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Amicable matrices and orthogonal designs

Department of Mathematics and Computer Science

AMICABLE MATRICES AND ORTHOGONAL DESIGNS

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Dedication

To

my parents.

Abstract

This thesis is mainly concerned with the orthogonal designs of Baumert-Hall array type, $OD(4n;n,n,n,n)$ where $n=2k$, k is odd integer. For every odd prime power p^r, we construct an infinite class of amicable T-matrices of order $n=p^r+1$ in association with negacirculant weighing matrices $W(n,n-1)$. In particular, for $p^r \equiv 1 \pmod 4$, we construct amicable T-matrices of order $n \equiv 2 \pmod 4$ and application of these matrices allows us to generate infinite class of orthogonal designs of type $OD(4n;n,n,n,n)$ and $OD(4n;2,2,2n-2,2n-2)$ where $n=2k$; k is odd integer. For a special class of T-matrices of order n where each of T_i is a weighing matrix of weight $w_i; 1 \leq i \leq 4$ and Williamson-type matrices of order m, we establish a theorem which produces four circulant matrices in terms of four variables. These matrices are additive and can be used to generate a new class of orthogonal design of type $OD(4mn;w_1s,w_2s,w_3s,w_4s)$; where $s=4m$. In addition to this, we present some methods to find amicable matrices of odd order in terms of variables which have an interesting application to generate some new orthogonal designs as well as generalized orthogonal designs.

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Chapter 1

Introduction

While Hadamard conjecture has remained open for around a century, which methods would be the good ones for constructing new Hadamard matrices? The answer must include orthogonal designs, a well known combinatorial object. A special category of orthogonal designs is Baumert-Hall array which is an interesting area of study in the field of combinatorial mathematics. In this thesis, our endeavor results in a new infinite class of Baumert-Hall arrays, $OD(4n;n,n,n,n)$ where $n = 2k$; k is odd integer and some Baumert-Hall-Welch arrays with reduced parameter of order $4n;n$ is odd.

We cover necessary pre-requisite materials throughout the Chapters 2 to 4. In Chapter 2 which laid the foundation of our research, we study some fundamental properties of Hadamard matrices and some well known construction methods which include the Kronecker product construction, Paley construction [36] and Williamson construction [42]. Chapter 3 mainly focuses on some different (± 1) sequences having aperiodic auto-correlation function as well as periodic auto-correlation function zero. As T-sequences are important tools to construct Hadamard matrices and orthogonal designs, we include some basic rules and formulas [39] that generate this sequences from some other sequences such as Golay sequences [18], Base sequences and Turyn type sequences [41]. Chapter 4 is mostly concerned with balanced generalized weighing matrices with classical parameters over cyclic groups. This class of matrices includes conference matrices, weighing matrices, ω-circulant matrices and negacirculant matrices. In chapter 5, we mostly concentrate on amicable T-matrices of composite order. Using the idea of existence of negacirculant weighing matrices of order $p^r + 1$ [15] for odd prime power p^r, we construct an infinite class of amicable T-matrices of order $n \equiv 2 \pmod 4$ where $n = p^r + 1$. Also we present a multiplication theorem for amicable T-matrices which enables us to generate composite order T-matrices of larger length. Chapter 6 includes the application of amicable T-matrices which results in orthogonal design of type $OD(4n; n, n, n, n)$. Using Williamson-type matrices of order m and a special class of T-matrices of order n where each of T_i is a weighing matrix of weight w_i, a new multiplication theorem is established which results in four circulant matrices A, B, C, D in terms of variables a, b, c, d satisfying $AA^T + BB^T + CC^T + DD^T = (w_1sa^2 + w_2sb^2 + w_3sc^2 + w_4sd^2)I_{mn}$ where $s = 4m$. This matrices generates a new class of orthogonal design of type $OD(4sn; w_1m, w_2m, w_3m, w_4m)$. Additionally, sets of four circulant amicable matrices of odd order in terms of variables are constructed and generalized orthogonal design has been discussed as consequences of these matrices. Finally, we present a new method to construct a Hadamard matrix of order 48 found by standard constructions.

Chapter 2

Hadamard Matrices

Definition 2.1. A square matrix H of order n with the entries from the set $\{\pm 1\}$ satisfying the property

$$HH^T = nI_n$$

is known as a *Hadamard matrix*. More explicitly, the inner product of any two distinct rows or columns of this matrix is zero and all rows are linearly independent and all columns are linearly independent. A few examples of Hadamard matrix are given below.

Example 2.2. $H(2)$, $H(4)$ and $H(8)$ are the examples of Hadamard matrices of order 2, 4 and 8 respectively, where

$$H(2) = \begin{pmatrix} 1 & 1 \\ 1 & - \end{pmatrix}, \qquad H(4) = \begin{pmatrix} 1 & - & - & - \\ - & 1 & - & - \\ - & - & 1 & - \\ - & - & - & 1 \end{pmatrix},$$

and

$$H(8) = \begin{pmatrix} 1 & - & - & - & 1 & - & - & - \\ - & 1 & - & - & - & 1 & - & - \\ - & - & 1 & - & - & - & 1 & - \\ - & - & - & 1 & - & - & - & 1 \\ 1 & - & - & - & - & 1 & 1 & 1 \\ - & 1 & - & - & 1 & - & 1 & 1 \\ - & - & 1 & - & 1 & 1 & - & 1 \\ - & - & - & 1 & 1 & 1 & 1 & - \end{pmatrix}.$$

"$-$" stands for -1.

2.1 Equivalence of Hadamard Matrices

Though superficially Hadamard matrices seem to be a random arrangement of 1 and -1, its configuration admits some basic matrix operations which yield new Hadamard matrices of the same order. So from an existing Hadamard matrix one get many more of the same kind.

Definition 2.3. Two Hadamard matrices H_1 and H_2 of the same order are called *equivalent* if one can be obtained from the other by performing one or more of the following operations.

(i) Permutation of the rows,

(ii) Permutation of columns,

(iii) Multiplying some row or column by -1.

Since every row or column in a Hadamard matrix H is orthogonal to all others, it follows the transpose of a Hadamard matrix is Hadamard matrix as we show now. The basic requirement, $HH^T = nI_n$ leads us to write,

$$\begin{aligned}(H^T)(H^T)^T &= H^T H \\ &= (nH^{-1})(H) \\ &= nI_n.\end{aligned}$$

This proves:

Proposition 2.4. *If H is a Hadamard matrix then H^T is also Hadamard matrix.*

Among many of underlying features of Hadamard matrices, our interest is to present one of these that restricts all of the entries in the first row and first column to be the same kind.

Definition 2.5. If all of the entries in the first row and in the first column of a Hadamard matrix H are $+1$, then this Hadamard matrix is called a *normalized* Hadamard matrix.

Example 2.6. For $n = 4$, the following is a normalized $H(4)$.

$$H(4) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & - & - \\ 1 & - & 1 & - \\ 1 & - & - & 1 \end{pmatrix}.$$

In a normalized Hadamard matrix H of order $4n$, we see that there are exactly equal number of $+1$'s and -1's in every row and every column except the first row and the first column. Moreover, n number of -1's in any row(column) overlap with n number of -1's in any other row(column) except the first row(column).

Multiplication of any row or column in a Hadamard matrix by -1 yields a new Hadamard matrix. So one can easily transform an arbitrary Hadamard matrix into a normalized Hadamard matrix by negating the rows and columns whose leading entry is -1. This shows:

Lemma 2.7. *Every Hadamard matrix is equivalent to a normalized Hadamard matrix.*

Example 2.8. Here $H(4)$ is equivalent to $H'(4)$ which is normalized.

$$H(4) = \begin{pmatrix} - & 1 & 1 & 1 \\ 1 & - & 1 & 1 \\ 1 & 1 & - & 1 \\ 1 & 1 & 1 & - \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & - & - & - \\ 1 & - & 1 & 1 \\ 1 & 1 & - & 1 \\ 1 & 1 & 1 & - \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & - & - \\ 1 & - & 1 & - \\ 1 & - & - & 1 \end{pmatrix} = H'(4).$$

As equivalence operations on a Hadamard matrix generate a class of new Hadamard matrices, the number of inequivalent Hadamard matrices of a particular order is a good topic of study. Searching those is very time consuming and more computational work. H. Kharaghani and Tayfeh-Rezaie [30] were able to classify the highest order Hadamard matrices of order 32. The following list presents the number of inequivalent Hadamard matrices corresponding to all orders found up to the present.

Table 2.1: No. of inequivalent Hadamard matrices

Order (n) [Ref.]	No. of matrices
1	1
2	1
4	1
8	1
12	1
16 [20]	5
20 [21]	3
24 [31]	60
28 [32]	487
32 [30]	13710027

2.2 Existence of Hadamard Matrices

In 1867 [40], J. J. Sylvester was the first person to study (± 1) matrices which would later be known as Hadamard matrices. After around 25 years, J. Hadamard was trying to find the maximal determinant of a matrix with entries from the complex unit disk and eventually found two (± 1) square matrices of orders 12 and 20 as a solution of the problem. Though these solution matrices did not fit with matrices found by J. J. Sylvester, the internal structure of both matrices were same. Later in 1893, J. Hadamard pointed out a hypothesis about the existence of these solution matrices and this hypothesis became known as the Hadamard conjecture. Now we recall a theorem from Sylvester's paper, published in 1867 [40] which allows the recursive use of a existing Hadamard matrix of order n to construct a new Hadamard matrix of order $2n$. An immediate corollary of this theorem generates a class of Hadamard matrice of order $2^t; t \geq 0$.

Theorem 2.9. *[40] If H is a Hadamard matrix then*

$$\begin{pmatrix} H & H \\ H & -H \end{pmatrix}$$

is also a Hadamard matrix.

Starting with $H = \begin{pmatrix} 1 \end{pmatrix}$ and iterating this theorem we obtain

Corollary 2.10. *There exist a Hadamard matrix of order $2^t; t \geq 0$.*

2.2.1 Hadamard Conjecture

Conjecture 2.11. There exist a Hadamard matrix of order $4n$ for all $n \in \mathbb{N}$.

This conjecture remains open since the last century. Solutions of some different open problems in combinatorial design theory depend on a positive reply to this conjecture.

Considering this conjecture as a sufficient condition for the existence of Hadamard matrices we prove here a necessary condition for the existence of a Hadamard matrix.

Proposition 2.12. *If there exist a Hadamard matrix of order n, then* $n = 1,2$ *or* $4k; k \in \mathbb{N}$.

Proof. Suppose H is Hadamard matrix of order n and H' is a normalized Hadamard matrix corresponding to H. Decompose H' into four blocks p,q,r,s and permute the columns of H' in such a way that for the first three rows, each column in block p is $(1,1,1)$, in block q is $(1,1,-)$, in block r is $(1,-,1)$ and in block s is $(1,-,-)$. So H' has form:

$$\left(\begin{array}{ccccc|ccccc|ccccc|ccccc}
1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\
1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & - & - & \cdots & - & - & - & - & \cdots & - & - \\
1 & 1 & \cdots & 1 & 1 & - & - & \cdots & - & - & 1 & 1 & \cdots & 1 & 1 & - & - & \cdots & - & - \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}\right)$$

Since H' is Hadamard matrix, taking the orthogonality property of distinct rows into account we have

$$\begin{aligned}
p+q+r+s &= n \\
p+q-r-s &= 0 \\
p-q+r-s &= 0 \\
p-q-r+s &= 0
\end{aligned}$$

A simple calculation yields the unique solution of this system of equations i.e, $p = q = r = s = n/4$. Here n must be multiple of 4 because p,q,r,s and n are all integral. $\square$

2.3 Construction of Hadamard Matrices

There are many existence results of Hadamard matrices which include different sub-classes. In this section, we present some well known construction methods and start here with some necessary definitions.

Definition 2.13. For a prime number p, a non-zero element in a Galois field, $a \in \mathbb{F}_p = \{1, 2, \cdots, p-1\}$ is said to be *quadratic residue* if

$$x^2 \equiv a \pmod{p}$$

for any $x \in \mathbb{F}_p$. Otherwise it is called *non-quadratic residue*.

Definition 2.14. If a is any quadratic residue $\pmod{p}$, then the *Legendre symbol* χ [36] is defined by

$$\chi(a) = \begin{cases} 0, & \text{if a} \equiv 0 \pmod{p}\ ; \\ 1, & \text{if a is quadratic ;} \\ -1, & \text{otherwise.} \end{cases}$$

Definition 2.15. The *Kronecker product* of a $m \times n$ matrix $A = [a_{ij}]$ and a $m' \times n'$ matrix $B = [b_{ij}]$ is a $mm' \times nn'$ block matrix given by

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}$$

Definition 2.16. The *Hadamard product* of a $m \times n$ matrix, $A = [a_{ij}]$ and a $m' \times n'$ matrix, $B = [b_{ij}]$ is $m \times n$ matrix, given by

$$A * B = [a_{ij}b_{ij}]$$

Definition 2.17. For a given n-dimensional vector $V = (v_0, v_1, \ldots, v_{n-1})$, a square matrix $A = [a_{ij}]$ obtained by circulating this row vector, whose entries are given by

$$a_{ij} = v_{j-i}; (j-i) \pmod n$$

is called *circulant* matrix and is denoted by $circ(v_0, v_1, \ldots, v_{n-1})$.

2.3.1 Kronecker Product Construction

The earliest method, allowing the recurrence use of existing Hadamard matrices to construct new Hadamard matrices is the *Kronecker product construction*. In 1893, J. Hadamard used this method to produce a new Hadamard matrix from old ones.

Theorem 2.18. *If H_1 and H_2 are two Hadamard matrices of order m and n respectively, then $H_1 \otimes H_2$ is a Hadamard matrix of order mn.*

Proof. Using a fundamental property of Kronecker product, $(H_1 \otimes H_2)^T = H_1^T \otimes H_2^T$, we have

$$\begin{aligned}
(H_1 \otimes H_2)(H_1 \otimes H_2)^T &= H_1 H_1^T \otimes H_2 H_2^T \\
&= mI_m \otimes nI_n \\
&= mnI_{mn}
\end{aligned}$$

which proves the theorem. □

Though Hadamard's direct *Kronecker product* construction method is very simple to generate Hadamard matrices of many new orders, it skips many orders in between the order of new and old Hadamard matrices. In a lecture note published in 1985, Agayan-Sharukhanyan [1] introduced a different way to use Kronecker products which allows us to retrieve a Hadamard matrix of order reduced by half of the order Hadamard matrix constructed by direct Kronecker product method.

Theorem 2.19. *[1] Let H_1 and H_2 be Hadamard matrices of order $4h$ and $4k$ respectively. Then there is a Hadamard matrix of order $8hk$.*

Proof. Decompose H_1 and H_2 into two 2×2 arrays

$$H_1 = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \text{ and } H_2 = \begin{pmatrix} K & L \\ M & N \end{pmatrix}.$$

Since both H_1 and H_2 are Hadamard matrices, we have

$$\begin{aligned} PP^T + QQ^T &= 4hI_{2h} & KK^T + LL^T &= 4kI_{2k} \\ RR^T + SS^T &= 4hI_{2h} & MM^T + NN^T &= 4kI_{2k} \\ PR^T + QS^T &= 0 & KM^T + LN^T &= 0 \\ RP^T + SQ^T &= 0 & MK^T + NL^T &= 0 \end{aligned}$$

Let

$$\begin{aligned} \alpha &= \tfrac{1}{2}(P+Q)\otimes K + \tfrac{1}{2}(P-Q)\otimes M \\ \beta &= \tfrac{1}{2}(P+Q)\otimes L + \tfrac{1}{2}(P-Q)\otimes N \\ \gamma &= \tfrac{1}{2}(R+S)\otimes K + \tfrac{1}{2}(R-S)\otimes M \\ \delta &= \tfrac{1}{2}(R+S)\otimes L + \tfrac{1}{2}(R-S)\otimes N \end{aligned}$$

Now we define $H = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$. To prove H is a Hadamard matrix we compute the following quantities.

$$\begin{aligned} \alpha\alpha^T + \beta\beta^T &= \frac{1}{4}[((P+Q)\otimes K + (P-Q)\otimes M)((P+Q)^T \otimes K^T + (P-Q)^T \otimes M^T)) \\ &\quad +((P+Q)\otimes L + (P-Q)\otimes N)((P+Q)^T \otimes L^T + (P-Q)^T \otimes N^T)] \\ &= \frac{1}{4}[(4hI_{2h} + PQ^T + QP^T)\otimes 4kI_{2k} + (4hI_{2h} - PQ^T - QP^T)\otimes 4kI_{2k}] \\ &= \frac{1}{4}[8hI_{2h}\otimes 4kI_{2k}] \\ &= 8hkI_{4hk}. \end{aligned}$$

and

$$\gamma\alpha^T + \delta\beta^T = \frac{1}{4}[((R+S)\otimes K + (R-S)\otimes M)((P+Q)^T \otimes K^T + (P-Q)^T \otimes M^T) +$$

$$
\begin{aligned}
& ((R+S)\otimes L+(R-S)\otimes N)((P+Q)^T\otimes L^T+(P-Q)^T\otimes N^T). \\
= \; & \frac{1}{4}[(RQ^T+SP^T)\otimes 4kI_{2k}+(-RQ^T-SP^T)\otimes 4kI_{2k}] \\
= \; & 0.
\end{aligned}
$$

Similarly, $\gamma\gamma^T+\delta\delta^T=8hkI_{4hk}$ and $\alpha\gamma^T+\beta\delta^T=0$. Therefore, we have

$$HH^T=\begin{pmatrix} 8hkI_{4hk} & 0 \\ 0 & 8hkI_{4hk} \end{pmatrix}=8hkI_{8hk}$$

i.e, H is a Hadamard matrix of order $8hk$. □

R. Craigen, J. Seberry and X. M Zhang gave a theorem in 1992 [9] which might be regarded as an extension of Theorem 2.19. This theorem allows us to compute a Hadamard matrices whose order is one fourth of the order of Hadamard matrices computed by using theorem 2.19. Now we present this theorem.

Theorem 2.20. *[9] If there exist Hadamard matrices of orders $4m,4m,4p,4q$, then there exist a Hadamard matrices of order $16mnpq$.*

Proof. Let H,K,L,M be four Hadamard matrices of order $4m,4n,4p,4q$ respectively. Decomposing these matrices into four blocks we write

$$H=\begin{pmatrix} H_1 \\ \hline H_2 \\ \hline H_3 \\ \hline H_4 \end{pmatrix} K=\begin{pmatrix} K_1 \\ \hline K_2 \\ \hline K_3 \\ \hline K_4 \end{pmatrix} L=\begin{pmatrix} L_1 \\ \hline L_2 \\ \hline L_3 \\ \hline L_4 \end{pmatrix} M=\begin{pmatrix} M_1 \\ \hline M_2 \\ \hline M_3 \\ \hline M_4 \end{pmatrix}$$

where $j\times 4j$ is the size of the blocks H_i,K_i,L_i,M_i for $1\le i\le 4$ and $j\in\{m,n,p,q\}$. Here we note that $\sum_{i=1}^{4}H_iH_i^T=HH^T$ and $H_iH_j^T=0; i\ne j$ which is also similarly applicable for K,L and M. We define

$$R \;=\; \frac{1}{2}(H_1+H_2)^T\otimes K_1+\frac{1}{2}(H_1-H_2)^T\otimes K_2$$

$$\begin{aligned}
S &= \frac{1}{2}(H_3+H_4)^T \otimes K_3 + \frac{1}{2}(H_3-H_4)^T \otimes K_4 \\
U &= \frac{1}{2}(L_1+L_2)^T \otimes M_1 + \frac{1}{2}(L_1-L_2)^T \otimes M_2 \\
V &= \frac{1}{2}(L_3+L_4)^T \otimes M_3 + \frac{1}{2}(L_3-L_4)^T \otimes M_4
\end{aligned}$$

Now compute

$$\begin{aligned}
RR^T + SS^T &= \frac{1}{2}(H_1H_1^T + H_2H_2^T + H_3H_3^T + H_4H_4^T) \otimes 4nI_n \\
&= \frac{1}{2}(4mI_{4m} \otimes 4nI_n) \\
&= 8mnI_{4mn}.
\end{aligned}$$

and a simple calculation yields $SR^T = RS^T = UV^T = VU^T = 0$. Here R,S,U,V are all (± 1) matrices. Suppose $X = \frac{1}{2}(R+S)$ and $Y = \frac{1}{2}(R-S)$ and then we obtain,

$$\begin{aligned}
XX^T &= \frac{1}{4}(RR^T + SS^T + SR^T + RS^T) \\
&= 2mnI_{4mn}
\end{aligned}$$

Similarly $YY^T = 2mnI_{4mn}$.

