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Quantum mechanical spectra of charged black holes

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Abstract

It has been argued by several authors, using different formalisms, that the quantum mechanical spectrum of black hole horizon area is discrete and uniformly spaced. Recently it was shown that two such approaches, namely the one involving quantization on a reduced phase space, and the algebraic approach of Bekenstein and Gour are equivalent for spherically symmetric, neutral black holes (hep-th/0202076). That is, the observables of one can be mapped to those of the other. Here we extend that analysis to include charged black holes. Once again, we find that the ground state of the black hole is a Planck size remnant.

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Black holes, in addition to being fascinating objects in our universe, serve as theoretical laboratories where many predictions of quantum gravity can be tested. It is well known that quantum mechanics plays a crucial role in many phenomena involving black holes, e.g., Hawking radiation and Bekenstein–Hawking entropy. Thus it is important to explore the quantum mechanical spectra of black hole observables such as horizon area, charge and angular momentum. It has been argued by various authors, using widely different approaches, that the spectra of above observables are discrete [1–14]. In particular, the horizon area of a black hole has been shown to have a uniformly spaced spectrum. Though the spectrum found in [15] is not strictly

uniformly spaced, it is effectively equally spaced for large areas.

As distinct as they may seem, since the different approaches attempt to address similar questions and predict similar spectra, it is expected that there is an underlying connection between them. In [16] we examined this issue for two of the above approaches, namely that advocated in [5,6] and that in [2], for black holes which are *spherically symmetric* and *neutral*. A direct mapping of the operators in the two approaches was found in that article, which was essential to get a physical interpretation of the abstract operators in the second approach. Moreover, we showed that the exact ‘quantum’ of horizon area (which turns out to be the square of Planck length) cannot be determined without this mapping.

In view of the above results, it is important to see how robust the results in the two approaches and the mapping between the two are. In this Letter, we try to address this question by relaxing the assumption that

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the black hole is uncharged, and consider black holes carrying an electric (or magnetic) charge instead. The two approaches whose underlying connections we will study are:

(1) The reduced phase space quantization of spherically symmetric black hole configurations of gravity [5,6]. To make it amenable to quantization, a canonical transformation is performed. We will call this *Approach I*.

(2) An algebra of black hole observables postulated by Bekenstein and Gour [2,3] giving uniformly spaced area spectrum. We will call this *Approach II*.

It was shown in [16] that for *neutral* black holes, the observables of one approach can be rigorously mapped to those of the other. Here, we show that the mapping can be extended to incorporate charged black holes. We also show that inclusion of charge leaves the important features of the spectrum unchanged, namely that the spectrum is still discrete and uniformly spaced. However, a new quantum number enters the picture, associated with the $U(1)$ charge, and the corresponding horizon area now depends linearly on two quantum numbers, instead of just one.

In approach II, one of the starting points is the assumption that horizon area is an adiabatic invariant, and from Bohr–Sommerfeld quantization rule which stipulates that adiabatic invariants must be quantized [17], it follows that the area spectrum is discrete and uniformly spaced. In approach I, on the other hand, a result which is similar to the above conjecture, was explicitly proven for spherically symmetric black holes which are away extremality. In particular, it was shown that the horizon area above extremality is an adiabatic invariant. We shall return to this issue later.

First we will briefly review the two methods. It follows from the analysis of [18,19] that the dynamics of static spherically symmetric charged black hole configurations in *any* classical theory of gravity in d -spacetime dimensions is governed by an effective action of the form

$$I = \int dt (P_M \dot{M} + P_Q \dot{Q} - H(M, Q)), \quad (1)$$

where M and Q are the mass and the charge of the black hole respectively and P_M, P_Q the corresponding conjugate momentum. This is essentially a con-

sequence of the no-hair theorem. The boundary conditions imposed on these variables are those of [20, 21]. It can be shown that P_M has the physical interpretation of asymptotic Schwarzschild time difference between the left and right wedges of a Kruskal diagram [22–24]. Note that H is independent of P_M and P_Q , such that from Hamilton's equations, M and Q are constants of motion.

