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Aspects of quantum gravity

Ali, Ahmed Farag

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ASPECTS OF QUANTUM GRAVITY

AHMED FARAG ALI

Master of Science, Benha University, 2008

A Thesis

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Department of Physics and Astronomy
University of Lethbridge
LETHBRIDGE, ALBERTA, CANADA

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Dedication

To Ebtessam, Ibrahim, and my parents.

Abstract

We propose a Generalized Uncertainty Principle(GUP) consistent with String Theory, Black Hole Physics and Doubly Special Relativity. This modifies all quantum mechanical Hamiltonians and predicts quantum gravity corrections. We compute corrections to the Lamb shift, simple harmonic oscillator, Landau levels, and the tunneling current. When applied to an elementary particle, it suggests that the space must be quantized, and that all measurable lengths are quantized in units of a fundamental length. We investigated whether GUP can explain the violation of the equivalence principle at small length scales that was observed experimentally. We investigated the consequences of GUP on Liouville theorem. We examined GUP effect on post inflation preheating, and show that it predicts an increase or a decrease in parametric resonance and a change in particle production. The effect of GUP on the creation of black holes is investigated to justify the experimental results from the Large Hadron Collider.

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Chapter 1

Introduction

Quantum gravity (QG) is the field of theoretical physics which attempts to unify quantum mechanics with the general theory of relativity. It is considered to be the holy grail which may enable physicists to complete the unification of all fundamental laws of physics.

The primary approach to quantum gravity leads to a theory with unsolvable divergences [1]. However, there are different approaches which cured these divergences such as string theory [2] and loop quantum gravity [3]. In this chapter, we will first outline the nature of general relativity and quantum mechanics.

1.1 General theory of relativity

The general theory of relativity is Einstein's theory of gravity. It is based on two fundamental principles:

- The principle of relativity; which states that all systems of reference are equivalent with respect to the formulation of the fundamental laws of physics.
- The principle of equivalence; the weak one states that the local effects of motion in a curved space (gravitation) are indistinguishable from those of an accelerated

observer in flat space, without exception. The strong one states that the outcome of any local experiment (gravitational or not) in a freely falling laboratory is independent of the velocity of the laboratory and its location in spacetime.

Einstein thought about the consequences of these principles for many years using many thought experiments. He then realized the importance of Riemannian geometry to construct a new theory in which the gravitational force was a result of the curvature of space-time. In Newtonian gravity, the source of gravity is the mass. In general theory of relativity, the mass turns out to be part of a more general quantity called the energy-momentum tensor ($T_{\mu\nu}$), which includes both energy and momentum densities as well as stress (that is, pressure and shear). It is natural to assume that the field equation for gravity involves this tensor. The energy-momentum tensor is divergence free where its covariant derivative in the curved spacetime is zero ($\nabla^\mu T_{\mu\nu} = 0$). By finding a tensor on other side which is divergence free, this yields the simplest set of equations which are called Einstein's (field) equations:

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}, \quad (1.1.1)$$

where G is the Newton constant, and c is the speed of light. $g_{\mu\nu}$ is defined as the spacetime metric. The spacetime metric captures all the geometric and causal structure of spacetime.

$R_{\mu\nu}$ is called the Ricci tensor and it is defined as

$$R_{\mu\nu} = \partial_\rho \Gamma_{\nu\mu}^\rho - \partial_\nu \Gamma_{\rho\mu}^\rho + \Gamma_{\rho\lambda}^\rho \Gamma_{\nu\mu}^\lambda - \Gamma_{\nu\lambda}^\rho \Gamma_{\rho\mu}^\lambda, \quad (1.1.2)$$

where $\partial_\rho = \frac{\partial}{\partial x^\rho}$, and Γ is known as a Christoffel symbol and is defined in terms of the space-time metric (g_{ij}) as

$$\Gamma^i_{kl} = \frac{1}{2} g^{im} \left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m} \right). \quad (1.1.3)$$

Note that the covariant derivative for any vector (e.g V^ν) is defined as

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\alpha} V^\alpha. \quad (1.1.4)$$

We have R which is called the Ricci scalar and is defined as,

$$R = g^{ab} \left(\frac{\partial \Gamma^c_{ab}}{\partial x^c} - \frac{\partial \Gamma^c_{ac}}{\partial x^b} + \Gamma^d_{ab} \Gamma^c_{cd} - \Gamma^d_{ac} \Gamma^c_{bd} \right). \quad (1.1.5)$$

Note that the left hand side in Eq. (1.1.1), $R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu}$ is called the Einstein tensor ($G_{\mu\nu}$) which is divergence free ($\nabla^\mu G_{\mu\nu} = 0$) [4].

There are a number of experimental confirmations of general relativity that have been found but there still a possibility that it does not hold exactly on very large scales, or in very strong gravitational forces. In any case, the theory breaks down at the Big Bang where quantum gravity effects became dominant. According to a naive interpretation of general relativity that ignores quantum mechanics, the initial state of the universe, at the beginning of the Big Bang, was a singularity.

1.2 Quantum mechanics

Quantum Theory was discovered with Planck's theory of quanta in the spectrum of black body radiation which classical theories can not explain [5–7].