Clearly X,Y are disjoint $(0,\pm 1)$ matrices. Now defining $H = X \otimes U + Y \otimes V$ we get

$$\begin{aligned}
HH^T &= (X \otimes U + Y \otimes V)(X \otimes U + Y \otimes V)^T \\
&= (XX^T \otimes UU^T + YY^T \otimes VV^T) \\
&= 2mnI_{4mn} \otimes (UU^T + VV^T) \\
&= 16mnpqI_{16mnpq}
\end{aligned}$$

Therefore, H is a Hadamard matrices of order $16mnpq$. $\square$

2.3.2 Paley Construction

Aside from Hadamard and Sylvester work, Paley's work is well known for finding Hadamard matrices of order $4n$ where n is odd. In 1933 [36], Paley constructed an infinite class of Hadamard matrices of order $(p+1)$ and $2(p+1)$ according as $p \equiv 3 \pmod 4$ and $p \equiv 1 \pmod 4$. To present these theorems with immediate examples we start here by giving the definition of Jacobsthal matrix.

Definition 2.21. Let p be an odd prime power and χ be the Legendre symbol [36] over the Galois field $GF(p) = \{a_1, a_2, \ldots, a_{p-1}$. Then a $p \times p$ matrix, $Q = [p_{ij}]$ with the entries given by

$$p_{ij} = \chi(i-j); (i-j) \pmod p$$

is known as *Jacobsthal matrix*.

Theorem 2.22. *[36] Let $p \equiv 3 \pmod 4$ be an odd prime power. Then*

$$H = \begin{pmatrix} 1 & j \\ j^T & Q-I \end{pmatrix}$$

is a Hadamard matrix of order $p+1$, where j is the all ones vector of length p and I is the identity matrix of order p.

Example 2.23. Over Z_7; 1, 2, 4 are non-zero squares and 3, 5, 6 are non squares. Then we have the following Hadamard matrix of order 8.

$$H(8) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & - & - & 1 & - & 1 & 1 \\ 1 & 1 & - & - & - & 1 & - & 1 \\ 1 & 1 & 1 & - & - & - & 1 & - \\ 1 & - & 1 & 1 & - & - & - & 1 \\ 1 & 1 & - & 1 & 1 & - & - & - \\ 1 & - & 1 & - & 1 & 1 & - & - \\ 1 & - & - & 1 & - & 1 & 1 & - \end{pmatrix}$$

These type of Hadamard matrices are known an Paley-type Hadamard matrices.

Theorem 2.24. *[36] Let* $p \equiv 1 \pmod 4$ *be an odd prime power. Then*

$$H = \left(\begin{array}{cc|cc} 1 & j & -1 & j \\ j^T & Q+I & j^T & Q-I \\ \hline -1 & j & -1 & -j \\ j^T & Q-I & -j^T & -Q-I \end{array}\right)$$

is a Hadamard matrix of order $2(p+1)$ *where* j *is the all ones vector of length* p *and* I *is the identity matrix of order* p*.*

Example 2.25. Over Z_5, 1, 2 are squares and 3, 4 are non-squares. Then we have the following Hadamard matrix of order 12.

$$H(12) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & - & - & 1 & 1 & - & 1 & - & - & 1 \\ 1 & 1 & 1 & 1 & - & - & 1 & 1 & - & 1 & - & - \\ 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & - & 1 & - \\ 1 & - & - & 1 & 1 & 1 & 1 & - & - & 1 & - & 1 \\ 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & 1 & - \\ - & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - \\ 1 & - & 1 & - & - & 1 & - & - & - & 1 & 1 & - \\ 1 & 1 & - & 1 & - & - & - & - & - & - & 1 & 1 \\ 1 & - & 1 & - & 1 & - & - & 1 & - & - & - & 1 \\ 1 & - & - & 1 & - & 1 & - & 1 & 1 & - & - & - \\ 1 & 1 & - & 1 & 1 & - & - & - & 1 & 1 & - & - \end{pmatrix}$$

2.3.3 Williamson Construction

In 1944, Williamson found a class of Hadamard matrices by plugging a set of typical matrices into a pre-designed array, called Williamson array. This method is very simple to construct Hadamard matrices while the challenge is to find those typical plugging matrices of order $t \in \mathbb{N}$. Just before turning to this construction, we recall the definition of these plugging matrices.

Definition 2.26. A set of $n \times n$ matrices $\{A, B, C, D\}$ satisfying the following conditions

(i) all are symmetric and circulant with entries from the set $\{\pm 1\}$,

(ii) $AA^T + BB^T + CC^T + DD^T = 4nI_n$.

are called *Williamson matrices*. However, if $\{A, B, C, D\}$ is pairwise amicable for A, B, C, D and $AA^T + BB^T + CC^T + DD^T = 4nI_n$ then these are known as Williamson-type matrices.

Williamson matrices are known for all orders $n \leq 63$ [7] (section V) while for $n \in \{35, 47, 53, 59\}$ there are no Williamson matrices [24].

Theorem 2.27. *[42] If there exist $n \times n$ Williamson matrices A, B, C and D then there exist a Hadamard matrix of order $4n$ given by*

$$\begin{pmatrix} A & B & C & D \\ -B & A & D & -C \\ -C & -D & A & B \\ -D & C & -B & A \end{pmatrix}$$

Example 2.28. Picking Williamson matrices as $A = circ(1, -, -, -, -)$, $B = circ(1, -, -, -, -)$, $C = circ(1, 1, -, -, 1)$ and $D = circ(1, -, 1, 1, -)$ we get a Hadamard matrix of order 20.

$$
\left(\begin{array}{ccccc|ccccc|ccccc|ccccc}
1 & - & - & - & - & - & 1 & 1 & 1 & 1 & - & - & 1 & 1 & - & - & 1 & - & - & 1 \\
- & 1 & - & - & - & 1 & - & 1 & 1 & 1 & - & - & - & 1 & 1 & 1 & - & 1 & - & - \\
- & - & 1 & - & - & 1 & 1 & - & 1 & 1 & 1 & - & - & - & 1 & - & 1 & - & 1 & - \\
- & - & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & 1 & - & - & - & - & - & 1 & - & 1 \\
- & - & - & - & 1 & 1 & 1 & 1 & 1 & - & - & 1 & 1 & - & - & 1 & - & - & 1 & - \\
\hline
1 & - & - & - & - & 1 & - & - & - & - & - & 1 & - & - & 1 & 1 & 1 & - & - & 1 \\
- & 1 & - & - & - & - & 1 & - & - & - & 1 & - & 1 & - & - & 1 & 1 & 1 & - & - \\
- & - & 1 & - & - & - & - & 1 & - & - & - & 1 & - & 1 & - & - & 1 & 1 & 1 & - \\
- & - & - & 1 & - & - & - & - & 1 & - & - & - & 1 & - & 1 & - & - & 1 & 1 & 1 \\
- & - & - & - & 1 & - & - & - & -1 & 1 & - & - & - & - & 1 & - & - & 1 & 1 & \\
\hline
1 & 1 & - & - & 1 & 1 & - & 1 & 1 & - & 1 & - & - & - & - & - & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & - & - & - & 1 & - & 1 & 1 & - & 1 & - & - & - & 1 & - & 1 & 1 & 1 \\
- & 1 & 1 & 1 & - & 1 & - & 1 & - & 1 & - & - & 1 & - & - & 1 & 1 & - & 1 & 1 \\
- & - & 1 & 1 & 1 & 1 & 1 & - & 1 & 1 & - & - & - & 1 & - & 1 & 1 & 1 & - & 1 \\
1 & - & - & 1 & 1 & 1 & 1 & 1 & - & 1 & - & - & - & - & 1 & 1 & 1 & 1 & 1 & - \\
\hline
1 & - & 1 & 1 & - & - & - & 1 & 1 & - & 1 & - & - & - & - & 1 & - & - & - & - \\
- & 1 & - & 1 & 1 & - & - & - & 1 & 1 & - & 1 & - & - & - & - & 1 & - & - & - \\
1 & - & 1 & - & 1 & 1 & - & - & - & 1 & - & - & 1 & - & - & - & - & 1 & - & - \\
1 & 1 & - & 1 & - & 1 & 1 & - & - & - & - & - & - & 1 & - & - & - & - & 1 & - \\
- & 1 & 1 & - & 1 & - & 1 & 1 & - & - & - & - & - & - & 1 & - & - & - & - & 1
\end{array}\right)
$$

Hadamard matrices obtained in this way are known as *Williamson-type Hadamard matrices*. Employing this idea Baumert et.al. showed the existence of Hadamard matrices of order 92 [4] and order 116 [2] which were unknown in the list of Hadamard matrices of order less than 200 made by Paley. In addition to these methods, the Baumert Hall array is another powerful tool for constructing Hadamard matrices which we cover in Chapter 6.

Chapter 3

Sequences

As far as study of different combinatorial objects like orthogonal designs and Hadamard matrices are concerned, a set of sequences with commuting variables are of great interest. Due to different generic properties, these sequences have versatile applications in computational mathematics, infrared multislit spectrometry [17], optical time-domain reflectometry or orthogonal frequency division multiplexing (OFDM) [37] etc. These sequences include Golay sequences, Base sequences, Turyn sequences, T-sequences etc. In this chapter we study T-sequences among others.

3.1 T-Sequences

We start this section with some trivial properties of sequences.

Definition 3.1. If $X = \{x_{11}, \dots, x_{1n}\}$ is a sequence in commuting variables and is of length n, then its *aperiodic auto-correlation function* is defined by

$$N_X(j) = \sum_{i=1}^{n-j}(x_{1,i}x_{1,i+j});\ 1 \le j \le n-1.$$

For a set of sequences $Y = \{\{x_{11}, \dots, x_{1n}\}, \{x_{21}, \dots, x_{2n}\}, \dots, \{x_{m1}, \dots, x_{mn}\}\}$ of length n we define,

$$N_Y(j) = \sum_{i=1}^{n-j}(x_{1,i}x_{1,i+j} + x_{2,i}x_{2,i+j} + \cdots + x_{m,i}x_{m,i+j}); 1 \le j \le n-1.$$

Similarly, the *periodic auto-correlation function* is defined by

$$P_Y(j) = \sum_{i=1}^{n}(x_{1,i}x_{1,i+j} + x_{2,i}x_{2,i+j} + \cdots + x_{m,i}x_{m,i+j}); 1 \le j \le n-1.$$

where the second subscript is from the set of residues $\pmod n$.

Definition 3.2. A set of sequences having aperiodic auto-correlation function zero is called *complementary sequences* (CS).

Here we note that the number of non zero entries in a sequence is called the weight of that sequence and if at most one sequence is non zero at any position among all these sequences then these sequences are called *disjoint sequences* (DS).

Example 3.3. A and B are disjoint complementary sequences of length 8.

$$A = (11000000)$$
$$B = (001-11-1)$$

where "$-$" stands for -1.

Definition 3.4. Four disjoint sequences $M = \{T_1, T_2, T_3, T_4\}$ with entries from the set $\{0, \pm 1\}$ and of length n satisfying

$$N_M(j) = 0; 1 \leq j \leq n-1$$

are called T-sequences (TS).

Example 3.5. Here T_1, T_2, T_3 and T_4 are T-sequences of length 9.

$$T_1 = 100000000, \qquad T_2 = 011000000,$$
$$T_3 = 0001-11-1, \qquad T_4 = 000000000.$$

These T-sequences have a close relation with many other sequences having aperiodic auto-correlation function zero. In the next few sections, we study some of these sequences.

3.2 Golay Sequences

Definition 3.6. Two (± 1) sequences $\{X = \{x_{11}, \cdots, x_{1n}\}, Y = \{y_{11}, \cdots, y_{1n}\}\}$ of length n satisfying,

$$N_X(j) + N_Y(j) = 0 \text{ for } 1 \leq j \leq n-1$$

are called Golay sequences. Sometimes X, Y are also referred to as a Golay pair.

Example 3.7. $X = 111-$ and $Y = 11-1$ are the Golay sequences of length 4.

Golay originally found four pairs of sequences of lengths 2, 10 and 26 which include one pair of length 2, two pairs of length 10 and one pair of length 26. Now we present these four pairs of sequences.

Table 3.1: Original Golay-sequences

Order(n)	Sequences
2	$(11);(1-)$.
10	$(-11-1-111-),(-111111--1)$. $(1-1-1111--),(1111-11--1)$.
26	$(1-11--1----1-1----11---1-1)$, $(-1--11-1111--------11---1-1)$.

These sequences have many properties some of which we present here.

3.2.1 Symmetry Properties

The symmetry properties are one of the most important characteristics of Golay sequences. This properties permits one or more of the following operations which were originally given by Golay [18].

(i) Reversing either or both sequences.

(ii) Negating either or both sequences.

(iii) Transforming by $x_i \to (-1)^i x_i, y_i \to (-1)^i y_i$.

(iv) Interchanging the sequences in the pair.

3.2.2 Other Properties

Due to versatile importance in many engineering applications, many people are interested in characterizing Golay sequences. Here we mention some characteristics of Golay sequences found by different authors [15], [18], [41].

(i) $\sum_{i=1}^{n-j}(x_i x_{i+j} + y_i y_{i+j}) = 0$; for $1 \leq j \neq 0 \leq n-1$.

(ii) They exist for all lengths $n \in 2^a 10^b 26^c; a,b,c \geq 0$ [41]. No Golay sequences are known to exist for other than these lengths.

(iii) Length n is even and is the sum of two squares [18].

(iv) $x_{n-i+1} = e_i x_i \Leftrightarrow y_{n-i+1} = -e_i y_i$ where $e_i = \pm 1$ [15] .

S. Eliahou, M. Kervaire and B. Saffari's [12] outstanding work is very helpful to study the non existence of Golay sequence. They showed that there are no Golay sequences of length n having a prime factor $p \equiv 3 \pmod 4$. This work also includes the works of M. Griffin [19] and C. Koukouvinos et.al. [35] where they proved that Golay sequences do not exist for the lengths $2 \cdot 9^t$ and $2 \cdot 49^t$ respectively where t is a positive integer. In order to prove this theorem here, we recall first the idea of Hall polynomial [7] and then a lemma whose proof is given in the same paper of S. Eliahou et.al. [12].

Definition 3.8. If $A = (a_0, a_1, \cdots, a_{n-1})$ is a sequence of length n, then its associated Hall polynomial is defined as

$$A(x) = \sum_{i=0}^{n-1} a_i x^i.$$

For a given Hall polynomial $A(x)$, we define $A(x^{-1}) = \sum_{i=0}^{n-1} a_i x^{-i}$ and $|A|^2 = A(x)A(x^{-1})$. Then we get

$$|A|^2 = \left(\sum_{i=0}^{n-1} a_i x^i\right)\left(\sum_{i=0}^{n-1} a_i x^{-i}\right)$$

$$= \sum_{i=0}^{n-1} a_i^2 + \sum_{j=1}^{n-1} \sum_{k=0}^{n-1-j} a_k a_{k+j} (x^j + x^{-j})$$
$$= \sum_{k=0}^{n-1} a_k^2 + \sum_{k=0}^{n-1} N(k)(x^k + x^{-k})$$

where $N(k)$ is the aperiodic auto-correlation function. So for a (± 1) sequence A of length n we have, $|A|^2 = n + \sum_{k=0}^{n-1} N(k)(x^k + x^{-k})$ and for a pair of (± 1)-complementary sequences A and B of length n, we get $|A|^2 + |B|^2 = 2n$.

Lemma 3.9. *[12] Let p be a prime number satisfying* $p \equiv 3 \pmod 4$. *Let N be a positive integer* ζ *be the primitive* p^N*-th root of unity. Suppose* α, β *satisfy the condition*

$$\alpha \cdot \overline{\alpha} + \beta \cdot \overline{\beta} \in p^2 \mathbb{Z}[\zeta]$$

where bar denotes the complex conjugation. Then $\alpha, \beta \in p\mathbb{Z}[\zeta]$.

Theorem 3.10. *[12] There are no Golay sequences of length n, divisible by a prime factor p of the form* $p = 4k + 3$.

Proof. Assume that P and Q are Golay sequences of length n. If $P(x)$ and $Q(x)$ are the corresponding Hall polynomials then we must have

$$P(x)P(x^{-1}) + Q(x)Q(x^{-1}) = 2n$$

in the Laurent polynomial ring $\mathbb{Z}[x, x^{-1}]$. Putting $x = 1$, we obtain

$$P(1)^2 + Q(1)^2 = 2n$$

i.e $2n$ is the sum of two integral squares which implies that if n is divisible by a prime $p \equiv 3 \pmod 4$, then n must be divisible by p^2. Indeed, $P(1)^2 + Q(1)^2 = 0 \pmod{p\mathbb{Z}}$ is possible only when $P(1) \equiv Q(1) \equiv 0 \pmod{p\mathbb{Z}}$. Then p^2 must divide $P(1)^2 + Q(1)^2$.
Now by establishing a contradiction with the assumption $n \equiv 0 \pmod{p^2}; p \equiv 3 \pmod 4$, we prove the theorem.
Suppose $q = p^N; N$ is large and ζ be a primitive q'th root of unity. Then we obtain

$$P(\zeta)\overline{P(\zeta)} + Q(\zeta)\overline{Q(\zeta)} = 2n \in p^2\mathbb{Z}[\zeta]$$

where $P(\zeta^{-1}) = P(\overline{\zeta}) = \overline{P(\zeta)}$; $\overline{\zeta}$ is the complex conjugation of ζ.

Therefore, using the lemma above, we can write $P(\zeta), Q(\zeta) \in p\mathbb{Z}[\zeta]$. However,

$$P(\zeta) = a_0 + a_1\zeta + \cdots + a_{n-1}\zeta^{n-1} \text{ where } a_i = \pm 1; \text{ for } 0 \leq i \leq n-1.$$

Now, for prime power $q = p^N$, let $\Phi_q(X) = \sum_{i=0}^{p-1} X^{i(p^{N-1})}$ is a cyclotomic polynomial over $Z[X]$ and so $\mathbb{Z}[\zeta]$ is a free Z-module whose rank is $\Phi = \phi(p^N) = p^{N-1}(p-1)$. If $P(\zeta) \in p\mathbb{Z}[\zeta]$ then there exist integers $s_0, s_1, \ldots, s_{n-1} \in \mathbb{Z}$ such that

$$P(\zeta) = \sum_{i=0}^{n-1} a_i\zeta^i = \sum_{j=0}^{\phi(q)-1} ps_j\zeta^j \tag{3.1}$$

Assume that N is big enough and satisfies the inequality $n < \phi = \phi(p^N) = p^{N-1}(p-1)$. Since $\{1, \zeta, \zeta^2, \cdots, \zeta^{\phi-1}\}$ form a basis over $\mathbb{Z}$, we deduce from Equation 3.1

$$a_i = ps_i; 0 \leq i \leq n-1$$

which is impossible since $a_i = \pm 1$. Hence the theorem follows. □

To construct new Golay sequences, one way is to construct them from already existing Golay sequences. Before shifting into this construction we review some basic operations on sequences.

For a pair of sequences $X = (x_1, x_2, x_3, ..., x_m)$ and $Y = (y_1, y_2, y_3, ..., y_m)$,

(i) *Concatenation* is defined as:

$$X|Y = (x_1, x_2, ..., x_m, y_1, y_2, ..., y_m).$$

(ii) *Interleaving* is defined as:

$$X \sim Y = (x_1, y_1, x_2, y_2, ..., x_m, y_m).$$

If $X = (x_1, x_2, x_3, ..., x_m)$ and $Y = (y_1, y_2, y_3, ..., y_m)$ are Golay sequences of length n then $(X|Y, X|(-Y))$ and $(X \sim Y, X \sim (-Y))$ are Golay sequences of length $2n$ [27]. Golay showed how to double the length of existing Golay sequences, which is known as Golay's direct construction method [18]. Using this method one can compute a Golay pair of length $2n$ from a existing pair of length n. These sequences are very helpful to construct a set of four complementary sequences. Here we mention a lemma for such a construction.

Lemma 3.11. *If $X;Y$ are Golay sequences of length n_1 and $U;V$ are Golay sequences of length n_2, then $A = (X|U)$, $B = (X|-U)$, $C = (Y|V)$ and $D = (Y|-V)$ are four complementary sequences of length $(n_1 + n_2)$.*

Since our interest is the construction of T-sequences, we mention here a theorem that generates T-sequences from Golay sequences.

Theorem 3.12. *If X and Y are Golay sequences of lengths n, and if ϕ_n denotes the vector of n zeroes then $T = \{\{1, \phi_n\}, \{\phi, \frac{1}{2}(X+Y)\}, \{\phi, \frac{1}{2}(X-Y)\}, \{\phi_{n+1}\}\}$ are T-sequences of length $n+1$.*

Proof. Obvious. □

See table 3.2 for examples.

Table 3.2: T-sequences (TS) from Golay-sequences (GS)

GS of length (n)	TS length (n+1)
$11, 1-$	$100, 010, 001, 000.$
$111-, 11-1$	$10000, 01100,$ $0001-, 00000.$
$111-11-1, 111---1-$	$100000000, 0111-0000,$ $0000011-1, 000000000.$

3.3 Base Sequences

Definition 3.13. If $\{X_1, X_2, X_3, X_4\}$ are all $\{\pm 1\}$ sequences of length $(m+p), (m+p), m, m$ respectively, where p is an odd and if the aperiodic auto-correlation function, is zero then the sequences X_i's; $1 \leq i \leq 4$; form *Base sequences* (BS).

