STRINGS ON COMPLEX MULTIPLICATION TORI AND RATIONAL CONFORMAL FIELD THEORY WITH MATRIX LEVEL

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Dedication

To the soul of my father.

Abstract

Conformal invariance in two dimensions is a powerful symmetry. Two-dimensional quantum field theories which enjoy conformal invariance, i.e., conformal field theories (CFTs) are of great interest in both physics and mathematics. CFTs describe the dynamics of the world sheet in string theory where conformal symmetry arises as a remnant of reparametrization invariance of the world-sheet coordinates. In statistical mechanics, CFTs describe the critical points of second order phase transitions. On the mathematics side, conformal symmetry gives rise to infinite dimensional chiral algebras like the Virasoro algebra or extensions thereof. This gave rise to the study of vertex operator algebras (VOAs) which is an interesting branch of mathematics.

Rational conformal theories are a simple class of CFTs characterized by a finite number of representations of an underlying chiral algebra. The chiral algebra leads to a set of Ward identities which gives a complete non-perturbative solution of the RCFT. Identifying the chiral algebra of an RCFT is a very important step in solving it. Particularly interesting RCFTs are the ones which arise from the compactificatin of string theory as σ-models on a target manifold M. At generic values of the geometric moduli of M, the corresponding CFT is not rational. Rationality can arise at particular values of the modui of M. At these special values of the moduli, the chiral algebra is extended. This interplay between the geometric picture and the algebraic description encoded in the chiral algebra makes CFTs/RCFTs a perfect link between physics and mathematics. It is always useful to find a geometric interpretation of a chiral algebra in terms of a σ-model on some target manifold M. Then the next step is to figure out the conditions on the geometric moduli of M which gives a RCFT.

In this thesis, we limit ourselves to the simplest class of string compactifications, i.e., strings on tori. As Gukov and Vafa proved, rationality selects the complex-multiplication tori.

On the other hand, the study of the matrix-level affine algebra $U_{m,K}$ is motivated

by conformal field theory and the fractional quantum Hall effect. Gannon completed the classification of $U_{m,K}$ modular-invariant partition functions. Here we connect the algebra $U_{2,K}$ to strings on 2-tori describable by rational conformal field theories. We point out that the rational conformal field theories describing strings on complex-multiplication tori have characters and partition functions identical to those of the matrix-level algebra $U_{m,K}$. This connection makes obvious that the rational theories are dense in the moduli space of strings on T^m, and may prove useful in other ways.

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Chapter 1

Introduction

The two pillars of modern physics are quantum field theory and general relativity. The former describes the interactions between elementary particles at subatomic scales. The latter describes the gravitational attraction between macroscopic objects on large scale. These two seemingly incompatible theories describe most phenomena in our universe from subatomic scales to cosmological scales. String theory is an attempt to reconcile the two theories into a unified theory of everything. By replacing the notion of a point particle by a string, a consistent quantum theory of gravity emerges which reproduces Einstein's equations as its first approximation. The other forces and fields of the standard model can also arise in a similar manner.

Two-dimensional conformal field theories (CFTs) are the building blocks of perturbative string theory. CFTs are also important because they describe statistical mechanical systems with second order phase transitions at criticality. They are solvable non-perturbatively, and so may teach us about the non-perturbative physics of other field theories.

The main subject of this thesis is rational conformal field theory. This is a particular class of quantum field theories which enjoys a large amount of symmetry. In this chapter, we discuss the idea of symmetry and how it is realized and described in physical systems.

Symmetry principles played a very important role in the study of physical phenomena. Most physical systems possess a certain amount of symmetry which is manifested in the shape or the dynamics of the system. The ideas of symmetry were paramount in the construction of general relativity, the standard model, and in string theory. A system is symmetric if it doesn't change after we perform a *symmetry transformation* on it. For example, an n-gon will look the same if we rotate it by $2\pi l/n(l = 0, 1, \cdots, n-1)$ around an axis through its center and perpendicular to its plane. Also, the Hamiltonian of the hydrogen atom depends only on the distance r between the electron and the proton and is an example of a system with spherical symmetry. A symmetry transformation can be continuous (e.g., translation, rotation, Lorentz boost) or discrete (e.g., space inversion or parity, time reversal, and lattice translation).

The mathematical description of symmetry and its realization in physics is done via group theory [1]. A set of symmetry transformations $S = \langle a, b, \ldots \rangle$ form a group if there is a product $\cdot$ which satisfies

- The product of any two transformations gives another transformation, i.e., the set is closed under the product.
- The product is associative, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
- There is a unit element $e \in S$ which amounts to the identity transformation or simply doing nothing to the system.
- For every element $a \in S$ there exists an inverse $a^{-1} \in S$, i.e., $a^{-1} \cdot a = e$.

A group is Abelian if the order of the transformations doesn't matter, $a \cdot b = b \cdot a$. If the order matters $a \cdot b \neq b \cdot a$; then we are talking about non-Abelian groups.

It is clear that translations, rotations, and Lorentz boosts are examples of continuous groups. They are subgroups of the *Poincaré group* $ISO(1,3)$, which is the symmetry group of Minkowski space-time. The symmetry transformations of the Poincaré group can be implemented by transforming the space-time coordinates of the system in the following way:

$$x^\mu \to \Lambda^\mu_\nu x^\nu + a^\mu, \tag{1.0.1}$$

where Λ^μ_ν and a^μ are constants which encode the parameters of the transformation. For $a^\mu = 0$, one recovers the Lorentz group $SO(1,3)$ which is the group of rotations of the space-time (spatial rotations and Lorentz boosts.)

The Poincaré group is the set of space-time coordinate transformations which preserve the relativistic infinitesimal interval

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = \eta'_{\mu\nu} dx'^\mu dx'^\nu, \tag{1.0.2}$$

where $\eta_{\mu\nu}$ is the metric of space-time. The invariance of ds^2 imposes

$$\eta_{\mu\nu} = \Lambda^\alpha_\mu \Lambda^\beta_\nu \eta_{\alpha\beta}, \qquad a^\mu = \text{constants}. \tag{1.0.3}$$

From the above equation, we learn that the Poincaré group is the set of space-time coordinate transformations which leave the metric *form invariant*. It is a 10 parameter group (4 translations, 3 spatial rotations, and 3 Lorentz boosts) which embodies the relativistic description of nature. We remark that any fundamental physical theory must remain invariant under the Poincaré group as required by the relativistic symmetry of the fundamental laws of physics. This relativistic symmetry signifies our ability to repeat experiments at different places, at different times, in different directions, and in different inertial reference frames without affecting the outcome of the experiment.

If we relax the condition of the form invariance of the metric and demand that the metric remains form invariant up to an overall function, we arrive at the *conformal group* of space-time. Invariance under the conformal group, however, is not always guaranteed and most physical systems break conformal invariance. An example of a conformal transformation which is not a Poincaré transformation is *scaling* which acts as $x^\mu \to \lambda x^\mu$. One would expect that scaling the dimensions of a physical system will lead to different outcomes of physical experiments. For example, by doubling the volume of a closed room the pressure will decrease (assuming the ideal gas law is valid). Invariance of a physical system under scaling or more generally under the conformal group requires intricate dynamics. One can get an idea of what kind of physical systems might enjoy conformal symmetry by looking at (1.0.2) for the case of light rays, i.e., the null interval $ds^2 = 0$. In this case, scaling the metric would leave the interval null. Hence light propagation or more generally massless particles propagation is conformally invariant. It turned out that *classical* Maxwell and Yang-Mills theories in 4D are conformally invariant.

The symmetries we have been discussing about so far are external symmetries acting on the background space-time in which the physical system lives. There is, however, a different kind of symmetry which corresponds to transformations of the internal degrees of freedom of the system. These internal symmetries are not directly related to space-time. An example of a continuous internal symmetry is the phase rotation of the wave function in quantum mechanics

$$\psi(x) \longrightarrow e^{i\alpha}\psi(x), \tag{1.0.4}$$

where α is a constant phase. The set of constant phase transformations in (1.0.4) form the *global* $U(1)$ group.

Another example is the isospin invariance of nuclear interactions. Nuclear forces between protons and neutrons does not depend on the electric charge and is blind to whether the particle is a proton or a neutron. The proton and the neutron are treated two different states of the same particle, the nucleon, with different isospin charge

$$\begin{pmatrix} P \\ N \end{pmatrix} \longrightarrow S \begin{pmatrix} P \\ N \end{pmatrix}, \tag{1.0.5}$$

where S is a 2×2 constant unitary matrix with unit determinant. This is the global $SU(2)$ group which played a very important role in the early days of the strong-interaction physics. In terms of the quark structure of hadrons, the $SU(2)$ symmetry of strong interactions is due to the approximate equality of the masses of the up and down quarks. If one also include the charm quark and ignore the small mass differences with the up and down quark, we arrive at the $SU(3)$ flavor group. This group underlies the eight-fold way scheme of the classification of baryons and mesons [2]. One can also have discrete internal symmetries like charge conjugation which replace every particle with its antiparticle.

One can allow the above symmetry transformations to depend on space-time by having independent symmetry transformations at each point in space-time, i.e., by *gauging the global symmetries*. Such *local* symmetries are called gauge symmetries, and the theories which describe them are called gauge theories. Such theories are known generically as Yang-Mills theories [3, 4] and they describe the interactions between massless spin-1 gauge bosons. The modern description of the fundamental interactions of nature relies on the gauging of the internal symmetries they respect. The Maxwell theory of the electromagnetic interactions is a $U(1)$ gauge theory. On the other hand, the weak and the strong interactions are based on gauging the $SU(2)$ and the $SU(3)$ groups, respectively. The standard model of particle physics which describes the non-gravitational forces of nature is based on a Yang-Mills theory with a gauge group $SU(3) \times SU(2) \times U(1)$.

Einstein's theory of gravity can be recast as the gauge theory of the Poincaré group. The classic example is the 4-potential A_μ of classical electrodynamics, where the $U(1)$ gauge symmetry tells us that A_μ and $A_\mu + \partial_\mu \Lambda$ give the same field strength, i.e., the same physics. In a gauge theory, physical observables must be gauge invariant, i.e., they transform trivially under the gauge group. We should stress that gauge symmetries are not symmetries in the usual sense but rather they are redundancies in the description of physical systems. In the case of gravity, where the gauge degrees of freedom are the space-time coordinates, this implies that local operators are not gauge invariant. In the standard model, gauge symmetry is realized as an internal symmetry acting on fields which live in a fixed space-time background. In gravity, gauge symmetry is an external symmetry acting on the space-time metric which is a dynamical field. This distinction between the way gauge symmetry is realized in the standard model and in gravity is one of the reasons why gravity is still hard to reconcile with the principles of quantum mechanics.

Supersymmetry (SUSY) is an extension of space-time symmetries which relates bosons and fermions [5, 6]. It predicts the existence of a superpartner with the same mass for every elementary particle in nature. Since no superpartners have been discovered so far, it is assumed that SUSY is broken at low energies which uplifts the degeneracy between bosons and fermions masses. One of the motivations for SUSY is that it gives a resolution of the hierarchy problem of the Standard Model. Without the extra superpartners, quantum corrections drives the Higgs boson mass close to the Planck mass and a very superficial fine tuning is required to keep the Higgs mass low. The mathematical description of SUSY requires the introduction of fermionic coordinates θ^α. These fermionic coordinates transform as spinors under the Lorentz group and together with the bosonic space-time coordinates x^μ they make up the superspace. SUSY transformations can be realized as translations and rotations of the superspace [7]. The relevant CFTs which arise in string theory are the ones which enjoys a certain amount of supersymmetry. These are superconformal CFTs which underlies the superstring perturbation theory.

To appreciate the importance of symmetry in the description of physical systems, we recall the Noether theorem: *For every global continuous symmetry, i.e., a transformation of a physical system which acts the same way everywhere and at all time, there exists an associated time-independent quantity (i.e., a conserved quantity).* Applying the Noether theorem to space-time translations leads to the energy-momentum conservation. Similarly, invariance under spatial rotations leads via Noether theorem to the conservation of angular momentum. Lorentz boosts, however, don't lead to any conserved quantity since they don't commute with time evolution. The Noether theorem can also be applied to internal symmetries. The global $U(1)$ symmetry of the electromagnetic interactions leads to the conservation of electric charge. The global

$SU(2)$ symmetry of the weak interaction and the global $SU(3)$ of the strong interactions leads to the conservation of global charges as well. Gauging a global symmetry doesn't lead to any new conserved charges. This supports the claim that local gauge invariance is not a symmetry in the usual sense. SUSY leads to conserved fermionic supercharges as well. These supercharges transform bosons into fermions and vice versa.

Classically, it is straightforward to tell whether a system is invariant under a certain symmetry by looking at the action

$$S = \int L[q_i(t), \dot{q}_i(t)]dt, \tag{1.0.6}$$

where $L[q_i(t), \dot{q}_i(t)]$ is the Lagrangian of the system. A symmetry of a classical system is a transformation of the dynamical variables $q_i(t) \to \mathcal{R}[q_i(t)]$ that leaves the action unchanged. In the Hamiltonian formalism, a symmetry is realized by a generator which has a vanishing Poisson bracket with the Hamiltonian

$$\{I_{\mathcal{R}}, H\} = 0. \tag{1.0.7}$$

This formalism can be generalized to classical field theories which are described by a Lagrangian density $\mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x))$.

In the quantum theory we no longer have the classical configuration space but rather a Hilbert space of quantum states. The classical symmetry is realized linearly by unitary operators acting on the Hilbert space. It could happen that a classical symmetry doesn't survive after quantization. In this case, we speak of an anomalous symmetry. To see how this could happen, let us recall that the quantum theory is described by an amplitude which can be given as a path integral

$$Z = \int \mathcal{D}q e^{iS[q,\dot{q}]}, \tag{1.0.8}$$

where $S[q, \dot{q}]$ is the classical action written in terms of the c-number classical variables. The only way a classical symmetry could be broken is if the measure $\mathcal{D}q$ fails to be symmetric. This can happen if the regularization prescription, which is needed to make sense of the above path integral, violates the symmetry. An example of a theory which is conformally invariant (*classically*) is Yang-Mills theory in 4D (e.g., the Maxwell theory of electromagnetism). However, the conformal symmetry in this case doesn't survive after quantization and the conformal symmetry is anomalous. Yang-Mills theories in 4 dimensions are conformally invariant but quantum Yang-Mills theories are not. It is known that classical Yang-Mills theory which describes massless particles develops a mass gap due to quantum mechanical corrections [8]. An example of a theory which is conformally invariant even on the quantum level is $\mathcal{N} = 4$ super Yang-Mills theory in 4D.

Another important feature about the way symmetry is realized in quantum field theory is the concept of *phases*. Quantum field theory can have different phases in which the symmetry of the fundamental Lagrangian is realized differently. For example, the high energy $SU(2) \times U(1)$ symmetry of the electroweak interactions is spontaneously broken in the Higgs phase at low energy to $U(1)$ symmetry. This however doesn't mean the symmetry is lost. It just means that the ground state of the system doesn't respect the symmetry.

The way symmetry is realized in string theory is different from point-particle quantum field theories. The fundamental objects in perturbative string theory are one-dimensional strings which can be open or closed. When the string moves in space-time it sweeps out a two-dimensional surface. This surface is called the world-sheet of the string.

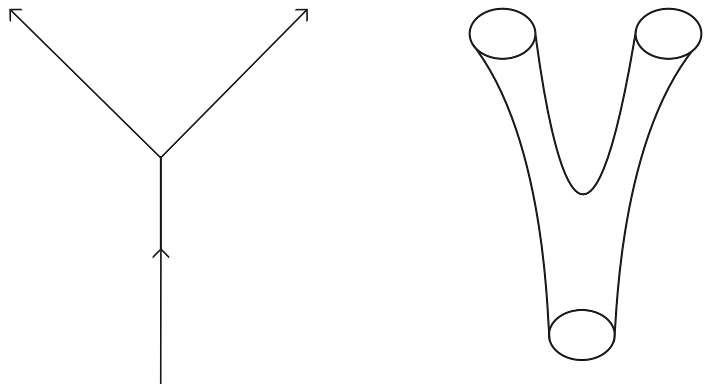

Figure 1.1: The classical trajectory of a string minimizes the area of the world sheet.

The surface is specified by the functions

$$X^\mu = X^\mu(\sigma_0, \sigma_1), \tag{1.0.9}$$

which describe the embedding of the string in the background space-time.

The starting point of perturbative string theory is the world-sheet action of a string moving in some background space-time with a metric g. The embedding of the string world-sheet Σ in a background D-dimensional space-time M is described by the maps $X : \Sigma \to M$. The action of the bosonic string is

$$S(X, g) = \frac{1}{4\pi\alpha'} \int_\Sigma d^2x \sqrt{h} h^{\alpha\beta} g_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu, \tag{1.0.10}$$

where $h^{\alpha\beta}$ is the world-sheet metric and α' is the only dimensional parameter of string theory and has the dimensions of length squared.

The above action can be supersymmetrized which leads to an $\mathcal{N} = 1$ supergravity theory on the world sheet. After gauge fixing it will give a theory with $\mathcal{N} = 1$ superconformal symmetry. We will only consider bosonic CFTs in this thesis. We now specialize to flat space-time backgrounds $g_{\mu\nu} = \delta_{\mu\nu}$ with Euclidean signature.

Diffeomorphism plus Weyl invariance of the above action enables one to chose [9]

$$h^{\alpha\beta} = \delta^{\alpha\beta}. \tag{1.0.11}$$

This is always possible locally on any world-sheet. The above action now boils down

$$S(X) = \frac{1}{4\pi\alpha'} \int_{\Sigma} d^2x\, \partial_\alpha X^\mu \partial_\alpha X^\mu, \tag{1.0.12}$$

This is the action of a two-dimensional quantum field theory living on the world-sheet where the string coordinates X^μ act as D massless bosons in two dimensions. The above action is scale invariant. If we scale $\tau \to \lambda\tau$ and $\sigma \to \lambda\sigma$ the action remains the same. The action is invariant under the much bigger conformal group in two dimensions and the two-dimensional quantum field theory which governs the fluctuations of the world sheet is conformal. The space-time physics of string theory, in particular the propagation of strings and their interactions, can be pulled back to the two-dimensional conformal field theory (CFT) on the world-sheet. The geometric characteristics of the target manifold follow from the properties of the world-sheet CFT. This world-sheet CFT is required to have a vanishing conformal anomaly[1]. As it turns out, the vanishing of the conformal anomaly translates to the Einstein's equations in 25 D in the case of the bosonic string and 10 D for the supersymmetric string. Thus a choice of a CFT on the world-sheet is equivalent to a choice of a background (or a classical solution) in string theory. The classification of 2D CFTs is equivalent to classifying the classical solutions of perturbative string theory.

There are five supersymmetric string theories in ten-dimensional flat space [9, 10]: Type I with an $SO(32)$ gauge group, type IIA and type IIB with no gauge symmetry and Heterotic $SO(32)$ and Heterotic $E_8 \times E_8$ with gauge groups $SO(32)$ and $E_8 \times E_8$,

[1]The vanishing of the conformal anomaly is required in order to preserve the 2D Weyl invariance.

respectively. One can engineer more general gauge groups by the inclusion of D-branes (for a comprehensive review see [11]). D-branes are hyperplanes on which open strings can end. They are dynamical objects in string theory which arise upon quantizing open strings subject to Dirichlet boundary conditions. In the world-sheet description of string theory, D-branes give rise to boundary conformal field theories [12–14]. Gauge symmetry can also arise by compactification on manifolds with isometries [15] or from singularities in the compactification manifold [16, 17]. The aforementioned gauge groups have an elegant world-sheet description in terms of spin-1 currents which generate an affine Kac-Moody algebra [18–21]. In this thesis, we will study similar algebras which arises from compactification of strings on tori.

It is worth mentioning that there are no global symmetries in string theory. It can be shown using a simple world-sheet argument [9, 10] that any symmetry in string theory must be gauged. This is consistent with the observation that there can't be global conserved charges in a theory with black holes. Global charges will disappear behind the horizon and there is no way for them to be measured. However, if the global symmetry is gauged, then one can measure the charge while staying away from the horizon by measuring the flux of the field strength. This doesn't violate the *No-hair theorem* since the charges which characterize a black hole correspond to a global symmetry which is gauged [22]. The fact that string theory doesn't allow global symmetries makes it a viable candidate for a theory of quantum gravity.

In studying the simplest compactifications of string theory one considers a space-time which has product form

$$M^{10} = M^{3,1} \times X^6, \tag{1.0.13}$$

where $M^{3,1}$ is our Minkowski space-time and M^6 is the a compact six-dimensional space.

In terms of the CFT description of the string

$$\mathrm{CFT}_{10} = \mathrm{CFT}_4^{\mathrm{ext}} \times \mathrm{CFT}_6^{\mathrm{int}}, \tag{1.0.14}$$

where $\mathrm{CFT}_6^{\mathrm{int}}$ is the internal CFT which corresponds to the compactified dimensions. Different choices for $\mathrm{CFT}_6^{\mathrm{int}}$ give different space-time physics. Supersymmetry in space-times requires that the world-sheet CFT to have $\mathcal{N} = 2$ supersymmetry. Classical solutions of string theory are given in terms of $\mathcal{N} = 2$ superconformal field theories. We will not consider superconformal theories in this thesis and will only consider the bosonic sectors. Adding the fermions will not change any of the results in this thesis and the bosonic theories we will consider can be uplifted to $\mathcal{N} = 2$ superconformal field theories which are relevant in string theory.

