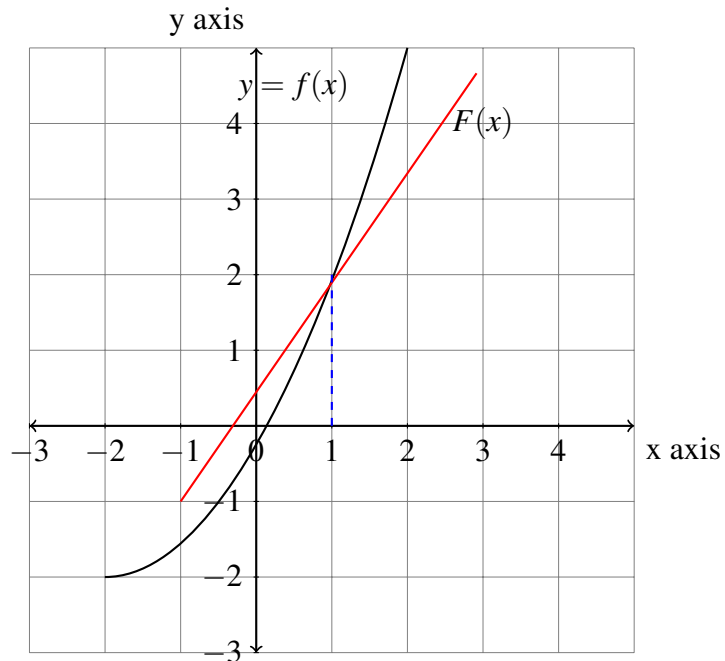


Figure 2.2: Taylor's Polynomial of a function $f(x)$ at $x_0 = 1$

knowing the function.

Taylor n^{th} Degree Polynomial: For a function $f(x)$ if there exists n^{th} derivative at $x = x_0$. So, the n^{th} degree polynomial can be written as,

$$F_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \dots + \frac{f^n(x_0)}{n!}(x - x_0)^n \quad (2.3)$$

Figure 2.2 shows the behavior of a function $f(x)$ and its Taylor polynomial $F(x)$ at a specific point $x_0 = 1$.

Spectral Radius

Consider a matrix A of size $n \times n$ and $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of the matrix. The spectral radius of matrix $A = \max_{i=1}^n |\lambda_i|$

Eigenvalue and Eigenvector

Let, A be a matrix of size $n \times n$. A scalar named λ is called an eigenvalue of the matrix A if there is a nonzero vector $x \in \mathbb{R}^n$ such that

$$Ax = \lambda x \quad (2.4)$$

is satisfied. Then x is called an eigenvector of A for the corresponding eigenvalue λ . Note that, I is an identity matrix of $n \times n$ dimension. The equation 2.4 is equivalent to,

$$(A - \lambda I)x = 0 \quad (2.5)$$

Characteristic Polynomial

From the equation 2.5, we know x is a non-zero eigenvector. To get a nontrivial solution for x , the matrix $A - \lambda I$ needs to be singular. Otherwise, λ will not be an eigenvalue of A . This condition gives us the determinant of the matrix $A - \lambda I$ to be zero. We can derive the characteristic polynomial and equation from this condition. In general, we can write the characteristic polynomial of the matrix A as,

$$P(x) = (x - \lambda_1)(x - \lambda_2) \dots (x - \lambda_n) \quad (2.6)$$

where, x is a variable and $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of A .

2.2 Gershgorin Circle Theorem

For a complex square matrix, Gershgorin circle theorem [4] gives a lower and upper bound on the spectrum. Given a square matrix H , the theorem tells us that, each eigenvalue of H will be in at least one of the Greshgorin's disks. A disk is $D(x_{ii}, r_i)$ where $r_i = \sum_{j \neq i} |x_{ij}|$ is the summation of absolute non-diagonal entries at i^{th} row. It has a radius r_i and center $(x_{ii}, 0)$. An example is given for the following 4×4 matrix

$$H = \begin{bmatrix} 3 & 2 & 1 & 1 \\ 2 & 3 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

According to the theorem, the disks are $D_1(3,4)$, $D_2(3,4)$, $D_3(1,3)$, $D_4(1,3)$. In addition, the eigenvalues of the matrix H are 0, 6, 1, 1. The disks along with eigenvalues are plotted in figure 2.3. The eigenvalues are inside the disks or intersection of the disks (red dots).

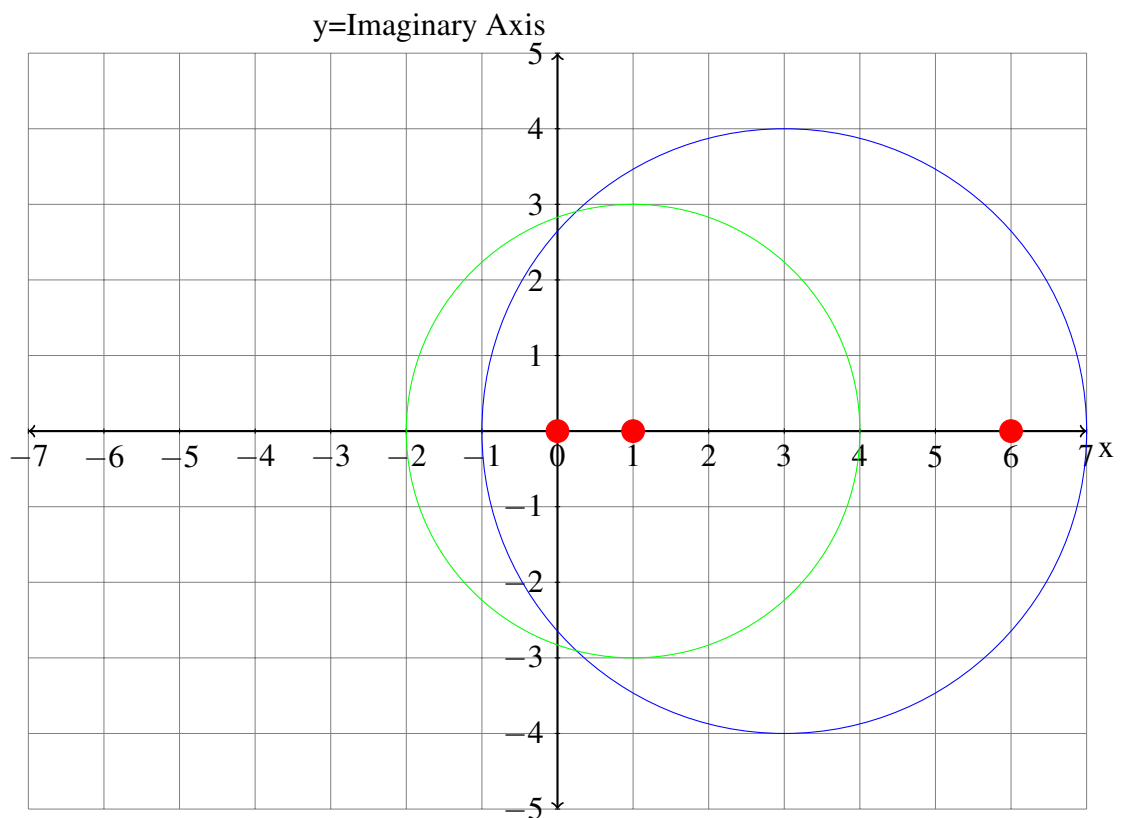


Figure 2.3: An example on Gershgorin circle theorem

It gives us an estimation that, what could be the lower bound and upper bound for the spectrum of the matrix H . For the matrix above, we get the lower and upper bound estimate of -2 and 7 accordingly.

2.3 Adiabatic Theorem

A state $|\alpha(t)\rangle$ evolves in a quantum system according to the Schrödinger's equation,

$$i\frac{d}{dt}|\alpha(t)\rangle = H(t)|\alpha(t)\rangle \quad (2.7)$$

where $\frac{d}{dt}$ is a first order partial derivative with respect to t . $H(t)$ is a time dependent Hamiltonian at any time t with,

$$H(0) = H_0 \quad (2.8)$$

and

$$H(1) = H_1 \quad (2.9)$$

Now, we consider a function $s = \frac{t}{T}$ linear in time t where T is the total time. At time $t = 0$, s will produce the Hamiltonian H_0 . Similarly, at time, $t = T$, we will get H_1 . In general, the Hamiltonian can be defined as a convex combination of H_0 and H_1 ,

$$H(s) = (1-s)H_0 + sH_1 \quad (2.10)$$

We have an initial ground state $|\alpha(0)\rangle = |\Phi_0(0)\rangle$ of Hamiltonian H_0 . According to the Adiabatic theorem [24], there are two limiting cases for T .

Case 1: If $T \rightarrow 0$, the state of the system will remain unchanged.

Case 2: If $T \rightarrow \infty$, the system will remain in the ground state of the Hamiltonian $H(s)$ at any time t .

The eigenvectors are the energy states and eigenvalues are the energies of the system. The state with lower energy is called a ground state. The Hamiltonian $H(s)$ must be slow so that it remains very close to the ground state at any time t . Question is how slow the Hamiltonian has to be to reach a final state near to the ground state $|\Phi_0(1)\rangle$ of the final Hamiltonian.

We need to get a lower bound on T . $|\Phi_0(t)\rangle$ is always the ground state of the Hamil-

tonian $H(s)$ at any time t . In notation, if $|\Phi_x(t)\rangle$ is the eigenstate of Hamiltonian $H(s)$ and $\lambda_x(t)$ is the corresponding eigenvalue then

$$H(s) |\Phi_x(t)\rangle = \lambda_x(t) |\Phi_x(t)\rangle \quad (2.11)$$

The adiabatic theorem gives the condition,

$$|\langle \Phi_0(T) | \alpha(T) \rangle|^2 \geq 1 - \sigma^2 \quad (2.12)$$

with,

$$\frac{|\langle \frac{dH}{dt} \rangle|_{\max}}{g_{\min}^2} \leq \sigma \quad (2.13)$$

where σ is a constant $\ll 1$. In the above equation g_{\min} is the minimum eigengap of the two smallest eigenvalues of $H(s)$. If the eigenvalues are defined as $\lambda_0(s) \leq \lambda_1(s) \leq \dots \leq \lambda_{N-1}(s)$ then

$$g_{\min} = \min_{0 \leq s \leq 1} (\lambda_1(s) - \lambda_0(s))$$

According to [24], an existence of nonzero gap between two eigenvalues will always keep $|\alpha(t)\rangle$ close to the ground state until reaching the final Hamiltonian $H(1)$ when T is large enough.

2.4 Deutsch Algorithm

We describe the Deutsch algorithm [7, 34] for the given one bit function,

$$f\{0, 1\} \rightarrow f\{0, 1\}$$

we need to determine whether the function f is a constant or a balanced function. Figure 2.4 is the quantum circuit for determining the property of a one bit function. Now, with time we analyze the circuit from the left to right. Input of the circuit is in an initial state, $|0\rangle |1\rangle$. After applying Hadamard transformation, the state is changed to,

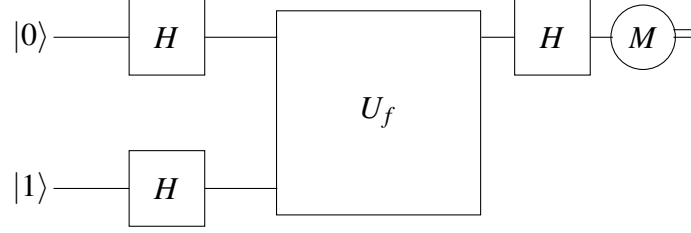


Figure 2.4: Quantum circuit for Deutsch algorithm [34]

$$\frac{1}{2} |00\rangle - \frac{1}{2} |01\rangle + \frac{1}{2} |10\rangle - \frac{1}{2} |11\rangle$$

equivalent to,

$$\frac{1}{2} |0\rangle (|0\rangle - |1\rangle) + \frac{1}{2} |1\rangle (|0\rangle - |1\rangle)$$

Next, we perform U_f on the present state of qubits which is defined as,

$$U_f |p\rangle |q\rangle = |p\rangle |q \oplus f(p)\rangle$$

So, the state will be transformed to,

$$\frac{1}{2} |0\rangle (|0 \oplus f(0)\rangle - |1 \oplus f(0)\rangle) + \frac{1}{2} |1\rangle (|0 \oplus f(1)\rangle - |1 \oplus f(1)\rangle)$$

Using the fact,

$$|0 \oplus b\rangle - |1 \oplus b\rangle = (-1)^b (|0\rangle - |1\rangle) \text{ for } b \in \{0, 1\}$$

We can rewrite the above U_f transformation as,

$$\begin{aligned} & \left(\frac{1}{2} (-1)^{f(0)} |0\rangle (|0\rangle - |1\rangle) + \frac{1}{2} (-1)^{f(1)} |1\rangle (|0\rangle - |1\rangle) \right) \\ &= \left(\frac{1}{\sqrt{2}} (-1)^{f(0)} |0\rangle + \frac{1}{\sqrt{2}} (-1)^{f(1)} |1\rangle \right) \left(\frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle \right) \end{aligned}$$

After performing the U_f , the state of second qubit is not changed. So, we can consider the state independent of the second qubit and it will give us the state,

$$\begin{aligned} & \left(\frac{1}{\sqrt{2}} (-1)^{f(0)} |0\rangle + \frac{1}{\sqrt{2}} (-1)^{f(1)} |1\rangle \right) \\ &= (-1)^{f(0)} \left(\frac{1}{\sqrt{2}} (|0\rangle + (-1)^{f(0) \oplus f(1)} |1\rangle) \right) \end{aligned}$$

The last Hadamard transformation gives us the state,

$$(-1)^{f(0)} |f(0) \oplus f(1)\rangle$$

In the end of the analysis, Deutsch quantum circuit gives state $|f(0) \oplus f(1)\rangle$ as 0 (1) for the constant (balanced) functions accordingly in the measurement.