Here we list some Base sequences of odd lengths.

Table 3.3: Base-sequences of odd lengths

Lengths	Sequences
$3,3,2,2$	$111, 11-, 1-, 1-.$
$5,5,4,4$	$11-11, 1111-, 11--, 1-1-$.
$7,7,6,6$	$11-1-11, 11---1-, -111111, --11-1.$
$9,9,8,8$	$1111-11-1, -111-11-1, 111---1-, 111---1-.$

One can regard Golay sequences as appropriate tools to construct Base sequences. We see, if X, Y are Golay sequences of length n, then the set $\{\{1, X\}, \{1, -X\}, \{Y\}, \{Y\}\}$ forms Base sequences of length $(n+1), (n+1), n$ and n respectively. Base sequences exist for all lengths with $m+p; m \in \{1, \ldots, 30\} \cup 2^a 10^b 26^c$ and $p = 1$.
Base sequences are also effective tools to find four complementary sequences(CS) as well as T-sequences. To illustrate this we state the following theorem from the monograph by J. Seberry et.al. [39].

Theorem 3.14. *If there are Base sequences of lengths $m+1, m+1, m, m$, there are*

(i) 4 T-sequences of length $(2m+1)$.

(ii) 4 complementary sequences of length $(2m+1)$.

Proof. Suppose $A = \{X, Y, Z, W\}$ are Base sequences of lengths $(m+1), (m+1), m, m$. Then the set of sequences defined as:

$$\{\{X,W\},\{X,-W\},\{Y,Z\},\{Y,-Z\}\}$$

are 4-complementary sequences of length $(2m+1)$ and the set of sequences defined as:

$$\{\{\tfrac{1}{2}(X+Y),\phi_m\},\{\tfrac{1}{2}(X-Y),\phi_m\},\{\phi_{m+1},\tfrac{1}{2}(W+Z)\},\{\phi_{m+1},\tfrac{1}{2}(W-Z)\}\}$$

are T-sequences of length $(2m+1)$. □

Table 3.4 lists examples.

Table 3.4: T-sequences and Complementary sequences from Base sequences

BS of lengths $m+1,m+1,m,m$	TS of length $(2m+1)$	CS of length $(2m+1)$
$111,11-,1-,1-.$	$11000,00100,0001-,00000.$	$1111-,111-1,11-1-,11--1.$
$11-1,11--,$ $111,1-1.$	$11-0000,0001000,$ $0000101,0000010.$	$11-1111,11-1---,$ $11--1-1,11---1-.$

3.4 Turyn Type Sequences

Definition 3.15. A set $\{X,Y,Z,W\}$ of four $\{\pm1\}$ sequences of lengths n,n,n and $(n-1)$ respectively is said to be of Turyn type if the following condition is holds.

$$N_X(j)+N_Y(j)+2N_Z(j)+2N_W(j)=0; j\geq 1$$

Turyn type sequences first appeared in a paper of R. J. Turyn [41]. In that paper, he gave three examples of these sequences for lengths $n=4,6,8$. Except these lengths all other Turyn type sequences were found by computer search. Koukouvinos et.al. constructed these sequences for lengths $n=10,12,\ldots,24$ in [33] and the credit for the other lengths $n=26,28,\ldots,34$ goes to Kounias et.al. [34]. H. Kharaghani and B. Tayfeh-Rezaie found Turyn type sequences of length $n=36$ and verified the existence of a Hadamard matrix of order 428 [29]. In the same paper [41], Turyn established a theorem to extract Base sequences from existing Turyn type sequences. Now we present this well known theorem.

Theorem 3.16. *[41] If X,Y,Z,W are Turyn type sequences of lengths $n,n,n,(n-1)$, then the sequences $A=(Z|W),B=(Z|-W),C=X,D=Y$ are Base sequences of lengths $(2n-1),(2n-1),n,n$.*

See Table 3.5 for examples.

Table 3.5: Base sequences (BS) from Turyn sequences (TrS)

TrS of lengths n,n,n,(n-1)	BS of lengths (2n-1),(2n-1),n,n
$1-1-,1---,1--1,111.$	$1--1111,1--1---,1-1-,1---.$
$1-11-1,111--1,$ $1-111-,1----.$	$1-111-1----,1-111--1111,$ $1-11-1,111--1.$
$1--1---1,111----1,$ $1111--1-,111-111.$	$1111--1-111-111,1111--1----1---,$ $1--1---1,111----1.$

Yang's [43, 44] work on Base sequences is an crucial step to find odd order T-sequences of larger lengths. He showed if there is a number $y \in \{3,7,13,...,2g+1\}$ where $g \in 2^a10^b26^c; a,b,c \geq 0$, termed as Yang's number and there are Base sequences of lengths $m+p,m+p,m,m$ then there are T-sequences of length $y(2m+p)$ which is of great interest for the case when $(2m+p)$ is odd.

Theorem 3.17. *(Yang) Let A,B,C,D be sequences of lengths $(m+1),(m+1),m,m$, respectively and $G=(g_k)$ and $F=(f_k)$ be Golay sequences of lengths s. Then the following Q,R,S,T are 4-complementary sequences. Using X^* to denote the reverse of X:*

$$\begin{aligned}
Q &= (Af_s,Cg_1;0,0,Af_{s-1},Cg_2;0,0;,...,Af_1,Cg_s;0,0;-B^*,0);\\
R &= (Bf_s,Dg_s;0,0;Bf_{s-1},Dg_{s-1},0,0,...,Bf_1,Dg_1;0,0;A^*,0);\\
S &= (0,0;Ag_s,-CF_1;0,0,Ag_{s-1},-Cf_2;...,0,0;Ag_1,-Cf_s;0,-D^*);\\
T &= (0,0;Bg_1,-Df_1;0,0;Bg_2,-Df_2;...;0,0;Bg_s,-Df_s;0,C^*);
\end{aligned}$$

Furthermore, if we define sequences

$X_1 = (Q+R)/2, \quad X_2 = (Q-R)/2, \quad X_3 = (S+T)/2, \quad X_4 = (S-T)/2,$

then the sequences become T-sequences of lengths $t(2s+1), t = (2m+p)$.

Proof. Let $P(x), Q(x), R(x), S(x)$ be the associated Hall polynomials of the sequences X_1, X_2, X_3 and X_4, having length $(2s+1)(2m+p)$, and weights w_1, w_2, w_3, w_3 respectively. Defining $|P|^2 = P(x)P(x^{-1})$, we compute

$$|P|^2 + |Q|^2 + |R|^2 + |S|^2 = (w_1 + w_2 + w_3 + w_4)$$

where $P(x^{-1})$ is the involution of $P(x)$ and all these sequences are disjoint $\{0, \pm 1\}$. So the theorem follows. □

Although T-sequences can be obtained from many infinite class of sequences, they do not exist for all lengths. At the end of this chapter, we present a conjecture relating the existence of T-sequences [7].

Conjecture 3.18. There are T-sequences of lengths n for each odd integer $n \geq 1$.

This conjecture has been verified for all odd integers $n < 200$ [7] (section V) except for $n = 97, 103, 109, 113, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199$.

Chapter 4

Balanced Generalized Weighing Matrices

In the field of combinatroics, much research has been done on the multiple application of balanced generalized weighing matrices. This class of matrices includes Hadamard matrices and conference matrices. In this chapter, we study balanced generalized weighing matrices with classical parameter that includes ω-circulant as well as negacirculant weighing matrices.

4.1 Difference Sets

We begin here with the notion of set theory over a finite cyclic group G.

Definition 4.1. Let G be a multiplicative group of order m. Then a k-subset $S(m,k,\mu)$ of G is said to be a *difference set* over G if every $g \in G \backslash \{id\}$ can be expressed as $s_1 s_2^{-1}$ exactly μ ways for $s_1, s_2 \in S$.

A simple computation shows that there are exactly $(k^2 - k)$ pairs of elements in S which produce non-identity elements and so the difference set satisfies the following equation

$$k^2 - k = (m-1)\mu$$

This becomes more clear from the following examples.

Example 4.2. $\{0,1,3\}$ is a $(7,3,1)$ difference set in Z_7.

Example 4.3. Over Galois field, $GF(2^3) = \{0,1,x,x^2,x+1,x^2+1,x^2+x,x^2+x+1\}$, $\alpha = x+1$ is a primitive element of $GF(2^3)^*$. Let $S = \{\alpha, \alpha^2, \alpha^4\}$ is a 3-subset of $GF(2^3)^*$. Then we compute

$$\{s_i s_j^{-1}\} = \{\alpha^{-1}, \alpha^{-3}, \alpha, \alpha^{-2}, \alpha^3, \alpha^2\}$$

$$= \{x, x^2, x+1, x^2+1, x^2+x, x^2+x+1\}; (1 \le i \ne j \le 3)$$

where $\alpha^{-1} = \alpha^6$. Hence, S is a (7,3,1) difference set over the multiplicative group $GF(2^3)^*$.

Definition 4.4. Let H be a normal subgroup of order m of a group G of order mn. Then a k-subset $D = \{g_1, g_2, ..., g_k\}$ of distinct elements of G with parameters (m, n, k, λ), is said to be a *relative difference set* (RDS) if the set

$$\{g_i g_j^{-1}; 1 \le i \ne j \le k\}$$

contains each element of the set $G \backslash H$ exactly λ times and no element of H exists within this set.

Example 4.5. Let $S = \{1, \alpha, \alpha^2, \alpha^4\}$ be 4-subset of $GF(2^3)^*$. Then we compute

$$\begin{aligned} D = \{g_i g_j^{-1}\} &= \{\alpha^{-1}, \alpha^{-2}, \alpha^{-4}, \alpha, \alpha^{-1}, \alpha^{-3}, \alpha^2, \alpha, \alpha^{-2}, \alpha^4, \alpha^3, \alpha^2\} \\ &= \{\alpha, \alpha, \alpha^2, \alpha^2, \alpha^3, \alpha^3, \alpha^4, \alpha^4, \alpha^5, \alpha^5, \alpha^6, \alpha^6\} \\ &= \{x, x, x^2, x^2, x+1, x+1, x^2+1, x^2+1, \\ &\quad x^2+x, x^2+x, x^2+x+1, x^2+x+1\}; (1 \le i \ne j \le 3) \end{aligned}$$

Here $|GF(2^3)^*| = 7$ and D contains each element of $GF(2^3)^*$ exactly 2 times except the identity element. Thus S is a (7, 1, 4, 2) RDS with respect to the trivial subgroup.

However, in case of trivial subgroup H in G, an RDS in G with respect to H is a difference set in G. Though there are several kinds of relative difference sets, we limit ourselves in the study of relative difference set with classical parameters.

Definition 4.6. If q is a prime power and d is positive integer then a RDS with parameters

$$((q^{d+1}-1)/(q-1); q-1; q^d; q^{d-1})$$

in a cyclic group is said to be RDS with classical parameters.

4.2 Balanced Generalized Weighing Matrices

Definition 4.7. A square $m \times m$ matrix $W = (w_{ij})$ over a multiplicative group G is called Balanced Generalized Weighing Matrix $BGW(m,k,\mu)$ [25] if

(i) $w_{ij} \in \{G \cup 0\}$,

(ii) There are exactly k non zero elements in each row and each column in W,

(iii) The multiset $\{w_{ai}w_{bi}^{-1}; 1 \leq i \leq m, w_{ai}, w_{bi} \neq 0\}$ contains precisely $\mu/|G|$ copies of each element of G for $a,b \in \{1,2,...,m\}$.

Example 4.8. A BGW(5, 4, 3) over a cyclic group $G = <\omega>$ of order 3 is as follows:

$$W = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & \omega & \omega^2 \\ 1 & 1 & 0 & \omega^2 & \omega \\ 1 & \omega & \omega^2 & 0 & 1 \\ 1 & \omega^2 & \omega & 1 & 0 \end{pmatrix}$$

Definition 4.9. If α is a generator of the cyclic group G and $W = (w_{ij})$ is an $m \times m$ square matrix over the set $G \cup \{0\}$ then W is called ω- circulant if $w_{i+1,j+1} = w_{i,j}$ and $w_{i+1,1} = \omega w_{i,m}$ for $1 \leq j \leq m-1$, where $\omega = \alpha^{-1}$.

This class of matrices include BGW matrices. Jungnikel et.al. [26] gave a method for constructing these class of matrices over Galois field(GF).

Example 4.10. Over $GF(3^2)^*$, $R = \{\alpha^4, \alpha^5, \alpha^7\}$ is $(4,2,3,1)$ relative difference set with respect to $N = GF(3)^*$ where $\alpha = x+1$. Then the corresponding ω-circulant BGW(4, 3, 1) is

$$W = \begin{pmatrix} \alpha^4 & \alpha^4 & 0 & \alpha^4 \\ 1 & \alpha^4 & \alpha^4 & 0 \\ 0 & 1 & \alpha^4 & \alpha^4 \\ 1 & 0 & 1 & \alpha^4 \end{pmatrix}$$

where $\omega = \alpha^4 = 2$.

4.3 Negacirculant Matrices

A special feature of ω-circulant matrices covers the notion of negacirculant matrices. This class of matrices is very useful because of its converging nature to balanced generalized weighing matrices, conference matrices and many others. In particular, the theory behind this class of matrices opens the door to our research.

Definition 4.11. A weighing matrix C of order ν with off diagonal elements from the set $\{\pm 1\}$ and diagonal elements are all zero, satisfying

$$CC^T = (\nu - 1)I_\nu$$

is known as a *conference matrix*.

Example 4.12. A conference matrix of order 6 and weight 5, $C = C(6,5)$ is

$$C = \begin{pmatrix} 0 & - & - & - & - & - \\ - & 0 & - & 1 & 1 & - \\ - & - & 0 & - & 1 & 1 \\ - & 1 & - & 0 & - & 1 \\ - & 1 & 1 & - & 0 & - \\ - & - & 1 & 1 & - & 0 \end{pmatrix}$$

For $i = 0, 1, \ldots, \nu - 1$, define a square matrix N of order ν with entries given by :

$$N_{i,i+1} = 1,$$
$$N_{\nu-1,0} = -1 \text{and}$$
$$N_{i,j} = 0; \text{otherwise.}$$

N generates a matrix group of order 2ν where $N^\nu = -I_\nu$, $N^T = -N^{\nu-1}$ and $NN^T = I_\nu$.

Definition 4.13. Any square matrix A of order ν is said to be negacirculant [10] if $AN = NA$. For $i = 0, 1, ..., \nu - 2$, its entries $a_{i,j}; 0 \le i, j \le \nu - 1$ are of the form:

$$a_{i+1,0} = -a_{i,\nu-1}$$

$$a_{i+1,j} = a_{i,j-1}; j = 1, 2, ..., \nu - 1.$$

Example 4.14.

$$N = \begin{pmatrix} 0 & 1 & 1 & - & 1 & - & - & - & - & 1 \\ - & 0 & 1 & 1 & - & 1 & - & - & - & - \\ 1 & - & 0 & 1 & 1 & - & 1 & - & - & - \\ 1 & 1 & - & 0 & 1 & 1 & - & 1 & - & - \\ 1 & 1 & 1 & - & 0 & 1 & 1 & - & 1 & - \\ 1 & 1 & 1 & 1 & - & 0 & 1 & 1 & - & 1 \\ - & 1 & 1 & 1 & 1 & - & 0 & 1 & 1 & - \\ 1 & - & 1 & 1 & 1 & 1 & - & 0 & 1 & 1 \\ - & 1 & - & 1 & 1 & 1 & 1 & - & 0 & 1 \\ - & - & 1 & - & 1 & 1 & 1 & 1 & - & 0 \end{pmatrix}$$

In [36], Paley constructed so called C_K-conference matrices of order $q+1$ over $GF(q)$ for an odd prime power $q = p^k$. If K is the set of cardinal number $q+1$ that includes pair-wise independent vectors from the 2- dimensional vector space $V(2,q)$ then the associated Paley matrix is defined [10] as

$$C_K = [\chi det(x_i, x_j); x_i, x_j \in K] \text{ for } i, j \in \{0, 1, ..., q\}$$

where χ is the Legendre symbol over $GF(q)$. Over the $GF(5)$, C_K is the Paley matrix associated with the set $K = \{(0,1), (1,0)\ (2,1), (3,1), (3,2), (1,1)\}$ such that $C_K C_K^T = 5I_6$.

$$C_K = \begin{pmatrix} 0 & 1 & - & - & - & 1 \\ 1 & 0 & 1 & 1 & - & 1 \\ - & 1 & 0 & 1 & 1 & 1 \\ - & 1 & 1 & 0 & - & - \\ - & - & 1 & - & 0 & 1 \\ 1 & 1 & 1 & - & 1 & 0 \end{pmatrix}$$

Chapter 5

Amicable T-Matrices

Earlier in this thesis, we saw that Williamson matrices of order n imply the existence of Hadamard matrices of order $4n$. Like Williamson matrices, there is another interesting class of matrices called T-matrices whose size is directly related to the size of many combinatorial objects such as orthogonal designs and Hadamard matrices. J. Cooper and J. Wallis were the first in giving the formal definition of T-matrices in 1972 [8]. In this chapter, we form an infinite class of amicable T-matrices of order $n = q+1$, where q is odd prime power and in particular, of order $n \equiv 2 \pmod 4$. Moreover, we introduce a multiplication theorem for amicable T-matrices which is very useful to lengthen the size of T-matrices.

5.1 T-Matrices

We start here with few basic definitions.

Definition 5.1. Two $\{0, \pm 1\}$ matrices A and B of the same size are said to be *disjoint* if there is at most one nonzero element in the same position of the both matrices for every position. More explicitly, if there is a nonzero entry at the (i, j)-th position of A then the entry at the same position in B must be zero, i.e, $A * B = 0$.

Example 5.2. A and B are disjoint.

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

Definition 5.3. A set of matrices $\{A_1, A_2, \cdots, A_{2n}\}$ is said to be *amicable* [28] if it satisfies

$$\sum_{i=1}^{n}(A_{2i-1}A_{2i}^T - A_{2i}A_{2i-1}^T) = 0$$

If this set consisting of only two matrices then these matrices are called amicable i.e, if $A_1A_2^T = A_2A_1^T$ then A_1, A_2 are amicable. We call a pair of matrices (A_1, A_2), for which amicability happens in a set, is amicable in matching (A_1, A_2). In the definition above, for $n = 4$, A_1, A_2, A_3, A_4 are amicable in matching (A_1, A_2) and (A_3, A_4)

Definition 5.4. A set of quadruples $\{T_1, T_2, T_3, T_4\}$ of square matrices of order n with entries from the set $\{0, \pm 1\}$ are called T-matrices if

(i) they are pairwise disjoint,

(ii) the commutative property holds for every pair of matrices in the set $K = \{T_1, T_2, T_3, T_4, T_1^T, T_2^T, T_3^T, T_4^T\}$,

(iii) there exist a monomial matrix (matrix having a single nonzero entry in each row and each column) R of order n such that AR is amicable to B for all $A, B \in K$,

(iv) $T_1T_1^T + T_2T_2^T + T_3T_3^T + T_4T_4^T = tI_n$ for some t.

Example 5.5. T_1, T_2, T_3 and T_4 are the T-matrices of order 5.

$$T_1 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad T_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$T_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & - \\ - & 0 & 0 & 0 & 1 \\ 1 & - & 0 & 0 & 0 \\ 0 & 1 & - & 0 & 0 \\ 0 & 0 & 1 & - & 0 \end{pmatrix} \qquad T_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

A set of T-matrices satisfying the condition in Definition 5.3 with matching (T_1, T_2) and (T_3, T_4) are called amicable T-matrices. In a study [46], G.X Zuo and M. Y. Xia used the matching (T_1, T_4^T) and $(T_3, -T_2^T)$ in the definition of amicability and called them suitable T-matrices.

Definition 5.6. A set of T-matrices $\{T_1, T_2, T_3, T_4\}$ satisfying the amicablility condition with matching $(T_1, T_3 + T_4)$ and $(T_2, T_3 - T_4)$ i.e,

$$T_1(T_3 + T_4)^T + T_2(T_3 - T_4)^T = (T_3 + T_4)T_1^T + (T_3 - T_4)T_2^T$$

is called *weak amicable T-matrices* [23].

In a paper published in 2006 [23], Holzmann and Kharaghani found weak amicable T-matrices for orders $\{3, 5, \ldots, 21\}$. We use these matrices to construct amicable matrices in terms of variables in Chapter 6.

Example 5.7. $T_1 = circ(1,0,0), T_2 = circ(0,0,0), T_3 = circ(0,1,0)$ and $T_4 = circ(0,0,1)$ are weak amicable T-matrices of order 3 in matching $(T_1, T_3 + T_4)$ and $(T_2, T_3 - T_4)$.