The idea was to consider a distribution of the electromagnetic energy over modes of charged oscillators in matter. Planck's Law was formulated when Planck assumed

1.2. *Quantum mechanics*

that the energy of these oscillators was limited to a set of discrete, integer multiples of a fundamental unit of energy, E , proportional to the oscillation frequency (ν):

$$E = h\nu, \quad (1.2.1)$$

where $h = 6.626 \times 10^{-34}$ J . s is Planck's constant.

There are many contributions from Bohr, Heisenberg, Schrödinger, Pauli, Dirac, and many other physicists who formulated the theory of quantum mechanics.

One of the differences between quantum mechanics and classical mechanics is the uncertainty principle discovered by W. Heisenberg. The uncertainty relation implies that it is impossible to simultaneously measure the position while also determining the motion of a particle, or of any system small enough to require quantum mechanical treatment [8]. In quantum mechanics, physical observables are described by operators acting on the Hilbert space of states. The most fundamental ones, the position operator \hat{x} and momentum operator \hat{p} satisfy the canonical commutation relation

$$[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar, \quad (1.2.2)$$

where $\hbar = \frac{h}{2\pi}$. As a consequence, for the position and momentum uncertainties Δx and Δp of a given state, the Heisenberg uncertainty relation holds:

$$\Delta x \Delta p \geq \frac{\hbar}{2}. \quad (1.2.3)$$

In 1926, Erwin Schrödinger discovered the Schrödinger equation which describes how the quantum state of a physical system changes in time [9]. The general form is the time-dependent Schrödinger equation which gives a description of a system evolving with time:

$$i\hbar\frac{\partial}{\partial t}\psi(\vec{r}, t) = H\psi(\vec{r}, t) = \left(\frac{-\hbar^2}{2m}\nabla^2 + V(r)\right)\psi(\vec{r}, t), \quad (1.2.4)$$

where $\frac{-\hbar^2}{2m}\nabla^2$ is the kinetic energy operator, m is the particle mass, ∇^2 is the Laplace operator in three dimensional space, $V(r)$ is the potential energy, $\psi(\vec{r}, t)$ is the quantum state or wave function, which could be considered as the amplitude to find a particle with probability $|\psi|^2$ in a given volume centered at the point \vec{r} at an instant of time t , and H is the Hamiltonian operator for a single particle in a potential. For the time independent case, the Schrödinger equation takes the form:

$$\left(\frac{-\hbar^2}{2m}\nabla^2 + V(r)\right)\psi(\vec{r}) = E\psi(\vec{r}), \quad (1.2.5)$$

where E are the energy eigenvalues.

Further applications of quantum theory have led to successful models of nuclear physics and as a consequence, many physical phenomena can now be described using quantum interactions. The electromagnetic and weak nuclear interactions are unified into one force while the strong nuclear interaction is a force of a similar nature known as a gauge theory. Together these forces and all observed particles are combined into one self consistent theory known as the Standard Model of particle physics.

Despite the success of the relativistic version of quantum mechanics which was confirmed in high energy accelerator experiments, quantum theory is still criticized by some physicists who consider it an incomplete because of its indeterministic nature and its dependency on the role of the observer.

An unsolvable problem emerges when one tries to incorporate quantum mechanics into the general theory of relativity. This problem is the appearance of infinities when calculating quantum corrections. This problem is known as non-renormalizability [1]. We shall discuss this problem briefly in chapter 2. Resolving this problem will result

in a a complete theory of quantum gravity which is considered as the holy grail for physicists which may enable them to answer many of fundamental questions about the universe.

1.3 Outline of thesis

Formulating a quantum theory of gravitation, and unifying it with the three other forces of nature have remained as two of the most important problems in theoretical physics. Promising approaches such as String Theory and Loop Quantum Gravity have made significant advances in this direction. In Chapter 2, we shall review briefly the main approaches to quantum gravity. We shall review the canonical approach, loop quantum gravity, graviton approach and string theory approach.

These approaches have made some predictions which are not testable in current laboratories. This, as we know, is essential for any scientific theory.

The primary goal of my thesis is to try to suggest ways of extracting potential experimental signatures from various approaches to Quantum Gravity. Most such approaches predict some form of modification of the Heisenberg Uncertainty Principle near the Planck scale, in what is collectively known as the Generalized Uncertainty Principle, or GUP [10–20]. We do a brief review about the generalized uncertainty principle in Chapter 2. We shall discuss the different forms of GUP which were proposed in the different approaches of quantum gravity in chapter 3. We shall discuss our proposed GUP which is consistent with string theory, black hole physics and doubly special relativity at the end of chapter 3 [21].

In some recent papers [21–23], we have proposed a GUP consistent with all such approaches, and have shown that it induces a universal term proportional to $\ell_{Pl}p^3$

(where ℓ_{Pl} =Planck length= $\sqrt{\frac{\hbar G}{c^3}}$, Planck mass= $M_p = \sqrt{\frac{\hbar c}{G}}$, Planck time= $t_p = \sqrt{\frac{\hbar G}{c^5}}$ and p =momentum of the system) in all quantum mechanical Hamiltonians, which in turn predicts small corrections in measurable quantities in several condensed matter and atomic physics experiments (such as in the Lamb shift, Landau levels, the simple harmonic oscillator and the Scanning Tunneling Microscope (STM)). Thus, there is hope that quantum gravity effects may be observable in the laboratory. We discuss different proposals to test quantum gravity in the laboratory in chapter 4 [23].

