EXPLICIT RESULTS ON PRIMES.

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Abstract

Let p be a prime number. In this thesis we prove the following theorems.

Theorem -1.0.1. *Let $x_0 \geq 4 \cdot 10^{18}$ be a fixed constant and let $x > x_0$. Then there exists at least one prime p such that $(1 - \Delta^{-1})x < p < x$, where Δ is a constant depending on x_0 and is given in Table 2.2.*

Theorem -1.0.1 improves on the 2003 paper of Ramaré and Saouter [67]. This work has been refereed and is to appear in the scholarly journal Integers [38]. Theorem -1.0.1 allows one to deduce the following verification of the odd Goldbach conjecture:

Corollary -1.0.2. *Every odd number larger than 5 and smaller than*

$$1\ 966\ 196\ 911 \times 4 \cdot 10^{18} = 7.864 \ldots \cdot 10^{27}$$

is the sum of at most three primes.

The second part of this thesis investigates the error term in the prime number theorem in arithmetic progressions. Let a and q be positive relatively prime integers. Let

$$\psi(x; q, a) = \sum_{\substack{n \leq x \\ n \equiv a (\mathrm{mod}\, q) \\ n = p^k \\ k \geq 1}} \log n$$

and

$$E(x; q, a) = \left| \frac{\psi(x; q, a) - \frac{x}{\varphi(q)}}{\frac{x}{\varphi(q)}} \right|,$$

be the error term for the prime number theorem in arithmetic progressions. We prove

Theorem -1.0.3. *For b a fixed positive constant and $x \geq 10^b$, there exists $\epsilon_{q,b} > 0$ such that $E(x; q, a) \leq \epsilon_{q,b}$ where $\epsilon_{q,b}$ is given explicitly in Theorem 3.7.8 and is computed in*

in Table 3.10.

Theorem -1.0.3 improves the 1996 work of Ramaré and Rumely [66], which was the article of reference for explicit bounds on $E(x; q, a)$ for nearly 20 years.

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Contents

List of Tables

Chapter 0

Notation

We introduce some basic notations and definitions for reference.

- Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ be the set of natural numbers, integers, real numbers and complex numbers respectively.
- In this thesis p is used to denote a prime, q, a, n are integers, x is a real number and $s = \sigma + it$ is a complex number.
- $\gamma_0 = 0.5772156\ldots$ is the Euler-Mascheroni constant.
- The notation $n \equiv a \pmod{q}$ means there exists $k \in \mathbb{Z}$ such that $n = kq + a$.
- $(a, q) = \gcd(a, q)$ is the greatest common divisor of a and q.
- The Euler phi function $\varphi(q)$ is given by $\varphi(q) = \#\{n \in \mathbb{N} \mid n \leq q \text{ and } (n, q) = 1\}$.
- We say f is *asymptotic* to g, and write $f(x) \sim g(x)$, to mean $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$.
- We say f is big oh of g, and write $f(x) = O(g(x))$ or $f(x) \ll g(x)$ to mean that there is a constant C such that for all sufficiently large x, $|f(x)| \leq Cg(x)$.
- We say f is little oh of g, write $f(x) = o(g(x))$, to mean $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$.
- An *arithmetic function* is any function $f : \mathbb{N} \to \mathbb{C}$.

- The von Mangoldt function $\Lambda(n)$ is an arithmetic function given by

$$\Lambda(n) = \begin{cases} \log(p) & \text{if } n = p^k_, p \text{ prime}, k \in \mathbb{N} \\ 0 & \text{else.} \end{cases} \tag{0.0.1}$$

- We consider the following prime counting functions:

$$\pi(x) = \sum_{p \leq x} 1, \ \theta(x) = \sum_{p \leq x} \log(p), \quad \text{and} \quad \psi(x) = \sum_{n \leq x} \Lambda(n).$$

- More generally, for $(a, q) = 1$, we have

$$\pi(x, q, a) = \sum_{\substack{p \leq x \\ p \equiv a (\text{mod } q)}} 1, \ \theta(x, q, a) = \sum_{\substack{p \leq x \\ p \equiv a (\text{mod } q)}} \log(p) \ \text{ and } \ \psi(x, q, a) = \sum_{\substack{n \leq x \\ n \equiv a (\text{mod } q)}} \Lambda(n).$$

- The *summatory function* of an arithmetic function f is given by

$$F(x) = \sum_{n \leq x} f(n). \tag{0.0.2}$$

For instance, the summatory function for the sequence $p(n)$ defined by $p(n) = 1$ if n is prime and $p(n) = 0$ otherwise is $\pi(x)$. The summatory function of $\Lambda(n)$ is $\psi(x)$.

- A *Dirichlet series* is a function of a complex variable s given by

$$g(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

for $\{a_n\}_{n \in \mathbb{N}}$ a complex valued sequence.

- Any Dirichlet series $g(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ has an *abscissa of convergence* σ_c, where σ_c has the property that $g(s)$ is convergent for all $\sigma > \sigma_c$ and divergent for $\sigma < \sigma_c$.

- If f is an arithmetic function we have a *Dirichlet series associated to* f given by

$$g(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

Essential examples of Dirichlet series are the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

and the Dirichlet L-functions

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

where χ is defined below.

- Let G be a finite group. A *character* of $(G,\cdot)$ is a homomorphism

$$\chi : G \longrightarrow \mathbb{C}^*$$

such that for any $g_1, g_2 \in G$ we have

$$\chi(g_1 g_2) = \chi(g_1)\chi(g_2).$$

- Let $q \in \mathbb{N}$. A *Dirichlet character* is a group character defined on $((\mathbb{Z}/q\mathbb{Z})^*, \cdot)$. We call q the modulus of χ.

- For χ a Dirichlet character modulo q, the superscript $*$, written on χ^*, denotes the *primitive character* which induces χ. It has conductor q^* and $q^*|q$.

- χ_0 denotes the *principal character* modulo q and is defined by

$$\chi_0(n) = \begin{cases} 1 & \text{if } (n,q) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

- $\mathfrak{a}$ represents the *parity* of the character χ, given by

$$\mathfrak{a} = \frac{1-\chi(-1)}{2} = \begin{cases} 0 & \text{if } \chi(-1) = 1, \\ 1 & \text{if } \chi(-1) = -1. \end{cases}$$

- Let f be a complex valued locally integrable function on $(0, \infty)$. The *Mellin transform* of f, is given by

$$M_f(s) = \int_0^\infty x^{s-1} f(x) dx, \tag{0.0.3}$$

which converges in a vertical strip depending on the analytic structure of $f(x)$ as $x \to 0^+$ and $x \to \infty$. If we suppose that

$$f(x) = \begin{cases} O(x^{-a-\epsilon}) & \text{as } x \to 0^+, \\ O(x^{-b-\epsilon}) & \text{as } x \to \infty, \end{cases}$$

where $\epsilon > 0$ and $a < b$, then the integral (1.2.1) converges absolutely in the strip $a < \mathfrak{Re}(s) < b$. The *inverse Mellin transform* is

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} M_f(s) x^{-s} ds,$$

where $a < c < b$.

Let $q \in \mathbb{N}$, $q \geq 3$ and χ a Dirichlet character modulo q. The following are related to the zeros of $\zeta(s)$ and $L(s, \chi)$:

- For $\beta, \gamma \in \mathbb{R}$, $\rho = \beta + i\gamma$ denotes a zero of either $\zeta(s)$ or $L(s, \chi)$,
- Z is defined to be the set of non-trivial zeros for $\zeta(s)$,

$$Z = \{\rho = \beta + i\gamma \mid \zeta(\rho) = 0, 0 < \beta < 1\}. \tag{0.0.4}$$

- $Z(\chi)$is defined to be the set of non-trivial zeros for $L(s,\chi)$,

$$Z(\chi) = \{\rho = \beta + i\gamma \mid L(\rho,\chi) = 0, 0 < \beta < 1\}. \tag{0.0.5}$$

- Let $T > 0$.

$$N(T) = \#\{\rho = \beta + i\gamma \mid \zeta(\rho) = 0,\ 0 < \beta < 1 \ \text{ and } 0 < \gamma \leq T\}. \tag{0.0.6}$$

- Let $0 < \sigma < 1$, $T > 0$.

$$N(\sigma, T) = \#\{\rho = \beta + i\gamma \mid \zeta(\rho) = 0,\ \sigma < \beta < 1 \ \text{ and } 0 < \gamma \leq T\}. \tag{0.0.7}$$

- Let $T \geq 0$.

$$N(T,\chi) = \#\{\rho = \beta + i\gamma \mid L(\rho,\chi) = 0,\ 0 < \beta < 1 \ \text{ and } \ |\gamma| \leq T\}. \tag{0.0.8}$$

- Let $t > 0$. We introduce

$$\mathbf{N}(t,q) = \sum_{\substack{r|q \\ r\neq 1}} \sum_{\chi^* (\mathrm{mod}\ r)} N(t,\chi).$$

- Let $t > 0$. We introduce the sums over the zeros of $L(s,\chi)$

$$S_0(t,\chi) = \sum_{\substack{1<|\gamma|\leq t \\ \rho\in Z(\chi)}} \frac{1}{|\gamma|}, \quad \mathbf{S}_0(t,q) = \sum_{\substack{r|q \\ r\neq 1}} \sum_{\chi^* (\mathrm{mod}\ r)} S_0(t,\chi).$$

- For $t > 0$ we introduce the notation GRH(t) to mean that the generalized Riemann hypothesis has been verified for all $\rho \in Z(\chi)$ such that $|\mathfrak{Im}(\rho)| \leq t$.

We make use of the following special functions:

- Let $\nu \geq 1$ be an integer, $z > 0, w \geq 0$ be real numbers, then the *modified Bessel* *function of the second kind* with parameter ν is given by

$$K_\nu(z,w) = \frac{1}{2}\int_w^\infty u^{\nu-1}\exp\left(-\frac{z}{2}(u+\frac{1}{u})\right)du.$$

- Let $s \in \mathbb{C}$, the *Gamma function* is given by

$$\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}dx.$$

- Let $a, b, x \in \mathbb{R}$, such that $a, b > 0$ and $0 \leq x \leq 1$. The *regularized incomplete beta function* is given by [7, p. 142, Eqn 5.12.1]

$$I_x(a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^x t^{a-1}(1-t)^{b-1}dt.$$

- Let $x > 0$ be a real number, then the *complementary error function* is given by

$$\text{erfc}(x) = \int_x^\infty e^{-t^2}dt.$$

Chapter 1

Introduction

1.1 Prime Counting Functions

It was proven by Euclid that there are an infinite number of primes. Hence a natural question to ask about them is how many primes are there up to some finite bound x. To answer questions like this we define the following prime counting functions,

$$\pi(x) = \#\{p \leq x \mid p \text{ is a prime}\} = \sum_{p \leq x} 1,$$
$$\theta(x) = \sum_{\substack{p \leq x \\ p \text{ prime}}} \log(p), \quad \text{and} \quad \psi(x) = \sum_{n \leq x} \Lambda(n) = \sum_{\substack{p^k \leq x \\ k \in \mathbb{N} \\ p \text{ prime}}} \log(p), \tag{1.1.1}$$

where $\Lambda(n)$ is the von Mangoldt function defined by

$$\Lambda(n) = \begin{cases} \log(p) & \text{if } n = p^k_, p \text{ prime, } k \in \mathbb{N} \\ 0 & \text{else.} \end{cases} \tag{1.1.2}$$

In 1808, Legendre conjectured that

$$\pi(x) \sim \frac{x}{\log(x) + A}, \quad \text{where} \quad A = -1.08366.$$

Legendre made this guess based on numerical computations. In 1849 Gauss also estimated that within a neighbourhood of the value x the density of primes should be

$\frac{1}{\log(x)}$. Based on this one expects that

$$\sum_{p\le x} f(p) \sim \int_2^x \frac{f(y)}{\log(y)} dy.$$

In particular, this predicts

$$\pi(x) = \sum_{p\le x} 1 \sim \mathrm{li}(x) = \int_2^x \frac{dy}{\log(y)} \sim \frac{x}{\log x}. \tag{1.1.3}$$

It may be shown that the following statements are equivalent:

$$\pi(x) \sim \frac{x}{\log(x)},\ \theta(x) \sim x,\ \text{and}\ \psi(x) \sim x.$$

Consequently, in order to prove (1.1.3) it suffices to show $\psi(x) \sim x$. Nearly 100 years later, in 1896 Hadamard and de la Vallée Poussin independently showed that

$$\psi(x) \sim x,$$

and Legendre's conjecture became known as the Prime Number Theorem. Throughout the rest of this thesis we abbreviate Prime Number Theorem to PNT. The method used to show $\psi(x) \sim x$ encouraged a new way of looking at the problem of estimating the size of these prime counting functions. Through the proof we see that it is possible to get a more precise expression for the size of $\psi(x)$:

$$\psi(x) = x + E_0(x), \tag{1.1.4}$$

where,

$$E_0(x) = o(x) \quad \text{as} \quad x \to \infty.$$

A generalization of this idea allows us to study primes in arithmetic progressions. Primes in arithmetic progressions are of the form $p \equiv a \pmod q$ with $(a, q) = 1$. It was conjectured that there are infinitely many such primes. This was proven by Dirichlet in 1837 through the introduction of what are now known as Dirichlet characters and L-functions.

Theorem 1.1.1 (Dirichlet, 1837)**.** *Let a, q be two relatively prime integers. There exist infinitely many primes p such that $p = nq + a$, for some $n \in \mathbb{Z}$.*

We may ask again, how many primes of this form occur below some finite bound x. To study this question we define the functions

$$\pi(x; q, a) = \#\{p \leq x \mid p \text{ is a prime and } p = nq + a, \text{ for some } n \in \mathbb{N}\} = \sum_{\substack{p \leq x \\ p \equiv a (\text{mod } q) \\ p \text{ prime}}} 1,$$

$$\theta(x; q, a) = \sum_{\substack{p \leq x \\ p \equiv a (\text{mod } q) \\ p \text{ prime}}} \log(p) \quad \text{and} \quad \psi(x; q, a) = \sum_{\substack{n \leq x \\ n \equiv a (\text{mod } q)}} \Lambda(n), \tag{1.1.5}$$

with $\Lambda(n)$ as in (1.1.2).

We expect that the primes $p = nq + a$ are distributed evenly amongst the $\varphi(q)$ possible values of a, which leads one to believe that

$$\psi(x; q, a) \sim \frac{\psi(x)}{\varphi(q)}.$$

In the case where q is a fixed value, the prime number theorem in arithmetic progressions (PNT in AP) asserts that

$$\pi(x; q, a) \sim \frac{x}{\varphi(q) \log(x)}.$$

Note that when $q = 1$, we recover the original PNT. These prime counting functions defined in (1.1.5) have the following relationship with one another:

$$\pi(x; q, a) \sim \frac{x}{\varphi(q) \log(x)} \iff \theta(x; q, a) \sim \frac{x}{\varphi(q)} \iff \psi(x; q, a) \sim \frac{x}{\varphi(q)}.$$

The proof for PNT in AP mimics closely the proof for PNT. Hence, as with the primes, we may provide a more precise expression for the size of $\psi(x; q, a)$ and rewrite the theorem as

$$\psi(x;q,a) = \frac{x}{\varphi(q)} + E_0(x;q,a), \tag{1.1.6}$$

where

$$E_0(x;q,a) = o(x) \quad \text{as} \quad x \to \infty.$$

1.2 Analytic Tools

The first proof of the prime number theorem used complex analysis. Riemann invented a groundbreaking method which allows one to study prime numbers via complex functions and complex analysis. In this section we provide some of the necessary tools for making the transition. We begin with the definition of a Dirichlet series.

Definition 1.2.1. *A Dirichlet series is a function of a complex variable s given by*

$$g(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

with $\{a_n\}_{n\in\mathbb{N}}$ a complex valued sequence.

Each Dirichlet series has an abscissa of convergence σ_c. A Dirichlet series can be represented using Mellin transforms.

Definition 1.2.2. *[39, Ch 1] Let f be a complex valued locally integrable function on $(0,\infty)$. The Mellin transform of f, is given by*

$$M_f(s) = \int_0^{\infty} x^{s-1} f(x)dx, \tag{1.2.1}$$

which converges in a vertical strip depending on the analytic structure of $f(x)$ as $x \to 0^+$ and $x \to \infty$. If we suppose that

$$f(x) = \begin{cases} O(x^{-a-\epsilon}) & \text{as } x \to 0^+, \\ O(x^{-b-\epsilon}) & \text{as } x \to \infty, \end{cases}$$

where $\epsilon > 0$ and $a < b$, then the integral (1.2.1) converges absolutely in the strip

$a < \mathfrak{Re}(s) < b$. *The inverse Mellin transform is*

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} M_f(s) x^{-s} ds,$$

where $a < c < b$.

As a consequence of [59, Theorem 1.3] we can write the Dirichlet series associated to $\{a_n\}_{n\in\mathbb{N}}$ as an integral:

$$g(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = s \int_1^{\infty} A(x) x^{-s-1} dx,$$

where $\mathfrak{Re}(s) > \sigma_c$ and $A(x)$ is the summatory function of $a(n)$:

$$A(x) = \sum_{n\leq x} a(n).$$

Similarly, Perron's formula allows us to write a finite sum $\sum_{n\leq x} a_n$ in terms of an integral of the Dirichlet series, $g(s) = \sum_{n\geq 1} \frac{a_n}{n^s}$, associated to the finite sum.

Theorem 1.2.1 (Perron's formula)**.** *[59, Theorem 5.1] Let* $g(s) = \sum_{n\geq 1} \frac{a_n}{n^s}$ *with abscissa of convergence* σ_c*. If* $\sigma_0 > \max(0, \sigma_c)$ *and* $x > 0$*, then*

$$\sideset{}{'}\sum_{n\leq x} a_n = \frac{1}{2\pi i} \int_{\sigma_0-i\infty}^{\sigma_0+i\infty} g(s) \frac{x^s}{s} ds.$$

Here the prime on the summation indicates that if x *is an integer,* a_x *is replaced with* $a_x/2$*.*

Hence we see from Perron's formula that for $x \notin \mathbb{Z}$

$$A(x) = \frac{1}{2\pi i} \int_{\sigma_0-i\infty}^{\sigma_0+i\infty} g(s) \frac{x^s}{s} ds.$$

If $a_n = 1$ for all $n \in \mathbb{N}$, then the Dirichlet series associated to $\{a_n\}_{n\in\mathbb{N}}$ gives the

Riemann zeta-function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Note that $\zeta(s)$ has abscissa $\sigma_c = 1$. Using the fact that each integer can be written in a unique way as the product of powers of primes, $\zeta(s)$ can be expressed as the Euler product

$$\zeta(s) = \prod_{\substack{p \\ p \text{ prime}}} \left(1 - \frac{1}{p^s}\right)^{-1} \text{ for } \mathfrak{Re}(s) > 1.$$

The Euler product shows the relationship between the primes and the complex function $\zeta(s)$. Euler used this expression for $\zeta(s)$ to prove that there exist infinitely many primes, by noting that $\lim_{s \to 1^+} \zeta(s)$ diverges. In 1837, Dirichlet used special arithmetic functions that characterize the congruence condition, $p \equiv a (\bmod q)$, which allowed him to express the primes in arithmetic progressions in terms of an Euler product. Using these arithmetic functions he proved that there exist infinitely many primes $p \equiv a \pmod q$.

1.3 Dirichlet Characters and L-functions.

Definition. *Let $(G, \cdot)$ be a finite group. A character of $(G, \cdot)$ is a homomorphism*

$$\chi : G \longrightarrow \mathbb{C}^*$$

such that for any $g_1, g_2 \in G$ we have

$$\chi(g_1 g_2) = \chi(g_1)\chi(g_2).$$

Let $q \in \mathbb{N}$. A Dirichlet character is a group character defined on $((\mathbb{Z}/q\mathbb{Z})^*, \cdot)$. We call q the modulus of χ.

Let $\widehat{(\mathbb{Z}/q\mathbb{Z})^*}$ be the set of all characters χ for the group $(\mathbb{Z}/q\mathbb{Z})^*$. Then the following hold. We denote the principal character as χ_0, whose value is 1 for all $n \in (\mathbb{Z}/q\mathbb{Z})^*$.

For χ a Dirichlet character we have $\chi(1) = 1$ and multiplication of two characters is given by $\chi_1\chi_2(n) = \chi_1(n)\chi_2(n)$. We note that for all $\chi \in \widehat{(\mathbb{Z}/q\mathbb{Z})^*}$ we have $\chi_0\chi(n) = \chi\chi_0(n) = \chi(n)$, so that χ_0 is the multiplicative identity for $\widehat{(\mathbb{Z}/q\mathbb{Z})^*}$.

Let $\overline{\chi}(n) = \overline{\chi(n)}$, where $\overline{\chi(n)}$ means the complex conjugate of $\chi(n)$, we have $\chi\overline{\chi}(n) = \chi_0(n) = 1$.

We have $\widehat{(\mathbb{Z}/q\mathbb{Z})^*}$ is a multiplicative group which is isomorphic to $(\mathbb{Z}/q\mathbb{Z})^*$, hence $\left|\widehat{(\mathbb{Z}/q\mathbb{Z})^*}\right| = \varphi(q)$.

Let $n \in \mathbb{N}$ such that $(n, q) = 1$. Since $n^{\varphi(q)} \equiv 1 \pmod q$, then $\chi(n^{\varphi(q)}) = 1$ and consequently $\chi(n)^{\varphi(q)} = 1$. From this we see that the values taken by characters are precisely the $\varphi(q)$-th roots of unity.

We now state the orthogonality relation for Dirichlet characters which characterizes integers congruent to a modulo q:

Lemma 1.3.1. *[59, Corollary 4.5] For a fixed $a \in (\mathbb{Z}/q\mathbb{Z})^*$ we have*

$$\frac{1}{\varphi(q)} \sum_{\chi (\mathrm{mod}\ q)} \overline{\chi}(a)\chi(n) = \begin{cases} 1 \text{ if } n \equiv a (\mathrm{mod}\ q), \\ 0 \text{ otherwise.} \end{cases}$$

We now extend the definition of Dirichlet characters from integers co-prime to q to all integers:

$$\chi(n) = \begin{cases} \chi(n(\mathrm{mod}\ q)) & \text{if } (n, q) = 1, \\ 0 & \text{else.} \end{cases}$$

Lemma 1.3.1 allows us to encode the condition $n \equiv a \pmod q$ in the definition (1.1.5) of $\psi(x; q, a)$:

$$\psi(x; q, a) = \sum_{n \le x} \frac{1}{\varphi(q)} \sum_{\chi (\mathrm{mod}\ q)} \overline{\chi}(a)\chi(n)\Lambda(n) = \frac{1}{\varphi(q)} \sum_{\chi (\mathrm{mod}\ q)} \overline{\chi}(a) \sum_{n \le x} \chi(n)\Lambda(n).$$

Suppose that $d \mid q$ and that χ^* is a character for the group $(\mathbb{Z}/d\mathbb{Z})^*$. Then set

$$\chi(n) = \begin{cases} \chi^*(n) & \text{if } (n, q) = 1, \\ 0 & \text{else.} \end{cases}$$

It is easy to see that χ is completely multiplicative and has period q. Then by [59, Theorem 4.7] we have that χ is a Dirichlet character modulo q. In this situation we say that χ^* induces χ.

Lemma 1.3.2. *[59, Lemma 9.1] Let χ be a character modulo q. We say that d is a quasi-period of χ if $\chi(m) = \chi(n)$ whenever $(mn, q) = 1$ and $m \equiv n \pmod{d}$. The least quasi-period of χ is called the conductor of χ and is a divisor of q.*

Definition. *We say that a character χ modulo q is primitive if its conductor is q.*

For each Dirichlet character χ modulo q, the associated Dirichlet series, also called a Dirichlet L-function, is given by:

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \quad \text{for } \mathfrak{Re}(s) > \sigma_c.$$

If $\chi = \chi_0$ then $L(s, \chi)$ has an abscissa of convergence $\sigma_c = 1$ and if $\chi \neq \chi_0$ then $\sigma_c = 0$. We also obtain an Euler product representation

$$L(s, \chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}, \quad \text{for } \mathfrak{Re}(s) > \sigma_c.$$

Showing that $L(1, \chi) \neq 0$ for each character is a key step in the proof of Dirichlet's theorem (Theorem 1.1.1). We refer the reader to [13, Chapter 1, p. 5-10.] for reference. Considering the Euler product of $L(s, \chi)$ we may write the Dirichlet series associated to χ in terms of χ^*

$$L(s, \chi) = L(s, \chi^*) \prod_{p|q} \left(1 - \frac{\chi^*(p)}{p^s}\right).$$

Similarly for $L(s, \chi_0)$ we have

$$L(s, \chi_0) = \zeta(s) \prod_{p|q} \left(1 - \frac{1}{p^s}\right),$$

and hence, in some sense we see that $\zeta(s)$ induces χ_0 for any q.

1.4 Zeros of the Riemann Zeta Function.

In 1859, in his famous memoir, Riemann showed that $\zeta(s)$ has a simple pole at $s = 1$. He also proved that $\zeta(s)$ admits a meromorphic continuation to the complex plane. To do this he established the following functional equation

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s). \tag{1.4.1}$$

Since $\Gamma(s/2)$ has simple poles at $s = -2, -4, -6, \ldots$ and the right hand side of the functional equation is well-defined $\zeta(s)$ vanishes at $s = -2, -4, -6, \ldots$ (these are called the *trivial zeros*). The remaining zeros are called *non-trivial* and are located in the strip $0 < \mathfrak{Re}(s) < 1$. We denote Z the set of such zeros:

$$Z = \{\rho = \beta + i\gamma \mid \zeta(\rho) = 0, 0 < \beta < 1\}. \tag{1.4.2}$$

We have the following facts about Z. $|Z|$ is countably infinite and there are finitely many $\rho \in Z$ such that $|\gamma| \le T$, with $T > 0$. In fact, for $N(T)$ the number of zeros $\rho = \beta + i\gamma$ such that $0 < \beta < 1$ and $0 < \gamma \le T$, Riemann conjectured (proven by von Mangoldt) that

$$N(T) = \frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + \frac{7}{8} + O(\log(T)). \tag{1.4.3}$$

From the functional equation (1.4.1) we see the zeros are symmetric with respect to both the real axis and the line $\mathfrak{Re}(s) = \frac{1}{2}$. Riemann conjectured that if $\rho \in Z$ then ρ lies on the vertical line $\mathfrak{Re}(s) = \frac{1}{2}$. His conjecture is now known as the Riemann Hypothesis. This is considered by many to be the greatest unsolved problems in mathematics.

Conjecture 1.4.1 (Riemann Hypothesis (RH), 1859)**.** *Let Z be defined as in* (1.4.2)*. If $\rho = \beta + i\gamma \in Z$, then $\beta = \frac{1}{2}$.*

To date there has not been a proof or disproof of RH but there are a number of results which seem to hint at its truth.

In 1914 Hardy showed that infinitely many of the zeros lie on the $\frac{1}{2}$-line and, in 1942, Selberg showed that, of the total number of zeros, a positive proportion of them must lie on the $\frac{1}{2}$-line. In 1974, Levinson [49] showed that at least 36.5% of the zeros must lie on the critical line and, in 1989, Conrey [11] improved this to 40.7%. In 2012, Feng [21] improved this result to 41.28%.

We also have a number of numerical results that can be considered as indicators of its truth. For example, a number of authors have worked on the partial verification of the Riemann Hypothesis. So far all the zeros that have been found lie on the line $\mathfrak{Re}(s) = \frac{1}{2}$ and are simple. Let $H > 0$ be a constant, Table 1.1 lists authors who have shown for $\rho \in Z$ if $\gamma \leq H$ then $\beta = \frac{1}{2}$ for larger and larger H values. Each of these results required extensive numerical calculations on computer networks.

Table 1.1: History of Partial Verification of RH

Year	Authors	Verification Height H
1859	Riemann	25.010
1903	Gram [26]	65.112
1914	Backlund [3]	198.015
1925	Hutchinson [32]	299.840
1935	Titchmarsh [78]	388.846
1936	Titchmarsh [79]	1,467.477
1953	Turing [83]	1,540.030
1956	Lehmer [47]	9,878.910
1956	Lehmer [46]	21,942.593
1958	Meller [57]	29,750.745
1966	Lehman [44]	170,570.745
1968	Rosser, Schoenfeld & Yohe [71]	1,893,194.452
1977	Brent [8]	18,114,537.803
1979	Brent [9]	35,018,261.243
1982	Brent, van de Lune, te Riele, Winter [10]	81,702,130.190
1983	van de Lune, te Riele [53]	119,590,809.282
1986	van de Lune, te Riele &Winter [54]	545,439,823.215
2003	Wedeniwski* [86]	57,292,877,670.307
2004	Gourdon* [25]	2,445,999,556,030.000
2011	Platt [63]	30,610,046,000.000

(*unpublished)

On the other hand we know that these zeros do not lie “too close” to the 1-line either. This result is known as the zero free region. For instance, the fact that $\zeta(s)$ does not vanish for $\mathfrak{Re}(s) \geq 1$ was enough to prove the PNT. Extending this region to the left provides a more precise size of the error term. We state here the classical form for the zero free region.

Theorem 1.4.2. *[13, §13, pg. 86] There exists a positive numerical constant R_0 such that $\zeta(s)$ has no zero in the region*

$$\mathfrak{Re}(s) \geq 1 - \frac{1}{R_0 \log(|\mathfrak{Im}(s)|)}, \ \mathfrak{Im}(s) \geq 2.$$

We give a history of the explicit constant R_0 in Table 1.2.

Table 1.2: History of Improving R_0 (Zero Free Region Constant).

Year	Authors	Value of R_0
1899	de la Vallée Poussin [15]	34.82
1939	Rosser [68]	19.00
1962	Rosser & Schoenfeld [69]	17.52
1975	Rosser & Schoenfeld [70]	9.65
2002	Ford [23] or [24]	8.47
2005	Kadiri [35]	5.70

Lastly, we also have some information about the density of zeros which are close to the line $\mathfrak{Re}(s) = 1$. Such zeros are sparse and one can use this information to further reduce the error term for the PNT. This was done for the first time in Faber and Kadiri [20]. We detail this in Section 2.2.6, specifically equation (2.2.37) and (2.2.38). Define the function $N(\sigma, T)$ to be the number of zeros $\rho = \beta + i\gamma$ in Z such that $\sigma \leq \beta < 1$ and $0 < \gamma \leq T$. [80, Theorem 9.15 (A)] asserts that for any $\sigma > \frac{1}{2}$,

$$N(\sigma, T) = O(T).$$

This can be made more precise and there is a rich history of results on this topic. For instance we can prove that

$$N(\sigma, T) = O(T^{4\sigma(1-\sigma)+\epsilon}), \text{ for any } \frac{1}{2} < \sigma < 1 \text{ and for any } \epsilon > 0.$$

On the other hand, there are only a few explicit bounds for $N(\sigma, T)$. In Chapter 2 we make use of the following theorem due to Kadiri [37] in order to reduce the length of an interval containing a prime.

Theorem 1.4.3. *[37, Theorem 1.1] Let* $\sigma \geq 0.55$ *and* $T \geq H_0$. *Let* σ_0 *and* H *such that* $0.5208 < \sigma_0 < 0.9723$, $\sigma_0 < \sigma$ *and* $10^3 \leq H \leq H_0$. *Then there exist* b_1, b_2, b_3, *positive constants depending on* σ, σ_0, H, *such that:*

$$N(\sigma, T) \leq b_1(T - H) + b_2 \log(TH) + b_3.$$

The b_i*'s are defined in [37, Equation 6.3].*

We provide a sample of explicit estimates of $N(\sigma, T)$ in Table 2.1.

1.5 Explicit Prime Number Theorem.

We may estimate the size of the error term in the PNT by considering the following formula, conjectured by Riemann 1859 and proven by von Mangoldt in 1895.

Theorem 1.5.1. *Suppose that* $\psi_0(x)$ *is defined as*

$$\psi_0(x) = \begin{cases} \psi(x) & \text{if } x \text{ is not a prime power,} \\ \psi(x) - \frac{\Lambda(x)}{2} & \text{if } x \text{ is a prime power.} \end{cases} \tag{1.5.1}$$

Then for $x \geq 2$ *we have*

$$\psi_0(x) = x - \sum_{\rho \in Z} \frac{x^\rho}{\rho} - \log(2\pi(1 - x^{-2})^{\frac{1}{2}}), \tag{1.5.2}$$

where Z *is defined as in* (1.4.2).

The infinite sum in (1.5.2) is interpreted as

$$\sum_{\rho \in Z} \frac{x^\rho}{\rho} = \lim_{T \to \infty} \sum_{|\mathfrak{Im}(\rho)| \leq T} \frac{x^\rho}{\rho}.$$

Using the explicit formula (1.5.2) a version of the PNT for $\psi(x)$ is given below.

Theorem 1.5.2 (Prime Number Theorem, 1896, Hadamard & de la Vallée Poussin)**.**

$$\psi(x) = x + E_0(x),$$

where $E_0(x) = o(x)$ *as* $x \to \infty$.

Since this original announcement the error term $E_0(x)$ has been widely investigated, and more precise results have been shown. In 1899 de la Vallée Poussin [15] showed there exists a constant $c_1 > 0$ such that

$$E_0(x) = O\left(x \exp(-c_1\sqrt{\log x})\right). \tag{1.5.3}$$

In 1922 Littlewood [50] showed that there exists a constant $c_2 > 0$ such that

$$E_0(x) = O\left(x \exp(-c_2\sqrt{\log\log x \log x})\right). \tag{1.5.4}$$

Finally, in 1958 Korobov [42] and Vinogradov [84] showed that there exists a constant $c_3 > 0$ such that

$$E_0(x) = O\left(x \exp\left(-c_3 \frac{(\log x)^{\frac{3}{5}}}{(\log\log x)^{\frac{1}{5}}}\right)\right). \tag{1.5.5}$$

It was shown, by von Koch [40] in 1901, that if the Riemann Hypothesis holds then

$$E_0(x) = O\left(\sqrt{x}(\log x)^2\right).$$

Note that an equiovalent statement forthe PNT is for all $\epsilon > 0$, there exists $x_0 = x_0(\epsilon)$ such that for all $x \geq x_0$ we have $E_0(x) \leq \epsilon x$. Instead we focus on the following consequence of the PNT: Let x_0 be a fixed positive constant. If $x \geq x_0$ then there exists a computable ϵ_{x_0} such that

$$E_0(x) \leq \epsilon_{x_0} x.$$

Explicit bounds of this type have a number of applications including Diophantine approximation, cryptography, and computer science.

There are a number of different ways of estimating $\psi(x)$, depending on the size of x. We discuss techniques dependent on three ranges for the size of x. For the first values of x we compute $\psi(x)$ exactly: this can be done by using either lists of primes, or known information about $\theta(x)$. Rosser and Schoenfeld [69] give a description of this calculation. The authors also provide a table of values for $\theta(x)$ up to $x = 16\,000$, consequently, it is possible to achieve the same for $\psi(x)$. When x becomes asymptotically large we make use of purely analytic tools. Korobov and Vinogradov's bounds (1.5.5) give the sharpest error term in this case. See Ford [23] for an effective version of (1.5.5). However, in the middle range ($10^8 < x \leq e^{10000}$) (1.5.5) is no longer the best possible, and a combination of exact computations and analytic techniques is required to provide the best bound for $\psi(x)$.