2.5 Deutsch-Jozsa Algorithm

For the generalization of an input function over n bits, an extended algorithm is given by Deutsch and Jozsa [8]. We consider the function (with a promise that the function is either constant or balanced) to describe the Deutsch-Jozsa algorithm [34].

$$f : \{0, 1\}^n \rightarrow \{0, 1\}$$

The goal is to determine the above function over n bits is either constant or balanced. The initial state is, $|0\rangle^n |1\rangle$. After $n + 1$ Hadamard transformation on $|0\rangle^n |1\rangle$ the state will be,

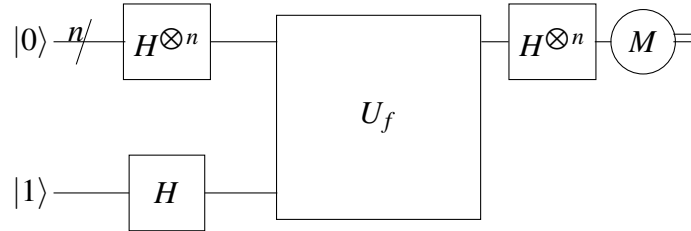


Figure 2.5: Deutsch-Jozsa quantum circuit [34]

$$\frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle \left(\frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle \right)$$

Now, the

$$U_f |p\rangle |q\rangle = |p\rangle |q \oplus f(p)\rangle$$

transformation gives us the state where the last qubit is independent of the first n qubits.

$$\left(\frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} |x\rangle \right) \left(\frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle \right)$$

We can ignore the last qubit and the superposition will be,

$$\frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} |x\rangle$$

We perform one more Hadamard transformation before the measurement. The final hadamard transformation gives the superposition of the resulting state $|y\rangle$ as,

$$\frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} \left(\frac{1}{\sqrt{2^n}} \sum_{y \in \{0,1\}^n} (-1)^{x \cdot y} |y\rangle \right)$$

equivalent to,

$$\frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{f(x)+x \cdot y} |y\rangle$$

The above superposition gives us the amplitude of observing a state $|y\rangle$. So, the probability to observe the state $|0^n\rangle$ is,

$$\left| \frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} \right|$$

The probability will be 1 if the given function is constant and 0 for the balanced function.

Chapter 3

Literature Review

3.1 Previous Work

Parameterized complexity finds the complexity of hard problems as a function of some parameters. Such parameterized complexity is studied in a classical setting [10]. It studies the idea of parameterized complexity for different classical problems. It includes different lower bound methods. It also discuss the pros and cons of different methods as well as variation of solutions for a problem.

It takes $2^{n-1} + 1$ queries in classical settings to determine a constant or balanced function over n inputs. Deutsch-Jozsa proposed a quantum algorithm [8] that illustrates the advantage of quantum computation. The algorithm takes 1 query and n measurements to determine whether a function is constant or balanced. This was a generalization of Deutsch's algorithm [7], proposed for a single bit input function that in a single query and single measurement can determine whether the function is constant or balanced. In a classical setting, it always requires 2 queries.

The question regarding computational power whether the adiabatic and standard quantum computation are equivalent or not is answered in [2, 20, 11, 28]. A five body Hamiltonian described in [21] is reduced to a two body Hamiltonian where the eigengap is lower bounded. An alternative proof of the equivalence of the two types of computation is given in [26].

An implementation of Deutsch algorithm on an adiabatic quantum computer was given by Das *et. al.*[6]. Similar adiabatic computation is also known for the Deutsch-Jozsa algo-

rithm [8]. A time-dependent Hamiltonian was chosen and it was shown that global adiabatic evolution takes $O(N)$ time where $N = 2^n$ is the total number of possible quantum states and n is the total number of qubits. Using local adiabatic computation they obtain a quadratic speed up of $O(\sqrt{N})$. Further improvement to this result is due to Wei *et al.* [35] who gave an adiabatic evolution for the Deutsch-Jozsa algorithm with time complexity $O(1)$.

Shor gave the quantum algorithm [31] for two problems. The problems were to find a discrete logarithm and to factorize a number. Peng *et al.* [29] gave an adiabatic algorithm for factorization that takes a fewer number of qubits than Shor's algorithm. The algorithm was implemented in a quantum processor and the authors factorized the number 21.

Given an unsorted database with N elements, the task is to find an item from the database. Classically it will require to test $\frac{N}{2}$ items in the average case. Grover [16] gave a quantum algorithm that takes $O(\sqrt{N})$ time. The global and the local adiabatic evolution for Grover's algorithm is given by Roland *et al.* [30]. The global adiabatic evolution takes N in time steps where N is the total number of elements in the database. For the problem, the eigengap was calculated analytically and found to be independent of the solution. A quadratic speedup can be shown for the local adiabatic evolution of this problem and it gives an $O(\sqrt{N})$ time complexity. They no longer considered time a linear function; rather divided it into infinitesimal small interval and applied all the adiabatic condition locally. Such adiabatic computation is shown for a problem where there are more than one solution. For M solutions, global adiabatic evolution gives $O(\frac{N}{M})$ time complexity. In the local evolution it obtains $O(\sqrt{\frac{N}{M}})$ time complexity.

Farhi *et al.* [14] gave a quantum adiabatic algorithm to solve the satisfiability problem. The adiabatic evolution encodes the solution to the problem in the ground state of the final Hamiltonian. They showed that the algorithm is a polynomial time algorithm for some instances. For the exact cover which is an NP-complete problem, Farhi *et al.* [13] tested the behavior of a quantum algorithm on some random instances.

The power of the adiabatic quantum evolution is examined in [32]. For a set of mini-

mization problems a lower bound on the delay to reach the final Hamiltonian is shown to be exponential in the size of input of the problem. An open question is posed in the paper is whether the adiabatic computation can always surpass the computational power of classical computation or not?

3.2 Contribution of the Thesis

Parameterization of quantum algorithm such as Deutsch-Jozsa seems interesting to us. We perform such parameterization of the Deutsch-Jozsa algorithm to find out the number of queries needed to determine ϵ -balancedness and δ -duality of a function.

Adiabatic theorem gives the theme to develop quantum algorithms to solve a problem. In terms of parameterization, it is promising to see how the Deutsch-Jozsa algorithm behaves with respect to the parameter. Specifically, we are motivated to determine the lower bound on the total evolution time to reach the solution state for an adiabatic quantum computer to test for ϵ -balanced and δ -dual functions. We show that local adiabatic quantum algorithms take less time than classical algorithms.

Chapter 4

Parameterized Query Complexity

4.1 Introduction

In this chapter, we analyze the query complexity of two different problems in terms of a parameter. At first, we give the definition of the ϵ -balanced function, δ -dual function. Our first problem is to determine if a function is ϵ -balanced or not. The second problem is determining δ -duality of a function. We will give quantum algorithms for solving these two problems. Then we establish two theorems. Empirical results are also included at the end of each proof.

4.2 ϵ -balanced Function

Given a boolean function,

$$f : \{0, 1\}^n \rightarrow \{0, 1\}$$

A pair of input (x, y) is called balanced if $f(x) \neq f(y)$. A function is called ϵ -balanced if the maximum number of balanced pairs is $(1 - \epsilon)2^{n-1}$. Therefore, the number of unbalanced pairs will be $\epsilon 2^{n-1}$.

For a constant function, the number of balanced pairs is 0 and the number of unbalanced pairs is 2^{n-1} . Therefore, the function is called 1-balanced function.

For a balanced function, the number of balanced pairs is 2^{n-1} and the number of unbalanced pairs is 0. So, function is a 0-balanced function in terms of ϵ parameter. Any boolean function is $\frac{i}{2^{n-1}}$ balanced for some $i \in \{1, 2^{n-1}\}$. We show a proof of contradiction for the

minimality of ϵ .

Fact 1: *If f is ϵ -balanced and ϵ is minimum then for all unbalanced pairs (x, y) , $f(x) = f(y) = b \in \{0, 1\}$. Stated otherwise, it is not the case that there are two pairs (x_1, y_1) and (x_2, y_2) such that $f(x_1) = f(y_1) = 0$ and $f(x_2) = f(y_2) = 1$. If such was the case then consider the pairs (x_1, x_2) and (y_1, y_2) , the pairs are balanced. So the total number of unbalanced pairs is smaller. This contradicts the minimality of ϵ .*

4.3 Self Dual Function

A function g is self-dual if for all inputs x , $g(x) \neq g(\bar{x})$. If $g(x)$ is restricted to be a constant or a balanced function then the Deutsch-Jozsa circuit can distinguish the property of self-duality. Note that, every self-dual function is balanced but every balanced function is not self-dual. So, it is obvious that the direct application of Deutsch-Jozsa algorithm can't differentiate self-dual functions from non self-dual functions. We now construct a function f and claim a fact as follows,

$$f(x) = g(x) \oplus g(\bar{x}) \tag{4.1}$$

Fact 2: *$f(x)$ will be constant at one if and only if $g(x)$ is self-dual.*

This fact promise that for an input function $f(x)$ in the Deutsch-Jozsa circuit, we will always observe 0 after the measurement if the function $g(x)$ is self-dual.

4.4 Quantum Algorithms

In this section, we give two quantum algorithms. The first algorithm will determine whether a function is ϵ -balanced or not. The other quantum algorithm will determine whether a function is δ -dual or not. We will establish two theorems at the end of this chapter to show the number of tries required for the above algorithms with a constant probability of

success $> \frac{1}{2}$.

4.4.1 Quantum Algorithm for ϵ -Balanced Function

We consider the number of queries needed to determine if an input function is balanced as a function of ϵ in the classical settings. It is notable that, the simple randomized algorithm that works when the input functions are restricted to be either constant or self-dual (balanced) does not work with the same error probability if we expand the class of functions to be ϵ -balanced. If we choose any l inputs such that $x_1, x_2, \dots, x_l \in \{0, 1\}^n$ and find that $f(x_1) = f(x_2) = \dots = f(x_l)$ then we can say the function f is constant. Otherwise, it is a self-dual function. The algorithm always shows correctness for the constant functions. For the self-dual functions it has the probability of error $2^{-(l-1)}$. If the input function is expanded in the class of ϵ -balanced then the number of balanced and unbalanced pairs will be different for each ϵ . So, the error probability will also be different.

f is an input function with a promise that it is ϵ -balanced. We want to determine whether the function f is balanced with a constant probability of error $< \frac{1}{2}$. Note that any constant error probability can be amplified to $\frac{1}{n}$ using $\log n$ independent trials. This guarantees a success probability of 1 almost always asymptotically. The quantum algorithm is direct application of the Deutsch Jozsa circuit.

Algorithm 1 Quantum Algorithm for ϵ -balanced function

```

Run Deutsch-Jozsa circuit  $t$  times
Let  $m$  be the number of times state  $|0\rangle$  is observed
if  $m > 0$  then
     $f$  is not  $\epsilon$ -balanced
else
     $f$  is  $\epsilon$ -balanced
end if

```

If the function is constant then we observe state $|0\rangle$ all the t times, and we say that the function is not balanced (constant) and it is a correct decision. If the function is balanced then state $|0\rangle$ is observed 0 times during the run and we correctly report that the function is balanced. So we recover the separation guaranteed by Deutsch-Jozsa circuit.