In papers [21,22] , we have also arrived at an intriguing possibility that the continuum nature of space-time breaks down at the fundamental/Planck level, giving way to an intrinsically discrete structure (this also was suggested earlier by other approaches). In chapter 5, we shall discuss the discreteness of space in the non-relativistic case and in the relativistic case. We shall discuss the validity of discreteness of space for one dimensional space and three dimensional space.

The violation of weak equivalence principle that was discovered in neutron interferometry experiment cannot be explained by quantum mechanics [24,25]. We found in [26] that the generalized uncertainty principle is necessary to explain the experimental results. These results shall be discussed in chapter 6. If GUP is a necessary mechanism to explain such important experimental results, a naturally arising question is whether the number of states inside a volume of phase space does not change with time in the presence of GUP. We shall discuss the answer to this question and its implications for some physical phenomenon like the holographic entropy bound at the end of chapter 6.

Further phenomenological study has been done of cosmology in [27], where we examined GUP effects in cosmology, especially in the preheating phase of our universe,

where based on calculations, we found significant corrections. A detailed study is presented in chapter 7.

Some recent experimental results at LHC [28] suggest that there are no signs of black holes at the range 3.5 – 4.5 TeV, which disagree with the predictions of large extra dimensions theories. In [29], we suggest that the generalized uncertainty principle is a mechanism beside the large extra dimensions that may explain these experimental results. A detailed study is given in chapter 8.

We understand that a number of additional things would have to be taken into account, including other effects which may be comparable to the Quantum Gravity predictions, and in the end even if our predictions turn out to be too minuscule to measure, we would still learn about the limitations of this approach. On the other hand, in an optimistic scenario, we hope some of the effects may be measurable, that the current approach might open up a phenomenological window to Quantum Gravity, and that this would strengthen the synergy among experimentalists and theorists.

The last chapter is to give conclusions and an overview of the whole thesis. We shall discuss the important ideas of the thesis and summarize the results.

Chapter 2

Approaches to quantum gravity

There have been many attempts to formulate a theory of quantum gravity. One of the main differences between different approaches to quantum gravity is the issue of background independence, according to which no particular background should enter into the definition of the theory. The spacetime metric is quantized in quantum gravity, and thus the theory is formulated on a fixed manifold without a background metric. This is known as background independence. Some theories like string theory are formulated on flat space or anti-de Sitter space and are thus not background independent. All approaches are assumed to respect diffeomorphism invariance. In the present chapter we shall briefly review some of the main approaches to quantum gravity. Most such approaches predict some form of modification of the Heisenberg Uncertainty Principle near the Planck scale, in what is collectively known as the Generalized Uncertainty Principle, or GUP [10–20]. We do a brief review about the generalized uncertainty principle at the end of this chapter.

2.1 The canonical approach

The canonical approach assumes (3 + 1) decomposition of spacetime with its invariance under general coordinate transformations. So the canonical approach respects the diffeomorphism of the given manifold $\Sigma \times R$ that is the foliation into hypersurfaces Σ that is fixed for each time $t \in R$. This invariance is called general covariance (also known as diffeomorphism invariance or general invariance) which is the invariance of the form of physical laws under arbitrary differentiable coordinate transformations. The essential idea is that coordinates do not exist a priori in nature, but are only artifices used in describing nature, and hence should play no role in the formulation of fundamental physical laws. Let us now explore this approach, following the work of DeWitt [30].

Consider a spacelike surface, say

$$y^\alpha = f^\alpha(x^i). \quad (2.1.1)$$

The tangent vector to this surface is

$$\xi_i \equiv \partial_i f^\alpha \partial_\alpha, \quad (2.1.2)$$

and the induced metric (that is, the pull-back to the surface of the spacetime metric) is

$$h_{ij} \equiv g_{\alpha\beta} \xi_i^\alpha \xi_j^\beta. \quad (2.1.3)$$

The unit normal is then defined as,

$$\begin{aligned} g_{\alpha\beta} n^\alpha \xi_i^\beta &= 0, \\ n^2 \equiv g_{\alpha\beta} n^\alpha n^\beta &= 1. \end{aligned} \quad (2.1.4)$$

