

# Gravitational anomalies: a recipe for Hawking radiation\*

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## Abstract

We explore the method of Robinson and Wilczek for deriving the Hawking temperature of a black hole. In this method, the Hawking radiation restores general covariance in an effective theory of near-horizon physics which otherwise exhibits a gravitational anomaly at the quantum level. The method has been shown to work for broad classes of black holes in arbitrary spacetime dimensions. These include static black holes, accreting or evaporating black holes, charged black holes, rotating black holes, and even black rings. In the cases of charged and rotating black holes, the expected super-radiant current is also reproduced.

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In this essay, we will review a method developed by Robinson and Wilczek [1], and subsequently by others [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21], in which Hawking radiation [22] enters the theory as a mechanism that cancels a would-be gravitational anomaly, thus saving the unitarity of quantum mechanics near a black hole from potential violation. For purposes of illustration, we will first show the method for the simplest case, following [1], and then describe subsequent developments. Consider a  $d$ -dimensional, static, spherically symmetric black hole spacetime with metric:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{(d-2)}^2, \quad (1)$$

where  $d\Omega_{(d-2)}$  is the line element on  $\mathbb{S}^{(d-2)}$  and  $f(r)$  is an arbitrary function of the radial coordinate  $r$  only. The more general case where  $g_{tt} \neq g^{rr}$  is considered in [2, 6]. For a black hole, we require the existence of a horizon at coordinate  $r = r_h$  determined by the condition  $f(r_h) = 0$ . The surface gravity of the horizon is given by  $\kappa \equiv \frac{1}{2} \partial_r f(r) \Big|_{r=r_h}$ . For this simple spacetime, the varying definitions of ‘‘horizon’’ (event, Killing, *etc.*) all coincide. For a discussion of the role of horizon taxonomy in black hole thermodynamics, see [23]. Because the horizon is a Killing horizon for translations in  $t$ , the associated conserved quantity fails to behave like a proper energy inside the horizon where  $\partial_t$  becomes spacelike. As a symptom of this pathology, the vacuum state constructed to have zero Fock space occupation number with respect to  $\partial_t$  (the so-called Boulware state [24]) contains divergences due to a pile up of outgoing high frequency modes at the horizon.

The essence of the Robinson-Wilczek method is to take the lessons of effective field theory seriously: the physics observed by a given experiment should be describable by an effective theory of only those degrees of freedom accessible to the experiment. The effective theory is derived from the fundamental theory by integrating out inaccessible degrees of freedom. In a black hole spacetime, degrees of freedom inside the horizon are inaccessible to the outside observer. Outgoing near-horizon modes are also inaccessible due to their diverging energy, as discussed above. To form the effective theory of the outside observer, we must remove these modes from all fields in the theory. Upon doing so, however, we will encounter a problem. Arguments that theories of quantum gravity should be formulated in terms of the effective degrees of freedom accessible to a given observer have also appeared in [25, 26, for example].

As a probe of this background geometry, we consider a scalar field with arbitrary self-

interactions. By expanding the field in partial wave modes using  $(d-2)$ -dimensional spherical harmonics and taking the near-horizon limit, we see that the action for each partial wave mode reduces to that of a free, massless,  $(1+1)$ -dimensional scalar field on the  $r$ - $t$  section of the original spacetime. The only remnant of  $d$ -dimensional physics is the degenerate angular momentum quantum numbers, which are now just labels on otherwise identical fields. This effective dimensional reduction is demonstrated explicitly in [2], but has been seen in earlier work [27, 28].

Since we have eliminated the outgoing modes, the effective near-horizon theory is chiral. As shown in [29],  $(1+1)$ -dimensional chiral theories exhibit a gravitational anomaly and therefore fail to covariantly conserve the energy-momentum tensor. For the case of an ingoing scalar field, the anomaly takes the form [30]:

$$\nabla_\mu T_\nu^\mu = \frac{1}{\sqrt{-g}} \partial_\mu N_\nu^\mu \equiv A_\nu, \quad (2)$$

where

$$N_\nu^\mu = \frac{1}{96\pi} \varepsilon^{\beta\mu} \partial_\alpha \Gamma_{\nu\beta}^\alpha, \quad (3)$$

