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Quantum mechanics of charged black holes

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Abstract

We quantize the spherically symmetric sector of generic charged black holes. Thermal properties are incorporated by imposing periodicity in Euclidean time, with period equal to the inverse Hawking temperature of the black hole. This leads to an exact quantization of the area (A) and charge (Q) operators. For the Reissner–Nordström black hole, $A = 4\pi G\hbar(2n + p + 1)$ and $Q = me$, for integers n, p, m . Consistency requires the fine structure constant to be quantized: $e^2/\hbar = p/m^2$. Remarkably, vacuum fluctuations exclude extremal black holes from the spectrum, while near extremal black holes are highly quantum objects. We also prove that horizon area is an adiabatic invariant.

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Bekenstein and Hawking [1] showed almost thirty years ago that black holes possess intriguing thermodynamic properties which make them rich theoretical laboratories for testing theories of quantum gravity. Although candidate theories for quantum gravity exist (such as string theory and quantum geometry), the microscopic origin of thermodynamic behaviour is still largely the subject of conjecture. It is therefore important to learn as much as possible about the quantum behaviour of black holes without assuming a specific underlying microscopic theory. One very natural question that arises in this context concerns the quantum mechanical spectrum of the observables associated with charged black holes. Based on the conjecture that horizon area is an adiabatic invariant, as well as from other considerations, it was postulated that for neutral black holes, the area spectrum is discrete and uniformly spaced; i.e., $A \propto n$, where n is an integer [2].

In this Letter we follow an argument originally presented in [3] to quantize the spherically symmetric sector of generic charged black holes. Our starting point is the assumption that it is possible to incorporate the thermodynamic behaviour of black holes into a quantum description by imposing periodicity in Euclidean time, with period equal to the inverse Hawking temperature of the black hole. This single assumption allows us to (i) derive an exact quantized area spectrum; (ii) derive the spectrum of electric charge and (iii) show that black hole quantization places stringent restrictions on the fine structure constant; (iv) prove that horizon area is an adiabatic invariant. We emphasize that our analysis is quite general and is not tied to a specific model or theory of gravity, in contrast with other derivations of qualitatively similar spectra [4]. It applies, for example, to charged as well as neutral black holes in Einstein–Maxwell theory in any dimension, and in fact, even to the 3-dimensional rotating BTZ black hole (where the angular momentum plays the role of the electric charge).

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Note that our goal here is not to explain the microscopic source of the thermodynamic behaviour. We simply encode it into the boundary conditions and observe the consequences. The formalism of Euclidean quantum field theory, as is well known, can originate from two distinctively different physical situations — from the description of thermodynamical ensemble (statistical, i.e., not pure, state) or from the description of classically forbidden transitions between pure states — quantum mechanical underbarrier tunneling. Quite amazingly, in quantum gravity these two functions of the Euclidean formalism are not clearly separated. Indeed, the Euclidean section of the Schwarzschild solution can, on one hand, be regarded as a saddle point of the path integral for the statistical partition function and, on the other hand, can be viewed as a classical configuration interpolating in the imaginary time between the two causally disconnected spacetime domains: the right and left wedges of the Kruskal diagram. Our requirement of periodicity in imaginary time can be viewed as a kind of consistency of quantum states in these two domains, or the finiteness of the semiclassical underbarrier transition amplitude between them (remember that the Hawking periodicity requirement is based on the absence of conical singularity which is, in its turn, motivated by the regularity of the semiclassical distribution). So amplitudes not satisfying this periodicity requirement can be regarded as suppressed.

We restrict consideration to black hole spacetimes that are static and can be parametrized by only two coordinate invariant parameters, which we choose to be the mass M and charge Q . This basically assumes a Birkhoff-like theorem, and forbids the presence of monopole gravitational or electromagnetic radiation. With this assumption, there exists a coordinate system in which the metric takes form:

$$ds^2 = -f(x; M, Q) dt^2 + \frac{dx^2}{f(x; M, Q)} + r^2(x) d\Omega^{(d-2)}, \quad (1)$$

where x is radial coordinate. The function $f(x; M, Q)$ is uniquely determined by the requirement that $-g_{tt}$ and g_{xx} are inverse proportional to one another, and the location of the horizon $x_h = x_h(M, Q)$ is given implicitly by $f(x_h; M, Q) = 0$.