We are interested computing the number of tries t needed so that we have a constant probability of success when we report that an input function is balanced. If the function is balanced then state $|0\rangle$ is never observed [8]. So the above algorithm never makes a mistake when it says that the function is not balanced. It possibly makes a mistake when the output is balanced but the function is not balanced. The probability of the mistake is the probability with which state $|0\rangle$ is not observed.

The first theorem is stated as follows.

Theorem 4.1. *If g is an ε -balanced function with an efficient unitary operator U_f , then the algorithm above is correct with probability $> \frac{1}{2}$ and uses $\frac{\log \frac{1}{2}}{2 \log(1-\varepsilon^2)}$ queries.*

Proof: From the analysis of Deutsch-Jozsa algorithm, for n qubits, the amplitude of a quantum state $|y\rangle$ is,

$$\frac{1}{2^n} \left(\sum_{x \in \{0,1\}^n} (-1)^{f(x)+x \cdot y} \right)$$

State $|y\rangle = |0^n\rangle$ will have the amplitude,

$$\frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{f(x)}$$

In terms of pairs (x_i, y_i) , $i \in \{1, \dots, 2^{n-1}\}$, the amplitude can be written as,

$$\frac{1}{2^n} \sum_{i=1}^{2^{n-1}} ((-1)^{f(x_i)} + (-1)^{f(y_i)})$$

Given that, f is ε -balanced, only $\varepsilon 2^{n-1}$ pairs contribute a sum of $(-1)^b \varepsilon 2^{n-1}$. b is either 0 or 1 for all the unbalanced pairs. The balanced pairs cancel each other out in the sum. The amplitude for state $|0^n\rangle$ is,

$$\frac{1}{2^n} (\varepsilon (-2)^b) 2^{n-1} = (-1)^b \varepsilon$$

So, the probability of observing state $|0\rangle^n$ is ε^2 . Now, the probability of error in a single iteration is the probability of not observing the state $|0^n\rangle$. So, we can write the probability of error after t iterations as,

$$Pr[Error] = (1 - \epsilon^2)^t$$

We put an upper bound on the error probability that it will be strictly $< \frac{1}{2}$. So,

$$(1 - \epsilon^2)^t < \frac{1}{2}$$

$$t \log(1 - \epsilon^2) < \log \frac{1}{2}$$

$$t > \frac{\log \frac{1}{2}}{\log(1 - \epsilon^2)}$$

Figure 4.1 shows the plot of $\frac{\log \frac{1}{2}}{\log(1 - \epsilon^2)}$ as a function of ϵ in steps of 0.01.

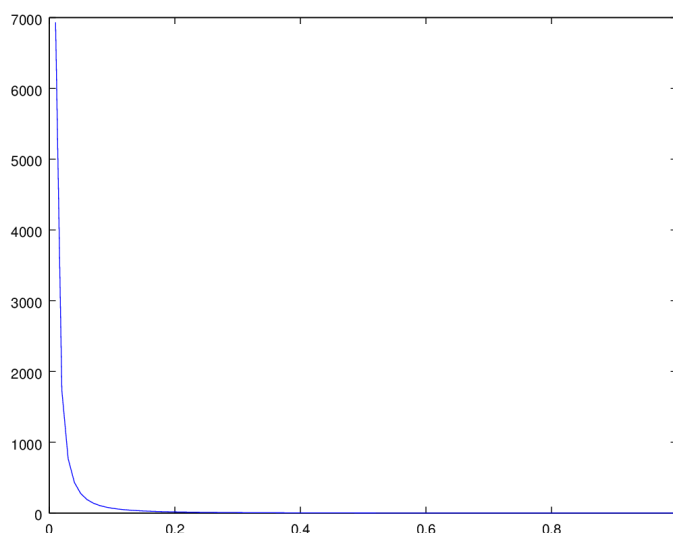


Figure 4.1: ϵ -balanced: Number of queries as a function of ϵ

4.4.2 Quantum Algorithm for δ -Dual Function

We give a quantum algorithm to determine whether a given input function is δ -dual or not. In particular, we are interested in the query complexity. Before describing the algorithm, a new definition is given to measure how far a function is from being self-dual.

From the equation 4.1, let g_b be the number of inputs x such that $g(x) = b$ for $b \in \{0, 1\}$. We call the function f , δ -dual if

$$f_0 = \delta(f_0 + f_1) \tag{4.2}$$

An example is given below for two qubits for different values of δ . Using the equation 4.1,

Table 4.1: Different value of δ for $n = 2$

x	$f(x_1)$	$f(x_2)$	$f(x_3)$	$f(x_4)$
00	0	0	0	1
01	0	0	1	1
10	0	1	1	1
11	0	1	1	1
δ	1	$\frac{1}{2}$	$\frac{1}{4}$	0

we construct the function, $f(x)$. Note that, in 4.2 g_1, g_5 are constant functions and g_2 is a

Table 4.2: Behavior of $f(x)$ for $n = 2$

x	g_1	f_1	g_2	f_2	g_3	f_3	g_4	f_4	g_5	f_5
00	0	0	0	1	0	1	0	0	1	0
01	0	0	0	1	1	0	1	0	1	0
10	0	0	1	1	1	0	1	0	1	0
11	0	0	1	1	1	1	0	0	1	0

balanced, self-dual function. g_4 is balanced but not a self-dual function. We estimate the number of queries as a function of δ with the fact that $f(x)$ will be constant at 1 if $g(x)$ is self-dual.

First, we examine the query complexity of determining if a function is self-dual as a function of δ . If no error is allowed then the function f has to be evaluated on all the inputs. It is equivalent to evaluating g on all the 2^{n-1} pairs. If one sided error is allowed then the probability that for a randomly chosen x , $f(x) = 1$ is $1 - \delta$. After t queries, the error probability is $(1 - \delta)^t$. Choosing $t = \frac{1}{\delta}$, we get the probability of error,

$$(1 - \delta)^{\frac{1}{\delta}} = \frac{1}{e}$$

$\frac{1}{\delta}$ trials are needed to obtain constant probability of success. Now, we give the quantum algorithm to determine the δ -duality of a function. Now, we give the second theorem to determine the number of queries with a certain probability.

Theorem 4.2. *If f has a efficient unitary operator U_f and f is δ -dual then the algorithm above is correct with probability $> \frac{1}{2}$ after $t = \frac{\log \frac{1}{2}}{\log(2\delta-1)^2}$ tries.*

Algorithm 2 Quantum Algorithm for δ -dual function

```

Construct  $f(x) = g(x) \oplus g(\bar{x})$ 
Run the Deutsch-Jozsa circuit  $t$  times on the function  $f$ 
Let,  $m$  be the number of times state  $|0\rangle$  is observed
if  $m = t$  then
    The function  $f$  is  $\delta$ -dual
else
     $f$  is not  $\delta$ -dual
end if

```

Proof: If a state is observed other than $|0^n\rangle$ in any iteration then the algorithm reports that g is not self-dual. Then, f is not a constant function. So the algorithm does not make any error in this case. The algorithm makes one sided error if the function f is not constant but state $|0\rangle$ is observed in all the t iterations. We compute the error probability in a single iteration. This probability is equivalent to the probability of observing state $|0\rangle$. Deutsch Jozsa algorithm gives us the amplitude of a quantum state $|y\rangle$ being observed is,

$$\frac{1}{2^n} \left(\sum_{x \in \{0,1\}^n} (-1)^{f(x)+x \cdot y} \right)$$

For $|y\rangle = |0^n\rangle$, the amplitude is,

$$\frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{f(x)}$$

There are $\delta 2^n$ inputs on which $f(x)$ is 0 and $(1 - \delta) 2^n$ inputs on which it is 1. So the amplitude of observing $|0^n\rangle$ is,

$$\frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^0 \delta 2^n + (-1)^1 (1 - \delta) 2^n$$

So, the amplitude of observing $|0^n\rangle$ is $(2\delta - 1)$. So, the probability of observing state $|0\rangle$ is

$$Pr[|0^n\rangle] = (2\delta - 1)^2$$

After n iterations, the error probability $< \frac{1}{2}$. By taking \log on both sides of the inequality,

$$t \log(2\delta - 1)^2 < \log \frac{1}{2}$$

After multiplying both sides by -1 we get,

$$t > \frac{\log \frac{1}{2}}{\log(2\delta - 1)^2}$$

Each query to f is implemented by two queries to t . So, the number of queries in g is $2t$. The following figure 4.2 shows the number of queries on the y-axis as a function of delta in steps of 0.01 to determine self-duality of a function f . Dashed blue curve indicates $(2\delta - 1)$, dot green curve shows the number of queries for the quantum algorithm and the red curve shows the number of queries for the classical randomized algorithm. In comparison to the

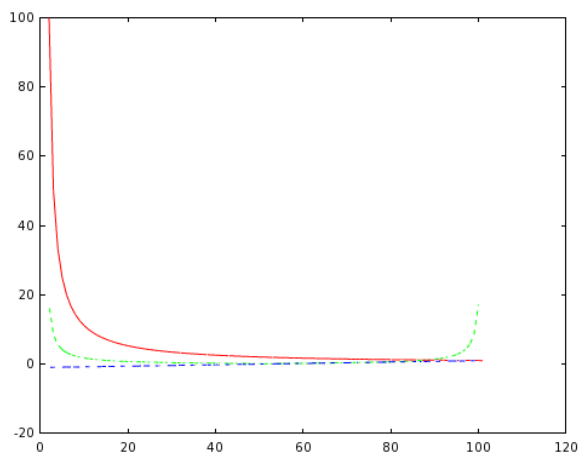


Figure 4.2: δ -Dual: Number of queries as a function of δ

classical algorithm, there is a reduction in the number of queries needed.

Chapter 5

Adiabatic Quantum Computation (ϵ -balanced Function)

5.1 Introduction

In this chapter, we describe adiabatic quantum computation for ϵ -balanced functions. We describe the properties of the corresponding Hamiltonians such as eigengap, the multiplicity of eigenvalues etc. We give a lower bound on the time required to reach from initial to final state under adiabatic evolution. Numerical results are shown in graphs as well as in tabular form.

5.2 Adiabatic Computation

We will consider a system of n qubits of the dimension $N = 2^n$. We define the state $|\psi\rangle = \alpha_1 |0\rangle + \alpha_2 |1\rangle + \dots + \alpha_N |N\rangle$ as a column or row vector shown below,

$$|\psi\rangle = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdot \\ \alpha_N \end{bmatrix}_{(N \times 1)}, \quad \langle\psi| = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdot & \cdot & \cdot & \alpha_N \end{bmatrix}_{(1 \times N)}$$

Inner product $\langle\psi|\psi\rangle$ and outerproduct $|\psi\rangle\langle\psi|$ have the dimensions $(1 \times 1), (N \times N)$ respectively.

We define a time dependent Hamiltonian as a convex combination of initial Hamiltonian H_0 and final Hamiltonian H_1 ,

$$H(s) = (1 - s)H_0 + sH_1 \quad (5.1)$$

where, $s = \frac{t}{T}$ is a linear function in time t . Let $\Phi_0(0)$ and $\Phi_0(1)$ are the ground state of initial and final Hamiltonians. Now, H_0 and H_1 are the initial and final Hamiltonians defined as,

$$H_0 = I - |\Phi_0(0)\rangle \langle \Phi_0(0)| \quad (5.2)$$

$(N \times N)$

$$H_1 = I - |\Phi_0(1)\rangle \langle \Phi_0(1)| \quad (5.3)$$

$(N \times N)$

The initial ground state $\Phi_0(0)$ can be written as,

$$|\Phi_0(0)\rangle = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} |i\rangle \quad (5.4)$$

We assume a final ground state $|\Phi_0(1)\rangle$ as follows,

$$|\Phi_1(0)\rangle = \varepsilon|0\rangle + \sqrt{1 - \varepsilon^2}|k\rangle \quad (k \neq 0) \quad (5.5)$$

To justify the assumption, let us analyze the probability of state $|0\rangle$ if the function is assured to be ε -balanced. From the analysis of Deutsch-Jozsa algorithm, we get the probability of observing state $|0^n\rangle$ is ε^2 in theorem 4.1. The probability of observing any other state is assumed to be $\sqrt{1 - \varepsilon^2}$. We can rewrite the Equation 5.1 as,

$$H(s) = (1 - s(t))(I - |\Phi_0(0)\rangle \langle \Phi_0(0)|) + s(t)(I - |\Phi_0(1)\rangle \langle \Phi_0(1)|) \quad (5.6)$$

Substituting equations 5.4, 5.5 in 5.1, $H(s)$ can be written as,

$$H(s) = (1-s) \left[I - \frac{1}{N} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} |i\rangle \langle j| \right] + s \left[I - (\epsilon^2 |0\rangle \langle 0| + (1-\epsilon^2) |k\rangle \langle k| + \epsilon \sqrt{1-\epsilon^2} (|k\rangle \langle 0| + |0\rangle \langle k|)) \right] \quad (5.7)$$

Each element in the matrix can be represented by the following notation.

$$H_{mn} = \langle m | (1-s) \left[I - \frac{1}{N} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} |i\rangle \langle j| \right] + s \left[I - (\epsilon^2 |0\rangle \langle 0| + (1-\epsilon^2) |k\rangle \langle k| + \epsilon \sqrt{1-\epsilon^2} (|k\rangle \langle 0| + |0\rangle \langle k|)) \right] | n \rangle \quad (5.8)$$

where, $m, n \in [0, 1, \dots, N-1]$.