We will study the CFTs which result from string compactification on a two torus T^2. Toridal compactification is the harmonic oscillator of string compactification due to its simplicity and exact solvablity [23, 24]. A torus is a manifold with an $U(1)^2$ isometry which corresponds to the compact translations around the two cycles of the torus. We will describe toroidal compactification in more details in Chapter 2. The $U(1)^2$ symmetry gives rise to two conserved currents on the world-sheet CFT of the string and one expects the existence of two conserved charges. These are the quantized momenta which correspond to translations in the compact dimensions. However, there are two other conserved charges which have stringy origins. Strings can *wind* around the compact dimensions and the winding numbers behaves like a conserved charge. These *topological* charges come from the dual $U(1)^2$ currents and they characterize the winding states. The momentum and winding charges live on a self-dual integer Narain lattice $\Gamma^{2,2}$ which for the case of T^2 has a signature $(2, 2)$ [23].

We will study these lattices in detail later.

It is always useful to find a geometric interpretation of a given CFT in terms of a σ-model on some target manifold M. The interplay between the algebraic and geometrical pictures have been very fruitful in revealing many aspects of CFTs and the target manifolds. This is most obvious in the case of strings on Calabi-Yau manifolds M which are described by an $N = 2$ superconformal field theory. The $N = 2$ superconformal algebra admits an automorphism which reverses the sign of a $U(1)$ charge. This operation which gives an isomorphic superconformal field theory gives a completely different Calabi-Yau target W. The cohomology groups $H^{1,1}$ and $H^{2,1}$ of M and W are interchanged under the aforementioned automorphism. This lead the authors of [25, 26] to propose the existence of a *mirror manifold* W for each Calabi-Yau manifold M with $H^{1,1}(M) = H^{2,1}(W)$ and $H^{2,1}(M) = H^{1,1}(W)$. The main motivation for this proposal is the fact that both M and W give isomorphic superconformal field theories. An explicit construction of mirror pairs of Calabi-Yau manifolds was given in [27].

A conformal field theory is rational when the chiral algebra generated by the conserved currents is large enough so that the Hilbert space contains a finite number of representations. For CFTs based on a target space M rationality can arise at special values of the geometric moduli of M. To see how a CFT can become rational at certain points of its moduli space, we recall the simplest example of an RCFT: the $c = 1$ CFT of a compact free boson on a circle of rational square radius $R^2 = r/s$. The radius R plays the role of a geometric modulus. At a generic value of R, the number of primary fields is infinite. At $R^2 = r/s$, the chiral algebra of this theory is extended. The infinite list of representations reconstitute themselves into finitely many irreducible representations of the extended chiral algebra (see Chapter 2).

More interesting RCFTs arise is in string compactifications on Calabi-Yau manifolds at special values of their moduli, e.g., in Gepner models [28]. For example, the Fermat quintic in $\mathbb{CP}^4$

$$P(z) = \sum_{a=1}^{5} (z_a)^5 = 0. \tag{1.0.15}$$

This quintic corresponds to a Gepner point where the corresponding CFT is rational and is described by a product of $N = 2$ minimal models [28].

It is important to understand the conditions for rationality for CFTs based on a target manifold M. For example, the simplest compactifications of a string theory are on tori T^m–when are they described by RCFTs? This question has been studied in [29–32]. In particular, Wendland [30] derived rationality conditions valid for all torus dimensions m. More importantly for us, in the case of 2-tori, Gukov and Vafa [32] found a simple, geometric criterion for rationality. For T^2, the modular parameter τ and the Kähler parameter ρ must take special values; they must belong to an imaginary quadratic number field. Such 2-tori have the property of *complex multiplication*, and are known as complex-multiplication (CM) tori. For them, a Gauss product exists, and was used in [33] to classify the corresponding RCFTs.

We will study strings on (CM) tori which are a particular class of tori with more symmetry. CM tori are characterized by the values of their modular parameter τ which satisfies

$$a\tau^2 + b\tau + c = 0 \longrightarrow \tau = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \tag{1.0.16}$$

that is, τ belongs to a quadratic number field $\mathbb{Q}(D)$, where $D = b^2 - 4ac < 0$. The quadratic number field $\mathbb{Q}(D)$ is the set of complex numbers z which can be written as

$$z = \alpha + \beta\sqrt{D}, \tag{1.0.17}$$

where $\alpha, \beta \in \mathbb{Q}$ are rational number. It is obvious that the modular parameter of a CM torus belongs to $\mathbb{Q}(D)$. This is also true for the the Kähler parameter ρ which is exchanged with τ under mirror symmetry. As was shown in [32] rationality picks CM tori.

In earlier work, Gannon [34] gave a complete classification of the modular invariant partition functions of the Abelian matrix-level affine algebra $U_{m,K}$. This algebra is important in the description of the Fractional Quantum Hall Effect (FQHE). The algebra $U_{m,K}$ also appears in the 2D CFTs which are the holographic dual of 3D Chern-Simons theories which describe topological membranes [35–37].

We give a geometric connection of the $U_{m,K}$ algebra for any K and for $m = 2$. Using the Gauss product [33], we relate the $U_{2,K}$ algebra to RCFTs which arise from strings on complex multiplication tori. We will uncover a connection between the matrix-level algebras $U_{2,K}$ and the CM tori, and so to the extended Moore-Seiberg algebras that make strings on them describable by RCFTs. We will use the *Gauss product* [33] to construct the matrix levels for the algebra $U_{2,K}$ using the geometric data encoded in τ and ρ which correspond to strings on CM tori. We will show that the RCFTs which arise from strings on CM-tori and the RCFTs based on the $U_{m,K}$ algebra have the same set of characters and the same partition function. One can write down a Moore-Seiberg algebra for the RCFTs which result from strings on CM tori. But as we will see in the next chapter, this algebra is already complicated for case of a rational circle. In this thesis, we propose the algebra $U_{m,K}$ as an alternative description of the extended chiral algebra for strings on CM tori.

Using this connection between the $U_{2,K}$ algebra and strings on complex multiplication tori, we will show that RCFTs are dense inside Narain moduli space. This confirms the results in [32] about the density of the RCFTs which arise from strings on CM tori.

Since the algebra $U_{m,K}$ also appears in the study of the FQHE, we use the connection between $U_{2,K}$ and strings on CM tori to relate the topological order of fractional quantum hall states to the number of $D0$ branes allowed on CM tori. We will also propose a generalization of the matrix-level algebra to the non-Abelian case.

This thesis is organized as follows: In Chapter 2, we first give a brief introduction to CFT then we discuss the algebra $U_{m,K}$ and its modular invariant partition functions. In Chapter 3, we give a physicist's introduction to complex multiplication tori and we describe various examples. In Chapter 4, we present our main results. Chapter 5 is the conclusion.

Chapter 2

Rational conformal field theory

Two-dimensional conformal field theories (CFTs) [38] (for reviews see [39, 40]) are quite well understood. They are probably the best understood among all quantum field theories. Two-dimensional conformal systems are very special, because only in two dimensions is the conformal algebra infinite-dimensional[1]. The local conformal symmetry described by the Virasoro algebra, or one of its extensions, is in many cases powerful enough to lead to a complete determination of the operator spectrum, as well as to explicit formulas for the correlation functions. This constitutes a complete non-perturbative solution of the theory.

On the physics side CFTs appear in two contexts. First, they describe the behavior of many statistical mechanical systems at a renormalization group fixed point. When we look at these systems at larger and larger scales (or by going to the deep infrared) we find that fluctuations occur at all length scales. At the critical point (the phase transition point), the system is covariant under scale transformations. It was shown [38, 41] that scale invariance plus rotational invariance (or Lorentz invariance in Minkowski signature) implies the invariance under the full conformal group.

[1] In higher dimensions, the conformal group is finite dimensional.

Hence, these systems can be described by a CFT. These theories thus give concrete instances of non-trivial fixed points of the renormalization group, many of which have a realization in statistical mechanical systems.

An example of a critical phenomenon is the ferromagnetic phase transition at the *Curie temperature* T_c in ferromagnetic materials (like iron). Above the Curie temperature, the dipole moments of the individual atoms are randomly aligned, resulting in a vanishing total dipole moment and the system is in the paramagnetic phase. When the temperature is decreased below the Curie temperature, the individual dipole moments pick a preferred direction in order to minimize their total energy. This results in a non-vanishing total dipole moment (magnetization) and the system is in the ferromagnetic phase. The paramagnetic phase is a disordered phase where the system is invariant under three-dimensional rotations, i.e., the system enjoys an $SO(3)$ symmetry. The ferromagnetic phases is an ordered phase and the rotational symmetry is broken. Ordered-disordered phases are a characteristic of second order phase transitions and it is always characterized by an *order parameter* which takes a non-zero value in the broken phase. In the case of the ferromagnetic phase transition, the total magnetization is an order parameter.

The ferromagnetic phase transition can be described theoretically using the *Ising model* [42]. The 2D Ising model is a square lattice with a discrete variable s_i (spin) at each lattice site. The variable s_i can can take only the values $+1$ or -1. Neighboring spins interact with each other through an exchange coupling

$$E = -J \sum_{<i,j>} s_i s_j, \tag{2.0.1}$$

where J is the exchange energy. In the high temperature phases, $\langle s \rangle = 0$ and in the low temperature phase $\langle s \rangle \neq 0$.

An important quantity is the correlation length $\xi(T)$

$$\langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle \sim e^{-\frac{|i-j|}{\xi(T)}}. \tag{2.0.2}$$

For generic T, the correlation length $\xi(T)$ is finite and the statistical fluctuations of the spins are correlated over a finite length. As T approaches its critical value, the correlation length diverges

$$\xi(T) \sim \frac{1}{|T - T_c|}$$

and the system starts to fluctuate over all length scales (see Figure 2.1). At the critical point, the correlators follow a power law in contrast to the exponential decay in (**??**). This self-similar behavior and the divergence of the correlation length is a manifestation of the scale invariance of the system at the critical temperature. It turns out that scale invariance is a part of a bigger symmetry, conformal symmetry, which emerges at the critical point. For statistical models in two dimensions, the continuum description at a second-order phase transition is given by a conformal field theory. The prime example is the Ising model which is a two-dimensional model of ferromagnetism. The 2D Ising model was analytically solved by Onsager [43] where he showed that correlation functions and the free energy of the model are given by a theory of free lattice fermions.

This is an example of a second-order phase transition which separates a high temperature disordered phase (with the expectation value $\langle s \rangle = 0$) and a low temperature ordered phase (with $\langle s \rangle \neq 0$).

The second major application of CFT in physics is string theory. The classical solutions of perturbative string theory (or the vacuum configurations) are described by 2D CFTs. Classifying and understanding these solutions amounts to understanding the classifications of CFTs.

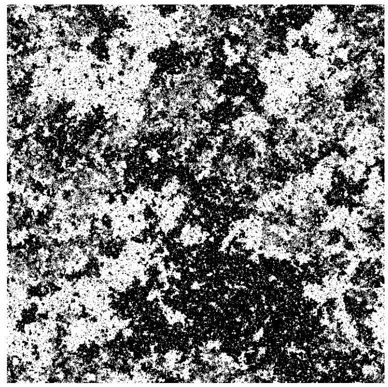

Figure 2.1: A snapshot of the Ising model at T_c. There are fluctuations on all length scales (self-similarity) [44].

This chapter is a review of conformal field theory. We start with the definition of conformal transformations, then we go on and develop the basic tools used in CFT, focusing on those aspects of CFT which will be used in the thesis. Our discussion will be based on [9, 39, 40]

The conformal group is the set of transformations which preserve the angle between vectors [40]

$$\cos\theta_{XY} = \frac{X \cdot Y}{\sqrt{\|X\|^2}\sqrt{\|Y\|^2}} = \frac{\eta_{\mu\nu}X^\mu Y^\nu}{\sqrt{\eta_{\mu\nu}X^\mu X^\nu}\sqrt{\eta_{\mu\nu}Y^\mu Y^\nu}}. \tag{2.0.3}$$

The generators of the conformal group are

$$\begin{aligned}
P_\mu &= -i\partial_\mu && \text{translation} \\
D &= -ix^\mu\partial_\mu && \text{dilation} \\
L_{\mu,\nu} &= i\big(x_\mu\partial_\nu - x_\nu\partial_\mu\big) && \text{rotation} \\
K_\mu &= -i\big(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu\big) && \text{special conformal transformation.}
\end{aligned} \tag{2.0.4}$$

Now we define the generators $J_{a,b} = -J_{b,a}$, $a, b \in \{-1, 0, 1, \ldots, d\}$ as follows

$$\begin{aligned} L_{\mu,\nu} &= J_{\mu,\nu}, \qquad J_{-1,\mu} = \frac{1}{2}\big(P_\mu - K_\mu\big), \quad \mu, \nu = 1, \cdots, d, \\ J_{-1,0} &= D, \qquad J_{0,\mu} = \frac{1}{2}\big(P_\mu + K_\mu\big). \end{aligned} \tag{2.0.5}$$

They satisfy the algebra

$$\big[J_{ab}, J_{cd}\big] = i\big(\eta_{ad}J_{bc} + \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac}\big), \tag{2.0.6}$$

where $\eta_{ab} = \mathrm{diag}(-1, 1, 1, \ldots, 1)$. This is the algebra of $\mathrm{SO}(d+1, 1)$, the conformal group in d-dimensions. The dimension of $\mathrm{SO}(d+1, 1)$ is $(d+2)(d+1)/2$, i.e., the conformal group in d dimensions is $(d+2)(d+1)/2$-parameter group. Quantum field theories in $d > 2$ with conformal symmetry are interesting, and although the conformal symmetry leads to some simplifications, it is still not big enough to give exact solutions.

The situation in $d = 2$ changes dramatically, since the conditions on ϵ give

$$\partial_1\epsilon_1 = \partial_2\epsilon_2, \qquad \partial_1\epsilon_2 = -\partial_2\epsilon_1. \tag{2.0.7}$$

If we use complex coordinates

$$\begin{aligned} z &= x^1 + ix^2, \qquad \bar{z} = x^1 - ix^2, \\ \partial &= \frac{1}{2}\big(\partial_1 - i\partial_2\big), \qquad \bar{\partial} = \frac{1}{2}\big(\partial_1 + i\partial_2\big), \\ \epsilon(z) &= \epsilon_1 + i\epsilon_2, \qquad \bar{\epsilon}(\bar{z}) = \epsilon_1 - i\epsilon_2, \end{aligned} \tag{2.0.8}$$

then (2.0.7) are the *Cauchy-Riemann equations* for the real and imaginary parts of $\epsilon(z)$. Hence, conformal transformations in $d = 2$ coincide with the set of holomorphic (anti-holomorphic) coordinate transformations (conformal mappings) of the complex plane

$$z \longrightarrow f(z), \qquad \bar{z} \longrightarrow \bar{f}(\bar{z}). \tag{2.0.9}$$

It is known that the set of holomorphic coordinate transformations in $d = 2$ is infinite dimensional and so that the conformal algebra in $d = 2$ is infinite dimensional. It is worth mentioning that the conformal group of $\mathbb{C}$ is not infinite-dimensional since only the transformations corresponding to L_0, L_1, L_{-1} are globally defined (see [45]). If, however, we work locally (or on the level of the algebra) then we do get an infinite dimension algebra.

Locally, a holomorphic transformation can be written as

$$f(z) = z + \epsilon(z) = z + \sum_{n\in\mathbb{Z}} \big(-\epsilon_n z^n \big), \qquad \bar{f}(\bar{z}) = \bar{z} + \bar{\epsilon}(\bar{z}) = \bar{z} + \sum_{n\in\mathbb{Z}} \big(-\bar{\epsilon}_n \bar{z}^n \big). \tag{2.0.10}$$

The generators of the conformal transformations are

$$l_n := -z^{n+1}\partial, \qquad \bar{l}_n := -\bar{z}^{n+1}\bar{\partial}. \tag{2.0.11}$$

They act on the space of functions and satisfy the *classical Witt algebra*

$$\big[l_m, l_n\big] = (m-n)l_{m+n}, \qquad \big[\bar{l}_m, \bar{l}_n\big] = (m-n)\bar{l}_{m+n}, \quad [l_n, \bar{l}_m] = 0 \tag{2.0.12}$$

which is a direct sum $\mathcal{A} \oplus \bar{\mathcal{A}}$ of two commuting subalgebras. In what follows we only treat the holomorphic part, with the understanding that the analysis also applies to the anti-holomorphic part. The classical Witt algebra is realized quantum mechanically with a central extension (Virasoso algebra). The central term is related to a quantum mechanical anomaly in the transformation of the energy-momentum tensor.

The Virasoro algebra $\mathrm{Vir} \oplus \overline{\mathrm{Vir}}$ is

$$\begin{aligned}
&[L_n, L_m] = (m-n)L_{n+m} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \quad m, n \in \mathbb{Z}, \\
&[\bar{L}_n, \bar{L}_m] = (m-n)\bar{L}_{n+m} + \frac{\bar{c}}{12}(m^3 - m)\delta_{m+n,0}, \\
&[L_n, \bar{L}_m] = 0.
\end{aligned} \tag{2.0.13}$$

This is the chiral algebra which underlies any conformal field theory. It is the quantum extension of the classical Witt algebra and is characterized by the central terms proportional to c and $\bar{c}$. The central terms vanish when restricted to the generators of the global conformal group, L_0, L_1, L_{-1} and $\bar{L}_0, \bar{L}_1, \bar{L}_{-1}$ implying that this group is non-anomalous. It is worth mentioning that a CFT can have a bigger chiral algebra than $\mathrm{Vir} \otimes \overline{\mathrm{Vir}}$, e.g., W-algebras, current algebras, superconformal algebras.

The relevant representations of the Virasoro algebra which are important in physical applications involve operators which have a fixed scaling dimension Δ. These are the conformal primary operators $\mathcal{O}_i$, which are eigenfunctions of the scaling operator D with the eigenvalue $\Delta_i = h_i + \bar{h}_i$. Here, h_i denotes the *holomorphic conformal dimension* and $\bar{h}_i$ denotes the *anti-holomorphic conformal dimension.* The normalization of the primary operators is arbitrary and we chose the basis of the primary operators such that the two-point function takes the form

$$\langle \mathcal{O}_i(z_1, \bar{z}_1) \mathcal{O}_j(z_2, \bar{z}_2) \rangle = \frac{\delta_{ij}}{|z_1 - z_2|^{2\Delta_i}}. \tag{2.0.14}$$

CFTs admit an operator product expansion (OPE) in which two local operators inserted at nearby points can be closely approximated by a string of operators at one of these points

$$\mathcal{O}_i(z, \bar{z}) \mathcal{O}_j(z, \bar{z}) = \sum_k C_{ij}^k (z - w, \bar{z} - \bar{w}) \mathcal{O}_k(w, \bar{w}), \tag{2.0.15}$$

where $\{\mathcal{O}_i(z)\}$ is a set of local fields which can be decomposed into a direct sum of conformal families or conformal towers

$$\{\mathcal{O}_i(z)\} = \sum_n [\phi_n]. \tag{2.0.16}$$

On top of each one of these towers sits a highest weight state which corresponds to a primary field. The other members of the conformal family $[\phi_n]$ are called descendant fields and they have conformal dimensions

$$h_n^{(k)} = h_n + k, \qquad \bar{h}_n^{(\bar{k})} = \bar{h}_n + \bar{k}. \tag{2.0.17}$$

where $k, \bar{k} \in \mathbb{Z}$ by construction.

The OPE can be written down for any set of local operators in any quantum field theory, e.g., QCD. The novel thing about CFT is the absence of any dimensional parameter l. It was argued that the above OPE converges in a CFT since harmful terms of form $e^{-l/|z-w|}$ are absent.

The OPE of the energy-momentum tensor is of utmost importance

$$\begin{aligned} T(z)T(w) &\sim \frac{c/2}{(z-w)^4} + \frac{2T(z)}{(z-w)^2} + \frac{\partial T(z)}{(z-w)} + \text{reg} \\ \bar{T}(\bar{z})\bar{T}(\bar{z}) &\sim \frac{\bar{c}/2}{(\bar{z}-\bar{w})^4} + \frac{2\bar{T}(\bar{z})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}\bar{T}(\bar{z})}{(\bar{z}-\bar{w})} + \text{reg} \\ T(z)\bar{T}(\bar{z}) &\sim \text{reg}, \end{aligned} \tag{2.0.18}$$

where $c, \bar{c} \in \mathbb{N}$ are the holomorphic and anti-holomorphic central charges, respectively. The regularity of the $T(z)\bar{T}(\bar{z})$ OPE is a manifestation of the decoupling of the holomorphic and anti-holomorphic sectors. They encode the quantum anomally in the transformation of $T(z)$ and $\bar{T}(\bar{z})$. One-loop modular invariance imposes

$$c - \bar{c} = 0 \mod 24. \tag{2.0.19}$$

This condition can be realized by taking $c = \bar{c}$ as in bosonic and type II string theory.

