Gravitational non-commutativity and Gödel-like spacetimes

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We derive general conditions under which geodesics of stationary spacetimes resemble trajectories of charged particles in an electromagnetic field. For large curvatures (analogous to strong magnetic fields), the quantum mechanicical states of these particles are confined to gravitational analogs of *lowest Landau levels*. Furthermore, there is an effective non-commutativity between their spatial coordinates. We point out that the Som-Raychaudhuri and Gödel spacetime and its generalisations are precisely of the above type and compute the effective non-commutativities that they induce. We show that the non-commutativity for Gödel spacetime is identical to that on the fuzzy sphere. Finally, we show how the star product naturally emerges in Som-Raychaudhuri spacetimes.

I. INTRODUCTION

It is conceivable that the as-yet-undiscovered fundamental theory that underlies (quantum) gravity and the interactions of particles will be characterized by a non-commutative geometry. Indeed, there is some evidence in string theory that this is the case [1]. The origins of this non-commutativity seem to be associated with strong magnetic-like fields that occur in various string and supergravity theories [2]. At a seemingly less fundamental level, the quantum theory of a charged particle in an external magnetic field - the Landau atom - is associated with an effective spatial non-commutativity.

But if spacetime is in some sense non-commutative, then it should show up in the structures that are collectively know as geometry. Similarities between the geodesic equation for gravity and the Lorentz force equation for electromagnetism are well known. The analogy is further strengthened for stationary spacetimes, where not only is there an exact correspondence between different components of the metric and electric and magnetic fields (such that the geodesic equation can be exactly cast in the form of Lorentz force equation), but that given certain special stationary spacetimes, particles in them exhibit interesting phenemonena such as Landau levels and spatial non-commutativity. One such class of spacetimes was discovered by Som and Raychaudhuri and later re-derived using simple matter field configurations of electromagnetic and scalar fields by one of us (JG) and A. Das [11, 12]. We will refer to them as SR spacetimes. We examine some aspects of these spacetimes and compute the Landau levels as well as spatial non-commutativities that they induce. Similarities between the Landau atom and the above spacetimes was discovered by Hikida et al [3] and Drukker et al [4]. Here we present an underlying reason for this, and as well we further extend their results and show that magnetic fields responsible for ordinary Landau levels are naturally associated with these spacetimes as their sources, and that their spatial coordinates exhibit an effective non-commutativity.

What we obtain is the backbone of a cosmology with effective spatial non-commutativity occuring at a large scale. The scale is given by the rotation parameter Ω, which also can be interpreted as the scale of a magnetic-like, or twist, gravitational field. It also describes the boundary of the 'causally safe region' of the spacetime. That is in the region where the radial coordinate satisfies $0 < r < 1/\Omega$, there are no closed timelike or null curves. In the neighbourhood of $r = 0$, the spacetime appears Minkowskian.

We want to begin to explore the idea here that effective spatial non-commutativity emerges from the behaviour of geodesics in a realm where quantum theory is relevant. We believe that there is an analogy here to black hole thermodynamics. In the latter the existence of analogies between the properties of geometrical quantities associated with black hole geometry and classical thermodynamics is often interpreted as revealing underlying quantum field theoretic issues associated with gravity. In our case, what is revealed is that in the case of very strong gravito-magnetic fields, effective spatial non-commutativity can be observed in a sector of the quantum mechanics of particles in that gravitational background. Furthermore, such non-commutativity is closely related to that on the fuzzy sphere, and naturally gives rise to the star product in these spacetimes.

This article is organised as follows: in the next section, we recast the geodesic equations of a charged particle moving in electromagnetic and stationary gravitational fields in the form of a combined Lorentz force-like equation, in which the particle is subjected to 'effective' eletromagnetic fields which include contributions from spacetime curvature. In section (III) we review literature where it was shown that charges in strong magnetic fields experience an effective non-commutativity of spatial coordinates, and extend the results to include gravitational fields. That is, just as electromagnetism, gravity can induce non-commutativity. In section (IV), we discuss SR spacetime and its generalization as Gödel/AdS metrics as solutions of the Einstein equations with appropriate sources. In section (V) we further show that for strong enough curvatures of these metrics, energetic particles are confined to analogs of lowest Landau levels and that their spatial coordinates become non-commutative. In section (VI), we show that non-commutativity on Gödel spacetime is identical to that on the fuzzy sphere, and in section (VII), we show that the star-product emerges for SR spacetimes. Finally we conclude in section (VIII) with a summary and some open questions.