Now to explicitly incorporate the thermodynamic properties of these black holes, motivated by Euclidean quantum gravity [25], we assume that the conjugate momentum P_M is periodic with period equal to inverse Hawking temperature. That is,

$$P_M \sim P_M + \frac{1}{T_H(M, Q)}. \quad (2)$$

Similar assumptions were made in the past using different arguments [8–10]. Note that the above identification implies that the (M, P_M) phase subspace has a wedge removed from it, which makes it difficult, if not impossible to quantize on the full phase-space. Thus, one can make a canonical transformation $(M, Q, P_M, P_Q) \rightarrow (X, Q, \Pi_X, \Pi_Q)$, which on the one hand ‘opens up’ the phase space, and on the other hand, naturally incorporates the periodicity (2) [5,6]:

$$X = \sqrt{\frac{A - A_0}{4\pi G_d}} \cos(2\pi P_M T_H), \quad (3)$$

$$\Pi_X = \sqrt{\frac{A - A_0}{4\pi G_d}} \sin(2\pi P_M T_H), \quad (4)$$

$$Q = Q, \quad (5)$$

$$\Pi_Q = P_Q + \Phi P_M + \frac{(d[A_0(Q)]/dQ)P_M T_H}{4G_d}, \quad (6)$$

where A is the black hole horizon area, G_d the d -dimensional Newton's constant. Note that both A and T_H are functions of M and Q . $A_0(Q)$ is the value of area at extremality when the mass of the black hole approaches its charge. For a d -dimensional Reissner–Nordström black hole, the value of $A_0(Q)$ is

$$A_0(Q) = k_d Q^{(d-2)/(d-3)}, \quad (7)$$

where

$$k_d = (1/4)(A_{d-2}/G_d)^{(d-4)/2(d-3)} \times (8\pi/(d-2)(d-3))^{(d-2)/2(d-3)}$$

with $A_{d-2} = 2\pi^{(d-1)/2}/\Gamma((d-1)/2)$ (area of unit S^{d-2}). Also, Φ is the electrostatic potential at the horizon and it will be treated as a c -number in the following. The validity of the first law of black hole thermodynamics ensures that the above set of transformations is indeed canonical [5,6]. Squaring and adding (3) and (4), we get

$$A_1 \equiv A - A_0(Q) = 4\pi G_d (X^2 + \Pi_X^2). \quad (8)$$

The r.h.s. is nothing but the Hamiltonian of a simple harmonic oscillator defined on the (X, Π_X) phase space with mass μ and angular frequency ω given by $\mu = 1/\omega = 1/8\pi G_d$. Upon quantization, the ‘position’ and ‘momentum’ variables are replaced by the operators

$$X \rightarrow \hat{X}, \quad \Pi_X \rightarrow \hat{\Pi}_X = -i \frac{\partial}{\partial X}, \quad (9)$$

and the spectrum of the operator A_1 follows immediately:

$$\text{Spec}\{\hat{A}_1\} \equiv a_n = n\bar{a} + a_{\text{Pl}}, \quad n = 0, 1, 2, \dots, \quad (10)$$

where $\bar{a} = 8\pi G_d = 8\pi \ell_{\text{Pl}}^{d-2}$ is the basic quantum of area, $a_{\text{Pl}} = \bar{a}/2$ is its ‘zero-point’ value (ℓ_{Pl} is the d -dimensional Planck length).

To complete the analysis of the spectrum, we use the following result from [18]:

$$\delta P_Q = -\Phi \delta P_M + \delta \lambda,$$

where Φ is the electrostatic potential on the boundary under consideration, and variation refers to small change in boundary conditions, λ being the gauge parameter at the boundary. This in turn implies that for compact $U(1)$ gauge group, $\chi \equiv e\lambda = e(P_Q + \Phi P_M)$ is periodic with period 2π (where e is the fundamental unit of electric charge). Also, we saw earlier from thermodynamic arguments that $\alpha \equiv 2\pi P_M T_H(M, Q)$ has period 2π . In terms of these ‘angular’ coordinates, the momentum Π_Q in (6) can be written as

$$\Pi_Q = \frac{\chi}{e} + \frac{(d[A_0(Q)]/dQ)\alpha}{8\pi G_d}.$$

Thus, the following identification must hold in the (Q, Π_Q) subspace:

$$(Q, \Pi_Q) \sim \left(Q, \Pi_Q + \frac{2\pi n_1}{e} + \frac{n_2(d[A_0(Q)]/dQ)}{4G_d} \right) \quad (11)$$

for any two integers n_1, n_2 . Now, wavefunctions of charge eigenstates are of the form:

$$\psi_Q(\Pi_Q) = \exp(i Q \Pi_Q),$$

which is single valued under the identification (11), provided there exists another integer n_3 such that

$$n_1 \frac{Q}{e} + \frac{n_2 Q(d[A_0(Q)]/dQ)}{8\pi G_d} = n_3.$$

Now, it can be easily shown that the above conditions is satisfied if and only if the following two quantization conditions hold:

$$\frac{Q}{e} = m, \quad m = 0, \pm 1, \pm 2, \dots, \quad (12)$$

$$\frac{Q}{8\pi G_d} (d[A_0(Q)]/dQ) = p, \quad p = 0, 1, 2, \dots \quad (13)$$

For d -dimensional Reissner–Nordström black holes, using the expression for $A_0(Q)$ from Eq. (7), and combining (10), (12) and (13) we get its final area spectrum:

$$\begin{aligned} \text{Spec}\{\hat{A}\} &= \text{Spec}\{\hat{A}_0 + \hat{A}_1\} \equiv a_{nm} \\ &= \left[n + \left(\frac{d-3}{d-2} \right) p \right] \bar{a} + a_{\text{Pl}}, \\ n, p &= 0, 1, 2, \dots, \end{aligned} \quad (14)$$

where m and p are related by Eqs. (12), (13). Hawking radiation takes place when the black hole jumps from a higher to a lower area level, the difference in quanta being radiated away. The above spectrum shows that the black hole does not evaporate completely, but a Planck size remnant is left over at the end of the evaporation process. It may be noted that the periodic classical orbits in the phase space under consideration admit of an adiabatic invariant. From (10), it can be seen that the adiabatic invariant in this case is

$$\text{Adiabatic invariant} = \oint \Pi_X dX = \frac{A_1}{4G}. \quad (15)$$

Thus for $A \gg A_0(Q)$ (i.e., far from extremality), the horizon area is indeed an adiabatic invariant (as conjectured in [2]). However, close to extremality, the above relation suggests that it is the area above extremality which is an adiabatic invariant. The advantage of relation (15) is that on the one hand it is consistent with the discrete spectra (14), and on the other hand, it ensures that the extremality bound $A \geq A_0(Q)$ is automatically obeyed.

The above result indicates that the relevant operator in the algebra of approach II for a generic non-extremal black hole is \hat{A}_1 , which along with the charge operator \hat{Q} , and the black hole creation operator $\hat{\mathcal{R}}_{nms_{nm}}$ forms a closed algebra (we follow the notation of [2]). The operator $\hat{\mathcal{R}}_{nms_{nm}}$ creates a single black hole state from the vacuum $|0\rangle$ with \hat{A}_1 eigenvalue a_n and \hat{Q} eigenvalue me in an internal quantum state s_{nm} :

$$\hat{\mathcal{R}}_{nms_{nm}}|0\rangle = |nms_{nm}\rangle, \quad (16)$$

$$\hat{A}_1|nms_{nm}\rangle = a_n|nms_{nm}\rangle, \quad (17)$$

$$\hat{Q}|nms_{nm}\rangle = me|nms_{nm}\rangle. \quad (18)$$

We choose $s_{nm} \in \{0, 1, \dots, k_{nm} - 1\}$ as in [26] such that the degeneracy of states with same total area eigenvalue a_{nm} , obeys $\ln k_{nm} \propto a_{nm}$. All these states have the same area and charge, which ensures that the Bekenstein–Hawking area law for black hole entropy is obeyed.

We shall denote area above extremality in Bekenstein's algebra as \hat{A}'_1 with eigenvalues a'_n such that the lowest eigenvalue is $a'_0 = 0$. We will shortly see the relation between the operators \hat{A}_1 and \hat{A}'_1 and their respective eigenvalues a_n and a'_n . From symmetry, linearity, closure and gauge invariance of area operator, the algebra satisfied by the charged black hole operators will be [2]

$$[\hat{Q}, \hat{A}'_1] = 0, \quad (19)$$

$$[\hat{A}'_1, \hat{\mathcal{R}}_{nms_{nm}}] = a'_n \hat{\mathcal{R}}_{nms_{nm}}, \quad (20)$$

$$[\hat{Q}, \hat{\mathcal{R}}_{nms_{nm}}] = me \hat{\mathcal{R}}_{nms_{nm}}, \quad (21)$$

$$[\hat{\mathcal{R}}_{nms_{nm}}, \hat{\mathcal{R}}_{n'm's'_{n'm'}}] = \epsilon_{nn'mm'}^{n''m''} \hat{\mathcal{R}}_{n''m''s''_{n''m''}} \\ (\epsilon_{nn'mm'}^{n''m''} \neq 0 \text{ iff } a'_n + a'_{n'} = a'_{n''} \text{ and } m + m' = m''), \quad (22)$$

$$[\hat{A}'_1, [\hat{\mathcal{R}}_{n'm's'_{n'm'}}^\dagger, \hat{\mathcal{R}}_{nms_{nm}}]] \\ = (a'_n - a'_{n'}) [\hat{\mathcal{R}}_{n'm's'_{n'm'}}^\dagger, \hat{\mathcal{R}}_{nms_{nm}}] \text{ iff } a'_n > a'_{n'}. \quad (23)$$