An elegant way to extract thermodynamic information about black hole spacetimes is to euclideanize the solution and concentrate on the ‘near-horizon’ region. By a suitable choice of Euclidean time $t_E = -it$ and spatial coordinate

$$R(x) = \sqrt{\frac{f(x; M, Q)}{f'(x_h; M, Q)}},$$

one can put the metric near the horizon ($R \rightarrow 0$) into the form, $ds_E^2 = dR^2 + R^2 d\alpha^2$, where $\alpha := t_E f'(x_h; M, Q)/2$. To avoid a conical singularity at the horizon, α must have the period 2π , implying that t_E must be periodic with range $0 \leq t_E \leq 4\pi/f'(x_h; M, Q)$. It follows from finite temperature quantum field theory that the (Hawking) temperature associated with this black hole is the inverse of the Euclidean period, i.e., $T_H(M, Q) = \hbar f'(x_h; M, Q)/4\pi$. This will play a key role in what follows. The Bekenstein–Hawking entropy, $S_{\text{BH}}(M, Q)$, of the black hole is defined generically by requiring it to obey the first law of thermodynamics:

$$\delta M = T_H(M, Q) \delta S_{\text{BH}}(M, Q) + \Phi(M, Q) \delta Q, \quad (2)$$

where $\Phi(M, Q)$ is the electrostatic potential at the horizon. Given T_H and the electrostatic potential, this determines $S_{\text{BH}}(M, Q)$ up to an additive constant, which is fixed by requiring the Bekenstein–Hawking entropy to vanish when the mass and charge both vanish. For spherically symmetric black holes in any dimension, this yields the usual relationship between the entropy and the area of the outer horizon: $S_{\text{BH}} = A/4G\hbar$. For example, in the case of the Reissner–Nordström black hole,

$$T_H = 2\hbar\sqrt{M^2 - Q^2}/A, \quad A = 4\pi r_+^2$$

and $\Phi = Q/r_+$, where

$$r_+ = (GM + \sqrt{G^2 M^2 - GQ^2}).$$

Since M and Q are assumed to be the only diffeomorphism invariant parameters, the reduced action governing the dynamics of the spherically symmetric sector of isolated, generic charged black holes in any theory must be of the form [5,6]:

$$I^{\text{red}} = \int dt (P_M \dot{M} + P_Q \dot{Q} - H(M, Q)), \quad (3)$$

where P_M and P_Q are the conjugates to M and Q , respectively. The exact expression for the Hamiltonian

is irrelevant: the fact that it is independent of P_M and P_Q alone ensures that M and Q are constants of motion. Of course, one can also arrive at this reduced action via a rigorous Hamiltonian analysis of the spherically symmetric charged black hole spacetimes. For details, see [5,6].

For boundary conditions which preserve the so called Schwarzschild form (1) of the metric at either end of a spatial slice, P_M can be shown to be proportional to the difference between the Schwarzschild times at either end of the slice [7–9]. Moreover, the momentum P_Q is related to P_M by means of the following relation:

$$\delta P_Q = -\Phi \delta P_M + \delta \lambda, \tag{4}$$

where δP_Q and δP_M refer to variations under a change in boundary conditions and $\delta \lambda$ is the variation in $U(1)$ gauge transformation λ at the horizon.

In order to quantize we need to know the boundary conditions on the phase space variables. We require $M > 0$, $T_H(M, Q) \geq 0$ and Q to be real. Using the expressions derived in [5], it can be shown that positivity of the Hawking temperature leads generically to a condition of the form:

$$S_{\text{BH}}(M, Q) \geq S_0(Q), \tag{5}$$

where the equality is achieved in the limit of extremal black holes. The lower bound $S_0(Q)$ on the Bekenstein–Hawking entropy is a uniquely determined function of Q for each theory in the class under consideration. For example, for the Reissner–Nordström black hole, $S_0(Q) = \pi Q^2/\hbar$.

Until this point our analysis has been more or less standard. We now go to the Euclidean sector where the time difference P_M becomes imaginary as well as periodic, with period given by $T_H^{-1}(M, Q)$. Although it is possible to derive a black hole spectrum by imposing periodicity of the Lorentzian time coordinate [4], the motivation for the periodicity is more problematic than in the Euclidean sector. In the present case, the procedure is well defined and consistent. Essentially, we start with the reduced Hamiltonian and action as given in the Lorentzian sector, which is of precisely the same form as Eq. (3), and analytically continuing to Euclidean time before quantizing. As a direct consequence the momenta conjugate to M and Q are pure imaginary. However, since the Hamiltonian is independent of these conjugate momenta, it does not

change its form. Thus, euclideanization merely generates an overall factor of i in front of the reduced action and keeps the dynamics unaltered. Ultimately, the physical relevance of our derived spectra will rest on the connection between the charge and mass eigenstate wave functions that we construct in the Euclidean sector using Hamiltonian techniques, and their counterparts in the Euclidean path integral formulation of quantum gravity.