II. STATIONARY SPACETIMES AND ELECTROMAGNETISM

We start with the general case of a particle of mass m and charge e in a curved spacetime with metric $g_{\mu\nu}$ and electromagnetic field $F_{\mu\nu}$. For the most part, we are interested in the uncharged case, but we consider the charged case in order to see most clearly the gravitational analogue of charged particle motion in an external magnetic field.

The action functional is

$$S[x^0, x^a] := \int_{\tau_1}^{\tau_2} d\tau \left[\frac{mc^2}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \frac{e}{c} A_\mu \dot{x}^\mu \right]. \tag{1}$$

We write $\dot{x}^\mu := dx^\mu / cd\tau$. In the above, in general, the metric $g_{\mu\nu}$ and 4-potential A_μ depend on all the coordinates. We assume that the metric and the electromagnetic fields are stationary. That is, there exists a Killing vector field k^μ which is timelike everywhere, except perhaps in a compact closed region, such that $\mathcal{L}_k F_{\mu\nu} = 0$. We can choose coordinates adapted to the Killing vector field such that the metric is in ADM form:

$$ds^2 = -c^2 d\tau^2 = -h(x)(dx^0 - g_a(x)dx^a)^2 + \gamma_{ab}(x) dx^a dx^b, \tag{2}$$

where $-h(x) = g_{00}$ and $g_{0a} = g_{0i}/h(x)$. In the above, (x) now denotes dependence on the spatial coordinates $x^a, a, b, ... = 1, 2, 3$ only. The electromagnetic 4-potential $A_\mu(x^0, x^a)$ is of the form $(A_0(x), A_a(x))$, i.e., it does not depend on the coordinate x^0. In these coordinates, the Killing vector $k^\mu = [1, 0, 0, 0]$.

We now compute the functional derivatives of $S[x^0, x^a]$. We fix the coordinates x^0, x^a at τ_1 and τ_2. First, varying x^0 only we obtain

$$mc^2 h\omega = \frac{e}{c} A_0 + const. \tag{3}$$

We have written $\omega := \dot{x}^0 - g_a(x)\dot{x}^a$. We restrict to observers who measure time in the timelike Killing direction, that is, the unit infinitesimal time interval is $\sqrt{h}(dx^0 - g_a dx^a)$. Hence, the 3-velocity is given by

$$v^a := \frac{c}{\sqrt{h}\omega} \dot{x}^a. \tag{4}$$

Writing $\beta^a := v^a/c, \beta^2 := \gamma_{ab}\beta^a\beta^b$, we find

$$\sqrt{h}\omega = (1 - \beta^2)^{-1/2}. \tag{5}$$

If we now vary the space coordinate x^b, we get

$$\begin{aligned} m\gamma_{ab} \left[\sqrt{1-\beta^2} \frac{d}{d\tau} \left(\frac{v^a}{\sqrt{1-\beta^2}} \right) + \frac{1}{\sqrt{1-\beta^2}} \lambda^a_{cd} v^c v^d \right] &= \frac{mc^2}{\sqrt{1-\beta^2}} \left[-\partial_b \left(\log \sqrt{h} \right) - \sqrt{h} \frac{v^a}{c} f_{ab} \right] \\ &+ \frac{e}{c} \left[\frac{1}{\sqrt{h}} \partial_b A_0 - \frac{v^a}{c} \left(F_{ab} + g_b \partial_a A_0 - g_a \partial_b A_0 \right) \right], \end{aligned} \tag{6}$$

where $f_{ab} := \partial_a g_b - \partial_b g_a$ and $F_{ab} := \partial_a A_b - \partial_b A_a$. The quantities λ^a_{bc} are the components of the Christoffel connection for the 3D spatial metric $\gamma_{ab}(x)$.

The first term on the right hand side above is the Newtonian part of the gravitational force; while the second term is the twist, or gravito-magnetic part. These terms can be defined covariantly as, respectively, the field strengths associated with the scalar potential $\phi := \ln\sqrt{-g_{\mu\nu}k^\mu k^\nu}$ and the vector potential $g_\mu := e^{-2\phi}g_{\mu\nu}k^\nu$. We see that the twist field is trivial if the 1-form $\hat{g} := g_\alpha dx^\alpha$ is closed, that is $d\hat{g} = 0$. These 1-form fields are effectively three dimensional. That is, $H_\mu = [0, \gamma_{cd}\epsilon^{abc}\partial_a g_b/\sqrt{\gamma}]$.