Eq. (22) implies that the black hole state created by a commutator of two black hole creation operators, $[\hat{\mathcal{R}}_{nms_{nm}}, \hat{\mathcal{R}}_{n'm's'_{n'm'}}]$, will be another single black hole state ($\hat{\mathcal{R}}_{n''m''s''_{n''m''}}|0\rangle$) provided $a'_n + a'_{n'} = a'_{n''}$ and $m + m' = m''$. Clearly, \hat{A}'_1 is a positive definite operator because the area above extremality cannot be negative.

Incorporating this, and adjoint relation of Eq. (20), we require the inequality condition $a'_n > a'_{n'}$ in Eq. (23). Clearly, the spectrum of the above algebra involves both addition and subtraction of \hat{A}'_1 eigenvalues which is possible if and only if the \hat{A}'_1 eigenvalues are equally spaced, i.e., $a'_n = n\bar{b}$ where \bar{b} is an unknown positive constant with dimensions of area.

Now, it can also be seen that the above algebra (19)–(23) is unchanged under a constant shift of the \hat{A}'_1 operator. Allowing this possibility, we redefine:

$$\hat{A}'_1 \rightarrow \hat{A}'_1 + \bar{c}\hat{I} \equiv \hat{A}_1, \quad (24)$$

where \bar{c} is an arbitrary constant. The above relation implies that the eigenvalues a'_n and a_n are related as follows:

$$a_n = a'_n + \bar{c}. \quad (25)$$

Equivalently, the lowest eigenvalue a_0 is non-zero, $a_0 = \bar{c}$. Comparing the algebraic approach with reduced phase approach, the constant $\bar{c} = a_{P1} = 4\pi \ell_{P1}^{d-2}$ and the unknown area spacing $\bar{b} = \bar{a}$ so that the \hat{A}_1 spectrum is the same as Eq. (10). This fixes the spectrum $\{a_n\}$ of \hat{A}_1 uniquely. Therefore the spectrum of the total area operator for the charged black hole takes the form

$$a_{nm} = a_n + f(m), \quad (26)$$

where $f(m)$ corresponds to the contribution from area at extremality $A_0(Q)$. In order to determine the exact form of $f(m)$, first note that it has to be proportional to \bar{a} , on dimensional grounds. Secondly, the extremality bound for a charged black hole has to be satisfied, at least for macroscopic black holes. This unambiguously establishes the factor of proportionality to be $(d-3)p/(d-2)$, such that

$$f(m) = \left(\frac{d-3}{d-2}\right)p\bar{a}, \quad (27)$$

where m and p are implicitly related by Eqs. (12) and (13). Thus, the area spectra (14) and (26) become identical.

Our next step is to find a realization of the operators in approach II in terms of the fundamental degrees of freedom (M, Π_M, Q, Π_Q) in approach I. We proceed in two steps. First, we propose a representation of the algebra (19)–(23) with the following form for the

operators $\widehat{\mathcal{R}}_{ns_n}$, \widehat{A}_1 and \widehat{Q} :

$$\widehat{\mathcal{R}}_{nms_n} = (P^\dagger)^n [\theta(m)(\hat{q}^\dagger)^m + \theta(-m)(\hat{q}^\dagger)^{-m}] \hat{g}_{s_{nm}}, \quad (28)$$

$$\widehat{A}_1 = (\widehat{P}^\dagger \widehat{P} + 1/2) \bar{a}, \quad (29)$$

$$\widehat{Q} = e(\hat{q}^\dagger \hat{q} - \hat{q}^\dagger \hat{q}), \quad (30)$$

where \widehat{P}^\dagger (\widehat{P}) raises (lowers) the A_1 eigenlevels from n to $n+1$ ($n+1$ to n). \hat{q}^\dagger , and \hat{q} are the usual charge raising and lowering operators for particle states (i.e., positive charge states) and \hat{q}^\dagger , \hat{q} correspond to charge raising and lowering of antiparticle states (negative charge states). The hermitian internal operator $\hat{g}_{s_{nm}}$, similar to secret operator in [26], transforms the internal quantum state within the same \widehat{A}_1 eigenstate n and charge eigenstate m . Next, we postulate that these operators satisfy the following commutation relations such that (19)–(23) are satisfied:

$$[\widehat{P}, \widehat{P}^\dagger] = [\hat{q}, \hat{q}^\dagger] = [\hat{q}, \hat{q}^\dagger] = 1, \quad (31)$$

$$[\hat{q}, \hat{q}] = [\hat{q}^\dagger, \hat{q}^\dagger] = 0, \quad (32)$$

$$[\widehat{P}, \hat{g}_{s_{nm}}] = [\widehat{P}^\dagger, \hat{g}_{s_{nm}}] = [\hat{q}, \hat{g}_{s_{nm}}] = 0, \quad (33)$$

$$[\hat{g}_{s_{n'm'}}, \hat{g}_{s_{nm}}] = \epsilon_{nn'mm'}^{n''m''} \hat{g}_{s_{n''m''}},$$

where $\epsilon_{nn'mm'}^{n''m''} \neq 0$ iff $n'' = n + n'$ and

$$m'' = m + m'. \quad (34)$$

Also, the area creation (annihilation) operators \widehat{P}^\dagger (\widehat{P}) commute with the charge creation (annihilation) operators \hat{q} , \hat{q}^\dagger (\hat{q}^\dagger , \hat{q}) and $\hat{g}_{s_{nm}}$ commutes with all other operators. Eq. (34) ensures the validity of Eq. (22); however it should be remembered that the operators $\hat{g}_{s_{nm}}$ have a meaning only within the product form $(\widehat{P}^\dagger)^n [\theta(m)(\hat{q}^\dagger)^m + \theta(-m)(\hat{q}^\dagger)^{-m}] g_{s_{nm}}$. Comparison with the operators of reduced phase space approach (9) shows that the form of \widehat{P}^\dagger should be as follows:

$$\widehat{P}^\dagger = \frac{1}{\sqrt{2}} [\widehat{X} - i\widehat{\Pi}_X]. \quad (35)$$

Thus we have an explicit form for \widehat{P}^\dagger in terms of canonically conjugate variables (X, Π_X) in reduced phase space approach. Note that since observables in the reduced phase space approach consist of macroscopic quantities like M and Q alone, the operators \hat{q} and \hat{q}^\dagger are ‘hidden’ just as the operator $g_{s_{nm}}$. This is

perfectly consistent with the well-known fact that the same eigenvalue of \widehat{Q} can be obtained in many possible ways as the sum of particle $\hat{q}^\dagger \hat{q}$ and antiparticle $\hat{q}^\dagger \hat{q}$ charges. Equivalently, from the point of view of an asymptotic observer, the microscopic details of the particle–antiparticle charge composition for a given charge state are unobservable. Similarly, the microscopic quantum state determined by the secret operator $\hat{g}_{s_{nm}}$ cannot be accessible to the asymptotic observer. These arguments are equivalent to the no hair theorem.

We see that approaches I and II are equivalent in the spherically symmetric sector from the asymptotic observer viewpoint. The algebra studied by Bekenstein is similar to the problems of single particle quantum mechanics where non-trivial zero point energy always exists except for a free particle. Hence it is not surprising that the vacuum area is non-zero. However note that the precise value of the remnant (as well as the quantum of area) remains undetermined in this approach. In the reduced phase space approach on the other hand, the remnant (and area quantum) is explicitly determined to be a multiple of the Planck area in the relevant dimension. Since the latter is the only natural length scale in quantum gravity, this seems satisfactory. Moreover, this comparison makes the physical significance of the abstract operators of approach II clear. They are simply the canonically transformed version of the macroscopic gravitational degrees of freedom. This significance has also been recently emphasized by Gour [27]. Finally, note that the discrete area spectrum (14) means that Hawking radiation would consist of discrete spectrum lines, enveloped by the semi-classical Planckian distribution. As argued in [1,5,6], for Schwarzschild black holes of mass M , the gap is order $1/M$, which is comparable to the frequency at which the peak of the Planckian distribution takes place. Hence the spectrum would be far from continuum, and can potentially be tested if and when Hawking radiation becomes experimentally measurable. It would also be interesting to explore the implications of the Planck size remnant to the problem of information loss, since the presence of the former can considerably influence black hole evolution near its end stage [28].

A further test of the correspondence elucidated in this Letter would be to apply it to axi-symmetric rotating black holes. However, for this, one has to first

extend the reduced phase space formalism to situations involving angular momentum, since the former has not been explored beyond the realm of spherical symmetry.

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