5.2.1 Hamiltonian in Matrix Form

The Hamiltonian H can be represented in a matrix form of n dimension. Now we find the entries of the H matrix from equation 5.8.

- $H_{00} = 1 + \frac{s-1}{N} - s\epsilon^2$
- $H_{0k} = \frac{s-1}{N} - s\epsilon\sqrt{1-\epsilon^2} \quad (k \geq 1)$
- $H_{k0} = \frac{s-1}{N} - s\epsilon\sqrt{1-\epsilon^2} \quad (k \geq 1)$
- $H_{kk} = 1 + \frac{s-1}{N} + (\epsilon^2 - 1)s \quad (k \geq 1, k \neq m, k \neq n)$
- $H_{0m} = H_{m0} = \frac{s-1}{N} \quad (m \geq 1, m \neq k)$
- $H_{1m} = H_{m1} = \frac{s-1}{N} \quad (m \geq 1, m \neq k)$
- $H_{mm} = 1 + \frac{s-1}{N} \quad (m > 1, n > 1, k \neq m)$
- $H_{mn} = \frac{s-1}{N} \quad (m > 1, n > 1, m \neq n, m \neq k, n \neq k)$

We assume to observe state $|1\rangle$ ($k = 1$) with the probability $\sqrt{1 - \epsilon^2}$. So, the resulting Hamiltonian matrix has the following form,

$$H(s) = \begin{bmatrix} 1 + \frac{s-1}{N} - s\epsilon^2 & \frac{s-1}{N} - s\epsilon\sqrt{1 - \epsilon^2} & \frac{s-1}{N} & \dots & \frac{s-1}{N} \\ \frac{s-1}{N} - s\epsilon\sqrt{1 - \epsilon^2} & 1 + \frac{s-1}{N} + (\epsilon^2 - 1)s & \frac{s-1}{N} & \dots & \frac{s-1}{N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{s-1}{N} & \frac{s-1}{N} & \frac{s-1}{N} & \dots & \frac{s-1}{N} + 1 \end{bmatrix} \quad (5.9)$$

By Adiabatic Theorem [24],

$$\frac{|\langle \frac{dH}{dt} \rangle|}{g_{\min}^2} \leq \sigma \quad (5.10)$$

The numerator can be written as,

$$\left\langle \frac{dH}{dt} \right\rangle = \left\langle \frac{dH}{ds} \right\rangle \cdot \frac{ds}{dt} \quad (5.11)$$

and

$$\left\langle \frac{dH}{ds} \right\rangle = \left\langle v_0 \left| \frac{dH}{ds} \right| v_1 \right\rangle \quad (5.12)$$

where, $|v_0\rangle$ and $|v_1\rangle$ are two normalized eigenvector of $H(s)$ corresponding to the lowest two eigenvalues that gives us the minimum gap g_{\min} . $s = \frac{t}{T}$ is a linear function of t where T is the total time for the system to evolve. So,

$$\frac{ds}{dt} = \frac{1}{T} \quad (5.13)$$

Differentiation of 5.1 gives us,

$$\frac{dH}{ds} = H_1 - H_0 \quad (5.14)$$

After routine simplification, we can rewrite the adiabatic condition,

$$\frac{|\langle v_0 | H_1 - H_0 | v_1 \rangle| \frac{1}{T}}{g_{\min}^2} \leq \sigma \quad (5.15)$$

To get a lower bound on T the above inequality is written as,

$$T \geq \frac{|\langle v_0 | H_1 - H_0 | v_1 \rangle|}{g_{\min}^2 \sigma} \quad (5.16)$$

This is a minimization problem. To obtain a lower bound on T , the numerator is maximized and the denominator is minimized. Note that, $\sigma \ll 1$. Finally the expression looks like,

$$T \geq \frac{\max\{|\langle v_0 | H_1 - H_0 | v_1 \rangle|\}}{g_{\min}^2 \sigma} \quad (5.17)$$

Let the matrix-vector multiplication, $(H_1 - H_0) |v_1\rangle$ will give us a new vector $|v_2\rangle$. Now the numerator looks like $\langle v_0, v_2 \rangle$. Applying Cauchy-Schwarz inequality,

$$\langle v_0, v_2 \rangle \leq \|v_0\|_2 \cdot \|v_2\|_2 \quad (5.18)$$

As $|v_0\rangle$ is normalized to 1, $\|v_0\|_2 = 1$. Now, inequality 5.18 looks like following,

$$\langle v_0, v_2 \rangle \leq 1 \cdot \|v_2\|_2 \quad (5.19)$$

According to the property of induced norm [25] we write $|v_2\rangle$ as follows,

$$\|(H_1 - H_0)v_1\|_2 \leq \|H_1 - H_0\|_2 \cdot \|v_1\|_2 \quad (5.20)$$

Now, $|v_1\rangle$ is normalized and $H_1 - H_0$ is a Hermitian operator. The norm of the matrix $H_1 - H_0$ equals to the spectral radius [1] named $|\lambda_{\max}|$. So,

$$T \geq \frac{|\lambda_{\max}|}{g_{\min}^2 \sigma} \quad (5.21)$$

5.3 Upper Bound on λ_{max}

We want an upper bound on λ_{max} in the equation 5.21. The matrix $H_1 - H_0$ has the following form,

$$H_1 - H_0 = \begin{bmatrix} \frac{1}{N} - \varepsilon^2 & \frac{1}{N} - \varepsilon\sqrt{1 - \varepsilon^2} & \frac{1}{N} & \cdots & \frac{1}{N} \\ \frac{1}{N} - \varepsilon\sqrt{1 - \varepsilon^2} & \frac{1}{N} + (\varepsilon^2 - 1) & \frac{1}{N} & \cdots & \frac{1}{N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{N} & \frac{1}{N} & \frac{1}{N} & \cdots & \frac{1}{N} \end{bmatrix} \quad (5.22)$$

If we consider $N \rightarrow \infty$, the $H_1 - H_0$ matrix is,

$$H_1 - H_0 = \begin{bmatrix} -\varepsilon^2 & -\varepsilon\sqrt{1 - \varepsilon^2} & 0 & \cdots & 0 \\ -\varepsilon\sqrt{1 - \varepsilon^2} & \varepsilon^2 - 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \quad (5.23)$$

The matrix 5.22 has the following Gershgorin disks,

- $D_1(\frac{1}{N} - \varepsilon^2, \frac{N-1}{N} - \varepsilon\sqrt{(1 - \varepsilon^2)})$
- $D_2(\frac{1}{N} + \varepsilon^2 - 1, \frac{N-1}{N} - \varepsilon\sqrt{(1 - \varepsilon^2)})$
- $D_3(\frac{1}{N}, \frac{N-1}{N})$

Note that the term $\frac{1}{N}, \frac{N-1}{N}$ will be maximum at $n = 1$ and it is $\frac{1}{2} = \frac{1}{2}$. So, the disks can be written as,

- $D_1(\frac{1}{2} - \varepsilon^2, \frac{1}{2} - \varepsilon\sqrt{(1 - \varepsilon^2)})$
- $D_2(\frac{1}{2} + \varepsilon^2 - 1, \frac{1}{2} - \varepsilon\sqrt{(1 - \varepsilon^2)})$
- $D_3(\frac{1}{2}, \frac{1}{2})$

For $N \rightarrow \infty$, we get the following disks for the matrix 5.23.

- $D_4(-\varepsilon^2, \varepsilon\sqrt{1-\varepsilon^2})$
- $D_5(\varepsilon^2 - 1, \varepsilon\sqrt{1-\varepsilon^2})$

The center of the circles are $(\frac{1}{2} - \varepsilon^2, 0)$, $(\frac{1}{2} + \varepsilon^2 - 1, 0)$, $(\frac{1}{2}, 0)$, $(-\varepsilon^2, 0)$, $(\varepsilon^2 - 1, 0)$ with the radius $\frac{1}{2} - \varepsilon\sqrt{1-\varepsilon^2}$, $\frac{1}{2} - \varepsilon\sqrt{1-\varepsilon^2}$, $\frac{1}{2}$, $\varepsilon\sqrt{1-\varepsilon^2}$, $\varepsilon\sqrt{1-\varepsilon^2}$ accordingly. The ε parameter varies from 0 to 1. Moreover, the matrix $H_1 - H_0$ is a real hermitian matrix. So, all the eigenvalues will be real. We need to obtain the lowest and the highest coordinates on the x -axis intersected by the circumference of a circle. For the first and second circle we have to find the minimum and maximum of $(\frac{1}{2} - \varepsilon^2 \pm (\frac{1}{2} - \varepsilon\sqrt{1-\varepsilon^2}))$, $(\frac{1}{2} + \varepsilon^2 - 1 \pm (\frac{1}{2} - \varepsilon\sqrt{1-\varepsilon^2}))$. The third circle's circumference will intersect the x -axis at $(0,0)$ and $(1,0)$. For the fourth and fifth circle we minimize and maximize the quantity $(-\varepsilon^2 \pm \varepsilon\sqrt{1-\varepsilon^2})$, $(\varepsilon^2 - 1 \pm \varepsilon\sqrt{1-\varepsilon^2})$. Local maxima and minima are written as a pair as $(1, \frac{1}{2} - \frac{1}{\sqrt{2}})$, $(-\frac{1}{2} + \frac{1}{\sqrt{2}}, -1)$, $(1, \frac{1}{2} - \frac{1}{\sqrt{2}})$, $(-\frac{1}{2} + \frac{1}{\sqrt{2}}, -1)$, $(-\frac{1}{2} + \frac{1}{\sqrt{2}}, 0)$, $(-1, -\frac{1}{2} - \frac{1}{\sqrt{2}})$. We take the largest absolute value of the global maximum or minimum that gives us an upper bound on the eigenvalue. Figure 5.1 shows the bound on the eigenvalue of H . Applying the idea of

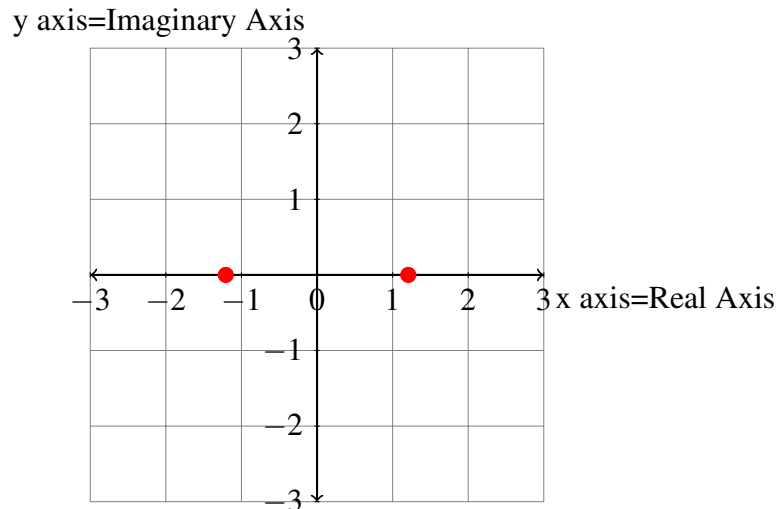


Figure 5.1: Gershgorin circle theorem for upper bounding λ_{max}

Greshgorin circle theorem we get the upper bound on λ_{max} ,

$$\boxed{|\lambda_{\text{max}}| \leq 1.20711} \quad (5.24)$$

The upper bound on λ_{max} can be improved by solving the characteristics polynomial equation for 5.23, We write the equation and solve it for the roots.

$$\begin{aligned} x^2 - (-\varepsilon^2 + \varepsilon^2 - 1)x + ((-\varepsilon^2(\varepsilon^2 - 1)) - (\varepsilon^2(1 - \varepsilon^2))) &= 0 \\ \Rightarrow x^2 + x &= 0 \\ \Rightarrow x(x + 1) &= 0 \end{aligned} \quad (5.25)$$

The roots of the equation are $x = 0, -1$. So the absolute maximum eigenvalue, $\lambda_{\text{max}} = 1$. So, the upper bound on the numerator,

$$\boxed{|\lambda_{\text{max}}| \leq 1} \quad (5.26)$$

5.3.1 Eigenvalues of the Hamiltonian

For the matrix 5.9 in $N \times N$ dimension, there are three distinct eigenvalues $\lambda_0, \lambda_1, \lambda_2$. We can derive the lowest two eigenvalues for smaller N as shown next.