The sum over the non-trivial zeros of $\zeta(s)$ in (1.5.2) is not absolutely convergent. The standard technique to bypass this problem was developed by de la Vallée Poussin and exploited by Rosser in 1941 [68] to obtain the first explicit bounds for $E_0(x)$. This idea consisted of applying the classical explicit formula (1.5.2) to an integral average of $\psi(x)$ over a small interval containing $[0, x]$. Bounds for $\psi(x)$ could be obtained via the following formula. Let $\delta > 0$. Then for $h = \pm x\delta)$ we have

$$\int_0^{x\delta} \psi(x+z) - (x+z) + \log(2\pi) + \frac{1}{2}\log(1-(x+z)^{-2})dz = \sum_{\rho\in Z} \frac{x^{\rho+1} - (x+h)^{\rho+1}}{\rho(\rho+1)}, \tag{1.5.6}$$

where Z is the set of non-trivial zeros of $\zeta(s)$, defined in (1.4.2). Note that the further to the left of the line $\mathfrak{Re}(s) = 1$ that the zeros ρ are the smaller the left side of (1.5.6) will be. Hence after obtaining the explicit formula, Rosser splits the sum over the zeros into two pieces. The goal was to prove that the sum in (1.5.6) was of size $o(x)$. The first piece corresponds to the region of the complex plane in which RH had been verified up to a certain height H. Doing this forces the size of this sum to be $O(x^{\frac{1}{2}})$. In the second piece, Rosser made use of the fact that the remaining zeros were far enough from the line $\mathfrak{Re}(s) = 1$, so that this second piece also became negligible compared to x. The main theorem of [68] is stated below.

Theorem 1.5.3. *[68, Theorem 21] For b a fixed positive constant and $x \geq 10^b$ there exists $\epsilon_b > 0$ such that*

$$x(1-\epsilon_b) - 1.84 < \psi(x) < x(1+\epsilon_b) - \frac{1}{2}\log(1-1/x^2),$$

where numerical values for ϵ_b are listed in [68, Table II].

The first result Rosser needed about the zeros of the Riemann zeta function was a partial verification of the RH. We recall that $N(T)$ is a function counting the zeros of $\zeta(s)$ and is defined in (1.4.3).

Theorem 1.5.4. *[68, Theorem 18] let $T > 0$. Let*

$$P(T) = \frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + \frac{7}{8}.$$

Let $H_0 > 0$ such that $N(H_0) = P(H_0) = 1041$ (that is $H_0 \approx 1467.47747$). For $0 < T \leq H_0$, we have $|N(T) - P(T)| < 2$; and for $\rho \in Z$ such that $0 < \gamma \leq H_0$ we have $\beta = \frac{1}{2}$.

The following provides an explicit estimate for $N(T)$. It was proven by Rosser in 1941 [68].

Theorem 1.5.5. *[68, Theorem 19] For $T \geq 2$, $|N(T) - P(T)| < R(T)$ where*

$$P(T) = \frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + \frac{7}{8},\quad R(T) = a_1\log(T) + a_2\log\log(T) + a_3,$$

with $a_1 = 0.137$, $a_2 = 0.443$ and $a_3 = 1.588$.

Recently this was slightly improved by T. Trudgian [82, Corollary 1] with $a_1 = 0.111,\ a_2 = 0.275,\ a_3 = 2.450$.

Rosser establishes the following zero free region.

Theorem 1.5.6. *[68, Theorem 20] Let $s = \sigma + it$, $t \geq H_0$ and $R_0 = 17.72$, then $\zeta(s)$*

does not vanish for

$$\sigma > 1 - \frac{1}{R_0 \log(t)}.$$

Later, in 1962, Rosser and Schoenfeld [69] updated Theorem 1.5.3 with [69, Theorem 28]. They explained [69, p.78,§6, line 8] that the main improvement of their results came from an extended verification of the Riemann Hypothesis. They used Lehmer's verification [46] and [47] for zeros $\rho = \beta + i\gamma$ with $|\gamma| \le H = e^{9.99} \approx 21\ 807.29879$. Moreover, the authors proved a larger zero free region than Theorem 1.5.6: [69, Theorem 11] gives $R_0 = 17.51631$.

In 1975, Rosser and Schoenfeld [70] provided similar explicit results. They introduced further averaging. Let $x, \delta > 0$ with $x \notin \mathbb{N}$ and $m \in \mathbb{N}$. Let $h(z) = \psi(x+z) - (x+z)$, then

$$h(z) + z \le \frac{1}{(\delta x/m)^m} \int_0^{\delta x/m} \dots \int_0^{\delta x/m} (h(y_1 + \dots + y_m) + (y_1 + \dots + y_m))\, dy_1 \dots dy_m.$$

They applied the explicit formula (1.5.2) as was done in (1.5.6). Their main theorem was presented in a slightly different way (given below). Here they improved the zero free region as given in Theorem 1.5.6, obtaining $R_0 = 9.645908801$. The also used the numerical verifications done up to height $H = 1\,894\,438$ by Rosser, Schoenfeld and Yohe [71] and by Lehman [45].

Theorem 1.5.7. *[70, Theorem 2] For b a fixed positive constant and $x \ge e^b$ there exists $\epsilon_b > 0$ such that*

$$E_0(x) \le \epsilon_b x,$$

with numerical values for ϵ_b given in [70, Table 1].

For the next 50 years, subsequent authors follow essentially the same method of [70] combining it with improvements of the verification of Riemann Hypothesis and of the zero free region. Dusart [16] uses $R_0 = 9.645908801$ and $H = 545\,439\,823$ (from van de Lune [54]). Two new results which remain unpublished are due to Dusart [18] and Nazardonyavi and Yakubovich [61]. Both authors use Kadiri's zero free region

$R_0 = 5.69693$ [35] and Gourdon's numerical verification with $H = 2\ 445\ 999\ 556\ 030$ [25].

The latest results are due to Faber and Kadiri [20] who change the method used to obtain the explicit formula. Rather than using the integral average as was described in [69], the authors instead appeal to the Mellin Transform (defined in 1.2.2) of a generic smooth function f. The method compares the function $\psi(x)$ to a smoothed variant $\psi_f(x)$:

$$\psi_f(x) = \sum_{n\geq 1} \Lambda(n) f\left(\frac{n}{x}\right) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} x^s F(s)\Big(-\frac{\zeta'}{\zeta}(s)\Big)ds. \qquad (1.5.7)$$

The function f is later optimized to minimize the size of the error term (where RH is unknown). This is described in Section 3.9. In the same section we explain how the method of Rosser and Schoenfeld [70] is a special case of this idea. In [20] the authors used an improvement on Theorem 1.5.6 with $R_0 = 5.69693$ and gave results for both Platt's verification height $H = 3.061 \cdot 10^{10}$ [63] and Gourdon's $H = 2\,445\,999\,556\,030$ [25]. Platt's value is computed more rigorously as he employs interval arithmetic. However for comparison with [18, 61] we present the values from [20] which rely on Gourdon's H in Table 1.3. In addition to this optimization the authors also consider explicit results for $N(\sigma, T)$ to give further improvements. [20, Theorem 1.1] is stated as in Theorem 1.5.7 with the numerical values of ϵ_b instead provided in [20, Table 3]. We present a collection of the numerical bounds given for $\frac{1}{x}|\psi(x) - x|$ in Table 1.3.

Table 1.3: For all $x \geq e^b$ we have $E_0(x) < \epsilon_b x$.

$b \setminus \epsilon_b$	[70, 1975]	[16, 1998]	[18, 61, 2010,2013]*	[20, 2014]
20	$6.5941 \cdot 10^{-4}$	$6.3020 \cdot 10^{-4}$	$6.1230 \cdot 10^{-4}$	$5.3688 \cdot 10^{-4}$
100	$1.6993 \cdot 10^{-5}$	$8.8427 \cdot 10^{-8}$	$2.9456 \cdot 10^{-11}$	$2.4618 \cdot 10^{-11}$
500	$1.2407 \cdot 10^{-5}$	$8.0001 \cdot 10^{-8}$	$2.6691 \cdot 10^{-11}$	$2.2506 \cdot 10^{-11}$
1000	$7.0482 \cdot 10^{-6}$	$6.3372 \cdot 10^{-8}$	$2.3299 \cdot 10^{-11}$	$1.9921 \cdot 10^{-11}$
3000	$5.1018 \cdot 10^{-8}$	$1.3761 \cdot 10^{-8}$	$1.0530 \cdot 10^{-11}$	$9.5728 \cdot 10^{-12}$

(*unpublished)

1.6 Zeros of Dirichlet L-functions.

When considering primes in arithmetic progressions it is possible to follow a similar thread as Riemann did with $\zeta(s)$. In 1882 Hurwitz developed the functional equation for Dirichlet L-functions. As in the case with $\zeta(s)$ this equation extended the definition of the Dirichlet L-functions meromorphically to the rest of the complex plane. It can also be seen in (1.6.2) that if $L(s,\chi) = 0$ then so does $L(1-s,\overline{\chi})$. The functional equation below is only valid for primitive characters, and takes on a different form depending on its parity:

$$\mathfrak{a} = \begin{cases} 0 & \text{if } \chi(-1) = 1, \\ 1 & \text{if } \chi(-1) = -1. \end{cases} \tag{1.6.1}$$

Theorem 1.6.1. *For χ a primitive character modulo q we have the following functional equation*

$$\pi^{-\frac{1-s+\mathfrak{a}}{2}} q^{\frac{1-s+\mathfrak{a}}{2}} \Gamma\left(\frac{1-s+\mathfrak{a}}{2}\right) L(1-s,\overline{\chi}) = i^{\mathfrak{a}} \frac{q^{\frac{1}{2}}}{\tau(\chi)} \pi^{-\frac{s+\mathfrak{a}}{2}} q^{\frac{s+\mathfrak{a}}{2}} \Gamma\left(\frac{s+\mathfrak{a}}{2}\right) L(s,\chi), \tag{1.6.2}$$

for $\mathfrak{a}$ defined in (1.6.1).

As a consequence of (1.6.2) we see that $L(s,\chi)$ vanishes at $s = -2n - \mathfrak{a}$, for $n = 0, 1, 2, \ldots$ (these are called the *trivial zeros* of $L(s,\chi)$). The remaining zeros are called *non-trivial* and we define the set of these zeros as

$$Z(\chi) = \{\rho = \beta + i\gamma \mid L(\rho,\chi) = 0, 0 < \beta < 1\}. \tag{1.6.3}$$

We have the following facts about $Z(\chi)$. $|Z(\chi)|$ is countably infinite, and there are finitely many $\rho \in Z(\chi)$ such that $|\gamma| \leq T$, with $T > 0$.

Theorem 1.6.2. *[13, Eq. 1, p. 101] Let $N(T,\chi)$ be the number of zeros in the rectangle*

$0 < \beta < 1$ *and* $|\gamma| \leq T$ *then we have*

$$\frac{1}{2}N(T,\chi) = \frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + O(\log(qT)). \tag{1.6.4}$$

From (1.6.2) we again see symmetry on the line $\mathfrak{Re}(s) = \frac{1}{2}$: if $\rho \in Z(\chi)$ then $1-\rho \in Z(\overline{\chi})$. The generalized Riemann Hypothesis (GRH), says that each such $\rho \in Z(\chi)$ has $\beta = \frac{1}{2}$, where the line $\mathfrak{Re}(s) = \frac{1}{2}$ is referred to as the critical line. This was possibly first suggested in 1884 by Adolf Piltz [13, p. 124].

Conjecture 1.6.3 (Generalized Riemann Hypothesis GRH)**.** *Let* $Z(\chi)$ *be defined as in* (1.6.3)*. If* $\rho \in Z(\chi)$ *then we have* $\beta = \frac{1}{2}$*.*

As in the case of RH there has yet to be a proof or disproof of this conjecture but there is some supporting evidence for its truth. In 2000 Bauer [5] showed that for a fixed q at least 36.5815% of the zeros lie on the critical line.

In addition, there are a number of numerical results that can be considered as indicators of the truth of GRH. For example, a number of authors have verified GRH for the first zeros of Dirichlet L-functions for a finite list of moduli q and every character χ modulo q: if $\rho \in Z(\chi)$ and $|\gamma| \leq H_q$, then $\beta = \frac{1}{2}$, where H_q is positive. We give a history of the verification heights H_q and the moduli $q \leq Q$ under consideration in Table 1.4.

Table 1.4: Verification heights H_q for varying moduli q.

Year	Authors	Character type	Moduli q	Height H_q
1961	Davies & Haselgrove [14]	Real Complex	$3,4,5,7,8,11,12,13,15,24$ $5,7,11,19,43,67,163$	$2000/\varphi(q)$ ≤ 105
1969	Spira [75]	All	≤ 24	24
1993	Rumely [72]	All	≤ 13 $13 < q \leq 72$ prime $73 < q \leq 113$ 48 others ≤ 486	10 000 2 500 2 500 2 500
2001	Bennett [6]	All	prime $73 \leq q \leq 347$	1 000
2013	Platt [64]	All	$\leq 400\,000$	$10^8/q$

The most extensive and rigorous result to date is due to Platt [64].

Theorem 1.6.4 (Theorem 7.1,[64]). *GRH holds for Dirichlet L-functions of primitive character modulus $q \leq 400,000$ and to height $H_q = \max\left(\frac{10^8}{q}, \frac{7.5\cdot 10^7}{q} + 200\right)$ for even q and to height $H_q = \max\left(\frac{10^8}{q}, \frac{3.75\cdot 10^7}{q} + 200\right)$ for odd q.*

His results are on nearly the same scale as with $\zeta(s)$. For example if $q = 3$ then $H = \frac{10^8}{3}$.

On the other hand we also may say that the elements of $Z(\chi)$ are not "too close" to the 1-line, with some exceptions depending on whether or not χ is real or complex valued. Let $s = \sigma + it$. If χ is a primitive complex valued character then one can generalize the zero-free region by mimicking the proof for $\zeta(s)$ as closely as possible. That is there exists an absolute positive constant c such that $L(s, \chi)$ does not vanish in the region [13, p. 90]

$$\sigma \geq 1 - \frac{c}{\log(q(|t| + 2))}. \tag{1.6.5}$$

However, if χ is a real valued character then $\chi^2 = \chi_0$, this is a problem since $L(s, \chi_0)$ has a pole at $s = 1$. One needs to modify the argument when t is small, namely $t < \frac{\delta}{\log q}$, for some positive constant $\delta > 0$. One can prove in this case that there exists at most one zero such that

$$\sigma \geq 1 - \frac{c'}{\log(q(|t| + 2))}, \tag{1.6.6}$$

for some positive constant c'. Moreover, this zero if it exists must be real. Lastly, when $t \geq \frac{\delta}{\log q}$, one recovers the same region as (1.6.5). Combining the inequalities (1.6.5) and (1.6.6) one has the classical zero free region below.

Theorem 1.6.5. *[13, §14, pg. 93] There exists an absolute positive constant R_0 with the following property. If χ is a complex character modulo q, then $L(s, \chi)$ has no zero in the region defined by*

$$\sigma \geq 1 - \frac{1}{R_0 \log(q \max(1, |t|))}.$$

If χ is a non-principal real character, the only possible zero of $L(s, \chi)$ in the above region is a single (simple) real zero. It is called an exceptional zero, and the modulus and character associated to it are also called exceptional.

This result is due partially to Gronwall [27] and Titchmarsh [76, 77]. As before we are able to give an explicit value for the constant R_0, we provide a history of explicit results in Table 1.5. One of the most important open problems in analytic number

Table 1.5: History of the increasing zero free region

Year	Authors	R_0
1984	McCurley [56]	9.65
2005	Kadiri [34]	6.41

theory is Siegel's conjecture:

Conjecture 1.6.6. *There are no exceptional zeros.*

In 1935, Siegel [74] established a range for the exceptional zeros. We give the version stated in Montgomery and Vaughan's [59].

Theorem 1.6.7. *[59, Theorem 11.11] For any $\epsilon > 0$ there exists a positive number $C(\epsilon)$ such that, if χ is any real non-principal character, with modulus q, then $L(s, \chi) \neq 0$ for*

$$s > 1 - C(\epsilon)q^{-\epsilon}.$$

Unfortunately, the constant $C(\epsilon)$ is ineffective. That is, the proof does not give a method for computing $C(\epsilon)$, given ϵ.

1.7 Explicit Prime Number Theorem in Arithmetic Progressions

As with the Prime Number Theorem we may provide an explicit result for its counterpart in arithmetic progressions. For a fixed q and a we recall

Theorem 1.7.1 (Prime Number Theorem in Arithmetic Progressions PNT in AP, de la Vallée Poussin, 1899)**.**

$$\psi(x; q, a) = \frac{x}{\varphi(q)} + E_0(x; q, a),$$

where $E_0(x; q, a) = o(x)$ as $x \to \infty$.

Again, we appeal to an explicit formula to provide precise estimates for the size of the error term in Theorem 1.7.1. Suppose χ is a primitive character modulo q. Then define

$$\psi(x,\chi) = \sum_{n\le x} \chi(n)\Lambda(n).$$

Let $x \in \mathbb{R}$ such that it is not a prime power. We have from [13, §19 Eqns 2 & 3] that

$$\psi(x,\chi) = \begin{cases} -\displaystyle\sum_{\rho\in Z(\chi)} \frac{x^\rho}{\rho} - \frac{L'}{L}(0,\chi) + \sum_{n=1}^{\infty} \frac{x^{1-2n}}{2n-1} & \text{if } \chi(-1) = -1, \\ -\displaystyle\sum_{\rho\in Z(\chi)} \frac{x^\rho}{\rho} - \log(x) - b(\chi) + \sum_{n=1}^{\infty} \frac{x^{-2n}}{2n} & \text{if } \chi(-1) = 1, \end{cases} \tag{1.7.1}$$

where $b(\chi)$ is the constant term in the Laurent expansion of $\frac{L'}{L}(s,\chi)$ at $s = 0$, and $Z(\chi)$ is defined in (1.6.3). The infinite sum in (1.7.1) is interpreted as

$$\sum_{\rho\in Z(\chi)} \frac{x^\rho}{\rho} = \lim_{T\to\infty} \sum_{|\mathfrak{Im}(\rho)|\le T} \frac{x^\rho}{\rho}.$$

The relationship between $\psi(x;q,a)$ and $\psi(x,\chi)$ is given by the following [13, Eq. 2, p. 121]

$$\psi(x;q,a) = \frac{1}{\varphi(q)} \sum_{\chi (\text{mod } q)} \overline{\chi}(a)\psi(x,\chi).$$

Let $\delta > 0$ be a fixed constant. If $q \le (\log x)^{1-\delta}$, then we have an effective positive constant c such that

$$E_0(x;q,a) = O(x\exp(-c\sqrt{\log x})).$$

This result is first attributed to de la Vallée Poussin [15]. In the wider range for the modulus q, namely $q < Q$ when

$$Q = \exp(C\sqrt{\log x}) \quad \text{and } C > 0 \text{ is fixed,} \tag{1.7.2}$$

then, in 1935, Page [62] showed that there exists at most one exceptional modulus q_1 not exceeding Q. In [62, Theorem 1], Page stated a result on the error term for

$\pi(x;q,a) - \frac{\text{li}(x)}{\varphi(q)}$. We state his theorem in terms of $\psi(x;q,a)$ below.

Theorem 1.7.2. *[13, §20, p. 124] Let $C > 0$ be any constant. Then, except possibly if q is a multiple of q_1 as in (1.7.2), we have that*

$$E_0(x;q,a) = O(x\exp(-c'\sqrt{\log x})),$$

for a positive constant c' depending only on C, and this holds uniformly with respect to $q < Q$. The integer q_1 also satisfies $\log x \ll q_1 \log^4(q_1)$.

In 1936 Walfisz applied Siegel's Theorem (stated here in Theorem 1.6.7) in order to provide the sharpest results so far known for $E_0(x;q,a)$. Walfisz's theorem was in terms of $\pi(x;q,a)$ and can be found in [85, Theorem 3]. We again state it below in terms of $\psi(x;q,a)$.

Theorem 1.7.3 (Siegel-Walfisz Theorem)**.** *[13, §22 p. 133] Let N be any positive constant. Then there exists a positive constant $C(N)$, depending only on N, such that if $q \leq (\log x)^N$ then*

$$E_0(x;q,a) = O(x\exp(-C(N)\sqrt{\log x})),$$

uniformly in q.

It follows from [59, Corollary 13.8] that if GRH is true, then we have

$$E_0(x;q,a) = O(\sqrt{x}\log^2(x)).$$

As in the case with the Prime Number Theorem we consider a history of explicit results regarding primes in arithmetic progressions.

An explicit statement of the PNT in AP is written in the following way. For $x \geq x_0(q) = x_0$ there exists a computable $\epsilon_{x,q}$ such that

$$E_0(x;q,a) \leq \epsilon_{x,q} x.$$

Similar sorts of issues arise when trying to obtain precise upper bounds for $\psi(x;q,a)$ as in the case of $\psi(x)$. The method of estimation is chosen based on the size of x, with an additional consideration based on the size of q. As it can be seen from Theorems 1.7.2 and 1.7.3, the effectiveness of an error term for $\psi(x;q,a)$ is dependent on the size of q. In Chapter 3 we adapt the method of Faber and Kadiri [20] to be used in terms of primes in arithmetic progressions. We give a brief history on obtaining explicit error terms for the PNT in AP.

In 1984 McCurley [56], in his doctoral thesis, adapts the method of Rosser and Schoenfeld [70] to primes in arithmetic progressions. He applies the same integral averaging technique as Rosser and Schoenfeld. At the time, there were few explicit results known about the non-trivial zeros of the Dirichlet L functions. The main results of [56] are given below.

Theorem 1.7.4. *[56, Theorem 1.2] Let q be a non-exceptional modulus, and let $(q,a) = 1$. For various values of ϵ and b, [56, Table 1] gives values of c such that*

$$\left|\psi(x;q,a) - \frac{x}{\varphi(q)}\right| < \frac{\epsilon x}{\varphi(q)},$$

and

$$\left|\theta(x;q,a) - \frac{x}{\varphi(q)}\right| < \frac{\epsilon x}{\varphi(q)},$$

provided that $q \geq 10^b$ and $x \geq \exp(c\log^2(k))$.

Instead of fixing the size of x when providing bounds, he fixes ϵ and provides a method by which one can show that $|E_0(x;q,a)| < \frac{\epsilon x}{\varphi(q)}$ provided that q and x are large enough. A sample of his results are listed in Table 1.6.

As in the case of with the primes, McCurley required explicit results on the zeros of the Dirichlet L-functions. He establishes the following estimate of $N(T,\chi)$ defined in (1.6.4).

Theorem 1.7.5. *[56, Theorem 2.1] Let $T \geq 1$ and χ be a primitive non-principal*

Table 1.6: For $q \geq 10^b$, $x \geq x_0(b,c)$, we have $|E_0(x;q,a)| \leq \frac{\epsilon_{b,c}x}{\varphi(q)}$.

b	c	x_0	$\epsilon_{b,c}$
1	208.3	$4.25 \cdot 10^{479}$	0.001
2	78.34	$3.46 \cdot 10^{721}$	0.001
3	48.74	$1.13 \cdot 10^{1010}$	0.001
4	36.49	$2.20 \cdot 10^{1344}$	0.001

character modulo q. If $0 < \eta \leq \frac{1}{2}$, then

$$\left| N(T,\chi) - \frac{T}{\pi} \log \frac{qT}{2\pi e} \right| < C_1 \log kT + C_2,$$

where

$$C_1 = \frac{1+2\eta}{\pi \log 2},$$

and

$$C_2 = 0.3058 - 0.268\eta + \frac{4 \log \zeta(1+\eta)}{\log(2)} - \frac{2 \log \zeta(2+2\eta)}{\log 2} + \frac{2}{\pi} \log \zeta \left(\frac{3}{2} + 2\eta \right).$$

McCurley [56] establishes the following explicit zero free region.

Theorem 1.7.6. *[56, Theorem 1.1] Let $\mathscr{L}_q(s) = \prod_{\chi \pmod q} L(s,\chi)$. There exists at most a single zero of $\mathscr{L}_q(s)$ in the region*

$$\sigma \geq 1 - \frac{1}{R \log(\max(q, q|t|, 10))},$$

where $R = 9.645908801$. The only possible zero in this region is a simple real zero arising from an L-function formed with a real non-principal character modulo q.

For nearly 10 years no new results on the zeros or improvements on the bounds of $\psi(x;q,a)$ were published, aside from a follow up paper of McCurley which focused on giving bounds for the modulus $q = 3$ [55]. In 1993 Rumely [72] provided numerical verification of the generalized Riemann Hypothesis for some moduli q.

Theorem 1.7.7. *[72, Theorem 4] For all $q \leq 13$, the GRH holds for all primitive*

Dirichlet L-functions $L(s,\chi)$ with modulus q and $|T| \leq 10\ 000$. For all $q \leq 72$, all composite $q \leq 112$, and some other moduli with $q \leq 486$ the GRH holds for all primitive Dirichlet L-functions $L(s,\chi)$ with modulus q and $|T| \leq 2\ 500$.

Making use of this result, along with a slight modification of the zero free region, Ramaré and Rumely [66] adapted McCurley's method so that they may use the values of H provided in Theorem 1.7.7. The theorem in [66] returns to the familiar form it has been expressed in as in the case of $\psi(x)$. We state it below, with some of their numerical results given in Table 1.7.

Theorem 1.7.8. *[66, Theorem 1] For any triple (q,ϵ,x_0) given by [66, Table 1],and any a prime to q, we have*

$$\max_{1\leq y\leq x}\left|\theta(x;q,a)-\frac{x}{\varphi(q)}\right| \leq \frac{\epsilon x}{\varphi(q)} \quad \textit{for } x \geq x_0,$$

and

$$\max_{1\leq y\leq x}\left|\psi(x;q,a)-\frac{x}{\varphi(q)}\right| \leq \frac{\epsilon x}{\varphi(q)} \quad \textit{for } x \geq x_0.$$

Table 1.7: Values of ϵ, where for each q and $x \geq x_0$ we have $|E_0(x;q,a)| < \frac{\epsilon x}{\varphi(q)}$

$q \setminus x_0$	10^{10}	10^{13}	10^{30}	10^{100}
3	$2.238 \cdot 10^{-3}$	$1.951 \cdot 10^{-3}$	$1.813 \cdot 10^{-3}$	$1.310 \cdot 10^{-3}$
5	$2.785 \cdot 10^{-3}$	$2.250 \cdot 10^{-3}$	$2.105 \cdot 10^{-3}$	$1.606 \cdot 10^{-3}$
100	$1.647 \cdot 10^{-2}$	$1.031 \cdot 10^{-2}$	$9.631 \cdot 10^{-3}$	$7.737 \cdot 10^{-3}$
210	$1.770 \cdot 10^{-2}$	$1.043 \cdot 10^{-2}$	$9.708 \cdot 10^{-3}$	$7.787 \cdot 10^{-3}$
486	$4.383 \cdot 10^{-2}$	$1.287 \cdot 10^{-2}$	$1.145 \cdot 10^{-2}$	$9.740 \cdot 10^{-3}$

The results they obtained are very uniform for all moduli under consideration and give values for smaller x than McCurley [56]. We note that Ramaré and Rumely provided a very slight improvement to Theorem 1.7.6. They took $\eta = \frac{1}{2}$ in Theorem 1.7.5 and noted [66, p. 410 line 6] that this choice was not optimal.

In 2001 Bennett [6] used Rumely's algorithm to fill in some gaps for moduli $73 \leq q \leq 347$, and then utilized the method outlined in [66] to give more explicit bounds for $\theta(x;q,a)$. He then used these values to give a measure of the irrationality of an algebraic number. In 2002 Dusart [17] provided an explicit formula for $\theta(x;q,a)$ as well as $\psi(x;q,a)$.

Theorem 1.7.9. *[17, §1 p. 1137] Let* $X = \sqrt{\frac{\log x}{R_0}}$, *where* R_0 *is given in Theorem 1.7.6. For* $x \geq x_0(q)$, *where* $x_0(q)$ *can be easily computed, we have*

$$|\theta(x;q,a) - x/\varphi(q)| \text{ and } |\psi(x;q,a) - x/\varphi(q)| < x\epsilon(x),$$

where

$$\epsilon(x) = 3\sqrt{\frac{q}{\varphi(q)C_1(q)}}X^{\frac{1}{2}}\exp(-X),$$

for an explicit value of $C_1(q)$ *which is defined in [66, Theorem 3.6.3].*

These works were done in an environment without any new information regarding the verification of GRH. Then, in 2013, David Platt [64] provided an extension to the verification to GRH, given in Theorem 1.6.4. We also have access to the latest zero free region for Dirichlet L-functions given by Kadiri [34] which gives $R = 6.41$. These are two of the tools used to obtain explicit results for $E_0(x;q,a)$.

1.8 Statement of Results

In Chapter 2 we provide short explicit intervals containing at least one prime number. The work in this chapter will be published in Integers [38]. We prove :

Theorem 1.8.1. *Let* $x_0 \geq 4 \cdot 10^{18}$ *be a fixed constant and let* $x > x_0$. *Then there exists at least one prime* p *such that* $(1 - \Delta^{-1})x < p < x$, *where* Δ *is a constant depending on* x_0 *and is given in Table 2.2.*

This theorem improves the work from 2003 of Ramaré and Saouter [67]: we are able to take a much larger value for Δ. For example, when $x_0 = e^{59}$, Ramaré and Saouter [67] found that the interval gap was given by $\Delta = 209\,257\,759$, where Theorem 1.8.1 provides $\Delta = 1\ 946\ 282\ 821$. Theorem 1.8.1 also allows us to derive the following corollary about the Odd Goldbach conjecture. This conjecture was recently settled by Harold Helfgott [28, 29, 30]. A more precise history of this is given in Chapter 2.

Corollary 1.8.2. *Every odd number larger than* 5 *and smaller than*

$$1\ 966\ 196\ 911 \times 4 \cdot 10^{18} = 7.864\ldots \cdot 10^{27}$$

is the sum of at most three primes.

The proof of Theorem 1.8.1 begins with using $\psi(x)$ to detect if an interval contains a prime. Then we relate this condition to the zeros of $\zeta(s)$ via the following explicit formula for $\psi(x)$. Let $2 \leq b \leq c$, and let g be a continuously differentiable function on $[b, c]$. We have

$$\int_b^c \psi(u)g(u)du = \int_b^c ug(u) - \sum_{\rho \in Z} \int_b^c \frac{u^\varrho}{\varrho} g(u)du + \int_b^c \left(\log 2\pi - \frac{1}{2}\log(1-u^{-2}) \right) g(u)du,$$

where Z is the set of the non-trivial zeros of of $\zeta(s)$, defined in (1.4.2).

The main contribution to this expression comes from the sum over the zeros, and we use explicit information about the non-trivial zeros of $\zeta(s)$ to bound it. For instance we use partial verification of RH with $H = 3.061 \cdot 10^{10}$ [63] which allows us to split our sum along the imaginary axis into two pieces. For those zeros $\rho = \beta + i\gamma$ such that $|\gamma| \leq H$ we obtain an error term $O(x^{1/2})$. The remaining piece is usually bounded by using a zero free region for $\zeta(s)$ in order to give an error term slightly less than $O(x)$. Instead we introduce further splitting of the sum, this time along the real axis. Fix $\frac{3}{5} < \sigma_0 < 1$. For those zeros $\rho = \beta + i\gamma$ with $\frac{1}{2} \leq \beta < \sigma_0$ we use estimates for $N(T)$ due to Rosser [68] to count the zeros. We recall that $N(T) = O(T \log T)$ and note that this term has size $O(x^{\sigma_0})$. In the remaining region, that is those zeros $\rho = \beta + i\gamma$ such that $\sigma_0 < \beta < 1$ we employ two explicit results due to Kadiri. The first is a classical tool used for bounding the sum over the zeros, an explicit zero free region [35] which gives the term size $O(x^{1-\frac{1}{R_0 \log H}})$, R_0 a constant. We note that the previous authors did not employ this result. The second is a new result that has not been applied in this context before. Instead of counting the number of zeros using $N(T)$, we instead make use of an explicit estimate of $N(\sigma_0, T)$ due to Kadiri [37] (note the values provided are for $\frac{3}{5} < \sigma_0 < 1$), this result gives us that $N(\sigma_0, T) = O_{\sigma_0}(T)$, which provides a savings of size at least $\log(T)$. This last result drastically reduces the size of the error term by providing a more accurate measurement of the number of zeros near the line $\mathfrak{Re}(s) = 1$. In addition, we make use of the Brun-Titchmarsh inequality given in [58] in order to narrow the search space required for detecting a prime.

In chapter 3 we shift the focus to answer questions about primes in arithmetic progressions. We define here

$$E(x; q, a) = \frac{\varphi(q)}{x} E_0(x; q, a) = \left| \frac{\psi(x; q, a) - \frac{x}{\varphi(q)}}{\frac{x}{\varphi(q)}} \right|. \tag{1.8.1}$$

We prove :

Theorem 1.8.3. *For b a fixed positive constant and $x \geq 10^b$ there exists $\epsilon_{q,b} > 0$ such that $E(x; q, a) \leq \epsilon_{q,b}$ where $\epsilon_{q,b}$ is given explicitly in Theorem 3.7.8 and is computed in Table 3.10.*

This chapter provides improvements over the results from 1996 of Ramaré and Rumely [66]. A sample of their results can be found in Table 1.7. For nearly 20 years this was the article of reference for this type of result which was cited. For some moduli q we are able to improve the bounds by a factor of almost 10^4. For example, when $q = 3$ and $x = 10^{100}$, [66] gives $E(x; q, a) \leq 1.310 \cdot 10^{-3}$, while Theorem 1.8.3 gives that $E(x; q, a) \leq 6.299 \cdot 10^{-7}$. It is important to note that Theorem 1.8.3 provides, for the first time, a result that is on the same scale as the results for the primes. For example, Faber and Kadiri [20], which provides latest bounds for $E(x) = \left|\frac{\psi(x)-x}{x}\right|$, finds that when $x = 10^{100}$, $E(x) \leq 1.5274 \cdot 10^{-9}$. We provide a sample of the numerical results given by Theorem 1.8.3 in Table 1.8.

Additionally, we extend the computations for $E(x; q, a)$ to a number of moduli which previously did not have bounds. We give a sample of these results in Table 1.9.

One of the ideas to prove Theorem 1.8.3, is to adapt the smoothing argument of Faber and Kadiri [20] to our case. We appeal to a smooth function f which has a Mellin transform so that we can obtain an explicit formula of the form (1.5.7). We compare

Table 1.8: For each q and $x \geq 10^b$ we give $\epsilon_{q,b}$ such that $E(x; q, a) \leq \epsilon_{q,b}$.

$q \setminus 10^b$	10^{10}	10^{13}	10^{30}	10^{100}
3	$4.282 \cdot 10^{-4}$	$1.979 \cdot 10^{-5}$	$6.723 \cdot 10^{-7}$	$6.136 \cdot 10^{-7}$
5	$1.007 \cdot 10^{-3}$	$3.970 \cdot 10^{-5}$	$1.140 \cdot 10^{-6}$	$1.044 \cdot 10^{-6}$
100	$8.458 \cdot 10^{-3}$	$3.348 \cdot 10^{-4}$	$2.209 \cdot 10^{-5}$	$2.037 \cdot 10^{-5}$
210	$8.733 \cdot 10^{-3}$	$3.959 \cdot 10^{-4}$	$4.529 \cdot 10^{-5}$	$4.179 \cdot 10^{-5}$
486	$2.569 \cdot 10^{-2}$	$1.161 \cdot 10^{-3}$	$1.061 \cdot 10^{-4}$	$9.815 \cdot 10^{-5}$

Table 1.9: For each q and $x \geq x_0$ we give ϵ such that $|E(x; q, a)| < \epsilon_{q,b}$.