The electric and magnetic fields can be expressed as:

$$E_a := F_{0a} = -\partial_a A_0(x); \tag{7}$$

$$B^a := -\frac{1}{2}\eta^{abc}F_{bc} = -\frac{1}{2}\eta^{abc}\left(\partial_b A_c(x) - \partial_c A_b(x)\right). \tag{8}$$

In the above, η^{abc} is the spatial LeviCivita *tensor*. In addition to the usual electromagnetic gauge transformations

$$A'_\mu = A_\mu - \frac{\partial f}{\partial x^\mu}\ , \tag{9}$$

one can also define ‘gravitational gauge transformation’ which preserves the stationarity of the metric, leaving the equations of motion unchanged:

$$x^a \to x^a\ ,\ x^0 \to x^0 + f(x^a), \tag{10}$$

under which:

$$h \to h\ ,\ \gamma_{ab} \to \gamma_{ab}\ ,\ g_a \to g_a - \frac{\partial f}{\partial x^a}\ ,\ f_{ab} \to f_{ab}\ . \tag{11}$$

We now set the charge $e = 0$. The uncharged massive particle will now see an effective gravitational force with Newtonian and twist components, given in 3D form as follows (in the $v/c << 1$ limit) [24]:

$$\vec{f} = mc^2\left[-\vec{\nabla}\left(\ln\sqrt{h}\right) + \frac{\sqrt{h}\ \vec{v}}{c}\times\left(\vec{\nabla}\times\vec{g}\right)\right]. \tag{12}$$

We see that motion in a stationary gravitational field is analogous to that of a charged particle in an electromagnetic field in a precise sense and that the geodesic equation resembles the Lorentz force law. The gravitational analogue of the electric charge is the energy mc^2 of the particle. The effective scalar and 3-vector potentials are

$$\Psi := \ln\sqrt{h},$$

$$\vec{\mathcal{A}} := \vec{g}. \tag{13}$$

Hence the effective 3-force is

$$\vec{f} = mc^2(\vec{\mathcal{E}}_g + \sqrt{h}\vec{\mathcal{B}}_g), \tag{14}$$

where $\vec{\mathcal{E}}_g := -\vec{\nabla}\Psi, \vec{\mathcal{B}}_g := c\vec{\nabla}\times\vec{g}$.

III. LANDAU LEVELS AND NON-COMMUTATIVITY IN GRAVITY

Now let us return to the motion of charged particle in an electromagnetic field. The Lagrangian for the system is:

$$L = \frac{1}{2}m\dot{\vec{x}}^2 + \frac{e}{c}\ \dot{\vec{x}}\cdot\vec{A} - e\Phi\ , \tag{15}$$

where $A^\mu = [\Phi, \vec{A}]$ is the gauge potential. Assuming that the charge is in a constant magnetic field along the x^3-axis, one can choose a symmetric gauge:

$$A^i = -\frac{B}{2}\ \epsilon^{ij}x^j \tag{16}$$

Now, if the magnetic field is very strong, such that $eA/mc \gg v$, the characteristic velocity of the particle, then as shown in [5] (see also [6, 7]). the kinetic term can be dropped from (15) and one obtains:

$$L \to \frac{eB}{2c}\left[-y\dot{x} + x\dot{y}\right] - e\Phi\ \ , \tag{17}$$

Poisson brackets can be calculated using which the prescription of [8]. Writing (17) as

$$L dt = a_i dx^i - e\ \Phi dt \tag{18}$$

with $(x^1, x^2) = (x, y)$ and computing $f_{ij} \equiv a_{[j,i]}$, we get: $f_{xy} = eB/2c$. Then:

$$\{x, y\} = f_{yx}^{-1} = -\frac{2c}{eB}\ , \tag{19}$$

from which their commutator follows [6, 7]:

$$[x, y] = -i\frac{2\hbar c}{eB}\ . \tag{20}$$

We note here that if we switch to polar coordinates (θ, r) in space, then the commutator is

$$[r, \theta] = -i\frac{2\hbar c}{eB}\frac{1}{r}. \tag{21}$$

Hence for very large r, spatial commutativity appears to be restored.