The Virasoro algebra (2.0.13) is derived from the above OPE of T with itself

$$
\begin{aligned}
[L_n, L_m] &= \frac{1}{(2\pi i)^2}\oint_0 dw \oint_w dz w^{m+1} z^{n+1} T(z)T(w) \\
&= \frac{1}{(2\pi i)^2}\oint_0 dw \oint_w dz w^{m+1} z^{n+1} \left(\frac{c/2}{(z-w)^4} + \frac{2T(z)}{(z-w)^2} + \frac{\partial T(z)}{(z-w)} \right) \\
[\bar{L}_n, \bar{L}_m] &= \frac{1}{(2\pi i)^2}\oint_0 d\bar{w} \oint_w d\bar{z} \bar{w}^{m+1} \bar{z}^{n+1} \bar{T}(\bar{z})\bar{T}(\bar{w}) \\
&= \frac{1}{(2\pi i)^2}\oint_0 d\bar{w} \oint_w d\bar{z} \bar{w}^{m+1} \bar{z}^{n+1} \left(\frac{\bar{c}/2}{(\bar{z}-\bar{w})^4} + \frac{2\bar{T}(\bar{z})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}\bar{T}(\bar{z})}{(\bar{z}-\bar{w})} \right) \\
[\bar{L}_n, L_m] &= \frac{1}{(2\pi i)^2}\oint_0 d\bar{w} \oint_w dz z^{n+1} \bar{w}^{m+1} \bar{T}(\bar{z})T(w).
\end{aligned}
\tag{2.0.20}
$$

2.1 One-loop partition function

So far, we have been studying the CFT on the complex plane, which after adding the point at infinity, is equivalent to a sphere (the Riemann sphere, a genus $h = 0$ surface). The two chiral algebras (the holomorphic and anti-holomorphic) were considered independent and in principle one can have different chiral algebras in the two sectors. However, in order to have single-valued and monodromy-free correlation functions on a genus $h = 0$ surface, one has to couple the two sectors together and impose various consistency conditions on the irreducible representations in each sector. It turns out that a set of strong constraints appear when one studies the CFT on a Riemann surface with higher genus. In particular, on a Riemann surface of genus $h = 1$ or a torus. The requirement of *modular invariance* puts a stringent constraint on the spectrum coming from the interaction of the holomorphic and antiholomorphic sectors [46].

The one-loop partition function, or the zero-point function on the torus, encodes the spectrum and the multiplicity of the different primary fields which appear in the

CFT. In the context of string theory, the one-loop partition function is the one-loop stringy Feynman diagram and it encodes the different fields which propagate in the closed string channel together with their multiplicities. For application in critical phenomena, the torus geometry appears when one considers the plane with periodic boundary conditions in two directions. The partition function is defined as

$$Z(q,\bar{q}) = \mathrm{Tr}_{\mathcal{H}}\Big(q^{L_0-c/24}\bar{q}^{\bar{L}_0-\bar{c}/24}\Big), \tag{2.1.1}$$

where

$$q = \exp 2\pi i t, \qquad \bar{q} = \exp -2\pi i\bar{t}. \tag{2.1.2}$$

The parameter t is the modular parameter of the torus[2].

The Hilbert space of the CFT is decomposed as

$$\mathcal{H} = \sum_{\lambda,\bar{\mu}\in\mathcal{E}} \mathcal{M}_{\lambda\bar{\mu}}\mathcal{H}_{\lambda}\otimes\mathcal{H}_{\bar{\mu}}, \tag{2.1.3}$$

where we assumed that the CFT is compact so the spectrum of conformal dimensions is discrete. The set $\mathcal{E}$ could in principle be infinite, however, when the CFT is rational $\mathcal{E}$ is finite.

The one-loop partition function $Z = \mathrm{Tr}e^{-\beta H}$ can be written as

$$Z(t,\bar{t}) = \sum_{\lambda,\bar{\mu}\in\mathcal{E}} \mathcal{M}_{\lambda\bar{\mu}}\ \chi_{\lambda}(t)\,\bar{\chi}_{\bar{\mu}}(\bar{t}), \tag{2.1.4}$$

where $\chi_\lambda(t)$ is the genus-1 character of an irreducible representation h_λ

$$\chi_{\lambda}(\tau) = \mathrm{Tr}_{\mathcal{H}_\lambda}\big(q^{L_0-\frac{c}{24}}\big), \tag{2.1.5}$$

where $\mathcal{H}_\lambda$ denotes the Verma module built from the highest weight state $|h_\lambda\rangle$.

[2]We point out that we are using t instead of the more familiar τ since we reserve τ to denote the modular parameter of the target space torus to be discussed later.

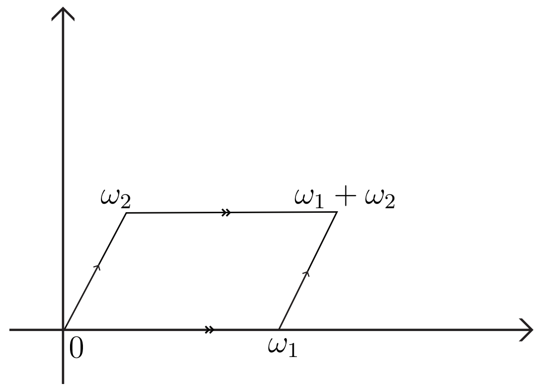

Figure 2.2: The torus as a parallelogram with opposite sides identified.

The matrix $\mathcal{M}_{\lambda\bar{\mu}}$ in (2.1.4) incorporates the multiplicities of the different fields which appear in $Z(t,\bar{t})$ and as such $\mathcal{M}_{\lambda\bar{\mu}} \in \mathbb{Z}_+$. Uniqueness of the vacuum forces $\mathcal{M}_{0\bar{0}} = 1$. Modular invariance will put more constraints on $\mathcal{M}_{\lambda\bar{\mu}}$ and to see that we will need to study the geometry of tori in more detail.

The torus can be described as a parallelogram in $\mathbb{C}$ generated by two linearly independent lattice vectors ω_1, ω_2 with the opposite sides of the parallelogram being identified

$$z \sim z + m\omega_1, \qquad z \sim z + n\omega_2, \quad m, n \in \mathbb{Z}. \tag{2.1.6}$$

Since we have a scaling symmetry at our disposal we can rescale one of the lattice vectors, say ω_1 to 1. The two lattice vectors will be denoted by $1, t \in \mathbb{C}$, where $t = \omega_1/\omega_2$ is the scale invariant parameter which is relevant in the CFT. The linear independence of ω_1 and ω_2 requires $\Im(t) > 0$, i.e., $t \in \mathbb{H}^+$. The parameter t is called the *modular parameter* of the torus and it defines the shape and the size of the torus.

Any integer linear combination of the lattice vectors will give an equivalent basis for the lattice. One should make sure that the description of the torus, or the CFT thereof, is independent of the particular choice of the basis. A change of the basis will take the form

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}, \quad a, b, c, d \in \mathbb{Z}. \tag{2.1.7}$$

The above matrix should be invertible and the requirement that the unit lattice cell should have the same area in both basis imposes

$$ab - cd = 1. \tag{2.1.8}$$

This restricts us to the group $\mathrm{SL}(2, \mathbb{Z})$ of 2×2 invertible matrices with integer entries and unit determinant. The above action on ω_1 and ω_2 induces the following $\mathrm{SL}(2, \mathbb{Z})$ action on t

$$t \mapsto \frac{at + b}{ct + d} \qquad \text{with} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}), \quad ab - cd = 1 \tag{2.1.9}$$

which gives an equivalent torus. Since the above action on t remains the same if we change the signs of a, b, c, d, then only $\mathrm{PSL}(2, \mathbb{Z}) = \mathrm{SL}(2, \mathbb{Z})/\mathbb{Z}_2$ acts faithfully. The group $\mathrm{PSL}(2, \mathbb{Z})$ is the *modular group* of the torus. It can be shown [47], that the modular group is generated by the following two operations

$$\begin{aligned} T : t \longrightarrow t + 1 \quad &\text{or } T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ S : t \longrightarrow -\frac{1}{t} \quad &\text{or } S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{2.1.10}$$

The T and S transformations satisfy

$$(ST)^3 = S^2 = \mathbb{I}. \tag{2.1.11}$$

The set of inequivalent tori is parametrized by $\mathbb{H}^+$ modulo the T and S transformations. It can be shown that the fundamental domain of t is

$$\mathcal{F}_0 = \Big\{ -\frac{1}{2} < \mathcal{R}(t) \leq \frac{1}{2},\ \Im(t) > 0,\ |t| \geq 1 \Big\}. \tag{2.1.12}$$

The values of $t \in \mathcal{F}_0$ parametrize inequivalent tori, i.e., tori which can't be transformed into one another by $\mathrm{PSL}(2,\mathbb{Z})$. The one-loop partition function defined at a particular value of t needs to remain the same if we change t by a $\mathrm{PSL}(2,\mathbb{Z})$ transformation, i.e., $Z(t,\bar{t})$ is a modular group invariant. If the CFT has a Lagrangian description then one can ensure modular invariance by integrating over $\mathcal{F}_0$ with the correct measure.

2.2 Rational conformal field theory (RCFT)

The number of primary fields which appear in a given CFT will generically be infinite. However, there are certain classes of CFTs in which the infinite number of primary fields can reorganize themselves into a finite number of blocks corresponding to the an extended chiral algebra. These are *rational* conformal field theories (RCFTs) [46], and they will be the main subject of this thesis. One arena where RCFTs arise is in string compactifications on Calabi-Yau manifolds at special values of their moduli, e.g., in Gepner models [28]. It is important to understand the conditions for rationality. For example, the simplest compactifications of a string theory are on tori T^m–when are they described by RCFTs? This question has been studied in [29–32]. In particular, Wendland [30] derived rationality conditions valid for all torus dimensions m. More importantly for us, in the case of 2-tori, Gukov and Vafa [32] found a simple, geometric criterion for rationality. For T^2, the modular parameter τ and the Kähler parameter ρ must take special values; they must belong to an imaginary quadratic number field. Such 2-tori have the property of complex multiplication, and are known as CM tori. For them, a Gauss product exists (see Chapter 3), and was used in [33] to classify the corresponding RCFTs.

The special values of the moduli indicate RCFTs where the infinite number of fields is organized into a finite set that is primary with respect to an extended chiral algebra [48]. In the T^m case, the generic boson algebra $U(1)^m = U(1)_{k_1} \times \cdots \times U(1)_{k_m}$ is extended by vertex operators defined by vectors of the lattice Λ_m describing $T^m \cong \mathbb{R}^m/\Lambda_m$. The extended algebra is well understood; it was written explicitly for the $m = 1$ case in [48], for example.

To see how a CFT can become rational at special points of its moduli space, we revisit the simplest example of an RCFT: the $c = 1$ CFT of a compact free boson on a circle of rational square raduis $R^2 = r/s$. The radius R plays the role of a geometric modulus. At a generic value of R, the number of primary fields is infinite. At $R^2 = r/s$, the chiral algebra of this theory is extended. The infinite list of representations reconstitute themselves into finitely many irreducible representations of the extended chiral algebra. In the present case, the chiral algebra is the Kac-Moody algebra generated by a $U(1)$ current $J(z) = \partial X$ extended by including charged fields $E^{\pm}(z) = \exp[\pm i\sqrt{2k}X(z)]$ of dimension k and charge $\pm 2k$, where $k = rs$. The representations of $U(1)_k$ which are local with respect to all vertex operators are labelled by an integer defined modulo $2k$. Defining the even integral lattice $\Gamma_k = \mathbb{Z}\sqrt{2k}$, the extended chiral algebra is the chiral vertex operator algebra $\mathcal{A}\big(\Gamma_k\big) = \langle J(z), E^{\pm}(z)\rangle$. The representations of the chiral algebra $\mathcal{A}(\Gamma_k)$ are labeled by $a \in \Gamma_k^*/\Gamma_k \cong \mathbb{Z}/2k\mathbb{Z}$, where Γ_k^* is the dual lattice of Γ_k. We define the mode expansions

$$J(z) = \sum_n j_n z^{n-1}, \qquad E^{\pm}(z) = \sum_n E_n^{\pm} z^{n-k}, \tag{2.2.1}$$

The extended chiral algebra in this case is the Moore-Seiberg algebra

$$\begin{aligned}
[J_n, J_m] &= n\delta_{m+n,0}, \\
[J_n, E^{\pm}_m] &= \pm k E^{\pm}_{m+n}, \\
\left[E^+_n, E^+_m\right] &= \left[E^-_n, E^-_m\right] = 0, \\
\left[E^+_n, E^-_m\right] &= \frac{(m^2-(k-1)^2)\cdots(m^2-1)m}{(2k-1)!}\delta_{m+n,0} + \cdots \\
&\quad + \frac{(\sqrt{2k}i)^{2k-1}}{(2k-1)!}\sum_{q_1+\cdots+q_{2k-1}=n+m} : J_{q_1}\cdots J_{2k-1} : .
\end{aligned} \tag{2.2.2}$$

Another thing to notice about this simplified model is that the chiral algebra depends only on k while the σ-model modulus in this case is $R^2 = r/s$. Starting from the algebra $U(1)_k$, there are many candidate rational boson theories, one for each factorization of k into two coprime integers $k = rs$. The different factorizations of k give different geometric realizations corresponding to a σ-model on a circle of radius square $R^2 = r/s$. We will encounter a similar feature when we study the case of $U_{m,K}$ algebra.

As we mentioned before, an RCFT is specified by a chiral algebra $\mathcal{A}$ which is the Virasoro algebra $\mathrm{Vir} \oplus \overline{\mathrm{Vir}}$ or one of its extensions. Examples of extended chiral algebras are

- Superconformal algebras. These are supersymmetric extensions of the Virasoro algebra by fermionic currents (of half-integral spins), e.g., the $N = 1$ and $N = 2$ superconformal algebras. Space-time supersymmetry in string theory requires an $N = 2$ superconformal symmetry on the world-sheet of the superstring [28, 49].

- The semidirect product of the Virasoro algebra with affine Kac-Moody Lie algebras [50–52]. They describe the propagation of strings on group manifolds [21, 53].

- $\mathcal{W}$-algebras. These are higher-spin extensions of the Virasoro algebra by bosonic currents [54, 55].

Rationality means that $\mathcal{A}$ has a finite set $\mathcal{E}$ of irreducible representations $\mathcal{V}_\mu$, $\mu \in \mathcal{E}$. The partition function in this case is a sesquilinear form with finite number of terms

$$Z(t,\bar{t}) = \sum_{\lambda,\bar{\mu}\in\mathcal{E}} \mathcal{M}_{\lambda\bar{\mu}}\, \chi_\lambda(t)\, \bar{\chi}_{\bar{\mu}}(\bar{t}). \tag{2.2.3}$$

The positive integers $\mathcal{M}_{\lambda\bar{\mu}}$ express the fact that a representation $(\lambda, \bar{\mu})$ of the left and right copies of $\mathcal{A}$ can appear with some multiplicity. Rationality can be stated in terms of $\mathcal{M}_{\lambda\bar{\mu}}$: A CFT is rational if $\mathcal{M}_{\lambda\bar{\mu}}$ has a finite rank. We point out that the the partition function (zero-point function) (2.2.3) of RCFT factorizes into a finite sum of holomorphic times anti-holomorphic expressions in the modular parameter of the torus. This *holomorphic* factorization is a basic property of RCFTs and it continues to hold for higher-point correlation functions and for RCFTs on higher genus Riemann surfaces [56–59]. It was shown in [60] that holomorphic factorization in RCTTs implies that the central charge c and the conformal dimensions h are rational numbers.

RCFTs are consistently described by a finite set of representations $\lambda \in \mathcal{E}$ of a certain chiral algebra. Moreover the corresponding genus-1 characters $\chi_\lambda(q)$ form a finite dimensional unitary representation of the modular group $\mathrm{PSL}(2,\mathbb{Z})$

$$\chi_\lambda\left(\frac{at+b}{ct+d}\right) = \sum_{\mu\in\in\mathcal{E}} M_\lambda^\mu \chi_\mu(t), \tag{2.2.4}$$

where $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$.

The T and S transformations are represented on the space of characters as

$$\begin{aligned}\chi_\lambda(t+1) &= \sum_{\mu\in\in\mathcal{E}} \mathcal{T}_\lambda^\mu \chi_\mu(t)\\ \chi_\lambda\Big(-\frac{1}{t}\Big) &= \sum_{\mu\in\in\mathcal{E}} \mathcal{S}_\lambda^\mu \chi_\mu(t).\end{aligned} \tag{2.2.5}$$

The requirement of on-loop modular invariance of the partition function boils down to the constraints

$$[\mathcal{T},\mathcal{M}] = [\mathcal{S},\mathcal{M}] = 0. \tag{2.2.6}$$

Every nonnegative integer matrix $\mathcal{M}$ with $\mathcal{M}_{0\bar{0}} = 1$ which satisfies (2.2.6) defines a *physical modular invariant*. Classifying the physical modular invariant partition functions in RCFT amounts to finding all non-negative integer matrices $\mathcal{M}_{\lambda\bar{\mu}}$ with $\mathcal{M}_{0\bar{0}} = 1$ subject to (2.2.6). This classification has been completed for the case $A_1^{(1)}$ for all k in [61,62] which gave rise to the ADE pattern. The ADE classification of $A_1^{(1)}$ also leads to the classification of the $c < 1$ minimal models and their supersymmetric extension. The other classification which have been found are $A_2^{(1)}$ for all k; $A_l^{(1)}$, $B_l^{(1)}$, and $D_l^{(1)}$ for all $k \leq 3$; $(A_1 \oplus A_1)^{(1)}$ for all level (k_1, k_1) (See [63] for references). The case which is important for us in this thesis is the algebra $U_{m.K} = (u(1), \cdots, u(1))^{(1)}$ for all matrix valued level K [34].

2.3 Lattices

To define a lattice Γ, we start with a vector space V which we take to be $\mathbb{R}^{p,q}$ (that is, $\mathbb{R}^{p+q}$ with the standard orthonormal basis and a Lorentzian inner product). The lattice is made of a discrete set of points of the form

$$\Gamma = \Big\{ \sum_{i=1}^{m=p+q} n_i e_i \,|\, n_i \in \mathbb{Z} \Big\}. \tag{2.3.1}$$

The lattice Γ has a dimension $m = p + q$ and we assume that e_i are linearly independent, so that Γ spans $\mathbb{R}^{p+q}$. In this case, m is the rank of Γ. The signature of Γ is (p, q).

The metric on Γ, or the intersection form, is given by

$$K_{ij} = \langle e_i | e_j \rangle, \quad i, j = 1, \ldots, m, \tag{2.3.2}$$

where the inner product is computed using the Lorentzian metric on $\mathbb{R}^{p,q}$. The intersection form encodes the lengths of the basis vectors and the angles between them.

The unit cell of the lattice Γ is the set of points

$$U = \sum_{i=1}^{m} t_i e_i, \quad 0 \leq t_i \leq 1. \tag{2.3.3}$$

For example, for a rank-2 lattice spanned by e_1 and e_2, the unit cell represents the parallelogram with the sides at $0, e_1, e_2, e_1 + e_2$. The volume of the unit cell is given by $\sqrt{\det K}$.

The dual lattice Γ^* of a lattice Γ is the set of vectors which have integer inner products with all vectors of Γ

$$\Gamma^* = \{x \in \mathbb{R}^{p+q} \mid \langle x | \lambda \rangle \in \mathbb{Z} \ \forall \ \lambda \in \Gamma\}. \tag{2.3.4}$$

The basis vectors of Γ^* will be denoted by e_i^* and are defined by

$$\langle e_i^* | e_j \rangle = \delta_{ij}. \tag{2.3.5}$$

The dual lattice Γ^* is

$$\Gamma^* = \left\{ \sum_{i=1}^{m=p+q} n_i e_i^* \,|\, n_i \in \mathbb{Z} \right\}. \tag{2.3.6}$$

The metric or the intersection form of the dual lattice Γ^* is given by

$$K_{ij}^* = \langle e_i^* | e_j^* \rangle = K_{ij}^{-1}. \tag{2.3.7}$$

It follows that the volume of the unit cell of Γ^* is given by

$$|\Gamma^*| = \frac{1}{\sqrt{\det K}}. \tag{2.3.8}$$

An integral lattice Γ is a lattice for which all the inner products between all lattice vectors is integer, $\langle\lambda|\lambda\rangle \in \mathbb{Z}$. For an integral lattice one can immediately conclude that $\Gamma \subseteq \Gamma^*$. An integral lattice is *even* when the products between all lattice vectors are even integers, $\langle\lambda|\lambda\rangle \in 2\mathbb{Z}$. Otherwise the lattice is called *odd*.

The discriminant of a lattice is the determinant of its intersection matrix, Disc $\Gamma = \det(K)$. For an integral lattice we have

$$\text{Disc}\ \Gamma = \Big|\Gamma^*/\Gamma\Big| \tag{2.3.9}$$

A self-dual lattice is one for which $\Gamma^* = \Gamma$ and hence it satisfies Disc $\Gamma = 1$, i.e., a self-dual lattice is unimodular. Even self-dual lattices $\Gamma^{p,q}$ with signature (p, q) can exist only for

$$p - q = 0 \mod 8\,, \tag{2.3.10}$$

a crucial fact in the construction of the allowed heterotic string theories in 10D, for example. For $p = q$, there is a unique even-self dual lattice up to isomorphism.

The intersection form K specifies the lattice Γ up to $GL(r, \mathbb{Z})$ transformations which preserve the volume of the unit cell or the discriminant. The $GL(r, \mathbb{Z})$ transformations act on the basis of Γ as

$$e'_i = G_{ij} e_j. \tag{2.3.11}$$

This action takes the following form on the intersection matrix

$$K' = G^T K G. \tag{2.3.12}$$

Fixing the discriminant Disc $\Gamma = \det K = D$ of Γ, then K will define an equivalence class of lattices. Two members of the same class are related by the $GL(r,\mathbb{Z})$ transformation (2.3.12). Similarly, we can define equivalence classes under $SL(r,\mathbb{Z})$ transformations. These equivalence classes and their construction will be explained in more detail in Chapter 3.