Definition 5.8. A set of T-matrices $\{T_1, T_2, T_3, T_4\}$ satisfying

$$(T_1 + T_2) * (T_3 + T_4)^T = 0$$

is called *strongly disjoint* (SD) T-matrices [46] where $*$ is the Hadamard product of two matrices.

Example 5.9.

For $n = 5$, the following $\{T_1, T_2, T_3, T_4\}$ are SD T-matrices.

$$T_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \qquad T_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$T_3 = \begin{pmatrix} 0 & 0 & 1 & - & 0 \\ 0 & 0 & 0 & 1 & - \\ - & 0 & 0 & 0 & 1 \\ 1 & - & 0 & 0 & 0 \\ 0 & 1 & - & 0 & 0 \end{pmatrix}, \qquad T_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Definition 5.10. A set of amicable T-matrices with matching (T_1, T_4^T) and (T_2, T_3^T) is called a *proper amicable T-matrices* [16]. Equivalently the condition is

$$(T_1T_4 + T_2T_3) = (T_1T_4 + T_2T_3)^T$$

Example 5.11. $T_1 = circ(1,0,0), T_2 = circ(0,-,0), T_3 = circ(0,0,1)$ and $T_4 = circ(0,0,0)$ is proper amicable in matching (T_1, T_4^T) and (T_2, T_3^T).

This class of T-matrices is known for order $3,5,7,9,11,13,17,21$ [16].

T-matrices satisfying disjointedness and proper amicability properties have a useful role in the construction of four circulant amicable matrices in terms of variables. But such a set of T-matrices of odd order does not exist. Now we present a theorem from Hamed Gholamiangonabadi's M.Sc thesis [16] showing that it is not possible to find odd order T-matrices that have both of the conditions.

Theorem 5.12. *There are no proper amicable T-matrices of odd order that satisfy the disjointedness property.*

Proof. Consider a set of T-matrices $\{T_1, T_2, T_3, T_4\}$ of odd order n which meets the following conditions simultaneously.

$$(T_1T_4 + T_2T_3) = (T_1T_4 + T_2T_3)^T$$
$$(T_1 + T_2) * (T_3 + T_4)^T = 0$$

Let $X = T_1 + T_4^T$ and $Y = T_2 + T_3^T$, then we compute

$$\begin{aligned} XX^T + YY^T &= (T_1 + T_4^T)(T_1^T + T_4) + (T_2 + T_3^T)(T_2^T + T_3) \\ &= (T_1T_4 + T_2T_3 + T_1^TT_4^T + T_2^TT_3^T) + (T_1T_1^T + T_2T_2^T + T_3T_3^T + T_4T_4^T) \\ &\equiv I_n \pmod 2 \end{aligned}$$

where we use the fact of amicability in matching (T_1, T_4^T) and (T_2, T_3^T) and matrix $A \equiv -A$, (mod 2). Since they are strongly disjoint we have, $X + Y = J(\text{mod } 2)$ and so, $Y = X + J(\text{mod } 2)$. Then we compute

$$XX^T + YY^T = XX^T + XX^T + XJ^T + JX^T + JJ^T \equiv J \pmod 2.$$

since for circulant matrices, $AJ^T = JA^T$. Hence, $XX^T + YY^T = I_n = J$, a contradiction. □

In the next section, we make progress towards one of our main research objective and show that amicable T-matrices exist for order $n \equiv 2 \pmod 4$.

5.2 Amicable T-Matrices of Order $n \equiv 2 \pmod 4$

In a study [10], P. Delsarte et.al. determined that the only negacirculant $W(n, n-1)$ of order $n < 1000$ have $n = p^r + 1$ where p^r is an odd prime power. An important theorem by Delsarte et.al. [10] states that there exist an infinite class of negacirculant weighing matrix of order $n+1$ where n is odd prime power. Proof of this theorem come directly from the proofs of a theorem and a corollary which are given in the paper [10]. We give this significant theorem with an immediate example.

Theorem 5.13. *[15] There is a negacirculant $W(p^r+1, p^r)$ whenever p^r is an odd prime power.*

Example 5.14.

$$W(4,3) = \begin{pmatrix} 0 & 1 & - & - \\ 1 & 0 & 1 & - \\ 1 & 1 & 0 & 1 \\ - & 1 & 1 & 0 \end{pmatrix}$$

5.2.1 Construction of Amicable T-Matrices of Order $n = p^r + 1$; p^r is Odd Prime Power

We know that the sum of periodic auto-correlation function of a identity matrix and a negacirculant weighing matrix of same order is zero. Let

$$T_1 = I_n, \quad T_2 = W(n, n-1), \quad T_3 = 0_n, \quad T_4 = 0_n;$$

where $n = p^r + 1$, (p^r is odd prime power) and 0_n is zero matrix of order n. All of T_i's; $1 \leq i \leq 4$; are disjoint and cyclic, so form a commutative set with the all its transpose and meet the requirement of amicability condition in pair. Moreover they satisfy

$$T_1 T_1^T + T_2 T_2^T + T_3 T_3^T + T_4 T_4^T = nI_n.$$

Consequently, T_1, T_2, T_3 and T_4 form a set of amicable T-matrices.

Example 5.15.

$$T_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad T_2 = \begin{pmatrix} 0 & 1 & 1 & 1 & - & 1 \\ - & 0 & 1 & 1 & 1 & - \\ 1 & - & 0 & 1 & 1 & 1 \\ - & 1 & - & 0 & 1 & 1 \\ - & - & 1 & - & 0 & 1 \\ - & - & - & 1 & - & 0 \end{pmatrix}$$

$$T_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad T_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

By the above and Theorem 5.13, we have

Theorem 5.16. *For every odd prime power p^r, there exist amicable T-matrices of order $n = p^r + 1$.*

A particular case for $p^r \equiv 1 \pmod 4$ in the above theorem gives

Corollary 5.17. *There exist amicable T-matrices of order $n \equiv 2 \pmod 4$.*

5.3 Multiplication Theorem for Amicable T-Matrices

In this section, we introduce a multiplication theorem that leads us to generate infinite class of composite order T-matrices using amicable T-matrices. Yang used two different types of complementary sequences to obtain T-sequences (equivalently T-matrices) of larger order. Later Zuo and Xia [45] derived T-matrices of order $t(2m+p)$ from Base sequences (BS) of lengths $m+p, m+p, m, m$ (p odd) and suitable T-matrices of order t. In 2012, the same authors proved a theorem to find composite order T-matrices using directly strongly disjoint T-matrices and suitable T-matrices [46]. Now we present a theorem that allows generating new T-matrices using only amicability condition.

Theorem 5.18. *Suppose T_1, T_2, T_3, T_4 are arbitrary T-matrices of order t and A_1, A_2, A_3, A_4 are amicable T-matrices of order n with matching (A_1, A_3) and (A_2, A_4). Then the following matrices*

$$C_1 \quad = \quad T_1 \otimes A_1 - T_2 \otimes A_3 - T_3 \otimes A_2^T - T_4 \otimes A_4^T$$

$$
\begin{aligned}
C_2 &= T_1 \otimes A_3 + T_2 \otimes A_1 + T_3 \otimes A_4^T - T_4 \otimes A_2^T \\
C_3 &= T_1 \otimes A_2 - T_2 \otimes A_4 + T_3 \otimes A_1^T + T_4 \otimes A_3^T \\
C_4 &= T_1 \otimes A_4 + T_2 \otimes A_2 - T_3 \otimes A_3^T + T_4 \otimes A_1^T
\end{aligned}
$$

are T-matrices of order tn.

Proof. Since T_i's and A_i's are all T- matrices, it clearly follows that

$$C_i * C_j = 0; \text{ for } 1 \leq i \neq j \leq 4.$$

A simple calculation shows that

$$
\begin{aligned}
\sum_{i=1}^{4} C_i C_i^T &= (T_1T_1^T + T_2T_2^T + T_3T_3^T + T_4T_4^T) \otimes (A_1A_1^T + A_2A_2^T + A_3A_3^T + A_4A_4^T) \\
&\quad + (T_1T_2^T - T_2T_1^T + T_3T_4^T - T_4T_3^T) \otimes (A_3A_1^T - A_1A_3^T + A_4A_2^T - A_2A_4^T) \\
&= tnI_{tn}.
\end{aligned}
$$

Therefore the theorem follows. □

The multiplication theorem by Xia et.al. [46] is useful to find odd composite order T-matrices using suitable T-matrices but this type of T-matrices are only known for orders $n \in \{3, 5, \cdots, 21, 25\}$ [45, 46]. However, our approach has greater scope since we impose only the condition of amicability on one set of T-matrices keeping the other free from any condition and showed the existence of an infinite class of amicable T-matrices in Theorem 5.16.

Example 5.19. An obvious use of Theorem 5.18 with T-matrices $T_1 = circ(1,0,0), T_2 = circ(0,1,0)$, $T_3 = circ(0,0,1)$, $T_4 = circ(0,0,0)$ and amicable T-matrices as in Example

5.15 generate T-matrices of order 18.

$$C_1 = \left(\begin{array}{cccccc|cccccc|cccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & - & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & 0 & 1 & - & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & - & 0 & 1 & - & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & - & - & 0 & 1 & - \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & - & - & - & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & - & 1 & - & - & - & 0 \\
\hline
0 & 1 & - & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
- & 0 & 1 & - & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
- & - & 0 & 1 & - & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
- & - & - & 0 & 1 & - & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & - & - & - & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
- & 1 & - & - & - & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & - & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & - & 0 & 1 & - & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & - & - & 0 & 1 & - & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & - & - & - & 0 & 1 & - & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & - & - & - & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & - & 1 & - & - & - & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right)$$

$$C_2 = \left(\begin{array}{cccccc|cccccc|cccccc}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

$$C_3 = \left(\begin{array}{cccccc|cccccc|cccccc}
0 & 1 & 1 & 1 & - & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
- & 0 & 1 & 1 & 1 & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & - & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
- & 1 & - & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
- & - & 1 & - & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
- & - & - & 1 & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & - & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & - & 0 & 1 & 1 & 1 & - & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & - & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & - & 1 & - & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & - & - & 1 & - & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & - & - & - & 1 & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & - & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & - & 0 & 1 & 1 & 1 & - \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & - & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & - & 1 & - & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & - & - & 1 & - & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & - & - & - & 1 & - & 0
\end{array}\right)$$

$$C_4 = \left(\begin{array}{cccccc|cccccc|cccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & - & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & - & 0 & 1 & 1 & 1 & - & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & - & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & - & 1 & - & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & - & - & 1 & - & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & - & - & - & 1 & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & - & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & 0 & 1 & 1 & 1 & - \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & - & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & 1 & - & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & - & 1 & - & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & - & - & 1 & - & 0 \\
\hline
0 & 1 & 1 & 1 & - & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
- & 0 & 1 & 1 & 1 & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & - & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
- & 1 & - & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
- & - & 1 & - & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
- & - & - & 1 & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

Chapter 6

New Orthogonal Designs

In the literature that we reviewed in the previous chapters, our purpose was to find some helpful tools like T-matrices, amicable matrices for constructing some well known arrays called orthogonal designs. These arrays capture BH-arrays, BHW-arrays and essentially result in Hadamard matrices. This chapter mainly includes the application of amicable T-matrices as well as amicable matrices in terms of variables. We begin this chapter with the definition of orthogonal design.

6.1 Orthogonal Designs

Definition 6.1. Any square matrix A of order n, with entries from a set of commuting variables, $X = \{\pm x_1, \pm x_2, \ldots, \pm x_k\}$ satisfying the property

$$AA^T = \sum_{i=1}^{k}(s_i x_i^2) I_n$$

is called an *orthogonal design* of type $(n; s_1, s_2, ..., s_k)$ and is denoted by $OD(n; s_1, s_2, ..., s_k)$ where we call $\{s_1, s_2, \ldots, s_n\}$ is the parameter set.

Example 6.2. A is a orthogonal design of type $(4;1,1,1)$ where

$$A = \begin{pmatrix} 0 & x_1 & x_2 & x_3 \\ -x_1 & 0 & -x_3 & x_2 \\ -x_2 & x_3 & 0 & -x_1 \\ -x_3 & -x_2 & x_1 & 0 \end{pmatrix}$$

The Williamson array is a well known example of an orthogonal design of type $(4;1,1,1,1)$. Depending on the look of the parameter set, orthogonal design has different name such as Baumert-Hall arrays, Baumert-Hall-Welch arrays, Plotkin arrays etc. As our research revolves around the construction of Baumert-Hall arrays, we recall the definition here.

Definition 6.3. An orthogonal design with four variables and of same parameter set is called a *Baumert Hall* array or BH-array. More explicitly, an array A of size $4n; n \in \mathbb{N}$ will be called a *Baumert Hall* array if it satisfies

$$AA^T = n(x_1^2 + x_2^2 + x_3^2 + x_4^2)I_{4n}.$$

and is denoted by $\mathrm{OD}(4n; n, n, n, n)$.

Definition 6.4. A special class of Baumert-Hall array, consisting of 16 circulant matrices is known as Baumert-Hall-Welch array and is denoted by BHW-array.

In 1965, Baumert and Hall [3] gave the idea of such array by introducing an orthogonal design of type $(12;3,3,3,3)$ for the first time. The generalization of the idea of Baumert-Hall arrays remained incomplete until the discovery of T-matrices, originally found by R. J. Turyn. Now we present well known Goethals-Seidal array which has an extensive use in generating orthogonal designs.

$$GS = \begin{pmatrix} A & BR & CR & DR \\ -BR & A & -D^TR & C^TR \\ -CR & D^TR & A & -B^TR \\ -DR & -C^TR & B^TR & A \end{pmatrix}$$

where R is the back diagonal identity matrix of order n i.e. $R = [r_{ij}]$ where $r_{ij} = 1$ if $i + j = n + 1$ and 0 otherwise. In addition, if A, B, C, D are four circulant amicable matrices with matching (A, C) and (B, D) then the following array H [16] generates a BH-array, consisting of 16 block circulant matrices.

$$H = \begin{pmatrix} A & C & B & D \\ -C & A & -D & B \\ -B^T & D^T & A^T & -C^T \\ -D^T & -B^T & C^T & A^T \end{pmatrix}$$

Now we present a theorem of Cooper et.al. [15] with a slight variation which has an extensive use in constructing orthogonal designs, especially BHW-arrays in our approach.

Theorem 6.5. *(Cooper-J. Wallis [15]) Let T_i's; $1 \leq i \leq 4$ be T-matrices of order n and a, b, c and d be commuting variables. Then*

$$\begin{aligned} A &= aT_1 + bT_2 + cT_3 + dT_4 \\ B &= -bT_1 + aT_2 + dT_3 - cT_4 \\ C &= -cT_1 - dT_2 + aT_3 + bT_4 \\ D &= -dT_1 + cT_2 - bT_3 + aT_4 \end{aligned}$$

can be used in Goethals-Seidal array to obtain a Baumert Hall array of order n.

In the next few sections, we pay attention to some new orthogonal designs such as BH-arrays and BHW-arrays using amicable T-matrices formed in Chapter 5 and explore some circulant amicable matrices in terms of variables that allow us to generate new orthogonal designs.

6.2 Orthogonal Designs From Amicable T-Matrices

Using the amicable T-matrices of order $n \equiv 2 \pmod 4$ in Theorem 6.5 we obtain circulant matrices A, B, C, D in terms of variables a, b, c and d. Then plugging these matrices into Goethals-Seidal array generates our desired orthogonal design of type $(4n; n, n, n, n)$ where $n = 2k$; k is odd.

Moreover, Seberry et.al. showed that BHW-array can be used to construct an infinite class of orthogonal designs [38, 41].

Theorem 6.6. *(Seberry-Yamada-Turyn [38, 41]) Suppose there are T-matrices of order t and an orthogonal design $OD(4s; u_1, \dots, u_n)$ constructed from 16 circulant $s \times s$ blocks in variables $x_1, \dots, x_n$. Then there is an $OD(4st; tu_1, \dots, tu_n)$. In particular, if there is a BH-array, $OD(4s; s, s, s, s)$ constructed from 16 circulant $s \times s$ blocks then there is an $OD(4st; st, st, st, st)$.*

Proof. Let $N_{ij}; 1 \le i, j \le 4$, be the 16 circulant blocks of the orthogonal design $OD(4s; u_1, \dots, u_n)$. Since this design is orthogonal then we have,

$$N_{i1}N_{j1}^T + N_{i2}N_{j2}^T + N_{i3}N_{j3}^T + N_{i4}N_{j4}^T = \begin{cases} \sum_{k=1}^{4} u_k x_k^2 I_s & ; i = j \\ 0 & ; i \neq j. \end{cases}$$

Suppose T_1, T_2, T_3, T_4 are the T-matrices of order t. Then we form the following matrices

$$P = T_1 \otimes N_{11} + T_2 \otimes N_{21} + T_3 \otimes N_{31} + T_4 \otimes N_{41},$$

$$Q = T_1 \otimes N_{12} + T_2 \otimes N_{22} + T_3 \otimes N_{32} + T_4 \otimes N_{42},$$

$$R = T_1 \otimes N_{13} + T_2 \otimes N_{23} + T_3 \otimes N_{33} + T_4 \otimes N_{43},$$

$$S = T_1 \otimes N_{14} + T_2 \otimes N_{24} + T_3 \otimes N_{34} + T_4 \otimes N_{44}$$

which yields $PP^T + QQ^T + RR^T + SS^T = t \sum_{k=1}^{4} u_k x_k^2 I_{st}$. Plugging these matrices into the Goethals-Seidal array produce $OD(4st; tu_1, \cdots, tu_n)$. □

However, amicability in T-matrices allows us to find circulant amicable matrices using Theorem 6.5. These matrices have a major role in constructing BHW-arrays and BH-array of order $4s$ and $4st$.

Theorem 6.7. *[16] Suppose T_1, T_2, T_3, T_4 are amicable T-matrices of order n. Then there is a BHW-array of type $OD(4n; n, n, n, n)$ consisting of 16 circulant matrices.*

Proof. Assume T_1, T_2, T_3, T_4 are amicable T-matrices of order n with matching (T_1, T_3) and (T_2, T_4). We have

$$A = aT_1 + bT_2 + cT_3 + dT_4,$$

$$B = -bT_1 + aT_2 + dT_3 - cT_4,$$

$$C = -cT_1 - dT_2 + aT_3 + bT_4,$$

$$D = -dT_1 + cT_2 - bT_3 + aT_4$$

which are Seberry-Cooper matrices [15]. Now we compute

$$AC^T - CA^T + BD^T - DB^T = (a^2 + b^2 + c^2 + d^2)(T_1T_3^T - T_3T_1^T + T_2T_4^T - T_4T_2^T) = 0$$

and

$$AA^T + BB^T + CC^T + DD^T = n(a^2 + b^2 + c^2 + d^2)I_n$$

which reveals that A, B, C, D are amicable with matching (A, C) and (B, D). Plugging these matrices into the array H we obtain $HH^T = (AA^T + BB^T + CC^T + DD^T)I_{4n}$. Therefore, H is a BHW-array of type $OD(4n; n, n, n, n)$, consisting of 16 circulant matrices. □

Now we have the following result

Theorem 6.8. *For every odd prime power $p^r \equiv 1$, there is a BHW-array of type $OD(4n; n, n, n, n)$ where $n = p^r + 1$ and also a BH-array of type $OD(4nt; nt, nt, nt, nt)$ for any T-matrices of order t.*

Proof. For every odd prime power $p^r \equiv 1$, we established that there is a set of amicable T-matrices of order $n = p^r + 1$ (Corollary 5.17). Then the Theorem 6.7 gives a BHW-array of type $OD(4n; n, n, n, n)$. Suppose T_1, T_2, T_3, T_4 are any T-matrices of order t and then the Theorem of Seberry-Yamada-Turyn (Theorem 6.6) gives four matrices P, Q, R, S

of order nt. Plugging these matrices into Goethals-Seidal array result in a BH-array of type $OD(4nt;nt,nt,nt,nt)$. $\square$

J. Bell and D. Z. Djokovic [6] found $OD(4n;n,n,n,n)$ where $n=2k;k\in\{1,3,5,\cdots,17\}$. All these are a subclass of Baumert-Hall array, known as BHW-array. Now we tabulate some new Baumert-Hall arrays corresponding to T-matrices of order $n\equiv 2 \pmod 4) < 100$, which are different from those found by Bell and Djokovic [6].