For $N = 2$, the 2×2 matrix from 5.9 has the following spectrum,

$$\lambda = \begin{bmatrix} \frac{1}{2} - \frac{1}{2} \sqrt{1 + \frac{4(s^2 - s)(2 - 1 - 2\varepsilon\sqrt{1 - \varepsilon^2})}{2}} \\ \frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{4(s^2 - s)(2 - 1 - 2\varepsilon\sqrt{1 - \varepsilon^2})}{2}} \end{bmatrix} \quad (5.27)$$

For $N = 4$, we have the following spectrum for 5.9,

$$\lambda = \begin{bmatrix} \frac{1}{2} - \frac{1}{2} \sqrt{1 + \frac{4(s^2-s)(4-1-2\varepsilon\sqrt{1-\varepsilon^2})}{4}} \\ \frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{4(s^2-s)(4-1-2\varepsilon\sqrt{1-\varepsilon^2})}{4}} \\ 1 \\ 1 \end{bmatrix} \quad (5.28)$$

In general, matrix 5.9 has the following spectrum,

$$\lambda = \begin{bmatrix} \frac{1}{2} - \frac{1}{2} \sqrt{1 + \frac{4(s^2-s)(N-1-2\varepsilon\sqrt{1-\varepsilon^2})}{N}} \\ \frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{4(s^2-s)(N-1-2\varepsilon\sqrt{1-\varepsilon^2})}{N}} \\ 1 \\ \cdot \\ \cdot \\ 1 \end{bmatrix} \quad (5.29)$$

In the dimension N , the eigenvalue $\lambda_2 = 1$ has the algebraic multiplicity of $N - 2$. Now we give the proof for the algebraic multiplicity of eigenvalue 1.

Proof of Multiplicity: As $\lambda_2 = 1$ is an eigenvalue of the matrix 5.9, the determinant of the matrix $H(s) - \lambda I$ is zero. We now determine the nullspace of $|H(s) - \lambda I|$. We can write the matrix as following,

$$H(s) - \lambda I = \begin{bmatrix} \frac{s-1}{N} - s\varepsilon^2 & \frac{s-1}{N} - s\varepsilon\sqrt{1-\varepsilon^2} & \frac{s-1}{N} & \cdots & \frac{s-1}{N} \\ \frac{s-1}{N} - s\varepsilon\sqrt{1-\varepsilon^2} & \frac{s-1}{N} + (\varepsilon^2 - 1)s & \frac{s-1}{N} & \cdots & \frac{s-1}{N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{s-1}{N} & \frac{s-1}{N} & \frac{s-1}{N} & \cdots & \frac{s-1}{N} \end{bmatrix} \quad (5.30)$$

After performing the elementary row operations, we get the 5.30 matrix in the following

form,

$$H(s) - \lambda I = \begin{bmatrix} \varepsilon\sqrt{(1-\varepsilon^2)} & 1-\varepsilon^2 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & \dots & 1 \end{bmatrix} \quad (5.31)$$

The matrix has rank 2 and total number of columns of the matrix is N . Now according to the nullity theorem, Null space of the matrix is $N - 2$. The geometric multiplicity of a matrix is the dimension of the null space of the matrix $H(s) - \lambda I$ [15]. Also, the algebraic multiplicity bounds the geometric multiplicity. So, the bound on algebraic multiplicity of the eigenvalue $\lambda_2 = 1$ is $\geq N - 2$. The other two distinct eigenvalues $\lambda_0 = \frac{1}{2} - \frac{1}{2}\sqrt{1 + \frac{4(s^2-s)(N-1-2\varepsilon\sqrt{1-\varepsilon^2})}{N}}$ and $\lambda_1 = \frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4(s^2-s)(N-1-2\varepsilon\sqrt{1-\varepsilon^2})}{N}}$ has algebraic multiplicity 1. The gap of the two lowest eigenvalue gives us,

$$g = \sqrt{1 + \frac{4(s^2-s)(N-1-2\varepsilon\sqrt{(1-\varepsilon^2)})}{N}} \quad (5.32)$$

The plot of two distinct eigenvalues λ_0, λ_1 is shown in Figure 5.2.

5.4 Global Adiabatic Computation

According to Claim 7 in [1], the minimum spectral gap g_{\min} of a Hamiltonian can be lower bounded by $|\langle \alpha | \beta \rangle|$ where α and β are the initial and final quantum states. In our problem, $\Phi_0(0)$ and $\Phi_0(1)$ are the initial and final superposition. So,

$$g_{\min} \geq |\langle \Phi_0(0) | \Phi_0(1) \rangle| \quad (5.33)$$

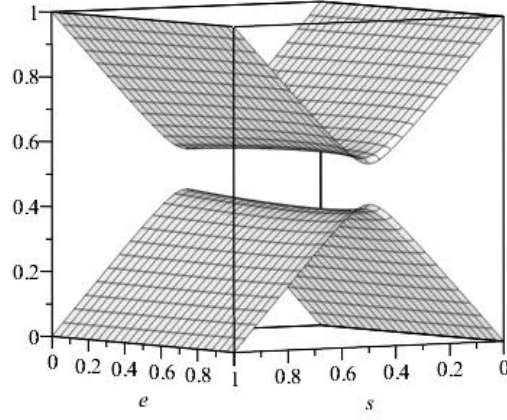


Figure 5.2: Two distinct eigenvalues of $H(s)$ for $N = 64$

The initial and final state can be written in the vector form as,

$$\Phi_0(0) = \left[\frac{1}{\sqrt{N}} \quad \frac{1}{\sqrt{N}} \quad \cdot \quad \cdot \quad \cdot \quad \frac{1}{\sqrt{N}} \right] \quad (5.34)$$

$$\Phi_0(1) = [\varepsilon \quad 0 \quad \cdot \quad \cdot \quad \sqrt{1-\varepsilon^2} \quad \cdot \quad \cdot \quad 0] \quad (5.35)$$

So, the lower bound on g_{\min} is,

$$g_{\min} \geq \frac{\sqrt{1+2\varepsilon\sqrt{1-\varepsilon^2}}}{\sqrt{N}} \quad (5.36)$$

Now, we have the general analytical Eigengap for the Hamiltonian $H(s)$ in the equation 5.32. Recall,

$$g = \sqrt{1 + \frac{4(s^2-s)(N-1-2\varepsilon\sqrt{1-\varepsilon^2})}{N}}$$

At $s = \frac{1}{2}$, the Eigengap is minimum and it gives us the quantity,

$$g_{\min} = \sqrt{1 - \frac{(N-1-2\varepsilon\sqrt{1-\varepsilon^2})}{N}}$$

$$g_{\min} = \sqrt{\frac{1+2\varepsilon\sqrt{1-\varepsilon^2}}{N}}$$
(5.37)

So, the lower bound on the minimum spectral gap is tight. We can replace both the numerator and denominator in equation 6.13. This gives us,

$$T \geq \frac{1}{\frac{1+2\varepsilon\sqrt{1-\varepsilon^2}}{N}\sigma}$$

$$T \geq \frac{N}{1+2\varepsilon\sqrt{1-\varepsilon^2}\sigma}$$
(5.38)

So, the global adaibatic evolution gives us the time complexity $O(N)$.

5.4.1 Numerical Experiments

Experimental Tool: For all the experiments, codes are implemented and run by octave-3.4.2. Some of the plots are generated using Maple 2017.0.

Experimental Setup: We generate initial and final vectors for the Hamiltonians. The final state is generated using Deutsch-Jozsa quantum algorithm. The Hamiltonian $H_1 - H_0$ is multiplied by the two normalized eigenvectors corresponding to the lowest eigenvalues of $H(s)$. Again, we compute the square of the gap of the two lowest eigenvalues. Finally, we calculate the ratio that gives us the bound on T . Figure 5.3 is the plot of T as a function of ε for a fixed $N = 8$ and at $s = 0.5$. Next we calculate the bound on evolution time T at $s = 0.5$ for different N . The results are in Table 5.1. The plot is in Figure 5.4. Another plot is in Figure 5.5 that shows T as a function of s for a fixed N and ε . Note that, we can see T reaches highest point in the plot for $s = 0.5$. Our experiments match with the theoretical estimates that T is $O(N)$.

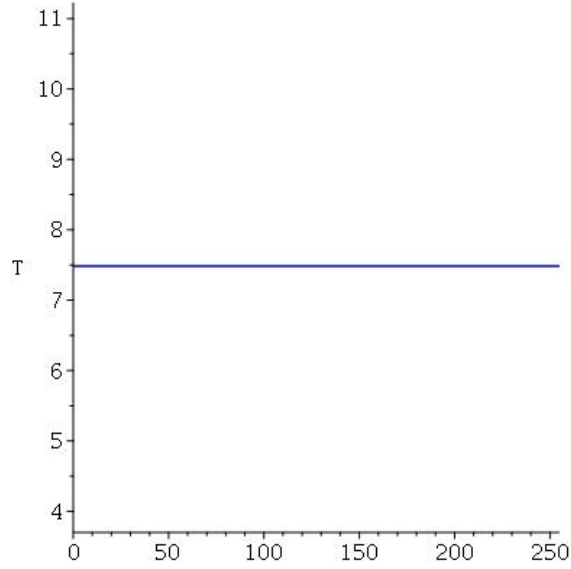


Figure 5.3: T as a function of ϵ at $N = 8, s = 0.5$

Table 5.1: T as a function of N at $s = 0.5$

n	1	2	3	4	5	6	7	8
N	2	4	8	16	32	64	128	256
T	1.4142	3.4641	7.4843	15.4919	31.4960	63.4980	127.4990	255.4995

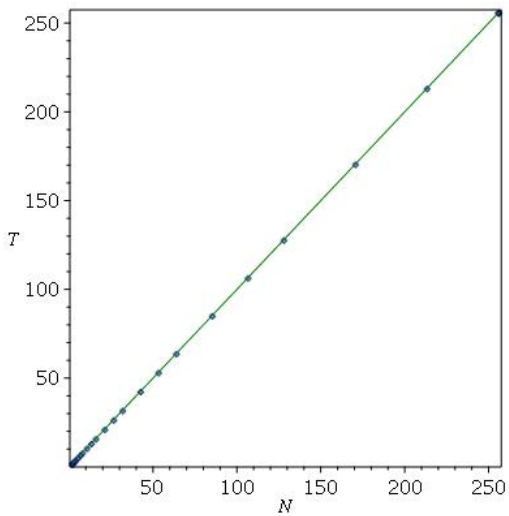
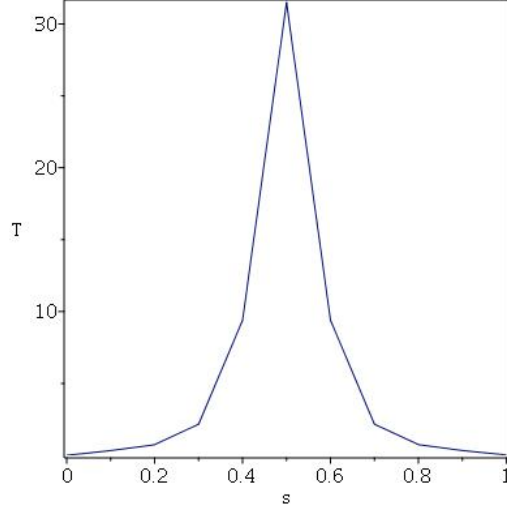


Figure 5.4: T as a function of N at $s = 0.5$

5.5 Local Adiabatic Computation

In the global computation, we considered s to be a linear function of time t . Now, we consider t divided into small intervals and apply the local adiabatic conditions on each