2.1. *The canonical approach*

The approach uses a set of surfaces that covers all spacetime; that is, a spacetime foliation, namely a one-parameter family of spacelike disjoint hypersurfaces

$$\Sigma_t \equiv \{y^\alpha = f^\alpha(x^i, t)\}. \quad (2.1.5)$$

In a classical analysis Arnowitt, Deser and Misner (ADM) [31] characterized the embedding via two functions: the lapse and the shift [3]: the vector is defined

$$N^\alpha \equiv \frac{\partial f^\alpha}{\partial t}, \quad (2.1.6)$$

in terms of which the lapse, N , is just the projection in the direction of the normal, and the shift, N^i the (three) projections tangent to the hypersurface:

$$N^\alpha = N n^\alpha + N^i \xi_i^\alpha. \quad (2.1.7)$$

All this amounts to a particular splitting of the full spacetime metric:

$$g_{\mu\nu} = h^{ij} \xi_{i\mu} \xi_{j\nu} + n_\mu n_\nu. \quad (2.1.8)$$

The extrinsic curvature distinguishes between equivalent surfaces from the intrinsic point of view but they are embedded in different ways:

$$K_{ij} = -\xi_i^\alpha \nabla_\rho n_\alpha \xi_j^\rho. \quad (2.1.9)$$

Using the aforementioned splitting, the Einstein–Hilbert action reads:

$$L_{EH} \equiv \sqrt{g} R[g] = N \sqrt{h} \left(R[h] + K_{ij} K^{ij} - K^2 \right) - \partial_\alpha V^\alpha, \quad (2.1.10)$$

with

$$V^\alpha = 2\sqrt{g} \left(n^\beta \nabla_\beta n^\alpha - n^\alpha \nabla_\beta n^\beta \right). \quad (2.1.11)$$

2.1. *The canonical approach*

Primary constraints appear when defining canonical momenta:

$$p^\mu \equiv \frac{\partial L}{\partial \dot{N}_\mu} \sim 0. \quad (2.1.12)$$

The momenta conjugate to the spatial part of the metric are:

$$\pi^{ij} \equiv \frac{\delta L}{\delta \dot{h}_{ij}} = -\sqrt{h} (K^{ij} - Kh^{ij}). \quad (2.1.13)$$

The canonical commutation relations yield:

$$[\pi^{ij}(x), h_{kl}(x')] = -\delta(x - x') \frac{1}{2} (\delta_k^i \delta_l^j + \delta_k^j \delta_l^i), \quad (2.1.14)$$

where $\delta(x - x') = \begin{cases} +\infty, & x = x' \\ 0, & x \neq x' \end{cases}$ is the Dirac delta function.

The total Hamiltonian reads:

$$H \equiv \int d^3x (\pi_\mu \dot{N}^\mu + \pi^{ij} \dot{h}_{ij} - L) = \int d^3x (N\mathcal{H} + N^i \mathcal{H}_i), \quad (2.1.15)$$

where

$$\mathcal{H}(h, \pi) = h^{-1/2} \left(\pi^{ij} \pi_{ij} - \frac{1}{2} \pi^2 \right) - h^{1/2} R[h], \quad (2.1.16)$$

and

$$\mathcal{H}_i(h, \pi) = -2h_{ik} \partial_j \pi^{kj} - (2\partial_j h_{ki} - \partial_i h_{kj}) \pi^{kj} = -2\nabla[h]_j \pi_i^j. \quad (2.1.17)$$

The whole Hamiltonian analysis boils down to the two constraint equations

$$\mathcal{H} = 0 \quad (2.1.18)$$

$$\mathcal{H}_i = 0 \quad (2.1.19)$$

2.1. *The canonical approach*

Physical states in the Hilbert space are provisionally defined according to Dirac

$$\hat{\mathcal{H}}|\psi\rangle = 0 \quad (2.1.20)$$

$$\hat{\mathcal{H}}_i|\psi\rangle = 0. \quad (2.1.21)$$

It has long been realized that this whole approach suffers from the frozen time problem, i.e., the Hamiltonian reads

$$H \equiv \int d^3x \left(N\mathcal{H} + N^i\mathcal{H}^i \right), \quad (2.1.22)$$

so that it acts on physical states

$$\hat{H}|\psi\rangle = 0, \quad (2.1.23)$$

in such a way that Schrödinger's equation

$$i\frac{\partial}{\partial t}|\psi\rangle = \hat{H}|\psi\rangle \quad (2.1.24)$$

seemingly forbids any time dependence.

We can proceed further using the Schrödinger representation defined in such a way that

$$(\hat{h}_{ij}\psi)[h] \equiv h_{ij}(x)\psi[h], \quad (2.1.25)$$

and

$$(\hat{\pi}^{ij}\psi)[h] \equiv -i\hbar\frac{\delta\psi}{\delta h_{ij}(x)}[h]. \quad (2.1.26)$$

If we assume that diffeomorphisms act on wave functionals as,

$$\psi[f^*h] = \psi[h], \quad (2.1.27)$$

2.1. *The canonical approach*

then the whole construction for the quantum dynamics of the gravitational field lies in Wheeler's superspace (nothing to do with supersymmetry) which is the set of three-dimensional metrics modulo three-dimensional diffeomorphisms: $\text{Riem}(\Sigma)/\text{Diff}(\Sigma)$. The Hamiltonian constraint then implies the renowned Wheeler–DeWitt equation.

$$-\hbar^2 2\kappa^2 G_{ijkl} \frac{\delta^2 \psi}{\delta h_{ik} \delta h_{jl}}[h] - \frac{\hbar}{2\kappa^2} R^{(3)}[h] \psi[h] = 0, \quad (2.1.28)$$

where the De Witt tensor is

$$G_{ijkl} \equiv \frac{1}{\sqrt{h}} \left(h_{ij} h_{kl} - \frac{1}{2} h_{ik} h_{jl} \right). \quad (2.1.29)$$

The Wheeler–DeWitt equation could be possibly written in the form which is similar to the Klein–Gordon equation [30, 32]:

$$\left(\frac{-\partial^2}{\partial \zeta^2} + \frac{32}{3\zeta^2} g^{ab} \partial_a \partial_b + \frac{3}{32} \zeta^2 R^{(3)} \right) \psi = 0, \quad (2.1.30)$$

where $\zeta = \sqrt{\frac{32}{3}} h^{\frac{1}{4}}$.