Table 6.1: Orthogonal Design of type $OD(4n;n,n,n,n);n=2k$, k is odd

Order n	$n =$ p^r+1	Order of T-matrices from Theorem 5.18	OD(4n; n, n, n, n) n=2k; k is odd
6	5+1	2*3	OD(24; 6, 6, 6, 6)
10	9+1	2*5	OD(40; 10, 10, 10, 10)
14	13+1	2*7	OD(56; 14, 14, 14, 14)
18	17+1	6* 3	OD(72; 18, 18, 18, 18)
22		2*11	OD(88; 22, 22, 22, 22)
26	25+1	2*13	OD(104; 26, 26, 26, 26)
30	29+1	6*5	OD(120; 30, 30, 30, 30)
34		2*17	OD(136: 34, 34, 34, 34)
38	37+1	2*19	OD(152; 38, 38, 38, 38)
42	41+1	6*7	OD(168; 42, 42, 42, 42)
46		2*23	OD(184; 46, 46, 46, 46)
50	49+1	10*5	OD(200; 50, 50, 50, 50)
54	53+1	6*9	OD(216; 54, 54, 54, 54)
62	61+1		OD(248; 62, 62, 62, 62)
66		6*11	OD(264; 66, 66, 66, 66)
70		10*7	OD(280; 70, 70, 70, 70)
74	73+1		OD(296; 74, 74, 74, 74)
78		6*13	OD(312; 78, 78, 78, 78)
82		2*41	OD(328; 82, 82, 82, 82)
90	89+1	10*9	OD(360; 90, 90, 90, 90)
98	97+1		OD(392; 98, 98, 98, 98)

Moreover, this type of amicable T-matrices of order $n \equiv 2 \pmod 4$ can be used to generate orthogonal designs of type $OD(4n;2,2,2n-2,2n-2)$.

Theorem 6.9. *For every odd prime power p^r, there is an orthogonal design of type $OD(4n;2,2,2n-2,2n-2)$ where $n = p^r+1$.*

Proof. For the amicable T-matrices constructed as in Section 5.3, set

$$
\begin{aligned}
A &= aT_1 + bT_2,\\
B &= -aT_1 + bT_2,\\
C &= cT_1 + dT_2,\\
D &= -cT_1 + dT_2
\end{aligned}
$$

and plugging these circulant matrices into Goethals-Seidal array yields an orthogonal design of type $OD(4n;2,2,2n-2,2n-2)$ where $n = p^r+1$. □

Since there are no odd order amicable T-matrices [5], finding odd order circulant amicable matrices in terms of variables is a challenging job. However, we employ a different approach which help us to construct odd order amicable matrices in two variables and also in four variables. Here we use the amicability of Williamson matrices to construct four circulant amicable matrices in order to generate infinite class of orthogonal designs of BH-array type.

Theorem 6.10. *Suppose W_1,W_2,W_3,W_4 are Williamson-matrices of order n. Then there is an orthogonal design of type $OD(4n;2n,2n)$ and also an orthogonal design of type $OD(4nt;2nt\ ,2nt)$ for any T-matrices of order t.*

Proof. Let W_1,W_2,W_3,W_4 be Williamson matrices of order n and a,b be two indeterminate. Setting $T_1 = (W_1+W_2)/2$, $T_2 = (W_1-W_2)/2$, $T_3 = (W_3+W_4)/2$, $T_4 = (W_3-W_4)/2$ and

$$A = aT_1 + bT_2,$$

$$B = bT_1 - aT_2,$$

$$C = aT_3 - bT_4,$$

$$D = bT_3 + aT_4$$

we obtain circulant amicable matrices A,B,C,D with matching (A,C) and (B,D). Then substitution of these matrices into the array H produces $OD(4n;2n,2n)$ and plugging the new matrices formed with A,B,C,D and T-matrices as in Theorem 6.6, into Goethals-Seidal array leads an orthogonal design of type $OD(4nt;2nt,2nt)$. □

Example 6.11. Let $W_1 = circ(1,1,1)$, $W_2 = circ(-,1,1)$, $W_3 = circ(-,1\ ,1)$, $W_4 = circ(-,1,1)$ be Williamson matrices of order 3. Then $T_1 = circ(0,1,1)$, $T_2 = circ(1,0,0)$, $T_3 = circ(-,1,1)$, $T_4 = circ(0,0,0)$ are four $(0,\pm1)$ circulant amicable matrices. Thus we have

$$A = \begin{pmatrix} b & a & a \\ a & b & a \\ a & a & b \end{pmatrix}, \qquad B = \begin{pmatrix} -a & b & b \\ b & -a & b \\ b & b & -a \end{pmatrix},$$

$$C = \begin{pmatrix} -a & a & a \\ a & -a & a \\ a & a & -a \end{pmatrix}, \qquad D = \begin{pmatrix} -b & b & b \\ b & -b & b \\ b & b & -b \end{pmatrix}.$$

A simple calculation shows $AC^T - CA^T + BD^T - DB^T = 0$ and $AA^T + BB^T + CC^T + DD^T = 6(a^2+b^2)I_3$ and putting these matrices into the array H produces an $OD(12;6,6)$.

In Theorem 6.5, we saw that amicability in A,B,C,D happens if T_1,T_2,T_3,T_4 are amicable which is not possible for odd order T-matrices in general. However, we notice if T_1 and T_4 are all zero matrices then the amicability with matching (T_1,T_3) and (T_2,T_4) is trivial. But it is hard to find such a set of T-matrices while it is easy to find a set of T-matrices where T_1 is identity and T_4 is all zero matrix. Since $II^T = I$, we substitute 1 by 0 in $T_1 = I$

and so that the T-matrices $T_1 = I_n, T_2, T_3, T_4 = 0$ of order n allows us to construct four circulant amicable matrices of order n. We know this class of T-matrices exist for order $n+1$ where $n \in 2^a 10^b 26^c; a,b,c \geq 0$.

Theorem 6.12. *There is an orthogonal design of type $OD(4(n+1);n,n,n,n)$ consisting of 16 circulant matrices for every Golay length n. Moreover, for any T-matrices of order t there exist an $OD(4t(n+1);nt,nt,nt,nt)$.*

Proof. Suppose P,Q be Golay sequences of length n. Then circulating the sequences $(1,0_n)$, $(0,\frac{P+Q}{2})$, $(0,\frac{P-Q}{2})$, (0_{n+1}), we have T-matrices T_1,T_2,T_3,T_4 of order $n+1$ where 0_n is the zero sequence of length n. Substituting 1 by 0 in T_1 and then applying Theorem 6.5, we obtain four circulant amicable matrices of order $n+1$. Using these matrices into the array H and in Theorem 6.6 suffices to the proof. □

Example 6.13. Let $P = (11); Q = (1-)$ be Golay sequences of length 2. Then we form $T_1 = circ(0,0,0), T_2 = circ(0,1,0)$, $T_3 = circ(0,0,1)$, $T_4 = circ(0,0,0)$ and obtain

$$A = \begin{pmatrix} 0 & b & c \\ c & 0 & b \\ b & c & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & -d & a \\ a & 0 & -d \\ -d & a & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & a & d \\ d & 0 & a \\ a & d & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} 0 & -c & b \\ b & 0 & -c \\ -c & b & 0 \end{pmatrix}.$$

Since A,B,C,D yield $AC^T - CA^T + BD^T - DB^T = 0$ and $AA^T + BB^T + CC^T + DD^T = 2(a^2+b^2+c^2+d^2)I_3$, putting these matrices into the array H produces an $OD(12;2,2,2,2)$.

Repeat that $(p,q,r,s) = -(a,b,c,d)$. In Appendix A, we list $OD(72;18,18,18,18)$, $OD(88;22,22,22,22)$ and $OD(104;26,26,26,26)$. These arise from the Theorem 6.8 in a manner similar to Example 6.13.

Aside from Golay sequences of lengths $n \in 2^a 10^b 26^c; a, b, c \geq 0$, Georgiou et.al. found periodic Golay sequences of lengths $n \in \{34, 50, 58, 74, 82, 122, 136, 202, 226\}$ [13] in 2014. In same year, Djokovic et.al. [11] published a paper giving a unique example of periodic Golay sequences of length 72. At present these are the only periodic Golay sequences having an order with factor congruent to 3 modulo 4. From these periodic Golay sequences we can construct amicable T-matrices of size n as follows

$$T_1 = circ(\tfrac{A+B}{2}), \quad T_2 = circ(\tfrac{A-B}{2}), \quad T_3 = 0n, \quad T_4 = 0_n$$

where A, B are periodic Golay sequences of length n and T_3, T_4 are zero matrices of order n. Using these matrices we have orthogonal designs of type $OD(4n; n, n, n, n)$; $n \in \{34, 50, 58, 74, 82, 122, 136, 202, 226\}$

6.3 Orthogonal Designs From Special Class of T-Matrices

For any T-matrices of order n, Cooper-Seberry matrices [15] A, B, C, D satisfy the property

$$AA^T + BB^T + CC^T + DD^T = q(a^2 + b^2 + c^2 + d^2)I_n \tag{6.1}$$

where each of the variables appears same number of times. In this section, we introduce a theorem which results in four circulant matrices in terms of four variables satisfying the property 6.1 where each variable appears varied number of times.

Theorem 6.14. *Let $\{T_1, T_2, T_3, T_4\}$ be a set of T-matrices of order t where each T_i is a weighing matrix of weight w_i and A, B, C, D are Williamson-type matrices of order n. Then the following matrices*

$$C_1 = aA \otimes T_1 + bB \otimes T_2 + cC \otimes T_3 + dD \otimes T_4,$$
$$C_2 = -bA \otimes T_2 + aB \otimes T_1 + dC \otimes T_4 - cD \otimes T_3,$$
$$C_3 = -cA \otimes T_3 - dB \otimes T_4 + aC \otimes T_1 + bD \otimes T_2,$$
$$C_4 = -dA \otimes T_4 + cB \otimes T_3 - bC \otimes T_2 + aD \otimes T_1$$

satisfy $\sum_{i=1}^{4} C_i C_i^T = 4n(w_1a^2 + w_2b^2 + w_3c^2 + w_4d^4)I_{nt}$.

Proof. Since A,B,C,D are pairwise amicable, we have

$$\begin{aligned}
\sum_{i=1}^{4} C_i C_i^T &= w_1a^2(AA^T + BB^T + CC^T + DD^T) \otimes I_t + \\
&\quad w_2b^2(AA^T + BB^T + CC^T + DD^T) \otimes I_t + \\
&\quad w_3c^2(AA^T + BB^T + CC^T + DD^T) \otimes I_t + \\
&\quad w_4d^2(AA^t + BB^T + CC^T + DD^T) \otimes I_t. \\
&= (w_1a^2 + w_2b^2 + w_3c^2 + w_4d^4) \\
&\quad (AA^T + BB^T + CC^T + DD^T) \otimes I_t \\
&= 4n(w_1a^2 + w_2b^2 + w_3c^2 + w_4d^4)I_n \otimes I_t \\
&= 4n(w_1a^2 + w_2b^2 + w_3c^2 + w_4d^4)I_{nt}.
\end{aligned}$$

□

This Theorem is very useful in generating a new and interesting class of orthogonal designs in four variables where each variable appears varied number of times.

Corollary 6.15. *Suppose m be the order of Williamson-type matrices and* $\{T_1,T_2,T_3,T_4\}$ *be a set of T-matrices of order n where each* T_i *is a weighing matrix of weight* w_i. *Then there exist an orthogonal design of type* $OD(4mn; w_1s, w_2s, w_3s, w_4s)$ *where* $s = 4m$.

Proof. Constructing C_i's with T-matrices matrices of weight w_1,w_2,w_3,w_4 and plugging these matrices into Goethals- Seidal array yields orthogonal design of type
$OD(4n; w_1s, w_2s, w_3s, w_4s)$ where $s = 4m$. □

Example 6.16. Let $T_1 = circ(1,0,0)$, $T_2 = circ(0,1,0)$, $T_3 = circ(0,0,1)$, $T_4 = circ(0,0,0)$ and A,B,C,D be Williamson-type matrices of order 3. Then

$$\sum_{i=1}^{4} C_i C_i^T = 12(a^2 + b^2 + c^2)$$

and generate an OD(36;12,12,12).

Moreover for $T_1 = circ(0,0,-,0,-,-,1)$, $T_2 = circ(0,0,0,1,0,0,0)$,

$T_3 = circ(0,1,0,0,0,0,0)$, $T_4 = circ(-,0,0,0,0,0,0)$ and A,B,C,D are Williamson-type matrices of order 3, we have

$$\sum_{i=1}^{4} C_i C_i^T = (48a^2 + 12b^2 + 12c^2 + 12d^2)$$

and generate an OD(84;12,12,12,48). Also if A,B,C,D are Williamson-type matrices of order 5 then we have

$$\sum_{i=1}^{4} C_i C_i^T = (80a^2 + 20b^2 + 20c^2 + 20d^2)$$

and generate an OD(140;20,20,20,80). In Appendix A we include the design $OD(84;12,12,12,48)$.

Remark 6.17. Corollary 6.15 produces new classes of orthogonal designs, so it would be an interesting topic for future research to classify T-matrices where each of T_i is a weighing matrix of weight w_i.

6.4 Orthogonal Designs From Amicable Matrices With Composite Entries

This section is quite different from the preceding one. Here we construct circulant amicable matrices with composite entries, specially with linear combination of variables. Application of these matrices produces *generalized orthogonal designs*. However, generalized orthogonal design is another interesting topic for research and is beyond our present research scope. To present the application of these amicable matrices, we gently start this section with the formal definition of generalized orthogonal design.

Definition 6.18. Suppose $\{x_1, x_2, \ldots, x_t\}$ be a set of commuting variables and D be a $m \times n$ matrix whose entries are of the form $\pm k_{ij} x_i$ for $k_{ij} \geq 0, i = 1,2,\ldots,t;\ j = 1,2,\ldots,u_i$ (u_i is the number of times x_i appears) and $\sum_{i=0}^{t} u_i = n$ where u_0 is the number of zeros in each row or column. Set $s_i = \sum_{j=1}^{u_i} k_{ij}^2$. Then D is said to be *generalized orthogonal design* (GOD) if

$$DD^T = (\sum_{i=1}^{t} s_i x_i^2) I_m$$

and is denoted by

$$D = GOD(m; n; k_{1,1}, k_{1,2}, \ldots, k_{1,u_1}; k_{2,1}, k_{2,2}, \ldots, k_{2,u_2}; \ldots; k_{t,1}, k_{t,2}, \ldots, k_{t,u_t}).$$

Also an alternative way to denote GOD is

$$D = GOD(m; n; < k_{1,1}, a_{1,1} >, \ldots, < k_{1,u_1}, a_{1,u_1} >; \ldots; < k_{t,1}, a_{t,1} >, \ldots, < k_{t,u_t}, a_{t,u_t} >$$

where k_{ij} denotes the times of repetition of the variables in the coefficient a_{ij}. If $m = n$ then the generalized orthogonal design is denoted by

$$GOD(n; k_{11}, k_{12}, \ldots, k_{1u_1}; \ldots; k_{t1}, k_{t2}, \ldots, k_{tu_t}).$$

Moreover, if $m = n$ and $k_{ij} = 1$ for all $i = 1, 2, \ldots, t$; $j = 1, 2, \ldots, u_i$ then generalized orthogonal design turns into orthogonal design $OD(n; u_1, u_2, \ldots, u_t)$. Thus orthogonal designs are a special case of generalized orthogonal designs. Here we present here two different examples of generalized orthogonal design.

Example 6.19. The following array D is a $GOD(4; 1, 1; 1; 1)$ as well as $OD(4; 2, 1, 1)$ with $n = 4$, $u_1 = 2$, $u_2 = u_3 = 1$ and $a_{11} = a_{12} = a_{21} = a_{31} = 1$.

$$D = \begin{pmatrix} b & a & b & c \\ -a & b & c & -b \\ -b & -c & b & a \\ -c & b & -a & b \end{pmatrix}$$

Furthermore,

$$D = \begin{pmatrix} 3a & -2b & -c & -2a & -7b & -c & -3b & -4a \\ 2b & 3a & -2a & c & -c & 7b & 4a & -3b \\ c & 2a & 3a & -2b & -3b & -4a & 7b & c \\ 2a & -c & 2b & 3a & -4c & 3b & -c & 7b \\ 7b & c & 3b & 4a & 3a & -2b & -c & -2a \end{pmatrix}$$

is a $GOD(5;8;2,3,4;2,3,7;1,1)$ with $t=3, u_1=u_2=3, u_3=2$.

Although amicability in odd order T-matrices is not possible, Holzmann and Kharaghani gave the definition of weak amicability in T-matrices and found ones for odd orders $\{3,5,\ldots,21\}$ [23]. Use of weak amicable T-matrices in Theorem 6.5 with a substitution $T_3 = T_3+T_4$ and $T_4 = T_3-T_4$ help us to compute circulant amicable matrices of odd order some of whose entries are linear combination of variables. Application of these matrices generates an infinite class of generalized orthogonal designs.

Theorem 6.20. *If T_1,T_2,T_3,T_4 are weak amicable T-matrices of order n then there is a generalized orthogonal design of type $GOD(4n;k_{11},\ldots k_{1u_1};k_{21},\ldots,k_{2u_2})$. Furthermore, for any T-matrices of order t there exist a generalized orthogonal design of type $GOD(4nt;k_{11},\ldots k_{1u_1};k_{21},\ldots,k_{2u_2})$.*

Proof. Suppose T_1,T_2,T_3,T_4 are weak amicable T-matrices of order n with matching (T_1,T_3+T_4), (T_2,T_3-T_4). Then the following matrices

$$A = aT_1+bT_2+c(T_3+T_4)+d(T_3-T_4),$$

$$B = -bT_1+aT_2-d(T_3+T_4)+c(T_3-T_4),$$

$$C = -cT_1+dT_2+a(T_3+T_4)-b(T_3-T_4),$$

$$D = -dT_1-cT_2+b(T_3+T_4)+a(T_3-T_4)$$

are amicable in matching (A,C) and (B,D) with some composite entries. Plugging these matrices into the array H yields an orthogonal array of order $4n$. An appropriate substitution in the variables leading each composite entry into either 0 or in the form $\pm k_{ij}x_i$ turns the array H into generalized orthogonal design of type $GOD(4n;k_{11},\ldots k_{1u_1};k_{21},\ldots,k_{2u_2})$. The second part is obvious in accordance the Theorem 6.8. □

Example 6.21. We have $T_1 = circ(1,0,0), T_2 = circ(0,0,0)$, $T_3 = circ(0,1,0)$, $T_4 = circ(0,0,1)$ are weak amicable T-matrices with matching (T_1,T_3+T_4) and $(T_2,T_3-$

T_4). Then we compute

$$A = \begin{pmatrix} a & c+d & c-d \\ c-d & a & c+d \\ c+d & c-d & a \end{pmatrix}, \qquad B = \begin{pmatrix} -b & d-c & d+c \\ d+c & -b & d-c \\ d-c & d+c & -b \end{pmatrix},$$

$$C = \begin{pmatrix} -c & a+b & a-b \\ a-b & -c & a+b \\ a+b & a-b & -c \end{pmatrix}, \qquad D = \begin{pmatrix} -d & a-b & -b-a \\ -b-a & -d & a-b \\ a-b & b-a & -d \end{pmatrix}.$$

Here A,B,C,D satisfy $AC^T - CA^T + BD^T - DB^T = 0$ and $AA^T + BB^T + CC^T + DD^T = 5(a^2+b^2+c^2+d^2)I_3$. Plugging these matrices into the array H produces an orthogonal array of order 12.

Over and above, array H in the above Theorem can be used to generate an infinite class of weighing matrices.

Corollary 6.22. *If W_1, W_2, W_3, W_4 are Williamson matrices of order n and T_1, T_2, T_3, T_4 are weak amicable T-matrices of order t then there exist a weighing matrix of weight $4nt-2n$.*

Proof. In the array constructed in the proof of Theorem 6.20, an appropriate substitution of each entry by any of $(W_1+W_2)/2$, $(W_1-W_2)/2$, $(W_3+W_4)/2$ and $(W_3-W_4)/2$ such that each of the composite entries must one of $\pm\{W_1,W_2,W_3,W_4\}$, leads a weighing matrix of weight $4nt-2n$. □

Amicability is a generic feature in Williamson-type matrices that allows us to generate circulant amicable matrices where each entry is a linear combination of four variables.