It is well known that the system under consideration has discrete energy eigenvalues ('Landau levels'), which are given by:

$$E_n = \frac{eB\hbar}{mc}\left(n + \frac{1}{2}\right)\ . \tag{22}$$

Thus, the above result also shows that for very strong magnetic fields, the gap between ground and first excited state is large and the charge is effectively confined to the lowest Landau level.

Motion in a Newtonian gravity field is described by the Lagrangian Eq.(15) if $e = mc^2$ and $\vec{A} = 0$. However, the presence of the factor $\sqrt{h}$ in the 3-force contribution from the twist potential prevents using that Lagrangian to describe motion in the case when gravity has both a Newtonian and a twist part. In the remainder of this paper, we consider only the case when there is no Newtonian limit (and no Coulomb electromagnetic field). In a the case of a curved 3-space, with metric $\gamma_{ab}(x)$, we generalize Eq.(15) to

$$L = \frac{1}{2}m\gamma_{ab}\dot{x}^a\dot{x}^b + \frac{e}{c}\dot{x}^a A_a. \tag{23}$$

The charge e is either an electric charge or in the case of gravity, the energy mc^2, in which case A_a is the twist part of the gravitational field.

Now we see that an analogous effect takes place in the presence of a strong stationary gravitational fields, if its magnetic analog $\vec{\nabla} \times \vec{g}$ is constant. Consider first the case of a flat 3-space metric $\gamma_{ab} = \delta_{ab}$. The effective Lagrangian for the system is (15) with $e/c = -2mc$ and $\vec{A} = \vec{g}$. A particle in this spacetime would be confined to the lowest energy level given by (22) with $\vec{B} \to \vec{\mathcal{B}}_g = c\vec{\nabla} \times \vec{g}$ a constant twist field. Spatial non-commutativity follows as well from (20).

Extension of the above results to charged particle confined to S^2 or two dimensional hyperbolic plane (H^2) goes as follows [9, 10]: For S^2, the magnetic field is assumed to be perpendicular to the spherical surface at every point and of constant magnitude. The effective 2-space is a sphere of radius R with metric $R^2(d\theta^2 + \sin^2\theta)$. The corresponding monopole potential on S^2 is given by:

$$A_\phi = 2BR^2\sin^2(\theta/2) \quad , \quad A_r = A_\theta = 0\ , \tag{24}$$

where B is the magnetic field and R^2, the radius of S^2 [25]. In this case, the effective Lagrangian for strong magnetic fields is:

$$L = 4mcBR^2\ \dot{\phi}\sin^2\frac{\theta}{2}\ , \tag{25}$$

from which it follows that $f_{\theta\phi} = -2mcBR^2\sin\theta$ and

$$[\theta, \phi] = -\frac{i\hbar}{2mcBR^2}\frac{1}{\sin\theta}\ . \tag{26}$$

Note that, in this case the non-commutativity is not a constant. In fact, the non-commutativity is maximal near the poles $\theta = 0, \pi$. Similarly, for constant magnetic fields on a two dimensional hyperbolic manifold H_2, $R^2 \to -R^2$, and the signature in the RHS of (26) is reversed. The energy levels in these cases are respectively:

$$E_n = \frac{eB\hbar}{mc}\left(n + \frac{1}{2}\right) \pm \frac{\hbar^2}{2m}\ \frac{n(n+1)}{R^2}\ . \tag{27}$$

For large B, the lowest Landau level effectively dominates the spectrum. It is interesting to note however, that B drops out of the second term. The latter is an effect of the curvature of 3-space, and seems analogous to the Casimir energy. Eqs.(24) and (27) reduce to (16) and (22) respectively in the limit $R^2 \to \infty$, with the identification $\theta = r/R$.

IV. RAYCHAUDHURI-SOM GEOMETRIES

We now pose the following question: are there stationary spacetimes whose 'magnetic' parts $\vec{g}$ are of the same functional form as (16) or (24), such that particle in them would exhibit Landau levels? Remarkably, the answer is in the positive. These spacetimes were discovered by Som and Raychaudhuri in 1968 [11] as a class of solutions of Einstein-Maxwell equations with a charged dust source. These geometries have the same sort of causal pathologies as the Gödel metric, and like the latter, have sources which obey physically reasonable energy conditions. This family of solutions was rediscovered later by one of us together with A. Das [12], but now in the form of a solution of the Einstein-Maxwell equations with a massive Klein-Gordon scalar field as source.