$q \setminus x$	10^{10}	10^{13}	10^{30}	10^{100}
1000	$6.456 \cdot 10^{-2}$	$2.556 \cdot 10^{-3}$	$2.191 \cdot 10^{-4}$	$2.034 \cdot 10^{-4}$
2000	$1.118 \cdot 10^{-1}$	$4.707 \cdot 10^{-3}$	$4.374 \cdot 10^{-4}$	$4.077 \cdot 10^{-4}$
5000	$2.728 \cdot 10^{-1}$	$1.112 \cdot 10^{-2}$	$1.091 \cdot 10^{-3}$	$1.017 \cdot 10^{-3}$
10000	$5.783 \cdot 10^{-1}$	$2.309 \cdot 10^{-2}$	$2.180 \cdot 10^{-3}$	$2.035 \cdot 10^{-3}$

$\psi(x; q, a)$ with $\psi_f(x; q, a)$ and use the Mellin transform given by

$$\psi_f(x; q, a) = \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \overline{\chi}(a) \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) \left(-\frac{L'}{L}(s, \chi^*) \right) x^s ds, \tag{1.8.2}$$

to obtain an explicit formula. We note that f is taken to be a generic function such that it has a Mellin transform. The function f is chosen by solving an optimization problem. This idea introduces more flexibility in controlling the size of the error term which comes from the sum over the zeros of $L(s, \chi)$. In fact, since we know the main term comes from the region in the complex plane where GRH is not known, we choose f such that it optimizes this error term. Section 3.9.3 demonstrates the significance of the smoothing function.

Another new idea consists of using the bounds in [20] for $\psi(x)$ to control the size of the term arising from the principal character χ_0 in (1.8.2). At last we use a third new idea: we introduce a sieving argument, which relies on the Brun-Titchmarsh inequality given in [58]. This argument was used in the case of short intervals (as described earlier). In this case it allows us to reduce the difference between $\psi(x; q, a)$ and $\psi_f(x; q, a)$. We explain the significance of these ideas in Section 3.9.1.

Finally, we provide bounds for the sum over the zeros of $L(s, \chi)$ by appealing to some classical tools. For example, we make use of an extensive partial verification of GRH, done by Platt [64], stated as Theorem 1.6.4. As before this gives us an error term of size $O(x^{\frac{1}{2}})$. His theorem allows for us to split our sum along the imaginary axis at height H_q, where the H_q provided is much larger than the previous result due to Rumely [72]. Theorem 1.6.4 also provides a verification for many more moduli than [72] so that we could in fact extend our computations to further moduli q. An explicit zero free region due to Kadiri [34], which gives $R_0 = 6.41$, where the previous authors used McCurley's [56] $R_0 = 9.65$, provides an error term of $O(x^{1-\frac{1}{R_0 \log(qH_q)}})$. To count the number of zeros in each piece of the sum we use explicit estimates for $N(T,\chi)$ given by Trudgian [81].

Chapter 2

Primes in Short Intervals

2.1 Introduction.

In this chapter, we address the problem of finding short intervals containing primes. In 1845 Bertrand conjectured that for any integer $n > 3$, there always exists at least one prime number p with $n < p < 2n - 2$. This was proven by Chebyshev in 1850, using elementary methods. Since then other intervals of the form $(kn, (k+1)n)$ have been investigated, and we refer the reader to [2] for $k = 2$, [52] for $k = 3$. Assuming that x is arbitrarily large, the length of intervals containing primes can be drastically reduced. To date, the record is held by Baker, Harman, and Pintz [4] as they prove that there is at least one prime between x and $x + x^{0.525+\varepsilon}$. This is an impressive result since under the Riemann Hypothesis the exponent 0.525 can only be reduced to 0.5. On the other hand, maximal gaps for the first primes have been checked numerically up to $4 \cdot 10^{18}$ by Oliveria e Silva et al. [31]. In particular, they find that the largest prime gap before this limit is 1 476 and occurs at $1\,425\,172\,824\,437\,699\,411 = e^{41.8008\cdots}$. The purpose of this chapter is to obtain an effective result of the form: for all $x \geq x_0$, there exists $\Delta > 0$ such that the interval $(x(1-\Delta^{-1}), x)$ contains at least one prime. In 1976 Schoenfeld's [73, Theorem 12] gave this for $x_0 = 2\,010\,881.1$ and $\Delta = 16\,598$. In 2003 Ramaré and Saouter improved on Schoenfeld's method by using a smoothing argument. They also extended the computations to many other values for x_0 ([67, Theorem 2 and Table 1]). In [36], Kadiri generalized this theorem to primes in arithmetic progression and applied this to Waring's seven cube problem. Here, our theorem improves [67] by making use of a new explicit zero-density for the zeros of the Riemann zeta function:

Theorem 2.1.1. *Let $x_0 \geq 4 \cdot 10^{18}$ be a fixed constant and let $x > x_0$. Then there exists* *at least one prime p such that $(1-\Delta^{-1})x < p < x$, where Δ is a constant depending on x_0 and is given in Table 2.2.*

In Section 2.2, we prove a general theorem (Theorem 2.2.7) which provides conditions for intervals of the form $((1-\Delta^{-1})x, x)$ to contain a prime. In Section 2.3, we apply this theorem to compute explicit values for Δ.

We present an example of numerical improvement this theorem allows, for instance when $x_0 = e^{59}$. Ramaré and Saouter [67] found that the interval gap was given by $\Delta = 209\,257\,759$. In [28, page 74], Helfgott mentioned an improvement of Ramaré using Platt's latest verification of the Riemann Hypothesis [63]: $\Delta = 307\,779\,681$. Our Theorem 2.1.1 leads to $\Delta = 1\ 946\ 282\ 821$.

We now mention an application to the verification of the Ternary Goldbach conjecture. This conjecture was known to be true for sufficiently large integers (by Vinogradov), and Liu and Wang [51] prove it for all integers $n \geq e^{3100}$. On the other hand, the conjecture was verified for the first values of n. In [67, Corollary 1], Ramaré and Saouter verified it for $n \leq 1.132 \cdot 10^{22}$. Very recently, Oliveria e Silva et. al. [31, Theorem 2.1] extended this limit to $n \leq 8.370 \cdot 10^{26}$. In [28, Proposition A.1.], Helfgott applied the above result on short intervals containing primes ($\Delta = 307\,779\,681$) and found $n \leq 1.231 \cdot 10^{27}$. This allowed him to complete his proof [28] [29] of the Ternary Goldbach conjecture for the remaining integers. Here our main theorem gives:

Corollary 2.1.2. *Every odd number larger than 5 and smaller than*

$$1\ 966\ 196\ 911 \times 4 \cdot 10^{18} = 7.864\ldots \cdot 10^{27}$$

is the sum of at most three primes.

As of today, Helfgott and Platt [30] have announced a verification up to $8.875 \cdot 10^{30}$.

2.2 Proof of Theorem 2.1.1.

We recall the definition of the classical Chebyshev functions:

$$\theta(x) = \sum_{p\le x} \log p, \quad \psi(x) = \sum_{n\le x} \Lambda(n), \text{ with } \Lambda(n) = \begin{cases} 1 & \text{if } n = p^k \text{ for some } k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

For each x_0, we want to find the largest $\Delta > 0$ such that, for all $x > x_0$, there exists a prime between $x(1-\Delta^{-1})$ and x. This happens as soon as

$$\theta(x) - \theta(x(1-\Delta^{-1})) > 0.$$

2.2.1 Introduction of parameters

We list here the parameters we will be using throughout the proof.

$$\begin{aligned}
&* \ m \text{ integer with } m \ge 2,\\
&* \ 0 \le u \le 0.0001,\ \delta = mu \text{ and } 0 \le \delta \le 0.0001,\\
&* \ 0 \le a \le 1/2,\\
&* \ \Delta = \left(1 - (1+\delta a)(1+\delta(1-a))^{-1} e^{-u}\right)^{-1}, \qquad (2.2.1)\\
&* \ X \ge X_0 \ge e^{38},\\
&* \ x = e^u X(1+\delta(1-a)) \ge x_0 = e^u X_0(1+\delta(1-a)),\\
&* \ y = X(1+\delta a) = x\left(1-\Delta^{-1}\right).
\end{aligned}$$

2.2.2 Smoothing the difference $\theta(x) - \theta(y)$

We follow here the smoothing argument of [67]. Let f be a positive function integrable on $(0,1)$. We denote

$$\|f\|_1 = \int_0^1 f(t)dt, \tag{2.2.2}$$

$$\nu(f,a) = \int_0^a f(t)dt + \int_{1-a}^1 f(t)dt, \tag{2.2.3}$$

$$\text{and } I_{\delta,u,X} = \frac{1}{\|f\|_1}\int_0^1 \left(\theta(e^u X(1+\delta t)) - \theta(X(1+\delta t))\right) f(t)dt. \tag{2.2.4}$$

Note that for all $a \leq t \leq 1-a$, $\theta(e^u X(1+\delta t)) - \theta(X(1+\delta t)) \leq \theta(x) - \theta(y)$. We integrate with the positive weight f and obtain:

$$\int_a^{1-a} \left(\theta(e^u X(1+\delta t)) - \theta(X(1+\delta t))\right) f(t)dt \leq (\theta(x) - \theta(y)) \int_a^{1-a} f(t)dt. \tag{2.2.5}$$

We extend the left integral to the interval $(0,1)$ and use a Brun-Titchmarsh inequality to control the primes on the extremities $(0,a)$ and $(1-a,1)$ of the interval (see [67, page 16, line -5] or [58, Theorem 2]):

$$\int_{t\in(0,a)\cup(1-a,1)} \left(\theta(e^u X(1+\delta t)) - \theta(X(1+\delta t))\right) f(t)dt$$
$$\leq 2(1+\delta)(e^u-1)\frac{\log(e^u X)}{\log(X(e^u-1))}\nu(f,a)X. \tag{2.2.6}$$

Note that [67] uses the slightly larger bound

$$2.0004u\frac{\log X}{\log(uX)}\nu(f,a)X.$$

Combining (2.2.5) and (2.2.6) gives for $I_{\delta,u,X}$:

$$I_{\delta,u,X} \leq (\theta(x)-\theta(y))\frac{\int_a^{1-a} f(t)dt}{\|f\|_1} + 2(1+\delta)(e^u-1)\frac{\log(e^u X(1+\delta))}{\log(X(e^u-1))}\frac{\nu(f,a)}{\|f\|_1}X. \tag{2.2.7}$$

Thus $\theta(x) - \theta(y) > 0$ when

$$I_{\delta,u,X} - 2(1+\delta)(e^u - 1)\frac{\log(e^u X(1+\delta))}{\log(X(e^u-1))}\frac{\nu(f,a)}{\|f\|_1}X > 0. \tag{2.2.8}$$

It remains to establish a lower bound for $I_{\delta,u,X}$. To do so, we first approximate $\theta(x)$ with $\psi(x)$. This will allow us to translate our problems in terms of the zeros of the zeta function. We use approximations proven by Costa in [12, Theorem 5]:

Lemma 2.2.1. *Let $x \geq e^{38}$. Then*

$$0.999\sqrt{x} + \sqrt[3]{x} < \psi(x) - \theta(x) < 1.001\sqrt{x} + \sqrt[3]{x}. \tag{2.2.9}$$

Then we have that for all $0 < t < 1$,

$$\begin{aligned}&(\psi(e^u X(1+\delta t)) - \theta(e^u X(1+\delta t))) - (\psi(X(1+\delta t)) - \theta(X(1+\delta t)))\\ &< \sqrt{X}\sqrt{1+\delta}\left(1.001e^{u/2} - 0.999 + X^{-1/6}(1+\delta)^{-1/6}(e^{u/3}-1)\right) < \omega\sqrt{X},\end{aligned} \tag{2.2.10}$$

where we can take, under our assumptions (2.2.1),

$$\omega = 2.05022 \cdot 10^{-3}. \tag{2.2.11}$$

We denote

$$J_{\delta,u,X} = \frac{1}{\|f\|_1}\int_0^1 (\psi(e^u X(1+\delta t)) - \psi(X(1+\delta t)))\, f(t)dt. \tag{2.2.12}$$

It follows from (2.2.10) that

$$I_{\delta,u,X} \geq J_{\delta,u,X} - \omega\sqrt{X}. \tag{2.2.13}$$

Note that [67] used older approximations from [73], which lead to $\omega = 0.0325$. To summarize, we want to find conditions on m, δ, u, a so that

$$J_{\delta,u,X} - \omega\sqrt{X} - 2(1+\delta)(e^u-1)\frac{\log(e^uX(1+\delta))}{\log(X(e^u-1))}\frac{\nu(f,a)}{\|f\|_1}X > 0. \tag{2.2.14}$$

We are now left with evaluating $J_{\delta,u,X}$, which we shall do by relating it to the zeros of zeta through an explicit formula.

2.2.3 An explicit inequality for $J_{\delta,u,X}$

Lemma 2.2.2. *[67, Lemma 4] Let $2 \le b \le c$, and let g be a continuously differentiable function on $[b,c]$. We have*

$$\int_b^c \psi(u)g(u)du = \int_b^c ug(u) - \sum_\rho \int_b^c \frac{u^\rho}{\rho}g(u)du + \int_b^c \left(\log 2\pi - \frac{1}{2}\log(1-u^{-2})\right)g(u)du. \tag{2.2.15}$$

We apply this identity to respectively $g(t) = f\left(\delta^{-1}\left(e^{-u}X^{-1}t-1\right)\right), b = e^uX, c = e^uX(1+\delta)$ and $g(t) = f\left(\delta^{-1}\left(X^{-1}t-1\right)\right), b = X, c = X(1+\delta)$. It follows that

$$\begin{aligned} J_{\delta,u,X} = {} & \frac{1}{\|f\|_1}\int_0^1 (e^u-1)\,X(1+\delta t)f(t)dt - \frac{1}{\|f\|_1}\sum_\rho \int_0^1 \frac{(e^{u\rho}-1)\,(X(1+\delta t))^\rho f(t)}{\rho}dt \\ & - \frac{1}{2\|f\|_1}\int_0^1 \left(\log(1-(e^uX(1+\delta t))^{-2}) - \log(1-(X(1+\delta t))^{-2})\right)f(t)dt. \end{aligned}$$

Combining the terms in the last expression and taking the first term of the taylor expansion for $\log(1+z)$ we see the last term is $\ge -\frac{u}{2X}$. We obtain

$$\frac{J_{\delta,u,X}}{(e^u-1)X} \ge \frac{\int_0^1(1+\delta t)f(t)dt}{\|f\|_1} - \sum_\rho \left|\frac{(e^{u\rho}-1)}{(e^u-1)\rho}\frac{\int_0^1(1+\delta t)^\rho f(t)dt}{\|f\|_1}\right| X^{\Re\mathfrak{e}\rho-1} - \frac{u}{2(e^u-1)X^2}. \tag{2.2.16}$$

Note that we obtain some small savings by directly computing the first term whereas [67, equation (13)] use the following bound in (2.2.16) instead:

$$\frac{\int_0^1(1+\delta t)f(t)dt}{\|f\|_1} \ge \frac{u}{e^u-1}.$$

Let s be a complex number. We denote $G_{m,\delta,u}(s)$ the summand

$$G_{m,\delta,u}(s) = \frac{(e^{us}-1)}{(e^u-1)s}\frac{\int_0^1(1+\delta t)^s f(t)dt}{\|f\|_1}, \tag{2.2.17}$$

and we rewrite inequality (2.2.16) as

$$\frac{J_{\delta,u,X}}{(e^u-1)X} \geq G_{m,\delta,u}(1) - \sum_\rho |G_{m,\delta,u}(\rho)|\, X^{\mathfrak{Re}\rho-1} - \frac{u}{2(e^u-1)X^2}. \tag{2.2.18}$$

Since the right term increases with X, we can replace X with X_0 for $X \geq X_0$. For simplicity we denote

$$\Sigma = \Sigma_{m,\delta,u,X} = \sum_{\rho=\beta+i\gamma} |G_{m,\delta,u}(\rho)|\, X^{\beta-1}. \tag{2.2.19}$$

The following Proposition gives a first inequality in terms of the zeros of zeta and conditions on m, u, δ, a (and thus Δ) so that $\theta(x) - \theta(x(1-\Delta^{-1})) > 0$:

Proposition 2.2.3. *Let $m, u, \delta, a, \Delta, X_0$ satisfy (2.2.1). If $X \geq X_0$ and*

$$G_{m,\delta,u}(1) - \Sigma_{m,\delta,u,X_0} - \frac{u}{2(e^u-1)}X_0^{-2} - \frac{\omega}{(e^u-1)}X_0^{-1/2} - \frac{2\nu(f,a)(1+\delta)}{\|f\|_1}\frac{\log(e^u X_0(1+\delta))}{\log(X_0(e^u-1))} > 0, \tag{2.2.20}$$

then there exists a prime number between $x(1-\Delta^{-1})$ and x.

We are now going to make this Lemma more explicit by providing computable bounds for the sum over the zeros Σ_{m,δ,u,X_0}.

2.2.4 Evaluating $G_{m,\delta,u}$.

Let f be an m-admissible function over $[0,1]$. We recall the properties it entitles according to the definition of [67]:

- f is an m-times differentiable function,
- $f^{(k)}(0) = f^{(k)}(1) = 0$ for $0 \leq k \leq m-1$,
- $f \geq 0$,

- f is not identically 0.

Let $k = 0, \ldots, m$, $s = \sigma + i\tau$ be a complex number with $\tau > 0, 0 \le \sigma \le 1$. We denote

$$F_{k,m,\delta} = \frac{\int_0^1 (1+\delta t)^{1+k} |f^{(k)}(t)| dt}{\|f\|_1}. \tag{2.2.21}$$

We provide here finer estimates than [67] for $G_{m,\delta,u}$. Observe that

$$\left|\frac{e^{us}-1}{s}\right| = \left|\int_1^u e^{xs} dx\right| \le \int_1^u e^{x\sigma} dx = \frac{e^{u\sigma}-1}{\sigma}, \tag{2.2.22}$$

$$\left|\frac{e^{us}-1}{s}\right| \le \frac{e^{u\sigma}+1}{\tau}, \tag{2.2.23}$$

$$\text{and } \left|\int_0^1 (1+\delta t)^s f(t) dt\right| \le \frac{1}{\delta^k \tau^k} F_{k,m,\delta}. \tag{2.2.24}$$

We easily deduce bounds for $G_{m,\delta,u}(s)$ by combining (2.2.22) and (2.2.24) respectively with parameters $k = 0$, $k = 1$, $k = m$, and lastly by combining (2.2.23) and (2.2.24) with $k = m$:

$$|G_{m,\delta,u}(s)| \le F_{0,m,\delta} \frac{e^{u\sigma}-1}{(e^u-1)\sigma}, \tag{2.2.25}$$

$$|G_{m,\delta,u}(s)| \le F_{1,m,\delta} \frac{e^{u\sigma}-1}{(e^u-1)\sigma\delta\tau}, \tag{2.2.26}$$

$$|G_{m,\delta,u}(s)| \le F_{m,m,\delta} \frac{e^{u\sigma}-1}{(e^u-1)\sigma\delta^m\tau^m}, \tag{2.2.27}$$

$$|G_{m,\delta,u}(s)| \le F_{m,m,\delta} \frac{e^{u\sigma}+1}{(e^u-1)\delta^m\tau^{m+1}}. \tag{2.2.28}$$

2.2.5 Zeros of the Riemann-zeta function

We denote each zero of zeta $\rho = \beta + i\gamma$, $N(T)$ the number of zeros in the rectangle $0 < \beta < 1, 0 < \gamma < T$, and $N(\sigma_0, T)$ the number of those in the rectangle $\sigma_0 < \beta < 1, 0 < \gamma < T$. We assume that we have the following information.

Theorem 2.2.4.

1. *A numerical verification of the Riemann Hypothesis:*

   There exists $H > 2$ such that if $\zeta(\beta + i\gamma) = 0$ at $0 \le \beta \le 1$ and $0 \le \gamma \le H$, then

$\beta = 1/2$.

2. *A direct computation of some finite sums over the first zeros:*

   Let $0 < T_0 < H$ and $S_0 > 0$ satisfy

$$\sum_{\substack{0<\gamma\le T_0\\ \beta=1/2}} 1 \le N_0 = N(T_0), \tag{2.2.29}$$

$$\textit{and} \quad \sum_{\substack{0<\gamma\le T_0\\ \beta=1/2}} \frac{1}{\gamma} \le S_0. \tag{2.2.30}$$

3. *A zero-free region:*

   There exists $R_0 > 0$ constant, such that $\zeta(\sigma + it)$ does not vanish in the region

$$\sigma \ge 1 - \frac{1}{R_0 \log |t|} \quad \textit{and} \quad |t| \ge 2. \tag{2.2.31}$$

4. *An estimate for $N(T)$:*

   There exist a_1, a_2, a_3 positive constants such that, for all $T \ge 2$,

$$|N(T) - P(T)| \le R(T),$$
$$\textit{where } P(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8}, \quad R(T) = a_1 \log T + a_2 \log\log T + a_3. \tag{2.2.32}$$

5. *An upper bound for $N(\sigma_0, T)$:*

   Let $3/5 < \sigma_0 < 1$. Then there exist c_1, c_2, c_3 constants such that, for all $T \ge H$,

$$N(\sigma_0, T) \le c_1 T + c_2 \log T + c_3. \tag{2.2.33}$$

Note that [67] did not use any information of the type (2.2.30), (2.2.31), or (2.2.33). Instead they used (2.2.29), the fact that all nontrivial zeros satisfied $\beta < 1$, and the classical bound (2.2.32) for $N(T)$ as given in [68][Theorem 19]. Our improvement will mainly come from using a new zero-density of the form of (2.2.33).

2.2.6 Evaluating the sum over the zeros Σ_{m,δ,u,X_0}.

We assume Theorem 2.2.4. We split the sum Σ_{m,δ,u,X_0} vertically at heights $\gamma = 0$ (so as to use the symmetry with respect to the x-axis) and consider

$$\tilde{G}_{m,\delta,u}(\beta + i\gamma) = |G_{m,\delta,u}(\beta + i\gamma)| + |G_{m,\delta,u}(\beta - i\gamma)|.$$

We then split at $\gamma = H$ (so as to take advantage of the fact that all zeros below this horizontal line satisfy $\beta = 1/2$), and again at $\gamma = T_0$ and $\gamma = T_1$ (where T_1 will be chosen between T_0 and H), and consider:

$$\Sigma_0 = \sum_{0<\gamma\leq T_0} \tilde{G}_{m,\delta,u}(1/2 + i\gamma)X_0^{-1/2}, \tag{2.2.34}$$

$$\Sigma_1 = \sum_{T_0<\gamma\leq T_1} \tilde{G}_{m,\delta,u}(1/2 + i\gamma)X_0^{-1/2}, \tag{2.2.35}$$

$$\text{and} \quad \Sigma_2 = \sum_{T_1<\gamma\leq H} \tilde{G}_{m,\delta,u}(1/2 + i\gamma)X_0^{-1/2}. \tag{2.2.36}$$

For the remaining zeros (those with $\gamma > H$), we make use of the symmetry with respect to the critical line, and we split at $\beta = \sigma_0$ for some fixed $\sigma_0 > 1/2$ (we will consider $9/10 \leq \sigma_0 \leq 99/100$ for our computations). We denote

$$\begin{aligned}\Sigma_3 = &\sum_{\substack{\gamma>H\\ \beta=1/2}} \tilde{G}_{m,\delta,u}(1/2 + i\gamma)X_0^{-1/2}\\ &+ \sum_{\substack{\gamma>H\\ 1/2<\beta\leq\sigma_0}} \left(\tilde{G}_{m,\delta,u}(\beta + i\gamma)X_0^{\beta-1} + \tilde{G}_{m,\delta,u}(1-\beta + i\gamma)X_0^{-\beta}\right),\end{aligned} \tag{2.2.37}$$

$$\Sigma_4 = \sum_{\substack{\gamma>H\\ \sigma_0<\beta<1}} \left(\tilde{G}_{m,\delta,u}(\beta + i\gamma)X_0^{\beta-1} + \tilde{G}_{m,\delta,u}(1-\beta + i\gamma)X_0^{-\beta}\right). \tag{2.2.38}$$

As a conclusion, we have

$$\Sigma_{m,\delta,u,X_0} = \Sigma_0 + \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4. \tag{2.2.39}$$

We state here some preliminary results (see [20, equations (2.18), (2.19), (2.20), (2.21), (2.26)]).

Lemma 2.2.5. *Let T_0, H, R_0, σ_0 be as in Theorem 2.2.4. Let $m \geq 2, X_0 > 10$, and T_1 between T_0 and H. We define*

$$S_1(T_1) = \left(\frac{1}{2\pi} + q(T_0)\right)\left(\log\frac{T_1}{T_0}\log\frac{\sqrt{T_1T_0}}{2\pi}\right)\frac{2R(T_0)}{T_0}, \tag{2.2.40}$$

$$S_2(m, T_1) = \left(\frac{1}{2\pi} + q(T_1)\right)\left(\frac{1 + m\log\frac{T_1}{2\pi}}{m^2T_1^m} - \frac{1 + m\log\frac{H}{2\pi}}{m^2H^m}\right) + \frac{2R(T_1)}{T_1^{m+1}}, \tag{2.2.41}$$

$$S_3(m) = \left(\frac{1}{2\pi} + q(H)\right)\left(\frac{1 + m\log\frac{H}{2\pi})}{m^2H^m}\right) + \frac{2R(H)}{H^{m+1}}, \tag{2.2.42}$$

$$S_4(m, \sigma_0) = \left(c_1\left(1 + \frac{1}{m}\right) + \frac{c_2\log H}{H} + \left(c_3 + \frac{c_2}{m+1}\right)\frac{1}{H}\right)\frac{1}{H^m}, \tag{2.2.43}$$

$$S_5(X_0, m, \sigma_0) = \left(c_1 + \frac{c_2\log H}{H} + \frac{c_3}{H} + \left(c_1 + \frac{c_2}{H}\right)\frac{R_0}{2\log X_0}\frac{(\log H)^2}{(\frac{mR_0}{\log X_0})(\log H)^2 - 1}\right)\frac{1}{H^m}. \tag{2.2.44}$$

We assume Theorem 2.2.4. Then

$$\sum_{T_0<\gamma\leq T_1}\frac{1}{\gamma} \leq S_1(T_1), \tag{2.2.45}$$

$$\sum_{T_1<\gamma\leq H}\frac{1}{\gamma^{m+1}} \leq S_2(m, T_1), \tag{2.2.46}$$

$$\sum_{\gamma>H}\frac{1}{\gamma^{m+1}} \leq S_3(m), \tag{2.2.47}$$

$$\sum_{\substack{\gamma>H\\ \sigma_0<\beta<1}}\frac{1}{\gamma^{m+1}} \leq S_4(m, \sigma_0). \tag{2.2.48}$$

Moreover, if $\log X_0 < R_0m(\log H)^2$, then

$$\sum_{\substack{\gamma>H\\ \sigma_0<\beta<1}}\frac{X_0^{\frac{-1}{R_0\log\gamma}}}{\gamma^{m+1}} \leq S_5(X_0, m, \sigma_0)X_0^{\frac{-1}{R_0\log H}}. \tag{2.2.49}$$

Lemma 2.2.6. *Let m, δ, X_0 satisfy (2.2.1). We assume Theorem 2.2.4. If $\log X_0 <$*

$R_0 m(\log H)^2$, *then*

$$\Sigma_{m,\delta,u,X_0} \leq B_0(m,\delta)X_0^{-1/2} + B_1(m,\delta,T_1)X_0^{-1/2} + B_2(m,\delta,T_1)X_0^{-1/2} + B_3(m,\delta)\left(X_0^{\sigma_0-1} + X_0^{-\sigma_0}\right) + B_{41}(X_0,m,\delta,\sigma_0)X_0^{-\frac{1}{R_0\log(H)}} + B_{42}(m,\delta,\sigma_0)X_0^{-1+\frac{1}{R_0\log H}}, \quad (2.2.50)$$

where the B_i's are defined in (2.2.52), (2.2.55), (2.2.58), (2.2.60),(2.2.62), (2.2.63).

Proof. We investigate two ways to evaluate Σ_0 and Σ_1. For Σ_0, we can either combine (2.2.25) with (2.2.29) which computes $\sum_{0<\gamma\leq T_0} 1$, or (2.2.26) with (2.2.30) which computes $\sum_{0<\gamma\leq T_0}\gamma^{-1}$. We obtain respectively

$$X_0^{1/2}\Sigma_0 \leq \Sigma_{02}(m,\delta) = \frac{4F_{0,m,\delta}}{(e^{u/2}+1)}N_0, \quad \text{and} \quad X_0^{1/2}\Sigma_0 \leq \Sigma_{01}(m,\delta) = \frac{4F_{1,m,\delta}}{(e^{u/2}+1)\delta}S_0. \quad (2.2.51)$$

We denote

$$B_0(m,\delta) = \min(\Sigma_{01}(m,\delta), \Sigma_{02}(m,\delta)). \quad (2.2.52)$$

Thus

$$\Sigma_0 \leq B_0(m,\delta)X_0^{-1/2}. \quad (2.2.53)$$

For Σ_1, we can either combine (2.2.25) with the bound (2.2.32) for $N(T)$ from Theorem 2.2.4, or (2.2.26) with the bound (2.2.45) for $\sum_{T_0<\gamma\leq T_1}\gamma^{-1}$. We obtain

$$X_0^{1/2}\Sigma_1 \leq \Sigma_{12}(m,\delta) = \frac{4F_{0,m,\delta}}{e^{u/2}+1}(N(T_1)-N_0), \quad \text{and}$$
$$X_0^{1/2}\Sigma_1 \leq \Sigma_{11}(m,\delta) = \frac{4F_{1,m,\delta}}{(e^{u/2}+1)\delta}S_1(T_1). \quad (2.2.54)$$

We denote

$$B_1(m,\delta,T_1) = \min(\Sigma_{11}(m,\delta,T_1), \Sigma_{12}(m,\delta,T_1)). \quad (2.2.55)$$

Thus

$$\Sigma_1 \leq B_1(m,\delta,T_1)X_0^{-1/2}. \quad (2.2.56)$$

It follows from (2.2.28) and (2.2.46) that

$$\Sigma_2 \le B_2(m,\delta,T_1)X_0^{-1/2}, \tag{2.2.57}$$

with

$$B_2(m,\delta,T_1) = \frac{2F_{m,m,\delta}}{(e^{u/2}-1)\delta^m} S_2(m,T_1). \tag{2.2.58}$$

We use (2.2.28) to bound $\tilde{G}$ in Σ_3:

$$\Sigma_3 \le \frac{2F_{m,m,\delta}}{(e^u-1)\delta^m} \sum_{\substack{\gamma>H\\ 1/2\le\beta\le\sigma_0}} \frac{(e^{u\beta}+1)X_0^{\beta-1} + (e^{u(1-\beta)}+1)X_0^{-\beta}}{\gamma^{m+1}}.$$

Note that since $\log X_0 > u$, then $(e^{u\beta}+1)X_0^{\beta-1} + (e^{u(1-\beta)}+1)X_0^{-\beta}$ increases with $\beta \ge 1/2$. Moreover, we use (2.2.47) to bound the sum $\sum_{\substack{\gamma>H\\ \beta\ge 1/2}} \gamma^{-(m+1)}$, and obtain

$$\Sigma_3 \le B_3(m,\delta,\sigma_0)X_0^{\sigma_0-1} + B_3(m,\delta,1-\sigma_0)X_0^{-\sigma_0}, \tag{2.2.59}$$

where

$$B_3(m,\delta,\sigma) = \frac{2F_{m,m,\delta}}{\delta^m}\frac{e^{u\sigma}+1}{e^u-1} S_3(m). \tag{2.2.60}$$

For Σ_4 we use again (2.2.28) to bound $\tilde{G}$ and the fact that $X_0^{\beta-1} + X_0^{-\beta}$ increases with β. Since $\beta \le 1 - \frac{1}{R_0 \log\gamma}$ and $\gamma \ge H$ we obtain

$$\Sigma_4 \le \frac{2(e^u+1)F_{m,m,\delta}}{(e^u-1)\delta^m}\Bigg(\sum_{\substack{\gamma>H\\ \sigma_0<\beta<1}} \frac{X_0^{-\frac{1}{R_0\log\gamma}}}{\gamma^{m+1}} + X_0^{-1+\frac{1}{R_0\log H}} \sum_{\substack{\gamma>H\\ \sigma_0<\beta<1}} \frac{1}{\gamma^{m+1}}\Bigg).$$

We apply (2.2.48) and (2.2.49) to bound the above sums over the zeros and obtain

$$\Sigma_4 \le B_{41}(X_0,m,\delta,\sigma_0)X_0^{-\frac{1}{R_0\log(H)}} + B_{42}(m,\delta,\sigma_0)X_0^{-1+\frac{1}{R_0\log H}}, \tag{2.2.61}$$

with

$$B_{41}(X_0,m,\delta,\sigma_0)=\frac{2(e^u+1)F_{m,m,\delta}}{(e^u-1)\delta^m}S_5(X_0,m,\sigma_0), \tag{2.2.62}$$

$$B_{42}(X_0,m,\delta,\sigma_0)=\frac{2(e^u+1)F_{m,m,\delta}}{(e^u-1)\delta^m}S_4(m,\sigma_0). \tag{2.2.63}$$

□

Note that $G_{m,\delta,u}(1)=F_{0,m,\delta}$. We apply Proposition 2.2.3 and Lemma 2.2.6:

2.2.7 Main Theorem.

Theorem 2.2.7. *Let m,u,δ,a,Δ,X_0, and x satisfy (2.2.1). Let T_0,H,R_0,σ_0 be as in Theorem 2.2.4. We assume Theorem 2.2.4. If $X\geq X_0$ and*

$$\begin{aligned}F_{0,m,\delta}&-B_0(m,\delta)X_0^{-1/2}-B_1(m,\delta,T_1)X_0^{-1/2}-B_2(m,\delta,T_1)X_0^{-1/2}-B_3(m,\delta,\sigma_0)X_0^{\sigma_0-1}\\&-B_3(m,\delta,1-\sigma_0)X_0^{-\sigma_0}-B_{41}(X_0,m,\delta,\sigma_0)X_0^{-\frac{1}{R_0\log(H)}}-B_{42}(m,\delta,\sigma_0)X_0^{-1+\frac{1}{R_0\log H}}\\&-\frac{u}{2(e^u-1)}X_0^{-2}-\frac{\omega}{(e^u-1)}X_0^{-1/2}-\frac{2\nu(f,a)(1+\delta)}{\|f\|_1}\frac{\log(e^uX_0(1+\delta))}{\log(X_0(e^u-1))}>0,\end{aligned} \tag{2.2.64}$$

then there exists a prime number between $x(1-\Delta^{-1})$ and x.

2.3 Computations.

2.3.1 Introducing the Smooth Weight f

We choose the same weight as [67], that is

$$f_m(t)=(4t(1-t))^m \text{ if } 0\leq t\leq 1, \text{ and 0 otherwise.}$$

In [20] it was shown that a primitive form of f_m was providing a close to optimum weight to estimate $\psi(x)$. Thus, we believe that the above weight should also be close to optimal to evaluate $\psi(y) - \psi(x)$ when y is close to x. We recall [67, Lemma 6]:

$$\|f_m\|_1 = \frac{2^{2m}(m!)^2}{(2m+1)!}, \tag{2.3.1}$$

$$\|f_m^{(m)}\|_2 = \frac{2^{2m} m!}{\sqrt{2m+1}}. \tag{2.3.2}$$

We now provide estimates for $F_{k,m,\delta}$ as defined in (2.2.21).