Theorem 6.23. *If W_1, W_2, W_3, W_4 are Williamson-type matrices order n then there is a generalized orthogonal design of type $GOD(4n; k_{11}, \ldots, k_{1u_1})$. Further, for any T-matrices of order t there exist generalized orthogonal design of type $GOD(4nt; k_{11}, \ldots, k_{1u_1})$.*

Proof. Let W_1, W_2, W_3, W_4 be Williamson-type matrices order n and a, b, c, d be four variables. Then the following matrices

$$A = aW_1 + bW_2 + cW_3 + dW_4,$$

$$B = -bW_1 + aW_2 - dW_3 + cW_4,$$

$$C = -cW_1 + dW_2 + aW_3 - bW_4,$$

$$D = -dW_1 - cW_2 + bW_4 + aW_4$$

are four circulant amicable matrices with matching (A, C) and (B, D) where each entry is a linear combination of variables a, b, c, d. Plugging these matrices into the array H produce an orthogonal array of order $4n$ and theorem 6.8 gives an orthogonal array of order $4nt$. Then an appropriate substitution in variables of these orthogonal arrays where each entry turns into either 0 or in the form $\pm k_{ij}x_i$, turns the array H into a generalized orthogonal design of type $GOD(4n; k_{11}, \ldots, k_{1u_1})$ and $GOD(4nt; k_{11}, \ldots, k_{1u_1})$ respectively. $\square$

Example 6.24. Using the Williamson-type matrices $W_1 = circ(1, 1, 1)$, $W_2 = circ(-, 1, 1)$, $W_3 = circ(-, 1, 1)$, $W_4 = circ(-, 1, 1)$ of order 3 we get,

$$A = \begin{pmatrix} a-b-c-d & a+b+c+d & a+b+c+d \\ a+b+c+d & a-b-c-d & a+b+c+d \\ a+b+c+d & a+b+c+d & a-b-c-d \end{pmatrix},$$

$$B = \begin{pmatrix} -a-b+c-d & a-b-c+d & a-b-c+d \\ a-b-c+d & -a-b+c-d & a-b-c+d \\ a-b-c+d & a-b-c+d & -a-b+c-d \end{pmatrix},$$

$$C = \begin{pmatrix} -a-b-c+d & a+b-c-d & a+b-c-d \\ a+b-c-d & -a-b-c+d & a+b-c-d \\ a+b-c-d & a+b-c-d & -a-b-c+d \end{pmatrix},$$

$$D = \begin{pmatrix} -a+b-c-d & a-b+c-d & a-b+c-d \\ a-b+c-d & -a+b-c-d & a-b+c-d \\ a-b+c-d & a-b+c-d & -a+b-c-d \end{pmatrix}$$

that produce $AC^T - CA^T + BD^T - DB^T = 0$ and $AA^T + BB^T + CC^T + DD^T = 12(a^2+b^2+c^2+d^2)I_3$.

These types of generalized orthogonal designs have extensive use in statistics, specially in construction of orthogonal latin hypercube designs (LHD) [14] and in coding theory [22].

However each orthogonal array of order $4n$, constructed in Theorem 6.23, can be split into four coefficient matrix of order $4n$ with respect to variables. More explicitly, $H = aA + bB + cC + dD$ where A, B, C, D are pairwise anti-amicable. Use of a pair of anti-amicable coefficient matrices in the following array generates a orthogonal design of order $8n$ in two variables.

$$K = \begin{pmatrix} aA & bB \\ bB & aA \end{pmatrix}$$

In addition to this, using these anti-amicable coefficient matrices in the following array we obtain orthogonal design of order $16n$ in four variables consist of 16 blocks, where each block is constructed from 16 symmetric circulant matrices.

$$M = \begin{pmatrix} aA & bB & cC & dD \\ bB & aA & dD & cC \\ cC & dD & aA & bB \\ dD & cC & bB & aA \end{pmatrix}$$

6.5 A New Method to Construct a Hadamard Matrix of Order 48

A Hadamard matrix H can be constructed by $H = A \otimes B$ where A and B are two Hadamard matrices of order 12 and 4 respectively. Using negacirculant weighing matrix of order 4, equivalently amicable T-matrices of order 4 we construct an orthogonal design of type $\mathrm{OD}(48;12,12,12,12)$. Substituting the variables of this design by 1 we get a Hadamard matrix K of order 48. Using computer algebraic software MAGMA we determined that the Hadamard matrix $H(48)$ is inequivalent to $K(48)$. Thus the Hadamard matrix $K(48)$, arising from $\mathrm{OD}(48;12,12,12,12)$ is inequivalent to the Hadamard matrices constructed from Kronecker product method using Hadamard matrices of order 12 and 4 respectively.

Since we started the thesis with three examples of long known Hadamard matrices, it seems appropriate to end the thesis with this example of a new construction for a Hadamard matrix of order 48.

$$H=\begin{pmatrix}
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1 & 1 & 1 & - & - & - & - & 1 & - & - & - & 1 & - & - & - & 1 & - & - & - & 1 & - & - & - & 1 & - & - & - & 1 & - & - & - & 1 & - & - & - & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & - & - & - & - & 1 \\
1 & - & - & - & 1 & - & - & - & - & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & - & - & - & 1 & - & - & - & - & 1 & 1 & 1 & 1 & - & - & - & 1 & - & - & - & 1 & - & - & - & - & 1 & 1 & 1 & - & 1 & 1 & 1 \\
- & 1 & - & - & - & 1 & - & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & - & - & - & 1 & - & - & 1 & - & 1 & 1 & - & 1 & - & - & - & 1 & - & - & - & 1 & - & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 \\
- & - & 1 & - & - & - & 1 & - & 1 & 1 & - & 1 & 1 & 1 & - & 1 & - & - & 1 & - & - & - & 1 & - & 1 & 1 & - & 1 & - & - & 1 & - & - & - & 1 & - & - & - & 1 & - & 1 & 1 & - & 1 & 1 & 1 & - & 1 \\
- & - & - & 1 & - & - & - & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & - & - & - & - & 1 & - & - & - & 1 & 1 & 1 & 1 & - & - & - & - & 1 & - & - & - & 1 & - & - & - & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & - \\
- & 1 & 1 & 1 & 1 & - & - & - & 1 & - & - & - & - & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & - & - & - & - & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & - & - & - & - & 1 & 1 & 1 \\
1 & - & 1 & 1 & - & 1 & - & - & - & 1 & - & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & - & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & - & - & 1 & - & 1 & 1 & - & 1 & - & - & 1 & - & 1 & 1 \\
1 & 1 & - & 1 & - & - & 1 & - & - & - & 1 & - & 1 & 1 & - & 1 & 1 & 1 & - & 1 & - & - & 1 & - & 1 & 1 & - & 1 & 1 & 1 & - & 1 & - & - & 1 & - & 1 & 1 & - & 1 & - & - & 1 & - & 1 & 1 & - & 1 \\
1 & 1 & 1 & - & - & - & - & 1 & - & - & - & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & -
\end{pmatrix}$$

$$
K = \begin{pmatrix}
1&1&-&-&1&1&-&-&1&-&-&1&-&1&1&-&-&-&1&1&1&1&-&-&-&1&1&1&1&1&-&1&-&-&1&-&-&1&1&1&-&-&1&-&-&-&1&- \\
1&1&1&-&1&1&1&-&-&1&-&-&1&1&-&1&-&1&1&1&1&-&-&-&1&1&1&1&1&-&1&-&-&1&-&1&1&1&1&1&-&1&-&1&-&1&-&1 \\
1&1&1&1&1&1&1&1&1&-&1&-&1&-&1&-&1&1&1&1&-&-&-&-&1&1&1&-&-&1&-&-&1&-&1&1&1&1&1&-&1&-&1&1&1&-&1&1 \\
-&1&1&1&-&1&1&1&1&1&-&1&-&1&-&-&1&1&1&-&-&-&-&1&1&1&-&-&1&-&-&1&-&1&1&-&1&1&-&-&-&1&1&-&-&1&1&- \\
1&-&-&1&1&1&-&-&1&1&-&-&-&-&1&1&1&1&-&-&-&1&1&-&1&1&-&1&-&-&1&-&-&1&1&1&-&-&1&-&-&-&1&-&-&1&1&1 \\
-&1&-&-&1&1&1&-&1&1&1&-&-&1&1&1&1&-&-&-&1&1&-&1&1&-&1&-&-&1&-&1&1&1&1&1&-&1&-&1&-&1&-&1&1&1&1&1 \\
1&-&1&-&1&1&1&1&1&1&1&1&1&1&1&1&-&-&-&-&1&-&1&-&-&1&-&-&1&-&1&1&1&1&1&-&1&-&1&1&1&-&1&1&1&1&1&- \\
1&1&-&1&-&1&1&1&-&1&1&1&1&1&1&-&-&-&-&1&-&1&-&-&1&-&-&1&-&1&1&-&1&1&-&-&-&1&1&-&-&1&1&-&1&1&-&- \\
1&1&-&-&1&-&-&1&1&1&-&-&1&1&-&-&-&1&1&-&-&-&1&1&-&-&1&-&-&1&1&1&1&1&-&1&-&-&1&-&-&1&1&1&-&-&1&- \\
1&1&1&-&-&1&-&-&1&1&1&-&1&-&-&-&1&1&-&1&-&1&1&1&-&1&-&1&1&1&1&1&1&-&1&-&-&1&-&1&1&1&1&1&-&1&-&1 \\
1&1&1&1&1&-&1&-&1&1&1&1&-&-&-&-&1&-&1&-&1&1&1&1&1&-&1&1&1&1&1&-&-&1&-&-&1&-&1&1&1&1&1&-&1&-&1&1 \\
-&1&1&1&1&1&-&1&-&1&1&1&-&-&-&1&-&1&-&-&1&1&1&-&-&1&1&-&1&1&-&-&1&-&-&1&-&1&1&-&1&1&-&-&-&1&1&- \\
1&-&-&1&1&1&-&-&-&-&1&1&1&1&-&-&1&1&-&-&1&-&-&1&1&-&-&1&1&1&-&-&1&-&-&1&1&-&-&1&-&-&1&1&-&1&1&- \\
-&-&1&-&1&-&-&-&-&1&1&1&1&1&1&-&1&1&1&-&-&1&-&-&-&-&1&-&1&-&-&-&-&-&1&-&-&-&1&-&-&1&1&1&1&1&-&1 \\
-&1&-&1&-&-&-&-&1&1&1&1&1&1&1&1&1&1&1&1&1&-&1&-&-&1&-&1&-&-&-&-&-&1&-&1&-&1&-&1&1&1&1&1&1&-&1&- \\
1&-&1&1&-&-&-&1&1&1&1&-&-&1&1&1&-&1&1&1&1&1&-&1&1&-&1&1&-&-&-&1&1&-&1&1&1&-&1&1&1&1&1&-&-&1&-&- \\
1&1&-&-&-&-&1&1&1&-&-&1&1&-&-&1&1&1&-&-&1&1&-&-&1&1&-&-&1&-&-&1&1&-&-&1&-&-&1&1&-&1&1&-&1&-&-&1 \\
1&-&-&-&-&1&1&1&-&-&1&-&-&1&-&-&1&1&1&-&1&1&1&-&1&-&-&-&-&-&1&-&-&-&1&-&-&1&1&1&1&1&-&1&-&-&1&- \\
-&-&-&-&1&1&1&1&-&1&-&1&1&-&1&-&1&1&1&1&1&1&1&1&-&-&-&-&-&1&-&1&-&1&-&1&1&1&1&1&1&-&1&-&-&1&-&1 \\
-&-&-&1&1&1&1&-&1&-&1&1&1&1&-&1&-&1&1&1&-&1&1&1&-&-&-&1&1&-&1&1&1&-&1&1&1&1&1&-&-&1&-&-&1&-&1&1 \\
-&-&1&1&1&-&-&1&1&1&-&-&1&1&-&-&1&-&-&1&1&1&-&-&1&-&-&1&1&-&-&1&1&1&-&-&-&1&1&-&1&-&-&1&-&-&1&1 \\
-&1&1&1&-&-&1&-&1&-&-&-&1&1&1&-&-&1&-&-&1&1&1&-&-&-&1&-&-&-&1&-&1&-&-&-&1&1&-&1&-&-&1&-&-&1&1&1 \\
1&1&1&1&-&1&-&1&-&-&-&-&1&1&1&1&1&-&1&-&1&1&1&1&-&1&-&1&-&1&-&1&-&-&-&-&1&-&1&-&-&1&-&1&1&1&1&1 \\
1&1&1&-&1&-&1&1&-&-&-&1&-&1&1&1&1&1&-&1&-&1&1&1&1&-&1&1&1&-&1&1&-&-&-&1&-&1&-&-&1&-&1&1&1&1&1&- \\
1&-&-&-&-&-&1&-&1&1&-&1&-&1&1&-&-&-&1&1&-&1&1&-&1&1&-&-&1&1&-&-&1&-&-&1&1&-&-&-&1&1&-&1&-&1&1&1 \\
-&-&-&-&-&1&-&1&1&-&1&-&1&1&-&1&-&1&1&1&1&1&-&1&1&1&1&-&1&1&1&-&-&1&-&-&-&-&-&-&1&-&1&-&1&1&1&1 \\
-&-&-&1&1&-&1&1&-&1&-&-&1&-&1&-&1&1&1&1&1&-&1&-&1&1&1&1&1&1&1&1&1&-&1&-&-&-&-&1&-&1&-&-&1&1&1&- \\
-&-&1&1&-&1&1&-&1&-&-&1&-&1&-&-&1&1&1&-&-&1&-&-&-&1&1&1&-&1&1&1&1&1&-&1&-&-&1&1&1&-&-&1&1&1&-&- \\
-&-&1&-&1&1&-&1&1&-&-&-&-&-&1&1&-&1&1&-&-&1&1&-&1&-&-&1&1&1&-&-&1&1&-&-&1&1&-&1&-&1&1&1&1&-&-&- \\
-&1&-&1&1&-&1&-&-&-&-&-&-&1&1&1&1&1&-&1&1&1&-&1&-&1&-&-&1&1&1&-&1&1&1&-&1&-&1&-&1&1&1&1&-&-&-&- \\
1&-&1&1&-&1&-&-&-&-&-&1&1&1&1&1&1&-&1&-&1&-&1&-&1&-&1&-&1&1&1&1&1&1&1&1&-&1&-&-&1&1&1&-&-&-&-&1 \\
-&1&1&-&1&-&-&1&-&-&1&1&1&1&1&-&-&1&-&-&-&1&-&-&1&1&-&1&-&1&1&1&-&1&1&1&1&-&-&1&1&1&-&-&-&-&1&1 \\
1&1&-&1&1&-&-&-&-&-&1&-&-&1&1&-&-&1&1&-&-&-&1&1&1&1&-&-&1&-&-&1&1&1&-&-&-&1&1&1&1&-&-&-&1&1&-&1 \\
1&-&1&-&-&-&-&-&-&1&-&1&1&1&-&1&1&1&-&1&-&1&1&1&1&1&1&-&-&1&-&-&1&1&1&-&1&1&1&1&-&-&-&-&1&-&1&- \\
-&1&-&-&-&-&-&1&1&-&1&1&1&-&1&-&1&-&1&-&1&1&1&1&1&1&1&1&1&-&1&-&1&1&1&1&1&1&1&-&-&-&-&1&-&1&-&- \\
1&-&-&1&-&-&1&1&-&1&1&-&-&1&-&-&-&1&-&-&1&1&1&-&-&1&1&1&1&1&-&1&-&1&1&1&1&1&-&-&-&-&1&1&1&-&-&1 \\
1&-&-&-&1&1&-&1&1&1&-&1&-&1&1&-&1&1&-&-&1&-&-&1&-&1&1&1&-&-&1&-&1&-&-&-&1&1&-&-&1&1&-&-&1&-&-&1 \\
-&-&-&-&1&-&1&-&1&-&1&-&1&1&-&1&1&-&-&-&-&-&1&-&1&1&1&1&-&1&-&1&-&-&-&-&1&1&1&-&1&1&1&-&-&1&-&- \\
-&-&-&1&-&1&-&-&-&1&-&-&1&-&1&-&-&-&-&-&-&1&-&1&1&1&1&-&1&-&1&1&-&-&-&1&1&1&1&1&1&1&1&1&1&-&1&- \\
-&-&1&1&1&-&-&1&1&-&-&1&-&1&-&-&-&-&-&1&1&-&1&1&1&1&-&-&-&1&1&-&-&-&1&1&-&1&1&1&-&1&1&1&1&1&-&1 \\
1&1&-&1&1&1&-&1&1&-&-&-&1&1&-&-&1&-&-&1&-&1&1&-&-&-&1&-&1&-&-&-&-&1&1&1&1&-&-&1&1&1&-&-&1&1&-&- \\
1&-&1&-&1&-&1&-&-&-&-&-&1&-&-&-&-&-&1&-&1&1&-&1&-&1&-&1&-&-&-&-&1&1&1&1&-&1&-&-&1&1&1&-&1&1&1&- \\
-&1&-&-&-&1&-&-&-&-&-&1&-&-&-&-&-&1&-&1&1&-&1&-&1&-&1&1&-&-&-&1&1&1&1&-&1&-&1&-&1&1&1&1&1&1&1&1 \\
1&-&-&1&1&-&-&1&-&-&1&1&-&-&-&1&1&-&1&1&-&1&-&-&-&1&1&-&-&-&1&1&1&1&-&-&1&1&-&1&-&1&1&1&-&1&1&1 \\
1&1&-&1&1&-&-&-&1&1&-&1&1&-&-&1&-&1&1&-&1&1&-&-&1&-&-&-&-&1&1&1&-&-&1&-&1&1&-&-&1&-&-&1&1&1&-&- \\
1&-&1&-&-&-&-&-&1&-&1&-&-&-&1&-&1&1&-&1&1&-&-&-&-&-&-&-&1&1&1&1&-&1&-&1&1&1&1&-&-&1&-&-&1&1&1&- \\
-&1&-&-&-&-&-&1&-&1&-&-&-&1&-&1&1&-&1&-&-&-&-&-&-&-&-&1&1&1&1&-&1&-&1&1&1&1&1&1&1&-&1&-&1&1&1&1 \\
1&-&-&1&-&-&1&1&1&-&-&1&1&-&1&1&-&1&-&-&-&-&-&1&-&-&1&1&1&1&-&-&-&1&1&-&-&1&1&1&1&1&-&1&-&1&1&1
\end{pmatrix}
$$

6.6 Conclusion

As we discussed in abstract, we established that of for every odd prime power, $p^r \equiv 1 \pmod 4$ there exist a BHW-array of type $OD(4n;n,n,n,n)$; $n = p^r + 1$, which can be used to generate BH-array of type $OD(4nt;nt,nt,nt,nt)$ for any T-matrices of order t and an orthogonal design of type $OD(4n;2,2,2n-2,2n-2)$. A set of T-matrices of order n where each T_i is a weighing matrix of weight w_i $(1 \leq i \leq 4)$, can be used to generate an orthogonal design of type $OD(4nm;w_1s,w_2s,w_3s,w_4s)$ where $s = 4m; m$ is the order of any Williamson-type matrices. Moreover, we constructed generalized orthogonal designs of type $GOD(4n;k_{11},\dots k_{1u_1};k_{21},\dots,k_{2u_2})$ and $GOD(4n;k_{11},\dots k_{1u_1})$ from weak amicable T-matrices of order n and Williamson-type matrices order n respectively where each GOD is consisting of 16 circulant matrices.