The metric in cylindrical coordinates is of the form

$$ds^2 = -(cdt + \Omega r^2 d\phi)^2 + dr^2 + r^2 d\phi^2 + dz^2, \tag{28}$$

This solution has a charged source which satisfies a Weyl-Majumdar-Papapetrou relation between the charge q and mass μ of the source. In the units we use, $\Omega = \mu c/\hbar = qc/4\sqrt{\pi}\hbar$. Also, the Klein-Gordon field ψ is chosen in a gauge so that $\psi = \sqrt{2}$ and the non-zero components of the electromagnetic potential are

$$A_t = \frac{1}{c^3\sqrt{2G}}, \tag{29}$$

$$A_\theta = \frac{\Omega c^2 r^2}{\sqrt{2G}}, \tag{30}$$

where G is the Newton's constant. The electric field and the 'Newtonian part' of the gravitational field are zero; the magnetic field and the 'twist part' of the gravitational field are constant and in the $z-$direction. We see that these solutions have one free parameter, which we can assign as the mass of the source Klein-Gordon scalar field.

The metric (28) is homogeneous. We will see below that the isometry group of the metric is the direct product of translations in the $z-$direction with the nilpotent Lie group, which has 4 parameters. These geometries have recently reappeared in the literature, in the context of string/M-theory. This is because they are special cases of a class of geometries described by so-called Gödel/AdS metrics of the form [4, 13]:

$$ds^2 = -\left(cdt + 4\Omega R^2 \sin^2(\theta/2) d\phi\right)^2 + R^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) + dz^2\ . \tag{31}$$

If $z = constant$, it can be shown that the above metric is a solution of three dimensional Einstein-Maxwell equations with a positive cosmological constant $1/R^2$ and a monopole potential of the form $A_0 = k;\ A_\phi = 4k\Omega R^2 \sin^2(\theta/2)$, where $k^2 = 1 + 1/(2\Omega R)^2$. It is also a solution of the Einstein equations with more complicated matter field sources in four dimensions ([13]). For $1/R^2 = 2\Omega^2, 1/R^2 = 4\Omega^2$, the metric becomes, respectively, the 'pure Gödel', $AdS_3 \times R$. On the other hand, for $\Omega \to 0$ it reduces to a metric on an S^2 of radius R, whereas for $R \to \infty$ (and the identification $\theta = r/R$), it reduces to the SR metric (28). The corresponding metric for two dimensional hyperbolic space H_2 can be obtained by the substitution $R^2 \to -R^2$ [14].

Also, we note that these geometries, together with appropriate background fields, are exact backgrounds in string theory, in the sense that a sigma model in the background is an exact conformal field theory [15]. Furthermore, there are black holes whose metrics are asymptotically Gödel/AdS or SR [16, 17].

V. LANDAU LEVELS AND NON-COMMUTATIVITY IN SR GEOMETRIES

It is remarkable that the 'magnetic part' of the gravitational fields of the stationary spacetimes described in the previous sections precisely match that real magnetic fields which give rise to Landau levels in those backgrounds.

A. SR Spacetimes

The SR metric (28) in Cartesian coordinates is:

$$ds^2 = -\left[dt - \Omega(ydx - xdy)\right]^2 + dx^2 + dy^2 + dz^2 \ , \tag{32}$$

from which one can read-off the effective potentials and fields: implying

$$h = 1 \quad , \quad \vec{g} = \Omega \ \ (y, -x, 0) \tag{33}$$

Note that $\vec{g}$ is precisely of the form of the vector potential in Landau problem, Eq.(16). Furthermore, there are no electric or Newtonian gravitational fields. Thus, the energy of the test-charge would be quantised ('gravitational Landau levels') as:

$$E_n = 2\Omega mc \left(n + \frac{1}{2}\right) \quad . \tag{34}$$

Further, if the gravito-magnetic fields are strong enough and the charge is confined to the lowest Landau level, then from (20) it follows that the spatial non-commutativity experienced by the test particle is:

$$[x, y] = -i\frac{\hbar}{mc\Omega}. \tag{35}$$

B. Gödel Spacetimes

Next, for the Gödel like spacetimes (31) and its accompanying gauge potential, the electric and magnetic parts can be read off:

$$h = 1 \quad , \quad g_\phi = -4\Omega R^2 \sin^2(\theta/2) \ , \ g_r = g_\theta = 0, \tag{36}$$

for which:

$$\Psi \ = \ \vec{\mathcal{E}} = 0; \tag{37}$$

$$\mathcal{A}_\phi \ = \ 4\Omega\left(k - \frac{mc^2}{e}\right) R^2 \sin^2\left(\frac{\theta}{2}\right) \quad . \tag{38}$$