Lemma 2.3.1. *Let $m \geq 2, \delta > 0,$ and $0 < \sigma < 1$. We define*

$$\lambda_0(m,\delta) = \frac{(2m+1)!}{2^{2m-1}(m!)^2},$$

$$\lambda_1(m,\delta) = \frac{(1+\delta)^2(2m+1)!}{2^{2m-1}(m!)^2},$$

$$\lambda(m,\delta) = \sqrt{\frac{(1+\delta)^{2m+3}-1}{\delta(2m+3)}}\,\frac{(2m+1)!}{m!\sqrt{2m+1}}.$$

Then

$$1 \leq F_{0,m,\delta} \leq 1+\delta, \tag{2.3.3}$$

$$\lambda_0(m,\delta) \leq F_{1,m,\delta}(\sigma) \leq \lambda_1(m,\delta), \tag{2.3.4}$$

$$F_{m,m,\delta}(\sigma) \leq \lambda(m,\delta). \tag{2.3.5}$$

Proof. Inequalities (2.3.3) follow trivially from the fact $1 \leq (1+\delta t) \leq 1+\delta$.

To bound $F_{1,m,\delta}$, we note that

$$\frac{\|f_m'\|_1}{\|f_m\|_1} \leq F_{1,m,\delta} \leq \frac{(1+\delta)^2\|f_m'\|_1}{\|f_m\|_1}.$$

Since $f_m'(t)$ has same sign as $1-2t$, we have

$$\|f_m'\|_1 = \int_1^{1/2} f_m'(t)dt - \int_{1/2}^1 f_m'(t)dt = 2f_m(1/2) - f_m(0) - f_m(1) = 2.$$

Inequality (2.3.4) follows from this and (2.3.1).

Lastly, for $F_{m.m,\delta}$, we apply (2.3.2) together with the Cauchy-Schwarz inequality:

$$F_{m,m,\delta}(\sigma) \leq \frac{\sqrt{\int_0^1 (1+\delta t)^{2(m+1)} dt}\sqrt{\int_0^1 |f_m^{(m)}(t)|^2 dt}}{\|f_m\|_1} = \sqrt{\frac{(1+\delta)^{2m+3}-1}{\delta(2m+3)}}\frac{\|f_m^{(m)}\|_2}{\|f_m\|_1}.$$

□

Note that while $F_{0,m,\delta}$ and $F_{1,m,\delta}$ can be easily computed as integrals, it is not the case for $F_{m,m,\delta}$. The following observation helps us to compute $F_{m,m,\delta}$ directly. We recognize in the definition of $f_m^{(m)}$ the analogue of Rodrigues' formula for the shifted Legendre polynomials:

$$f_m^{(m)}(t) = 4^m m! P_m(1-2t),$$

where $P_m(x)$ is the m^{th} Legendre polynomial, and

$$P_m(1-2t) = (-1)^m \sum_{k=0}^{m} \binom{m}{k}\binom{m+k}{k}(-t)^k.$$

For each each $P_m(1-2t)$, we denote $r_{j,m}$, with $j = 0, \ldots, m$, its $m+1$ roots. Since $P_m(1-2t)$ alternates sign between each of them, we have

$$F_{m,m,\delta} = \frac{\int_0^1 (1+\delta t)^{m+1} |P_m(1-2t)| dt}{\|f\|_1} = \frac{1}{\|f\|_1}\sum_{j=0}^{m-1}(-1)^j \int_{r_j}^{r_{j+1}} (1+\delta t)^{m+1} P_m(1-2t) dt,$$

and GP-Pari is able to compute quickly this sum of polynomial integrals.

2.3.2 Explicit results about the zeros of the Riemann zeta function

We provide here the latest values for the constants appearing in Theorem 2.2.4:

Theorem 2.3.2. *1. A numerical verification of the Riemann Hypothesis (Platt [63]):*

$$H = 3.061 \cdot 10^{10}.$$

2. A direct computation of some finite sums over the first zeros (using A. Odlyzko's *list of zeros):*

$$\text{For } T_0 = 1\,132\,491,\ N_0 = N(T_0) = 2\,001\,052,\ \text{and } S_0 = 11.637732363.$$

3. *A zero-free region (Kadiri [35, Theorem 1.1]):*

$$R_0 = 5.69693.$$

4. *An estimate for $N(T)$ (Rosser [68, Theorem 19]):*

$$a_1 = 0.137,\ a_2 = 0.443,\ a_3 = 1.588.$$

5. *An upper bound for $N(\sigma_0, T)$ (Kadiri [37]): For all $T \geq H$,*

$$N(\sigma, T) \leq c_1 T + c_2 \log T + c_3,$$

where the c_i's are given in Table 2.1.

Table 2.1: $N(\sigma, T) \leq c_1 T + c_2 \log T + c_3$.

σ	c_1	c_2	c_3
0.90	5.8494	0.4659	$-1.7905 \cdot 10^{11}$
0.91	5.6991	0.4539	$-1.7444 \cdot 10^{11}$
0.92	5.5564	0.4426	$-1.7007 \cdot 10^{11}$
0.93	5.4206	0.4318	$-1.6592 \cdot 10^{11}$
0.94	5.2913	0.4215	$-1.6196 \cdot 10^{11}$
0.95	5.1680	0.4116	$-1.5819 \cdot 10^{11}$
0.96	5.0503	0.4023	$-1.5458 \cdot 10^{11}$
0.97	4.9379	0.3933	$-1.5114 \cdot 10^{11}$
0.98	4.8304	0.3848	$-1.4785 \cdot 10^{11}$
0.99	4.7274	0.3766	$-1.4470 \cdot 10^{11}$

Note that [68, Theorem 19] was recently improved by T. Trudgian in [82, Corollary 1] with $a_1 = 0.111,\ a_2 = 0.275,\ a_3 = 2.450$. Our results are valid with both Rosser's or Trudgian's bounds.

2.3.3 Understanding the contribution of the low lying zeros

We assume Theorem 2.3.2 and that

$$m \geq m_0 = 5, \delta < \delta_0 = 2 \cdot 10^{-8}, \text{ and } T_1 > t_1 = 10^9 \tag{2.3.6}$$

(this would be consistent with the values we choose in Table 2.2). We observe that

$$B_0(m,\delta) = \Sigma_{02} \text{ and } B_1(m,\delta,T_1) = \Sigma_{12}.$$

where Σ_{02} and Σ_{12} are defined in (2.2.51) and (2.2.54) respectively. In other words, it turns out that we obtain a smaller bound for the sum over the small zeros ($0 < \gamma < T$) by using $N(T)$ directly instead of evaluating $\sum_{0<\gamma<T} \gamma^{-1}$. This essentially comes from the fact that our choice of parameters insures us with $\delta \ll \frac{F_{1,m,\delta} S_0}{F_{0,m,\delta} N_0}$ and $\delta \ll \frac{F_{1,m,\delta} S_1(T_1)}{F_{0,m,\delta}(N(T_1)-N_0)}$. We first prove the inequality

$$\frac{S_1(t)}{N(t)} \geq c_0 \frac{\log t}{t}. \tag{2.3.7}$$

Proof. We denote

$$\begin{aligned}
w_1 &= \frac{1}{2}\left(\frac{1}{2\pi} + q(T_0)\right) = 0.0795\ldots, \; w_2 = -\log(2\pi)\left(\frac{1}{2\pi} + q(T_0)\right) = -0.2925\ldots, \\
w_3 &= \left(\frac{1}{2\pi} + q(T_0)\right)\left(\frac{-\log^2(T_0)}{2} + \log(T_0)\log(2\pi)\right) + \frac{2R(T_0)}{T_0} = -11.3860\ldots, \\
v_1 &= \frac{1}{2\pi} = 0.1591\ldots, \; v_2 = \frac{-\log(2\pi)}{2\pi} - 1 = -1.2925\ldots, \; v_3 = a_1 = 0.137, \\
v_4 &= a_2 = 0.443, \; v_5 = a_3 + \frac{7}{8} = 2.463.
\end{aligned}$$

and

$$S_1(t) = w_1(\log t)^2 + w_2 \log t + w_3, \; P(t) + R(t) = v_1 t \log t + v_2 t + v_3 \log t + v_4 \log\log t + v_5.$$

We have from (2.2.40) and Theorem 2.3.2 (d) that

$$\frac{S_1(t)}{N(t)} \geq \frac{S_1(t)}{P(t)+R(t)} = \frac{w_1(\log t)^2 + w_2 \log t + w_3}{v_1 t \log t + v_2 t + v_3 \log t + v_4 \log\log t + v_5}.$$

Since $t > t_1 = 10^9$, we deduce the bound

$$\frac{S_1(t)}{N(t)} \geq c_0 \frac{\log t}{t}, \tag{2.3.8}$$

where

$$c_0 = \frac{w_1 + \frac{w_2}{\log t_1} + \frac{w_3}{(\log t_1)^2}}{v_1 + \frac{v_3}{t_1} + \frac{v_4 \log\log t_1}{t_1 \log t_1} + \frac{v_5}{t_1 \log t_1}} \geq 0.7508. \tag{2.3.9}$$

□

We now establish that $\Sigma_{01} + \Sigma_{11}, \Sigma_{01} + \Sigma_{12}$, and $\Sigma_{02} + \Sigma_{11}$ are all larger than $\Sigma_{02} + \Sigma_{12}$. We make use of Lemma 2.3.1 to provide estimates for the $F_{k,m,\delta}$'s, of (2.3.8), and of the assumptions (2.3.6) on m, δ, T_1.

Proof. We have

$$(\Sigma_{01} + \Sigma_{11}) - (\Sigma_{02} + \Sigma_{12}) = \frac{4}{e^{u/2}+1}\left(\frac{F_{1,m,\delta}}{\delta}(S_0 + S_1(T_1)) - F_{0,m,\delta} N(T_1)\right)$$
$$> \frac{4(1+\delta)N(T_1)}{e^{u/2}+1}\left(\frac{(2m_0+1)!}{2^{2m_0-1}(m_0!)^2}\frac{1}{\delta_0(1+\delta_0)}\left(\frac{S_0}{P(t_1)+R(t_1)} + c_0\frac{\log t_1}{t_1}\right) - 1\right) > 0,$$

since the right term between brackets is $> 2.4796 - 1 > 0$. We have

$$(\Sigma_{01} + \Sigma_{12}) - (\Sigma_{02} + \Sigma_{12}) = \left(\frac{S_0}{\delta}F_{1,m,\delta} - N_0 F_{0,m,\delta}\right)\frac{4}{e^{u/2}+1}$$
$$> \frac{4(1+\delta)N_0}{e^{u/2}+1}\left(\frac{(2m_0+1)!}{2^{2m_0-1}(m_0!)^2}\frac{1}{\delta_0(1+\delta_0)}\frac{S_0}{N_0} - 1\right) > 0$$

since the right term between brackets is $> 1574 - 1$. Finally,

$$(\Sigma_{02} + \Sigma_{11}) - (\Sigma_{02} + \Sigma_{12}) = \frac{4}{e^{u/2}+1}\left(\frac{F_{1,m,\delta}}{\delta} S_1(T_1) - F_{0,m,\delta}(N(T_1) - N_0)\right)$$
$$> \frac{4(1+\delta)(N(T_1) - N_0)}{e^{u/2}+1}\left(\frac{(2m_0+1)!}{2^{2m_0-1}(m_0!)^2}\frac{1}{\delta_0(1+\delta_0)}\frac{S_1(t_1)}{(\frac{S(t_1)t_1}{c_0 \log t_1} - N_0)} - 1\right) > 0$$

since the right term between brackets is $> 1.3737 - 1$.

□

The values for T_1 and a given in the next table are rounded down to the last digit.

Table 2.2: For all $x \geq x_0$, there exists a prime between $x(1 - \Delta^{-1})$ and x.

$\log x_0$	m	δ	T_1	σ_0	a	Δ
$\log(4 \cdot 10^{18})$	5	$3.580 \cdot 10^{-8}$	272519712	0.92	0.2129	36 082 898
43	5	$3.349 \cdot 10^{-8}$	291 316 980	0.92	0.2147	38 753 947
44	6	$2.330 \cdot 10^{-8}$	488 509 984	0.92	0.2324	61 162 616
45	7	$1.628 \cdot 10^{-8}$	797 398 875	0.92	0.2494	95 381 241
46	8	$1.134 \cdot 10^{-8}$	1 284 120 197	0.92	0.2651	148 306 019
47	9	$8.080 \cdot 10^{-9}$	1 996 029 891	0.92	0.2836	227 619 375
48	11	$6.000 \cdot 10^{-9}$	3 204 848 430	0.93	0.3050	346 582 570
49	15	$4.682 \cdot 10^{-9}$	5 415 123 831	0.93	0.3275	518 958 776
50	20	$3.889 \cdot 10^{-9}$	8 466 793 105	0.93	0.3543	753 575 355
51	28	$3.625 \cdot 10^{-9}$	12 399 463 961	0.93	0.3849	1 037 917 449
52	39	$3.803 \cdot 10^{-9}$	16 139 006 408	0.93	0.4127	1 313 524 036
53	48	$4.088 \cdot 10^{-9}$	18 290 358 817	0.93	0.4301	1 524 171 138
54	54	$4.311 \cdot 10^{-9}$	19 412 056 863	0.93	0.4398	1 670 398 039
55	56	$4.386 \cdot 10^{-9}$	19 757 119 193	0.93	0.4445	1 770 251 249
56	59	$4.508 \cdot 10^{-9}$	20 210 075 547	0.93	0.4481	1 838 818 070
57	59	$4.506 \cdot 10^{-9}$	20 219 045 843	0.93	0.4496	1 886 389 443
58	61	$4.590 \cdot 10^{-9}$	20 495 459 359	0.93	0.4514	1 920 768 795
59	61	$4.589 \cdot 10^{-9}$	20 499 925 573	0.93	0.4522	1 946 282 821
60	61	$4.588 \cdot 10^{-9}$	20 504 393 735	0.93	0.4527	1 966 196 911
150	64	$4.685 \cdot 10^{-9}$	21 029 543 983	0.96	0.4641	2 442 159 714

$(\log(4 \cdot 10^{18}) = 42.8328\ldots)$

2.3.4 Verification of the Ternary Goldbach conjecture

Proof of Corollary 2.1.2. Let $N = 4 \cdot 10^{18}$. We follow Oliveira e Silva, Herzog and Pardi [31]'s argument where the authors computed all the prime gaps up to $4 \cdot 10^{18}$.

From Table 2.2, we have that for $x = e^{60}$ and $\Delta = 1\ 966\ 090\ 061$, there exists at least one prime in the interval $(x - x/\Delta, x]$. this one has length $5.8082 \cdot 10^{16}$. Then $N\Delta = 7.8647 \cdot 10^{27}$ and we may infer that the gap between consecutive primes up to $N\Delta$ can be no larger than N (since $N\Delta/\Delta = N$). The corollary follows by using all the odd primes up to $N\Delta$ to extend the minimal Goldbach partitions (method of computation explained in [31, Section 1])of $4, 6, \ldots, N$ up to $N\Delta$. We also note that $N + 2 = 211 + (N - 209)$ and $N + 4 = 313 + (N - 309)$, where $211, 313, N - 209$, and $N - 309$ are all prime. Thus, there is at least one way to write each odd number greater than 5 and smaller than $N\Delta$ as the sum of at most 3 primes. $\square$

Chapter 3

New Explicit Bounds for $\psi(x; q, a)$

3.1 Introduction

3.1.1 Main Theorem and History.

Let a and q be relatively prime positive integers. In this chapter we study primes of the form $an + q$. We recall that for $x \geq 0$,

$$\psi(x; a, q) = \sum_{\substack{n \leq x \\ n \equiv a (\mathrm{mod}\, q)}} \Lambda(n),$$

where $\Lambda(n) = \log p$ if $n = p^k, k \geq 1$ and $\Lambda(n) = 0$ otherwise. The prime number theorem in arithmetic progressions (PNT in AP) is equivalent to

$$\psi(x; q, a) \sim \frac{x}{\varphi(q)} \text{ as } x \to \infty,$$

where $\varphi(q)$ is the Euler phi function. We shall establish explicit bounds for the error term

$$E(x; q, a) = \left| \frac{\varphi(q)\psi(x; q, a)}{x} - 1 \right|.$$

Bounds for $E(x; q, a)$ are useful in a wide range of problems. Applications include rational approximation to algebraic numbers of small height due to Bennett [6], finding solutions to Thue equations of degree $3, 4$ and 6 due to Lettl, Pethő and Voutier [48], Chebychev's bias for composite numbers due to Moree [60], and representing integers as the sum of 7 cubes as in Ramaré [65] and Elkies [19]. We prove in this chapter

Theorem 3.1.1. *For b a fixed positive constant and $x \geq 10^b$ there exists $\epsilon_{q,b} > 0$ such* *that* $E(x;q,a) \leq \epsilon_{q,b}$ *where* $\epsilon_{q,b}$ *is given explicitly in Theorem 3.7.8 and is computed in Table 3.10.*

The method of proof combines an optimized smoothing argument, sieving arguments, and some classical tools such as the partial verification of GRH due to Platt [64] and an explicit zero free region due to Kadiri [34]. We describe Rosser's argument which was written in terms of $\zeta(s)$ and was generalized in 1984 by McCurley to the setting of primes in arithmetic progressions. The technique relies on a classical explicit formula for $\psi(x;q,a)$ [13, §19, eqns 2 & 3] which relates the primes in arithmetic progressions to the nontrivial zeros of $L(s,\chi)$. Suppose χ is a primitive character modulo q, then define

$$\psi(x,\chi) = \sum_{n \leq x} \chi(n)\Lambda(n).$$

For x not a prime power, we have the following explicit formula from [13, §19, Eqns, 2 & 3]:

$$\psi(x,\chi) = \begin{cases} -\sum_{\rho} \frac{x^{\rho}}{\rho} - \frac{L'}{L}(0,\chi) + \sum_{n=1}^{\infty} \frac{x^{1-2n}}{2n-1} & \text{if } \chi(-1) = -1, \\ -\sum_{\rho} \frac{x^{\rho}}{\rho} - \log(x) - b(\chi) + \sum_{n=1}^{\infty} \frac{x^{-2n}}{2n} & \text{if } \chi(-1) = 1, \end{cases} \tag{3.1.1}$$

where ρ runs through the non-trivial zeros of $L(s,\chi)$ and $b(\chi)$ is the constant term in the Laurent expansion of $\frac{L'}{L}(s,\chi)$ about $s = 0$. As in the case with $\zeta(s)$, it is not possible to use this formula directly to bound the error term $E(x;q,a)$. This is because the sum over the nontrivial zeros is not absolutely convergent. To bypass this problem McCurley [56] adapts Rosser and Schoenfeld [69]'s argument in order to apply an explicit formula to an average of $\psi(x;q,a)$ on a small interval containing $[0,x]$. Authors following him obtain successively smaller bounds for the error term by relying on improved information regarding the location of the nontrivial zeros of $L(s,\chi)$. This information includes the verification of the generalized Riemann Hypothesis, for specific moduli q, up to a fixed height H and an explicit zero-free region of the form $\mathfrak{Re}(s) \geq 1 - \frac{1}{R_0 \log(q|\mathfrak{Im}(s)|)}$ where R_0 is a computable constant. On the other hand

Theorem 3.1.1 relies on new arguments. We introduce a smooth weight f and compare $\psi(x;q,a)$ to $\psi_f(x;q,a) = \sum_{n\leq x, n\equiv a(\mathrm{mod}\, q)} \Lambda(n) f\left(\frac{n}{x}\right)$. As in [20] we choose f in a close to optimal way to make the bounds for $E(x;q,a)$ as small as possible. Since the article of Rosser [68] there has been only one family of functions, F used to obtain these explicit bounds. In this chapter we place Rosser's proof into a general perspective which allows us to use the "best"(modulo approximation) family F in order to provide sharper estimates. None of the authors following Rosser (Schoenfeld [70], McCurley [56], Ramaré & Rumely [66] or Dusart [17]) do this. Additionally, we utilize the results from [20] in combination with the Brun-Titchmarsh inequality of [58] in order to give a better bound on the difference between $\frac{\psi(x)}{\varphi(q)}$ and $\psi_f(x)$.

3.1.2 Zeros of Dirichlet $L-$functions.

For χ a non-principal primitive character associated to a non-exceptional modulus q, the zeros of $L(s,\chi)$ are split between the "trivial" and "non-trivial" zeros. We define here a parameter $\mathfrak{a}$ which is given by

$$\mathfrak{a} = \mathfrak{a}(\chi) = \begin{cases} 0 & \chi(-1) = 1 \\ 1 & \chi(-1) = -1. \end{cases}$$

The trivial zeros have the form $-\mathfrak{a} - 2n$ for $n \geq 0$. The non-trivial zeros lie in the critical strip $(0 < \mathfrak{Res} < 1)$ and are denoted by $\varrho = \beta + i\gamma$. The generalized Riemann Hypothesis (GRH), asserts that $\beta = \frac{1}{2}$ for every such ϱ. Let $H_q > 0$. We say that $L(s,\chi)$ satisfies GRH(H_q) if for all of its nontrivial zeros with $|\gamma| \leq H_q$ we have that $\beta = \frac{1}{2}$. The following theorem of PLatt gives the latest admissible H_q values.

Theorem 3.1.2 (Theorem 7.1,[64]). *GRH holds for Dirichlet L-functions of primitive character modulus $q \leq 400{,}000$ and to height $H_q = \max\left(\frac{10^8}{q}, \frac{7.5\cdot 10^7}{q} + 200\right)$ for even q and to height $H_q = \max\left(\frac{10^8}{q}, \frac{3.75\cdot 10^7}{q} + 200\right)$ for odd q.*

Outside the above region we can assert that the remaining non-trivial zeros are not too close to the 1-line as given in the following theorem. Let $\mathcal{L}_q$ to be the product of the

$\varphi(q)$ Dirichlet L-functions modulo q.

Theorem 3.1.3 (Theorem 1.1, [34]). *The function $\mathcal{L}_q(\sigma + it)$ has, at most, a single zero in the region*

$$\sigma \geq 1 - \frac{1}{R_0 \log(\max(q, q|t|))} \text{ where } R_0 = 6.41.$$

Such a zero, if it exists, is real, simple and corresponds to a real non-principal character modulo q. We refer to it as an exceptional zero and q as an exceptional modulus.

Let $N(T, \chi)$ be the number of zeros $\varrho = \beta + i\gamma$ of $L(s, \chi)$ in the region $0 < \beta < 1$ and $|\gamma| \leq T$. Then

Theorem 3.1.4 (Theorem 1,[81]). *Let $T \geq 1$ and χ be a primitive character modulo q,*

$$P(q, T) = \frac{T}{\pi} \log\left(\frac{qT}{2\pi}\right) - \frac{T}{\pi}, \qquad R(q, T) = C_1 \log(qT) + C_2,$$

where $C_1 = 0.317$ and $C_2 = 6.401$ are admissible values. Then

$$|N(T, \chi) - P(q, T)| \leq R(q, T).$$

3.2 General form of an explicit inequality for $\psi(x; q, a)$.

3.2.1 Introducing a smooth weight f.

Definition 3.2.1. *Let $\alpha,\ \delta > 0, m \in \mathbb{N}$ and $m \geq 2$. We define a function f on $[0, \alpha+\delta]$ by $f(x) = 1$ if $0 \leq x \leq \alpha$, $f(x) = 0$ if $x \geq \alpha + \delta$, and $f(x) = g\left(\frac{x-\alpha}{\delta}\right)$ if $\alpha \leq x \leq \alpha + \delta$, where g is a function defined on $[0, 1]$ satisfying*

1. *$0 \leq g(x) \leq 1$ for $0 \leq x \leq 1$,*

2. *g is an m-times differentiable function on $(0, 1)$ such that for all $k = 1, \dots, m$,*

$$g^{(k)}(0) = g^{(k)}(1) = 0,$$

and there exist positive constants a_k such that

$$|g^{(k)}(x)| \le a_k \quad \textit{for all} \quad 0 < x < 1.$$

We now consider

$$\psi_f(x;q,a) = \sum_{\substack{n\ge 1 \\ n\equiv a (\bmod q)}} \Lambda(n) f\Big(\frac{n}{x}\Big) \quad \text{and} \quad E_f(x;q,a) = \left| \frac{\psi_f(x;q,a) - \frac{x}{\varphi(q)}}{\frac{x}{\varphi(q)}} \right|. \qquad (3.2.1)$$

Let $\delta > 0$. We denote f^- and f^+ for the functions f defined above with the parameters $\alpha = 1-\delta$ and $\alpha = 1$ respectively. We define ψ_f^- and ψ_f^+ the sums ψ_f associated to the weights f^- and f^+ respectively, and denote E_f^- and E_f^+ are their respective error terms. Observe that

$$\psi_f^-(x;q,a) \le \psi(x;q,a) \le \psi_f^+(x;q,a) \quad \text{and} \quad E(x;q,a) \le \max\Big(E_f^-(x;q,a), E_f^+(x;q,a)\Big). \qquad (3.2.2)$$

We consider the Mellin Transform of f given by

$$F(s) = \int_0^\infty f(t) t^{s-1} dt, \qquad (3.2.3)$$

and

$$M(\alpha, \alpha+\delta, k) = \int_0^1 |g^{(k+1)}(u)| \,(\delta u + \alpha)^{k+1}\, du \qquad \text{where } k \in \mathbb{N}. \qquad (3.2.4)$$

These were studied in [20, Section 2.1] and we restate here the properties we will need to study $\psi_f(x;q,a)$.

Lemma 3.2.1. *[20, Lemma 2.1]*

1. *F is analytic in $Re(s) > 0$.*

2. *F of f has a single pole at $s = 0$ with residue 1 and is analytic everywhere else.*

3.

$$F(1) = \alpha + \delta \int_0^1 g(u)du, \tag{3.2.5}$$

$$|F(s)| \leq \frac{M(\alpha, \alpha+\delta, k)}{\delta^k |s|^{k+1}}, \quad \textit{for all} \quad k = 0, \ldots, m, s \in \mathbb{C}. \tag{3.2.6}$$

We choose the same weight as [20], that is

$$g(x) = 1 - \frac{(2m+1)!}{(m!)^2} \int_0^x t^m (1-t)^m dt, \tag{3.2.7}$$

and recall that:

$$\int_0^1 g(u)du = \frac{1}{2}, \tag{3.2.8}$$

$$M(\alpha, \alpha+\delta, 0) = \frac{2\alpha+\delta}{2}. \tag{3.2.9}$$

Additionally, we note that

$$g^{(m+1)}(u) = -\frac{(2m+1)!}{m!} P_m(1-2u), \tag{3.2.10}$$

where P_m is the m^{th} Legendre polynomial, which has the following explicit expression (see [41, formula (0.2)]):

$$P_m(x) = 2^m \sum_{k=0}^{m} \binom{m}{k} \binom{\frac{m+k-1}{2}}{m} x^k.$$

The m roots of $P_m(x)$, r_k for $k = 1, 2, \ldots, m$, are easily computed with gp-pari, thus, letting $r_0 = 1$ and $r_{m+1} = -1$ we compute $M(\alpha, \alpha+\delta, m)$ directly from

$$\begin{aligned} M(\alpha, \alpha+\delta, m) &= \frac{(2m+1)!}{m!} \int_0^1 |P_m(1-2u)| \, (\delta u + \alpha)^{m+1} \, du \\ &= \sum_{a=0}^{m} (-1)^{m-k} \int_{(1-r_a)/2}^{(1-r_{a+1})/2} P_m(1-2u)(\delta u + \alpha)^{m+1} du. \end{aligned} \tag{3.2.11}$$

We recall from [20, Equation 3.5] that

$$M(\alpha, \alpha+\delta, m) \leq \lambda(\alpha, \alpha+\delta, m) = \sqrt{\frac{(\alpha+\delta)^{2m+3} - \alpha^{2m+3}}{\delta(2m+3)}} \cdot \frac{\sqrt{(2m)!(2m+1)!}}{m!}. \quad (3.2.12)$$

3.3 Handling the Imprimitive case

We recall that a sum over all integers $n \equiv a (\bmod\, q)$ can be interpreted as a sum over all characters modulo q by the orthogonality of characters. On the other hand we will make use of an explicit formula to relate this sum to the zeros of Dirichlet L-functions. As this formula is only valid for primitive characters, we handle the imprimitive characters before appealing to the explicit formula. To do so, we use an argument originally due to Lagarias and Odlyzko [43], though the form it is given in follows more closely the one provided in Ramaré and Rumely [66, p 414]. We reprove the result here in full detail. Throughout this chapter we use the following notation. For each non-principal Dirichlet character χ (mod q) we denote χ^* as the primitive character that induces it, and q^* its associated modulus. Define

$$\omega_q(a, n) = \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \chi^*(n)\overline{\chi}(a). \quad (3.3.1)$$

Let Q_n is the largest divisor of q such that $(n, Q_n) = 1$, in [66, §4.3 p.414] it was shown that:

$$\omega_q(a, n) = \begin{cases} \frac{\varphi(Q_n)}{\varphi(q)} & \text{if } n \equiv a (\bmod\, Q_n) \\ 0 & \text{otherwise .} \end{cases}$$

Hence it suffices to study

$$\psi_f^*(x; q, a) = \sum_{n \geq 1} \omega_q(a, n)\Lambda(n) f\left(\frac{n}{x}\right). \quad (3.3.2)$$

Lemma 3.3.1. *We have*

$$0 \leq \psi_f^*(x; q, a) - \psi_f(x; q, a) \leq \log(x) \sum_{p|q} \frac{1}{p-1}.$$

Proof. For the first inequality we note that if $n \equiv a (\bmod\ q)$ then $n \equiv a (\bmod\, Q_n)$. We also note that if $(n, q) = 1$ then $\omega_q(a, n) = 1$ and $\varphi(q) \geq \varphi(Q_n)$.

For the second inequality, we consider

$$\begin{aligned}
\psi_f^*(x; q, a) - \psi_f(x; q, a) &= \sum_{\substack{n \leq x \\ n \equiv a (\bmod\, Q_n)}} \frac{\varphi(Q_n)}{\varphi(q)} \Lambda(n) f\left(\frac{n}{x}\right) - \sum_{\substack{n \leq x \\ n \equiv a (\bmod\, q)}} \Lambda(n) f\left(\frac{n}{x}\right) \\
&= \sum_{\substack{n \leq x \\ n \equiv a (\bmod\, Q_n) \\ n \not\equiv a (\bmod\, q)}} \frac{\varphi(Q_n)}{\varphi(q)} \Lambda(n) f\left(\frac{n}{x}\right) + \sum_{\substack{n \leq x \\ n \equiv a (\bmod\ q)}} \left(\frac{\varphi(Q_n)}{\varphi(q)} - 1\right) \Lambda(n) f\left(\frac{n}{x}\right) \\
&= \sum_{\substack{n \leq x \\ n \equiv a (\bmod\, Q_n) \\ n \not\equiv a (\bmod\, q)}} \frac{\varphi(Q_n)}{\varphi(q)} \Lambda(n) f\left(\frac{n}{x}\right) + \sum_{\substack{n \leq x \\ n \equiv a (\bmod\ q) \\ Q_n \neq q}} \left(\frac{\varphi(Q_n)}{\varphi(q)} - 1\right) \Lambda(n) f\left(\frac{n}{x}\right).
\end{aligned}$$

Note that the quantity $\frac{\varphi(Q_n)}{\varphi(q)} - 1 < 0$,

$$\psi_f^*(x; q, a) - \psi_f(x; q, a) \leq \sum_{\substack{n \leq x \\ n \equiv a (\bmod\, Q_n) \\ n \not\equiv a (\bmod\, q)}} \frac{\varphi(Q_n)}{\varphi(q)} \Lambda(n) f\left(\frac{n}{x}\right).$$

We have

$$\frac{\varphi(Q_n)}{\varphi(q)} = \frac{Q_n \prod_{p \mid Q_n} \left(1 - \frac{1}{p}\right)}{q \prod_{p \mid q} \left(1 - \frac{1}{p}\right)} = \frac{Q_n}{q} \prod_{\substack{p \mid q \\ p \nmid Q_n}} (1 - \frac{1}{p})^{-1}.$$

As we are summing over the prime powers for n ($n = p_n^k$), then $q = Q_n p_n^{\nu_p(q)}$, and thus

$$\frac{\varphi(Q_n)}{\varphi(q)} = \frac{1}{\varphi(p^{\nu_p(q)})} = \frac{1}{p^{\nu_p(q)-1}(p-1)} \leq \frac{1}{p-1}.$$

Hence for $l = p^k$, bounding f by 1 gives

$$\psi_f^*(x;q,a) - \psi_f(x;q,a) \leq \sum_{\substack{l^k \leq x \\ l^k \equiv a(\bmod Q_n) \\ l^k \not\equiv a(\bmod q) \\ p|q, p\nmid Q_n}} \frac{\log(l)}{p^{\nu_p(q)-1}(p-1)} \leq \log(x) \sum_{p|q} \frac{1}{p-1}.$$

□

It follows that

$$\psi_f^*(x;q,a) = \frac{1}{\varphi(q)} \sum_{n\geq 1} \Lambda(n) f\left(\frac{n}{x}\right) + \frac{1}{\varphi(q)} \sum_{\substack{\chi(\bmod q) \\ \chi\neq\chi_0}} \overline{\chi}(a) \sum_{n\geq 1} \chi^*(n)\Lambda(n) f\left(\frac{n}{x}\right). \quad (3.3.3)$$

It is convenient to define

$$\mathscr{S}(x) = \sum_{n\geq 1} \Lambda(n) f\left(\frac{n}{x}\right) \text{ and } \mathscr{S}(x;q,a) = \sum_{\substack{\chi(\bmod q) \\ \chi\neq\chi_0}} \overline{\chi}(a) \sum_{n\geq 1} \chi^*(n)\Lambda(n) f\left(\frac{n}{x}\right). \quad (3.3.4)$$

3.4 Bounding $\mathscr{S}(x)$.

We want to use previous bounds computed for $\psi(x)$ to estimate our $\mathscr{S}(x)$. There are two ways in which we can do this, which are described below. We consider both methods and our computation will take the one that minimizes the bound for $\mathscr{S}(x)$.

Lemma 3.4.1. *[20, Theorem 1.1] Let $b_0 \leq 9963$ be a fixed positive constant. Let $x \geq e^{b_0}$. Then there exists $\epsilon_x > 0$ such that $|\psi(x) - x| \leq \epsilon_x x$, where numerical values for ϵ_x are given in [20, Table 3]*

We give a sample of values in Table 3.1.

3.4.1 Partial Summation

Here we use a simple application of partial summation, combined with properties of our smoothing function to get a handle on the difference between $\mathscr{S}(x)$ and $\psi(x)$.

Table 3.1: Bounds for $\psi(x)$ from [20].

x	ϵ_x
$\frac{10^{10}}{2}$	$1.7156 \cdot 10^{-4}$
10^{10}	$1.2293 \cdot 10^{-4}$
$\frac{10^{13}}{2}$	$7.6034 \cdot -6$
10^{13}	$5.8239 \cdot 10^{-6}$
$\frac{10^{30}}{2}$	$1.6090 \cdot 10-9$
10^{30}	$1.6085 \cdot 10^{-9}$
$\frac{10^{100}}{2}$	$1.5277 \cdot 10^{-9}$
10^{100}	$1.5274 \cdot 10^{-9}$

Lemma 3.4.2. *For $x \geq 10^8$ we have*

$$\sum_{n\geq 1} \Lambda(n) f\left(\frac{n}{x}\right) \leq x + xJ_1(x,\delta) = x + x\left(\frac{\delta}{2} + \epsilon_{(\alpha+\delta)x}\left(\alpha + \frac{\delta}{2}\right)\right). \tag{3.4.1}$$

Proof. We use partial summation and the fact that f is decreasing in $[\alpha, \alpha+\delta]$ to obtain

$$\begin{aligned}\sum_{n\geq 1} \Lambda(n) f\left(\frac{n}{x}\right) &= -\int_{\alpha}^{\infty} f'(u) \sum_{n\leq ux} \Lambda(n) du \\ &= -\int_{\alpha}^{\alpha+\delta} f'(u)\psi(ux)du.\end{aligned}$$

The second equality holds by the definitions of $\psi(x)$ and f respectively.