Periodic boundary conditions on phase space variables are familiar in classical mechanics. Akin to the action-angle formulation of the harmonic oscillator, we can ‘unwrap’ our gravitational phase space, by transforming to a set of unrestricted variables. Consider the following transformation $(M, Q, P_M, P_Q) \rightarrow (X, Q, \Pi_X, \Pi_Q)$, which directly incorporates the correct periodicity of P_M :

$$\begin{aligned} X &= \sqrt{\hbar B(M, Q)/\pi} \cos(2\pi P_M T_H(M, Q)/\hbar), \\ \Pi_X &= \sqrt{\hbar B(M, Q)/\pi} \sin(2\pi P_M T_H(M, Q)/\hbar), \\ Q &= Q, \\ \Pi_Q &= \Pi_Q(M, P_M, Q, P_Q), \end{aligned} \tag{6}$$

where the functions $B(M, Q)$ and $\Pi_Q(M, P_M, Q, P_Q)$ will be determined shortly. Direct calculation shows that this transformation is canonical if and only if:

$$\frac{\partial B}{\partial M} = \frac{1}{T_H(M, Q)}, \quad P_Q = \Pi_Q + P_M T_H \frac{\partial B}{\partial Q}. \tag{7}$$

From the first law of black hole mechanics we know that $\partial S_{\text{BH}}/\partial M = T_H^{-1}(M, Q)$. Thus we conclude:

$$B(M, Q) = S_{\text{BH}}(M, Q) + F(Q), \tag{8}$$

where $F(Q)$ is an arbitrary function of the charge. Combining (8) and (6) we get:

$$S_{\text{BH}}(M, Q) + F(Q) = \frac{2\pi}{\hbar} \left(\frac{1}{2} X^2 + \frac{1}{2} \Pi_X^2 \right), \tag{9}$$

which shows that the subspace (X, Π_X) has a ‘hole’ of radius $[S_0(Q) + F(Q)]^{1/2}$, the interior of which is inaccessible. To remove potential quantization ambiguities, we choose $F(Q) = -S_0(Q)$, thus removing the perforation and rendering the phase space complete. As a bonus, this automatically ensures that the inequality (5) is satisfied in a natural way. The extremal limit now gets mapped to the origin of the new

phase space. With this choice, Π_Q is uniquely determined to be:

$$\Pi_Q = \frac{\hbar}{e}\chi + \frac{\hbar}{2\pi}S'_0(Q)\alpha, \tag{10}$$

where $' = d/dQ$, $\chi = (e/\hbar)(P_Q + \Phi P_M)$ and $\alpha = 2\pi P_M T_H(M, Q)/\hbar$.

From (9) it follows that the operator $S_{BH} - S_0(Q)$ is precisely the Hamiltonian of a simple harmonic oscillator with the mass and frequency both equal to unity. Since $-\infty \leq X, \Pi_X \leq \infty$, standard quantization yields the spectrum:

$$S_{BH} = 2\pi\left(n + \frac{1}{2}\right) + S_0(Q), \quad n = 0, 1, 2, \dots, \tag{11}$$

where we have assumed the usual harmonic oscillator factor ordering for the operators X and P_X in constructing the quantum version of (9). A remarkable feature of (11) above is that vacuum fluctuations exclude extremal black holes ($S_{BH} = S_0$) from the quantum spectrum. Another important result, that is independent of the choice of factor ordering, is that near-extremal states are highly quantum-mechanical objects ($n \sim 0$), even for large values of M and Q .

To quantize the electromagnetic sector, we note from (4) that for compact gauge group $U(1)$, $\chi := e\lambda/\hbar = e(P_Q + \Phi P_M)/\hbar$ has period 2π , where e is the electromagnetic coupling. Thus from (10), Π_Q is a function of two angular coordinates χ and α which, according to arguments given above are both periodic with period 2π . We must therefore identify the phase space points

$$(Q, \Pi_Q) \sim (Q, \Pi_Q + 2\pi n_1 \hbar/e + n_2 \hbar S'_0(Q)) \tag{12}$$

for arbitrary integers n_1 and n_2 . In the coordinate representation, $\widehat{Q} = -i\hbar\partial/\partial\Pi_Q$, the wave functions for charge eigenstates take the form $\psi_Q(\Pi_Q) = (\text{const}) \times \exp(iQ\Pi_Q/\hbar)$. The spectrum of Q is restricted by the requirement that the wave function be single valued under the identification (12): for each admissible Q , for all integers n_1 and n_2 there must exist a third integer n_3 such that:

$$\frac{n_1 Q}{e} + \frac{n_2 Q S'_0(Q)}{2\pi} = n_3. \tag{13}$$

This in turn requires that

$$\frac{Q}{e} = m, \quad \frac{Q}{2\pi}S'_0(Q) = p, \tag{14}$$

for integer m, p . To see this, suppose that there exists some values of n_1, n_2 and Q for which (14) holds for some n_3 . If we increase the value of n_1 by 1, the value of n_3 increases by Q/e , so it is necessary that $Q/e = m$, for some integer m in order that the shifted n_3 be an integer. Similarly, if we increase n_2 by one, the second relation in (15) emerges as necessary.