The corresponding twist (gravito-magnetic) field is

$$\vec{B} = 2\Omega\hat{R}. \tag{39}$$

The energy levels of the Landau-like system for an uncharged particle of mass m is

$$E_n = 2c\hbar\Omega\left(n + \frac{1}{2}\right) \pm \frac{\hbar^2}{2m} \ \frac{n(n+1)}{R^2} \ . \tag{40}$$

This results in a non-commutativity from Eq.(26): of

$$[\theta, \phi] = -\frac{i\hbar}{2mc\Omega}\frac{1}{\sin\theta} \tag{41}$$

Once again, the second terms in (39), (40) and in the denominator of (41) can be attributed to gravity alone. Similarities between Gödel and non-commutative spacetimes were also noticed in [18].

VI. RELATION TO THE FUZZY SPHERE

In this section, we show that the non-commutativity generated in the context of a charged particle on S^2 with a monopole at the centre, or equivalently for Gödel spacetimes, is identical to the non-commutativity on the fuzzy sphere. Let us consider Eq.(25) once again. Using

$$\phi \ = \ \arctan\frac{y}{x} \ , \ \dot{\phi} = \frac{x\dot{y} - y\dot{x}}{x^2 + y^2}, \tag{42}$$

$$\sin^2\phi \ = \ \frac{1}{2}\left(1 - \frac{z}{R}\right) \ , \ x^2 + y^2 + z^2 = R^2. \tag{43}$$

Eq.(25) can be written as:

$$L = \left(\frac{BR^2 e}{c}\right)\left(\frac{x\dot{y} - y\dot{x}}{x^2+y^2}\right)\left(1 - \frac{\sqrt{R^2-(x^2+y^2)}}{R}\right) \ , \tag{44}$$

implying

$$Ldt \ \equiv \ a_x dx + a_y dy, \tag{45}$$

$$\text{where} \quad a_x \ = \ \frac{-y}{x^2+y^2}\left(1 - \frac{\sqrt{R^2-(x^2+y^2)}}{R}\right) \ \frac{BR^2 e}{c}, \tag{46}$$

$$a_y \ = \ \frac{x}{x^2+y^2}\left(1 - \frac{\sqrt{R^2-(x^2+y^2)}}{R}\right) \ \frac{BR^2 e}{c} \ , \tag{47}$$

from which, one obtains:

$$f_{xy} \equiv a_{[x,y]} = -\frac{BRe}{c}\frac{1}{z} \ . \tag{48}$$

The above in turn implies the following Poisson and commutator brackets

$$\{x,y\} = f_{yx}^{-1} = -\frac{c}{BRe}z \ \ , \ [x,y] = -\frac{i\hbar c}{BRe}z \ . \tag{49}$$

By changing the coordinate labels $\{x,y,z\} \to \{y,z,x\}$ and $\{x,y,z\} \to \{z,x,y\}$ respectively, the commutators $[y,z]$ and $[z,x]$ can be obtained similarly, and the results can be summarised as:

$$[x_i, x_j] = i\epsilon_{ijk}\lambda \ x_k \ , \tag{50}$$

where $\lambda \equiv -\frac{\hbar c}{BRe}$. Note that Eq.(50) above is the algebra of coordinates on a fuzzy sphere [19]. This implies that the Landau atom on S^2, or equivalently the Gödel universe can be regarded as concrete realisations of the fuzzy sphere.

Finally, let us consider the stereographic projections of the fuzzy sphere defined by:

$$y_\pm \ = \ 2Rx_\pm(R-z)^{-1} \equiv y_1 \pm iy_2, \tag{51}$$

$$\text{where} \quad x_\pm \ = \ x \pm iy \tag{52}$$

In the limit $R \to \infty$ and $N \equiv 2R/\lambda \to \infty$, keeping $2R^2/N = \hbar c/eB$ fixed ($= \theta$ of ref.[20]), we get the non-commutative (fuzzy) plane defined by [20] :

$$[y_1, y_2] = -i\frac{\hbar c}{eB}, \tag{53}$$

which is identical to the commutation relation (20), upto a multiplicative factor of order unity.