We use Lemma 3.4.1, with ϵ_{ux} given in [20, Table 3], and integration by parts to obtain

$$\begin{aligned}\sum_{n\geq 1} \Lambda(n) f\left(\frac{n}{x}\right) &\leq x(1+\epsilon_{(\alpha+\delta)x}) \int_{\alpha}^{\alpha+\delta} (-f'(u))udu \\ &\leq -x(1+\epsilon_{(\alpha+\delta)x})\left(uf(u)|_{\alpha}^{\alpha+\delta} - \int_{\alpha}^{\alpha+\delta} f(u)du\right) \\ &= x(1+\epsilon_{(\alpha+\delta)x})(\alpha+\frac{\delta}{2}) \leq x + x\left(\frac{\delta}{2} + \epsilon_{(\alpha+\delta)x}\left(\alpha+\frac{\delta}{2}\right)\right).\end{aligned}$$

Where the second equality comes from the definition of f and (3.2.8). □

3.4.2 Brun-Titchmarsh

In this section we discuss a second method for bounding the difference between $\mathscr{S}(x)$ and $\psi(x)$. The effect of this technique will be explained in Section 3.9.1.

Lemma 3.4.3. *Let x, y and k be positive real numbers, with $x \geq 10^8$ and $1 < y \leq kx$. Let $C(k) = \frac{k}{2} + (0.0557)\sqrt{1+k} + 0.001$. Then*

$$\psi(x+y) - \psi(x) < \frac{2y\log(x+y)}{\log(y)} + C(k)\sqrt{x}.$$

Proof. We see from [58, Theorem 2] that we can write

$$\theta(x+y) - \theta(x) < \frac{2y\log(x+y)}{\log(y)}.$$

From [12, Theorem 4 & Theorem 5] we have that for all $x \geq 10^8$:

$$C_1\sqrt{x} < \psi(x) - \theta(x) < C_2\sqrt{x}$$

where $C_1 = 0.999$, $C_2 = 1.0557$. Thus

$$\begin{aligned}\psi(x+y) - \psi(x) &= \psi(x+y) - \theta(x+y) + \theta(x+y) - \theta(x) + \theta(x) - \psi(x)\\ &\leq C_2\sqrt{x+y} + \frac{2y\log(x+y)}{\log(y)} - C_1\sqrt{x}.\end{aligned}$$

Use the mean value theorem to bound $C_2\sqrt{x+y} - C_1\sqrt{x}$:

$$\begin{aligned}C_2\sqrt{x+y} - C_1\sqrt{x} &= \sqrt{x+y} - \sqrt{x} + (C_2-1)\sqrt{x+y} + (1-C_1)\sqrt{x}\\ &\leq \frac{y}{2\sqrt{x}} + (C_2-1)\sqrt{x+y} + (1-C_1)\sqrt{x}\\ &= \sqrt{x}\left(\frac{y}{2x} + (C_2-1)\sqrt{1+\frac{y}{x}} + 1 - C_1\right)\\ &\leq \sqrt{x}\left(\frac{k}{2} + (C_2-1)\sqrt{1+k} + 1 - C_1\right)\end{aligned}$$

□

Lemma 3.4.4. *Let $\frac{e}{x} \leq z \leq \frac{\delta}{2}$ and let C be the constant given in Lemma 3.4.3. Then* *for* $x \geq x_0 \geq 10^8$ *we have*

$$\sum_{n \geq 1} \Lambda(n) f\left(\frac{n}{x}\right) \leq x + xJ_2(x_0, m, \alpha, \delta, z))$$

where

$$\begin{aligned} J_2(x_0, m, \alpha, \delta, z) &= \epsilon_{x_0} + (-f'(\alpha+z)) \frac{2e\log(x_0(\alpha+\delta-z)+e)}{x_0^2} \\ &+ (\delta - 2z + \epsilon_{x_0/2}(2+\delta))(f(\alpha+z) - f(\alpha+d-z)) \qquad (3.4.2) \\ &+ (1-f(\alpha+z))\left(\frac{4z\log(x_0(\alpha+z))}{\log(zx_0)} + \frac{2}{\sqrt{x_0}} C\left(\frac{z}{\alpha}\right)\left(\sqrt{\alpha} + \sqrt{\alpha+\delta-z}\right)\right). \end{aligned}$$

Proof. In this proof we assume $\frac{e}{x} < z \leq \frac{\delta}{2}$. By definition

$$\sum_{n \geq 1} \Lambda(n) f\left(\frac{n}{x}\right) = \psi(x) + \sum_{\substack{n \in \mathbb{N} \\ \frac{n}{x} \in [\alpha, \alpha+\delta]}} \Lambda(n) f\left(\frac{n}{x}\right). \qquad (3.4.3)$$

By Lemma 3.4.1

$$\psi(x) \leq (1+\epsilon_x)x. \qquad (3.4.4)$$

In the second sum in (3.4.3) we split $[\alpha, \alpha+\delta]$ into five intervals:

$$\begin{aligned} I_1 = [\alpha, \alpha + \frac{e}{x}],\ I_2 = [\alpha + \frac{e}{x}, \alpha + z],\ I_3 = [\alpha+z, \alpha+\delta-z] \\ I_4 = [\alpha+\delta-z, \alpha+\delta-\frac{e}{x}], \quad \text{and} \quad I_5 = [\alpha+\delta-\frac{e}{x}, \alpha+\delta]. \end{aligned}$$

For $j = 1, 2, 3, 4, 5$ let $I_j = [a_j, b_j]$ and

$$S_j = \sum_{\substack{n \in \mathbb{N} \\ \frac{n}{x} \in I_j}} \Lambda(n) f\left(\frac{n}{x}\right).$$

By partial summation we have

$$S_j = \int_{a_j}^{b_j} (-f'(u))(\psi(ux) - \psi(a_j x))du.$$

We recall that the symmetry of f gives $f(t+\alpha) = 1 - f(\alpha+\delta-t)$ for $0 \leq t \leq \delta$. Hence making the variable change $v = 2\alpha + \delta - u$ in both S_4 and S_5 gives $I_4' = [\alpha, \alpha + z - \frac{e}{x}]$ and $I_5' = [\alpha + z - \frac{e}{x}, \alpha + z]$, such that for $j = 4, 5$, let $I_j' = [c_j, d_j]$ we have

$$S_j = \int_{c_j}^{d_j} (-f'(v))(\psi((2\alpha + \delta - v)x) - \psi(xc_j))dv.$$

In order to bound S_j, we require bounds for $\psi(ux) - \psi(a_j x)$ and $\psi(x(2\alpha+\delta-v)) - \psi(xc_j)$. This term is bounded trivially for S_1 and S_5. We bound S_2 and S_4 using the Brun-Titchmarsh inequality (Lemma 3.4.4). For S_3, we apply Faber and Kadiri's explicit bounds for $\psi(x)$ as in Lemma 3.4.1. We now bound the S_j's.

We have the trivial bounds

$$\psi(ux) - \psi(\alpha x) \leq \log(\alpha x + e) \quad \text{for } u \in I_1$$

and

$$\psi((2\alpha + \delta - v)x) - \psi((\alpha + \delta - z)x) \leq \log((\alpha + \delta - z)x + e) \quad \text{for } v \in I_5'.$$

Thus

$$S_1 \leq \frac{e}{x} \log(\alpha x + e)(-f'(\alpha + z))$$

and

$$S_5 \leq \frac{e}{x} \log((\alpha + \delta - z)x + e)(-f'(\alpha + z)).$$

Note that $\frac{\log((\alpha x+e)((\alpha+\delta-z)x+e))}{x} \leq \frac{\log((\alpha+\delta-z)x+e)}{x}$ so that

$$S_1 + S_5 \leq (-f'(\alpha + \delta)) \frac{2e \log((\alpha + \delta - z)x + e)}{x}. \tag{3.4.5}$$

Next, by Lemma 3.4.1, since $ux \geq (\alpha+z)x \geq \frac{x_0}{2}$, for $x \geq x_0$ we have

$$\begin{aligned}\psi(ux) - \psi(x(\alpha+z)) &\leq (1+\epsilon_{x_0/2})ux - (1-\epsilon_{x_0/2})(\alpha+z)x \\ &\leq x((u-\alpha-z) + \epsilon_{x_0/2}(u+\alpha+z)) \\ &\leq x(\delta - 2z + \epsilon_{x_0/2}(2+\delta)).\end{aligned}$$

So that

$$\begin{aligned}S_3 &\leq x(\delta - 2z + \epsilon_{x_0/2}(2+\delta)) \int_{\alpha+z}^{\alpha+\delta-z} (-f'(u))du \\ &\leq x(\delta - 2z + \epsilon_{x_0/2}(2+\delta))(f(\alpha+z) - f(\alpha+\delta-z)). \qquad (3.4.6)\end{aligned}$$

Finally, let $X = \alpha x$, $Y = (u-\alpha)x$ and $k = \frac{(u-\alpha)}{\alpha}$ as in Lemma 3.4.4 to obtain

$$S_2 \leq \int_{\alpha+\frac{e}{x}}^{\alpha+z} (-f'(u)) \left(\frac{2x(u-\alpha)\log(xu)}{\log(x(u-\alpha))} + C\left(\frac{u-\alpha}{\alpha}\right)\sqrt{\alpha x} \right) du.$$

Since $-f'$ is positive and the second function in the integrand is increasing in u we obtain

$$S_2 \leq \left(f\left(\alpha + \frac{e}{x}\right) - f(\alpha+z) \right) \left(\frac{2zx\log(x(\alpha+z))}{\log(zx)} + C\left(\frac{z}{\alpha}\right)\sqrt{\alpha x} \right). \qquad (3.4.7)$$

Similarly for S_4, let $X = (2\alpha+\delta-v)x$, $Y = (\alpha+z-v)x$ and $k = \frac{\alpha+z-v}{2\alpha+\delta-v}$ in Lemma 3.4.4 to obtain

$$S_4 \leq \int_{\alpha}^{\alpha+z-\frac{e}{x}} (-f'(v)) \left(\frac{2x(\alpha+z-v)\log(x(2\alpha+\delta-v))}{\log(\alpha+z-v)} + C\left(\frac{\alpha+z-v}{2\alpha+\delta-v}\right)\sqrt{x(\alpha+\delta-z)} \right).$$

Since $-f'$ is positive and the second function in the integrand is decreasing in u we obtain

$$S_4 \leq \left(f(\alpha) - f\left(\alpha+z-\frac{e}{x}\right) \right) \left(\frac{2zx\log(x(\alpha+\delta))}{\log(zx)} + C\left(\frac{z}{\alpha+\delta}\right)\sqrt{(\alpha+\delta-z)x} \right). \qquad (3.4.8)$$

Note that $f\left(\alpha + \frac{e}{x}\right) \leq f(\alpha) = 1$, $f\left(\alpha + z - \frac{e}{x}\right) \geq f(\alpha + z)$ and that $C(k)$ increases with k so that $C(z/(\alpha+\delta)) \leq C(z/\alpha)$. Hence we combine (3.4.7) and (3.4.8) to give

$$S_2 + S_4 \leq (1 - f(\alpha+z))\left(\frac{4zx\log((\alpha+z)x)}{\log(zx)} + 2C\left(\frac{z}{\alpha}\right)\left(\sqrt{\alpha x} + \sqrt{x(\alpha+\delta-z)}\right)\right). \tag{3.4.9}$$

So if we combine (3.4.4), (3.4.5), (3.4.6) and (3.4.9) and factor out an x we obtain

$$\begin{aligned} x + xJ_2(x,m,\alpha,\delta,z) = x\Bigg(&1 + \epsilon_x + (-f'(\alpha+z))\frac{2e\log(x(\alpha+\delta-z)+e)}{x^2} \\ &+(\delta - 2z + \epsilon_{x/2_0}(2+\delta))(f(\alpha+z) - f(\alpha+d-z)) \\ &+(1-f(\alpha+z))\left(\frac{4z\log(x(\alpha+z))}{\log(zx)} + \frac{2}{x}C\left(\frac{z}{\alpha}\right)\left(\sqrt{\alpha} + \sqrt{\alpha+\delta-z}\right)\right)\Bigg). \end{aligned}$$

We note that $\frac{d}{dx}J_2(x,m,\alpha,\delta,z) < 0$ so that for $x \geq x_0$ we have $J_2(x,m,\alpha,\delta,z) \leq J_2(x_0,m,\alpha,\delta,z)$ which completes the proof. □

3.4.3 Bounds for $\mathscr{S}(x)$.

Considering the bounds from Lemma 3.4.2 and Lemma 3.4.4 we have that

$$\left|\frac{\psi(x;q,a) - \frac{x}{\varphi(q)}}{\frac{x}{\varphi(q)}}\right| \leq \min\left(J_1(x,\delta), J_2(x,m,\alpha,\delta,z)\right) + \frac{1}{x}\left|\mathscr{S}(x;q,a)\right|. \tag{3.4.10}$$

It remains to prove bounds for $|\mathscr{S}(x;q,a)|$.

3.5 An Explicit Formula

Using the Inverse Mellin formula for f we can deduce the following

$$f\left(\frac{n}{x}\right) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} n^{-s}F(s)x^s ds, \tag{3.5.1}$$

for $c > 1$. By (3.3.1), (3.3.4) and (3.5.1) we have

$$\mathscr{S}(x;q,a) = \frac{1}{\varphi(q)}\sum_{\substack{\chi \bmod q \\ \chi \neq \chi_0}} \overline{\chi}(a)\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\left(-\frac{L'}{L}(s,\chi^*)\right)x^s ds. \tag{3.5.2}$$

Define $Z(\chi^*)$ as the set of zeros for $L(s,\chi^*)$ such that $0 < \mathfrak{Re}\rho < 1$.

3.5.1 The Explicit Formula

Poles of $F(s)x^s$

We recall that

$$F(s) = \frac{G(s)}{s}, \qquad \text{where} \qquad G(s) = -\int_{\alpha}^{\alpha+\delta} f'(t)t^s dt,$$

is an entire function such that $G(0) = 1$ (see [20, proof of Lemma 2.1]). Hence $F(s)x^s$ has a simple pole at $s = 0$ and its Laurent expansion at this point is

$$\frac{1}{s}(G(0)+G'(0)s+G''(0)\frac{s^2}{2}+\cdots)(1+\log(x)s+\cdots) = \frac{1}{s}\left(1 + \left(\log x + G'(0)\right)s + \mathcal{O}(s^2)\right). \tag{3.5.3}$$

We have

$$G'(0) = \log\alpha + \int_{\alpha}^{\alpha+\delta} \frac{f(t)}{t}dt. \tag{3.5.4}$$

Poles of $-\frac{L'}{L}(s,\chi^*)$

1. χ^* is even

   $-\frac{L'}{L}(s,\chi^*)$ has a simple pole at $s = 0$, at the non-trivial zeros $\rho \in Z(\chi^*)$, and at the trivial zeros $-2n, n \in \mathbb{N}$. The residues at the zeros are -1 and the following lemma presents the Laurent expansion at $s = 0$.

Lemma 3.5.1. *Let χ^* be a primitive character modulo q^* with $q^* > 1$ and $\chi^*(-1) = 1$ and s near 0 we have the Laurent expansion*

$$-\frac{L'}{L}(s,\chi^*) = -\frac{1}{s} + B(\chi^*) + \mathcal{O}(s), \tag{3.5.5}$$

where

$$B(\chi^*) = \log(q^*) - \gamma_0 - \log(2\pi) + \frac{L'}{L}(1,\overline{\chi^*}), \tag{3.5.6}$$

where γ_0 represents Euler's constant.

This result also appears in [29, §4.1, p. 33], however we provide a proof for completeness.

Proof. It follows from the functional equation of $L(s,\chi^*)$ that

$$-\frac{L'}{L}(s,\chi^*) = \log\left(\frac{q^*}{\pi}\right) + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1-s}{2}\right) + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right) + \frac{L'}{L}\left(1-s,\overline{\chi^*}\right).$$

Since $\frac{1}{s} + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right) = \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+1\right)$, then we can rewrite

$$-\frac{L'}{L}(s,\chi^*) + \frac{1}{s} = \log\left(\frac{q^*}{\pi}\right) + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1-s}{2}\right) + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+1\right) + \frac{L'}{L}\left(1-s,\overline{\chi^*}\right).$$

We denote $B(\chi^*)$ the limit of the above at $s=0$:

$$\begin{aligned} B(\chi^*) &= \lim_{s\to 0}\left[-\frac{L'}{L}(s,\chi^*) + \frac{1}{s}\right] = \log\left(\frac{q^*}{\pi}\right) + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}\right) + \frac{1}{2}\frac{\Gamma'}{\Gamma}(1) + \frac{L'}{L}(1,\overline{\chi^*}) \\ &= \log(q^*) - \gamma_0 - \log(2\pi) + \frac{L'}{L}(1,\overline{\chi^*}), \end{aligned}$$

since from [1, Eqn. 6.3.4 and Eqn 6.3.2] we respectively obtain $\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}\right) = -\gamma_0 - 2\log 2$, and $\frac{\Gamma'}{\Gamma}(1) = -\gamma_0$. □

2. χ^* is odd

   $-\frac{L'}{L}(s,\chi^*)$ has a simple pole at the non-trivial zeros $\rho \in Z(\chi^*)$, and at the trivial zeros $-2n+1, n \in \mathbb{N}$, all with residue -1.

The Explicit Formula

Proposition 3.5.2. *We have the explicit formula:*

$$\mathscr{S}(x;q,a) = S_1(q,x) + S_2(q,x) + S_3(q) + S_4(q,x). \tag{3.5.7}$$

with

$$S_1(q,x) = -\frac{1}{\varphi(q)} \sum_{\substack{\chi (\mathrm{mod}\ q) \\ \chi \neq \chi_0}} \overline{\chi}(a) \sum_{\rho \in Z(\chi^*)} F(\rho) x^{\rho},$$

$$S_2(q,x) = -\frac{\log(x) + G'(0)}{\varphi(q)} \sum_{\substack{\chi (\mathrm{mod}\, q) \\ \chi^* \ \textit{even} \\ \chi \neq \chi_0}} \overline{\chi}(a),$$

$$S_3(q) = \frac{1}{\varphi(q)} \sum_{\substack{\chi (\mathrm{mod}\ q) \\ \chi \neq \chi_0}} \overline{\chi}(a) B(\chi^*),$$

$$\textit{and } S_4(q,x) = -\frac{1}{\varphi(q)} \sum_{\substack{\chi (\mathrm{mod}\ q) \\ \chi \neq \chi_0}} \overline{\chi}(a) \sum_{n=1}^{\infty} x^{\mathfrak{a}-2n} F(\mathfrak{a} - 2n).$$

where $B(\chi^*) = \log(q^*) - \gamma_0 - \log(2\pi) + \frac{L'}{L}(1, \overline{\chi^*})$ *and* γ_0 *is Euler's constant.*

Proof of Proposition 3.5.2. We can express $\mathscr{S}(x;q,a)$ as an integral from (3.5.2):

$$\mathscr{S}(x;q,a) = \frac{1}{\varphi(q)} \sum_{\substack{\chi \bmod q \\ \chi \neq \chi_0}} \overline{\chi}(a) \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) \Big(-\frac{L'}{L}(s,\chi^*) \Big) x^s ds.$$

Fix $k \in \mathbb{R}$, $T \geq 2$ such that $k + \frac{1}{2} \in \mathbb{N}$ and T does not equal an ordinate of a zero of $L(s,\chi^*)$. Observe that the integrand has poles which depend on the parity of χ^*. For even characters $\chi^* \neq \chi_0$ it follows from (3.5.3) and (3.5.5) that there is a double pole at $s = 0$ with residue $B(\chi^*) - G'(0) - \log(x)$. We also have simple poles at the nontrivial zeros ρ with residue $-x^{\rho}F(\rho)$ and at the trivial zeros with residue $-x^{2n}F(-2n)$. So that the explicit formula for the even characters is given by

$$B(\chi^*) - G'(0) - \log(x) - \sum_{\substack{\rho \\ L(\varrho,\chi^*)=0}} x^{\varrho} F(\varrho) - \sum_{n=1}^{\infty} x^{-2n} F(-2n). \tag{3.5.8}$$

For odd characters we obtain a simple pole at $s = 0$ with residue $-\frac{L'}{L}(0,\chi^*)$ which we see through the functional equation is $B(\chi^*)$. In addition we have simple zeros at the nontrivial zeros ρ with residue $-x^{\rho}F(\rho)$ and the nontrivial zeros with residue

$-x^{-2n+1}F(-2n+1)$.

So the explicit formula for the odd characters is given by

$$B(\chi^*) - \sum_{\substack{\rho \\ L(\varrho,\chi^*)=0}} x^{\varrho}F(\varrho) - \sum_{n=1}^{\infty} x^{1-2n}F(1-2n). \tag{3.5.9}$$

We move the vertical line of integration $c - iT$ to $c + iT$ extending to the line of integration from $-k - iT$ to $-k + iT$ in order to form a rectangle $\mathcal{R}$. Thus, summing for all the characters modulo q except the principal and combining (3.5.8) and (3.5.9) gives

$$\mathscr{S}(x;q,a) = \frac{1}{\varphi(q)} \sum_{\substack{\chi \bmod q \\ \chi\neq\chi_0}} \overline{\chi}(a) \lim_{k,T\to\infty} (I_1(T,k) + I_2(T,k) + I_3(T,k))$$

$$-\frac{1}{\varphi(q)} \sum_{\substack{\chi (\mathrm{mod}\, q) \\ \chi\neq\chi_0}} \overline{\chi}(a)(\log(q^*)-\log(2\pi)-\gamma_0+\frac{L'}{L}(1,\overline{\chi}_1)) - \frac{1}{\varphi(q)} \sum_{\substack{\chi (\mathrm{mod}\, q) \\ \chi\neq\chi_0 \\ \chi(-1)=1}} \overline{\chi}(a)(G'(0)+\log(x))$$

$$-\frac{1}{\varphi(q)} \sum_{\substack{\chi (\mathrm{mod}\, q) \\ \chi\neq\chi_0}} \overline{\chi}(a) \sum_{\varrho\in Z(\chi^*)} x^{\varrho}F(\varrho) - \frac{1}{\varphi(q)} \sum_{\substack{\chi (\mathrm{mod}\, q) \\ \chi\neq\chi_0}} \overline{\chi}(a) \sum_{n=1}^{\infty} x^{\mathfrak{a}-2n}F(\mathfrak{a}-2n),$$

where I_1, I_2, I_3 are respectively integrating along the segments $[-k+iT, c+iT]$, $[-k+iT, -k-iT]$, $[-k-iT, c-iT]$. It remains to prove that for each $j = 1,2,3$, $\lim_{k,T\to\infty} |I_j(T,k)| = 0$. Consider the classical bounds (see [13, page 116])

$$\left|-\frac{L'}{L}(\sigma \pm iT)\right| \ll \begin{cases} \log^2(qT) & \text{if } -1 \leq \sigma \leq 2 \\ \log(q(|\sigma| + T)) & \text{if } \sigma \leq -1 \end{cases} \tag{3.5.10}$$

where the first estimate holds so long as T is not an ordinate of a zero and the second estimate holds if circles of a radius $\frac{1}{2}$ around the trivial zeros are excluded. We consider a sequence $T_\nu \to \infty$ as $\nu \to \infty$ with each T_ν not an ordinate of a zero. Thus we apply the first inequality together with (3.2.6) and obtain

$$|I_1(T_\nu,k)| \ll \frac{\log^2(qT_\nu)}{T_\nu^{m+1}}\frac{x^c}{\log(x)} + \frac{\log(qT_\nu)}{T_\nu^{m+1}x\log(x)} + \frac{x^{-T_\nu}}{T_\nu^{m-1}}.$$

We conclude that $\lim_{k,\nu\to\infty}|I_1(T_\nu,k)| = 0$. We note that $I_3(T,k) = I_1(-T,k)$ converges to 0 by a similar argument. For $I_2(T,k)$ we combine (3.2.6) with the second inequality in (3.5.10):

$$|F(-k+it)||-\frac{L'}{L}(-k+iT)| \ll \begin{cases} \frac{\log(qk)}{k^{m+1}} & \text{if } |t| \leq \frac{3}{2} \\ \frac{\log(q|t|)}{|t|^{m+1}} & \text{if } |t| > \frac{3}{2} \end{cases}.$$

As before in order to apply our bound we consider $k_\nu \to \infty$ as $\nu \to \infty$ such that each k_ν satisfies $k_\nu + \frac{1}{2} \in \mathbb{N}$. Thus $|I_2(T,k_\nu)| \ll x^{-k_\nu}\left(\frac{\log k_\nu}{k_\nu^{m+1}} + \frac{\log T}{T^{m+1}}\right)$, and $\lim_{T,\nu\to\infty}|I_2(T,k_\nu)| = 0$. □

3.6 Explicit Inequality

The following provides a list of lemmas necessary for bounding the terms not related to the nontrivial zeros of $L(s,\chi^*)$.

3.6.1 Lemmas for Bounding $S_3(q)$ and $S_4(q,x)$

Lemma 3.6.1. *[22, Lemma 5.5] Let $s = \sigma + iT$ with $\frac{1}{4} \leq \sigma \leq 1$. For any primitive character $\chi \pmod q$ with $q \geq 2$, if $L(s,\chi) \neq 0$ then*

$$\left|\frac{L'}{L}(s,\chi)\right| \leq \sum_{\substack{\rho \\ |T-\gamma|\leq 2}} \frac{1}{s-\varrho} + \sqrt{2}\log(0.609q(|T|+5)) + 4.48.$$

For convenience we define the quantity

$$\mathbf{N}(T,q) = \sum_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} N(T,\chi^*). \tag{3.6.1}$$

Lemma 3.6.2. *For any primitive character $\chi \pmod q$ with $q \geq 2$, if GRH(2) is* *satisfied then*

$$\left|\frac{1}{\varphi(q)}\sum_{\substack{\chi(\text{mod } q)\\ \chi\neq\chi_0}}\overline{\chi}(a)\frac{L'}{L}(1,\overline{\chi^*})\right| \leq \frac{2\mathbf{N}(2,q)}{\varphi(q)}+\frac{\varphi(q)-1}{\varphi(q)}(\sqrt{2}\log(q)+6.05473),$$

where $\mathbf{N}(2,q)$ *is defined in* (3.6.1).

Proof. We apply Lemma 3.6.1 letting $s=1$ and taking $\chi=\overline{\chi^*}$ we have that $L(1,\overline{\chi^*})\neq 0$ so we obtain

$$\left|\frac{1}{\varphi(q)}\sum_{\substack{\chi(\text{mod } q)\\ \chi\neq\chi_0}}\overline{\chi}(a)\frac{L'}{L}(1,\overline{\chi})\right| \leq \frac{1}{\varphi(q)}\sum_{\chi(\text{mod } q)}\sum_{\substack{\varrho\in Z(\chi)\\ |\gamma|\leq 2}}\frac{1}{1-\varrho}+\frac{\varphi(q)-1}{\varphi(q)}(\sqrt{2}\log(q)+6.05473).$$

Note that $GRH(2)$ is satisfied, so we use the bound

$$\left|\sum_{\substack{\rho\\ |\gamma|\leq 2}}\frac{1}{1-\varrho}\right| = \sum_{\substack{\rho\\ |\gamma|\leq 2}}\frac{1}{|1-\varrho|} = \sum_{\substack{\rho\\ |\gamma|\leq 2}}\frac{1}{\sqrt{(1-(1/2))^2+t^2}} \leq \sum_{\substack{\rho\\ |\gamma|\leq 2}} 2 = 2N(2,\chi).$$

□

Note that [29, Lemma 4.2] also provides a bound for $\frac{L'}{L}(1,\chi)$, however the result is slightly worse than the one stated above in Lemma 3.6.2. The final lemma in this section is used to bound the sum over the trivial zeros.

Lemma 3.6.3. *For* $q>1$ *we have*

$$|S_4(q,x)| \leq \frac{-\log(1-x^{-1})M(\alpha,\alpha+\delta,0)}{2}.$$

Proof. Note that

$$S_4(q,x)=\frac{1}{\varphi(q)}\sum_{\substack{\chi \bmod q\\ \chi\neq\chi_0}}\overline{\chi}(a)\sum_{n=1}^{\infty}x^{-2n+\mathfrak{a}}F(-2n+\mathfrak{a}).$$

We consider the cases for χ an even or an odd character separately:

For $\mathfrak{a} = 0$ we have

$$\left|\sum_{n=1}^{\infty} x^{-2n} F(-2n)\right| \leq M(\alpha, \alpha+\delta, 0) \sum_{n=1}^{\infty} \frac{x^{-2n}}{2n} = M(\alpha, \alpha+\delta, 0) \frac{-\log(1-x^{-2})}{2}.$$

The case $\mathfrak{a} = 1$ we find

$$\begin{aligned}
&\left|\sum_{n=1}^{\infty} x^{-2n+1} F(-2n+1)\right| \leq M(\alpha, \alpha+\delta, 0) \sum_{n=1}^{\infty} \frac{x^{-2n+1}}{2n-1} \\
&= M(\alpha, \alpha+\delta, 0) \left(\left(x^{-1} + \frac{x^{-2}}{2} + \frac{x^{-3}}{3} + \frac{x^{-4}}{4} \dots \right) - \frac{1}{2} \left(x^{-2} + \frac{x^{-4}}{2} + \dots \right) \right) \\
&= M(\alpha, \alpha+\delta, 0) \left(-\log(1-x^{-1}) - \frac{-\log(1-x^{-2})}{2} \right).
\end{aligned}$$

So combining the two cases together gives:

$$|S_4(q,x)| = \left| \frac{1}{\varphi(q)} \sum_{\substack{\chi \bmod q \\ \chi \neq \chi_0}} \overline{\chi}(a) \sum_{n=1}^{\infty} x^{-2n+\mathfrak{a}} F(-2n+\mathfrak{a}) \right| \leq \frac{M(\alpha, \alpha+\delta, 0)}{2} (-\log(1-x^{-1})).$$

□

3.6.2 Explicit Inequality

Proposition 3.6.4. *We have*

$$\begin{aligned}
E_{\psi_f^*}(x; q, a) \leq \min\left(J_1(x,\delta), J_2(x,m,\alpha,\delta,z)\right)) + \sum_{\chi (\bmod q)} \sum_{\substack{\varrho = \beta + i\gamma \\ \rho \in Z(\chi^*)}} |F(\varrho)| x^{\beta - 1} \\
+ \frac{B_1(q,x) + B_2(q) + B_3(q,x,\alpha,\delta)}{x},
\end{aligned}$$

where

$$B_1(q,x) = \left(2\mathbf{N}(2,q) + \frac{(\varphi(q)-2)|\log(x)+G'(0)|}{2}\right), \tag{3.6.2}$$

$$B_2(q) = (\varphi(q)-1)\Big(\gamma_0 + (1+\sqrt{2})\log(q) + \log(2\pi) + 6.05473\Big) \tag{3.6.3}$$

$$\text{and } B_3(q,x) = (\varphi(q)-1)\frac{\Big(-\log(1-x^{-1})M(\alpha,\alpha+\delta,0)\Big)}{2}. \tag{3.6.4}$$

We recall $J_i's$ are defined in (3.4.1) and (3.4.2) respectively, with $\mathbf{N}(2,q)$ given by (3.6.1), and γ_0 is Euler's constant.