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Appendix A

Here we include some orthogonal designs where $(p,q,r,s) = -(a,b,c,d)$.

$$
\setcounter{MaxMatrixCols}{72}
\begin{bmatrix}
a & c & c & c & r & c & b & d & d & d & s & d & c & a & p & a & a & a & q & q & q & b & q & s & c & r & c & c & c & a & s & d & s & s & s & q & r & r & r & c & r & a & q & b & q & q & q & d & a & p & a & a & a & r & s & s & s & d & s & b & a & p & a & a & a & r & b & q & b & b & b & s \\
r & a & c & c & c & r & s & b & d & d & d & s & p & c & a & p & a & a & q & q & b & q & s & b & r & c & c & c & a & r & d & s & s & s & q & d & r & r & c & r & a & c & b & q & q & q & d & b & p & a & a & a & r & p & s & s & d & s & b & d & p & a & a & a & r & p & q & b & b & b & s & q \\
c & r & a & c & c & c & d & s & b & d & d & d & p & p & c & a & p & a & q & b & q & s & b & b & c & c & c & a & r & c & s & s & s & q & d & s & r & c & r & a & c & c & q & q & q & d & b & q & a & a & a & r & p & a & s & d & s & b & d & d & a & a & a & r & p & a & b & b & b & s & q & b \\
r & c & r & a & c & c & s & d & s & b & d & d & p & p & p & c & a & p & b & q & s & b & b & b & c & c & a & r & c & r & s & s & q & d & s & d & c & r & a & c & c & c & q & q & d & b & q & b & a & a & r & p & a & p & d & s & b & d & d & d & a & a & r & p & a & p & b & b & s & q & b & q \\
r & r & c & r & a & c & s & s & d & s & b & d & a & p & p & p & c & a & q & s & b & b & b & q & c & a & r & c & r & r & s & q & d & s & d & d & r & a & c & c & c & r & q & d & b & q & b & b & a & r & p & a & p & p & s & b & d & d & d & s & a & r & p & a & p & p & b & s & q & b & q & q \\
r & r & r & c & r & a & s & s & s & d & s & b & p & a & p & p & p & c & s & b & b & b & q & b & a & r & c & r & r & r & q & d & s & d & d & d & a & c & c & c & r & c & d & b & q & b & b & b & r & p & a & p & p & p & b & d & d & d & s & d & r & p & a & p & p & p & s & q & b & q & q & q \\
c & a & p & a & a & a & a & c & c & c & r & c & b & d & d & d & s & d & c & r & c & c & c & a & s & d & s & s & s & q & q & q & q & b & q & s & q & b & q & q & q & d & a & p & a & a & a & r & r & r & r & c & r & a & a & p & a & a & a & r & b & q & b & b & b & s & s & s & s & d & s & b \\
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d & q & s & s & s & d & p & c & a & p & a & a & q & d & b & q & b & b & q & s & b & b & b & q & r & p & c & r & c & c & a & c & p & p & p & a & r & a & c & c & c & r & b & s & q & b & q & q & d & q & s & s & s & d & r & r & c & r & a & c & s & s & d & s & b & d & a & p & p & p & c & a \\
q & s & s & s & d & s & c & a & p & a & a & a & d & b & q & b & b & b & s & b & b & b & q & b & p & c & r & c & c & c & c & p & p & p & a & p & a & c & c & c & r & c & s & q & b & q & q & q & q & s & s & s & d & s & r & r & r & c & r & a & s & s & s & d & s & b & p & a & p & p & p & c \\
p & a & p & p & p & c & q & b & q & q & q & d & d & d & d & s & d & q & r & c & r & r & r & p & a & a & a & p & a & c & q & q & q & b & q & s & b & q & b & b & b & s & d & d & d & s & d & q & r & r & r & c & r & a & c & a & p & a & a & a & a & c & c & c & r & c & b & d & d & d & s & d \\
a & p & p & p & c & a & b & q & q & q & d & b & d & d & s & d & q & s & c & r & r & r & p & c & a & a & p & a & c & p & q & q & b & q & s & b & q & b & b & b & s & q & d & d & s & d & q & s & r & r & c & r & a & c & p & c & a & p & a & a & r & a & c & c & c & r & s & b & d & d & d & s \\
p & p & p & c & a & p & q & q & q & d & b & q & d & s & d & q & s & s & r & r & r & p & c & r & a & p & a & c & p & p & q & b & q & s & b & b & b & b & b & s & q & b & d & s & d & q & s & s & r & c & r & a & c & c & p & p & c & a & p & a & c & r & a & c & c & c & d & s & b & d & d & d \\
p & p & c & a & p & a & q & q & d & b & q & b & s & d & q & s & s & s & r & r & p & c & r & c & p & a & c & p & p & p & b & q & s & b & b & b & b & b & s & q & b & q & s & d & q & s & s & s & c & r & a & c & c & c & p & p & p & c & a & p & r & c & r & a & c & c & s & d & s & b & d & d \\
p & c & a & p & a & a & q & d & b & q & b & b & d & q & s & s & s & d & r & p & c & r & c & c & a & c & p & p & p & a & q & s & b & b & b & q & b & s & q & b & q & q & d & q & s & s & s & d & r & a & c & c & c & r & a & p & p & p & c & a & r & r & c & r & a & c & s & s & d & s & b & d \\
c & a & p & a & a & a & d & b & q & b & b & b & q & s & s & s & d & s & p & c & r & c & c & c & c & p & p & p & a & p & s & b & b & b & q & b & s & q & b & q & q & q & q & s & s & s & d & s & a & c & c & c & r & c & p & a & p & p & p & c & r & r & r & c & r & a & s & s & s & d & s & b \\
q & b & q & q & q & d & d & d & d & s & d & q & p & a & p & p & p & c & a & a & a & p & a & c & q & q & q & b & q & s & r & c & r & r & r & p & d & d & d & s & d & q & r & r & r & c & r & a & b & q & b & b & b & s & b & d & d & d & s & d & c & a & p & a & a & a & a & c & c & c & r & c \\
b & q & q & q & d & b & d & d & s & d & q & s & a & p & p & p & c & a & a & a & p & a & c & p & q & q & b & q & s & b & c & r & r & r & p & c & d & d & s & d & q & s & r & r & c & r & a & c & q & b & b & b & s & q & s & b & d & d & d & s & p & c & a & p & a & a & r & a & c & c & c & r \\
q & q & q & d & b & q & d & s & d & q & s & s & p & p & p & c & a & p & a & p & a & c & p & p & q & b & q & s & b & b & r & r & r & p & c & r & d & s & d & q & s & s & r & c & r & a & c & c & b & b & b & s & q & b & d & s & b & d & d & d & p & p & c & a & p & a & c & r & a & c & c & c \\
q & q & d & b & q & b & s & d & q & s & s & s & p & p & c & a & p & a & p & a & c & p & p & p & b & q & s & b & b & b & r & r & p & c & r & c & s & d & q & s & s & s & c & r & a & c & c & c & b & b & s & q & b & q & s & d & s & b & d & d & p & p & p & c & a & p & r & c & r & a & c & c \\
q & d & b & q & b & b & d & q & s & s & s & d & p & c & a & p & a & a & a & c & p & p & p & a & q & s & b & b & b & q & r & p & c & r & c & c & d & q & s & s & s & d & r & a & c & c & c & r & b & s & q & b & q & q & s & s & d & s & b & d & a & p & p & p & c & a & r & r & c & r & a & c \\
d & b & q & b & b & b & q & s & s & s & d & s & c & a & p & a & a & a & c & p & p & p & a & p & s & b & b & b & q & b & p & c & r & c & c & c & q & s & s & s & d & s & a & c & c & c & r & c & s & q & b & q & q & q & s & s & s & d & s & b & p & a & p & p & p & c & r & r & r & c & r & a
\end{bmatrix}
$$

OD(72; 18, 18, 18, 18)

a a c p b b d a b q b b b d q p p r b p a p b q b a d b b p c a a a p a q c a a b s q q c r c d p c c s q d d d s d r q d d c a r r d s d r q d d c a r r r c r s a r r d b s s
b a a c p b b d a b q p b b d q p p r b p a q b a d b b p c a a b p a q c a a b s q q a r c d p c c s q d d c s d r q d d c a r r d s d r q d d c a r r d c r s a r r d b s s r
q b a a c p b b d a b a p b b d q p p r b p b a d b b p c a a b q a q c a a b s q q a p c d p c c s q d d c r d r q d d c a r r d s d r q d d c a r r d s r s a r r d b s s r c
b q b a a c p b b d a p a p b b d q p p r b a d b b p c a a b q b q c a a b s q q a p a d p c c s q d d c r c r q d d c a r r d s d r q d d c a r r d s d s a r r d b s s r c r
a b q b a a c p b b d b p a p b b d q p p r d b b p c a a b q b a c a a b s q q a p a q p c c s q d d c r c d q d d c a r r d s d r q d d c a r r d s d r a r r d b s s r c r s
d a b q b a a c p b b r b p a p b b d q p p b b p c a a b q b a d a a b s q q a p a q c c c s q d d c r c d p d d c a r r d s d r q d d c a r r d s d r q r r d b s s r c r s a
b d a b q b a a c p b p r b p a p b b d q p b p c a a b q b a d b a b s q q a p a q c a c s q d d c r c d p c d c a r r d s d r q d d c a r r d s d r q d r d b s s r c r s a r
b b d a b q b a a c p p p r b p a p b b d q p c a a b q b a d b b b s q q a p a q c a a s q d d c r c d p c c c a r r d s d r q d d c a r r d s d r q d d d b s s r c r s a r r
p b b d a b q b a a c q p p r b p a p b b d c a a b q b a d b b p s q q a p a q c a a b q d d c r c d p c c s a r r d s d r q d d c a r r d s d r q d d c b s s r c r s a r r d
c p b b d a b q b a a d q p p r b p a p b b a a b q b a d b b p c q q a p a q c a a b s d d c r c d p c c s q r r d s d r q d d c a r r d s d r q d d c a s s r c r s a r r d b
a c p b b d a b q b a b d q p p r b p a p b a b q b a d b b p c a q a p a q c a a b s q d c r c d p c c s q d r d s d r q d d c a r r d s d r q d d c a r s r c r s a r r d b s
b b d q p p r b p a p a a c p b b d a b q b a p a q c a a b s q q b q b a d b b p c a a d s d r q d d c a r r c r c d p c c s q d d r c r s a r r d b s s d s d r q d d c a r r
p b b d q p p r b p a b a a c p b b d a b q p a q c a a b s q q a q b a d b b p c a a b s d r q d d c a r r d r c d p c c s q d d c c r s a r r d b s s r s d r q d d c a r r d
a p b b d q p p r b p q b a a c p b b d a b a q c a a b s q q a p b a d b b p c a a b q d r q d d c a r r d s c d p c c s q d d c r r s a r r d b s s r c d r q d d c a r r d s
p a p b b d q p p r b b q b a a c p b b d a q c a a b s q q a p a a d b b p c a a b q b r q d d c a r r d s d d p c c s q d d c r c s a r r d b s s r c r r q d d c a r r d s d
b p a p b b d q p p r a b q b a a c p b b d c a a b s q q a p a q d b b p c a a b q b a q d d c a r r d s d r p c c s q d d c r c d a r r d b s s r c r s q d d c a r r d s d r
r b p a p b b d q p p d a b q b a a c p b b a a b s q q a p a q c b b p c a a b q b a d d d c a r r d s d r q c c s q d d c r c d p r r d b s s r c r s a d d c a r r d s d r q
p r b p a p b b d q p b d a b q b a a c p b a b s q q a p a q c a b p c a a b q b a d b d c a r r d s d r q d c s q d d c r c d p c r d b s s r c r s a r d c a r r d s d r q d
p p r b p a p b b d q b b d a b q b a a c p b s q q a p a q c a a p c a a b q b a d b b c a r r d s d r q d d s q d d c r c d p c c d b s s r c r s a r r c a r r d s d r q d d
q p p r b p a p b b d p b b d a b q b a a c s q q a p a q c a a b c a a b q b a d b b p a r r d s d r q d d c q d d c r c d p c c s b s s r c r s a r r d a r r d s d r q d d c
d q p p r b p a p b b c p b b d a b q b a a q q a p a q c a a b s a a b q b a d b b p c r r d s d r q d d c a d d c r c d p c c s q s s r c r s a r r d b r r d s d r q d d c a
b d q p p r b p a p b a c p b b d a b q b a q a p a q c a a b s q a b q b a d b b p c a r d s d r q d d c a r d c r c d p c c s q d s r c r s a r r d b s r d s d r q d d c a r
q b q p s q q a r p p p a p b r p p q d b b a a c p b b d a b q b b b d q p p r b p a p c p r s s b c s d s c d q s c c p d c r c d d q s c c p d c r c d r a c d d q r d s d r
b q p s q q a r p p q a p b r p p q d b b p b a a c p b b d a b q p b b d q p p r b p a p r s s b c s d s c c q s c c p d c r c d d q s c c p d c r c d d a c d d q r d s d r r
q p s q q a r p p q b p b r p p q d b b p a q b a a c p b b d a b a p b b d q p p r b p r s s b c s d s c c p s c c p d c r c d d q s c c p d c r c d d q c d d q r d s d r r a
p s q q a r p p q b q b r p p q d b b p a p b q b a a c p b b d a p a p b b d q p p r b s s b c s d s c c p r c c p d c r c d d q s c c p d c r c d d q s d d q r d s d r r a c
s q q a r p p q b q p r p p q d b b p a p b a b q b a a c p b b d b p a p b b d q p p r s b c s d s c c p r s c p d c r c d d q s c c p d c r c d d q s c d q r d s d r r a c d
q q a r p p q b q p s p p q d b b p a p b r d a b q b a a c p b b r b p a p b b d q p p b c s d s c c p r s s p d c r c d d q s c c p d c r c d d q s c c q r d s d r r a c d d
q a r p p q b q p s q p q d b b p a p b r p b d a b q b a a c p b p r b p a p b b d q p c s d s c c p r s s b d c r c d d q s c c p d c r c d d q s c c p r d s d r r a c d d q
a r p p q b q p s q q q d b b p a p b r p p b b d a b q b a a c p p p r b p a p b b d q s d s c c p r s s b c c r c d d q s c c p d c r c d d q s c c p d d s d r r a c d d q r
r p p q b q p s q q a d b b p a p b r p p q p b b d a b q b a a c q p p r b p a p b b d d s c c p r s s b c s r c d d q s c c p d c r c d d q s c c p d c s d r r a c d d q r d
p p q b q p s q q a r b b p a p b r p p q d c p b b d a b q b a a d q p p r b p a p b b s c c p r s s b c s d c d d q s c c p d c r c d d q s c c p d c r d r r a c d d q r d s
p q b q p s q q a r p b p a p b r p p q d b a c p b b d a b q b a b d q p p r b p a p b c c p r s s b c s d s d d q s c c p d c r c d d q s c c p d c r c r r a c d d q r d s d
p a p b r p p q d b b q b q p s q q a r p p b b d q p p r b p a p a a c p b b d a b q b d q s c c p d c r c d c p r s s b c s d s c r a c d d q r d s d r d q s c c p d c r c d
a p b r p p q d b b p b q p s q q a r p p q p b b d q p p r b p a b a a c p b b d a b q q s c c p d c r c d d p r s s b c s d s c c a c d d q r d s d r r q s c c p d c r c d d
p b r p p q d b b p a q p s q q a r p p q b a p b b d q p p r b p q b a a c p b b d a b s c c p d c r c d d q r s s b c s d s c c p c d d q r d s d r r a s c c p d c r c d d q
b r p p q d b b p a p p s q q a r p p q b q p a p b b d q p p r b b q b a a c p b b d a c c p d c r c d d q s s s b c s d s c c p r d d q r d s d r r a c c c p d c r c d d q s
r p p q d b b p a p b s q q a r p p q b q p b p a p b b d q p p r a b q b a a c p b b d c p d c r c d d q s c s b c s d s c c p r s d q r d s d r r a c d c p d c r c d d q s c
p p q d b b p a p b r q q a r p p q b q p s r b p a p b b d q p p d a b q b a a c p b b p d c r c d d q s c c b c s d s c c p r s s q r d s d r r a c d d p d c r c d d q s c c
p q d b b p a p b r p q a r p p q b q p s q p r b p a p b b d q p b d a b q b a a c p b d c r c d d q s c c p c s d s c c p r s s b r d s d r r a c d d q d c r c d d q s c c p
q d b b p a p b r p p a r p p q b q p s q q p p r b p a p b b d q b b d a b q b a a c p c r c d d q s c c p d s d s c c p r s s b c d s d r r a c d d q r c r c d d q s c c p d
d b b p a p b r p p q r p p q b q p s q q a q p p r b p a p b b d p b b d a b q b a a c r c d d q s c c p d c d s c c p r s s b c s s d r r a c d d q r d r c d d q s c c p d c
b b p a p b r p p q d p p q b q p s q q a r d q p p r b p a p b b c p b b d a b q b a a c d d q s c c p d c r s c c p r s s b c s d d r r a c d d q r d s c d d q s c c p d c r
b p a p b r p p q d b p q b q p s q q a r p b d q p p r b p a p b a c p b b d a b q b a d d q s c c p d c r c c c p r s s b c s d s r r a c d d q r d s d d d q s c c p d c r c
r c r s a r r d b s s s d s c b s s r p c c r a c d d q r d s d r s b d r r a s r c r s a a c p b b d a b q b b b d q p p r b p a p p r a q q s p q b q p b d q p p r b p a p b
c r s a r r d b s s r d s c b s s r p c c s a c d d q r d s d r r b d r r a s r c r s s b a a c p b b d a b q p b b d q p p r b p a r a q q s p q b q p p d q p p r b p a p b b
r s a r r d b s s r c s c b s s r p c c s d c d d q r d s d r r a d r r a s r c r s s b q b a a c p b b d a b a p b b d q p p r b p a q q s p q b q p p r q p p r b p a p b b d
s a r r d b s s r c r c b s s r p c c s d s d d q r d s d r r a c r r a s r c r s s b d b q b a a c p b b d a p a p b b d q p p r b q s p q b q p p r a p p r b p a p b b d q
a r r d b s s r c r s b s s r p c c s d s c d q r d s d r r a c d r a s r c r s s b d r a b q b a a c p b b d b p a p b b d q p p r q s p q b q p p r a q p r b p a p b b d q p
r r d b s s r c r s a s s r p c c s d s c b q r d s d r r a c d d a s r c r s s b d r r d a b q b a a c p b b r b p a p b b d q p p s p q b q p p r a q q r b p a p b b d q p p
r d b s s r c r s a r s r p c c s d s c b s r d s d r r a c d d q s r c r s s b d r r a b d a b q b a a c p b p r b p a p b b d q p p q b q p p r a q q s b p a p b b d q p p r
d b s s r c r s a r r r p c c s d s c b s s d s d r r a c d d q r r c r s s b d r r a s b b d a b q b a a c p p p r b p a p b b d q q b q p p r a q q s p p a p b b d q p p r b
b s s r c r s a r r d p c c s d s c b s s r s d r r a c d d q r d c r s s b d r r a s r p b b d a b q b a a c q p p r b p a p b b d b q p p r a q q s p q a p b b d q p p r b p
s s r c r s a r r d b c c s d s c b s s r p d r r a c d d q r d s r s s b d r r a s r c c p b b d a b q b a a d q p p r b p a p b b q p p r a q q s p q b p b b d q p p r b p a
s r c r s a r r d b s c s d s c b s s r p c r r a c d d q r d s d s s b d r r a s r c r a c p b b d a b q b a b d q p p r b p a p b p p r a q q s p q b q b b d q p p r b p a p
s d s c b s s r p c c r c r s a r r d b s s s b d r r a s r c r s r a c d d q r d s d r b b d q p p r b p a p a a c p b b d a b q b b d q p p r b p a p b p r a q q s p q b q p
d s c b s s r p c c s c r s a r r d b s s r b d r r a s r c r s s a c d d q r d s d r r p b b d q p p r b p a b a a c p b b d a b q d q p p r b p a p b b r a q q s p q b q p p
s c b s s r p c c s d r s a r r d b s s r c d r r a s r c r s s b c d d q r d s d r r a a p b b d q p p r b p q b a a c p b b d a b q p p r b p a p b b d a q q s p q b q p p r
c b s s r p c c s d s s a r r d b s s r c r r r a s r c r s s b d d d q r d s d r r a c p a p b b d q p p r b b q b a a c p b b d a p p r b p a p b b d q q q s p q b q p p r a
b s s r p c c s d s c a r r d b s s r c r s r a s r c r s s b d r d q r d s d r r a c d b p a p b b d q p p r a b q b a a c p b b d p r b p a p b b d q p q s p q b q p p r a q
s s r p c c s d s c b r r d b s s r c r s a a s r c r s s b d r r q r d s d r r a c d d r b p a p b b d q p p d a b q b a a c p b b r b p a p b b d q p p s p q b q p p r a q q
s r p c c s d s c b s r d b s s r c r s a r s r c r s s b d r r a r d s d r r a c d d q p r b p a p b b d q p b d a b q b a a c p b b p a p b b d q p p r p q b q p p r a q q s
r p c c s d s c b s s d b s s r c r s a r r r c r s s b d r r a s d s d r r a c d d q r p p r b p a p b b d q b b d a b q b a a c p p a p b b d q p p r b q b q p p r a q q s p
p c c s d s c b s s r b s s r c r s a r r d c r s s b d r r a s r s d r r a c d d q r d q p p r b p a p b b d p b b d a b q b a a c a p b b d q p p r b p b q p p r a q q s p q
c c s d s c b s s r p s s r c r s a r r d b r s s b d r r a s r c d r r a c d d q r d s d q p p r b p a p b b c p b b d a b q b a a p b b d q p p r b p a q p p r a q q s p q b
c s d s c b s s r p c s r c r s a r r d b s s s b d r r a s r c r r r a c d d q r d s d b d q p p r b p a p b a c p b b d a b q b a b b d q p p r b p a p p p r a q q s p q b q
s d s c b s s r p c c c r c d p c c s q d d s b d r r a s r c r s c p r s s b c s d s c a c p b b d a b q b a q s b a a c q a p a q a a c p b b d a b q b b b d q p p r b p a p
d s c b s s r p c c s r c d p c c s q d d c b d r r a s r c r s s p r s s b c s d s c c c p b b d a b q b a a s b a a c q a p a q q b a a c p b b d a b q p b b d q p p r b p a
s c b s s r p c c s d c d p c c s q d d c r d r r a s r c r s s b r s s b c s d s c c p p b b d a b q b a a c b a a c q a p a q q s q b a a c p b b d a b a p b b d q p p r b p
c b s s r p c c s d s d p c c s q d d c r c r r a s r c r s s b d s s b c s d s c c p r b b d a b q b a a c p a a c q a p a q q s b b q b a a c p b b d a p a p b b d q p p r b
b s s r p c c s d s c p c c s q d d c r c d r a s r c r s s b d r s b c s d s c c p r s b d a b q b a a c p b a c q a p a q q s b a a b q b a a c p b b d b p a p b b d q p p r
s s r p c c s d s c b c c s q d d c r c d p a s r c r s s b d r r b c s d s c c p r s s d a b q b a a c p b b c q a p a q q s b a a d a b q b a a c p b b r b p a p b b d q p p
s r p c c s d s c b s c s q d d c r c d p c s r c r s s b d r r a c s d s c c p r s s b a b q b a a c p b b d q a p a q q s b a a c b d a b q b a a c p b p r b p a p b b d q p
r p c c s d s c b s s s q d d c r c d p c c r c r s s b d r r a s s d s c c p r s s b c b q b a a c p b b d a a p a q q s b a a c q b b d a b q b a a c p p p r b p a p b b d q
p c c s d s c b s s r q d d c r c d p c c s c r s s b d r r a s r d s c c p r s s b c s q b a a c p b b d a b p a q q s b a a c q a p b b d a b q b a a c q p p r b p a p b b d
c c s d s c b s s r p d d c r c d p c c s q r s s b d r r a s r c s c c p r s s b c s d b a a c p b b d a b q a q q s b a a c q a p c p b b d a b q b a a d q p p r b p a p b b
c s d s c b s s r p c d c r c d p c c s q d s s b d r r a s r c r c c p r s s b c s d s a a c p b b d a b q b q q s b a a c q a p a a c p b b d a b q b a b d q p p r b p a p b
c r c d p c c s q d d s d s c b s s r p c c c p r s s b c s d s c s b d r r a s r c r s q s b a a c q a p a q a c p b b d a b q b a b b d q p p r b p a p a a c p b b d a b q b
r c d p c c s q d d c d s c b s s r p c c s p r s s b c s d s c c b d r r a s r c r s s s b a a c q a p a q q c p b b d a b q b a a p b b d q p p r b p a b a a c p b b d a b q
c d p c c s q d d c r s c b s s r p c c s d r s s b c s d s c c p d r r a s r c r s s b b a a c q a p a q q s p b b d a b q b a a c a p b b d q p p r b p q b a a c p b b d a b
d p c c s q d d c r c c b s s r p c c s d s s s b c s d s c c p r r r a s r c r s s b d a a c q a p a q q s b b b d a b q b a a c p p a p b b d q p p r b b q b a a c p b b d a
p c c s q d d c r c d b s s r p c c s d s c s b c s d s c c p r s r a s r c r s s b d r a c q a p a q q s b a b d a b q b a a c p b b p a p b b d q p p r a b q b a a c p b b d
c c s q d d c r c d p s s r p c c s d s c b b c s d s c c p r s s a s r c r s s b d r r c q a p a q q s b a a d a b q b a a c p b b r b p a p b b d q p p d a b q b a a c p b b
c s q d d c r c d p c s r p c c s d s c b s c s d s c c p r s s b s r c r s s b d r r a q a p a q q s b a a c a b q b a a c p b b d p r b p a p b b d q p b d a b q b a a c p b
s q d d c r c d p c c r p c c s d s c b s s s d s c c p r s s b c r c r s s b d r r a s a p a q q s b a a c q b q b a a c p b b d a p p r b p a p b b d q b b d a b q b a a c p
q d d c r c d p c c s p c c s d s c b s s r d s c c p r s s b c s c r s s b d r r a s r p a q q s b a a c q a q b a a c p b b d a b q p p r b p a p b b d p b b d a b q b a a c
d d c r c d p c c s q c c s d s c b s s r p s c c p r s s b c s d r s s b d r r a s r c a q q s b a a c q a p b a a c p b b d a b q d q p p r b p a p b b c p b b d a b q b a a
d c r c d p c c s q d c s d s c b s s r p c c c p r s s b c s d s s s b d r r a s r c r q q s b a a c q a p a a a c p b b d a b q b b d q p p r b p a p b a c p b b d a b q b a