2.4 Toroidal compactification

In toroidal compactification on an n-torus T^n one assumes a space-time metric of the form

$$ds^2 = \sum_{\mu,\nu=0}^{d-1} \eta_{\mu\nu} dX^\mu dX^\nu + \sum_{I,J=1}^{n} G_{IJ} dX^I dX^J, \tag{2.4.1}$$

where $D = d + n$ is the total space-time dimensions and X^μ and Y^I represent the external and internal space-time coordinates, respectively. Here $\eta_{\mu\nu}$ is the Minkowski metric and G_{IJ} is the Ricci-flat metric on the torus. The geometry of the torus is encoded in the constant metric G_{IJ} and in the simplest case the metric takes the form

$$G_{IJ} = R_I^2 \delta_{IJ}. \tag{2.4.2}$$

This is a rectangular torus made up of a product of perpendicular circles with radii R_I, $T^n = S^1_{R_1} \times \cdots S^1_{R_n}$. More generally, the metric will have off-diagonal components corresponding to non-orthogonal circles.

The new feature of strings moving on tori is the existence of winding modes. This is because one can consider more general periodicity conditions on the embedding coordinates of the internal space $Y^I(\sigma,\tau)$

$$Y^I(\sigma + 2\pi, \tau) = Y^I(\sigma,\tau) + 2\pi W^I, \quad W^I \in \mathbb{Z}. \tag{2.4.3}$$

This corresponds to a closed string winding W^I times around the Ith circle.

By turning on a constant background B field[3], then the internal part of the action of the string takes the form

$$S = -\frac{1}{2\pi}\int d^2\sigma\big(G_{IJ}\eta^{\alpha\beta} - B_{IJ}\epsilon^{\alpha\beta}\big)\partial_\alpha Y^I \partial_\beta Y^J. \tag{2.4.4}$$

The left- and right-moving momenta derived from this action are

$$\begin{aligned} p_L^I &= W^I + G^{IJ}\Big(\frac{K_J}{2} - B_{JK}W^K\Big) \\ p_R^I &= -W^I + G^{IJ}\Big(\frac{K_J}{2} - B_{JK}W^K\Big), \end{aligned} \tag{2.4.5}$$

where $K_I \in \mathbb{Z}$ and G^{IJ} denotes the inverse metric.

The mass spectrum is given by

$$\frac{1}{2}M_0^2 = (W \quad K)\mathcal{G}^{-1}\begin{pmatrix} W \\ K \end{pmatrix}, \tag{2.4.6}$$

where the $2n \times 2n$ matrix $\mathcal{G}^{-1}$ is

$$\mathcal{G}^{-1} = \begin{pmatrix} 2(G - BG^{-1}B) & BG^{-1} \\ -G^{-1}B & \frac{1}{2}G^{-1} \end{pmatrix} \tag{2.4.7}$$

and $(W \quad K)$ is a $2n$ charge vector.

The mass spectrum is invariant under the the $O(n, n, \mathbb{Z})$ duality group acting as

$$\mathcal{G} \longrightarrow A\mathcal{G}A^T, \quad \begin{pmatrix} W \\ K \end{pmatrix} \longrightarrow \begin{pmatrix} W' \\ K' \end{pmatrix} = A\begin{pmatrix} W \\ K \end{pmatrix}. \tag{2.4.8}$$

This is the T-duality group of toroidal compactification which generalizes the $R \to 1/R$ duality of circle compactification.

The moduli space of toroidal compactification is generated by the n^2 truly marginal operators

$$\eta^{\alpha\beta}\partial_\alpha Y^I \partial_\beta Y^J, \qquad \epsilon^{\alpha\beta}\partial_\alpha Y^I \partial_\beta Y^J. \tag{2.4.9}$$

[3]The B field exists in all oriented string theories.

These operators are truly marginal since the model is Gaussian. The n^2 couplings can be organized in a background matrix

$$E = G + B. \tag{2.4.10}$$

The only restriction on E is that its symmetric part, G, be positive definite and we can represent this space of matrices as a coset

$$\mathcal{M}^0_{n,n} = O(n,n;\mathbb{R})/[O(n;\mathbb{R}) \times O(n;\mathbb{R})]. \tag{2.4.11}$$

To get the physical moduli space, we quotient the above moduli space by the T-duality group [64]

$$\mathcal{M}_{n,n} = \mathcal{M}^0_{n,n}/O(n,n;\mathbb{Z}). \tag{2.4.12}$$

2.5 Toroidal CFTs

The action of the string on an n-torus gives a toroidal CFT with central charge $c = n$. The n-torus can be represented as

$$T^n = \mathbb{R}^n/\Lambda, \tag{2.5.1}$$

where Λ is a rank n lattice.

The chiral algebra of this CFT is the affine $U(1)^n \times U(1)^n$ chiral algebra generated by the currents $j^1(z), \cdots, j^n(z)$ and $\bar{j}^1(\bar{z}), \cdots, \bar{j}^n(\bar{z})$ with the operator product expansion

$$j^m(z)j^n(w) \sim \frac{\delta^{mn}}{(z-w)^2}. \tag{2.5.2}$$

The momentum vector p^I of a string state in the toroidal compactification on T^n is the charge vector under the gauge group $U(1)^n \times U(1)^n$

$$p^I = (p^I_L, p^I_R). \tag{2.5.3}$$

This charge vector lives on a charge lattice $\Gamma^{n,n}$ with signature (n, n) and with a scalar product given by

$$p^2 = p_L^2 - p_R^2. \tag{2.5.4}$$

One-loop modular invariance forces the lattice $\Gamma^{n,n}$ to be an *even self-dual lattice*. The charge lattice uniquely determines the toroidal CFT and the moduli space of toroidal CFTs (2.4.12) is equivalent to the moduli space of even self-dual lattices [23, 24].

The set of primary fields up of the vertex operators

$$\mathcal{V}_p(z, \bar{z}) =: e^{ipX}(z, \bar{z}) : . \tag{2.5.5}$$

The energy momentum tensor is given by the Sugwara construction

$$T(z) = \frac{1}{2}\sum_{i=1}^{n} : j^i j^i : (z), \qquad \bar{T}(\bar{z}) = \frac{1}{2}\sum_{i=1}^{n} : \bar{j}^i \bar{j}^i : (\bar{z}). \tag{2.5.6}$$

Thus the conformal dimension of $\mathcal{V}_p(z, \bar{z})$ is given by

$$(h, \bar{h}) = \Big(\frac{p_L^2}{2}, \frac{p_R^2}{2}\Big). \tag{2.5.7}$$

The primary fields have simple fusions in which their charges add up. The operator product expansion of the primary fields takes the form

$$\mathcal{V}_p(z, w) \times \mathcal{V}_{p'}(\bar{z}, \bar{w}) \sim c_{pp''}(z - w)^{p_L p'_L}(\bar{z} - \bar{w})^{p_R p'_R}\mathcal{V}_{p+p'}(w, \bar{w}). \tag{2.5.8}$$

On the circle $(z - w)(\bar{z} - w) = 1$, the above OPE takes the form

$$\mathcal{V}_p(z, w) \times \mathcal{V}_{p'}(\bar{z}, \bar{w}) \sim c_{pp'}(z - w)^{p_L p'_L - p_R p'_R}\mathcal{V}_{p+p'}(w, \bar{w}) \tag{2.5.9}$$

which defines the natural metric on the lattice of charges $\Gamma^{n,n}$ to be

$$p \circ p' = p_L p'_L - p_R p'_R. \tag{2.5.10}$$

The partition function is

$$Z_{\Gamma^{n,n}}(t,\bar{t}) = \frac{1}{|\eta|^{2n}} \sum_{(p,\bar{p})\in\Gamma^{n,n}} q^{\frac{1}{2}p_L^2}\bar{q}^{\frac{1}{2}p_R^2}, \tag{2.5.11}$$

where $q = \exp(2\pi i t)$ and the Dedkind η function is defined as

$$\eta = q^{24} \prod_{n=1}^{\infty}(1-q^n). \tag{2.5.12}$$

The partition function is modular invariant by the even self-duality of the lattice $\Gamma^{n,n}$.

The charge lattice or the Narain lattice of the toroidal CFT is given by

$$\Gamma(G,B) = \Big\{ \big(p_L; p_R\big) := \frac{1}{\sqrt{2}}\big(\mu - \tilde{B}\lambda + \lambda; \mu - \tilde{B}\lambda - \lambda\big) \mid (\mu,\lambda) \in \Lambda^* \oplus \Lambda \Big\}, \tag{2.5.13}$$

where Λ^* and Λ are the momentum and winding lattices, respectively and $B = \Lambda^T\tilde{B}\Lambda$, where Λ also denotes the intersection form of the lattice Λ. The metric is given in terms of Λ as

$$G = \Lambda^T\Lambda, \tag{2.5.14}$$

The holomorphic and anti-holomorphic vertex operators are characterized by $p_R = 0$ and $p_L = 0$, respectively. They are parametrized by the values of their charges in $\Gamma_L = (p_L; 0)$ and $\Gamma_R = (0; p_R)$

$$\begin{aligned} \Gamma_L &= \big\{ \big(p_L; 0\big) := \frac{1}{\sqrt{2}}\big(\mu - \tilde{B}\lambda + \lambda; 0\big) | (\mu,\lambda) \in \Lambda^* \oplus \Lambda \big\} \\ \Gamma_R &= \Big\{ \big(0; p_R\big) := \frac{1}{\sqrt{2}}\big(0; \mu - \tilde{B}\lambda - \lambda\big) | (\mu,\lambda) \in \Lambda^* \oplus \Lambda \Big\}. \end{aligned} \tag{2.5.15}$$

We also define the following projections of the lattice $\Gamma(\tau,\rho)$:

$$\begin{aligned} \widetilde{\Gamma}_L &= (p_L; *) := \frac{1}{\sqrt{2}}\big(\mu - \tilde{B}\lambda + \lambda; *\big) | (\mu,\lambda) \in \Lambda^* \oplus \Lambda \Big\} \\ \widetilde{\Gamma}_R &= (*; p_R) := \frac{1}{\sqrt{2}}\big(*; \mu - \tilde{B}\lambda - \lambda\big) | (\mu,\lambda) \in \Lambda^* \oplus \Lambda \Big\}, \end{aligned} \tag{2.5.16}$$

where $p_R = *$ means p_R can take any value and doesn't constraint p_L. It can be shown that (see (2.7.6))

$$\tilde{\Gamma}_L = \Gamma_L^*. \tag{2.5.17}$$

For a generic point in the Narain moduli space (2.4.12), the number of primary fields is infinite and the toroidal CFT is not rational. However, for particular choices of the target space data G and B, the chiral algebra is extended by additional holomorphic vertex operators. With respect to this new $\mathcal{W}$-algebra, the infinite number of primary fields can be arranged into a finite number of blocks and one gets a rational CFT.

The conditions for rationality of a 2-torus CFT were analyzed in detail in [32, 33]. This is based on previous work by Moore [31] and Moore et al. [29]. Their conclusion is that a Narain lattice $\Gamma(\tau, \rho)$ is rational iff the following equivalent conditions are satisfied

(1) $\text{rank}\big(\Gamma_L\big) = \text{rank}\big(\Gamma_R\big) = 2$.

(2) $\tau, \rho \in \mathbb{Q}(D)$ for some integer $D < 0$.

In [32], it was shown that the above conditions are equivalent to the following statement

- A Narain lattice $\Gamma(\tau, \rho)$ is rational iff the lattice Γ_L is a finite-index sublattice of $\tilde{\Gamma}_L$.

In this case the number of primary fields of the rational theory or the dimension of the chiral ring is given by the index

$$|D| = [\tilde{\Gamma}_L : \Gamma_L] \tag{2.5.18}$$

Using (2.5.17), the group $\tilde{\Gamma}_L/\Gamma_L = \Gamma_L^*/\Gamma_L$ is the discriminant group of Γ_L [65]. It is a finite Abelian group of order $|D|$ and another way of writing the number of primary fields is

$$|D| = |\Gamma_L^*/\Gamma_L|. \tag{2.5.19}$$

The generalization to higher dimensional tori has been done in [30] where it was shown that if $\mathcal{C}(G, B)$ is a toroidal conformal field theory with target space data G and B and with a central charge $c = m$. Then $\mathcal{C}(G, B)$ is rational iff $G := \Lambda^T\Lambda \in \mathrm{GL}(d, \mathbb{Q})$ and $B \in \mathrm{Skew}(d) \cap \mathrm{Mat}(d, \mathbb{Q})$. The $\mathcal{W}$-algebra of $\mathcal{C}(G, B)$ is generated by the n $U(1)$ currents besides the extra holomorphic vertex operators $\mathcal{W}_{\lambda_i}$ which correspond to the generators λ_i of the lattice Γ_L.

It was shown in [30,31] that the points in Narain moduli space which correspond to RCFTs are dense. It is argued in [31] that a similar density property holds for RCFTs corresponding to string compactification on $K3$ surfaces. The generalization to the Calabi-Yau three-folds has been done in [32] where a criterion for the rationality of σ-models on Calabi-Yau three-folds has been conjectured. The authors of [32] conjectured that a CFT based on Calabi-Yau three-fold M is rational iff M and its mirror W admit complex multiplication over the same number field. Complex multiplication for Calabi-Yau manifolds will be explained in the next chapter.

2.6 $c = 1$

In this section we study the simplest example of a compact CFT of a real massless scalar field X which lives on a circle of radius R, i.e., $X \sim X + 2\pi R$. This is the $c = 1$ CFT of a compact free boson on a circle of radius $R \in \mathbb{R}_{\geq 0}$. The radius is the only geometric modulus in this model. The T-duality acts on R as $t : R \leftrightarrow 1/R$ (we are working in units where $\alpha' = 2$) and gives an isomorphic CFT and the moduli space of CFTs is given by

$$\mathcal{M} = \mathbb{R}^+/t. \tag{2.6.1}$$

The action is

$$S = \int d^2x \big[\partial_1 X \partial_1 X + \partial_2 X \partial_2 X\big] \tag{2.6.2}$$

In complex coordinates, the action takes the following form

$$S = \int dz\, d\bar{z}\; \partial X(z,\bar{z}) \bar{\partial} X(z,\bar{z}). \tag{2.6.3}$$

The equation of motion is

$$\bar{\partial}\partial X(z,\bar{z}) = 0 \tag{2.6.4}$$

implying that

$$\begin{aligned} j(z) &= i\partial X(z,\bar{z}) \qquad \text{holomorphic field} \\ \bar{j}(\bar{z}) &= i\bar{\partial} X(z,\bar{z}) \qquad \text{antiholomorphic field.} \end{aligned} \tag{2.6.5}$$

where $j(z)$ is the chiral current which generates the algebra $U(1)$.

The energy momentum tensor is

$$T(z) = -\frac{1}{2}\partial X \partial X = \frac{1}{2} j(z) j(z), \qquad \bar{T}(\bar{z}) = -\frac{1}{2}\bar{\partial} X \bar{\partial} X = \frac{1}{2}\bar{j}(\bar{z})\bar{j}(\bar{z}). \tag{2.6.6}$$

The left-moving and the right moving momenta are given by

$$p_l = \frac{1}{\sqrt{2}}\Big(\frac{n}{R} + mR\Big), \qquad p_r = \frac{1}{\sqrt{2}}\Big(\frac{n}{R} - mR\Big), \quad m, n \in \mathbb{Z}. \tag{2.6.7}$$

The vector (p_L, p_R) can be expressed as

$$(p_L, p_R) = m e_1 + n e_2, \tag{2.6.8}$$

where

$$e_1 = \frac{1}{\sqrt{2}}\Big(\frac{1}{R}, \frac{1}{R}\Big), \qquad e_2 = \frac{1}{\sqrt{2}}(R, -R). \tag{2.6.9}$$

With respect to the inner product (2.5.10), we have $e_1 \cdot e_1 = e_2 \cdot e_2 = 0$ and $e_1 \cdot e_2 = e_2 \cdot e_1 = 1$. The intersection form of the lattice $\Gamma^{1,1}$ spanned by e_1 and e_2 is

$$K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.6.10}$$

The lattice $\Gamma^{1,1}$ is an even self-dual lattice. This is the momentum-winding Narain lattice in which (p_L, p_R) lives.

The partition function is

$$Z_{\Gamma^{1,1}}(t, \bar{t}; R) = \frac{1}{|\eta|^2} \sum_{(p,\bar{p}) \in \Gamma^{1,1}} q^{\frac{1}{2}p_L^2} \bar{q}^{\frac{1}{2}\bar{p}_L^2}, \tag{2.6.11}$$

For generic values of R in $\mathcal{M}$, the models are not rational and the chiral algebra is the $U(1)$ algebra generated by the current $j(z)$. One can try to find extensions of the chiral algebra by looking for new holomorphic vertex operators which might extend it. A holomorphic vertex operator $\mathcal{V}_\gamma(z)$ will appear at $p_R = 0$. The new holomorphic vertex operators will have the form

$$\mathcal{V}_\gamma(z) = e^{i\gamma X}(z). \tag{2.6.12}$$

These are the only vertex operators allowed in the free boson theory. The conformal dimension of $\mathcal{V}_\gamma(z)$ is $\gamma^2/2$.

Looking at the second equation in (2.6.7), the constraint $p_R = 0$ gives

$$R^2 = \frac{n}{m} \tag{2.6.13}$$

which makes sense only when R^2 is rational, i.e., holomorphic vertex operators will appear only when R^2 is rational. We will put $R^2 = r/s$, where r and s are two coprime integers. The T duality $R \leftrightarrow 1/R$ now acts as $r \leftrightarrow s$. The conformal dimension must be invariant under $r \leftrightarrow s$. Since X is periodic, then $e^{i\gamma X}(z)$ must be well defined under

$$X \to X + 2\pi\sqrt{\frac{r}{s}}. \tag{2.6.14}$$

The above requirements forces γ to take the form

$$\gamma = \pm\sqrt{2rs} \tag{2.6.15}$$

where the factor of 2 is included to give $\mathcal{V}_\gamma(z)$ an integer conformal dimension.

The extended chiral algebra is generated by

$$j(z) = i\partial X(z), \qquad \mathcal{V}_{\pm\sqrt{2rs}} = e^{i\pm\sqrt{2rs}}X(z). \tag{2.6.16}$$

For $r = s = 1$, this gives the affine $su(2)_1$ extension of the free boson theory at the self-dual radius $R = 1$.

The primary fields are given by the vertex operators

$$\Phi_\alpha(z, \bar{z}) = e^{i\alpha X}(z, \bar{z}). \tag{2.6.17}$$

Since the operator product expansion of $\mathcal{V}_{\pm\sqrt{2rs}}$ and $\Phi_\alpha(z, \bar{z})$ can't have branch cuts, the value of α is fixed by locality to be

$$\alpha = \frac{l}{\sqrt{2rs}}, \quad l \in \mathbb{Z}. \tag{2.6.18}$$

The periodicity of X can be used to fix the fundamental domain of l to

$$l = -2rs + 1, -2rs + 2, \cdots, 2rs. \tag{2.6.19}$$

Let us formulate the above results in terms of even integral lattices. First we define the lattice

$$\Gamma_L = \mathbb{Z}e = \mathbb{Z}\sqrt{2rs}. \tag{2.6.20}$$

Since $\langle e|e\rangle = 2rs \in 2\mathbb{Z}$, then the lattice Γ_L is even integral. We will call it the lattice of holomorphic vertex operators. The dual lattice Γ_L^* is given by

$$\Gamma_L^* = \mathbb{Z}e^* = \frac{\mathbb{Z}}{\sqrt{2rs}}. \tag{2.6.21}$$

Looking at the weights of the primary fields (2.6.18) and their fundamental domain (2.6.19), one learns that the set of primary fields are parametrized by elements of the coset (or the discriminant group)

$$\Gamma_L^*/\Gamma_L. \tag{2.6.22}$$

The dimension of this coset is nothing but the determinant of the intersection matrix of the lattice Γ_L

$$D = 2rs. \tag{2.6.23}$$

This is the simplest realization of (2.5.19) which gives the dimensions of the chiral ring or the number of primary fields in the terms of the discriminant of the lattice of the holomorphic vertex operators.

In the language of the Lie algebra, the above results can be phrased in the following way. The lattice Γ_L^* is the weight lattice and the lattice Γ_L is the root lattice. This is so since the action of a holomorphic vertex operator, parametrized by an element in Γ_L, shifts the weight of the primary fields by an element of Γ_L^*. The ratio Γ_L^*/Γ_L has $|\Gamma_L|$ congruence classes which labels the set of admissible representations.

The above discussion shows that the $U(1)$ chiral algebra is extended at rational values of R^2 with the appearance of new holomorphic vertex operators. The requirement of locality with respect to the new holomorphic vertex operators selects only a finite number of primary fields and the theory becomes rational. Now let us see how can we formulate the above discussion in terms of the properties of the charge lattice $\Gamma^{1,1}$.