VII. EMERGENCE OF THE STAR PRODUCT

In this section, we follow the procedure outlined in [21] to show that the star product naturally emerges when one considers fluid flow in the background of Gödel-like spacetimes. We start with the Euler equation for a non-relativistic fluid in a general curved spacetime [22] :

$$\rho u^k u_{i;k} = p_{;i} - u_i\left(u^k p_{;k}\right) \ , \tag{54}$$

where $u^k = dx^k/ds$ is the 4-velocity field and ρ is the mass density of the fluid. With the stationary metric (2) and as before if we assume $h = 1$ and $v/c << 1$ (non-relativistic approximation) as well as $p_{,t} = 0$ and $-\vec{\nabla}p/\rho = \vec{f}$ as in [21], we obtain approximately:

$$\frac{\partial \vec{v}}{\partial t} + \left(\vec{v}\cdot\vec{\nabla}\right)\vec{v} = c \ \vec{v} \times \left(\vec{\nabla}\times\vec{g}\right) + \vec{f} \ , \tag{55}$$

which is identical to Eq(14) of [21], once when the latter is scaled by $1/m$ and $(e/mc)\vec{B}$ is identified with a *constant* gravito-magnetic field $c\vec{\nabla} \times \vec{g}$. The constancy of $\vec{B}$ ensures the identification with SR type metrics.

Further, from the covariant continuity equation

$$\frac{1}{\sqrt{-g}}\partial_i\left(\sqrt{-g}\rho u^i\right) = 0 \ , \tag{56}$$

can be written in this case as

$$\partial_0\left[\sqrt{det(\gamma)h}\left\{\rho h\left(\frac{1}{\sqrt{h}} + \frac{\vec{g}\cdot\vec{v}}{c}\right)\right\}\right] + \partial_\alpha\left(\sqrt{det(\gamma)h}\ \frac{\rho\ v^\alpha}{c\sqrt{1-v^2/c^2}}\right) = 0 \ . \tag{57}$$

Again, for $h = 1$ and $v/c << 1$ (such that the second term on the left hand side drops out) and defining:

$$\sqrt{det(\gamma)}\ \rho \equiv \rho_1 \ , \tag{58}$$

we get:

$$\dot{\rho}_1 + \vec{\nabla}\cdot(\rho_1\vec{v}) = 0 \ , \tag{59}$$

which is identical to Eq.(13) of [21]. Consequently, the analysis following Eq.(14) of [21] go through and a star product emerges. We sketch the steps in brief here. We assume that the force $\vec{f}$ can be derived from a potential of the form:

$$\vec{f}(r) = -\vec{\nabla}\frac{\delta}{\delta\rho(r)}\int d\vec{r}\ V \ . \tag{60}$$

In the large gravito-magnetic limit, Eqs.(55) and (59) can be derived from the Poisson brackets of ρ and its canonical momentum π with the Hamiltonian:

$$H = \int d^2r\left(\rho\ \frac{\pi^2}{2m} + V\right) \ , \tag{61}$$

with the following fundamental brackets (and with the identification: $eb/mc \to 2\sqrt{8}\pi c\mu/\hbar$):

$$\begin{aligned} \{\rho_1(\vec{r}), \rho_1(\vec{r}')\} &= -\frac{c}{eb}e^{ij}\partial_i\rho(\vec{r})\partial_i\delta\left(\vec{r}-\vec{r}'\right), & (62)\\ \{\tilde{\rho}(\vec{p}), \tilde{\rho}(\vec{q})\} &= -\frac{c}{eb}e^{ij}p^iq^j\tilde{\rho}\left(\vec{p}+\vec{q}\right), & (63)\end{aligned}$$

where

$$\tilde{\rho}(p) = \int d^2re^{i\vec{p}\cdot\vec{r}}\rho(\vec{r}) \ . \tag{64}$$