The first of the conditions (14) gives the expected result that Q must be an integer multiple of the fundamental charge e . However, the second condition can only be satisfied if e satisfies a subsidiary, and totally unexpected condition. The specific form of this condition depends on the theory under consideration. For Reissner–Nordström black holes in four dimensions, $S_0(Q) = \pi Q^2/\hbar$ and (11) and the second of equations (14) translate to:

$$S_{BH} = 2\pi n + \pi(p + 1), \quad Q^2 = p\hbar. \tag{15}$$

The integer p determines the charge of the quantum black holes and hence its minimum entropy $S_0 = \pi(p + 1)$, whereas n determines the excited level of the black hole over the “vacuum”, $n = 0$. Finally, the first of equations (14) requires

$$\frac{e^2}{\hbar} = \frac{p}{m^2}. \tag{16}$$

Thus, the fine structure constant e^2/\hbar must be a rational number. For the d -dimensional generalization of these results see [10].

For one dimensional periodic systems, it is well known that the integral $\mathcal{J}_X = \oint \Pi_X dX$ is an adiabatic invariant. Thus, in the present case, if we treat Q as a slowly varying parameter, it follows that $\mathcal{J}_X = \pi(A - 4G\hbar S_0(Q))/4G$ is an adiabatic invariant. Consequently, away from extremality ($A \gg 4G\hbar S_0(Q)$), this is consistent with Bekenstein’s conjecture [2] that the horizon area of a charged black hole is an adiabatic invariant.

The expression (11) has fascinating consequences. First of all, it implies that extremal black holes, for which $S_{BH} = S_0(Q)$, are not in the physical spectrum. Secondly, if we interpret the entropy in terms of statistical mechanics (11) tells us that the degeneracy of the n th level is: $g(n) = \exp[2\pi(n + 1/2) + S_0(Q)]$. Thus, the ground state is degenerate ($g(0) \neq 1$). It is tempting to conjecture that this Planck size remnant provides clues about the information loss problem associated with the endpoint of Hawking radi-

ation. Finally, (11) allows Hawking radiation to be emitted when a black hole jumps from one quantum entropy level to another. For a Reissner–Nordström black hole, the fundamental frequency of emission of a neutral quantum, ω_0 satisfies: $S(M + \hbar\omega_0, Q) - S(M, Q) = S(n + 1) - S(n) = \pi$, from which it follows that $\omega_0 = (r_+ - r_-)\pi/A$. In the Schwarzschild limit $Q \rightarrow 0$, $\omega_0 \sim 1/M$, agreeing with that found in [2]. Since the mean frequency of the Planck distribution of Hawking radiation lies at $T_H \sim 1/M$, the radiation spectrum consists of widely separated spectral lines, and deviates considerably from the continuum originally predicted by Hawking, no matter what the temperature. This turns out to be a generic feature of our spectra, valid for all black holes, charged or uncharged [10].

To summarize, we have looked at quantum gravity in the spherically symmetric, charged black hole sector. To incorporate the thermodynamic information, we assumed periodicity of the momentum conjugate to the black hole mass. It is particularly important to stress that we used only very general features of black hole dynamics and thermodynamics. Consequently, despite possible factor ordering ambiguities in our analysis, the following predictions are expected to be valid at least at the semi-classical level: (1) black hole area is an adiabatic invariant, hence its quantum spectrum is equally spaced, (2) near extremal black holes are highly quantum objects, and (3) the radiation spectrum of black holes is discrete, irrespective of the temperature. Finally, (11) and (14) imply that black holes emit and absorb quanta whose charges are multiples of e , which itself is not arbitrary, but quantized in terms of integers m and p . Thus, in analogy with the Dirac charge quantization condition in the presence of a magnetic monopole, the presence of charged black holes puts constraints on the fine structure constant. This is also reminiscent of the ‘big-fix mechanism’ advocated by Coleman, wherein the fundamental constants of nature are supposed to be fixed by the presence of wormholes and baby universes [11]. Although, a priori, it is not clear how the experimentally measured value of the fine structure constant $4\pi\hbar/e^2 = 137.03608\dots$ can be reproduced accurately as the ratio of integers that are not too large, as required by (16), we believe that our results reveal some intriguing features of the quantum mechanics of black holes and merit further study.

Note added

While this Letter was being completed, we became aware of two papers where the spectra of charged black holes was investigated [12,13]. Although their results bear qualitative resemblance to ours, their analysis and quantitative results differ considerably from ours.

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