First, let us study the projections of $\Gamma^{1,1}$ at $R^2 = r/s$. The left-moving momentum lattice which is spanned by the vectors p_L

$$\tilde{\Gamma}_L \equiv \frac{1}{\sqrt{2}}\Big(\frac{n}{R} + mR\Big), \quad m, n \in \mathbb{Z} \tag{2.6.24}$$

is a rank-2 lattice for generic R. However, for $R^2 = r/s$, one gets

$$\tilde{\Gamma}_L = \frac{\mathbb{Z}}{\sqrt{2rs}}. \tag{2.6.25}$$

This equation says that the vectors p_L live in an even integer lattice $\tilde{\Gamma}_L$ of rank 1 which is spanned by $e = 1/\sqrt{2rs}$. This is the weight lattice which we found earlier (2.6.18). The dual lattice $\tilde{\Gamma}^*_L$ of $\tilde{\Gamma}_L$ is given by

$$\tilde{\Gamma}^*_L = \mathbb{Z}e^* = \mathbb{Z}\sqrt{2rs}. \tag{2.6.26}$$

This the lattice of holomorphic vertex operators (2.6.20) which is a rank-1 lattice. The fact that

$$\tilde{\Gamma}^*_L = \Gamma_L \longrightarrow \tilde{\Gamma}_L = \Gamma^*_L \tag{2.6.27}$$

is easy to understand in terms of the projections $\Gamma^{1,1}$. First, since Γ_L parameterizes the holomorphic vertex operators then its vector will have the form

$$p = (p_L, 0). \tag{2.6.28}$$

The lattice $\tilde{\Gamma}_L$ being the lattice of the left-moving momentum its vectors will have the form

$$p' = (p'_L, *). \tag{2.6.29}$$

Now since

$$p \circ p' = p_L p'_L \in \mathbb{Z}. \tag{2.6.30}$$

Then (2.6.27) follows.

The number of primary fields, or the dimension of the chiral ring, now follow easily

$$|D| = |\Gamma_L^*/\Gamma_L| = |\tilde{\Gamma}_L/\Gamma_L| \tag{2.6.31}$$

This number is finite precisely because Γ_L is a finite-index sublattice of $\tilde{\Gamma}_L$ which is the condition for rationality alluded to earlier.

2.7 $c = 2$

From the simple example of the previous section, we learn that rationality emerges when Γ_L is a finite index sublattice of $\tilde{\Gamma}_L$ and in this case, the number of primary fields or the dimensions of the chiral ring is given by $|\Gamma_L^*/\Gamma_L|$. In this section, we review the analogous construction for strings on a 2-torus [32]. Rationality can be translated into a simple, geometric condition on the modular parameter τ and the Kähler parameter ρ of the 2-torus.

We consider a 2-torus, or an elliptic curve, with a metric

$$\begin{aligned} ds^2 &= G_{11}dx^2 + 2G_{12}dxdy + G_{22}dy^2 \\ &= \frac{\rho_2}{\tau_2}\big|dx + \tau dy\big|^2. \end{aligned} \tag{2.7.1}$$

The complex structure modulus and the Kähler modulus are given by

$$\begin{aligned} \tau &= \tau_1 + i\tau_2 = \frac{G_{12}}{G_{11}} + i\frac{\sqrt{\det G}}{G_{11}}, \\ \rho &= \rho_1 + i\rho_2 = B + i\sqrt{\det G}. \end{aligned} \tag{2.7.2}$$

The momentum-winding Narain lattice $\Gamma^{2,2}$ contains

$$\begin{pmatrix} p_L \\ p_R \end{pmatrix} = \frac{i}{\sqrt{2\tau_2\rho_2}}\mathbb{Z}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \oplus \begin{pmatrix} \bar{\rho} \\ \rho \end{pmatrix} \oplus \begin{pmatrix} \bar{\tau} \\ \tau \end{pmatrix} \oplus \begin{pmatrix} \bar{\rho}\tau \\ \rho\tau \end{pmatrix}. \tag{2.7.3}$$

We define the following projections of the lattice $\Gamma^{2,2}$ (here we consider only the left-moving part of $\Gamma^{2,2}$ but all the definition will apply analogously to the right-moving part)

$$\begin{aligned}\Gamma_L &= \Big\{ p_L \mid p_R = 0, \quad \big(p_L; 0\big) \in \Gamma^{2,2} \Big\} \\ \tilde{\Gamma}_L &= \Big\{ p_L \mid p_R = *, \quad \big(p_L; p_R\big) \in \Gamma^{2,2} \Big\}.\end{aligned} \tag{2.7.4}$$

It is clear that $\Gamma_L \subseteq \tilde{\Gamma}_L$. Also, since

$$(p_L, 0) \circ (q_L, *) = p_L q_L \in \mathbb{Z} \tag{2.7.5}$$

Then, Γ_L and $\tilde{\Gamma}_L$ are dual lattices

$$\Gamma_L \cong \tilde{\Gamma}_L^*. \tag{2.7.6}$$

The lattice Γ_L will in general have a rank zero since the constraint $p_L = 0$ will have no solutions except for special values of the moduli τ and ρ. For these special values of τ and ρ, the lattice Γ_L will become a rank-2 sublattice of $\Gamma^{2,2}$. Looking at the explicit basis of the Narain lattice (2.7.3), and since Γ_L is rank-2, then it is cut by two independent equations. Since Γ_L corresponds to $p_R = 0$, the two equations have the form

$$\begin{aligned} m_1 + m_2\rho + m_3\tau + m_4\tau\rho &= 0, \\ m_1' + m_2'\rho + m_3'\tau + m_4'\tau\rho &= 0. \end{aligned} \tag{2.7.7}$$

By solving for ρ from the second equation and plugging the result into the first, we arrive at

$$a\tau^2 + b\tau + c = 0, \quad a, b, c \in \mathbb{Z} \quad \gcd(a, b, c) = 1. \tag{2.7.8}$$

This implies that

$$\tau = \frac{-b + \sqrt{D}}{2a}, \quad D = b^2 - 4ac < 0, \tag{2.7.9}$$

where we have chosen the solution which gives τ a positive imaginary part. Substituting this value of τ back into the first equation in (2.7.7), we learn that ρ also satisfies a quadratic equation with the same discriminant D. Hence both τ and ρ belong to an imaginary quadratic number field $\mathbb{Q}(\sqrt{D})$.

The criteria for rationality found in [32] for a CFT based on an elliptic curve E is

$$\text{RCFT} \Longleftrightarrow \tau, \rho \in \mathbb{Q}(\sqrt{D}). \tag{2.7.10}$$

Elliptic curves for such values of τ, ρ are said to have complex multiplication or to be of CM type. We will study them in detail in the next chapter.

2.8 Matrix-level affine Kac-Moody algebra

In this section we introduce the matrix-level affine Kac-Moody algebras based on a group G. The $U_{m,K}$ algebra will arise as a special case when $G = U(1)$. Let G be a compact, simple, semi-simple Lie group with a Lie algebra g. The affine Kac-Moody algebra of g is $\hat{g}_k$, where $k \in \mathbb{Z}_{\geq 0}$ is the level. We study the affine extension of $g^{\oplus m}$ with matrix-valued level K_{AB}. We take K_{AB} to be the intersection form of a lattice Γ_K. The inverse K^{AB} is the intersection form of the dual lattice Γ_K^*. We will denote the currents by J_A^a, where a is the Lie algebra index $a = 1, \cdots, \dim(g)$ and $A = 1, \cdots, N = \text{rank}(K)$. The Lie algebra of $g^{\oplus m}$ takes the form

$$\left[J_A^a, J_B^b\right] = i f_{abc} J_B^c \delta_{AB}. \tag{2.8.1}$$

We assume the following OPE of the currents

$$J_A^a(z) J_B^b(w) \sim \frac{i f_{abc} J_B^c(w) \delta_{AB}}{z-w} + \frac{\delta^{ab} K_{AB}}{(z-w)^2}. \tag{2.8.2}$$

This is not the central extension of the loop group of $g^{\oplus m}$ since a cohomological obstruction prevents the construction of a two-cycle on the loop group of $g^{\oplus m}$. This however doesn't mean that one can't construct a CFT corresponding to the above OPE.

We use the Sugawara construction to find the energy-momentum tensor

$$T(z) = G^{AB}(J^a_A J^a_B)(z), \tag{2.8.3}$$

where G^{AB} is such that the currents $J^a_A(z)$ have conformal dimension one

$$T(z)J^a_A(w) \sim \frac{J^a_A(w)}{(z-w)^2}. \tag{2.8.4}$$

The matrix G^{AB} turns out to be

$$G^{AB} = \frac{K^{AB}}{2(hK+1)}, \tag{2.8.5}$$

where $K^{AB} = K^{-1}_{AB}$, h is the dual Coxeter number of the group G and $K = \mathrm{Tr}(K^{AB})$. Using the energy-momentum tensor

$$T(z) = \frac{K^{AB}}{2(hK+1)}(J^a_A J^a_B)(z) \tag{2.8.6}$$

we can find the central charge from the most singular term of the OPE of $T(z)$ with itself

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \cdots \tag{2.8.7}$$

The value of c is

$$c(K) = \frac{N \dim g}{hK+1}. \tag{2.8.8}$$

For $N = 1$ and $K = k^{-1}$, we recover the usual value of c

$$c(k) = \frac{k \dim g}{h+k}. \tag{2.8.9}$$

On the other hand, in the diagonal case $K_{AB} = k_A \delta_{AB}$, the central charge is not a sum

$$c(K) \neq \sum_A c\big(k_A\big). \tag{2.8.10}$$

The Abelian case corresponds to the algebra $u(1) \oplus \cdots \oplus u(1)$ which we denote by $U_{m,K}$, where K is an $m \times m$ matrix-valued level. This algebra was studied in [34] where its modular invariant partition functions were classified. The operator product expansion of the currents takes the form

$$J_A(z) J_B(w) \sim \frac{K_{AB}}{(z-w)^2}. \tag{2.8.11}$$

The motivation for studying Kac-Moody algebras with matrix-valued level is the existence of Chern-Simons theories on a three-manifold N_3, based on a gauge group $G = U(1)^m$ with the level being an integer valued matrix [66]

$$S = K_{AB} \int_{M_4} F^A \wedge F^B, \tag{2.8.12}$$

where M_4 is the four manifold for which N_3 is a boundary. The independence of the partition function from the extension from N_3 to M_4 forces K_{AB} to be an integer-valued matrix. The existence of the above action relies on the fact that $H^4(BU(1), \mathbb{Z}) = \mathbb{Z}$, where $BU(1)$ is the classifying space of $U(1)$. For $G = U(1)^m$, $BG = \mathbb{Z}^m$.

The study of the matrix-level affine algebra $U_{m,K}$ is partly motivated by the effective description of the fractional quantum Hall effect, via a Chern-Simons theory with a gauge group $U(1)^m$ and a matrix valued level K [66, 67]. The Witten correspondence [48, 68] relates the Chern-Simons theory with a gauge group G canonically quantized on a manifold $M = \Sigma \times \mathbb{R}_t$, to the RCFT based on the affine Kac-Moody algebra $\hat{G}_k$ (the Wess-Zumino-Witten model). For any simple, compact gauge group G the Chern-Simons actions on a three-manifold are classified by an integer $k \in \mathbb{Z}$.[4] It was shown in [66] that Chern-Simons theories for the Abelian group $U(1)^m$ are

[4] If the three-manifold is a spin manifold then k can be half-integer [69].

classified by positive integer lattices with intersection forms K. Due to their potential application in the fractional quantum Hall effect, it is of interest to study the two-dimensional avatars of the $U(1)^m$ Chern-Simons theories based on matrix valued level K. This reduces to the study of two-dimensional RCFTs with the affine Kac-Moody algebra $U_{m,K}$.

Chern-Simons theories with matrix-valued level K gives an effective description of the Abelian Fractional Quantum Hall (FQH) liquids [66,67,70,71]. The matrix-valued level K appears in a class of multi-layer FQH states which generalizes the Laughlin state [72]. The wave function of those FQH multi-layer states takes the form

$$\Psi(z) = \prod_{a,b,i,j} (z_{ai} - z_{bj})^{K_{ab}/2} \exp\Big(- \sum |z_{ai}|^2 \Big), \tag{2.8.13}$$

where z_{ai} is the coordinate of the ith electron in the ath layer. The matrix K determines the patterns of the zeros of the wave function and the various monodromies which occur when electrons moves around each other. This can be related to the number of the flux quanta attached to the electrons in each layer. For example, a zero of order $K_{aa}/2$ in the above wave function means the phase of the wave function will acquire a phase $K_{aa}\pi$ when the electrons circle one another. This in return means there are $K_{aa}/2$ flux quanta attached to each electron and the matrix K determines how flux quanta are attached to the electrons.

Since the gauge group of the action (2.8.12) is compact then the charges which appear in the theory are quantized and they live in a charge lattice. The allowed field redefinitions on the field strength F^A in (2.8.12) are the ones which preserve the integrality of the charges. These field transformations are elements of the group $SL(m, \mathbb{Z})$ and they act on the matrix K as

$$K \longrightarrow SKS^T, \tag{2.8.14}$$

where $S \in SL(m, \mathbb{Z})$.

Two FQH states (2.8.13) with their K matrices related by the $SL(m, \mathbb{Z})$ transformation (2.8.14) are considered equivalent, i.e., they belong to the same universality class. Hence, multi-layer FQH states are classified by $SL(m, \mathbb{Z})$ equivalence classes of K matrices where the equivalence relation is (2.8.14). This fact will be used later to define a new product on the set of FQH states (see the discussion after (5.0.5)).

Chapter 3

Complex multiplication

3.1 Elliptic curves

In this chapter we describe in more detail complex multiplication for elliptic curves, higher-dimensional tori, and for Calabi-Yau manifold. Our aim is to set the stage for the conjecture by Gukov and Vafa which relates the rationality of σ-models on Calabi-Yau manifolds to the complex multiplication property of these manifolds. We will give a physicist-friendly introduction to some concepts in algebraic number theory and the theory of elliptic curves which will be important later. We will follow [31, 47, 73].

A elliptic curve is a genus-1 surface. When viewed as real manifolds, all elliptic curves are diffeomorphic to a product of two circles $S^1 \times S^1$. However, when they are given a complex structure, i.e., when viewed as complex manifolds, elliptic curves are not all equivalent. Actually, there is an infinite number of elliptic curves with different complex structures, as we will see later.

A rigorous definition of an elliptic curve requires some technical tools from algebraic geometry. We will start with the definition of an elliptic curve using the algebraic equation:

$$f(x, y) = y^2 - 4x^3 + g_2 x + g_3 = 0, \quad x, y \in \mathbb{C}, \tag{3.1.1}$$

where g_2 and g_3 are complex numbers. This is the Weierstrass equation of the elliptic curve. Later on we will relate this definition to complex tori.

In order for the elliptic curve (3.1.1) to be well-defined one needs to make sure it is non-singular (no cusps, no self-intersection, no isolated points). The curve will be non-singular iff there are no points for which $f(x, y) = 0$ and $df(x, y) = 0$ simultaneously. For (3.1.1), this translates to

$$\Delta = 4g_2^3 + 27g_3^2 \neq 0. \tag{3.1.2}$$

The polynomial Δ is known as the *discriminant* of the elliptic curve.

A complex torus T^2 can be constructed in the following way: start with a lattice Λ in the complex plane

$$\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2, \tag{3.1.3}$$

where $\{\omega_1, \omega_2\}$ are the basis vectors of the lattice. Then a complex torus is the quotient of the complex plane by the lattice Λ

$$\mathbb{C}/\Lambda = \{z \sim z + \Lambda : z \in \mathbb{C}\}. \tag{3.1.4}$$

This definition makes it obvious that the complex torus is an Abelian group under addition since both $\mathbb{C}$ and Λ are Abelian groups. As such, a complex torus is an example of an Abelian variety. It is also a Riemann surface of genus one.

By rescaling we can bring the lattice to the form

$$\Lambda = m1 + n\tau, \quad \tau = \frac{\omega_1}{\omega_2} \in \mathbb{H}^+, \tag{3.1.5}$$

where $\mathbb{H}^+$ is the upper-half of the complex plane. The holomorphic coordinate on the torus is

$$z = x + \tau y. \tag{3.1.6}$$

As we explained before in Chapter 2, the $\mathrm{SL}(2,\mathbb{Z})$ action on ω_1 and ω_2 induces the following $\mathrm{SL}(2,\mathbb{Z})$ action on τ

$$\tau \mapsto \frac{a\tau+b}{c\tau+d} \qquad \text{with} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}), \quad ab-cd=1 \tag{3.1.7}$$

which gives an equivalent torus. Since the above action on τ remains the same if we change the signs of a, b, c, d, then only $\mathrm{PSL}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})/\mathbb{Z}_2$ acts faithfully. The group $\mathrm{PSL}(2,\mathbb{Z})$ is the *modular group* of the torus. The modular group is generated by the two operations [47]

$$\begin{aligned} T : \tau \longrightarrow \tau + 1 \quad &\text{or } T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ S : \tau \longrightarrow -\frac{1}{\tau} \quad &\text{or } S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{3.1.8}$$

The T and S transformations satisfy

$$(ST)^3 = S^2 = 1. \tag{3.1.9}$$

The set of inequivalent tori is parametrized by $\mathbb{H}^+$ modulo the T and S transformations. The fundamental domain of τ is [47]

$$\mathcal{F}_0 = \Big\{ -\frac{1}{2} < \mathcal{R}(\tau) \leq \frac{1}{2},\ \Im(\tau) > 0,\ |\tau| \geq 1 \Big\}. \tag{3.1.10}$$

The values of $\tau \in \mathcal{F}_0$ parametrize inequivalent tori, i.e., tori which can't be transformed into one another by $\mathrm{PSL}(2,\mathbb{Z})$.

An elliptic curve is isomorphic to a complex torus. To establish this isomorphism we need to define the Weierstrass function. First, an elliptic function is a function on $\mathbb{C}$ which is doubly periodic for Λ

$$\begin{aligned} F(z+\omega_1) &= F(z) \\ F(z+\omega_2) &= F(z). \end{aligned} \tag{3.1.11}$$

The Weierstrass function is an elliptic function defined as

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda}{}' \Big[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \Big]. \tag{3.1.12}$$

where $\sum'$ means we only sum over non-zero lattice vectors. Using the notation

$$s_m(\Lambda) = \sum_{\omega \in \Lambda}{}' \frac{1}{\omega^m} \tag{3.1.13}$$

we get the expansion

$$\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty} (2n+1) s_{2n+2}(\Lambda) z^{2n}. \tag{3.1.14}$$

The first few terms are

$$\wp(z) = \frac{1}{z^2} + 3s_4(\Lambda)z^2 + +5s_6(\Lambda)z^4. \tag{3.1.15}$$

Now we state without proof the following theorem [47].

Theorem 3.1.1. *Let* $g_2 = g_2(\Lambda) = 60s_4$ *and* $g_3 = g_3(\Lambda) = 140s_6$*. Then*

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3. \tag{3.1.16}$$

This means that the points $(x, y) = (\wp(z), \wp'(z))$ lie on the elliptic curve

$$y^2 = 4x^3 - g_2x - g_3. \tag{3.1.17}$$

It can be shown that the discriminant of this elliptic curve $\Delta(\Lambda)$ doesn't vanish and the equation defines a non-singular elliptic curve.

The j-invariant of the lattice Λ is defined as

$$j(\Lambda) = 1728 \frac{g_2(\Lambda)}{g_2(\Lambda)^3 - 27g_3(\Lambda)^2} = 1728 \frac{g_2(\Lambda)}{\Delta(\Lambda)} \tag{3.1.18}$$

This is always well defined since $\Delta(\Lambda) \neq 0$.

One of the properties of the j-invariant is that it is invariant under scaling of the lattice Λ. Two lattices Λ and Λ' are said to be *homothetic* if there is a non-zero complex number λ such that

$$\Lambda' = \lambda\Lambda \tag{3.1.19}$$

For example any lattice $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ is homothetic to a lattice of the form $\Lambda = \mathbb{Z}\oplus\mathbb{Z}\tau$. Since we are only interested in lattices up to a homothety, then the j-invariant is useful because it is defined over the equivalence classes of homothetic lattices

$$j(\Lambda) = j(\lambda\Lambda). \tag{3.1.20}$$

This can be easily proven by noting that

$$\begin{aligned} g_2(\lambda\Lambda) &= \lambda^{-4} g_2(\Lambda) \\ g_3(\lambda\Lambda) &= \lambda^{-6} g_6(\Lambda). \end{aligned} \tag{3.1.21}$$

When defined for the lattice $\Lambda_\tau = \mathbb{Z} \oplus \mathbb{Z}\tau$, the j-invariant becomes $j(\tau)$. The function $j(\tau)$ is invariant under the $SL(2,\mathbb{Z})$ group

$$j\left(\frac{a\tau + b}{c\tau + d}\right) = j(\tau). \tag{3.1.22}$$

In particular

$$j(\tau + 1) = j(\tau). \tag{3.1.23}$$

This implies that $j(\tau)$ is holomorphic in $q = e^{2\pi i\tau}$, where $0 < |q| < 1$ since $\Im(\tau) > 0$. The j function has the following q-expansion [47]

$$j(\tau) = \frac{1}{q} + \sum_{n=0}^{\infty} c_n q^n = \frac{1}{q} + 744 + 196884q + \cdots\,, \tag{3.1.24}$$

where $\tau \in \mathbb{H}$ and $c_n \in \mathbb{Z}\forall n \geq 0$.

Now we state one of the most important theorems in the study of elliptic curves.