$OD(88;22,22,22,22)$

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$OD(104;26,26,26,26)$

$$
\left[\begin{array}{c}
d c p q p p a s r p b p p a s r p b p p a a p p q p c s a p p q p c s p a a q a r d a p p b p c d a p p b p c d p a a q a c s a p p q p r d a p p q p r d p a a b a c d \\
a d c p q p p a s r p b p p a s r p b p p p p q p c s a p p q p c s a a a q a r d p p p b p c d a p p b p c d a a a q a c s p p p q p r d a p p q p r d a a a b a c d p \\
p a d c p q p p a s r p b p p a s r p b p p q p c s a p p q p c s a p a q a r d p a p b p c d a p p b p c d a p a q a c s p a p q p r d a p p q p r d a p a b a c d p a \\
p p a d c p q p p a s r p b p p a s r p b q p c s a p p q p c s a p p q a r d p a a b p c d a p p b p c d a p p q a c s p a a q p r d a p p q p r d a p p b a c d p a a \\
q p p a d c p b p p a s r p b p p a s r p p c s a p p q p c s a p p q a r d p a a q p c d a p p b p c d a p p b a c s p a a q p r d a p p q p r d a p p q a c d p a a b \\
p q p p a d c p b p p a s r p b p p a s r c s a p p q p c s a p p q p r d p a a q a c d a p p b p c d a p p b p c s p a a q a r d a p p q p r d a p p q p c d p a a b a \\
c p q p p a d r p b p p a s r p b p p a s s a p p q p c s a p p q p c d p a a q a r d a p p b p c d a p p b p c s p a a q a c d a p p q p r d a p p q p r d p a a b a c \\
s r p b p p a d c p q p p a s r p b p p a a p p q p c s p a a q a r d a p p q p c s a p p b p c d p a a q a c s a p p b p c d a p p q p r d p a a b a c d a p p q p r d \\
a s r p b p p a d c p q p p a s r p b p p p p q p c s a a a q a r d p p p q p c s a p p b p c d a a a q a c s p p p b p c d a p p q p r d a a a b a c d p p p q p r d a \\
p a s r p b p p a d c p q p p a s r p b p p q p c s a p a q a r d p a p q p c s a p p b p c d a p a q a c s p a p b p c d a p p q p r d a p a b a c d p a p q p r d a p \\
p p a s r p b p p a d c p q p p a s r p b q p c s a p p q a r d p a a q p c s a p p b p c d a p p q a c s p a a b p c d a p p q p r d a p p b a c d p a a q p r d a p p \\
b p p a s r p q p p a d c p b p p a s r p p c s a p p q a r d p a a q p c s a p p q p c d a p p b a c s p a a q p c d a p p b p r d a p p q a c d p a a b p r d a p p q \\
p b p p a s r p q p p a d c p b p p a s r c s a p p q p r d p a a q a c s a p p q p c d a p p b p c s p a a q a c d a p p b p r d a p p q p c d p a a b a r d a p p q p \\
r p b p p a s c p q p p a d r p b p p a s s a p p q p c d p a a q a r s a p p q p c d a p p b p c s p a a q a c d a p p b p c d a p p q p r d p a a b a c d a p p q p r \\
s r p b p p a s r p b p p a d c p q p p a p a a q a r d a p p q p c s a p p q p c s p a a q a c s a p p b p c d a p p b p c d p a a b a c d a p p q p r d a p p q p r d \\
a s r p b p p a s r p b p p a d c p q p p a a q a r d p p p q p c s a p p q p c s a a a q a c s p p p b p c d a p p b p c d a a a b a c d p p p q p r d a p p q p r d a \\
p a s r p b p p a s r p b p p a d c p q p a q a r d p a p q p c s a p p q p c s a p a q a c s p a p b p c d a p p b p c d a p a b a c d p a p q p r d a p p q p r d a p \\
p p a s r p b p p a s r p b p p a d c p q q a r d p a a q p c s a p p q p c s a p p q a c s p a a b p c d a p p b p c d a p p b a c d p a a q p r d a p p q p r d a p p \\
b p p a s r p b p p a s r p q p p a d c p a r d p a a q p c s a p p q p c s a p p q a c s p a a q p c d a p p b p c d a p p b a c d p a a b p r d a p p q p r d a p p q \\
p b p p a s r p b p p a s r p q p p a d c r d p a a q a c s a p p q p c s a p p q p c s p a a q a c d a p p b p c d a p p b p c d p a a b a r d a p p q p r d a p p q p \\
r p b p p a s r p b p p a s c p q p p a d d p a a q a r s a p p q p c s a p p q p c s p a a q a c d a p p b p c d a p p b p c d p a a b a c d a p p q p r d a p p q p r \\
p a a b a r d p a a b a r d a p p b p c s d c p q p p a s r p b p p a s r p b p p a r p q p p a d r p q p p a d c a b a a p d r a q a a p s r a q a a p s r p b p p a d \\
a a b a r d p a a b a r d p p p b p c s a a d c p q p p a s r p b p p a s r p b p p p q p p a d r p q p p a d r a b a a p d c a q a a p s r a q a a p s r p b p p a d r \\
a b a r d p a a b a r d p a p b p c s a p p a d c p q p p a s r p b p p a s r p b p q p p a d r p q p p a d r p b a a p d c a q a a p s r a q a a p s r a b p p a d r p \\
b a r d p a a b a r d p a a b p c s a p p p p a d c p q p p a s r p b p p a s r p b p p a d r p q p p a d r p q a a p d c a b a a p s r a q a a p s r a q p p a d r p b \\
a r d p a a b a r d p a a b p c s a p p b q p p a d c p b p p a s r p b p p a s r p p a d r p q p p a d r p q a p a p d c a b a a p s r a q a a p s r a q a p a d r p b p \\
r d p a a b a r d p a a b a c s a p p b p p q p p a d c p b p p a s r p b p p a s r a d r p q p p a d r p q p p p d c a b a a p s r a q a a p s r a q a a a d r p b p p \\
d p a a b a r d p a a b a r s a p p b p c c p q p p a d r p b p p a s r p b p p a s d r p q p p a d r p q p p a d c a b a a p s r a q a a p s r a q a a p d r p b p p a \\
p a a b a r d a p p b p c s p a a b a r d s r p b p p a d c p q p p a s r p b p p a r p q p p a d c a b a a p d r p q p p a d r a q a a p s r p b p p a d r a q a a p s \\
a a b a r d p p p b p c s a a a b a r d p a s r p b p p a d c p q p p a s r p b p p p q p p a d r a b a a p d c p q p p a d r a q a a p s r p b p p a d r a q a a p s r \\
a b a r d p a p b p c s a p a b a r d p a p a s r p b p p a d c p q p p a s r p b p q p p a d r p b a a p d c a q p p a d r p q a a p s r a b p p a d r p q a a p s r a \\
b a r d p a a b p c s a p p b a r d p a a p p a s r p b p p a d c p q p p a s r p b p p a d r p q a a p d c a b p p a d r p q a a p s r a q p p a d r p b a a p s r a q \\
a r d p a a b p c s a p p b a r d p a a b b p p a s r p q p p a d c p b p p a s r p p a d r p q a p a p d c a b a p a d r p q a p a p s r a q a p a d r p b p a p s r a q a \\
r d p a a b a c s a p p b p r d p a a b a p b p p a s r p q p p a d c p b p p a s r a d r p q p p p d c a b a a a d r p q p p p s r a q a a a d r p b p p p s r a q a a \\
d p a a b a r s a p p b p c d p a a b a r r p b p p a s c p q p p a d r p b p p a s d r p q p p a d c a b a a p d r p q p p a s r a q a a p d r p b p p a s r a q a a p \\
a p p b p c s p a a b a r d p a a b a r d s r p b p p a s r p b p p a d c p q p p a c a b a a p d r p q p p a d r p q p p a d r p b p p a d r a q a a p s r a q a a p s \\
p p b p c s a a a b a r d p a a b a r d p a s r p b p p a s r p b p p a d c p q p p a b a a p d c p q p p a d r p q p p a d r p b p p a d r a q a a p s r a q a a p s r \\
p b p c s a p a b a r d p a a b a r d p a p a s r p b p p a s r p b p p a d c p q p b a a p d c a q p p a d r p q p p a d r p b p p a d r p q a a p s r a q a a p s r a \\
b p c s a p p b a r d p a a b a r d p a a p p a s r p b p p a s r p b p p a d c p q a a p d c a b p p a d r p q p p a d r p q p p a d r p b a a p s r a q a a p s r a q \\
p c s a p p b a r d p a a b a r d p a a b b p p a s r p b p p a s r p q p p a d c p a p d c a b a p a d r p q p p a d r p q p p a d r p b p a p s r a q a a p s r a q a \\
c s a p p b p r d p a a b a r d p a a b a p b p p a s r p b p p a s r p q p p a d c p d c a b a a a d r p q p p a d r p q p p a d r p b p p p s r a q a a p s r a q a a \\
s a p p b p c d p a a b a r d p a a b a r r p b p p a s r p b p p a s c p q p p a d d c a b a a p d r p q p p a d r p q p p a d r p b p p a s r a q a a p s r a q a a p \\
p a a q a r s p a a q a r s a p p b p r d c a b a a p s c a b a a p s r p q p p a s d c p q p p a s r p b p p a s r p b p p a c p q p p a s c p q p p a s r a q a a p d \\
a a q a r s p a a q a r s p p p b p r d a a b a a p s c a b a a p s c p q p p a s r a d c p q p p a s r p b p p a s r p b p p p q p p a s c p q p p a s c a q a a p d r \\
a q a r s p a a q a r s p a p b p r d a p b a a p s c a b a a p s c a q p p a s r p p a d c p q p p a s r p b p p a s r p b p q p p a s c p q p p a s c p q a a p d r a \\
q a r s p a a q a r s p a a b p r d a p p a a p s c a b a a p s c a b p p a s r p q p p a d c p q p p a s r p b p p a s r p b p p a s c p q p p a s c p q a a p d r a q \\
a r s p a a q a r s p a a q p r d a p p b a p s c a b a a p s c a b a p a s r p q p q p p a d c p b p p a s r p b p p a s r p p a s c p q p p a s c p q p a p d r a q a \\
r s p a a q a r s p a a q a r d a p p b p p s c a b a a p s c a b a a a s r p q p p p q p p a d c p b p p a s r p b p p a s r a s c p q p p a s c p q p p p d r a q a a \\
s p a a q a r s p a a q a r d a p p b p r s c a b a a p s c a b a a p s r p q p p a c p q p p a d r p b p p a s r p b p p a s s c p q p p a s c p q p p a d r a q a a p \\
p a a q a r s a p p b p r d p a a q a r s c a b a a p s r p q p p a s c a b a a p s s r p b p p a d c p q p p a s r p b p p a c p q p p a s r a q a a p d c p q p p a s \\
a a q a r s p p p b p r d a a a q a r s p a b a a p s c p q p p a s r a b a a p s c a s r p b p p a d c p q p p a s r p b p p p q p p a s c a q a a p d r p q p p a s c \\
a q a r s p a p b p r d a p a q a r s p a b a a p s c a q p p a s r p b a a p s c a p a s r p b p p a d c p q p p a s r p b p q p p a s c p q a a p d r a q p p a s c p \\
q a r s p a a b p r d a p p q a r s p a a a a p s c a b p p a s r p q a a p s c a b p p a s r p b p p a d c p q p p a s r p b p p a s c p q a a p d r a q p p a s c p q \\
a r s p a a q p r d a p p b a r s p a a q a p s c a b a p a s r p q a p s c a b a b p p a s r p q p p a d c p b p p a s r p p a s c p q p a p d r a q a p a s c p q p \\
r s p a a q a r d a p p b p r s p a a q a p s c a b a a a s r p q p p p s c a b a a p b p p a s r p q p p a d c p b p p a s r a s c p q p p p d r a q a a a s c p q p p \\
s p a a q a r d a p p b p r s p a a q a r s c a b a a p s r p q p p a s c a b a a p b p p a s c p q p p a d r p b p p a s s c p q p p a d r a q a a p s c p q p p a \\
a p p b p r d p a a q a r s p a a q a r s r p q p p a s c a b a a p s c a b a a p s s r p b p p a s r p b p p a d c p q p p a r a q a a p d c p q p p a s c p q p p a s \\
p p b p r d a a a q a r s p a a q a r s p p q p p a s r a b a a p s c a b a a p s c a s r p b p p a s r p b p p a d c p q p p a q a a p d r p q p p a s c p q p p a s c \\
p b p r d a p a q a r s p a a q a r s p a q p p a s r p b a a p s c a b a a p s c a p a s r p b p p a s r p b p p a d c p q a a p d r a q p p a s c p q p p a s c p \\
b p r d a p p q a r s p a a q a r s p a a p p a s r p q a a p s c a b a a p s c a b p p a s r p b p p a s r p b p p a d c p q a a p d r a q p p a s c p q p p a s c p q \\
p r d a p p b a r s p a a q a r s p a a q p a s r p q a p a p s c a b a a p s c a b a b p p a s r p b p p a s r p q p p a d c p a p d r a q a p a s c p q p p a s c p q p \\
r d a p p b p r s p a a q a r s p a a q a a s r p q p p p s c a b a a p s c a b a a p b p p a s r p b p p a s r p q p p a d c p d r a q a a a s c p q p p a s c p q p p \\
d a p p b p r s p a a q a r s p a a q a r s r p q p p a s c a b a a p s c a b a a p r p b p p a s r p b p p a s c p q p p a d d r a q a a p s c p q p p a s c p q p p a \\
p a a b a c s p a a b a c s a p p q p r s c p b p p a d c p b p p a d c a q a a p s r a b a a p d r a b a a p d c p b p p a s d c p q p p a s r p b p p a s r p b p p a \\
a a b a c s p a a b a c s p p p q p r s a p b p p a d c p b p p a d c a q a a p s c a b a a p d r a b a a p d r p b p p a s c a d c p q p p a s r p b p p a s r p b p p \\
a b a c s p a a b a c s p a p q p r s a p p b p p a d c p b p p a d c p q a a p s c a b a a p d r a b a a p d r a b p p a s c p p a d c p q p p a s r p b p p a s r p b p \\
b a c s p a a b a c s p a a q p r s a p p p p a d c p b p p a d c p b a a p s c a q a a p d r a b a a p d r a b p p a s c p b p p a d c p q p p a s r p b p p a s r p b \\
a c s p a a b a c s p a a b p r s a p p q p a d c p b p p a d c p b p a p s c a q a a p d r a b a a p d r a b a p a s c p b p q p p a d c p b p p a s r p b p p a s r p \\
c s p a a b a c s p a a b a r s a p p q p a d c p b p p a d c p b p p p s c a q a a p d r a b a a p d r a b a a a s c p b p p p q p p a d c p b p p a s r p b p p a s r \\
s p a a b a c s p a a b a c s a p p q p r d c p b p p a d c p b p p a s c a q a a p d r a b a a p d r a b a a p s c p b p p a c p q p p a d r p b p p a s r p b p p a s \\
p a a b a c s a p p q p r s p a a b a c s c p b p p a d c a q a a p s c p b p p a d r a b a a p d c p b p p a s r a b a a p d s r p b p p a d c p q p p a s r p b p p a \\
a a b a c s p p p q p r s a a a b a c s p p b p p a d c a q a a p s c p b p p a d c a b a a p d r p b p p a s c a b a a p d r a s r p b p p a d c p q p p a s r p b p p \\
a b a c s p a p q p r s a p a b a c s p a b p p a d c p q a a p s c a b p p a d c p b a a p d r a b p p a s c p b a a p d r a p a s r p b p p a d c p q p p a s r p b p \\
b a c s p a a q p r s a p p b a c s p a a p p a d c p b a a p s c a q p p a d c p b a a p d r a b p p a s c p b a a p d r a b p p a s r p b p p a d c p q p p a s r p b \\
a c s p a a b p r s a p p q a c s p a a b p a d c p b p a p s c a q a p a d c p b p a p d r a b a p a s c p b p a p d r a b a b p p a s r p q p p a d c p b p p a s r p \\
c s p a a b a r s a p p q p c s p a a b a a d c p b p p p s c a q a a a d c p b p p p d r a b a a a s c p b p p p d r a b a a p b p p a s r p q p p a d c p b p p a s r \\
s p a a b a c s a p p q p r s p a a b a c d c p b p p a s c a q a a p d c p b p p a d r a b a a p s c p b p p a d r a b a a p r p b p p a s c p q p p a d r p b p p a s \\
a p p q p r s p a a b a c s p a a b a c s c a q a a p s c p b p p a d c p b p p a d c p b p p a s r a b a a p d r a b a a p d s r p b p p a s r p b p p a d c p q p p a \\
p p q p r s a a a b a c s p a a b a c s p a q a a p s c p b p p a d c p b p p a d c p b p p a s c a b a a p d r a b a a p d r a s r p b p p a s r p b p p a d c p q p p \\
p q p r s a p a b a c s p a a b a c s p a q a a p s c a b p p a d c p b p p a d c p b p p a s c p b a a p d r a b a a p d r a p a s r p b p p a s r p b p p a d c p q p \\
q p r s a p p b a c s p a a b a c s p a a a a p s c a q p p a d c p b p p a d c p b p p a s c p b a a p d r a b a a p d r a b p p a s r p b p p a s r p b p p a d c p q \\
p r s a p p q a c s p a a b a c s p a a b a p s c a q a p a d c p b p p a d c p b p p a s c p b a p d r a b a a p d r a b a b p p a s r p b p p a s r p q p p a d c p \\
r s a p p q p c s p a a b a c s p a a b a p s c a q a a a d c p b p p a d c p b p p a s c p b p p p d r a b a a p d r a b a a p b p p a s r p b p p a s r p q p p a d c \\
s a p p q p r s p a a b a c s p a a b a c s c a q a a p d c p b p p a d c p b p p a s c p b p p a d r a b a a p d r a b a a p r p b p p a s r p b p p a s c p q p p a d
\end{array}\right]
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$OD(84;48,12,12,12)$