Proof. By (3.5.7) we have

$$\mathscr{S}(x;q,a) = S_1(q,x) + S_2(q,x) + S_3(q) + S_4(q,x)$$

so that by the triangle inequality we have

$$E_{\widetilde{\psi}^*}(x;q,a) \le \min(J_1(x,\delta), J_2(x,m,\alpha,\delta,z)) + \frac{1}{x}|S_1(q,x)| + \frac{1}{x}|S_2(q,x)| + \frac{1}{x}|S_3(q)| + \frac{1}{x}|S_4(q,x)|.$$

We bound trivially $S_1(q,x)$:

$$\frac{1}{x}|S_1(q,x)| \le \sum_{\substack{\chi(\bmod q)\\ \chi\neq\chi_0}} \sum_{\substack{\varrho=\beta+i\gamma\\ \varrho\in Z(\chi^*)}} |F(\varrho)|x^{\beta-1}. \tag{3.6.5}$$

Using $\Big|\sum_{\substack{\chi(\bmod q)\\ \chi\neq\chi_0 \text{ even}}} \overline{\chi}(a)\Big| \le \frac{\varphi(q)-2}{2}$, we have

$$\frac{1}{x}|S_2(q,x)| \le \frac{\varphi(q)-2}{2x}|\log(x)+G'(0)|. \tag{3.6.6}$$

We note that $\Big|\sum_{\substack{\chi(\bmod q)\\ \chi\neq\chi_0}} \overline{\chi}(a)\Big| \le \varphi(q)-1$. Hence, using Lemma 3.6.2 we have:

$$\frac{1}{x}|S_3(q)| \le \frac{\varphi(q)-1}{x}((1+\sqrt{2})\log(q) + \gamma_0 + \log(2\pi) + 6.05473) + \frac{2\mathbf{N}(2,q)}{x}. \tag{3.6.7}$$

Finally, from Lemma 3.6.3:

$$\frac{1}{x}|S_4(q,x)| \leq \frac{(\varphi(q)-1)(-\log(1-x^{-1}))M(\alpha,\alpha+\delta,0)}{2x}. \tag{3.6.8}$$

Combining (3.6.5), (3.6.6), (3.6.7), and (3.6.8) gives the desired result. □

3.7 Studying the Sum over the zeros

To study the sum over the zeros, $\sum_{\varrho \in Z(\chi^*)} x^{\beta-1}F(\varrho)$, we introduce the notation

$$\begin{aligned}
&* \ H = H_q > 0 \text{ such that if } L(\beta+i\gamma,\chi^*) = 0 \text{ and } 0<\gamma<H, \text{ then } \beta = 1/2,\\
&* \ T_0 = T_0(q) > 1 \text{ such that } \sum_{\substack{\varrho\in Z(\chi^*)\\ 1<|\gamma|<T_0}} \gamma^{-1} \text{ can be directly computed,}\\
&* \ T_1 = T_1(q) \text{ is a parameter satisfying } T_0 < T_1 < H,\\
&* \ R_0 \text{ is a constant so that } \mathcal{L}_q(\sigma+it,\chi^*) \text{ does not vanish in the region}\\
&\quad \sigma \geq 1 - \frac{1}{R_0 \log(\max(q,q|t|))}.
\end{aligned} \tag{3.7.1}$$

We separate the zeros vertically at H:

$$\sum_{\varrho\in Z(\chi^*)} x^{\beta-1}|F(\varrho)| = \Sigma_1 + \Sigma_2,$$

with

$$\Sigma_1 = x^{-\frac{1}{2}} \sum_{\substack{\varrho\in Z(\chi^*)\\ |\gamma|\leq H}} \left|F\left(\frac{1}{2}+i\gamma\right)\right|, \qquad \Sigma_2 = \sum_{\substack{\varrho\in Z(\chi^*)\\ |\gamma|>H}} x^{\beta-1}|F(\varrho)|. \tag{3.7.2}$$

We split Σ_1 vertically at $1, T_0$ and T_1. In order to bound the portion of Σ_1 with $|\gamma| \leq T_0$ we consider $q \leq 10^4$ and $q > 10^4$ differently, based on whether there is a computed value for the following quantity. Define

$$S_0(T_0,\chi^*) = \sum_{\substack{\rho\in Z(\chi^*)\\ 1<|\gamma|\le T_0}} \frac{1}{|\gamma|}. \tag{3.7.3}$$

For $q \le 10^4$ we compute $S_0(T_0,\chi^*)$, $N(1,\chi^*)$ and $N(T_0,\chi^*)$ exactly. For the purposes of this paper we use $T_0 = 200$ and the values are computed in the appendix by David Platt, select values may be found in Table A.1. Then we consider two possible bounds for $|F(\rho)|$. First taking absolute values in (3.2.3) and using $|f(x)| \le 1$, we obtain $|F(\rho)| \le 2(\alpha+\delta)^{1/2}$. The other inequality we apply is (3.2.6) with $k = 0$. Finally we take the minimum between

$$b_{01}(\alpha,\delta,T_0) = 2(\alpha+\delta)^{1/2}N(1,\chi^*) + M(\alpha,\alpha+\delta,0)S_0(T_0,\chi^*) \tag{3.7.4}$$

and

$$b_{02}(\alpha,\delta,T_0) = 2(\alpha+\delta)^{1/2}N(T_0,\chi^*). \tag{3.7.5}$$

When $q > 10^4$ we set $T_0 = 10$ and take the minimum of an estimate for $b_{01}(\alpha,\delta,T_0)$ and $b_{02}(\alpha,\delta,T_0)$ instead of an exact computation. For the rest of the bounds we proceed the same regardless of the moduli. In the region $T_0 < |\gamma| \le H$, use (3.2.6) with $k = 0$ when $T_0 < |\gamma| \le T_1$, and $k = m$ when $T_1 < \gamma \le H$ respectively. Thus

$$\Sigma_1 \le x^{-\frac{1}{2}}\Bigg(\min\Big(b_{02}(\alpha,\delta,T_0), b_{01}(\alpha,\delta,T_0)\Big) + M(\alpha,\alpha+\delta,0)\sum_{\substack{\varrho\in Z(\chi^*)\\ T_0<|\gamma|\le T_1}} \frac{1}{|\gamma|} + \frac{M(\alpha,\alpha+\delta,m)}{\delta^m}\sum_{\substack{\varrho\in Z(\chi^*)\\ T_1<|\gamma|\le H}} \frac{1}{|\gamma|^{m+1}}\Bigg).$$

For Σ_2 we note that we can actually achieve a better bound by considering the fact that $\rho \in Z(\chi) \Leftrightarrow 1-\overline{\varrho} \in Z(\chi)$. So we can write

$$\sum_{\substack{\rho\in Z(\chi)\\ |\gamma|>H}} x^{\beta-1}|F(\varrho)| = \frac{1}{2}\left\{\sum_{\substack{\varrho\in Z(\chi)\\ |\gamma|>H}} x^{\beta-1}|F(\varrho)| + \sum_{\substack{\varrho\in Z(\overline{\chi})\\ |\gamma|>H}} x^{\beta-1}|F(\varrho)|\right\}$$
$$= \frac{1}{2}\sum_{\substack{\rho\in Z(\chi)\\ |\gamma|>H}}\left\{x^{-\beta}|F(\varrho)| + x^{\beta-1}|F(1-\overline{\varrho})|\right\}.$$

Now either $-\beta$ or $\beta-1<-\frac{1}{2}$. If we suppose that $-\beta<-\frac{1}{2}$ then $\beta-1>-\frac{1}{2}$. We use (3.2.6) with $k=m$ together with the zero-free region in Theorem 3.1.3, to obtain

$$\Sigma_2 \leq \frac{M(\alpha,\alpha+\delta,m)}{2\delta^m}\left(x^{-\frac{1}{2}}\sum_{\substack{\rho\in Z(\chi)\\ |\gamma|>H}}\frac{1}{|\gamma|^{m+1}} + \sum_{\substack{\rho\in Z(\chi)\\ |\gamma|>H}}\frac{x^{-\frac{1}{R_0\log(q|\gamma|)}}}{|\gamma|^{m+1}}\right).$$

We denote

$$s_1(q,T_0,T_1) = \sum_{\substack{\varrho\in Z(\chi^*)\\ T_0<|\gamma|\leq T_1}}\frac{1}{|\gamma|}, \qquad s_2(q,T_1,H,m) = \sum_{\substack{\varrho\in Z(\chi^*),\\ T_1<|\gamma|\leq H}}\frac{1}{|\gamma|^{m+1}} \tag{3.7.6}$$
$$s_3(q,H,m) = \sum_{\substack{\rho\in Z(\chi)\\ |\gamma|>H}}\frac{1}{|\gamma|^{m+1}}, \qquad s_4(x,q,H,m) = \sum_{\substack{\rho\in Z(\chi)\\ |\gamma|>H}}\frac{x^{-\frac{1}{R_0\log(q|\gamma|)}}}{|\gamma|^{m+1}}.$$

Hence we have

$$\sum_{\varrho\in Z(\chi^*)} x^{\beta-1}F(\varrho) \leq x^{-\frac{1}{2}}\Big(\min\Big(b_{02}(\alpha,\delta,T_0), b_{01}(\alpha,\delta,T_0)\Big) + M(\alpha,\alpha+\delta,0)s_1(q,T_0,T_1)$$
$$+\frac{M(\alpha,\alpha+\delta,m)}{\delta^m}\left(s_2(q,T_1,H,m)+\frac{s_3(q,H,m)}{2}\right)\Big) + \frac{M(\alpha,\alpha+\delta,m)}{2\delta^m}s_4(x,q,H,m). \tag{3.7.7}$$

Placing (3.7.7) in (3.6.2) and recalling Lemma 3.3.1 gives:

Lemma 3.7.1. *Let $0<\alpha,\delta$ and $q,m\in\mathbb{N}$, with $q,m\geq 2$. Let f be a function satisfying Definition 3.2.1. Let H,T_0,T_1,R_0 satisfy (3.7.1). Then for all $x>0$,*

$E_f(x;q,a) \le K(x,\alpha,\alpha+\delta,m,q)$, *where*

$$K(x,\alpha,\alpha+\delta,m,q) = \min(J_1(x,\delta), J_2(x,m,\alpha,\delta,z)) + \frac{(\varphi(q)-1)M(\alpha,\alpha+\delta,m)}{2\delta^m} s_4(x,q,H,m)$$
$$+ x^{-\frac{1}{2}} \min\Bigg(\sum_{\substack{\chi (\mathrm{mod}\, q)\\ \chi\neq\chi_0}} b_{02}(\alpha,\delta,T_0), \sum_{\substack{\chi (\mathrm{mod}\, q)\\ \chi\neq\chi_0}} b_{01}(\alpha,\delta,T_0)\Bigg)$$
$$+ x^{-\frac{1}{2}} \left((\varphi(q)-1)M(\alpha,\alpha+\delta,0)s_1(q,T_0,T_1)\right)$$
$$+ x^{-\frac{1}{2}} \left(\frac{(\varphi(q)-1)M(\alpha,\alpha+\delta,m)}{\delta^m}\left(s_2(q,T_1,H,m) + \frac{s_3(q,H,m)}{2}\right)\right)$$
$$+ \frac{B_1(q,x,\alpha,\delta) + B_2(q) + B_3(q,x,\alpha,\delta) + \log(x)\sum_{p|q}\frac{1}{p-1}}{x}. \tag{3.7.8}$$

with b_{01} *defined by* (3.7.4), b_{02} *defined by* (3.7.5), *the* s_i*'s,* B_i*'s and* $M(\alpha,\alpha+\delta,m)$ *are defined in* (3.7.6), (3.6.2) *and* (3.2.4) *respectively.*

Note that for α,δ,m,q fixed constants, $K(x,\alpha,\alpha+\delta,m,q)$ decreases as x increases. Thus, for all $x \ge x_0$

$$E_f(x;q,a) \le K(x_0,\alpha,\alpha+\delta,m,q). \tag{3.7.9}$$

3.7.1 Bounding $s_1(q,T_0,T_1), s_2(q,T_1,H,m)$ and $s_3(q,H,m)$

We give a result similar to one from Rosser and Schoenfeld [70] which was generalized by Dusart in 2002 [17, Lemma 5]. We provide a proof for completeness.

Lemma 3.7.2. *[17, Lemma 5] Let* $1 < U \le V$*, and let* $\Phi(y)$ *be non-negative and differentiable for* $U < y < V$*. Let* $(W-y)\Phi'(y) \ge 0$ *for* $U < y < V$*, where* W *need not lie in* $[U,V]$*. Let* Y *be one of* U,V,W *which is neither greater than both the others or less than both the others. Choose* $j = 0$ *or* 1 *so that* $(-1)^j(V-W) \ge 0$*. Then*

$$\sum_{U<|\gamma|\le V} \Phi(|\gamma|) \le \frac{1}{\pi}\int_U^V \Phi(y)\log\frac{qy}{2\pi}dy + (-1)^j C_1 \int_U^V \frac{\Phi(y)}{y}dy + E_j(q,U,V),$$

where the error term $E_j(q,U,V)$ *is given by*

$$E_j(q,U,V) = (1+(-1)^j)R(q,Y)\Phi(Y) + (N(V,\chi) - P(q,V) \\ - (-1)^j R(q,V))\Phi(V) - (N(U,\chi) - P(q,U) + R(q,U))\Phi(U).$$

Proof. We have

$$\sum_{U<|\gamma|\le V} \Phi(|\gamma|) = \sum_{U<\gamma\le V} \Phi(\gamma) + \sum_{-V\le\gamma<-U} \Phi(-\gamma) \\ = \sum_{U<\gamma\le V} \Phi(\gamma) + \sum_{U<-\gamma\le V} \Phi(-\gamma)$$

Then by a result of Ingham [33, p. 18] we have

$$\sum_{U<|\gamma|\le V} \Phi(|\gamma|) = \frac{1}{2}\int_U^V \Phi(y)dN(y,\chi) + \frac{1}{2}\int_U^V \Phi(-y)dN(y,\chi) = \int_U^V \Phi(y)dN(y,\chi) \\ = -\int_U^V N(y,\chi)\Phi'(y)dy + N(V,\chi)\Phi(V) - N(U,\chi)\Phi(U). \tag{3.7.10}$$

- $j=1$: Let $Y = \min(V,W)$. By Theorem 3.1.4, $N(y,\chi) \ge P(q,y) - R(q,y)$, hence replacing $N(y,\chi)$ in (3.7.10) with $P(q,y) - R(q,y)$ and integrating by parts we obtain

$$\sum_{U<|\gamma|\le V} \Phi(|\gamma|) \le -\int_U^V P(q,y)\Phi'(y)dy + \int_U^V R(q,y)\Phi'(y)dy \\ + N(V,\chi)\Phi(V) - N(U,\chi)\Phi(U) \\ = \frac{1}{\pi}\int_U^V \log\left(\frac{qy}{2\pi}\right)\Phi(y)dy - C_1\int_U^V \frac{\Phi(y)}{y}dy + E_1(q,U,V).$$

- $j=0$: Let $Y = \max(U,W)$. Split the integral in (3.7.10) at Y. Then for $U \le y \le Y$ we have $\phi(y) \le 0$, hence we replace $N(y,\chi)$ with $P(q,y) - R(q,y)$. For $Y \le y \le W$ we have that $\phi(y) \ge 0$, hence replace $N(y,\chi)$ with $P(q,y) + R(q,y)$.

Integrate both pieces by parts to get

$$\sum_{U<|\gamma|\leq V} \Phi(|\gamma|) \leq -\int_U^Y P(q,y)\Phi'(y)dy + \int_U^Y R(q,y)\Phi'(y)dy - \int_Y^V P(q,y)\Phi(y)dy$$
$$- \int_Y^V R(q,y)\Phi(y)dy + N(V,\chi)\Phi(V) - N(U,\chi)\Phi(U). \qquad (3.7.11)$$

Using the fact that

$$C_1 \int_Y^V \frac{\Phi(y)}{y} dy \leq C_1 \int_U^V \frac{\Phi(y)}{y} dy,$$

we simplify (3.7.11) to obtain

$$\sum_{U<|\gamma|\leq V} \Phi(|\gamma|) \leq \frac{1}{\pi} \int_U^V \log\left(\frac{qy}{2\pi}\right) \Phi(y)dy + C_1 \int_U^V \frac{\Phi(y)}{y} dy + E_0(q,U,V).$$

□

Corollary 3.7.3. *[17, Corollary 1] Under the same hypothesis of Lemma 3.7.2, if in addition we also have* $\frac{2\pi}{q} < U$*, then*

$$\sum_{U<|\gamma|\leq V} \Phi(|\gamma|) \leq \left(\frac{1}{\pi} + (-1)^j Q(q,Y)\right) \int_U^V \Phi(y) \log\left(\frac{qy}{2\pi}\right) dy + E_j(q,U,V),$$

where

$$Q(q,Y) = \frac{C_1}{y \log\left(\frac{qy}{2\pi}\right)}.$$

Proof. Proceed as in the proof of Lemma 3.7.2 using that for y in the range of integration we have

$$(-1)^j \frac{R'(q,y)}{\log(\frac{yq}{2\pi})} \leq (-1)^j Q(q,Y).$$

□

Moreover if $W < U$ and $j = 0$ then

$$E_0(q,U,V) \leq 2R(q,Y)\Phi(Y).$$

Apply Corollary 3.7.3 to s_1, s_2 and s_3 respectively taking

- $\Phi(y) = y^{-1}, U = T_0, V = T_1,$
- $\Phi(y) = y^{-1-m}, U = T_1, V = H$ and
- $\Phi(y) = y^{-1-m}, U = H, V = \infty.$

In each case $\Phi'(y) \leq 0$ for all y, and we take $W < U$, $Y = U$ and $j = 0$. Since

$$\int_{T_0}^{T_1} \frac{\log\left(\frac{qy}{2\pi}\right)}{y} dy = \log\left(\frac{T_1}{T_0}\right) \log\left(\frac{q\sqrt{T_1T_0}}{2\pi}\right),$$
$$\int_{U}^{V} \frac{\log\left(\frac{qy}{2\pi}\right)}{y^{m+1}} dy = \frac{1 + m\log\left(\frac{qU}{2\pi}\right)}{m^2U^m} - \frac{1 + m\log\left(\frac{qV}{2\pi}\right)}{m^2V^m},$$

we obtain:

$$s_1(q, T_0, T_1) \leq b_1(q, T_0, T_1) = \left(\frac{1}{\pi} + Q(q, T_0)\right) \log\left(\frac{T_1}{T_0}\right) \log\left(\frac{q\sqrt{T_1T_0}}{2\pi}\right) + \frac{2R(q, T_0)}{T_0}, \tag{3.7.12}$$

$$s_2(q, T_1, H, m) \leq b_2(q, T_1, H, m) = \left(\frac{1}{\pi} + Q(q, T_1)\right) \left(\frac{1 + m\log\left(\frac{qT_1}{2\pi}\right)}{m^2T_1^m} - \frac{1 + m\log\left(\frac{qH}{2\pi}\right)}{m^2H^m}\right) + \frac{2R(q, T_1)}{T_1^{m+1}}, \tag{3.7.13}$$

$$s_3(q, H, m) \leq b_3(q, H, m) = \left(\frac{1}{\pi} + Q(q, H)\right) \left(\frac{1 + m\log\left(\frac{qH}{2\pi}\right)}{m^2H^m}\right) + \frac{2R(q, H)}{H^{m+1}}. \tag{3.7.14}$$

3.7.2 Bounding $s_4(q, H, m)$

We define

$$\widetilde{A} := \int_{H}^{+\infty} \left(\frac{\log\frac{qt}{2\pi}}{\pi} + \frac{C_1}{t}\right) \frac{\exp\left(-\frac{\log x_0}{R_0\log(qt)}\right)}{t^{m+1}} dt, \tag{3.7.15}$$
$$\widetilde{B} := \frac{2}{H^{m+1}} x_0^{-\frac{1}{R_0\log(qH)}} \left(C_1\log(qH) + C_2\right).$$

Lemma 3.7.4. *[66, Lemma 4.1.3] Let $m \geq 1$ be an integer and x, x_0 two real numbers such that $x \geq x_0 \geq 1$ and $R_0 m \log^2(qH) \geq \log x_0$. Then*

$$s_4(x, q, H, m) = \sum_{|\gamma|>H} \frac{x_0^{-\frac{1}{R_0 \log(q|\gamma|)}}}{|\gamma|^{m+1}} \leq \widetilde{A} + \widetilde{B}.$$

Note that computing $\widetilde{A}(\chi)$ is a matter of rewriting it in terms of the modified Bessel function given by

$$K_\nu(z, w) = \frac{1}{2} \int_w^\infty u^{\nu-1} \exp\left(-\frac{z}{2}(u + \frac{1}{u})\right) du.$$

We do the variable change $qt = e^{\frac{z}{2m}u}$, taking $z_m = 2\sqrt{\frac{m \log(x_0)}{R}}$, $w_m = \sqrt{\frac{mR}{\log(x_0)}} \log(qH)$ to show:

Lemma 3.7.5. *[66, Lemma 4.2.2] We have*

$$\widetilde{A} = \frac{2\log(x_0)}{\pi R m} q^m K_2(z_m, w_m) + \frac{2}{\pi} \log\left(\frac{1}{2\pi}\right) \sqrt{\frac{\log(x_0)}{Rm}} q^m K_1(z_m, w_m) + 2C_1 \sqrt{\frac{\log(x_0)}{R(m+1)}} q^{m+1} K_1(z_{m+1}, w_{m+1}),$$

where $C_1 = 0.317$ is given in Theorem 3.1.4

In order to compute these Bessel integrals we integrate up to some finite bound and compute the tails by expressing them in terms of the following integral

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt.$$

Lemma 3.7.6. *Let $y = \frac{\sqrt{W_m} - \frac{1}{\sqrt{W_m}}}{\sqrt{2}}$ for $W_m = 100 w_m$. Then*

$K_1(z,w_m) < \widetilde{K}_1(z,w_m)$ and $K_2(z,w_m) < \widetilde{K}_2(z,w_m)$ where

$$\widetilde{K}_1(z,w_m) = \frac{1}{2}\int_{w_m}^{W_m} \exp\left(-\frac{z}{2}(u+\frac{1}{u})\right) du + \frac{e^{-z}}{2z}\left(\left(1+\frac{3\sqrt{2}}{8}y\right)e^{-zy^2} + \left(\frac{3}{8\sqrt{z}}+\sqrt{z}\right)\sqrt{\frac{\pi}{2}}\mathrm{erfc}(y\sqrt{z})\right)$$

and

$$\widetilde{K}_2(z,w_m) = \frac{1}{2}\int_{w_m}^{W_m} u\exp\left(-\frac{z}{2}(u+\frac{1}{u})\right) du + \frac{e^{-z}}{2z}\left(\left(\frac{35\sqrt{2}}{64}y^3 + 2y^2 + \left(\frac{105}{128z}+\frac{15}{8}\right)\sqrt{2}y + 2 + \frac{2}{z}\right)e^{-zy^2} + \left(\frac{105}{128z}+\frac{15}{8}+z\right)\sqrt{\frac{\pi}{2}}\mathrm{erfc}(y\sqrt{z})\right).$$

Proof. We integrate $K_\nu(z,w_m)$ from w_m to $100w_m = W_m$ and then, since $W_m \geq 1$ we estimate the tails of the integral by using [70, Equations 2.30 & 2.31]. □

Finally applying Lemmas 3.7.4, 3.7.5 and 3.7.6 we have that

$$\begin{aligned} s_4(x_0,q,H,m) \leq b_4(x_0,q,H,m) &= \frac{2\log(x_0)}{\pi R m} q^m \widetilde{K}_2(z_m,w_m) \\ &+ \frac{2}{\pi}\log\left(\frac{1}{2\pi}\right)\sqrt{\frac{\log(x_0)}{Rm}} q^m \widetilde{K}_1(z_m,w_m) \\ &+ 2C_1\sqrt{\frac{\log(x_0)}{R(m+1)}} q^{m+1}\widetilde{K}_1(z_{m+1},w_{m+1}) \\ &+ \frac{2}{H^{m+1}} x_0^{-\frac{1}{R\log(qH)}}\left(C_1\log(qH)+C_2\right). \end{aligned} \tag{3.7.16}$$

3.7.3 Main Theorem

We deduce a new bound for $K(x,\alpha,\alpha+\delta,m,q)$ from (3.7.12),(3.7.13), (3.7.14) and (3.7.16).

Theorem 3.7.7. *Let $0 < \alpha, \delta$ be two real numbers, $q, m \in \mathbb{N}$, $m \geq 2$, $q \geq 1$, and f be a function satisfying Definition 3.2.1. If $L(s,\chi)$ satisfies GRH(H) for some $H \geq 2$ and if $1 \leq T_0 \leq T_1 \leq H$, χ a character modulus q with χ^* the primitive character that induces*

χ and x, x_0 be two real numbers such that $x \geq x_0 \geq 10^8$ and $R_0 mlog^2(qH) \geq \log(x_0)$. Then

$$
\begin{aligned}
&K(x_0, \alpha, \alpha+\delta, m, q) \leq \\
&\min\left(J_1(x_0, \delta), J_2(x_0, m, \alpha, \delta, z)\right) + \frac{(\varphi(q)-1)M(\alpha, \alpha+\delta, m)}{2\delta^m} b_4(x_0, q, H, m) \\
&+ x_0^{-\frac{1}{2}} \left[\min\Big(\sum_{\substack{\chi (\operatorname{mod} q) \\ \chi \neq \chi_0}} b_{02}(\alpha, \delta, T_0), \sum_{\substack{\chi (\operatorname{mod} q) \\ \chi \neq \chi_0}} b_{01}(\alpha, \delta, T_0) \Big) + M(\alpha, \alpha+\delta, 0)(\varphi(q)-1) b_1(q, T_0, T_1) \right. \\
&\left. + \frac{(\varphi(q)-1))M(\alpha, \alpha+\delta, m)}{\delta^m} \left(b_2(q, T_1, H, m) + \frac{b_3(q, H, m)}{2} \right) \right] \\
&+ \frac{B_1(q, x_0, \alpha, \delta) + B_2(q) + B_3(q, x_0, \alpha, \delta) + \log(x_0) \sum_{p|q} \frac{1}{p-1}}{x_0}, \qquad (3.7.17)
\end{aligned}
$$

where the J_i's are defined by (3.4.1) and (3.4.2), b_i's are defined in (3.7.12),(3.7.13),(3.7.14), and (3.7.16), the B_is are defined in (3.6.2) and b_{01}, b_{02} are respectively given by (3.7.4),(3.7.5).

The previous theorem is a simple way to provide values for Theorem 3.1.1, the results were tabulated using a stronger version which yields a slightly sharper result and is presented below.

Theorem 3.7.8. *Let $q \geq 1, m \geq 2$ be integers. Let α, δ be two positive real numbers, and f a function satisfying Definition 3.2.1. Let $H_\chi \geq 2$ such that $L(s, \chi)$ satisfies GRH(H_χ) and if $1 \leq T_0 \leq T_1 \leq H_\chi$, with χ a character modulo q such that χ^* is the primitive character that induces χ. Let $x \geq x_0 \geq 10^8$ be real numbers such that for* *each $r|q$ and each χ^* with conductor r, $R_0 m(\log(rH_\chi))^2 \geq \log(x_0)$. Then*

$$\begin{aligned}
&K(x_0,\alpha,\alpha+\delta,m,q) \leq \\
&\min\left(J_1(x_0,\delta), J_2(x_0,m,\alpha,\delta,z)\right) + \frac{M(\alpha,\alpha+\delta,m)}{2\delta^m}\sum_{\substack{r|q\\ r\neq 1}}\sum_{\chi^*(\mathrm{mod}\ r)} b_4(x_0,r,H_\chi,m) \\
&+ x_0^{-\frac{1}{2}}\left[\min\left(\sum_{\substack{r|q\\ r\neq 1}}\sum_{\chi^*(\mathrm{mod}\ r)} b_{02}(\alpha,\delta,T_0), \sum_{\substack{r|q\\ r\neq 1}}\sum_{\chi^*(\mathrm{mod}\ r)} b_{01}(\alpha,\delta,T_0)\right)\right. \\
&+M(\alpha,\alpha+\delta,0)\sum_{\substack{r|q\\ r\neq 1}}\sum_{\chi^*(\mathrm{mod}\ r)} b_1(r,T_0,T_1) \\
&\left.+\frac{M(\alpha,\alpha+\delta,m)}{\delta^m}\sum_{\substack{r|q\\ r\neq 1}}\sum_{\chi^*(\mathrm{mod}\ r)}\left(b_2(q,T_1,H,m)+\frac{b_3(q,H,m)}{2}\right)\right] \\
&+\frac{B_1(q,x_0,\alpha,\delta)+B_2(q)+B_3(q,x_0,\alpha,\delta)+\log(x_0)\sum_{p|q}\frac{1}{p-1}}{x_0},
\end{aligned} \tag{3.7.18}$$

where the J_i's are defined by (3.4.1) and (3.4.2), b_i's are defined in (3.7.12),(3.7.13),(3.7.14), and (3.7.16), the B_is are defined in (3.6.2) and b_{01}, b_{02} are respectively given by (3.7.4),(3.7.5).

3.8 The argument of McCurley [56] and Ramaré and Rumely [66].

McCurley [56] and Ramaré and Rumely [66] use the same smoothing argument as in Rosser and Schoenfeld [70] to obtain bounds for $\psi(x;q,a)$.

3.8.1 Controlling the difference between $\psi(x;q,a)$ and $\psi_f(x;q,a)$.

The first step of their argument consists in studying $\psi(x;q,a)$ on average over a small interval around a large x value. Let $x,\delta>0$ with $x\notin\mathbb{N}$. Let $m\in\mathbb{N}$. It follows from the First Mean Value Theorem for Integrals applied to $h(z)=\psi(x+z;q,a)-(x+z)$ that there exists $z\in(0,\delta x)$ such that:

$$h(z)+z \leq \frac{1}{(\delta x/m)^m}\int_0^{\delta x/m}\cdots\int_0^{\delta x/m}\left(h(y_1+\ldots+y_m)+(y_1+\ldots+y_m)\right)dy_1\ldots dy_m.$$

(In order to make Ramaré and Rumely's article consistent with our setup, we replace their δ with our δ/m.) Implementing the explicit formula (3.1.1) in the right integrals together with the fact that $\psi(x+z;q,a) \geq \psi(x;q,a)$ leads to [56, Lemmas 3.3 & 3.4]:

$$E(x;q,a) \leq \frac{\delta}{2} + \sum_{\chi(\bmod q)} \Sigma(m,\delta,x) + \mathcal{O}(x^{-1}), \tag{3.8.1}$$

where $\rho = \beta + i\gamma$ and

$$\Sigma(m,\delta,x) = \sum_{\rho} x^{\beta-1}|I_{m,\delta}(\rho)|, \quad \text{and} \quad I_{m,\delta}(\rho) = \frac{\sum_{j=0}^{m}(-1)^{j+m+1}\binom{m}{j}(1+j\delta/m)^{m+\rho}}{(\delta/m)^m \rho(\rho+1)\dots(\rho+m)}.$$

The difference between $\psi(x;q,a)$ and $\psi_f(x;q,a)$ is given by $\frac{\delta}{2}$. We next have to give bounds for $\Sigma(m,\delta,x)$. The idea is to split it into two regions. One where the generalized Riemann Hypothesis has been verified and one where it is unknown. That is we write

$$\begin{aligned}\Sigma(m,\delta,x) &= \sum_{\substack{\rho\in Z(\chi)\\ |\gamma|\leq H_q}} x^{\beta-1}|I_{m,\delta}(\rho)| + \sum_{\substack{\rho\in Z(\chi)\\ |\gamma|> H_q}} x^{\beta-1}|I_{m,\delta}(\rho)| \\ &= \Sigma_{RR1} + \Sigma_{RR2}.\end{aligned}$$

3.8.2 Sum over the zeros for which GRH is verified.

We have $GRH(H_q)$, which means if $\rho \in Z(\chi)$ and $|\gamma| \leq H_q$, then $\beta = \frac{1}{2}$. For this sum, the authors of [66] bound $I_{m,\delta}(\rho)$ with McCurley's [56, Theorem 3.6]:

$$|I_{m,\delta}(\rho)| \leq \frac{1}{|\gamma|}\left(1+\frac{\delta}{2}\right).$$

They used estimates for $N(T,\chi)$ and partial summation so that Σ_{RR1} is approximately

$$\frac{1}{2\pi}\log^2(H_q) + \frac{\log\left(\frac{q}{2\pi}\right)\log(H_q)}{\pi}. \tag{3.8.2}$$

So that $\sum_{\chi(\mathrm{mod}\, q)} \Sigma_{RR1}$ is approximately

$$x^{-1/2}\varphi(q)\left(1+\frac{\delta}{2}\right)\left(\frac{1}{2\pi}\log^2(H_q)+\frac{\log\left(\frac{q}{2\pi}\right)\log(H_q)}{\pi}\right). \tag{3.8.3}$$

3.8.3 Sum over the remaining zeros.

The authors in [66] make the observation that $\rho \in Z(\chi) \iff 1-\overline{\rho} \in Z(\chi)$. Doing this we may split the sum into two terms. One which has a factor of $x^{-1/2}$ and one with $x^{-1/R\log(q|\gamma|)}$. That is we have

$$\Sigma_{RR2} \leq \frac{1}{2}\sum_{\substack{\rho\in Z(\chi)\\ |\gamma|>H_q}} x^{-\frac{1}{2}}|I_{m,\delta}(\rho)| + \frac{1}{2}\sum_{\substack{\rho\in Z(\chi)\\ |\gamma|>H_q}} x^{-\frac{1}{2}}|I_{m,\delta}(\rho)|. \tag{3.8.4}$$

Rosser established (see [68, Theorem 22])

$$|I_{m,\delta}(\rho)| \leq \frac{((1+\delta/m)^{m+1}+1)^m}{(\delta/m)^m|\gamma|^{m+1}} = \frac{2^m m^m}{\delta^m}(1+o(1)).$$

To bound the first sum in (3.8.4) the authors recognize a modified Bessel function and compute it as exactly as possible. If $mR(\log(qH))^2 > \log(x)$ we can estimate its value using Rosser and Schoenfeld's [70, Lemma 4].

In comparison to this first term of (3.8.4) the second term in (3.8.4) is negligible and we have that $\sum_{\chi(\mathrm{mod}\, q)} \Sigma_{RR2}$ is approximately

$$x^{-\frac{1}{R\log(qH_q)}}\frac{\varphi(q)R(\log(qH_q))^3}{mR(\log(qH_q))^2-\log(x)}\frac{2^m m^m}{2\pi\delta^m H_q^m}(1+o(1)). \tag{3.8.5}$$

3.8.4 Final Error Term

Combining equations (3.8.3) and (3.8.5) and placing them into (3.8.1) gives $E(x;q,a)$ is approximately

$$\frac{\delta}{2} + x^{-1/2}\varphi(q)\left(1+\frac{\delta}{2}\right)\left(\frac{1}{2\pi}\log^2(H_q) + \frac{\log\left(\frac{q}{2\pi}\right)\log(H_q)}{\pi}\right)$$
$$+ x^{-\frac{1}{R\log(qH_q)}}\frac{\varphi(q)R(\log(qH_q))^3}{mR(\log(qH_q))^2-\log(x)}\frac{2^m m^m}{2\pi\delta^m H_q^m}(1+o(1)) + O(x^{-\frac{1}{2}}). \quad (3.8.6)$$

3.9 New Arguments.

The argument we use has a number of optimizations over the previous method. The first improvement comes from the way in which we smooth the function $\psi(x;q,a)$, which generalizes the method of Faber and Kadiri [20].

3.9.1 Controlling the difference between $\psi(x;q,a)$ and $\psi_f(x;q,a)$

The general definition allows us to choose an appropriate function in order to optimize the bounds on the sum where the zeros of the $L-$functions may have real part not equal to one half.

Before considering the benefit of this we discuss first another new argument introduced in order to reduce the main term $\frac{\delta}{2}$. We do this by noting that $\frac{\delta}{2}$ comes from the principal character alone, and so we may first separate this term from the other characters. This allows us to make use of the bounds in [20] for $\psi(x)$ as well as apply a sieving argument. Which leads us to

$$E(x;q,a) \leq \min\left(\frac{\delta}{2}, J_2(x,m,\alpha,\delta,z)\right) + \sum_{\substack{\chi (\mathrm{mod}\, q)\\ \chi\neq\chi_0}} \Sigma(m,\delta,x) + \mathcal{O}(x^{-1}), \quad (3.9.1)$$

where $J_2(x,m,\alpha,\delta,z)$ is defined in (3.4.2), $\rho = \beta + i\gamma$ and

$$\Sigma(m,\delta,x) = \sum_{\rho\in Z(\chi)} x^{\beta-1}|F(\rho)| \qquad \text{with} \qquad F(\rho) = \int_0^\infty f(t)t^{\rho-1}dt.$$

As before, in order to bound $\Sigma(m,\delta,x)$ we split it into two regions. One where we have the generalized Riemann Hypothesis and one where we do not.

$$\Sigma(m,\delta,x) = \sum_{\substack{\rho\in Z(\chi)\\ |\gamma|\le H_q}} x^{\beta-1}|F(\rho)| + \sum_{\substack{\rho\in Z(\chi)\\ |\gamma|> H_q}} x^{\beta-1}|F(\rho)|$$
$$= \Sigma_{KL1} + \Sigma_{KL2}.$$

We see that our error term (3.9.1) has the same general form as (3.8.1) with the exception of the main term, which we discuss below.

Heuristic analysis for the effect of the Brun-Titchmarsh inequality.

The main idea here is to reduce the interval $[\alpha, \alpha+\delta]$ to $[\alpha+z, \alpha+\delta-z]$. We recall that, in the definition of $J_2(x,m,\alpha,\delta,z)$, that $\frac{e}{x} \le z \le \frac{\delta}{2}$ is a parameter which we choose to minimize the contribution of J_2. We have that $J_2(x,m,\alpha,\delta,z)$ is approximately

$$(\delta-2z)(f(\alpha+z)-f(\alpha+\delta-z)) + (1-f(\alpha+z))\frac{4z\log(x(\alpha+z))}{\log(xz)}.$$

Assume that $\log(z) = o(\log(x))$, this is reflected in the computational results. Using the definition of our optimized f and properties of $\log(x)$, we see that $J_2(x,m,\alpha,\delta,z)$ is approximately

$$(\delta-2z)\frac{(2m+1)!}{m!^2}\left(\int_{z/\delta}^{(\delta-z)/\delta} t^m(1-t)^m dt\right) + 4z\frac{(2m+1)!}{m!^2}\int_0^{z/\delta} t^m(1-t)^m dt \tag{3.9.2}$$

Let $w(t) = t^m(1-t)^m$ and recall that

$$\frac{(2m+1)!}{m!^2}\int_0^x w(t)dt = I_x(m+1,m+1),$$

where

$$I_x(a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^x t^{a-1}(1-t)^{b-1}dt,$$

is the regularized incomplete Beta function with parameters a and b. The exact value of $I_x(m+1, m+1)$ is given by

$$\sum_{j=m+1}^{2m+1} \binom{2m+1}{j} x^j (1-x)^{2m+1-j}.$$

While this is useful for the computation of $\epsilon_{q,b}$ it does not yield any meaningful results for how to choose our value of z. We proceed to give an estimate of $J_2(x, m, \alpha, \delta, z)$ by finding the optimal z value.