Theorem 3.1.2 (Uniformization theorem)**.** *Let E be an elliptic curve given by the Weierstrass equation*

$$y^2 = 4x^3 - g_2 x - g_3, \tag{3.1.25}$$

where $g_2, g_3 \in \mathbb{C}$ and $g_2^3 - 27g_3^2 \neq 0$. Then there is a unique lattice Λ such that

$$\begin{aligned} g_2 &= g_2(\Lambda) \\ g_3 &= g_3(\Lambda) \end{aligned} \tag{3.1.26}$$

The isomorphism between complex tori and elliptic curves can be written as

$$\begin{aligned} \psi \ : \ \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}) &\longrightarrow E \\ z \mapsto \psi(z) &= [\wp(z), \wp'(z)]. \end{aligned} \tag{3.1.27}$$

An elliptic curve is also a one-dimensional Calabi-Yau manifold. It can be represented by a homogeneous degree-3 polynomial in $\mathbb{CP}^2$

$$a_1 Z_1^3 + a_2 Z_2^3 + a_3 Z_3^3 + \cdots + a_{10} Z_2 Z_3^2 = 0. \tag{3.1.28}$$

This is the natural description of the elliptic curve as a target space in string theory in terms of a gauged linear sigma model [74].

There are 10 parameters a_i in (3.1.28) corresponding to the 10 independent degree-3 monomials in 3 variables. One parameter can be removed by an overall scaling and 8 parameters can be removed by $GL(3, \mathbb{C})$ transformations. The remaining parameter characterizes the complex structure of the elliptic curve, i.e., the moduli space of complex structures on an elliptic curve is one-dimensional. We can go from the projective representation (3.1.28) to the Weierstrass form (3.1.1) by first rescaling out one of the projective coordinates, say Z_1. Then the Weierstrass form can be obtained by coordinate redefinitions of the remaining coordinates. The g_2 and g_3 which appear in the Weierstrass form will be functions of the one parameter which remains in (3.1.28). The projective form of the Weierstrass equation is

$$zy^2 = 4x^3 + g_2 z^2 x + z^3 g_3 \tag{3.1.29}$$

which reduces to the affine form once we scale z out using the projective equivalence $(x, y, z) \sim (\beta x, \beta y, \beta z)$, where β is a non-zero complex number.

The j-invariant allows one to read the value of τ from the projective form of the elliptic curve. First, we reduce equation (3.1.28) to the Weierstrass form (characterized by g_2 and g_3 which are now functions of the parameters a_i) using $GL(3, \mathbb{C})$. Then τ can be found from

$$j(\tau) = 1728 \frac{g_2(\Lambda)}{\Delta(\Lambda)}. \tag{3.1.30}$$

For example, consider the elliptic curve represented by the Fermat cubic

$$Z_1^3 + Z_2^3 + Z_3^3 = 0. \tag{3.1.31}$$

Using the change of coordinates

$$Z_1 = \frac{1}{6}z + \frac{\sqrt{3}}{18}y, \qquad Z_2 = \frac{1}{6}z - \frac{\sqrt{3}}{18}y, \qquad Z_3 = -\frac{1}{3}x \tag{3.1.32}$$

we get

$$zy^2 = 4x^3 - z^3. \tag{3.1.33}$$

This means $g_2 = 0$ and $g_3 = 1$ hence $j(\tau) = 0$. This value of j corresponds to $\tau = e^{i\pi/3}$. We we will see in the next section that elliptic curves with this value of τ have complex multiplication.

The complex torus admits a nowhere-vanishing holomorphic 1-form $\omega = dz$. The integrals of ω around the two periods of the torus A, B define the *period*

$$\tau = \frac{\int_B \omega}{\int_A \omega}. \tag{3.1.34}$$

It is more convenient to normalize ω such that $\int_A \omega = 1$ and as such

$$\tau = \int_B \omega. \tag{3.1.35}$$

The periods A, B are the basis of 1-cycles on the torus (a genus-1 surface). There is an $\mathrm{SL}(2, \mathbb{Z})$ ambiguity in the choice of the basis of 1-cycles which translates to the $\mathrm{SL}(2, \mathbb{Z})$ action on τ.

An n-dimensional complex torus T^{2n} with complex coordinates $z_1, \cdots, z_n$ is described using the identification

$$z_i \sim z_i + \delta_{ij}, \qquad z_i \sim z_i + \mathcal{T}_{ij}, \tag{3.1.36}$$

where the $n \times n$ period matrix $\mathcal{T}_{ij}$ satisfies $\Im(\mathcal{T}_{ij}) > 0$. The column vectors of $\mathcal{T}_{ij}$ define a rank-n lattice Λ. The torus T^n is also given as a quotient $T^{2n} = \mathbb{C}^n/\Lambda$.

To define the notion of complex multiplication for Calabi-Yau manifolds, we first need to define the Jacobian of a genus-n Riemann surface Σ_n. On Σ_n there are n holomorphic 1-forms ω_i and $2n$ 1-cycles $A_i, B_i, i = 1, \ldots, n$ with a symplectic pairing

$$A_i \cap B_j = \delta_{ij}, \qquad A_i \cap A_j = B_i \cap B_j = 0. \tag{3.1.37}$$

The holomorphic 1-forms ω_i are normalized relative to the A_i-cycles such that

$$\int_{A_i} \omega_j = \delta_{ij}. \tag{3.1.38}$$

The period matrix is defined as

$$\mathcal{T}_{ij} = \int_{B_i} \omega_j. \tag{3.1.39}$$

There is an $\mathrm{SL}(2n, \mathbb{Z})$ ambiguity in the choice of the basis of 1-cycles which translates to the $\mathrm{SL}(2n, \mathbb{Z})$ action on $\mathcal{T}$.

For Calabi-Yau manifolds, there is a simple formula for the period matrix $\mathcal{T}_{IJ}$ in terms of the prepotential $\mathcal{F}$ of the complex-structure moduli space. To define $\mathcal{F}$, we first describe the complex-structure moduli space in more detail. We look at the middle-homology cycles which for a Calabi-Yau three fold X are three (real)-dimensional and they live in $\mathbb{H}_3(X,\mathbb{Z})$. The dimension of $\mathbb{H}_3(X,\mathbb{Z})$ is given by the third Betti number $b_3(X)$. For a Calabi-Yau X, $b_3(X)$ can be refined in terms of the dimensions of the Dolbeault cohomology groups as

$$b_3(X) = h^{2,1} + h^{1,2} + h^{3,0} + h^{0,3} = 2h^{2,1} + 2. \tag{3.1.40}$$

A basis of three-cycles $A^I, B_J, I, J = 0, \cdots, h^{2,1}$ can be chosen such that

$$A^I \cap B_J = \delta_{IJ}, \qquad A^I \cap A^J = B_I \cap B_J = 0. \tag{3.1.41}$$

To define a set of coordinates on the complex-structure moduli space, we need a quantity which depends on the complex structure. This is provided by the unique nowhere-vanishing holomorphic 3-form Ω which exists on any Calabi-Yau manifold. The integrals of Ω over A and B give

$$X^I = \int_{A^I} \Omega, \qquad F_I = \int_{A_I} \Omega, \quad I = 0, \cdots, h^{2,1}. \tag{3.1.42}$$

Since the dimensions of the complex-structure moduli space is $h^{2,1}$, the above coordinates are overcomplete and in fact, F_I are functions of X^I.

The prepotential of the Calabi-Yau manifold is defined as

$$\mathcal{F} = \frac{1}{2} X^I F_I(X^I). \tag{3.1.43}$$

The period matrix is now given by

$$\mathcal{T}_{IJ} = \frac{\mathcal{F}}{\partial X^I \partial X^J}. \tag{3.1.44}$$

3.2 Quadratic number fields

Let $\mathbb{Q}(\theta)$ denote the set of numbers of the form

$$x = x_1 + x_2 \theta \tag{3.2.1}$$

where $x_1, x_2 \in \mathbb{Q}$. The set $\mathbb{Q}[x]$ will denote the ring of polynomials in x with coefficients from $\mathbb{Q}$. While the set $\mathbb{Z}[x]$ will denote the ring of polynomials in x with coefficients from $\mathbb{Z}$.

Definition 3.2.1. An algebraic number α is a complex number $\alpha \in \mathbb{C}$ which satisfies a polynomial equation with rational coefficients

$$f(\alpha) = 0, \quad f \in \mathbb{Q}[x]. \tag{3.2.2}$$

For example, $\sqrt{2}, i$ are algebraic numbers since they are solutions of $x^2 - 2 = 0$ and $x^2 + 1 = 0$, respectively.

If the polynomial $f(\alpha) = 0$ has integer coefficients, i.e., $f \in \mathbb{Z}[x]$ where f is degree m polynomial together with $a_m = 1$, then α is an algebraic integer.

Definition 3.2.2. An algebraic number field is a subfield L of $\mathbb{C}$ such that $[L : \mathbb{Q}] < \infty$, where $[L : \mathbb{Q}]$ is the dimension of L when viewed as a vector space over $\mathbb{Q}$. The number $[L : \mathbb{Q}]$ is called the degree of L over $\mathbb{Q}$.

Theorem 3.2.1. *If L is a number field then $L = \mathbb{Q}(\alpha)$ for some algebraic number α.*

Definition 3.2.3. An algebraic number field L is quadratic if $[L : \mathbb{Q}] = 2$. Hence, there is a number θ such $L = \mathbb{Q}(\theta)$, where

$$\theta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \tag{3.2.3}$$

i.e., θ satisfies

$$a\theta^2 + b\theta + c = 0. \tag{3.2.4}$$

Writing $\sqrt{b^2 - 4ac} = r\sqrt{D}$ with $r \in \mathbb{Z}$, then

$$L = \mathbb{Q}(D). \tag{3.2.5}$$

If $D < 0$, then L is an imaginary quadratic number field.

3.3 The endomorphism ring of elliptic curves

Consider an elliptic curve (or a torus) $E_\tau = \mathbb{C}/\Lambda$, where Λ is a lattice $\Lambda = (\mathbb{Z} + \tau\mathbb{Z})$ and τ is the complex structure modulus. The endomorphisms of E_τ are given by holomorphic maps $F : E_\tau \to E_\tau$. The only such holomorphic maps fixing 0 are induced by maps $G : \mathbb{C} \to \mathbb{C}$ of the form $G(z) = \lambda z$ where $\lambda \in \mathbb{C}$ is a constant such that $\lambda\Lambda \subseteq \Lambda$. Any elliptic curve will admit a set of trivial endomorphisms corresponding to multiplication by integers $\lambda \in \mathbb{Z}$, since multiplication by integers just moves one from one lattice point to another. However, some special elliptic curves could have non-trivial endomorphisms for particular values of τ. We want to find the conditions on τ which give non-trivial endomorphisms.

Theorem 3.3.1. *Let E_τ be an elliptic curve over $\mathbb{C}$ corresponding to the lattice Λ. Then*

$$\mathrm{End}(E_\tau) \cong \{\lambda \in \mathbb{C} : \lambda\Lambda \subset \Lambda\} \tag{3.3.1}$$

For $\lambda \in \mathrm{End}(E_\tau)$, then there exists $j, k, m, n \in \mathbb{Z}$ such that the action of λ on the lattice Λ is represented on the generators as

$$\begin{aligned} \lambda\omega_1 &= j\omega_1 + k\omega_2 \\ \lambda\omega_2 &= m\omega_1 + n\omega_2 \end{aligned} \tag{3.3.2}$$

here we recovered the lattice vectors ω_1, ω_2 to make the derivation clearer but we can as well have worked with the basis $1, \tau$.

Writing (3.3.2) as

$$\begin{pmatrix} \lambda - j & -k \\ -m & \lambda - n \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.3.3}$$

The existence of non-trivial solutions of the above equation implies that

$$\lambda^2 - (j+n)\lambda + (jn - km) = 0, \tag{3.3.4}$$

that is, λ is an algebraic integer.

Now, we assume that λ is real, $\lambda \in \mathbb{R}$. Using the fact that ω_1 and ω_2 are linearly independent over $\mathbb{R}$. Then the first equation in (3.3.2)

$$(\lambda - j)\omega_1 - k\omega_2 = 0 \tag{3.3.5}$$

implies that $\lambda = j$, that is $\lambda \in \mathbb{Z}$ which gives the set of trivial endomorphisms of any elliptic curve.

If, on the other hand, λ is complex. Then $\lambda \in \mathbb{C}$ is an algebraic integer given by

$$\lambda = \frac{(j+n) \pm \sqrt{(j-n)^2 + 4km}}{2} \tag{3.3.6}$$

Using this value of λ in (3.3.5) and after dividing by ω_1 we get

$$\tau = \frac{(n-j) + \sqrt{(j-n)^2 + 4km}}{2k} \tag{3.3.7}$$

Define $b = j - n$, $c = -m$, and $a = k$ and diving by the greatest common factor of $b = j - n$, $c = m$, and $a = k$ if necessary then we can write

$$\tau = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{-b + \sqrt{D}}{2a}, \tag{3.3.8}$$

where $D = b^2 - 4ac < 0$. Hence

$$\tau \in \mathbb{Q}(D). \tag{3.3.9}$$

From the above discussion we conclude that for generic τ, the endomorphism ring $\mathrm{End}(E_\tau)$ is given by multiplication by integers. If, however, τ belongs to an imaginary quadratic number field $\tau \in \mathbb{Q}(D)$ then the elliptic curve has a bigger endomorphism ring given by multiplication by numbers of the form (3.3.6). Elliptic curves (or tori) for these special values of τ are said to have *complex multiplication* (or to be of CM type).

We end this section with what is called *the first main theorem of complex multiplication*. This theorem says that for $\tau \in \mathbb{Q}(D)$, then $j(\tau)$ is an algebraic integer, i.e.,

$$j^h + a_1 j^{h-1} + \cdots + a_h = 0, \quad a_i \in \mathbb{Z}, \tag{3.3.10}$$

where $h = h(D)$ is the class number of the field $\mathbb{Q}(D)$[1]. We see that although $j(\tau)$ is a non-trivial function of τ, for elliptic curves with complex multiplication $j(\tau)$ has a very simple property.

3.4 Complex multiplication for higher-dimensional tori and Calabi-Yau manifolds

The non-trivial endomorphisms of higher-dimensional tori can be found by a direct generalization of (3.3.2). Let Z stand for the column vector made of the complex coordinates $z_i, i = 1, \cdots, n$. The torus T^n admits complex multiplication if it has non-trivial endomorphisms [32]

$$Z \longrightarrow AZ. \tag{3.4.1}$$

The condition on A can be derived in similar way as we did for the case of the 2-torus

$$\begin{aligned} A &= M + N\mathcal{T} \\ \mathcal{T}A &= M' + N'\mathcal{T}, \end{aligned} \tag{3.4.2}$$

where M, N, M', N' are integer matrices. This gives the following second order equation for the period matrix

$$\mathcal{T}N\mathcal{T} + \mathcal{T}M - N'\mathcal{T} - M' = 0. \tag{3.4.3}$$

For Calabi-Yau manifolds, we first think of the period matrix in (3.1.44) as representing some torus. The Calabi-Yau is said to have complex multiplication if the

[1]See the end of this chapter for the definition $h = h(D)$

torus represented by (3.1.44) admits complex multiplication. This simply means that $\mathcal{T}$ in (3.1.44) satisfies (3.4.3).

An example of a Calabi-Yau manifold which enjoys complex multiplication is the Fermat quintic

$$P(z) = \sum_{a=1}^{5} (z_a)^5 = 0. \tag{3.4.4}$$

The period matrix of the Fermat quintic is given by [75, 76]

$$\mathcal{T} = \begin{pmatrix} \alpha - 1 & \alpha + \alpha^3 \\ \alpha + \alpha^3 & -\alpha^4, \end{pmatrix} \tag{3.4.5}$$

where $\alpha^5 = 1$.

The matrix $\mathcal{T}$ satisfies (3.4.3) with

$$N = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad N' = \begin{pmatrix} -1 & 0 \\ -1 & 0 \end{pmatrix}, \qquad M' = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}. \tag{3.4.6}$$

The endomorphism matrix A is given by

$$A = \begin{pmatrix} \alpha - 1 & \alpha + \alpha^3 \\ 1 + \alpha + \alpha^3 & -\alpha^4 \end{pmatrix} \tag{3.4.7}$$

We note that α is an algebraic integer since it is a solution of a polynomial equation with integer coefficients

$$\alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1 = 0. \tag{3.4.8}$$

As such, the elements of $\mathcal{T}$ and A belongs to

$$S = \mathbb{Q}(\alpha) \tag{3.4.9}$$

which is the field of rational numbers extended by α.

In [32], Gukov and Vafa conjectured that *a σ-model on a Calabi-Yau manifold M gives rise to an RCFT if and only if M and its mirror W both have complex multiplication over the same number field.*

3.5 Quadratic Forms

An integral quadratic form $Q(a,b,c)$ in two variables x, y is defined as:

$$Q(a,b,c) \,:\, f(x,y) = ax^2 + bxy + cy^2 \qquad a,b,c \in \mathbb{Z}. \tag{3.5.1}$$

The quadratic form Q is called primitive if the greatest common divisor $\gcd(a,b,c) = 1$. The discriminant of the quadratic form Q is $D = b^2 - 4ac$.

We can associate to Q a 2×2 matrix

$$M(Q) = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}. \tag{3.5.2}$$

The discriminant of Q is now given by $D = -\det(M)$.

The matrix representation allows us to define an equivalence relation. Two forms Q and Q' of the same discriminant are equivalent $Q \sim Q'$ if there exists $S \in SL(2,\mathbb{Z})$ such that

$$M(Q') = S^T M(Q) S. \tag{3.5.3}$$

Since $S = \pm I$ acts trivially, then only the group $PSL(2,\mathbb{Z})$ acts effectively. The above action of $SL(2,\mathbb{Z})$ leaves D unchanged. Two quadratic forms are said to be properly equivalent if they are in the same $SL(2,\mathbb{Z})$ orbit. Now we can talk of equivalence classes of quadratic forms under the action of $SL(2,\mathbb{Z})$. We simply take all the matrices which are in the same orbit of Q under $SL(2,\mathbb{Z})$ as one equivalence class. The set of properly $SL(2,\mathbb{Z})$-equivalent classes is defined as [33]

$$Cl(D) = \big\{Q(a,b,c) : \text{primitive quadrartic form} \,\big|\, D = b^2 - 4ac < 0,\ a > 0\big\} / \sim SL(2,\mathbb{Z}). \tag{3.5.4}$$

The set $Cl(D)$ is finite and its cardinality $h(D) = |Cl(D)|$ is the class number of

$\mathbb{Q}(D)$. The set of classes in $Cl(D)$ will be written as

$$Cl(D) = \{\mathcal{C}_1, \cdots, \mathcal{C}_{h(D)}\}. \tag{3.5.5}$$

We can repeat the same construction to the group $GL(2, \mathbb{Z})$ and define the set of improperly $GL(2, \mathbb{Z})$-equivalent classes

$$\widetilde{Cl}(D) = \big\{Q(a, b, c) : \text{primitive quadrartic form} \,\big|\, D = b^2 - 4ac < 0,\ a > 0\big\} / \sim GL(2, \mathbb{Z}). \tag{3.5.6}$$

Now, let us study the quadratic form

$$Q = f(\tau, 1) = a\tau^2 + b\tau + c. \tag{3.5.7}$$

We only consider the values of τ in the upper-half plane

$$\mathcal{H}^+ = \{\tau \in \mathbb{C} \mid \Im(\tau) > 0\}, \tag{3.5.8}$$

that is

$$\tau = \frac{-b + \sqrt{D}}{2a} \tag{3.5.9}$$

We can associate with each quadratic form $Q(a, b, c)$ a complex number τ_Q

$$\tau_{Q(a,b,c)} = \frac{-b + \sqrt{D}}{2a}. \tag{3.5.10}$$

We have the map

$$Q(a, b, c) \mapsto \tau_{Q(a,b,c)} \in \mathbb{H}^+. \tag{3.5.11}$$

The $SL(2, \mathbb{Z})$ action on Q induces a fractional linear transformation on τ_Q. One can parametrize the equivalence classes of quadratic forms by the points in $\mathcal{H}^+/PSL(2, \mathbb{Z})$. The $SL(2, \mathbb{Z})$ action on quadratic forms is compatible with the $PSL(2, \mathbb{Z})$ on $\mathbb{H}^+$. The $PSL(2, \mathbb{Z})$ orbits of $\tau_{Q(a,b,c)} \in \mathbb{H}^+$ depends on the class $\mathcal{C} = [Q(a, b, c)] \in Cl(D)$.

Quadratic forms and their equivalence classes are related to rank-2 lattices. We consider an even integer lattice Γ with basis e_1 and e_2 and an intersection form K. The intersection form K can be written as

$$\begin{pmatrix} 2a & b \\ b & 2a \end{pmatrix} := \begin{pmatrix} \langle e_1|e_1\rangle & \langle e_1|e_2\rangle \\ \langle e_2|e_1\rangle & \langle e_2|e_2\rangle \end{pmatrix}, \tag{3.5.12}$$

where $a, b \in \mathbb{Z}$. Since Γ is positive definite, then $4ac - b^2 = -D > 0$ and $a > 0$. We say that Γ is primitive if $\gcd(a, b, c) = 1$ which is equivalent to a primitive quadratic form.

Chapter 4

Strings on CM tori

We have seen that CFTs based on CM tori are rational. Hence, one should expect that there is an extended chiral algebra underlying those RCFTs. A simple demonstration of this is the Moore-Seiberg algebra for the $c = 1$ boson on a rational circle (2.2.2). One can try to write down a Moore-Seiberg algebra for the rational $c = 2$ theory as for the $c = 1$ case. However this algebra is already complicated for the $c = 1$ case (see (2.2.2)). In this thesis, we propose using a simpler, more compact chiral algebra to describe the RCFTs which describe strings on CM tori. These are the matrix-level affine Kac-Moody algebras studied by Gannon [34].