The Wheeler–DeWitt equation consists of an operator which acts on a wave functional. The Wheeler–DeWitt equation is well defined in cosmology in minisuperspaces like the configuration space of cosmological theories. An example of such a study is the Hartle–Hawking analysis [32].

The interpretation of the symbols \hat{H} and $|\psi\rangle$ in the Wheeler–DeWitt equation is different from quantum mechanics. $|\psi\rangle$ is a *functional* of field configurations on all of spacetime. *This wave function contains all of the information about the geometry and matter content of the universe.* \hat{H} is still an operator that acts on the Hilbert space of wave functions, but it is not the same Hilbert space as in the nonrelativistic case, and the Hamiltonian no longer determines evolution of the system, so the Schrödinger

equation $\hat{H}|\psi\rangle = i\hbar\partial/\partial t|\psi\rangle$ no longer applies. This property is known as timelessness. The reemergence of time requires the tools of decoherence and clock operators.

2.2 Loop quantum gravity

Loop quantum gravity is a canonical theory which means studying the evolution of canonical variables defined classically through a foliation of spacetime that was mentioned in the previous section. The standard choice in this case, as we have mentioned in the previous section, is the three-dimensional metric, g_{ij} and its canonical conjugate, related to the extrinsic curvature. Ashtekar defined a set of variables [33] derived from the Einstein–Hilbert Lagrangian written in the form

$$\int e^a \wedge e^b \wedge R^{cd} \epsilon_{abcd}, \quad (2.2.1)$$

where e^a are the one-forms associated to the tetrad,

$$e^a \equiv e^a_\mu dx^\mu. \quad (2.2.2)$$

Tetrads are defined up to a local Lorentz transformation

$$(e^a)' \equiv L^a_b(x) e^b. \quad (2.2.3)$$

The associated $SO(1, 3)$ connection one-form ω^a_b is called the spin connection which is naturally thought of as a standard gauge connection related to local Lorentz symmetry [34] and is defined as

$$\omega^a_b = e^a_\nu \partial_\mu e^{\nu b} + e^a_\nu e^{\sigma b} \Gamma^\nu_{\sigma\mu}, \quad (2.2.4)$$

where $\Gamma^\nu_{\sigma\mu}$ is the Levi-Civita connection and the e^a_ν are the local lorentz frame fields or vierbein. Note that here, Latin letters denote the local Lorentz frame indices; Greek

2.2. Loop quantum gravity

indices denote general coordinate indices. Its field strength is the curvature expressed as a two form:

$$R^a{}_b \equiv d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \quad (2.2.5)$$

Ashtekar's variables are based on the $SU(2)$ self-dual connection

$$A^i{}_\mu[\omega] = \omega^i{}_\mu - \frac{1}{2}i\epsilon^{ij}{}_{mn}\omega_\mu^{mn}, \quad (2.2.6)$$

where $\epsilon^{ij}{}_{mn}$ is completely anti-symmetric tensor. Its field strength is

$$F \equiv dA + A \wedge A. \quad (2.2.7)$$

The dynamical variables are then $(A_i, E^i \equiv F^{oi})$. The main property of these variables is that constraints are linearized. One of these constraints is exactly analogous to Gauss' law:

$$D_i E^i = 0. \quad (2.2.8)$$

There is another one related to three-dimensional diffeomorphism invariance,

$$\text{Tr} F_{ij} E^i = 0, \quad (2.2.9)$$

and, finally, there is the Hamiltonian constraint,

$$\text{Tr} F_{ij} E^i E^j = 0. \quad (2.2.10)$$

From a mathematical point of view, there is no doubt that Ashtekar's variables are of great elegance. From the physical point of view, they are not real in general. This forces a reality condition to be imposed, which is awkward. For this reason it is preferred to use the Barbero-Immirzi [35, 36] formalism in which the connection depends on a free parameter γ ,

$$A_a^i = \omega_a^i + \gamma K_a^i, \quad (2.2.11)$$

where ω is the spin connection and K the extrinsic curvature. When $\gamma = i$, the Ashtekar formalism is recovered; for other values of γ the explicit form of the constraints is more complicated. Thiemann [37] has proposed a form for the Hamiltonian constraint which seems promising, although it is not clear whether the quantum constraint algebra is isomorphic to the classical algebra. A pedagogical reference is [3].