 Figure 5.5: T as a function of s for a fixed N, ϵ

interval. So, our new adiabatic condition is,

$$\left| \frac{ds}{dt} \right| \leq \frac{\sigma g^2}{|\lambda_{\max}|} \quad (5.39)$$

The equation can be written after replacing the g^2 and $|\lambda_{\max}|$ from the equation 5.32, 5.26,

$$\begin{aligned} \frac{ds}{dt} &\leq \sigma \left[1 + \frac{4(s^2 - s)(N - 1 - 2\epsilon\sqrt{1 - \epsilon^2})}{N} \right] \\ \Rightarrow dt &\geq \frac{ds}{\left[1 + \frac{4(s^2 - s)(N - 1 - 2\epsilon\sqrt{1 - \epsilon^2})}{N} \right] \sigma} \end{aligned} \quad (5.40)$$

Integrating over s in the interval $[0 \dots 1]$, we get,

$$T \geq - \frac{\arctan\left(\frac{2\epsilon\sqrt{1 - \epsilon^2} - N + 1}{\sqrt{4\epsilon^4 + 2N\epsilon\sqrt{1 - \epsilon^2} - 4\epsilon\sqrt{1 - \epsilon^2} - 4\epsilon^2 + N - 1}}\right)N}{\sqrt{4\epsilon^4 + 2N\epsilon\sqrt{1 - \epsilon^2} - 4\epsilon\sqrt{1 - \epsilon^2} - 4\epsilon^2 + N - 1}} \quad (5.41)$$

Taylor's expansion gives us,

$$T \geq - \frac{\operatorname{csgn}\left(\frac{2\epsilon\sqrt{1 - \epsilon^2} + \frac{1}{N} - 1}{\sqrt{\frac{4\epsilon^4}{N} - \frac{4\epsilon\sqrt{1 - \epsilon^2}}{N} - \frac{4\epsilon^2}{N} + 2\epsilon\sqrt{1 - \epsilon^2} - \frac{1}{N} + 1}}\right)\pi}{2\sqrt{2\epsilon\sqrt{1 - \epsilon^2} + 1}\sqrt{\frac{1}{N}}} - 1 + O\left(\sqrt{\frac{1}{N}}\right) \quad (5.42)$$

Now we consider the first term of the expansion in (5.42) as below,

$$T \geq - \frac{\left(\frac{\frac{2\varepsilon\sqrt{1-\varepsilon^2}}{N} + \frac{1}{N} - 1}{\sqrt{\frac{4\varepsilon^4}{N} - \frac{4\varepsilon\sqrt{1-\varepsilon^2}}{N} - \frac{4\varepsilon^2}{N} + 2\varepsilon\sqrt{1-\varepsilon^2} - \frac{1}{N} + 1}} \right) \pi}{2\sqrt{2\varepsilon\sqrt{1-\varepsilon^2} + 1}\sqrt{\frac{1}{N}}} \quad (5.43)$$

We further expand the above term using Taylor's series expansion that gives us,

$$T \geq \frac{\pi}{2(1 + 2\varepsilon\sqrt{1-\varepsilon^2})} \sqrt{N} + O\left(\sqrt{\frac{1}{N}}\right) \quad (5.44)$$

We can now estimate a lower bound on T . So, the local adiabatic evolution gives us an $O(\sqrt{N})$ lower bound on T which is a quadratic speed up.

Chapter 6

Adiabatic Computation (δ -dual Function)

6.1 Introduction

This chapter describes the adiabatic quantum computation for δ -dual functions. Recall a function $g : \{0, 1\}^n \rightarrow \{0, 1\}$ is self dual if $g(x) \oplus g(\bar{x}) = 1$ for all $x \in \{0, 1\}^n$. We consider another function $f(x) = g(x) \oplus g(\bar{x})$ which is constant at 1 if and only if $g(x)$ is self-dual. Let δ be defined as $\frac{f_0}{f_0+f_1}$ where f_0 and f_1 are the number of inputs for which the output of f is 0 and 1 respectively. The function f is δ -dual function for some δ . We give an adiabatic quantum algorithm to determine whether an arbitrary function is δ -dual or not.

6.2 Adiabatic Quantum Computation

We consider n qubits with $N = 2^n$ possible quantum states. A general Hamiltonian $H(s)$ can be defined in a convex combination of two other Hamiltonians H_0 and H_1 as,

$$H(s) = (1 - s)H_0 + sH_1 \quad (6.1)$$

Given the initial state $\Phi_0(0)$ and the final state $\Phi_0(1)$, the initial and final Hamiltonians are written as,

$$H_0 = I - |\Phi_0(0)\rangle \langle \Phi_0(0)| \quad (6.2)$$

$$H_1 = I - |\Phi_0(1)\rangle \langle \Phi_0(1)| \quad (6.3)$$

The ground state of our initial Hamiltonian is ,

$$|\Phi_0(0)\rangle = \frac{1}{\sqrt{2^n}} \sum_{i=0}^{2^n-1} |i\rangle \quad (6.4)$$

For a δ -dual function the amplitude of observing the state $|0\rangle$ is $2\delta - 1$. This follows from the analysis of Deutsch-Jozsa algorithm in Chapter 4. We assume that we observe a state other than $|0\rangle$ with the probability $2\sqrt{\delta(1-\delta)}$. We assume that the final state's superposition is $(2\delta - 1)|0\rangle + 2\sqrt{\delta(1-\delta)}|k\rangle$ for $k \in [1 \dots N - 1]$. The final state of the H_1 is,

$$|\Phi_0(1)\rangle = 2\delta - 1|0\rangle + 2\sqrt{\delta(1-\delta)}|k\rangle \quad (6.5)$$

Now, the Hamiltonian in 6.1 can be written as,

$$H = (1-s)(I - |\Phi_0(0)\rangle\langle\Phi_0(0)|) + s(t)(I - |\Phi_0(1)\rangle\langle\Phi_0(1)|) \quad (6.6)$$

Substituting equations 6.4, 6.5 in equation 6.6 gives us,

$$\begin{aligned} H = (1-s) & \left[I - \frac{1}{N} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} |i\rangle\langle j| \right] \\ & + s \left[I - ((2\delta - 1)^2 |0\rangle\langle 0| + (4(\delta - \delta^2)) |k\rangle\langle k| + (2(2\delta - 1)\sqrt{\delta - \delta^2})(|k\rangle\langle 0| + |0\rangle\langle k|)) \right] \end{aligned} \quad (6.7)$$

We can get each element of the Hamiltonian H_{mn} from the following equation,

$$H_{mn} = \langle m|H|n\rangle \quad (6.8)$$

where, $m, n \in [0, 1, \dots, N-1]$. So equation 6.7 can be written as,

$$\begin{aligned}
 H_{mn} = & \langle m | (1-s) \left[I - \frac{1}{N} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} |i\rangle \langle j| \right] |n\rangle \\
 & + \langle m | s \left[I - ((2\delta-1)^2 |0\rangle \langle 0| + (4(\delta-\delta^2)) |k\rangle \langle k| + (2(2\delta-1)\sqrt{\delta-\delta^2})(|k\rangle \langle 0| + |0\rangle \langle k|)) \right] |n\rangle
 \end{aligned} \tag{6.9}$$

6.2.1 Hamiltonian in Matrix

Equation 6.9 gives us the following entries in the Hamiltonian matrix.

- $H_{00} = 1 + \frac{s-1}{N} - s(2\delta-1)^2$
- $H_{0k} = \frac{s-1}{N} - s((2\delta-1)(2\sqrt{(\delta-\delta^2)})) \quad (k \geq 1)$
- $H_{k0} = \frac{s-1}{N} - s((2\delta-1)(2\sqrt{(\delta-\delta^2)})) \quad (k \geq 1)$
- $H_{kk} = 1 + \frac{s-1}{N} - s(4(\delta-\delta^2)) \quad (k \geq 1, k \neq m, k \neq n)$
- $H_{0m} = H_{m0} = \frac{s-1}{N} \quad (m \geq 1, m \neq k)$
- $H_{1m} = H_{m1} = \frac{s-1}{N} \quad (m \geq 1, m \neq k)$
- $H_{mm} = 1 + \frac{s-1}{N} \quad (m > 1, n > 1, k \neq m)$
- $H_{mn} = \frac{s-1}{N} \quad (m > 1, n > 1, m \neq n, m \neq k, n \neq k)$

The Hamiltonian written in the matrix form is given in 6.10. The probability of observing state $|0\rangle$ is $(2\delta-1)^2$. In our calculation, we considered the state $|k\rangle = |1\rangle$ to be observed with a probability $4\delta(1-\delta)$.

$H(s) =$

$$\begin{bmatrix} 1 + \frac{s-1}{N} - s(2\delta - 1)^2 & \frac{s-1}{N} - s((2\delta - 1)(2\sqrt{(\delta - \delta^2)})) & \frac{s-1}{N} & \dots & \frac{s-1}{N} \\ \frac{s-1}{N} - s((2\delta - 1)(2\sqrt{(\delta - \delta^2)})) & 1 + \frac{s-1}{N} - s(4\delta(1 - \delta)) & \frac{s-1}{N} & \dots & \frac{s-1}{N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{s-1}{N} & \frac{s-1}{N} & \frac{s-1}{N} & \dots & \frac{s-1}{N} + 1 \end{bmatrix} \quad (6.10)$$

A lower bound on T is given by (see Equation 5.17).

$$T \geq \frac{|\langle v_0 | H_1 - H_0 | v_1 \rangle|_{\max}}{g_{\min}^2 \sigma} \quad (6.11)$$

Using the Cauchy-Schwartz inequality [18] and induced norm [25] we can write the numerator as follows,

$$|\langle v_0 | H_1 - H_0 | v_1 \rangle|_{\max} = \{\|H_1 - H_0\|_2\}_{\max} \quad (6.12)$$

The Euclidean norm of the Hermitian operator $H_1 - H_0$ is equal to it's spectral radius [1], $|\lambda_{\max}|$. So, we can write the Equation 6.11 as,

$$T \geq \frac{|\lambda_{\max}|}{g_{\min}^2 \sigma} \quad (6.13)$$

where $\sigma \ll 1$.

6.3 Upper Bound on λ_{max}

The $H_1 - H_0$ matrix for the δ -dual functions is below,

$$H_1 - H_0 = \begin{bmatrix} \frac{1}{N} - (2\delta - 1)^2 & \frac{1}{N} - (2\delta - 1)(2\sqrt{(\delta - \delta^2)}) & \frac{1}{N} & \cdots & \frac{1}{N} \\ \frac{1}{N} - (2\delta - 1)(2\sqrt{(\delta - \delta^2)}) & \frac{1}{N} - 4(\delta - \delta^2) & \frac{1}{N} & \cdots & \frac{1}{N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{N} & \frac{1}{N} & \frac{1}{N} & \cdots & \frac{1}{N} \end{bmatrix} \quad (6.14)$$

The above matrix has the following Gershgorin disks,

- $D_1(\frac{1}{N} - (2\delta - 1)^2, \frac{N-1}{N} - (2\delta - 1)(2\sqrt{(\delta - \delta^2)}))$
- $D_2(\frac{1}{N} - 4(\delta - \delta^2), \frac{N-1}{N} - (2\delta - 1)(2\sqrt{(\delta - \delta^2)}))$
- $D_3(\frac{1}{N}, \frac{N-1}{N})$

$\frac{1}{N}$ and $\frac{N-1}{N}$ are maximum at $n = 1$ with value $\frac{1}{2}$. So, the disks can be rewritten as,

- $D_1(\frac{1}{2} - (2\delta - 1)^2, \frac{1}{2} - (2\delta - 1)(2\sqrt{(\delta - \delta^2)}))$
- $D_2(\frac{1}{2} - 4(\delta - \delta^2), \frac{1}{2} - (2\delta - 1)(2\sqrt{(\delta - \delta^2)}))$
- $D_3(\frac{1}{2}, \frac{1}{2})$

For $N \rightarrow \infty$, the matrix (6.14) has the following form,

$$H_1 - H_0 = \begin{bmatrix} -(2\delta - 1)^2 & -(2\delta - 1)(2\sqrt{(\delta - \delta^2)}) & 0 & \cdots & 0 \\ -(2\delta - 1)(2\sqrt{(\delta - \delta^2)}) & -4(\delta - \delta^2) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \quad (6.15)$$