In this chapter, we establish the correspondence between strings on a CM tori and the $U_{2,K}$ algebra. The algebra $U_{2,K}$ is characterized by the left and right matrix-valued levels K_L and K_R. We will use the Gauss product to construct K_L and K_R from the complex structure and the Kähler structure parameters τ and ρ which characterize strings on tori. For CM tori, τ and ρ are takes values in a quadratic number field $\tau, \rho \in \mathbb{Q}(D)$. This enables one to represent τ and ρ by binary quadratic forms where a Gauss product is defined.

The special values of the moduli indicate RCFTs where the infinite number of fields is organized into a finite set that is primary with respect to an extended chiral algebra [48]. In the T^m case, the generic boson algebra $U(1)^m = U(1)_{k_1} \times \cdots \times U(1)_{k_m}$ is extended by vertex operators defined by vectors of the lattice Λ_m describing $T^m \cong \mathbb{R}^m/\Lambda_m$. The extended algebra is well understood; it was written explicitly for the $m = 1$ case in [48], for example.

On the other hand, a different algebra has also been of interest. The Abelian algebra $U_{m,K}$ with an $m \times m$ matrix-valued level K was studied in [34], where its modular invariant partition functions were classified. This matrix-level algebra generalizes $U(1)^m = U(1)_{k_1} \times \cdots \times U(1)_{k_m}$, which is recovered for a diagonal matrix K. Here we consider the more general case, allowing K to be a non-negative integer-valued matrix.

This relation has already proven useful in the following way. As was noted in [34], the moduli space of the RCFTs based on the $U_{m,K}$ algebra is given in terms of the moduli space of rational points on the Grassmannian $G_{d,d}(\mathbb{R})$. This is similar to the Narain moduli space of compactifications of strings on tori. We show that the characterization of the $U_{m,K}$ partition functions in terms of rational points on a Grassmannian [34] is equivalent to specifying CM tori inside the Narain moduli space. This is another way to show that the set of RCFTs is dense in the Narain moduli space since the set of rational points is dense in the Grassmannian $G_{d,d}(\mathbb{R})$. We hope that the relation between matrix-level RCFTs and strings on tori will also be helpful in other ways. The results in this chapter are published in [77].

4.1 Matrix-level affine algebras

In this section, we study affine Abelian chiral algebras with matrix-valued level in more details. To provide evidence that they are the chiral algebras of a consistent class of RCFTs, we describe their modular invariant partition functions [34]. In Section 4.2 we relate them to σ-models on CM tori.

Let Γ_K be a Euclidean, even, integral lattice of rank r

$$\Gamma_K = \mathbb{Z}\mathbf{e}_1 + \cdots + \mathbb{Z}\mathbf{e}_r, \quad \langle \mathbf{e}_i | \mathbf{e}_j \rangle = K_{ij} \in \mathbb{Z}, \tag{4.1.1}$$

where positive definite integer-valued symmetric intersection matrix K_{ij} has a determinant $-D > 0$. In the following we will refer to a lattice Γ_K and its intersection form K interchangeably with the hope that this will not cause any confusion. The basis $\{\mathbf{e}_i\}$ of Γ_K is defined up to $GL(r,\mathbb{Z})$ transformations which preserve the determinant of K

$$\mathbf{e}_i \to A_i^j \mathbf{e}_j, \quad A \in GL(r,\mathbb{Z}) \ , \qquad \det A = \pm 1 \, . \tag{4.1.2}$$

There are r linearly independent $U(1)$ currents

$$J_i(z) = \sum_{n\in\mathbb{Z}} \frac{J_i^n}{z^{n+1}}. \tag{4.1.3}$$

They have the OPE

$$J_i(z) J_j(w) \sim \frac{K_{ij}}{(z-w)^2}. \tag{4.1.4}$$

This is the Abelian version of (2.8.2). In terms of the modes we have

$$\left[J_i^m, J_j^n \right] = m K_{ij} \delta_{m+n,0}. \tag{4.1.5}$$

The zero modes of the currents can be used to grade the Hilbert space into different superselection sectors with different charges $\mathbf{q} \in \Gamma_K$

$$\mathcal{H} = \oplus_{\mathbf{q}} \mathcal{H}_{\mathbf{q}}, \tag{4.1.6}$$

where

$$\mathcal{H}_{\mathbf{q}} = \left\{ |\mathbf{q}\rangle \ \mid \ J_i^0|\mathbf{q}\rangle = q_i|\mathbf{q}\rangle \right. \tag{4.1.7}$$

The energy momentum tensor is given by the Sugawara construction

$$T(z) = \frac{1}{2}K^{ij} : J_i(z)J_j(z) :, \tag{4.1.8}$$

where $K^{ij} = K_{ij}^{-1}$ is the intersection form of the dual lattice.

From the OPE of $T(z)$ with itself we can read off the central charge

$$c = K^{ij}K_{ij} = r. \tag{4.1.9}$$

The Virasoro generators are given by

$$L_n = \frac{1}{2}K^{ij}\sum_{m=-\infty}^{\infty} : J_i^{m+n}J_j^{-m} : . \tag{4.1.10}$$

The Virasoro ground state in the sector $\mathcal{H}_{\mathbf{q}}$ is defined by

$$L_0|\mathbf{q}\rangle = \frac{1}{2}q^2|\mathbf{q}\rangle, \qquad L_n|\mathbf{q}\rangle = 0, \tag{4.1.11}$$

Now we specialize to rank two lattices Γ_K with basis $\mathbf{e}_1, \mathbf{e}_2$. Since the lattice is even-integer lattice, then its intersection form can be written as

$$K_{ij} = \langle \mathbf{e}_i|\mathbf{e}_j\rangle = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}, \quad a, b, c \in \mathbb{Z}. \tag{4.1.12}$$

We assume that $\gcd(a, b, c) = 1$ which corresponds to primitive quadratic forms.

The $GL(2,\mathbb{Z})$ transformation on the basis of Γ_K gives an equivalent lattice. The set of equivalence classes of primitive, even lattices is defined as

$$\mathcal{L}^p(D) := \left\{\Gamma_K : D = -\det K\right\}/GL(2,\mathbb{Z}). \tag{4.1.13}$$

We will consider different matrix levels K_L for the holomorphic and K_R for the anti-holomorphic sectors which give rise to heterotic theories. The set of standard representations of the affine algebra $U_{m,K}$ are labelled by $a \in P_+^{K_L} = \Gamma^*_{K_L}/\Gamma_{K_L}$ and $b \in P_+^{K_R} = \Gamma^*_{K_R}/\Gamma_{K_R}$ where $|K_L| = |K_R|$ and $|K|$ is the determinant of K [34].

The RCFT data are given by $(\Gamma_{K_L}, \Gamma_{K_R}, \{\chi_a^{\Gamma_{K_L}}\}, \{\chi_a^{\Gamma_{K_R}}\})$ where $\chi_a^{\Gamma_{K_R}}$ are the characters which are proportional to the theta functions of the lattice

$$\chi_a^{\Gamma_{K_L}}(q) = \frac{\theta_a^{\Gamma_{K_L}}(q)}{\eta(q)^2} = \frac{1}{\eta(q)^2} \sum_{v \in \Gamma_{K_L}} q^{\frac{1}{2}(a+v)^2}, \tag{4.1.14}$$

where $\eta(q)$ is the Dedekind eta function.

We can compute the modular S matrix from transformation of the characters $(\chi_a^{\Gamma_{K_L}} = \theta_a^{\Gamma_{K_L}}/\eta^2)$ under $\tau \to -1/\tau$

$$\chi_a^{\Gamma_{K_L}}\left(\frac{-1}{\tau}\right) = \frac{\theta_a^{\Gamma_{K_L}}(-1/\tau)}{\eta(-1/\tau)^2}. \tag{4.1.15}$$

Using the transformation properties of the theta functions we find

$$S_{ab} = \frac{i}{\sqrt{D}} e^{-2i\pi(a \cdot b)}. \tag{4.1.16}$$

Now we can find the fusion rules using the Verlinde formula

$$\mathcal{N}_{ab}^c = \sum_d \frac{S_{ad} S_{bd} S_{cd}^{-1}}{S_{0d}} \tag{4.1.17}$$

Since the gluing map (see the discussion after (4.2.14)) satisfies $(a, b) = (\varphi(a), \varphi(b))$ then we see that

$$\mathcal{N}_{ab}^c = \mathcal{N}_{\varphi(a)\varphi(b)}^{\varphi(c)}, \tag{4.1.18}$$

i.e., φ is a symmetry of the fusion rules and hence can be used to build new modular invariants. The quantum dimension of a primary field labelled by a is

$$\frac{S_{a0}}{S_{00}} = 1, \tag{4.1.19}$$

i.e., all the primaries are simple currents. This simplifies the construction of all modular invariants of the algebra as was shown in [34].

The spectrum is encoded in the (genus-1) partition function

$$Z^{\Gamma_{K_L},\Gamma_{K_R}} = \sum_{a\in P_+^{K_L},\ b\in P_+^{K_R}} M_{a,b}\ \chi_a^{\Gamma_{K_L}} \overline{\chi_b^{\Gamma_{K_R}}}, \tag{4.1.20}$$

where the matrix $M_{a,b}$ is constrained to satisfy $M_{a,b} \in \mathbb{Z}_{\geq}$ and $M_{0,0} = 1$.

Modular invariance dictates that

$$SM = MS, \qquad TM = MT, \tag{4.1.21}$$

where the matrices S and T are unitary and symmetric and T is diagonal

$$S_{ab} = \frac{1}{\sqrt{|K_L|}} \exp[-2\pi i(a \cdot b)]. \tag{4.1.22}$$

In [34], all modular invariants of the algebra $U_{m,K}$ were constructed in terms of even self-dual lattices Γ that contain Γ_{K_L} and Γ_{K_R}. Here we are only concerned with physical modular invariants, i.e., those which satisfy (4.1.21)[1]. In terms of the matrix K, full modular invariance means $K_{ij} \in 2\mathbb{Z}$. This, together with the symmetry of K, translates to an even integer lattice Γ_K.

The argument in [34] goes as follows: consider the heterotic partition function (4.1.20). Using $SM = MS$ and the unitarity of the S matrix we find

$$\begin{aligned} M_{a,b} &= \sum_{c,d} S_{a,c} M_{c,d} S^*_{d,b} \\ &= \frac{1}{|K_L|} \sum_{c,d} \exp[2\pi i(b \cdot d - a \cdot c)] M_{c,d}. \end{aligned} \tag{4.1.23}$$

[1] In [34], the second constraint in (4.1.21) is relaxed which leads to a much bigger set of modular invaraints which include *weak modular invariants*. But since we are looking for a geometric interpretation in terms of a torus target we will require full modular invariance. The σ-model with a torus target is modular invariant by construction.

From this equation we derive the relation

$$\frac{M_{a,b}}{M_{0,0}} = \frac{\sum_{c,d} z_{ab,cd} M_{c,d}}{\sum_{c,d} M_{c,d}}, \tag{4.1.24}$$

where we defined

$$z_{ab,cd} = \exp[2\pi i(b\cdot d - a\cdot c)], \quad |z_{ab,cd}| = 1. \tag{4.1.25}$$

Using the triangle inequality, then 4.1.24 implies that $|M_{a,b}| \leq |M_{0,0}| = 1$ with the equality iff

$$M_{c,d} \neq 0 \Longrightarrow b\cdot d = a\cdot c \pmod 1 \tag{4.1.26}$$

for all $c \in P_+^{K_L}, d \in P_+^{K_R}$. The above equation means that any two labels which appear in the character decomposition of the partition function must be related by an isometry of the discriminant groups $b = \varphi(a)$ and $c = \varphi(d)$ and $M_{a,b} = \delta_{b,\varphi(a)}$. The partition function now can be written as

$$Z^{\Gamma_{K_L},\Gamma_{K_R}} = \sum_{a\in P_+^{K_L}} \chi_a^{\Gamma_{K_L}} \overline{\chi_{\varphi(a)}^{\Gamma_{K_R}}}, \tag{4.1.27}$$

which is an example of an automorphism modular invariant. We only consider automorphism modular invariant partition functions of the algebra $U_{m,K}$. If $U_{m,K}$ is the maximally extended chiral algebra of strings on CM-tori then the automorphism modular invariants will exhaust the set of all modular invariants. However, if $U_{m,K}$ is not the maximally extended chiral algebra then there will be other non-automorphism modular invariants.

Define the set

$$\Omega = \bigcup_{a\in P_+^{K_L},\, b\in P_+^{K_R}} (a\oplus ib) + \left(\Gamma_{K_L} \oplus i\Gamma_{K_R}\right) \tag{4.1.28}$$

which is an even self dual lattice. The matrix $M_{a,b}$ in (4.1.20) satisfies

$$M_{a,b} = \begin{cases} 1 & \text{if } (a, ib) \in \Omega \\ 0 & \text{otherwise.} \end{cases} \tag{4.1.29}$$

Hence, the modular invariant partition functions of the $U_{m,K}$ current algebra are in 1-to-1 correspondence with the even self-dual lattices Γ which contain $(\Gamma_{K_L}; \Gamma_{K_R})$.

4.2 Strings on a CM torus

We consider the compactification of string theory on an elliptic curve E_τ which has complex multiplication (or a CM torus). The σ-model on E_τ is specified by the parameters τ and ρ. The compactification of strings on E_τ is characterized by a momentum-winding Narain lattice, an even self-dual lattice $\Gamma(\tau, \rho)$ of rank 4, where the parameters τ and ρ live in the upper-half plane $\mathbb{H}^+$ subject to a group of discrete symmetries Ξ. The Narain moduli space of this compactification is [32]

$$\mathcal{M} = \frac{\mathbb{H}^+ \times \mathbb{H}^+}{\Xi}, \tag{4.2.1}$$

where Ξ is the group of discrete symmetries of E_τ

$$\Xi = PSL(2,\mathbb{Z})_\tau \times PSL(2,\mathbb{Z})_\rho \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2. \tag{4.2.2}$$

The first $\mathbb{Z}_2$ is a mirror symmetry which exchanges τ and ρ, i.e., $\mathbb{Z}_2 : (\tau, \rho) \mapsto (\rho, \tau)$. The second $\mathbb{Z}_2$ is a space-time parity transformation $\mathbb{Z}_2 : (\tau, \rho) \mapsto (-\bar{\tau}, -\bar{\rho})$, where the bar denotes complex conjugation. The last $\mathbb{Z}_2$ is a world-sheet orientation reversal $\mathbb{Z}_2 : (\tau, \rho) \mapsto (\tau, -\bar{\rho})$[2].

[2] The three $\mathbb{Z}_2$ symmetries are not really independent since the second $\mathbb{Z}_2$ is equivalent to applying the third $\mathbb{Z}_2$ then the first $\mathbb{Z}_2$ then the third $\mathbb{Z}_2$ then the first $\mathbb{Z}_2$ again.

For generic values of τ and ρ, the conformal field theory is not rational and has an infinite number of primary fields. For special values of τ and ρ the infinite number of primary-field representations reorganize into a finite set of representations of a bigger chiral algebra and the theory becomes rational. It was shown in [32] that CFTs based on E_τ are rational iff E_τ and its mirror are both of CM type, that is, $\tau, \rho \in \mathbb{Q}(D)$ which implies that ρ, like τ, satisfies a quadratic equation with integer coefficients

$$a'\rho^2 + b'\rho + c' = 0, \qquad \rho = \frac{-b' + \sqrt{D}}{2a'}, \quad D' = b'^2 - 4a'c' < 0. \tag{4.2.3}$$

An example of an RCFT which enjoys this property results when $\tau = \rho = e^{2\pi i/3}$. In this case the chiral vertex operator algebra is isomorphic to that of $SU(3)$ WZW model at level 1 [78]. Clearly, both τ and ρ satisfy

$$\tau^2 + \tau + 1 = 0. \tag{4.2.4}$$

In [32], the diagonal case was studied in detail where it was shown that the condition for a diagonal modular invariant is

$$\tau = fa\rho, \tag{4.2.5}$$

where $f \in \mathbb{Z}$, $\tau, \rho \in \mathbb{Q}(D)$, and a is the coefficient of τ^2 in

$$a\tau^2 + b\tau + c = 0. \tag{4.2.6}$$

The generalization to non-diagonal modular invariants was given in [33].

We can associate a quadratic form with the complex numbers τ and ρ. Write

$$Q(a,b,c) = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}, \qquad \tau_{Q(a,b,c)} = \frac{-b + \sqrt{D}}{2a} \tag{4.2.7}$$

where $D = b^2 - 4ac = -\det Q$ denotes the discriminant of the quadratic form Q. Q is called primitive if $\gcd(a, b, c) = 1$. The discriminant of the quadratic form Q is invariant under $SL(2, \mathbb{Z})$:

$$Q \longrightarrow S^t Q S, \quad S \in SL(2, \mathbb{Z}). \tag{4.2.8}$$

Now we can consider equivalence classes of quadratic forms under the action of $SL(2, \mathbb{Z})$ (or more precisely $PSL(2, \mathbb{Z})$, since $S = \pm\mathbb{I}$ acts trivially). The set of equivalence classes is denoted by

$$Cl(D) = \left\{ Q(a, b, c) \,\middle|\, D = b^2 - 4ac < 0,\ a > 0 \right\} / \sim SL(2, \mathbb{Z}). \tag{4.2.9}$$

It is known that $Cl(D)$ is a finite set and we will denote the number of its elements by $h(D)$ (see [33] and references therein)

$$Cl(D) = \left\{ \mathcal{C}_1, \ldots, \mathcal{C}_{h(D)} \right\}. \tag{4.2.10}$$

Since $D < 0$, the complex number τ_Q lies in the upper-half plane $\mathbb{H}^+$. The $SL(2, \mathbb{Z})$ action on Q will induce a fractional linear transformation on τ_Q. The $PSL(2, \mathbb{Z})$ action on quadratic forms is compatible with the $PSL(2, \mathbb{Z})$ on $\mathbb{H}^+$. The $PSL(2, \mathbb{Z})$ orbits of $\tau_{Q(a,b,c)} \in \mathbb{H}^+$ depend on the class $\mathcal{C} = [Q(a, b, c)] \in Cl(D)$. Using the above mapping, then we can label the classes in $Cl(D)$ by points $[\tau_Q] \in \mathcal{F} = \mathbb{H}^+/PSL(2, \mathbb{Z})$. Since τ is the complex structure of a torus then fractional linear transformations on τ gives an equivalent torus. The classes $[\tau_Q]$ will give inequivalent tori in the Narain moduli space.

The same goes for ρ where we can also define equivalence classes of quadratic forms parametrized by points $[\rho_{Q'}] \in \mathcal{F} = \mathbb{H}^+/PSL(2, \mathbb{Z})$. The equivalence class of Narain lattices corresponding to an RCFT will be denoted by $\Gamma(\tau_{\mathcal{C}}, \rho_{\mathcal{C}'})$, where $\mathcal{C}$ and $\mathcal{C}'$ are the equivalence classes corresponding to τ and ρ.

Similarly, the group $GL(2,\mathbb{Z})$ acts on quadratic forms by the same formula as (4.2.8). The set of *improper* equivalence classes under $GL(2,\mathbb{Z})$ is defined by [33]

$$\tilde{Cl}(D) = \left\{Q(a,b,c)\,\middle|\, D = b^2 - 4ac < 0,\ a > 0\right\}/ \sim GL(2,\mathbb{Z}). \tag{4.2.11}$$

and we have a surjection q

$$q : Cl(D) \to \tilde{Cl}(D), \quad \mathcal{C} \to \tilde{\mathcal{C}}, \tag{4.2.12}$$

where $q^{-1}(\tilde{\mathcal{C}})$ has either one or two classes.

One can associate with the lattice Γ_K a primitive quadratic form given by the intersection form K. Therefore, the equivalence classes of primitive lattices $[\Gamma_K]$ are in 1-to-1 correspondence with the equivalence classes of primitive, quadratic forms $[Q(a,b,c)]$. Hence, we can identify the set $\tilde{Cl}(D)$ with the set $\mathcal{L}^p(D)$ in (4.1.13).

We define the following projections of $\Gamma(\tau_{\mathcal{C}}, \rho_{\mathcal{C}'})$

$$\Pi_L := \Gamma(\tau_{\mathcal{C}}, \rho_{\mathcal{C}'}) \cap \mathbb{R}^{2,0}, \qquad \Pi_R := \Gamma(\tau_{\mathcal{C}}, \rho_{\mathcal{C}'}) \cap \mathbb{R}^{0,2}, \tag{4.2.13}$$

which correspond to the equivalence classes of the left and right momentum lattices, characterized by the vanishing of the right moving and left moving momenta, respectively. [3]

The modular invariant partition functions studied in [33] take the form:

$$\begin{aligned} Z^{\Pi_L,\Pi_R,\varphi}(q,\bar{q}) &= \frac{1}{|\eta(q)|^4} \sum_{a\in\Pi_L^*/\Pi_L} \theta_a^{\Pi_L}(q), \overline{\theta_{\varphi(a)}^{\Pi_R}(q)} \\ &= \sum_{a\in\Pi_L^*/\Pi_L} \chi_a^{\Pi_L}(q), \overline{\chi_{\varphi(a)}^{\Pi_R}(q)} \end{aligned} \tag{4.2.14}$$

where φ is a gluing map between the discriminant groups Π_L^*/Π_L and Π_R^*/Π_R. It satisfies $(\varphi(a), \varphi(b)) = (a, b)$, where $a, b \in \Pi_L^*/\Pi_L$ and $(\cdot,\cdot)$ is the rational bilinear form on Π_L^*/Π_L which is induced from the bilinear form on Π_L.