$\varepsilon^{\mu\nu}$  is the anti-symmetric unit tensor ( $\varepsilon^{01} = 1$ ), and  $\Gamma_{\nu\beta}^\alpha$  is the Christoffel connection on the  $(1+1)$ -dimensional spacetime. Equation (2) can be solved as

$$T_t^t = -\frac{(K+Q)}{f} - \frac{B(r)}{f} - \frac{I(r)}{f} + T_\alpha^\alpha(r), \quad (4)$$

$$T_r^r = \frac{(K+Q)}{f} + \frac{B(r)}{f} + \frac{I(r)}{f}, \quad (5)$$

$$T_t^r = -K + C(r) = -f^2 T_r^t, \quad (6)$$

where

$$B(r) \equiv \int_{r_h}^r f(x) A_r(x) dx, \quad (7)$$

$$C(r) \equiv \int_{r_h}^r A_t(x) dx, \quad (8)$$

$$I(r) \equiv \frac{1}{2} \int_{r_h}^r T_\alpha^\alpha(x) f'(x) dx. \quad (9)$$

The constants  $K$ ,  $Q$ , and the trace  $T_\alpha^\alpha(r)$  are undetermined.

However, Equation (2) does not hold over the entire spacetime. In constructing the effective theory, we imposed this condition only at the horizon. Equation (2) will hold only in an infinitesimal region about the horizon between  $r_h \pm \epsilon$  in the limit  $\epsilon \rightarrow 0$ . Since the

fundamental theory is generally covariant, the quantum effective action  $W$  must be invariant under a coordinate transformation with parameter  $\lambda^\nu$ :

$$\lim_{\epsilon \rightarrow 0} \delta_\lambda W = 0. \quad (10)$$

Performing this coordinate variation explicitly, we find that

$$\begin{aligned} -\delta_\lambda W &= \int d^2x \sqrt{-g} \lambda^\nu \nabla_\mu \{T_{i\nu}^\mu \Theta_+ + T_{o\nu}^\mu \Theta_- + T_{\chi\nu}^\mu H\} \\ &= \int d^2x \sqrt{-g} \lambda^t \{ \partial_r (N_t^r H) + (T_{ot}^r - T_{\chi t}^r + N_t^r) \partial_r \Theta_+ + (T_{it}^r - T_{\chi t}^r + N_t^r) \partial_r \Theta_- \} \\ &\quad + \int d^2x \sqrt{-g} \lambda^r \{ (T_{or}^r - T_{\chi r}^r) \partial_r \Theta_+ + (T_{ir}^r - T_{\chi r}^r) \partial_r \Theta_- \}. \end{aligned} \quad (11)$$

We have written the total energy-momentum tensor as the sum of *inside*, *outside* and *chiral* parts:

$$T_\nu^\mu = T_{i\nu}^\mu \Theta_+ + T_{o\nu}^\mu \Theta_- + T_{\chi\nu}^\mu H, \quad (12)$$

where  $\Theta_\pm \equiv \Theta(\pm(r - r_h) - \epsilon)$  are step functions and  $H = 1 - \Theta_+ - \Theta_-$  is a ‘‘top hat’’ function which is equal to unity between  $r_h \pm \epsilon$  and zero elsewhere. The quantities  $T_{i\nu}^\mu$  and  $T_{o\nu}^\mu$  are covariantly conserved inside and outside the horizon, respectively. However,  $T_{\chi\nu}^\mu$  is not covariantly conserved and expresses the anomalous chiral physics on the horizon.

Taking derivatives of the  $\Theta$  functions and expanding for small  $\epsilon$ , we find that

$$\begin{aligned} \delta_\lambda W &= \int d^2x \lambda^t \{ [K_o - K_i] \delta(r - r_h) - \epsilon [K_o + K_i - 2K_\chi - 2N_t^r] \partial_r \delta(r - r_h) + \dots \} \\ &\quad - \int d^2x \lambda^r \left\{ \left[ \frac{K_o + Q_o + K_i + Q_i - 2K_\chi - 2Q_\chi}{f} \right] \right. \\ &\quad \left. - \epsilon \left[ \frac{K_o + Q_o - K_i - Q_i}{f} \right] \partial_r \delta(r - r_h) + \dots \right\}. \end{aligned} \quad (13)$$