The disks for the above matrix are,

- $D_4(-(2\delta - 1)^2, (2\delta - 1)(2\sqrt{(\delta - \delta^2)}))$

$$\bullet D_5(-4(\delta - \delta^2), (2\delta - 1)(2\sqrt{(\delta - \delta^2)}))$$

From the above five disks, we have five circles with center $(\frac{1}{2} - (2\delta - 1)^2, 0)$, $(\frac{1}{2} - 4(\delta - \delta^2), 0)$, $(\frac{1}{2}, 0)$, $(-(2\delta - 1)^2, 0)$, $(-4(\delta - \delta^2), 0)$. The Gershgorin circles have the radius $\frac{1}{2} - (2\delta - 1)(2\sqrt{(\delta - \delta^2)})$, $\frac{1}{2} - (2\delta - 1)(2\sqrt{(\delta - \delta^2)})$, $\frac{1}{2}$, $(2\delta - 1)(2\sqrt{(\delta - \delta^2)})$ and $(2\delta - 1)(2\sqrt{(\delta - \delta^2)})$ accordingly. The Hermitian matrix $H_1 - H_0$ has real Eigenvalues. We need to find the minimum and maximum point intersected by the circumference of the above circles. We find the global minimum and maximum are -1.20711 , 1.20711 . We get an upper bound of 1.20711 on the spectral radius of $H_1 - H_0$.

$$|\lambda_{\max}| \leq 1.20711 \quad (6.16)$$

If we solve the characteristics polynomial of matrix 6.15 then we get a new upper bound of 1.

$$\boxed{|\lambda_{\max} \leq 1} \quad (6.17)$$

6.4 Eigenvalues of the Hamiltonian

We derive the Eigenvalues for $N \times N$ matrix of $H(s)$. For $n = 1$, we have a 2×2 matrix with the following spectrum,

$$\lambda = \begin{bmatrix} \frac{1}{2} - \frac{1}{2} \sqrt{1 + \frac{4(s^2-s)(2-1-2(2\delta-1)2\sqrt{\delta-\delta^2})}{2}} \\ \frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{4(s^2-s)(2-1-2(2\delta-1)2\sqrt{\delta-\delta^2})}{2}} \end{bmatrix} \quad (6.18)$$

For $n = 2$, we have the following spectrum,

$$\lambda = \begin{bmatrix} \frac{1}{2} - \frac{1}{2} \sqrt{1 + \frac{4(s^2-s)(4-1-2(2\delta-1)2\sqrt{\delta-\delta^2})}{4}} \\ \frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{4(s^2-s)(4-1-2(2\delta-1)2\sqrt{\delta-\delta^2})}{4}} \\ 1 \\ 1 \end{bmatrix} \quad (6.19)$$

The Eigenvalues of the $N \times N$ ($N = 2^n$) matrix are,

$$\lambda = \begin{bmatrix} \frac{1}{2} - \frac{1}{2} \sqrt{1 + \frac{4(s^2-s)(N-1-2(2\delta-1)2\sqrt{\delta-\delta^2})}{N}} \\ \frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{4(s^2-s)(N-1-2(2\delta-1)2\sqrt{\delta-\delta^2})}{N}} \\ 1 \\ \cdot \\ \cdot \\ 1 \end{bmatrix} \quad (6.20)$$

In the spectrum of $H(s)$ in 6.20, the first two Eigenvalues λ_0 and λ_1 are distinct. The Eigenvalue $\lambda_2 = 1$ has a multiplicity of $N - 2$. Now, we give a proof for the algebraic multiplicity of Eigenvalue $\lambda_2 = 1$.

Multiplicity Proof: As $\lambda_3 = 1$ is an Eigenvalue of 6.10, then the determinant of the matrix $H(s) - \lambda I$ is zero. The matrix $H(s) - \lambda I$ will have the following form after elementary row operations assuming $\lambda = 1$.

$$H(s) - \lambda I = \begin{bmatrix} 2\delta - 1 & 2\sqrt{\delta - \delta^2} & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & \dots & 1 \end{bmatrix} \quad (6.21)$$

The matrix 6.21 has rank 2. We know from the nullity theorem for the above matrix that,

$$\text{Rank}(H(s) - \lambda I) + \text{Nullity}(H(s) - \lambda I) = \text{Column}(H(s) - \lambda I)$$

Total column of the matrix is N . So, the nullity of $H(s) - \lambda I$ is $N - 2$. The dimension of the null space gives us the geometric multiplicity of the matrix [15]. Again, geometric multiplicity \leq algebraic multiplicity. So, the algebraic multiplicity of the eigen-

value $\lambda_2 = 1$ in the matrix 6.21 $\geq N - 2$. The other two smallest Eigenvalues $\lambda_1 = \frac{1}{2} - \frac{1}{2}\sqrt{1 + \frac{4(s^2-s)(N-1-2(2\delta-1)2\sqrt{\delta-\delta^2})}{N}}$ and $\lambda_2 = \frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4(s^2-s)(N-1-2(2\delta-1)2\sqrt{\delta-\delta^2})}{N}}$ are plotted in the following Figure 6.1. The difference in the two eigenvalues will give us a mini-

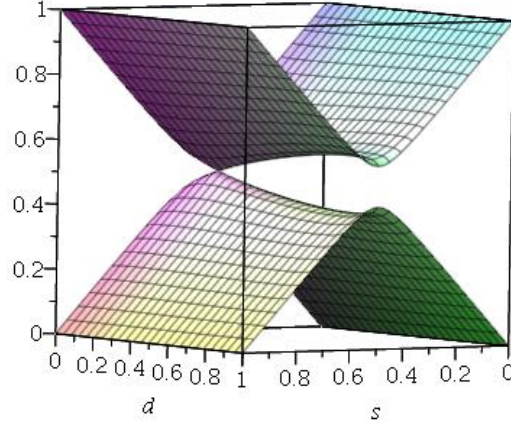


Figure 6.1: Eigenvalues of H as a function of s and δ at $N = 64$

imum gap at $s = \frac{1}{2}$. The Eigengap in general form is below,

$$g = \sqrt{1 + \frac{4(s^2 - s)(N - 1 - 2(2\delta - 1)2\sqrt{\delta - \delta^2})}{N}} \quad (6.22)$$

6.5 Global Adiabatic Computation

Claim 7 of [1] states that, the minimum spectral gap can be lower bounded and it is lower bounded by $\langle \Phi_0(0) | \Phi_1(0) \rangle$. The initial and the final superposition in vector form are,

$$\Phi_0(0) = \left[\frac{1}{\sqrt{N}} \quad \frac{1}{\sqrt{N}} \quad \frac{1}{\sqrt{N}} \quad \cdot \quad \cdot \quad \cdot \quad \frac{1}{\sqrt{N}} \right] \quad (6.23)$$

$$\Phi_0(1) = [(2\delta - 1) \quad 0 \quad \cdot \quad \cdot \quad 0 \quad 2\sqrt{\delta - \delta^2} \quad \cdot \quad \cdot \quad 0] \quad (6.24)$$

So, a lower bound on g_{\min} is,

$$g_{\min} \geq \frac{\sqrt{1 + 4(2\delta - 1)\sqrt{(\delta - \delta^2)}}}{\sqrt{N}} \quad (6.25)$$

In (6.22), the eigengap is minimum at $s = \frac{1}{2}$. It gives us that,

$$g_{\min} = \frac{\sqrt{1 + 4(2\delta - 1)\sqrt{(\delta - \delta^2)}}}{\sqrt{N}} \quad (6.26)$$

So the lower bound in 6.25 on g_{\min} is tight. Now, we replace both the numerator and denominator in the Equation (6.11) that gives us a new bound on T ,

$$T \geq \frac{N}{1 + 4(2\delta - 1)\sqrt{(\delta - \delta^2)}} \quad (6.27)$$

So, the global adiabatic evolution for δ -dual function has the time complexity $O(N)$.

6.5.1 Numerical Experiment

Experimental Tools and Setup:

Numerical experiments for the adiabatic computation of δ -dual functions are similar to the experiments performed for ϵ -balanced functions. We use Octave-3.4.2 and Maple 2017.0 as tools for running the experiments.

The input $f(x)$ is a δ -dual function. For the given function, the initial and the final superpositions are generated. Using the adiabatic theorem, we find out the lower bound on the evolution time. The first experiment shown in a tabular form (6.1) gives a lower bound on the time for different δ -dual functions. N is fixed at 8 and we calculate time T as s varies from 0.1 to 0.9. For $s = 0$ and $s = 1$, we plot the results in Figures 6.2, 6.3 to see

Table 6.1: T as a function of ϵ for $s = 0.1$ to 0.9

s	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
T	0.5833	1.1331	2.4243	5.1675	7.4833	5.1675	2.4243	1.1331	0.5833

how the lower bound on T behaves as a function of δ . The graphs 6.2, 6.3 show that, the

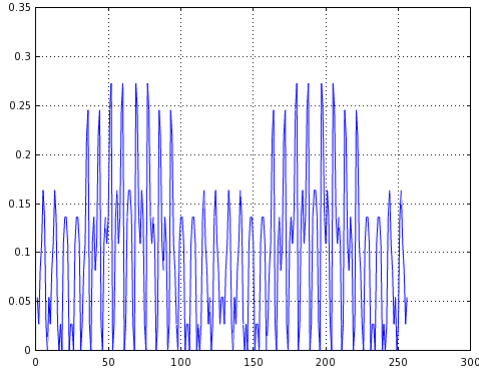


Figure 6.2: T at $s = 0$

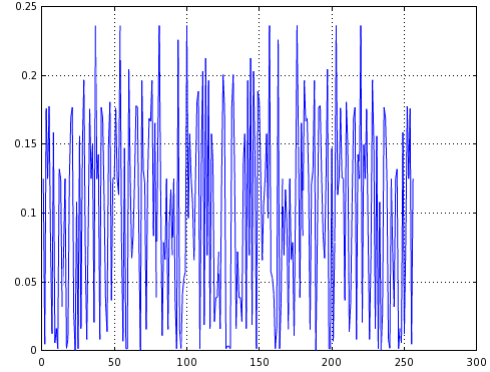


Figure 6.3: T at $s = 1$

T as a function of δ at $s = 0, 1$

global maximum on T varies for different ε -balanced functions and lower bound on T is not maximum compared to the bound for $s = \frac{1}{2}$.

Next, we plot T as a function of s for fixed N and δ (assume $\varepsilon = \delta$). The plot is similar to the figure 5.5. We show the graphical representation of T as a function of N where $s = 0.5$ and δ (let $\varepsilon = \delta$) is fixed in the figure 5.4.

6.6 Local Adiabatic Computation

We recall the adiabatic condition from 5.39 for the local adiabatic computation.

$$\left| \frac{ds}{dt} \right| \leq \frac{\sigma g^2}{|\lambda_{\max}|} \quad (6.28)$$

For the δ -dual function, we replace the numerator and denominator with the equations 6.17, 6.22 that gives us,

$$dt \geq \frac{ds}{\left[\sqrt{1 + \frac{4(s^2-s)(N-1-2(2\delta-1)2\sqrt{\delta-\delta^2})}{N}} \right]} \sigma \quad (6.29)$$

We do integration of the following term,

$$\int_0^T dt \geq \int_{s=0}^{s=1} \frac{ds}{\left[\sqrt{1 + \frac{4(s^2-s)(N-1-2(2\delta-1)2\sqrt{\delta-\delta^2})}{N}} \right]} \sigma \quad (6.30)$$

Let, $d_1 = 2\delta - 1$. We can rewrite the above equation (6.30) as below.

$$\int_0^T dt \geq \int_{s=0}^{s=1} \frac{ds}{\left[\sqrt{1 + \frac{4(s^2-s)(N-1-2d_1\sqrt{1-d_1^2})}{N}} \right]} \sigma \quad (6.31)$$

So the result of the integration gives us the following,

$$T \geq - \frac{\arctan\left(\frac{2d_1\sqrt{1-d_1^2}-N+1}{\sqrt{4\epsilon^4+2Nd_1\sqrt{1-d_1^2}-4d_1\sqrt{1-d_1^2}-4d_1^2+N-1}}\right)N}{\sqrt{4d_1^4+2Nd_1\sqrt{1-d_1^2}-4d_1\sqrt{1-d_1^2}-4d_1^2+N-1}} \quad (6.32)$$

we expand it using Taylor's series that gives us the following,

$$T \geq - \frac{\operatorname{csgn}\left(\frac{\frac{2d_1\sqrt{1-d_1^2}}{N} + \frac{1}{N} - 1}{\sqrt{\frac{4d_1^4}{N} - \frac{4d_1\sqrt{1-d_1^2}}{N} - \frac{4d_1^2}{N} + 2d_1\sqrt{1-d_1^2} - \frac{1}{N} + 1}}\right) \pi}{2\sqrt{2d_1\sqrt{1-d_1^2} + 1}\sqrt{\frac{1}{N}}} - 1 + O\left(\sqrt{\frac{1}{N}}\right) \quad (6.33)$$

Finally, we expand only the first term of the Equation (6.33) excluding csgn that gives us,

$$\begin{aligned} T &\geq \frac{\pi}{2(1+2d_1\sqrt{1-d_1^2})} \sqrt{N} + O\left(\sqrt{\frac{1}{N}}\right) \\ \Rightarrow T &\geq \frac{\pi\sqrt{N}}{2(8\delta\sqrt{\delta-\delta^2}-4(\delta-\delta^2)+1)} \end{aligned} \quad (6.34)$$

So, we can write the lower bound on T as, $O(\sqrt{N})$. For the δ -dual functions we get the lower bound of \sqrt{N} in local adiabatic computation.