We start by using the symmetry $w(t) = w(1-t)$, (3.9.2) can be expressed as

$$\frac{(2m+1)!}{m!^2} \left((\delta - 2z) \left(\int_0^1 w(t)dt - 2 \int_0^{z/\delta} w(t)dt \right) + 4z \int_0^{z/\delta} w(t)dt \right)$$
$$= \frac{(2m+1)!}{m!^2} \left((\delta - 2z) \int_0^1 w(t)dt + 2(4z - \delta) \int_0^{z/\delta} w(t)dt \right). \tag{3.9.3}$$

We choose $z = \frac{\delta}{2}(1-\epsilon)$ with $\frac{e}{\delta x} \leq \epsilon \leq \frac{1}{2}$. Thus

$$2z = \delta(1-\epsilon), \ \delta\epsilon = \delta - 2z, \ 4z - \delta = \delta(1-2\epsilon).$$

Then dividing (3.9.3) by $\delta \frac{(2m+1)!}{m!^2}$ we obtain

$$\epsilon \int_0^1 w(t)dt + 2(1-2\epsilon) \int_0^{1/2(1-\epsilon)} w(t)dt$$
$$= \epsilon \int_0^1 w(t)dt + 2(1-2\epsilon) \left(\int_0^{1/2} w(t)dt - \int_{1/2(1-\epsilon)}^{1/2} w(t)dt \right)$$
$$= \epsilon \int_0^1 w(t)dt + (1-2\epsilon) \int_0^1 w(t)dt - 2(1-2\epsilon) \int_{1/2(1-\epsilon)}^{1/2} w(t)dt$$
$$= (1-\epsilon) \int_0^1 w(t)dt - 2(1-2\epsilon) \int_{1/2(1-\epsilon)}^{1/2} w(t)dt$$
$$\approx (1-\epsilon) \int_0^1 w(t)dt - (1-2\epsilon)\epsilon w(1/2), \tag{3.9.4}$$

where the last approximation comes from the fact that $w(t)$ attains its maximum in the interval $[\frac{1}{2}(1-\epsilon), \frac{1}{2}]$ at $t = \frac{1}{2}$. This seems reasonable to use under the assumption that $\frac{1}{2}-\epsilon$ is close to $\frac{1}{2}$. The derivative of the last expression with respect to ϵ is

$$\begin{aligned} &-\int_0^1 w(t)dt + 2\epsilon w(1/2) - (1-2\epsilon)w(1/2) \\ =&-\int_0^1 w(t)dt + 4\epsilon w(1/2) - w(1/2). \end{aligned}$$

Solving the above for ϵ, we obtain

$$\epsilon = \frac{1}{4} + \frac{1}{4w(1/2)}\int_0^1 w(t)dt = \frac{1}{4} + 2^{2m-2}\frac{m!^2}{(2m+1)!} \sim \frac{1}{4} + \frac{\sqrt{\pi}}{8\sqrt{m}},$$

since

$$w(1/2) = 2^{-2m}, \quad \int_0^1 w(t)dt = \frac{m!^2}{(2m+1)!} \sim \frac{\sqrt{\pi}}{2^{2m+1}\sqrt{m}}$$

by Stirling's formula

$$k! \sim \sqrt{2\pi k}\left(\frac{k}{e}\right)^k.$$

This gives

$$z \sim \delta\left(\frac{3}{8} + \frac{\sqrt{\pi}}{16\sqrt{m}}\right).$$

Using this value for z in (3.9.3) we obtain $J_2(x,m,\alpha,\delta,z)$ is approximately

$$\begin{aligned} &\frac{(2m+1)!}{m!^2}\left(\left(\delta - 2\delta\left(\frac{3}{8}+\frac{\sqrt{\pi}}{16\sqrt{m}}\right)\right)\int_0^1 w(t)dt \right.\\ &\left.+2\left(4\delta\left(\frac{3}{8}+\frac{\sqrt{\pi}}{}16\sqrt{m}\right)-\delta\right)\int_0^{z/\delta} w(t)dt\right) \\ &= \delta\left(\frac{1}{4} - \frac{\sqrt{\pi}}{8\sqrt{m}} + \left(1+\frac{\sqrt{\pi}}{2\sqrt{m}}\right)I_{\frac{3}{8}+\frac{\sqrt{\pi}}{16\sqrt{m}}}(m+1,m+1)\right). \end{aligned} \tag{3.9.5}$$

In Table 3.2 we give an exact computation of (3.9.5) for some values of m which are observed in the computations.

We see from Table 3.2 that $J_2(x,m,\alpha,\delta,z)$ looks like $c(m)\delta$ where $c(m)$ is a constant depending on m. For $m \in [15,30]$ we have $c(m) \in [0.3616, 0.2657]$. We mention that for $2 \le m \le 7$, $c(m) > \frac{1}{2}$, for example if $m=2$ then $c(m) = 0.76515$. Hence our main

Table 3.2: Approximate values of $\frac{1}{\delta}J_2(x,m,\alpha,\delta,z)$

m	$\frac{1}{\delta}$(3.9.5)	m	$\frac{1}{\delta}$(3.9.5)
15	0.3616	23	0.2961
16	0.3506	24	0.2906
17	0.3406	25	0.2856
18	0.3315	26	0.2809
19	0.3232	27	0.2767
20	0.3155	28	0.2727
21	0.3085	29	0.2690
22	0.3020	30	0.2657

term $\min\left(\frac{\delta}{2}, J_2(x,m,\alpha,\delta,z)\right)$ is approximated by

$$\min\left(\frac{\delta}{2}, c(m)\delta\right). \tag{3.9.6}$$

We show that $c(m) \to \frac{1}{4}$ as $m \to \infty$. To do this we must give an approximation for $I_{z/\delta}(m+1,m+1)$. This is achieved by noting that for m such that $\frac{m+2}{m+1} \approx 1$ we may approximate the Beta distribution, whose probability distribution function is given by

$$f(x,m+1,m+1) = \frac{(2m+1)!}{m!^2}x^m(1-x)^m,$$

using the normal distribution. The condition $\frac{m+2}{m+1} \approx 1$ is achieved when we take $m \geq 10$, and numerically we observe that this estimate begins to make a difference for our main term when $m \geq 15$. So we compare the value of $I_{z/\delta}(m+1,m+1)$ to the left tail of the normal distribution.

It is well known that the percentage of area under the normal curve within c standard deviations of the mean is given by

$$1 - \text{erfc}\left(\frac{c}{\sqrt{2}}\right), \quad \text{where } \text{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt.$$

Thus the area in either tail (the parts more than c deviations from the mean) of the normal distribution is given by

$$\frac{\text{erfc}\left(\frac{c}{\sqrt{2}}\right)}{2}. \tag{3.9.7}$$

We note the Beta distribution with parameters $m+1$ and $m+1$ has mean $\frac{1}{2}$ and standard deviation $\frac{1}{2\sqrt{2m+3}}$. We compute how many standard deviations z/δ is from 1/2:

$$\frac{3}{8}+\frac{\sqrt{\pi}}{16\sqrt{m}}=\frac{1}{2}-\frac{c}{2\sqrt{2m+3}} \iff c=\sqrt{2m+3}\left(\frac{2\sqrt{m}-\sqrt{\pi}}{8\sqrt{m}}\right).$$

Hence

$$\begin{aligned} I_{z/\delta}(m+1,m+1) &\approx \frac{1}{2}\text{erfc}\left(\frac{\sqrt{2m+3}\left(\frac{2\sqrt{m}-\sqrt{\pi}}{8\sqrt{m}}\right)}{\sqrt{2}}\right) \\ &\approx \frac{1}{2}\text{erfc}\left(\frac{\sqrt{2m+3}}{4\sqrt{2}}\right). \end{aligned}$$

As m grows this decays exponentially, hence plugging this estimate into (3.9.5) we find $J_2(x,m,\alpha,\delta,z)$ is approximately $\frac{\delta}{4}$. Tables 3.3 and 3.4 demonstrate the effect of taking the minimum between $\frac{\delta}{2}$ and $J_2(x,m,\alpha,\delta,z)$.

Table 3.3: Contributions of terms Sieving and Smoothing

q	10^4		$4\cdot 10^4$	
x	10^{30}	10^{100}	10^{30}	10^{100}
m	26	24	26	25
Σ_1	$8.2271\cdot 10^{-11}$	$8.0859\cdot 10^{-46}$	$4.0368\cdot 10^{-10}$	$4.0167\cdot -45$
Σ_2	$8.1338\cdot 10^{-5}$	$8.3186\cdot 10^{-5}$	$3.1884\cdot 10^{-4}$	$2.4221\cdot 10^{-4}$
$\frac{\delta}{2}$	0.003724	0.003500	0.01510	0.01436
$J_2(x,m,\alpha,\delta,z)$	0.002114	0.002006	0.008525	0.008214
$\epsilon_{q,b}$	0.002196	0.002088	0.008844	0.008439

Table 3.4: Contributions of Terms No Sieving argument

q	10^4		$4\cdot 10^4$	
x	10^{30}	10^{100}	10^{30}	10^{100}
m	17	16	17	16
Σ_1	$7.4841\cdot 10^{-11}$	$7.3948\cdot 10^{-46}$	$3.7959\cdot 10^{-10}$	$3.7864\cdot -45$
Σ_2	$1.9613\cdot 10^{-4}$	$1.9736\cdot 10^{-4}$	$8.0049\cdot 10^{-4}$	$8.0447\cdot 10^{-4}$
$\frac{\delta}{2}$	0.003324	0.003146	0.013430	0.012705
$\epsilon_{q,b}$	0.003520	0.003343	0.014230	0.013509

3.9.2 Sum over the zeros for which GRH is verified.

In this chapter this sum is handled in a much more sophisticated way than the previous authors. We split the term into four pieces allowing for us to optimize each term depending on the location of the zero on the half line. The sum we end up with has the shape

$$\Sigma_{KL1} = x^{-1/2}\left(\sum_{|\gamma|\leq 1}|F(\rho)| + \sum_{1<|\gamma|\leq T_0}|F(\rho)| + \sum_{T_0<|\gamma|\leq T_1}|F(\rho)| + \sum_{T_1<|\gamma|\leq H}|F(\rho)|\right)$$

which for x large enough is approximately

$$x^{-1/2}\left(\sum_{T_0<|\gamma|\leq T_1}|F(\rho)| + \sum_{T_1<|\gamma|\leq H}|F(\rho)\right).$$

The approach for handling this term is to let T_1 vary in such a way that the contribution of these terms is minimized. We bound $|F(\rho)|$ with (3.2.6) with $k=0$ for the first sum and $k=m$ for the second to obtain

$$M(\alpha,\alpha+\delta,0)\sum_{\substack{\rho\in Z(\chi)\\ T_0<|\gamma|\leq T_1}}\frac{1}{|\gamma|} + \frac{M(\alpha,\alpha+\delta,m)}{\delta^m}\sum_{\substack{\rho\in Z(\chi)\\ T_1<|\gamma|\leq H}}\frac{1}{|\gamma|^m}. \tag{3.9.8}$$

From here we use Lemma 3.7.2 to obtain (3.9.8) is approximately

$$M(\alpha,\alpha+\delta,0)\frac{1}{\pi}\log\left(\frac{T_1}{T_0}\right)\log\left(\frac{q\sqrt{T_1T_0}}{2\pi}\right) + \frac{M(\alpha,\alpha+\delta,m)}{\delta^m}\left(\frac{1+m\log\left(\frac{qT_1}{2\pi}\right)}{\pi m^2T_1^m} - \frac{1+m\log\left(\frac{qH}{2\pi}\right)}{\pi m^2H^m}\right).$$

Using (3.2.9) for $M(\alpha,\alpha+\delta,0)$ and (3.2.12) to estimate $M(\alpha,\alpha+\delta,m)$ we obtain that

(3.9.8) is approximately

$$(1+\frac{\delta}{2})\frac{1}{\pi}\log\left(\frac{T_1}{T_0}\right)\log\left(\frac{q\sqrt{T_1T_0}}{2\pi}\right) + \frac{\sqrt{(2m)!(2m+1)!}}{m!\delta^m}\left(\frac{1+m\log\left(\frac{qT_1}{2\pi}\right)}{\pi m^2T_1^m}-\frac{1+m\log\left(\frac{qH}{2\pi}\right)}{\pi m^2H^m}\right). \tag{3.9.9}$$

So that $\sum_{\substack{\chi(\mathrm{mod}\, q)\\ \chi\neq\chi_0}} \Sigma_{KL1}$ is given by

$$x^{-1/2}(\varphi(q)-1)\left(\left(1+\frac{\delta}{2}\right)\frac{1}{\pi}\log\left(\frac{T_1}{T_0}\right)\log\left(\frac{q\sqrt{T_1T_0}}{2\pi}\right) + \frac{\sqrt{(2m)!(2m+1)!}}{m!\delta^m}\left(\frac{1+m\log\left(\frac{qT_1}{2\pi}\right)}{\pi m^2T_1^m}-\frac{1+m\log\left(\frac{qH}{2\pi}\right)}{\pi m^2H^m}\right)\right). \tag{3.9.10}$$

Taking $x = 10^{30}$, $H = 2\ 500$, $q = 40\ 000$ then we find $\delta = 0.0302$, if (3.8.2) is used the estimate is $32.0327\ldots$. Then using these same parameters we find $m = 26$ and $T_1 = \frac{12060572337605}{8589934592} \approx 1404.0354\ldots$ and using (3.9.9) we achieve $21.7614\ldots$, see Table 3.5 for further comparisons.

3.9.3 Sum over the remaining zeros.

As in [66] we make the observation that $\rho \in Z(\chi) \iff 1-\overline{\rho} \in Z(\chi)$. So we may also split the sum into two terms. One which has a factor of $x^{-1/2}$ and one with $x^{-1/R\log(q|\gamma|)}$. That is

$$\Sigma_{KL2} \leq \frac{1}{2}\sum_{\substack{\rho\in Z(\chi)\\ |\gamma|>H_q}} x^{-\frac{1}{2}}|F(\rho)| + \frac{1}{2}\sum_{\substack{\rho\in Z(\chi)\\ |\gamma|>H_q}} x^{-\frac{1}{2}}|F(\rho)|. \tag{3.9.11}$$

We estimate the sum in the same way with one major difference, the optimized smoothing argument. The function f was chosen in order to minimize the contribution of this term. We have

$$|F(\rho)| \leq \frac{M(\alpha, \alpha+\delta, m)}{\delta^m |\gamma|^{m+1}} \leq \frac{\sqrt{(2m)!(2m+1)!}}{m!\delta^m |\gamma|^{m+1}}(1+o(1)).$$

We use the same lemma of Rosser and Schoenfeld as in Section 3.8.3 so that $\sum_{\substack{\chi (\bmod q) \\ \chi \neq \chi_0}} \Sigma_{KL2}$ is approximately

$$x^{-\frac{1}{R\log(qH)}} \frac{(\varphi(q)-1)R(\log(qH))^3}{mR(\log(qH))^2 - \log(x)} \frac{\sqrt{(2m)!(2m+1)!}}{2\pi m!\delta^m H^m}(1+o(1)). \tag{3.9.12}$$

To compare with (3.8.5) we see that it follows from Stirling's Formula that the quotient $\frac{|F(\rho)|}{|I_{m,\delta}(\rho)|}$ goes to 0 as $m \to \infty$. For example this quotient is $0.003524\ldots$ for $q = 3, b = 30, m = 26$. Furthermore we compute the value of $M(\alpha, \alpha+\delta, m)$ exactly so that our savings are even sharper. We again have that the sum with $x^{-1/2}$ being extremely small.

3.9.4 Final Error Term

Combining (3.9.6), (3.9.10) and (3.9.12) and plugging it into (3.9.1) we obtain $E(x; q, a)$ is approximately

$$\begin{aligned} &\min\left(\frac{\delta}{2}, c(m)\delta\right) + x^{-1/2}(\varphi(q)-1)\left(\left(1+\frac{\delta}{2}\right)\frac{1}{\pi}\log\left(\frac{T_1}{T_0}\right)\log\left(\frac{q\sqrt{T_1T_0}}{2\pi}\right)\right. \\ &\quad \left. + \frac{\sqrt{(2m)!(2m+1)!}}{m!\delta^m}\left(\frac{1+m\log\left(\frac{qT_1}{2\pi}\right)}{\pi m^2 T_1^m} - \frac{1+m\log\left(\frac{qH}{2\pi}\right)}{\pi m^2 H^m}\right)\right) \\ &\quad + x^{-\frac{1}{R\log(qH)}} \frac{(\varphi(q)-1)R(\log(qH))^3}{mR(\log(qH))^2-\log(x)} \frac{\sqrt{(2m)!(2m+1)!}}{2\pi m!\delta^m H^m}(1+o(1)). \end{aligned} \tag{3.9.13}$$

In Table 3.5 we fix q, H, R_0 to be the same as in [66]:

Table 3.5: Comparing Results with [66].

q	13				486			
x	10^{30}		10^{100}		10^{30}		10^{100}	
Author	K & L	[66]	K & L	[66]	K & L	[66]	K & L	[66]
m	17	9	14	8	18	11	17	9
$\frac{\delta}{2}$	0.0023265	0.0022492	0.0019375	0.0017684	0.010480	0.010493	0.009500	0.008579
$J_2(x, m, \alpha, \delta, z)$	0.0016298	-	0.0014276	-	0.0071688	-	0.0065584	-
$\epsilon_{q,b}$	0.001726	0.002478	0.0015297	0.002020	0.0075694	0.011458	0.0069475	0.009740

3.10 Results

We present the results in a few different ways. The first set of tables will fix a b and present the parameters m, δ, and T_1 as well as the final $\epsilon_{q,b}$ for select moduli q. Afterward, we present results all values reported in Ramaré and Rumely's 1996 paper [66], with an extended list of values up to 40 000.

Table 3.6 presents select moduli q for $b = 10$.

Table 3.6: Let $x \geq 10^{10}$ then $E(x; q, a) \leq \epsilon_{q,10}$.

q	m	δ	T_1	$\epsilon_{q,10}$
3	20	$8.805 \cdot 10^{-6}$	3 703 235.305 . . .	$4.588 \cdot 10^{-4}$
4	17	$1.000 \cdot 10^{-5}$	2 809 020.583 . . .	$4.6040 \cdot 10^{-4}$
5	33	$2.405 \cdot 10^{-5}$	2 164 679.257 . . .	$1.1224 \cdot 10^{-3}$
7	36	$3.671 \cdot 10^{-5}$	1 539 986.437 . . .	$1.7863 \cdot 10^{-3}$
8	24	$2.808 \cdot 10^{-5}$	1 375 006.319 . . .	$1.0967 \cdot 10^{-3}$
100	2	$1.054 \cdot 10^{-2}$	200.000 . . .	$1.1452 \cdot 10^{-2}$
200	2	$1.372 \cdot 10^{-2}$	200.000 . . .	$1.930 \cdot 10^{-2}$
400	2	$1.791 \cdot 10^{-2}$	200.000 . . .	$3.4814 \cdot 10^{-2}$
600	2	$1.814 \cdot 10^{-2}$	200.000 . . .	$3.7161 \cdot 10^{-2}$
800	2	$2.385 \cdot 10^{-2}$	200.000 . . .	$6.9950 \cdot 10^{-2}$
1 000	2	$2.641 \cdot 10^{-2}$	200.000 . . .	$9.1906 \cdot 10^{-2}$
2 000	3	$3.651 \cdot 10^{-2}$	200.000 . . .	$2.2646 \cdot 10^{-1}$
3 000	3	$3.632 \cdot 10^{-2}$	200.000 . . .	$2.4660 \cdot 10^{-1}$
4 000	3	$4.277 \cdot 10^{-2}$	200.000 . . .	$4.3553 \cdot 10^{-1}$
5 000	3	$4.417 \cdot 10^{-2}$	200.000 . . .	$6.4939 \cdot 10^{-1}$
6 000	3	$4.285 \cdot 10^{-2}$	200.000 . . .	$5.0428 \cdot 10^{-1}$
7 000	3	$4.665 \cdot 10^{-2}$	200.000 . . .	$7.4949 \cdot 10^{-1}$
9 000	4	$5.126 \cdot 10^{-2}$	200.000 . . .	$7.4731 \cdot 10^{-1}$

Table 3.7 presents select moduli q for $b = 13$.

Table 3.7: Let $x \geq 10^{13}$ then $E(x;q,a) \leq \epsilon_{q,13}$.

q	m	δ	T_1	$\epsilon_{q,13}$
3	7	$3.287 \cdot 10^{-6}$	3 906 419.570 ...	$1.8260 \cdot 10^{-5}$
4	7	$4.324 \cdot 10^{-6}$	2 969 595.593 ...	$1.8809 \cdot 10^{-5}$
5	8	$6.174 \cdot 10^{-6}$	2 330 713.059 ...	$4.0907 \cdot 10^{-5}$
7	8	$8.644 \cdot 10^{-6}$	1 664 734.188 ...	$6.3492 \cdot 10^{-5}$
8	7	$8.766 \cdot 10^{-6}$	1 464 843.447 ...	$4.1541 \cdot 10^{-5}$
100	8	$1.160 \cdot 10^{-4}$	124 067.582 ...	$4.3750 \cdot 10^{-4}$
200	8	$2.468 \cdot 10^{-4}$	58 320.731 ...	$8.9251 \cdot 10^{-4}$
400	8	$4.933 \cdot 10^{-4}$	29 184.469 ...	$1.7865 \cdot 10^{-3}$
600	8	$6.180 \cdot 10^{-4}$	23 298.166 ...	$1.8511 \cdot 10^{-3}$
800	8	$9.853 \cdot 10^{-4}$	14 617.766 ...	$3.6474 \cdot 10^{-3}$
1 000	8	$1.230 \cdot 10^{-3}$	11 712.153 ...	$4.6869 \cdot 10^{-3}$
2 000	8	$2.453 \cdot 10^{-3}$	5 879.011 ...	$1.0610 \cdot 10^{-2}$
3 000	8	$3.109 \cdot 10^{-3}$	4 461.136 ...	$1.1238 \cdot 10^{-2}$
4 000	8	$4.872 \cdot 10^{-3}$	2 966.199 ...	$2.0092 \cdot 10^{-2}$
5 000	8	$6.070 \cdot 10^{-3}$	2 383.231 ...	$2.8045 \cdot 10^{-2}$
6 000	8	$6.216 \cdot 10^{-3}$	2 327.516 ...	$2.2223 \cdot 10^{-2}$
7 000	8	$8.233 \cdot 10^{-3}$	2 754.683 ...	$3.2269 \cdot 10^{-2}$
8 000	8	$9.610 \cdot 10^{-3}$	1 509.279 ...	$4.3617 \cdot 10^{-2}$
9 000	8	$9.965 \cdot 10^{-3}$	1 558.294 ...	$3.2328 \cdot 10^{-2}$
10 000	8	$1.180 \cdot 10^{-2}$	1 231.974 ...	$5.5393 \cdot 10^{-2}$

Table 3.8 presents select moduli q for $b = 30$.

Table 3.8: Let $x \geq 10^{30}$ then $E(x; q, a) \leq \epsilon_{q,30}$.

q	m	δ	T_1	$\epsilon_{q,30}$
3	26	$2.193 \cdot 10^{-6}$	18 971 195.912...	$6.7226 \cdot 10^{-7}$
4	26	$2.893 \cdot 10^{-6}$	14 380 903.341...	$8.8440 \cdot 10^{-7}$
5	27	$3.786 \cdot 10^{-6}$	11 384 039.814...	$1.1392 \cdot 10^{-6}$
7	28	$5.408 \cdot 10^{-6}$	8 246 156.383...	$1.6012 \cdot 10^{-6}$
8	26	$5.874 \cdot 10^{-6}$	7 082 777.192...	$1.7858 \cdot 10^{-6}$
100	26	$7.356 \cdot 10^{-5}$	565 611.725...	$2.2083 \cdot 10^{-5}$
200	27	$1.493 \cdot 10^{-4}$	288 710.195...	$4.4082 \cdot 10^{-5}$
400	25	$2.905 \cdot 10^{-4}$	138 089.588...	$8.7951 \cdot 10^{-5}$
600	25	$4.291 \cdot 10^{-4}$	93 494.474...	$1.2695 \cdot 10^{-4}$
800	26	$5.893 \cdot 10^{-4}$	70 625.667...	$1.7541 \cdot 10^{-4}$
1 000	26	$7.367 \cdot 10^{-4}$	56 499.896...	$2.1909 \cdot 10^{-4}$
2 000	27	$1.495 \cdot 10^{-3}$	28 856.110...	$4.3740 \cdot 10^{-4}$
3 000	26	$2.179 \cdot 10^{-3}$	19 119.024...	$6.4527 \cdot 10^{-4}$
4 000	25	$2.910 \cdot 10^{-3}$	13 807.600...	$8.7302 \cdot 10^{-4}$
5 000	26	$3.690 \cdot 10^{-3}$	11 300.589...	$1.0905 \cdot 10^{-3}$
6 000	25	$4.301 \cdot 10^{-3}$	9 350.118...	$1.2883 \cdot 10^{-3}$
7 000	25	$5.069 \cdot 10^{-3}$	7 937.290...	$1.5173 \cdot 10^{-3}$
8 000	26	$5.911 \cdot 10^{-3}$	7 064.222...	$1.7436 \cdot 10^{-3}$
9 000	25	$6.459 \cdot 10^{-3}$	6 234.572...	$1.9316 \cdot 10^{-3}$
10 000	26	$7.394 \cdot 10^{-3}$	5 652.588...	$2.1793 \cdot 10^{-3}$

Table 3.9 presents select moduli q for $b = 100$.

Table 3.9: Let $x \geq 10^{100}$ then $E(x; q, a) \leq \epsilon_{q,100}$.

q	m	δ	T_1	$\epsilon_{q,100}$
3	23	$1.990 \cdot 10^{-6}$	18 648 455.384 . . .	$6.1353 \cdot 10^{-7}$
4	22	$2.585 \cdot 10^{-6}$	13 775 864.307 . . .	$8.0692 \cdot 10^{-7}$
5	25	$3.503 \cdot 10^{-6}$	11 449 393.152 . . .	$1.0434 \cdot 10^{-6}$
7	24	$4.866 \cdot 10^{-6}$	7 934 562.100 . . .	$1.4688 \cdot 10^{-6}$
8	24	$5.420 \cdot 10^{-6}$	7 123 547.574 . . .	$1.6342 \cdot 10^{-6}$
100	23	$6.686 \cdot 10^{-5}$	555 080.605 . . .	$2.0364 \cdot 10^{-5}$
200	26	$1.401 \cdot 10^{-4}$	296 989.040 . . .	$4.0827 \cdot 10^{-5}$
400	24	$2.717 \cdot 10^{-4}$	142 129.510 . . .	$8.1386 \cdot 10^{-5}$
600	22	$3.889 \cdot 10^{-4}$	91 591.962 . . .	$1.2003 \cdot 10^{-4}$
800	24	$5.436 \cdot 10^{-4}$	71 050.684 . . .	$1.6269 \cdot 10^{-4}$
1 000	23	$6.695 \cdot 10^{-4}$	55 454.640 . . .	$2.0332 \cdot 10^{-4}$
2 000	26	$1.402 \cdot 10^{-3}$	29 700.760 . . .	$4.0762 \cdot 10^{-4}$
3 000	24	$2.010 \cdot 10^{-3}$	19 233.051 . . .	$5.9986 \cdot 10^{-4}$
4 000	23	$2.681 \cdot 10^{-3}$	13 865.650 . . .	$8.1286 \cdot 10^{-4}$
5 000	24	$3.403 \cdot 10^{-3}$	11 006.258 . . .	$1.0163 \cdot 10^{-3}$
6 000	22	$3.897 \cdot 10^{-3}$	9 160.745 . . .	$1.1995 \cdot 10^{-3}$
7 000	23	$4.666 \cdot 10^{-3}$	7 976.929 . . .	$1.4140 \cdot 10^{-3}$
8 000	23	$5.369 \cdot 10^{-3}$	6 935.539 . . .	$1.6267 \cdot 10^{-3}$
9 000	23	$5.941 \cdot 10^{-3}$	6 270.0537 . . .	$1.7997 \cdot 10^{-3}$
10 000	23	$6.716 \cdot 10^{-3}$	5 549.244 . . .	$2.0343 \cdot 10^{-3}$
40 000	23	$2.715 \cdot 10^{-2}$	1 390.563 . . .	$8.2133 \cdot 10^{-3}$

Table 3.10: For each q and $x \geq 10^b$ we give $\epsilon_{q,b}$ such that $|E(x; q, a)| < \epsilon_{q,b}$.

$q \setminus x$	10^{10}	10^{13}	10^{30}	10^{100}
3	$4.5880 \cdot 10^{-4}$	$1.8259 \cdot 10^{-5}$	$6.7226 \cdot 10^{-7}$	$6.1353 \cdot 10^{-7}$
4	$4.6039 \cdot 10^{-4}$	$1.8808 \cdot 10^{-5}$	$8.8440 \cdot 10^{-7}$	$8.0692 \cdot 10^{-7}$
5	$1.1224 \cdot 10^{-3}$	$4.0907 \cdot 10^{-5}$	$1.1391 \cdot 10^{-6}$	$1.0438 \cdot 10^{-6}$
6	$4.3148 \cdot 10^{-4}$	$1.8749 \cdot 10^{-5}$	$1.3017 \cdot 10^{-6}$	$1.1867 \cdot 10^{-6}$
7	$1.7862 \cdot 10^{-3}$	$6.3491 \cdot 10^{-5}$	$1.6012 \cdot 10^{-6}$	$1.4687 \cdot 10^{-6}$
8	$1.0966 \cdot 10^{-3}$	$4.1540 \cdot 10^{-5}$	$1.7858 \cdot 10^{-6}$	$1.6351 \cdot 10^{-6}$
9	$1.7403 \cdot 10^{-3}$	$6.3062 \cdot 10^{-5}$	$2.0365 \cdot 10^{-6}$	$1.8678 \cdot 10^{-6}$
10	$1.0382 \cdot 10^{-3}$	$4.0666 \cdot 10^{-5}$	$2.2109 \cdot 10^{-6}$	$2.0234 \cdot 10^{-6}$
11	$3.1165 \cdot 10^{-3}$	$1.0876 \cdot 10^{-4}$	$2.5204 \cdot 10^{-6}$	$2.3165 \cdot 10^{-6}$
12	$1.0236 \cdot 10^{-3}$	$4.1198 \cdot 10^{-5}$	$2.6325 \cdot 10^{-6}$	$2.4084 \cdot 10^{-6}$
13	$3.8240 \cdot 10^{-3}$	$1.3272 \cdot 10^{-4}$	$2.9799 \cdot 10^{-6}$	$2.7399 \cdot 10^{-6}$
14	$1.6448 \cdot 10^{-3}$	$6.2448 \cdot 10^{-5}$	$3.1147 \cdot 10^{-6}$	$2.8505 \cdot 10^{-6}$
15	$2.2421 \cdot 10^{-3}$	$8.2468 \cdot 10^{-5}$	$3.3656 \cdot 10^{-6}$	$3.0877 \cdot 10^{-6}$
16	$2.3399 \cdot 10^{-3}$	$8.6080 \cdot 10^{-5}$	$3.5781 \cdot 10^{-6}$	$3.2836 \cdot 10^{-6}$
17	$5.1798 \cdot 10^{-3}$	$1.7869 \cdot 10^{-4}$	$3.8965 \cdot 10^{-6}$	$3.5863 \cdot 10^{-6}$
18	$1.6026 \cdot 10^{-3}$	$6.3100 \cdot 10^{-5}$	$3.9555 \cdot 10^{-6}$	$3.6266 \cdot 10^{-6}$
19	$6.8408 \cdot 10^{-3}$	$2.3290 \cdot 10^{-4}$	$4.3581 \cdot 10^{-6}$	$4.0172 \cdot 10^{-6}$
20	$2.2147 \cdot 10^{-3}$	$8.4036 \cdot 10^{-5}$	$4.4324 \cdot 10^{-6}$	$4.0689 \cdot 10^{-6}$
21	$3.4616 \cdot 10^{-3}$	$1.2528 \cdot 10^{-4}$	$4.7233 \cdot 10^{-6}$	$4.3473 \cdot 10^{-6}$
22	$2.8608 \cdot 10^{-3}$	$1.0611 \cdot 10^{-4}$	$4.9018 \cdot 10^{-6}$	$4.4996 \cdot 10^{-6}$
23	$7.2000 \cdot 10^{-3}$	$2.4728 \cdot 10^{-4}$	$5.2705 \cdot 10^{-6}$	$4.8569 \cdot 10^{-6}$
24	$2.1781 \cdot 10^{-3}$	$8.4874 \cdot 10^{-5}$	$5.2818 \cdot 10^{-6}$	$4.8395 \cdot 10^{-6}$
25	$6.2946 \cdot 10^{-3}$	$2.1921 \cdot 10^{-4}$	$5.6891 \cdot 10^{-6}$	$5.2439 \cdot 10^{-6}$
26	$3.5108 \cdot 10^{-3}$	$1.2927 \cdot 10^{-4}$	$5.7958 \cdot 10^{-6}$	$5.3276 \cdot 10^{-6}$

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Table 3.10 – *Continued from previous page*

$q \setminus x$	10^{10}	10^{13}	10^{30}	10^{100}
27	$5.5565 \cdot 10^{-3}$	$1.9647 \cdot 10^{-4}$	$6.1104 \cdot 10^{-6}$	$5.6172 \cdot 10^{-6}$
28	$3.4152 \cdot 10^{-3}$	$1.2724 \cdot 10^{-4}$	$6.2208 \cdot 10^{-6}$	$5.7148 \cdot 10^{-6}$
29	$9.2336 \cdot 10^{-3}$	$3.1726 \cdot 10^{-4}$	$6.6479 \cdot 10^{-6}$	$6.1271 \cdot 10^{-6}$
30	$2.0573 \cdot 10^{-3}$	$8.3988 \cdot 10^{-5}$	$6.5369 \cdot 10^{-6}$	$5.9965 \cdot 10^{-6}$
31	$1.2446 \cdot 10^{-2}$	$4.2900 \cdot 10^{-4}$	$7.0998 \cdot 10^{-6}$	$6.5465 \cdot 10^{-6}$
32	$4.8715 \cdot 10^{-3}$	$1.7655 \cdot 10^{-4}$	$7.1499 \cdot 10^{-6}$	$6.5758 \cdot 10^{-6}$
33	$5.9694 \cdot 10^{-3}$	$2.1309 \cdot 10^{-4}$	$7.4264 \cdot 10^{-6}$	$6.8394 \cdot 10^{-6}$
34	$4.7526 \cdot 10^{-3}$	$1.7360 \cdot 10^{-4}$	$7.5769 \cdot 10^{-6}$	$6.9705 \cdot 10^{-6}$
35	$7.2570 \cdot 10^{-3}$	$2.5599 \cdot 10^{-4}$	$7.9150 \cdot 10^{-6}$	$7.2950 \cdot 10^{-6}$
36	$3.3663 \cdot 10^{-3}$	$1.2960 \cdot 10^{-4}$	$7.9137 \cdot 10^{-6}$	$7.2679 \cdot 10^{-6}$
37	$1.0932 \cdot 10^{-2}$	$4.1400 \cdot 10^{-4}$	$8.4706 \cdot 10^{-6}$	$7.8175 \cdot 10^{-6}$
38	$6.3573 \cdot 10^{-3}$	$2.2706 \cdot 10^{-4}$	$8.4684 \cdot 10^{-6}$	$7.7919 \cdot 10^{-6}$
39	$7.2284 \cdot 10^{-3}$	$2.5726 \cdot 10^{-4}$	$8.7793 \cdot 10^{-6}$	$8.0853 \cdot 10^{-6}$
40	$4.5868 \cdot 10^{-3}$	$1.7140 \cdot 10^{-4}$	$8.8581 \cdot 10^{-6}$	$8.1437 \cdot 10^{-6}$
41	$1.1810 \cdot 10^{-2}$	$4.6389 \cdot 10^{-4}$	$9.3839 \cdot 10^{-6}$	$8.6631 \cdot 10^{-6}$
42	$3.1696 \cdot 10^{-3}$	$1.2636 \cdot 10^{-4}$	$9.1720 \cdot 10^{-6}$	$8.4285 \cdot 10^{-6}$
43	$1.2093 \cdot 10^{-2}$	$4.8404 \cdot 10^{-4}$	$9.8405 \cdot 10^{-6}$	$9.0869 \cdot 10^{-6}$
44	$5.9986 \cdot 10^{-3}$	$2.1932 \cdot 10^{-4}$	$9.7887 \cdot 10^{-6}$	$9.0074 \cdot 10^{-6}$
45	$7.1156 \cdot 10^{-3}$	$2.5642 \cdot 10^{-4}$	$1.0070 \cdot 10^{-5}$	$9.2739 \cdot 10^{-6}$
46	$6.5954 \cdot 10^{-3}$	$2.3984 \cdot 10^{-4}$	$1.0250 \cdot 10^{-5}$	$9.4393 \cdot 10^{-6}$
47	$1.2908 \cdot 10^{-2}$	$5.3234 \cdot 10^{-4}$	$1.0753 \cdot 10^{-5}$	$9.9313 \cdot 10^{-6}$
48	$4.5112 \cdot 10^{-3}$	$1.7295 \cdot 10^{-4}$	$1.0546 \cdot 10^{-5}$	$9.6949 \cdot 10^{-6}$
49	$1.2039 \cdot 10^{-2}$	$4.7616 \cdot 10^{-4}$	$1.1157 \cdot 10^{-5}$	$1.0296 \cdot 10^{-5}$
50	$5.7631 \cdot 10^{-3}$	$2.1478 \cdot 10^{-4}$	$1.1063 \cdot 10^{-5}$	$1.0181 \cdot 10^{-5}$
51	$9.7538 \cdot 10^{-3}$	$3.4552 \cdot 10^{-4}$	$1.1480 \cdot 10^{-5}$	$1.0581 \cdot 10^{-5}$
52	$7.1268 \cdot 10^{-3}$	$2.6016 \cdot 10^{-4}$	$1.1570 \cdot 10^{-5}$	$1.0654 \cdot 10^{-5}$