There are some states that satisfy the Ashtekar constraints. These states are given by the loop representation depending both on the gauge field A and on a parameterized loop γ ,

$$W(\gamma, A) \equiv \text{Tr} P e^{\oint_{\gamma} A_{\mu} dx^{\mu}}, \quad (2.2.12)$$

and a functional transform mapping functionals of the gauge field $\psi(A)$ into functionals of loops, $\psi(\gamma)$:

$$\psi(\gamma) \equiv \int \mathcal{D}A W(\gamma, A) \psi(A). \quad (2.2.13)$$

The “loop transform” (2.2.13) can be viewed as a generalization of a Fourier transform that maps the representation of quantum states in x -space to the representation of quantum states in momentum space [38]. Here, the transform maps the $\psi(A)$ representation of quantum states in A space to the representation $\psi(\gamma)$ of quantum states in γ space, namely in loop space.

In the presence of a cosmological constant λ the Hamiltonian constraint reads:

$$\epsilon_{ijk} E^{ai} E^{bj} \left(F_{ab}^k + \frac{\lambda}{3} \epsilon_{abc} E^{ck} \right) = 0. \quad (2.2.14)$$

A particular class of solutions of the constraint [39] are self-dual solutions of the form

$$F_{ab}^i = -\frac{\lambda}{3} \epsilon_{abc} E^{ci}. \quad (2.2.15)$$

2.2. Loop quantum gravity

Kodama [40] showed that the Chern-Simons state,

$$\psi_{CS}(A) \equiv e^{\frac{3}{2\lambda} S_{CS}(A)}, \quad (2.2.16)$$

is a solution of the Hamiltonian constraint.

Loop states can be represented as spin network [41] states. There is also a path integral approach, known as *spin foam* [42], a topological theory of colored surfaces representing the evolution of a spin network. Spin foams can also be considered as an independent approach to the quantization of the gravitational field [43].

This approach has the same problem as all canonical approaches to covariant systems which is the problem of time. Dynamics are still somewhat mysterious; the Hamiltonian constraint does not show with respect to what the three-dimensional dynamics evolve.

One important success of the loop approach is the prediction that the area and the volume operators are quantized. This allows one to explain the formula for the black hole entropy. The result is obtained in terms of the Barbero-Immirzi parameter [44]. It has been pointed out [45] that there is a potential drawback in all theories in which the area (or mass) spectrum is quantized with eigenvalues A_n if the level spacing between eigenvalues δA_n is uniform because of the predicted thermal character of Hawking's radiation. The explicit computation yields,

$$\delta A_n \sim e^{-\sqrt{A_n}}, \quad (2.2.17)$$

which might avoid this set of problems. Loop quantum gravity succeeded in replacing the Big Bang spacetime singularity with a Big Bounce. Refs. [46–48], claimed that there is a solution for the Big Bang singularity which would give new weight to the oscillatory universe and Big Bounce theories. It purported to show that a previously

existing universe collapsed, not to the point of singularity, but to a point before that where the quantum effects of gravity become so strongly repulsive that the universe rebounds back out, forming a new branch. Throughout this collapse and bounce, the evolution is unitary.

2.3 Graviton approach

Another interesting approach is to study gravitons, gravitational force carriers, as ordinary (massless, spin two) particles in Minkowski space-time.

$$g_{\alpha\beta} = \bar{g}_{\alpha\beta} + kh_{\alpha\beta}. \quad (2.3.1)$$

This theory is one loop finite on shell as was shown in the brilliant calculations by Gerard 't Hooft and M. Veltman [49]. They computed the counterterm,

$$\Delta L^{(1)} = \frac{\sqrt{\bar{g}}}{\epsilon} \frac{203}{80} \bar{R}^2. \quad (2.3.2)$$

For higher loops, we get infinities as was studied by Goroff and Sagnotti [1] who calculated the two loop corrections.

$$\Delta L^{(2)} = \frac{209}{2880(4\pi)^4} \frac{1}{\epsilon} \bar{R}^{\alpha\beta} \bar{R}^{\gamma\delta} \bar{R}^{\rho\sigma} \bar{R}_{\alpha\beta}. \quad (2.3.3)$$

It has been concluded that the theory is not renormalizable with infinities which cannot be cured.

The general structure of perturbation theory is governed by the fact we have just mentioned that the coupling constant is dimensionful. A general diagram will then behave in the s-channel as $\kappa^n s^n$ and counterterms as,

$$\Delta L \simeq \sum \int \kappa^n R^{2+n/2}. \quad (2.3.4)$$

(where a symbolic notation has been used), packing all invariants with the same dimension; for example, R^2 stands for an arbitrary combination of R^2 , $R_{\alpha\beta}R^{\alpha\beta}$ and $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ conveying the fact that the theory is nonrenormalizable.

2.4 String theory

The main idea of string theory is to consider all elementary particles as quantized excitations of a one dimensional object, the string, which can be either open (free ends) or closed (a loop). Excellent books on String theory are available, such as [2,50].

String theories have enjoyed a rich history. Their origin can be traced to the Veneziano model of strong interactions. A crucial step was the reinterpretation by Scherk and Schwarz [51] of the massless spin two state in the closed sector (previously thought to be related to the Pomeron) as the graviton and consequently of the whole string theory as a potential theory of quantum gravity, and potential unified theories of all interactions. Now the wheel has made a complete turn, and we are perhaps back through the Maldacena conjecture [52] to a closer relationship than previously thought with ordinary gauge theories.