Chapter 7

Conclusion

7.1 Summary of the Thesis

Parameterized query complexity of balanced function and the self-dual functions is examined as a function of the parameter ϵ and δ . The ϵ -balanced and δ -dual functions are given as input to the Deutsch-Jozsa quantum circuit. The amplitude of observing state $|0^n\rangle$ is computed. The query complexity for the probability of success $> \frac{1}{2}$ with an assumed efficient unitary operator is established in theorems 4.1, 4.2.

We also analyze the time complexity of an adiabatic quantum computer to determine ϵ -balancedness and δ -duality of an arbitrary function. For both the properties we compute bound on T for the global and the local adiabatic evolution. For the global evolution, the lower bound on the evolution time is $O(N)$. The local adiabatic evolution has a lower bound of $O(\sqrt{N})$.

7.2 Future Research Direction

The final superposition of the ϵ -balanced and δ -dual functions depends on the parameter ϵ and δ . Octave code for the final state superposition and the final superposition chart for different ϵ -balanced, δ -dual functions ($N = 8$) is given in the appendix. For the different ϵ and δ , the probability of observing a non zero state is different. We performed our computation under the assumption that, we will have a nonzero state $|k\rangle$ with the probability $1 - \epsilon^2$, $4\delta(1 - \delta)$ for the ϵ -balanced and δ -dual functions respectively. Future research is needed to determine the evolution time if the amplitude are distributed as shown in the appendix.

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Appendix A

Source Code and Table for the Superposition

A.1 Source Code

This section includes the source code of the Deutsch-Jozsa quantum algorithm implemented in octave.

Source Code:

```
prompt=' Input _n ' ;
n=input ( prompt )
prompt=' Input _f ' ;
f=input ( prompt ) # e . g f=[0 0 0 0 0 0 0 1]
D=[0:2.^n-1]
B=dec2bin ( D )
x=0
for i = 1 : 2 . ^ n
for j = 1 : 2 . ^ n
for k = 1 : n
x=x+(str2num ( B ( i , k ) ) * str2num ( B ( j , k ) ) ) ;
end
E ( i , j ) = mod ( x , 2 ) ;
x=0 ;
end
end
E
for i = 1 : 2 . ^ n
for j = 1 : 2 . ^ n
E ( i , j ) = mod ( ( f ( i ) + E ( i , j ) ) , 2 ) ;
end
end
E=E ' ;
t=0 ;
for i = 1 : 2 . ^ n
for j = 1 : 2 . ^ n
t=t + ( ( - 1 ) . ^ E ( i , j ) ) ;
```

```

end
F(i)=t/2.^n;
t=0;
end
F #print the amplitude of the states

```

A.2 Table of the Superposition

This section includes the table for the final states superposition named S of different outputs of the function f . The superposition of first 32 functions are shown for $N = 8$.

Table A.1: Final State Superposition

f	S	f	S	f	S	f	S
0	1 000⟩	0	0.75 000⟩	0	0.75 000⟩	0	0.5 000⟩
0	0 001⟩	0	0.25 001⟩	0	-0.25 001⟩	0	0 001⟩
0	0 010⟩	0	0.25 010⟩	0	0.25 010⟩	0	0.5 010⟩
0	0 011⟩	0	-0.25 011⟩	0	0.25 011⟩	0	0 011⟩
0	0 100⟩	0	0.25 100⟩	0	0.25 100⟩	0	0.5 100⟩
0	0 101⟩	0	-0.25 101⟩	0	0.25 101⟩	0	0 101⟩
0	0 110⟩	0	-0.25 110⟩	1	-0.25 110⟩	1	-0.5 110⟩
0	0 111⟩	1	0.25 111⟩	0	-0.25 111⟩	1	0 111⟩

Table A.2: Final State Superposition

f	S	f	S	f	S	f	S
0	0.75 000⟩	0	0.5 000⟩	0	0.5 000⟩	0	0.25 000⟩
0	0.25 001⟩	0	0.5 001⟩	0	0 001⟩	0	0.25 001⟩
0	-0.25 010⟩	0	0 010⟩	0	0 010⟩	0	0.25 010⟩
0	0.25 011⟩	0	0 011⟩	0	0.5 011⟩	0	0.25 011⟩
0	0.25 100⟩	0	0.5 100⟩	0	0.5 100⟩	0	0.75 100⟩
1	-0.25 101⟩	1	-0.5 101⟩	1	0 101⟩	1	-0.25 101⟩
0	0.25 110⟩	0	0 110⟩	1	0 110⟩	1	-0.25 110⟩
0	-0.25 111⟩	1	0 111⟩	0	-0.5 111⟩	1	-0.25 111⟩

Table A.3: Final State Superposition

f	S	f	S	f	S	f	S
0	$0.75 000\rangle$	0	$0.5 000\rangle$	0	$0.5 000\rangle$	0	$0.25 000\rangle$
0	$-0.25 001\rangle$	0	$0 001\rangle$	0	$-0.5 001\rangle$	0	$-0.25 001\rangle$
0	$-0.25 010\rangle$	0	$0 010\rangle$	0	$0 010\rangle$	0	$0.25 010\rangle$
0	$-0.25 011\rangle$	0	$-0.5 011\rangle$	0	$0 011\rangle$	0	$-0.25 011\rangle$
1	$0.25 100\rangle$	1	$0.5 100\rangle$	1	$0.5 100\rangle$	1	$0.75 100\rangle$
0	$0.25 101\rangle$	0	$0 101\rangle$	0	$0.5 101\rangle$	0	$0.25 101\rangle$
0	$0.25 110\rangle$	0	$0 110\rangle$	1	$0 110\rangle$	1	$-0.25 110\rangle$
0	$0.25 111\rangle$	1	$0.5 111\rangle$	0	$0 111\rangle$	1	$0.25 111\rangle$

Table A.4: Final State Superposition

f	S	f	S	f	S	f	S
0	$0.5 000\rangle$	0	$0.25 000\rangle$	0	$0.25 000\rangle$	0	$0 000\rangle$
0	$0 001\rangle$	0	$0.25 001\rangle$	0	$-0.25 001\rangle$	0	$0 001\rangle$
0	$-0.5 010\rangle$	0	$-0.25 010\rangle$	0	$-0.25 010\rangle$	0	$0 010\rangle$
0	$0 011\rangle$	0	$-0.25 011\rangle$	0	$0.25 011\rangle$	0	$0 011\rangle$
1	$0.5 100\rangle$	1	$0.75 100\rangle$	1	$0.75 100\rangle$	1	$1 100\rangle$
1	$0 101\rangle$	1	$-0.25 101\rangle$	1	$0.25 101\rangle$	1	$0 101\rangle$
0	$0.5 110\rangle$	0	$0.25 110\rangle$	1	$0.25 110\rangle$	1	$0 110\rangle$
0	$0 111\rangle$	1	$0.25 111\rangle$	0	$-0.25 111\rangle$	1	$0 111\rangle$

Table A.5: Final State Superposition

f	S	f	S	f	S	f	S
0	$0.75 000\rangle$	0	$0.5 000\rangle$	0	$0.5 000\rangle$	0	$0.25 000\rangle$
0	$0.25 001\rangle$	0	$0.5 001\rangle$	0	$0 001\rangle$	0	$0.25 001\rangle$
0	$0.25 010\rangle$	0	$0.5 010\rangle$	0	$0.5 010\rangle$	0	$0.75 010\rangle$
1	$-0.25 011\rangle$	1	$-0.5 011\rangle$	1	$0 011\rangle$	1	$-0.25 011\rangle$
0	$-0.25 100\rangle$	0	$0 100\rangle$	0	$0 100\rangle$	0	$0.25 100\rangle$
0	$0.25 101\rangle$	0	$0 101\rangle$	0	$0.5 101\rangle$	0	$0.25 101\rangle$
0	$0.25 110\rangle$	0	$0 110\rangle$	1	$0 110\rangle$	1	$-0.25 110\rangle$
0	$-0.25 111\rangle$	1	$0 111\rangle$	0	$-0.5 111\rangle$	1	$-0.25 111\rangle$

Table A.6: Final State Superposition

f	S	f	S	f	S	f	S
0	$0.5 000\rangle$	0	$0.25 000\rangle$	0	$0.25 000\rangle$	0	$0 000\rangle$
0	$0.5 001\rangle$	0	$0.75 001\rangle$	0	$0.25 001\rangle$	0	$0.5 001\rangle$
0	$0 010\rangle$	0	$0.25 010\rangle$	0	$0.25 010\rangle$	0	$0.5 010\rangle$
1	$0 011\rangle$	1	$-0.25 011\rangle$	1	$0.25 011\rangle$	1	$0 011\rangle$
0	$0 100\rangle$	0	$0.25 100\rangle$	0	$0.25 100\rangle$	0	$0.5 100\rangle$
1	$0 101\rangle$	1	$-0.25 101\rangle$	1	$0.25 101\rangle$	1	$0 101\rangle$
0	$0.5 110\rangle$	0	$0.25 110\rangle$	1	$0.25 110\rangle$	1	$0 110\rangle$
0	$-0.5 111\rangle$	1	$-0.25 111\rangle$	0	$-0.75 111\rangle$	1	$-0.5 111\rangle$

Table A.7: Final State Superposition

f	S	f	S	f	S	f	S
0	$0.5 000\rangle$	0	$0.25 000\rangle$	0	$0.25 000\rangle$	0	$0 000\rangle$
0	$0 001\rangle$	0	$0.25 001\rangle$	0	$-0.25 001\rangle$	0	$0 001\rangle$
0	$0 010\rangle$	0	$0.25 010\rangle$	0	$0.25 010\rangle$	0	$0.5 010\rangle$
1	$-0.5 011\rangle$	1	$-0.75 011\rangle$	1	$-0.25 011\rangle$	1	$-0.5 011\rangle$
1	$0 100\rangle$	1	$0.25 100\rangle$	1	$0.25 100\rangle$	1	$0.5 100\rangle$
0	$0.5 101\rangle$	0	$0.25 101\rangle$	0	$0.75 101\rangle$	0	$0.5 101\rangle$
0	$0.5 110\rangle$	0	$0.25 110\rangle$	1	$0.25 110\rangle$	1	$0 110\rangle$
0	$0 111\rangle$	1	$0.25 111\rangle$	0	$-0.25 111\rangle$	1	$0 111\rangle$

Table A.8: Final State Superposition

f	S	f	S	f	S	f	S
0	$0.25 000\rangle$	0	$0 000\rangle$	0	$0 000\rangle$	0	$-0.25 000\rangle$
0	$0.25 001\rangle$	0	$0.5 001\rangle$	0	$0 001\rangle$	0	$0.25 001\rangle$
0	$-0.25 010\rangle$	0	$0 010\rangle$	0	$0 010\rangle$	0	$0.25 010\rangle$
1	$-0.25 011\rangle$	1	$-0.5 011\rangle$	1	$0 011\rangle$	1	$-0.25 011\rangle$
1	$0.25 100\rangle$	1	$0.5 100\rangle$	1	$0.5 100\rangle$	1	$0.75 100\rangle$
1	$0.25 101\rangle$	1	$0 101\rangle$	1	$0.5 101\rangle$	1	$0.25 101\rangle$
0	$0.75 110\rangle$	0	$0.5 110\rangle$	1	$0.5 110\rangle$	1	$0.25 110\rangle$
0	$-0.25 111\rangle$	1	$0 111\rangle$	0	$-0.5 111\rangle$	1	$-0.25 111\rangle$