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Table 3.10 – *Continued from previous page*

$q \setminus x$	10^{10}	10^{13}	10^{30}	10^{100}
53	$1.7435 \cdot 10^{-2}$	$7.0908 \cdot 10^{-4}$	$1.2122 \cdot 10^{-5}$	$1.1195 \cdot 10^{-5}$
54	$5.0910 \cdot 10^{-3}$	$1.9490 \cdot 10^{-4}$	$1.1863 \cdot 10^{-5}$	$1.0908 \cdot 10^{-5}$
55	$1.1294 \cdot 10^{-2}$	$4.3511 \cdot 10^{-4}$	$1.2442 \cdot 10^{-5}$	$1.1481 \cdot 10^{-5}$
56	$7.0232 \cdot 10^{-3}$	$2.5885 \cdot 10^{-4}$	$1.2423 \cdot 10^{-5}$	$1.1437 \cdot 10^{-5}$
57	$1.1579 \cdot 10^{-2}$	$4.2210 \cdot 10^{-4}$	$1.2826 \cdot 10^{-5}$	$1.1832 \cdot 10^{-5}$
58	$8.4851 \cdot 10^{-3}$	$3.0743 \cdot 10^{-4}$	$1.2921 \cdot 10^{-5}$	$1.1907 \cdot 10^{-5}$
59	$1.5393 \cdot 10^{-2}$	$6.7954 \cdot 10^{-4}$	$1.3490 \cdot 10^{-5}$	$1.2462 \cdot 10^{-5}$
60	$4.2563 \cdot 10^{-3}$	$1.7085 \cdot 10^{-4}$	$1.3059 \cdot 10^{-5}$	$1.2006 \cdot 10^{-5}$
61	$1.6255 \cdot 10^{-2}$	$7.1826 \cdot 10^{-4}$	$1.3946 \cdot 10^{-5}$	$1.2890 \cdot 10^{-5}$
62	$1.1915 \cdot 10^{-2}$	$4.1837 \cdot 10^{-4}$	$1.3810 \cdot 10^{-5}$	$1.2730 \cdot 10^{-5}$
63	$1.0524 \cdot 10^{-2}$	$3.8640 \cdot 10^{-4}$	$1.4121 \cdot 10^{-5}$	$1.3026 \cdot 10^{-5}$
64	$9.8960 \cdot 10^{-3}$	$3.5630 \cdot 10^{-4}$	$1.4274 \cdot 10^{-5}$	$1.3154 \cdot 10^{-5}$
65	$1.2737 \cdot 10^{-2}$	$5.2336 \cdot 10^{-4}$	$1.4710 \cdot 10^{-5}$	$1.3576 \cdot 10^{-5}$
66	$5.4664 \cdot 10^{-3}$	$2.1315 \cdot 10^{-4}$	$1.4437 \cdot 10^{-5}$	$1.3286 \cdot 10^{-5}$
67	$1.6837 \cdot 10^{-2}$	$7.7151 \cdot 10^{-4}$	$1.5313 \cdot 10^{-5}$	$1.4152 \cdot 10^{-5}$
68	$9.9365 \cdot 10^{-3}$	$3.5913 \cdot 10^{-4}$	$1.5129 \cdot 10^{-5}$	$1.3945 \cdot 10^{-5}$
69	$1.2129 \cdot 10^{-2}$	$4.8123 \cdot 10^{-4}$	$1.5531 \cdot 10^{-5}$	$1.4324 \cdot 10^{-5}$
70	$6.6378 \cdot 10^{-3}$	$2.5347 \cdot 10^{-4}$	$1.5383 \cdot 10^{-5}$	$1.4164 \cdot 10^{-5}$
71	$1.7604 \cdot 10^{-2}$	$8.1900 \cdot 10^{-4}$	$1.6225 \cdot 10^{-5}$	$1.5003 \cdot 10^{-5}$
72	$6.8549 \cdot 10^{-3}$	$2.6135 \cdot 10^{-4}$	$1.5807 \cdot 10^{-5}$	$1.4547 \cdot 10^{-5}$
74	$1.0936 \cdot 10^{-2}$	$4.0097 \cdot 10^{-4}$	$1.6478 \cdot 10^{-5}$	$1.5197 \cdot 10^{-5}$
75	$1.1703 \cdot 10^{-2}$	$4.4261 \cdot 10^{-4}$	$1.6758 \cdot 10^{-5}$	$1.5463 \cdot 10^{-5}$
76	$1.2007 \cdot 10^{-2}$	$4.3395 \cdot 10^{-4}$	$1.6908 \cdot 10^{-5}$	$1.5586 \cdot 10^{-5}$
77	$1.7321 \cdot 10^{-2}$	$7.3403 \cdot 10^{-4}$	$1.7442 \cdot 10^{-5}$	$1.6115 \cdot 10^{-5}$
78	$6.6140 \cdot 10^{-3}$	$2.5671 \cdot 10^{-4}$	$1.7066 \cdot 10^{-5}$	$1.5714 \cdot 10^{-5}$
80	$9.4145 \cdot 10^{-3}$	$3.4876 \cdot 10^{-4}$	$1.7678 \cdot 10^{-5}$	$1.6292 \cdot 10^{-5}$

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Table 3.10 – *Continued from previous page*

$q \setminus x$	10^{10}	10^{13}	10^{30}	10^{100}
81	$1.4133 \cdot 10^{-2}$	$6.0086 \cdot 10^{-4}$	$1.8247 \cdot 10^{-5}$	$1.6844 \cdot 10^{-5}$
82	$1.1814 \cdot 10^{-2}$	$4.4926 \cdot 10^{-4}$	$1.8255 \cdot 10^{-5}$	$1.6836 \cdot 10^{-5}$
84	$6.8642 \cdot 10^{-3}$	$2.6765 \cdot 10^{-4}$	$1.8321 \cdot 10^{-5}$	$1.6859 \cdot 10^{-5}$
85	$1.5671 \cdot 10^{-2}$	$7.0465 \cdot 10^{-4}$	$1.9224 \cdot 10^{-5}$	$1.7763 \cdot 10^{-5}$
86	$1.2098 \cdot 10^{-2}$	$4.6862 \cdot 10^{-4}$	$1.9144 \cdot 10^{-5}$	$1.7658 \cdot 10^{-5}$
87	$1.5097 \cdot 10^{-2}$	$6.3922 \cdot 10^{-4}$	$1.9563 \cdot 10^{-5}$	$1.8065 \cdot 10^{-5}$
88	$1.1623 \cdot 10^{-2}$	$4.4144 \cdot 10^{-4}$	$1.9534 \cdot 10^{-5}$	$1.8015 \cdot 10^{-5}$
90	$6.5132 \cdot 10^{-3}$	$2.5948 \cdot 10^{-4}$	$1.9572 \cdot 10^{-5}$	$1.8010 \cdot 10^{-5}$
91	$1.8023 \cdot 10^{-2}$	$8.2306 \cdot 10^{-4}$	$2.0613 \cdot 10^{-5}$	$1.9053 \cdot 10^{-5}$
92	$1.3033 \cdot 10^{-2}$	$5.0796 \cdot 10^{-4}$	$2.0459 \cdot 10^{-5}$	$1.8873 \cdot 10^{-5}$
93	$2.1148 \cdot 10^{-2}$	$8.5091 \cdot 10^{-4}$	$2.0909 \cdot 10^{-5}$	$1.9312 \cdot 10^{-5}$
94	$1.2914 \cdot 10^{-2}$	$5.1571 \cdot 10^{-4}$	$2.0920 \cdot 10^{-5}$	$1.9301 \cdot 10^{-5}$
95	$1.8097 \cdot 10^{-2}$	$8.2406 \cdot 10^{-4}$	$2.1482 \cdot 10^{-5}$	$1.9856 \cdot 10^{-5}$
96	$9.2325 \cdot 10^{-3}$	$3.5084 \cdot 10^{-4}$	$2.1055 \cdot 10^{-5}$	$1.9397 \cdot 10^{-5}$
98	$1.2045 \cdot 10^{-2}$	$4.6641 \cdot 10^{-4}$	$2.1698 \cdot 10^{-5}$	$2.0013 \cdot 10^{-5}$
99	$1.5210 \cdot 10^{-2}$	$6.6105 \cdot 10^{-4}$	$2.2203 \cdot 10^{-5}$	$2.0505 \cdot 10^{-5}$
100	$1.1452 \cdot 10^{-2}$	$4.3750 \cdot 10^{-4}$	$2.2082 \cdot 10^{-5}$	$2.0363 \cdot 10^{-5}$
102	$8.9217 \cdot 10^{-3}$	$3.4401 \cdot 10^{-4}$	$2.2316 \cdot 10^{-5}$	$2.0556 \cdot 10^{-5}$
104	$1.3299 \cdot 10^{-2}$	$5.3593 \cdot 10^{-4}$	$2.3087 \cdot 10^{-5}$	$2.1303 \cdot 10^{-5}$
105	$1.2465 \cdot 10^{-2}$	$5.0249 \cdot 10^{-4}$	$2.3300 \cdot 10^{-5}$	$2.1499 \cdot 10^{-5}$
106	$1.7444 \cdot 10^{-2}$	$6.9123 \cdot 10^{-4}$	$2.3586 \cdot 10^{-5}$	$2.1766 \cdot 10^{-5}$
108	$1.0614 \cdot 10^{-2}$	$4.0166 \cdot 10^{-4}$	$2.3679 \cdot 10^{-5}$	$2.1821 \cdot 10^{-5}$
110	$1.1303 \cdot 10^{-2}$	$4.2865 \cdot 10^{-4}$	$2.4196 \cdot 10^{-5}$	$2.2308 \cdot 10^{-5}$
111	$1.7361 \cdot 10^{-2}$	$7.9802 \cdot 10^{-4}$	$2.4946 \cdot 10^{-5}$	$2.3052 \cdot 10^{-5}$
112	$1.3014 \cdot 10^{-2}$	$5.2577 \cdot 10^{-4}$	$2.4788 \cdot 10^{-5}$	$2.2873 \cdot 10^{-5}$
114	$1.1102 \cdot 10^{-2}$	$4.2008 \cdot 10^{-4}$	$2.4939 \cdot 10^{-5}$	$2.2980 \cdot 10^{-5}$

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Table 3.10 – *Continued from previous page*

$q \setminus x$	10^{10}	10^{13}	10^{30}	10^{100}
116	$1.4699 \cdot 10^{-2}$	$6.2562 \cdot 10^{-4}$	$2.5785 \cdot 10^{-5}$	$2.3803 \cdot 10^{-5}$
118	$1.5403 \cdot 10^{-2}$	$6.5944 \cdot 10^{-4}$	$2.6249 \cdot 10^{-5}$	$2.4231 \cdot 10^{-5}$
120	$8.6716 \cdot 10^{-3}$	$3.4498 \cdot 10^{-4}$	$2.6076 \cdot 10^{-5}$	$2.4015 \cdot 10^{-5}$
122	$1.6266 \cdot 10^{-2}$	$6.9568 \cdot 10^{-4}$	$2.7136 \cdot 10^{-5}$	$2.5052 \cdot 10^{-5}$
124	$1.8215 \cdot 10^{-2}$	$7.5776 \cdot 10^{-4}$	$2.7560 \cdot 10^{-5}$	$2.5446 \cdot 10^{-5}$
125	$2.3348 \cdot 10^{-2}$	$1.1523 \cdot 10^{-3}$	$2.8293 \cdot 10^{-5}$	$2.6178 \cdot 10^{-5}$
126	$9.8994 \cdot 10^{-3}$	$3.8804 \cdot 10^{-4}$	$2.7451 \cdot 10^{-5}$	$2.5291 \cdot 10^{-5}$
128	$1.7504 \cdot 10^{-2}$	$7.5790 \cdot 10^{-4}$	$2.8481 \cdot 10^{-5}$	$2.6301 \cdot 10^{-5}$
130	$1.2749 \cdot 10^{-2}$	$5.1489 \cdot 10^{-4}$	$2.8597 \cdot 10^{-5}$	$2.6384 \cdot 10^{-5}$
132	$1.1318 \cdot 10^{-2}$	$4.3705 \cdot 10^{-4}$	$2.8819 \cdot 10^{-5}$	$2.6561 \cdot 10^{-5}$
134	$1.6851 \cdot 10^{-2}$	$7.4841 \cdot 10^{-4}$	$2.9794 \cdot 10^{-5}$	$2.7516 \cdot 10^{-5}$
138	$1.2143 \cdot 10^{-2}$	$4.7826 \cdot 10^{-4}$	$3.0187 \cdot 10^{-5}$	$2.7829 \cdot 10^{-5}$
140	$1.2985 \cdot 10^{-2}$	$5.2238 \cdot 10^{-4}$	$3.0710 \cdot 10^{-5}$	$2.8319 \cdot 10^{-5}$
142	$1.7620 \cdot 10^{-2}$	$7.9441 \cdot 10^{-4}$	$3.1646 \cdot 10^{-5}$	$2.9159 \cdot 10^{-5}$
143	$2.6454 \cdot 10^{-2}$	$1.3597 \cdot 10^{-3}$	$3.2407 \cdot 10^{-5}$	$3.0002 \cdot 10^{-5}$
144	$1.3061 \cdot 10^{-2}$	$5.2893 \cdot 10^{-4}$	$3.1669 \cdot 10^{-5}$	$2.9093 \cdot 10^{-5}$
150	$1.1360 \cdot 10^{-2}$	$4.4722 \cdot 10^{-4}$	$3.2609 \cdot 10^{-5}$	$3.0019 \cdot 10^{-5}$
154	$1.7340 \cdot 10^{-2}$	$7.2110 \cdot 10^{-4}$	$3.3986 \cdot 10^{-5}$	$3.1358 \cdot 10^{-5}$
156	$1.2824 \cdot 10^{-2}$	$5.2046 \cdot 10^{-4}$	$3.4098 \cdot 10^{-5}$	$3.1491 \cdot 10^{-5}$
162	$1.4153 \cdot 10^{-2}$	$5.9480 \cdot 10^{-4}$	$3.5496 \cdot 10^{-5}$	$3.2739 \cdot 10^{-5}$
163	$3.6946 \cdot 10^{-2}$	$1.9620 \cdot 10^{-3}$	$3.7172 \cdot 10^{-5}$	$3.4473 \cdot 10^{-5}$
168	$1.3082 \cdot 10^{-2}$	$5.2975 \cdot 10^{-4}$	$3.6569 \cdot 10^{-5}$	$3.3719 \cdot 10^{-5}$
169	$3.4642 \cdot 10^{-2}$	$1.8383 \cdot 10^{-3}$	$3.8407 \cdot 10^{-5}$	$3.5577 \cdot 10^{-5}$
170	$1.5694 \cdot 10^{-2}$	$6.9094 \cdot 10^{-4}$	$3.7452 \cdot 10^{-5}$	$3.4636 \cdot 10^{-5}$
174	$1.5120 \cdot 10^{-2}$	$6.3455 \cdot 10^{-4}$	$3.8068 \cdot 10^{-5}$	$3.5159 \cdot 10^{-5}$
180	$1.2885 \cdot 10^{-2}$	$5.2550 \cdot 10^{-4}$	$3.9072 \cdot 10^{-5}$	$3.6046 \cdot 10^{-5}$

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Table 3.10 – *Continued from previous page*

$q \setminus x$	10^{10}	10^{13}	10^{30}	10^{100}
182	$1.8050 \cdot 10^{-2}$	$8.0835 \cdot 10^{-4}$	$4.0167 \cdot 10^{-5}$	$3.7074 \cdot 10^{-5}$
186	$2.1175 \cdot 10^{-2}$	$8.4605 \cdot 10^{-4}$	$4.0745 \cdot 10^{-5}$	$3.7549 \cdot 10^{-5}$
190	$1.8127 \cdot 10^{-2}$	$8.1118 \cdot 10^{-4}$	$4.1783 \cdot 10^{-5}$	$3.8595 \cdot 10^{-5}$
198	$1.5241 \cdot 10^{-2}$	$6.5950 \cdot 10^{-4}$	$4.3217 \cdot 10^{-5}$	$3.9837 \cdot 10^{-5}$
210	$1.2498 \cdot 10^{-2}$	$5.1464 \cdot 10^{-4}$	$4.5282 \cdot 10^{-5}$	$4.1788 \cdot 10^{-5}$
216	$1.7817 \cdot 10^{-2}$	$8.0620 \cdot 10^{-4}$	$4.7283 \cdot 10^{-5}$	$4.3665 \cdot 10^{-5}$
222	$1.7402 \cdot 10^{-2}$	$7.9007 \cdot 10^{-4}$	$4.8518 \cdot 10^{-5}$	$4.4807 \cdot 10^{-5}$
234	$1.8214 \cdot 10^{-2}$	$8.1439 \cdot 10^{-4}$	$5.1024 \cdot 10^{-5}$	$4.7193 \cdot 10^{-5}$
242	$2.5496 \cdot 10^{-2}$	$1.2455 \cdot 10^{-3}$	$5.3536 \cdot 10^{-5}$	$4.9589 \cdot 10^{-5}$
243	$3.9994 \cdot 10^{-2}$	$1.9958 \cdot 10^{-3}$	$5.4591 \cdot 10^{-5}$	$5.0498 \cdot 10^{-5}$
250	$2.3404 \cdot 10^{-2}$	$1.1244 \cdot 10^{-3}$	$5.5061 \cdot 10^{-5}$	$5.0885 \cdot 10^{-5}$
256	$3.1024 \cdot 10^{-2}$	$1.5238 \cdot 10^{-3}$	$5.6823 \cdot 10^{-5}$	$5.2609 \cdot 10^{-5}$
286	$2.6531 \cdot 10^{-2}$	$1.3205 \cdot 10^{-3}$	$6.3044 \cdot 10^{-5}$	$5.8322 \cdot 10^{-5}$
326	$3.7053 \cdot 10^{-2}$	$1.8870 \cdot 10^{-3}$	$7.2282 \cdot 10^{-5}$	$6.6945 \cdot 10^{-5}$
338	$3.4756 \cdot 10^{-2}$	$1.7727 \cdot 10^{-3}$	$7.4727 \cdot 10^{-5}$	$6.9177 \cdot 10^{-5}$
360	$2.1806 \cdot 10^{-2}$	$1.0507 \cdot 10^{-3}$	$7.7966 \cdot 10^{-5}$	$7.2066 \cdot 10^{-5}$
420	$2.1729 \cdot 10^{-2}$	$1.0459 \cdot 10^{-3}$	$9.0361 \cdot 10^{-5}$	$8.3475 \cdot 10^{-5}$
432	$3.2245 \cdot 10^{-2}$	$1.6248 \cdot 10^{-3}$	$9.4295 \cdot 10^{-5}$	$8.7249 \cdot 10^{-5}$
486	$4.0230 \cdot 10^{-2}$	$1.9648 \cdot 10^{-3}$	$1.0604 \cdot 10^{-4}$	$9.8147 \cdot 10^{-5}$
500	$4.3727 \cdot 10^{-2}$	$2.2631 \cdot 10^{-3}$	$1.0981 \cdot 10^{-4}$	$1.0172 \cdot 10^{-4}$
700	$5.3024 \cdot 10^{-2}$	$2.7233 \cdot 10^{-3}$	$1.5268 \cdot 10^{-4}$	$1.4128 \cdot 10^{-4}$
900	$5.7545 \cdot 10^{-2}$	$2.8675 \cdot 10^{-3}$	$1.9428 \cdot 10^{-4}$	$1.7993 \cdot 10^{-4}$
1100	$9.2523 \cdot 10^{-2}$	$4.6728 \cdot 10^{-3}$	$2.4007 \cdot 10^{-4}$	$2.2275 \cdot 10^{-4}$
1500	$9.3555 \cdot 10^{-2}$	$4.7191 \cdot 10^{-3}$	$3.2399 \cdot 10^{-4}$	$2.9978 \cdot 10^{-4}$
1900	$1.7286 \cdot 10^{-1}$	$8.5819 \cdot 10^{-3}$	$4.1457 \cdot 10^{-4}$	$3.8533 \cdot 10^{-4}$
2300	$2.1076 \cdot 10^{-1}$	$1.0418 \cdot 10^{-2}$	$5.0173 \cdot 10^{-4}$	$4.6710 \cdot 10^{-4}$

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Table 3.10 – *Continued from previous page*

$q \setminus x$	10^{10}	10^{13}	10^{30}	10^{100}
2700	$2.1765 \cdot 10^{-1}$	$1.0004 \cdot 10^{-2}$	$5.8119 \cdot 10^{-4}$	$5.3987 \cdot 10^{-4}$
3100	$3.2427 \cdot 10^{-1}$	$1.5181 \cdot 10^{-2}$	$6.7598 \cdot 10^{-4}$	$6.2916 \cdot 10^{-4}$
3500	$3.3868 \cdot 10^{-1}$	$1.5603 \cdot 10^{-2}$	$7.6008 \cdot 10^{-4}$	$7.07267 \cdot 10^{-4}$
3900	$2.3019 \cdot 10^{-1}$	$1.1359 \cdot 10^{-2}$	$8.3553 \cdot 10^{-4}$	$7.7732 \cdot 10^{-4}$
4300	$4.6750 \cdot 10^{-1}$	$2.1323 \cdot 10^{-2}$	$9.3774 \cdot 10^{-4}$	$8.9951 \cdot 10^{-4}$
4700	$5.9566 \cdot 10^{-1}$	$2.5831 \cdot 10^{-2}$	$1.0246 \cdot 10^{-3}$	$9.5440 \cdot 10^{-4}$
5100	$3.9348 \cdot 10^{-1}$	$1.7642 \cdot 10^{-2}$	$1.0932 \cdot 10^{-3}$	$1.0168 \cdot 10^{-3}$
5900	$6.6947 \cdot 10^{-1}$	$2.9619 \cdot 10^{-2}$	$1.2859 \cdot 10^{-3}$	$1.1985 \cdot 10^{-3}$
6700	$9.8708 \cdot 10^{-1}$	$4.0405 \cdot 10^{-2}$	$1.4601 \cdot 10^{-3}$	$1.3612 \cdot 10^{-3}$
7500	$5.8359 \cdot 10^{-1}$	$2.5996 \cdot 10^{-2}$	$1.6099 \cdot 10^{-3}$	$1.4998 \cdot 10^{-3}$
9100	$8.3994 \cdot 10^{-1}$	$4.5325 \cdot 10^{-2}$	$1.9659 \cdot 10^{-3}$	$1.8328 \cdot 10^{-3}$
10000		$5.5393 \cdot 10^{-2}$	$2.1793 \cdot 10^{-3}$	$2.0343 \cdot 10^{-3}$
20000		$2.3547 \cdot 10^{-1}$	$4.3672 \cdot 10^{-3}$	$4.0849 \cdot 10^{-3}$
30000		$2.3359 \cdot 10^{-1}$	$6.4644 \cdot 10^{-3}$	$6.0354 \cdot 10^{-3}$
40000		$4.9031 \cdot 10^{-1}$	$8.7785 \cdot 10^{-3}$	$8.2133 \cdot 10^{-3}$

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Appendix A

Appendix A

The following computations were provided by David Platt.

Table A.1: Exact computations for the quantities $\mathbf{N}(1,q)$, $\mathbf{N}(2,q)$, $\mathbf{N}(200,q)$ and $\mathbf{S}_0(200,q)$.

q	$\frac{1}{2}\mathbf{N}(1,q)$	$\frac{1}{2}\mathbf{N}(2,q)$	$\frac{1}{2}\mathbf{N}(200,q)$	$\frac{1}{2}\mathbf{S}_0(200,q)$
3	0	0	114	$1.5901 \cdot 10^0$
4	0	0	122	$1.8093 \cdot 10^0$
5	0	0	388	$5.9140 \cdot 10^0$
6	0	0	114	$1.5901 \cdot 10^0$
7	0	0	701	$1.1298 \cdot 10^1$
8	0	0	411	$6.5085 \cdot 10^0$
9	0	0	706	$1.1411 \cdot 10^1$
10	0	0	388	$5.9140 \cdot 10^0$
11	0	1	1394	$2.4182 \cdot 10^1$
12	0	0	394	$6.0448 \cdot 10^0$
13	1	1	1759	$3.1449 \cdot 10^1$
14	0	0	701	$1.1298 \cdot 10^1$
15	0	0	995	$1.6134 \cdot 10^1$
16	0	1	1077	$1.8543 \cdot 10^1$
17	1	4	2526	$4.7924 \cdot 10^1$
18	0	0	706	$1.1411 \cdot 10^1$
19	1	5	2924	$1.0612 \cdot 10^2$
20	0	0	1032	$1.7203 \cdot 10^1$
21	0	1	1690	$2.9017 \cdot 10^1$
22	0	1	1394	$2.4182 \cdot 10^1$
23	2	6	3741	$7.2433 \cdot 10^1$
24	0	1	1041	$1.7482 \cdot 10^1$
25	1	5	3283	$6.2718 \cdot 10^1$
26	1	1	1759	$3.1449 \cdot 10^1$
27	1	5	2903	$5.5152 \cdot 10^1$
28	0	2	1745	$3.0808 \cdot 10^1$
29	3	10	5011	$1.0108 \cdot 10^2$
30	0	0	995	$1.6134 \cdot 10^1$
31	3	13	5441	$2.5147 \cdot 10^2$
32	1	4	2587	$5.0044 \cdot 10^1$
33	1	5	3213	$6.0030 \cdot 10^1$
34	1	4	2526	$4.7924 \cdot 10^1$
35	2	7	3963	$7.4685 \cdot 10^1$
36	1	1	1755	$3.1278 \cdot 10^1$
37	5	19	6760	$1.4587 \cdot 10^2$
38	1	5	2924	$1.0612 \cdot 10^2$
39	2	6	4018	$7.6258 \cdot 10^1$

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Table A.1 – *Continued from previous page*

q	$\frac{1}{2}\mathbf{N}(1,q)$	$\frac{1}{2}\mathbf{N}(2,q)$	$\frac{1}{2}\mathbf{N}(200,q)$	$\frac{1}{2}\mathbf{S}_0(200,q)$
40	1	3	2495	$4.5799 \cdot 10^1$
41	7	18	7664	$1.6956 \cdot 10^2$
42	0	1	1690	$2.9017 \cdot 10^1$
43	7	21	8120	$1.7605 \cdot 10^2$
44	1	6	3305	$6.8984 \cdot 10^1$
45	1	7	3983	$8.0406 \cdot 10^1$
46	2	6	3741	$7.2433 \cdot 10^1$
47	7	25	9041	$2.0201 \cdot 10^2$
48	1	4	2513	$4.6574 \cdot 10^1$
49	6	22	7980	$1.7711 \cdot 10^2$
50	1	5	3283	$6.2718 \cdot 10^1$
51	3	12	5691	$1.1325 \cdot 10^2$
52	2	8	4125	$8.0310 \cdot 10^1$
53	13	32	10442	$3.9470 \cdot 10^2$
54	1	5	2903	$5.5152 \cdot 10^1$
55	4	18	7343	$1.5425 \cdot 10^2$
56	2	9	4098	$7.9507 \cdot 10^1$
57	4	16	6557	$1.8296 \cdot 10^2$
58	3	10	5011	$1.0108 \cdot 10^2$
59	10	35	11865	$2.8020 \cdot 10^2$
60	1	3	2423	$4.3310 \cdot 10^1$
61	13	37	12341	$3.1100 \cdot 10^2$
62	3	13	5441	$2.5147 \cdot 10^2$
63	4	13	6487	$1.3149 \cdot 10^2$
64	4	15	5958	$1.2331 \cdot 10^2$
65	7	23	9116	$1.9098 \cdot 10^2$
66	1	5	3213	$6.0030 \cdot 10^1$
67	15	46	13790	$3.1850 \cdot 10^2$
68	3	14	5838	$1.3460 \cdot 10^2$
69	7	23	8328	$1.7344 \cdot 10^2$
70	2	7	3963	$7.4685 \cdot 10^1$
71	16	45	14771	$3.4354 \cdot 10^2$
72	2	7	4117	$8.1403 \cdot 10^1$
74	5	19	6760	$1.4587 \cdot 10^2$
75	4	16	7342	$1.7445 \cdot 10^2$
76	5	17	6721	$1.9977 \cdot 10^2$
77	9	35	11842	$3.7687 \cdot 10^2$
78	2	6	4018	$7.6258 \cdot 10^1$
80	3	13	5777	$1.2098 \cdot 10^2$
81	10	30	10757	$2.3151 \cdot 10^2$
82	7	18	7664	$1.6956 \cdot 10^2$
84	1	6	3989	$9.4884 \cdot 10^1$
85	9	34	12804	$2.8285 \cdot 10^2$
86	7	21	8120	$1.7605 \cdot 10^2$
87	9	32	11074	$2.7672 \cdot 10^2$
88	7	18	7569	$1.6131 \cdot 10^2$
90	1	7	3983	$8.0406 \cdot 10^1$
91	14	42	14671	$3.6796 \cdot 10^2$
92	7	23	8530	$2.1571 \cdot 10^2$
93	11	34	12011	$5.6120 \cdot 10^2$
94	7	25	9041	$2.0201 \cdot 10^2$
95	14	43	14696	$3.7130 \cdot 10^2$
96	5	14	5812	$1.1735 \cdot 10^2$
98	6	22	7980	$1.7711 \cdot 10^2$
99	11	36	11889	$2.6724 \cdot 10^2$

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Table A.1 – *Continued from previous page*

q	$\frac{1}{2}\mathbf{N}(1,q)$	$\frac{1}{2}\mathbf{N}(2,q)$	$\frac{1}{2}\mathbf{N}(200,q)$	$\frac{1}{2}\mathbf{S}_0(200,q)$
100	6	18	7525	$1.5527 \cdot 10^2$
102	3	12	5691	$1.1325 \cdot 10^2$
104	8	26	9389	$2.1340 \cdot 10^2$
105	6	21	8838	$1.8244 \cdot 10^2$
106	13	32	10442	$3.9470 \cdot 10^2$
108	6	16	6679	$1.4698 \cdot 10^2$
110	4	18	7343	$1.5425 \cdot 10^2$
111	14	46	14862	$3.3312 \cdot 10^2$
112	10	27	9335	$1.9577 \cdot 10^2$
114	4	16	6557	$1.8296 \cdot 10^2$
116	12	35	11331	$2.4794 \cdot 10^2$
117	15	42	14726	$3.7237 \cdot 10^2$
118	10	35	11865	$2.8020 \cdot 10^2$
120	3	13	5630	$1.1192 \cdot 10^2$
121	28	86	24494	$6.1711 \cdot 10^2$
122	13	37	12341	$3.1100 \cdot 10^2$
124	12	38	12285	$4.1035 \cdot 10^2$
125	29	74	21845	$5.3286 \cdot 10^2$
126	4	13	6487	$1.3149 \cdot 10^2$
128	15	42	13408	$3.5576 \cdot 10^2$
130	7	23	9116	$1.9098 \cdot 10^2$
132	4	18	7385	$1.6205 \cdot 10^2$
134	15	46	13790	$3.1850 \cdot 10^2$
138	7	23	8328	$1.7344 \cdot 10^2$
140	6	24	9062	$2.0526 \cdot 10^2$
142	16	45	14771	$3.4354 \cdot 10^2$
143	32	93	26542	$6.4220 \cdot 10^2$
144	8	25	9372	$2.0128 \cdot 10^2$
150	4	16	7342	$1.7445 \cdot 10^2$
154	9	35	11842	$3.7687 \cdot 10^2$
156	7	24	9173	$1.9370 \cdot 10^2$
162	10	30	10757	$2.3151 \cdot 10^2$
163	59	152	38712	$1.0282 \cdot 10^3$
168	7	24	9114	$2.0698 \cdot 10^2$
169	51	141	36551	$9.4234 \cdot 10^2$
170	9	34	12804	$2.8285 \cdot 10^2$
174	9	32	11074	$2.7672 \cdot 10^2$
180	8	24	9102	$1.9515 \cdot 10^2$
182	14	42	14671	$3.6796 \cdot 10^2$
190	14	43	14696	$3.7130 \cdot 10^2$
198	11	36	11889	$2.6724 \cdot 10^2$
210	6	21	8838	$1.8244 \cdot 10^2$
216	16	47	15028	$3.4844 \cdot 10^2$
222	14	46	14862	$3.3312 \cdot 10^2$
234	15	42	14726	$3.7237 \cdot 10^2$
242	28	86	24494	$6.1711 \cdot 10^2$
243	55	146	38099	$1.1880 \cdot 10^3$
250	29	74	21845	$5.3286 \cdot 10^2$
256	47	115	29715	$8.1454 \cdot 10^2$
286	32	93	26542	$6.4220 \cdot 10^2$
326	59	152	38712	$1.0282 \cdot 10^3$
338	51	141	36551	$9.4234 \cdot 10^2$
360	23	65	20400	$4.7351 \cdot 10^2$
420	21	61	19877	$4.7456 \cdot 10^2$
432	45	124	33316	$8.4586 \cdot 10^2$

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Table A.1 – *Continued from previous page*

q	$\frac{1}{2}\mathbf{N}(1,q)$	$\frac{1}{2}\mathbf{N}(2,q)$	$\frac{1}{2}\mathbf{N}(200,q)$	$\frac{1}{2}\mathbf{S}_0(200,q)$
486	55	146	38099	$1.1880 \cdot 10^3$
500	75	193	48176	$1.2630 \cdot 10^3$
600	49	133	36663	$1.0475 \cdot 10^3$
700	93	232	57925	$1.6200 \cdot 10^3$
800	144	353	81922	$2.2430 \cdot 10^3$
900	93	234	58122	$1.8210 \cdot 10^3$
1000	189	474	105260	$3.1253 \cdot 10^3$
2000	473	1116	228242	$8.8910 \cdot 10^3$
2500	629	1460	292410	$1.1726 \cdot 10^4$
3000	452	1079	224586	$9.9236 \cdot 10^3$
4000	1117	2579	491851	$1.7490 \cdot 10^4$
5000	1479	3371	629022	$2.7069 \cdot 10^4$
6000	1086	2500	484538	$2.0890 \cdot 10^4$
7000	1780	4044	755806	$3.1175 \cdot 10^4$
8000	2609	5863	1054376	$4.3420 \cdot 10^4$
9000	1785	4078	757807	$3.1050 \cdot 10^4$
10000	3390	7620	1346385	$5.6913 \cdot 10^4$