The string theory dynamics is determined by a two dimensional non-linear sigma model, which geometrically is a theory of embedding of a two-dimensional surface σ_2 (the world sheet of the string) to a (usually ten-dimensional) target space M_n :

$$x^\mu(\xi) : \Sigma_2 \rightarrow M_n \quad (2.4.1)$$

There are two types of interactions to consider. Sigma model interactions (in a given two-dimensional surface) are defined as an expansion in powers of momentum, where a new dimensionful parameter $\alpha' \equiv l_s^2$ sets the scale. This scale is a priori believed to be of the order of the Planck length. The first terms in the action always include a coupling to the massless backgrounds: the spacetime metric, the two-index Maxwell-like field known as the Kalb–Ramond or B-field, and the dilaton. To be specific,

$$S = \frac{1}{l_s^2} \int_{\Sigma_2} g_{\mu\nu}(x(\xi)) \partial_a x^\mu(\xi) \partial_b x^\nu(\xi) \gamma^{ab}(\xi) + \dots \quad (2.4.2)$$

There are also string interactions, (changing the two-dimensional surface) proportional to the string coupling constant, g_s , whose variations are related to the logarithmic variations of the dilaton field. Open strings (which have gluons in their spectrum) always contain closed strings (which have gravitons in their spectrum) as intermediate states in higher string order (g_s) corrections. This interplay open/closed is one of the most fascinating aspects of string theory.

It was discovered by Friedan [53] that in order for the quantum theory to be consistent with all classical symmetries (diffeomorphisms and conformal invariance), the beta function of the generalized couplings must vanish:

$$\beta(g_{\mu\nu}) = R_{\mu\nu} = 0. \quad (2.4.3)$$

This result remains until now one of the most important in string theory, hinting at a deep relationship between Einstein's equations and the renormalization group.

Polyakov [54] introduced the so called non-critical strings which have in general a two-dimensional cosmological constant (forbidden otherwise by Weyl invariance). The

dynamics of the conformal mode (often called Liouville in this context) is, however, poorly understood.

Fundamental strings live in $D = 10$ spacetime dimensions, and so a Kaluza–Klein mechanism of sorts must be at work in order to explain why we only see four non-compact dimensions at low energies. Strings have in general tachyons in their spectrum, and the only way to construct seemingly consistent string theories [55] is to project out those states, which leads to supersymmetry. This means in turn that all low energy predictions heavily depend on the supersymmetry breaking mechanisms.

Several stringy symmetries are believed to be exact: T-duality, relating large and small compactification volumes, and S-duality, relating the strong coupling regime with the weak coupling one. Besides, extended configurations (D branes), topological defects in which open strings can end, are known to be important [56]. They couple to Maxwell-like fields which are p-forms called Ramond-Ramond (RR) fields. These dualities [57] relate all five string theories (namely, Heterotic $E(8) \times E(8)$, Heterotic $SO(32)$, type I, IIA and IIB) and it is conjectured that there is a unified eleven-dimensional theory, dubbed M-theory of which $N = 1$ supergravity in $d = 11$ dimensions is the low energy limit.

2.4.1 **Important results**

Perhaps the main result is that graviton physics in flat space is well-defined for the first time, and this is no minor accomplishment. A graviton in string theory is a closed string. The scattering of gravitons in string theory can also be computed from the correlation functions in conformal field theory to give finite values. These graviton loops are finite because the string size acts like a natural cutoff which is not the same

case in point-particle theories.

Besides, there is evidence that at least some geometric singularities are harmless in the sense that strings do not feel them. Topology change amplitudes do not vanish in string theory.

The other important result [58] is that one can correctly count states of extremal black holes as a function of charges. This is at the same time astonishing and disappointing. It clearly depends strongly on the objects being BPS states (that is, on supersymmetry), and the result has not been extended to nonsupersymmetric configurations. On the other hand, as we have said, it exactly reproduces the entropy as a function of a sometimes large number of charges, without any adjustable parameter.

2.4.2 The Maldacena conjecture

Maldacena [52] proposed as a conjecture that IIB string theories in a background $AdS_5 \times S_5$ is equivalent to a four dimensional ordinary gauge theory in flat four-dimensional Minkowski space, namely $\mathcal{N} = 4$ super Yang-Mills with gauge group $SU(N)$ and coupling constant $g = g_s^{1/2}$.

Although there is much supersymmetry in the problem and the kinematics largely determine correlators (in particular, the symmetry group $SO(2, 4) \times SO(6)$ is realized as an isometry group on the gravity side and as an R-symmetry group as well as conformal invariance on the gauge theory side), the conjecture has passed many tests in the semiclassical approximation to string theory.