[3] More precisely it corresponds to an equivalence class of lattices.

The characters which enter the partition function above are given in terms of the theta function of the lattice Π_L:

$$\chi_a^{\Pi_L}(q) = \frac{\theta_a^{\Pi_L}(q)}{\eta(q)^2} = \frac{1}{\eta(q)^2}\sum_{v\in\Pi_L} q^{\frac{1}{2}(a+v)^2} \tag{4.2.15}$$

They are identical to the characters in (4.1.14).

4.2.1 The Gauss product

There is a binary operation which turns the set $Cl(D)$ of (4.2.9) into an Abelian group: the Gauss product (see [33], e.g.) takes two equivalence classes of quadratic forms of the same discriminant and produces a third with that discriminant.

Let $\mathcal{C} = [Q_1(a_1, b_1, c_1)]$ and $\mathcal{C}' = [Q_2(a_2, b_2, c_2)]$ be two such equivalence classes. We will restrict ourselves to primitive forms. We say that two quadratic forms $Q_1(a_1, b_1, c_1) \in \mathcal{C}$ and $Q_2(a_2, b_2, c_2) \in \mathcal{C}'$ are concordant if $a_1 a_2 \neq 0$, $\gcd(a_1, a_2) = 1$ and $b_1 = b_2$. Then the Gauss product of $\mathcal{C} \star \mathcal{C}'$ is defined as

$$\big[Q_1\big(a_1, b, c_1\big)\big] \star \big[Q_2\big(a_2, b, c_2\big)\big] = \big[Q_3\big(a_3, b_3, c_3\big)\big], \tag{4.2.16}$$

where $a_3 = a_1 a_2$, $b_3 = b$, and $c_3 = \frac{b^2 - D}{4a_1 a_2}$. It is important to mention that any pair of quadratic forms of the same discriminant can be $SL(2,\mathbb{Z})$-transformed to a concordant pair.

The unit $\mathbf{1}_D$ of $Cl(D)$ with respect to the product $\star$ is represented by

$$\mathbf{1}_D = \begin{cases} [1, 0, -\frac{D}{4}]; & \text{if } D \equiv 0 \mod 4 \\ \\ [1, 0, \frac{1-D}{4}]; & \text{if } D \equiv 1 \mod 4. \end{cases} \tag{4.2.17}$$

The quadratic form $Q_3(a_3, b_3, c_3)$ corresponds to a lattice with intersection form

$$Q_3\big(a_3, b_3, c_3\big) \equiv \begin{pmatrix} 2a_3 & b_3 \\ b_3 & 2c_3 \end{pmatrix}. \tag{4.2.18}$$

which gives an even integer lattice, an important fact which we will use when we construct the matrix levels K_L and K_R for the $U_{m,K}$ algebras.

The Gauss product is used to construct the intersection form of the lattices Π_L and Π_R in terms of the equivalence classes $\mathcal{C}$ and $\mathcal{C}'$, corresponding to $\tau_{Q(a_1,b_1,c_1)}$ and $\rho_{Q(a_2,b_2,c_2)}$, respectively [33]:

$$\Pi_L = q(\mathcal{C} \star \mathcal{C}'^{-1}), \qquad \Pi_R = q(\mathcal{C} \star \mathcal{C}')(-1). \tag{4.2.19}$$

Here q is the natural map $Cl(D) \rightarrow \tilde{C}l(D)$ and $q(\mathcal{C} \star \mathcal{C}')(-1)$ means we multiply the quadratic form $q(\mathcal{C} \star \mathcal{C}')$ by -1.

As was shown in [33], to prove the above result one first constructs the $\mathbb{Z}$-basis for Π_R of $\Gamma(\tau_\mathcal{C}, \rho_{\mathcal{C}'})$ in terms of the equivalence classes of $\mathcal{C}$ and $\mathcal{C}'$ and then compares the resulting quadratic form of Π_R with $Q_3(a_3, b_3, c_3)$ and similarly for Π_L.

In the special case when $\tau \propto \rho$, we have $\Pi_L = \Pi_R = \Pi$ and we get a diagonal modular invariant for $\varphi = \mathbb{I}$

$$Z^{\Pi,\Pi,\mathbb{I}}(q,\bar{q}) = \frac{1}{|\eta(q)|^4} \sum_{a \in \Pi^*/\Pi} \theta_a^\Pi(q) \overline{\theta_a^\Pi(q)}. \tag{4.2.20}$$

Now we have geometric data represented by the rational Narain lattice $\Gamma(\tau_\mathcal{C}, \rho_{\mathcal{C}'})$ which depends on the σ-model parameters τ and ρ (both τ and ρ are attached to an equivalence class of quadratic forms) and algebraic, RCFT data $(\Gamma_{K_L}, \Gamma_{K_R}, \{\chi_a\})$. The Gauss product can be used to relate them in the following way. First we will look at K_L and K_R as quadratic forms and hence we can talk about their respective equivalence classes under the $SL(2,\mathbb{Z})$ action, as we did with Q. Now, a rational point in the Narain moduli space specified by the special values $\tau_\mathcal{C}$ and $\rho_{\mathcal{C}'}$ defines two equivalence classes of quadratic forms $\mathcal{C}$ and $\mathcal{C}'$. The mapping between the two sets of data is

$$K_L = q\big(\mathcal{C} \star C'^{-1}\big), \qquad K_R = q\big(\mathcal{C} \star \mathcal{C}'\big)(-1). \tag{4.2.21}$$

We will need to apply a symmetrisation map (half the sum of the matrix and its transpose) to K_L and K_R if they are not symmetric or $SL(2,\mathbb{Z})$-equivalent to their symmetric forms.

The mapping (4.2.21) shows that the algebras $U_{m,K}$ can be given a geometric significance, by relating them to σ-models on CM tori. To justify this, we notice that the characters of the RCFTs in (4.2.15) which are proportional to the theta functions of the momentum-winding lattice are the same as the characters (4.1.14) of the algebra $U_{2,K}$. Both sets of characters are based on lattices which are constructed from the same set of geometric data using the Gauss product. Also, the modular invariant partition functions (4.2.14) are a subset of the modular invariant partition functions (4.1.20) of the algebra $U_{2,K}$ for which

$$M_{a,b} = \delta_{b,\varphi(a)}. \tag{4.2.22}$$

We note that the above mapping is not 1-to-1. We can't start from K_L and K_R and construct unique equivalence classes for τ and ρ. The matrices K_L and K_R are representatives of equivalence classes in the set $Cl(D)$ which have a finite number of elements. The algebraic description of the $U_{m,K}$ algebras depends only on K_L and K_R and as such is the same for all members of the set $Cl(D)$. On the other hand there is a geometric description for each member of the set $Cl(D)$. We conclude that the same $U_{m,K}$ algebra have many geometric avatars. This is similar to the case of a rational boson on a circle of radius square $R^2 = p/q$ which is described by a level $k = pq$ $U(1)_k$ algebra. On the other hand, starting from the algebra $U(1)_k$, there are many candidate rational boson theories, one for each factorization of k into two coprime integers $k = pq$.

4.3 Wen topological order

As a byproduct of the geometric connection of the $U_{m,K}$ algebras we can relate the Wen topological order [70, 71, 79] to the number of $D0$ branes in the models discussed in this chapter. It was shown in [32] that the number of $D0$ branes on T^2 is equal to the dimension of the chiral ring which is given by Γ^*_L/Γ_L.

Topological order is a new kind of order which arise in the topological phases of many systems, e.g., fractional quantum Hall systems. It is not related to any symmetry breaking, like the usual ordered-disordered transitions described by the Landau theory. There is no local order parameter which takes a non-zero vacuum expectation value in the broken phase. A new set of quantum numbers needs to be defined which characterizes topological order.

Wen topological order is the degeneracy of the quantum Hall ground state on the compact surface Σ_g of genus g and is one of the quantum numbers which characterizes the topological order. The low energy effective field theory of the quantum Hall ground state is the Chern-Simons theory on $M = \Sigma \times \mathbb{R}$. The degeneracy of the ground state is counted by the dimension of the Hilbert space of topologically non-trivial gauge fields on M [48, 80, 81]. This dimension can be readily computed from the RCFT fusion rules and the S modular transformation. The general formula for the dimension of the space of conformal blocks on a genus g surface with n punctures was found in [82, 83]

$$\dim\mathcal{H}(\Sigma_g;(P_1,i_1),\cdots,(P_n,i_n)) = \sum_{a\in\Pi_L^*/\Pi_L}\left(\frac{1}{S_0^a}\right)^{2(g-1)}\frac{S_{i_1}^k}{S_0^a}\cdots\frac{S_{i_n}^a}{S_0^a}. \qquad (4.3.1)$$

For the torus $g=1$ with no punctures and using the fact that all the primaries are simple currents we get

$$\dim\mathcal{H}(\Sigma_1) = \sum_{a\in\Pi_L^*/\Pi_L} 1 = D. \qquad (4.3.2)$$

which is simply the dimension of the chiral ring identified with the number of $D0$ branes in [32].

4.4 Rational points on Grassmannians and CM tori

In this section we study the rationality conditions of a Narain lattice in more detail. We formulate the rationality in terms of rational points on a Grassmannian and we show that these points are equivalent to tori of CM type. Our argument will be based on the results in [30].

We consider a generic Narain lattice $\Gamma(\tau,\rho)=(P_L;P_R)$ of the σ-model on T^d/Λ with a B-field

$$\Gamma(\tau,\rho) = \left\{\big(P_L;P_R\big) := \frac{1}{\sqrt{2}}\big(\mu-\tilde{B}\lambda+\lambda;\mu-\tilde{B}\lambda-\lambda\big) \mid (\mu,\lambda)\in\Lambda^*\oplus\Lambda\right\}, \qquad (4.4.1)$$

where Λ^* and Λ are the momentum and winding lattices, respectively and $B=\Lambda^T\tilde{B}\Lambda$.

The holomorphic and anti-holomorphic vertex operators are characterized by $P_R=0$ and $P_L=0$ and they are parametrized by the values of their charges in $\Pi_L=(P_L;0)$

and $\Pi_R = (0; P_R)$

$$\begin{aligned} \Pi_L &= \Big\{ \big(P_L; 0\big) := \frac{1}{\sqrt{2}}\big(\mu - \tilde{B}\lambda + \lambda; 0\big) | (\mu, \lambda) \in \Lambda^* \oplus \Lambda \Big\} \\ \Pi_R &= \Big\{ \big(0; P_R\big) := \frac{1}{\sqrt{2}}\big(0; \mu - \tilde{B}\lambda - \lambda\big) | (\mu, \lambda) \in \Lambda^* \oplus \Lambda \Big\}. \end{aligned} \tag{4.4.2}$$

We also define the following projections of the lattice $\Gamma(\tau, \rho)$:

$$\begin{aligned} \widetilde{\Pi}_L &= (P_L; *) := \frac{1}{\sqrt{2}}\big(\mu - \tilde{B}\lambda + \lambda; *\big) | (\mu, \lambda) \in \Lambda^* \oplus \Lambda \Big\} \\ \widetilde{\Pi}_R &= (*; P_R) := \frac{1}{\sqrt{2}}\big(*; \mu - \tilde{B}\lambda - \lambda\big) | (\mu, \lambda) \in \end{aligned} \tag{4.4.3}$$

where the $*$ means we forget about the corresponding component of P.

Note that

$$\Pi_L \subseteq \widetilde{\Pi}_L, \qquad \Pi_R \subseteq \widetilde{\Pi}_R. \tag{4.4.4}$$

Since the lattice $\Gamma(\tau, \rho)$ is even, self-dual and integral then its straightforward to show that

$$\Pi_L^* \cong \widetilde{\Pi}_L, \qquad \Pi_R^* \cong \widetilde{\Pi}_R, \tag{4.4.5}$$

i.e., $\widetilde{\Pi}_L$ is the dual of Π_L and the same for $\widetilde{\Pi}_R$.

Rationality can be expressed in terms of the rank of Π_L and Π_R . The Narain lattice $\Gamma(\tau, \rho)$ is rational if and only if [30]

$$\text{rank}\big(\Pi_L\big) = \text{rank}\big(\Pi_R\big) = d. \tag{4.4.6}$$

RCFTs are characterized by the appearance of extra holomorphic vertex operators which extend the chiral algebra. For a generic Narain lattice $\Gamma(\tau, \rho)$, the only holomorphic vertex operator is the one corresponding to the vacuum $P_L = P_R = 0$. Since any field is mutually local [4] to the vacuum, then the set of allowed representations

[4]Two fields are mutually local if their OPE doesn't contain branch cuts. In other words the $\circ$ product of their charges is integer.

$\mathcal{V} = \{V[P], P \in \Gamma(\tau, \rho)\}$ is infinite. However extra holomorphic vertex operators $\mathcal{W}[P]$ appear for $p \in \Pi_L$. The requirement of locality with respect to $\mathcal{W}[P]$ restricts the set of allowed irreducible representations to be $a \in \Pi_L^*/\Pi_L$, where Π_L^* is the lattice dual to Π_L so it contains charges which have integer product with Π_L. The condition for rationality translates to the requirement that Π_L be a finite index sublattice of Π_L^* so that the set Π_L^*/Π_L has a finite cardinality. This happens when Π_L have a finite rank which is the condition in (4.4.6).

The Narain moduli space of conformal field theories is isomorphic to the moduli space of even, self-dual lattices with signature (d, d)

$$\mathcal{M}_d = O(\Gamma^{d,d})\backslash O(d,d)/(O(d) \times O(d)), \tag{4.4.7}$$

where $\Gamma^{d,d}$ denotes the standard even self-dual lattice of signature (d, d) and $O(\Gamma^{d,d})$ its automorphism group. In the special case of $d = 2$ this gives the moduli space in (4.2.1). The above moduli space is also the Grassmannian of space-like d-planes in $\mathbb{R}^{d,d}$. The modular invariant partition functions in [34] are classified using rational points on the Grassmannian (4.4.7).

Note that

$$P_L \in W = \Pi_L \otimes \mathbb{R}, \qquad P_R \in W^\perp = \Pi_R \otimes \mathbb{R}, \tag{4.4.8}$$

where W is a space-like d-plane which correspond to a point on the Grassmannian (4.4.7) and $W^\perp$ is its orthogonal complement in $\mathbb{R}^{d,d}$.

The sets Π_L and Π_R in general are not lattices, since the set of vectors which span Π_L and Π_R will not remain linearly independent over $\mathbb{Z}$ when restricted to W and $W^\perp$. However, if W is a *rational* point on the Grassmannian (4.4.7) then Π_L and Π_R become lattices of rank d. A rational point on the Grassmannian (4.4.7) is a subspace W with basis $\{f_m\}$ which can be written over $\mathbb{Q}$ in terms of the preferred orthonormal basis e_i of $\mathbb{R}^{d,d}$

$$f_m = Q_{mi} e_i, \quad Q_{mi} \in \mathbb{Q}. \tag{4.4.9}$$

The above equation implies that the basis of W are rational vectors

$$\langle f_m | f_n \rangle \in \mathbb{Q}. \tag{4.4.10}$$

Since any group generated by rational vectors is a lattice, i.e., it can be generated by linearly independent vectors over $\mathbb{Z}$. Then Π_L which is the $\mathbb{Z}$-span of f_m is a lattice of rank $d = \dim(W)$ and the same for Π_R. It was shown in [30] that for $d = 2$

$$\mathrm{rank}\big(\Pi_L\big) = \mathrm{rank}\big(\Pi_R\big) = 2 \longleftrightarrow \tau, \rho \in \mathbb{Q}(D). \tag{4.4.11}$$

which in our case implies that rational points on the Grassmannian (4.4.7) correspond to CM tori. This is another way to see that RCFTs constitute a dense subset of the set of CFTs, since the set of rational points on a Grassmannian is dense. This is consistent with the finding in [32] that the values $\tau, \rho \in \mathbb{Q}(D)$ corresponding to RCFTs are dense in in the Narain moduli space.

Chapter 5

Conclusion

In summary, we studied the RCFTs based on the matrix-level algebra $U_{2,K}$ and we related them to strings on CM tori (corresponding to $\tau, \rho \in \mathbb{Q}(D)$) inside the Narain moduli space. The characters and modular-invariant partition functions were shown to be identical in the 2 types of theories.[1] Furthermore, the map between them was constructed explicitly: the Gauss product was used to write the matrix levels K_L and K_R in terms of the geometric data represented by τ and ρ.

The connection was shown to be useful in one way. By formulating the problem in terms of rational points on a Grassmannian we showed that the set of RCFTs is a dense subset in the Narain moduli space. This agrees with the observation in [32] that the values of $\tau, \rho \in \mathbb{Q}(D)$ which produce RCFTs are dense in the Narain moduli space. We anticipate that the relation between matrix-level algebras and RCFTs for strings on tori will prove useful in other ways. Using the relation between $U_{2,K}$ and strings on CM tori, Wen topological order for fractional quantum hall states was related to the number of $D0$ branes which are allowed on the CM torus.

The Gukov-Vafa criterion characterizes the rationality of the CFT in terms of the

[1]For characters, compare (4.2.15) to (4.1.14); for partition functions, (4.2.14) to (4.1.20,4.1.29).

geometry of the target space. In the Gannon classification, rationality is derived from the algebraic properties of the chiral algebra without reference to any target space interpretation. We showed that the geometric and the algebraic characterization of rationality are related and that the density property of RCFTs can be seen in both pictures.

We also proposed a generalization of $U_{m,K}$ to the non-Abelian case. This would be related to the Chern-Simons theories on a three-manifold N_3, based on a gauge group G with the level being an integer valued matrix

$$S = K_{AB} \int_{M^4} F^A \wedge F^B, \tag{5.0.1}$$

where M_4 is the four manifold for which N_3 is a boundary. The independence of the partition function from the extension from N_3 to M_4 forces K_{AB} to be an integer-valued matrix.

It would be interesting to see if the results described here might be generalized to higher dimensions $m > 2$, or to non-Abelian theories. The $m = 2$ case is special because only in this case one has a Gauss product which turns the set of equivalence classes of quadratic forms (or rank-2 lattices) into an Abelian group. For higher dimension tori which can be written as a product of elliptic curves

$$T^m = E_{\tau_1} \times \cdots \times E_{\tau_{m/2}}, \tag{5.0.2}$$

one can use the Gauss product for each individual factor to construct the algebra $U_{m,K}$. But a generic T^m will not factorize into a product of elliptic curves and the Gauss product can't be used directly.

Most exciting, perhaps, might be an explanation of the mysterious connection between bosons on a CM torus and matrix level. Does the chiral algebra of the former, extended as it is by vertex operators, have any more direct relation to the (simpler) Abelian matrix-level algebras, e.g.?

One could try to understand the Gannon classification [34] from a 3D perspective using the Chern-Simons dual theory. It would be interesting to derive the rationality condition

$$a\tau^2 + b\tau + c = 0 \tag{5.0.3}$$

from the Chern-Simons theory. The condition (5.0.3) which picks rational conformal field theories doesn't seem to be related to any physical property of the 2D CFT. Hopefully the 3D Chern-Simons dual would give a more physical explanation of (5.0.3). Our guess is that the condition (5.0.3) should arise as the minimum of a potential $V(\tau)$ over the complex structure moduli space parametrized by τ. If this the case, then we will learn that RCFTs are important in string compactification not only because of their simplicity (or their density in the moduli space of all CFTs but because they minimize some sort of energy on the space of CFTs.

Most importantly from the point of view of string theory is further investigation of the Gukov-Vafa conjecture. One could try to find other Gepner points in the moduli space of Calabi-Yau manifolds and check if these Calabi-Yau manifolds admit complex multiplication. The conjecture is obviously valid for Calabi-Yau manifolds which can be described as orbifolds of complex multiplication tori. It was also shown in [32] that the conjecture is also valid for Calabi-Yau manifolds which correspond to the Fermat point in the complex structure moduli space.

The connection with FQHE can be pursued further and the results in [77] can be applied to the characterization of topological order in the fractional quantum Hall states [70, 71, 79]. As we have seen in Section 2.8, the multi-layer FQH states are described by Laughlin wave functions which are labeled by a matrix K

$$\Psi_K(z) = \prod_{a,b,i,j} (z_{ai} - z_{bj})^{K_{ab}/2} \exp\Big(- \sum |z_{ai}|^2 \Big). \tag{5.0.4}$$

They are in 1-1 correspondence with $SL(m, \mathbb{Z})$ equivalence classes of K matrices. For $m = 2$, i.e., for a double layer FQH states, we can use the Gauss product to define a product on the set of Laughlin wave functions

$$\Psi_{K_1}(z) \odot \Psi_{K_2}(z) = \Psi_{K_3}(z), \tag{5.0.5}$$

where K_3 is given by the Gauss product of K_1 and K_2

$$K_3 = K_1 \star K_2. \tag{5.0.6}$$

The set of Laughlin wave functions together with the product $\odot$ now forms an Abelian group $\mathcal{G}$. The group property of $\odot$ follows directly from the group property of $\star$. As far as we know, this kind of group structure on the set of Laughlin wave functions didn't appear before in the study of FQHE. The dimension of $\mathcal{G}$ is equal to the number of equivalence classes of K with determinant $D = \det(K)$ which is nothing but the class number $h(D)$. As we have seen in (4.3.2), D gives the degeneracy of the ground state of the FQH systems which is a measure of topological order [70, 71, 79]. Could the number $h(D)$ be used as another index which characterizes the topological order?

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