It can be shown that (62) and (63) is satisfied by $\rho(\vec{r})$ of the form:

$$\rho_1(\vec{r}) = \sum_n \delta\left(\vec{r}-\vec{r}_n\right) \ , \tag{65}$$

where n labels the individual particles of the fluid, provided their coordinates satisfy:

$$\left\{r^i_m, r^j_n\right\} = \frac{c}{eb}\epsilon^{ij}\delta_{mn} \ . \tag{66}$$

Next, one quantises by writing the commutator bracket corresponding to (66) above:

$$\left[r^i_m, r^j_n\right] = -i\hbar\frac{c}{eb}\epsilon^{ij}\delta_{mn} \tag{67}$$

and assuming the following Weyl ordering:

$$\tilde{\rho}(\vec{p}) = \sum_n e^{i\vec{p}\cdot\vec{r}_n} \ . \tag{68}$$

From (67) and (68) and using the Baker-Campbell-Hausdorff formula, one obtains:

$$[\tilde{\rho}(p), \tilde{\rho}(q)] = 2i \sin\left(\frac{\hbar c}{2eb} e^{ij} p^i q^j\right) \tilde{\rho}(p+q) \ . \tag{69}$$

Finally defining:

$$\langle f \rangle = \int d^2 r \rho(\vec{r}) f(\vec{r}) = \frac{1}{(2\pi)^2} \int d^2 p \ \tilde{\rho}(\vec{p}) \tilde{f}(-\vec{p}) \ , \tag{70}$$

multiplying (69) by $\tilde{f}(-\vec{p})\tilde{g}(-\vec{q})$ and integrating gives:

$$[\langle f \rangle, \langle g \rangle] = \langle h \rangle \ , \tag{71}$$

with

$$h(r) = (f \star g)(\vec{r}) - (g \star f)(\vec{r}) \ , \tag{72}$$

where

$$(f \star g)(\vec{r}) \equiv \exp(\frac{i\hbar c}{2eb} \epsilon^{ij} \partial_i \partial'_j) f(\vec{r}) g(\vec{r}')|_{\vec{r}'=\vec{r}} \ . \tag{73}$$

Thus, we see how the star product emerges in the context of Gödel-like spacetimes, similar to its appearence in the Landau atom.

VIII. CONCLUSIONS

In this article we have shown that the Newtonian and twist parts of stationary metrics bear strong resemblance to ordinary electric and magnetic fields. The behaviour of particle geodesics in such backgrounds is similar to that of charged particles in electromagnetic fields. One important physical situation in the case of the latter is the so-called Landau problem, in which charges in a constant magnetic field have quantised energy levels. Morover, if the field is very strong then the charge is effectively confined to the lowest Landau levels. We observe that Gödel like spacetimes exhibit very similar behaviour for particle geodesics. Furthermore, the effective non-commutativities that result in these spacetimes are identical to that for the fuzzy sphere, under suitable identification of parameters, and the star-product emerges when one considers relativistc fluid motion in these spacetimes. It would be interesting to examine other physical implications of this result. To obtain the effective fields due to electromagnetism and gravity, we have made the simplifying assumption of non-relativistic particle velocities. Generalisation to arbitary velocities is expected to be straightforward and could provide interesting corrections to our results. Similarly, the θ dependence of the Poisson brackets in (26) merits further investigation. Finally, it would be interesting to see whether the non-commutativity studied here is related to non-commutativity in string theory, where the Neveu-Schwarz B field is taken to infinity in the presence of D-branes [1]. It is known that for ordinary field theories with quadratic actions, the star-product reduces to the ordinary product [23]. Thus quantities such as the two-point function should remain unchanged, not acquiring corrections from the underlying non-commutativity. We hope to report on these and related issues elsewhere.

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[24] The following definitions have been used: $(\vec{a}\times\vec{b})_a = \eta_{abc}a^b b^c$, $(\vec{a}\times\vec{b})^a = \frac{1}{2}\eta^{abc}a_b b_c$, $(\vec{\nabla}\times\vec{a})^a = \frac{1}{2}\eta^{abc}\left(\frac{\partial a_c}{\partial x_b} - \frac{\partial a_b}{\partial x_c}\right)$, $\vec{\nabla}\cdot\vec{a} = \frac{1}{\sqrt{\gamma}}\partial_a\left(\sqrt{\gamma}a^a\right)$
[25] Note that the potentials used here are related to the Wu-Yang potential (A'_i) as $A_i = \sqrt{g_{ii}}{A'}^i$