It is easily seen in Equation (13) that only the on-horizon values of the energy-momentum tensor will contribute to the gravitational anomaly. Since the parameters  $\lambda^t$  and  $\lambda^r$  are independent, each of the four terms in square brackets in Equation (13) must vanish simultaneously, but only needs to do so on the horizon. These conditions yield

$$K_o = K_i = K_\chi + \Phi, \quad (14)$$

$$Q_o = Q_i = Q_\chi - \Phi, \quad (15)$$

where

$$\Phi = N_t^r \Big|_{r_h} = \frac{\kappa^2}{48\pi} = \frac{\pi}{12} T_H^2 \quad (16)$$

and  $T_H$  is the Hawking temperature

$$T_H = \frac{\kappa}{2\pi}. \quad (17)$$

Equation (16) is exactly the flux per partial mode that would result from a thermal distribution at the Hawking temperature in the full  $d$ -dimensional theory. That is, this flux is necessary and sufficient to restore general covariance at the quantum level, although we have not shown that the full spectrum is in fact thermal.

The above construction was extended to the most general static spherically symmetric metrics in [2, 6]. The method was also successfully implemented in a number of special cases [7, 11, 12, 14, 15, 16, 17, 21]. Notable among the special cases are [16], where the spacetime in question exhibits a global deficit solid angle, and [17, 21], which study radiation from cosmological horizons (as opposed to black hole horizons) in de Sitter black hole spacetimes. Moreover, the time-dependent spherically symmetric spacetime, which includes evaporating and accreting black holes, was also studied in [6] by use of the Vaidya metric. In this dynamical spacetime, deviations from the purely thermal (blackbody) flux were derived as expected. To our knowledge, this is the only direct calculation of the Hawking flux per partial wave in a time dependent case.

An important generalization of the method was presented by Iso, Umetsu, and Wilczek [3] to charged black holes by using the gauge anomaly as well as the gravitational anomaly. This was further extended to four-dimensional Kerr-Newman black holes in [4, 5]. In these cases of charged and rotating black holes, we must take into account the energy flow and the super-radiant gauge currents. Despite the lack of spherical symmetry in the case of four-dimensional Kerr black holes, the essential ingredient of reduction to an  $r$ - $t$  theory still works. The angular isometry generates an effective  $U(1)$  gauge field in the  $1+1$  theory, with the  $m$  quantum number serving as the charge of each partial wave. At this point the analysis of [3] goes through, and the known result is obtained with angular momentum acting like a chemical potential for the effective charge.

Furthermore, the method was extended to  $d$  dimensions in [8, 9, 10] for Myers-Perry and Myers-Perry-(A)dS black holes. In this case, each independent angular momentum becomes a factor in a  $U(1)^N$  product gauge group. Then the previous analysis goes through.

We have described the Robinson-Wilczek method above as described in [1]: outgoing modes are eliminated only near the horizon. We should stress that some authors [3, for example] eliminate modes only outside the horizon. Moreover, they eliminate the ingoing

modes, which are irrelevant at the classical level to physics outside the horizon. While this makes no difference in the simple case described here, using only the ingoing exterior modes was found to be essential when either gauge symmetries or rotation are present, as explained in [9].

In this essay, we have presented a view of Hawking radiation in which it is a consequence of cancelation of gravitational anomalies that arise from following the philosophy of effective field theory. The broad successes of this approach, as outlined above, are sufficient to declare it legitimate, but open questions remain:

- 1) The method has not been applied to black objects of non-spherical topology.
- 2) No proof exists that anomaly cancelation induces a thermal radiation spectrum, although important steps have been taken in this direction [13, 20].
- 3) We constructed the effective theory “by hand”. Could it be constructed directly by integrating out modes in the path integral?
- 4) The method has remarkable qualitative similarity to the near-horizon conformal field theory approaches used in [27, 28, 31, 32, 33]. What is the quantitative connection?
- 5) The method uses only spacetime kinematics, not dynamics. Thus it seems unlikely that it can be used to calculate black hole entropy, but the similarities to the methods of [27, 28, 31, 32, 33], which can calculate entropy via the Cardy formula, leave this possibility open.

*Note added:* References [18, 19] appeared shortly after completing this manuscript, addressing the first open question listed above. The Robinson-Wilczek method has now been successfully applied to five-dimensional black ring spacetimes, showing that the method continues to work for black objects of non-spherical horizon topology. Again, the near-horizon physics reduces to a 1 + 1 theory with a  $U(1)$  gauge symmetry arising from the ring’s angular momentum. This work is significant because the nonseparable coordinates typical to black ring spacetimes have impeded previous detailed study of their thermodynamics. The simplicity inherent to near-horizon physics as employed in the Robinson-Wilczek method may lead to further advances in black ring thermodynamics.

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