The way the AdS/CFT works in detail [59] is that the supergravity action corresponding to fields with prescribed boundary values is related to gauge theory correlators of certain gauge invariant operators corresponding to the particular field studied:

$$e^{-S_{\text{Sugra}}[\Phi_i]} |_{\Phi_i|_{\partial\text{AdS}=\phi_i}} = \langle e^{\int \mathcal{O}_i \phi_i} \rangle_{\text{CFT}}. \quad (2.4.4)$$

This is the first time that a precise holographic description of spacetime in terms of a (boundary) gauge theory is proposed and, as such it is of enormous potential interest. It has been conjectured by 't Hooft [60] and further developed by Susskind [61] that there should be much fewer degrees of freedom in quantum gravity than previously thought. The conjecture claims that it should be enough with one degree of freedom per unit Planck surface in the two-dimensional boundary of the three-dimensional volume under study. The reason for that stems from an analysis of the Bekenstein-Hawking [62, 63] entropy associated to a black hole, given in terms of the two-dimensional area A^1 of the horizon by

$$\sigma = \frac{S}{k} = \frac{c^3}{4G\hbar} A. \quad (2.4.6)$$

This is a deep result indeed, still not fully understood. It is true on the other hand that the Maldacena conjecture has only been checked for the time being in some corners of parameter space, namely when strings can be approximated by supergravity in the appropriate background.

¹The area of the horizon for a Schwarzschild black hole is given by:

$$A = \frac{16\pi G^2}{c^4} M^2 \quad (2.4.5)$$

2.5 Quantum gravity approaches and generalized uncertainty principle

In string theory the Generalized Uncertainty Principle (GUP) was proposed since the works of Amati et al. [10]. There have been studied ultra high energy scatterings of strings in order to see how the theory tackles the inconsistencies of quantum gravity at the Planck scale. The authors find interesting effects, new compared to those found in usual field theories, originating from the soft short-distance behaviour of string theory. They studied particularly the hard processes excitable at short distance as high-energy fixed-angle scatterings, and find that it is not possible to test distances shorter than the characteristic string length $\lambda_s = (\hbar\alpha)^{1/2}$ (α is the string tension). Another scale is dynamically generated, the gravitational Schwarzschild radius $R(E) \sim (G_N E)^{\frac{1}{D-3}}$ and approaches towards λ_s depending on whether or not $R(E) > \lambda_s$. If the latter is true, new contributions at distances of the order of $R(E)$ appear, indicating a classical gravitational instability that can be attributed to black hole formation. If, on the contrary, $R(E) < \lambda_s$, those contributions are irrelevant: there are no black holes with a radius smaller than the string length. In this case, the analysis of short distances can go on and it has been shown that the larger momentum transfers do not always correspond to shorter distances. Precisely the analysis of the angle-distance relationship suggests the existence of a scattering angle θ_M such that when the scattering θ happened at $\theta < \theta_M$ the relation between interaction distance and momentum transfer is the classical one (i.e. follows the Heisenberg relation) with $q \sim \frac{\hbar}{b}$ (b is the impact parameter) while, when $\theta \gg \theta_M$ the classical picture is lost and becomes an important new regime where $\langle q \rangle \sim b$. This suggests a modification

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of the uncertainty relation at the Planck scale in the form of

$$\Delta x \sim \frac{\hbar}{\Delta p} + Y\alpha\Delta p, \quad (2.5.1)$$

(where Y is a suitable constant) and consequently the existence of a minimal observable [10] length of the order of string size λ_s .

Several other types of analysis have been performed about uncertainty relations and measurability bounds in quantum gravity. In [11], Maggiore obtained an expression of a GUP by analyzing a gedanken experiment for the measurement of the area of the apparent horizon of a black hole in quantum gravity. This rather model-independent approach provides a GUP which agrees in functional form with a similar result obtained in the framework of string theory. The gedanken experiment proceeds by observing the photons scattered by the studied black hole. The main physical hypothesis of the experiment is that the black hole emits Hawking radiation. Recording many photons of the Hawking radiation, it is obtained an ‘image’ of the black hole. Besides, measuring the direction of the propagation of photons emitted at different angles and tracing them back, we can (in principle) locate the position of the center of the hole. In this way we make a measurement of the radius R_h of the horizon of the hole. This measurement suffers two kinds of errors. The first one is, as in Heisenberg classical analysis, the resolving power of the microscope

$$\Delta x^{(1)} \sim \frac{\lambda}{\sin \theta}, \quad (2.5.2)$$

where θ is the scattering angle. Besides during the emission process the mass of the black hole varies from $M + \Delta M$ to M (with $\Delta M = \frac{h}{c\lambda}$) and the radius of the horizon changes accordingly. The corresponding error is intrinsic to the measurement and its

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value is

$$\Delta x^{(2)} \sim \frac{2G}{c^2} \Delta M = \frac{2G}{c^3} \frac{h}{\lambda}. \quad (2.5.3)$$

By means of the obvious inequality $\frac{\lambda}{\sin \theta} \geq \lambda$, the errors $\Delta x^{(1)}$ and $\Delta x^{(2)}$ are combined linearly to obtain

$$\Delta x \geq \lambda + k \frac{2G}{c^3} \frac{h}{\lambda} \sim \frac{h}{\Delta p} + k \frac{2G}{c^3} \Delta p, \quad (2.5.4)$$

which is the generalized uncertainty principle. The numerical constant k cannot be predicted by the model-independent arguments presented.

The generalized uncertainty principle has been found in the loop quantum gravity [64]. Hossain et al., found that the polymer quantization gives a modified uncertainty relation that resembles the one coming from string theory, and from black hole physics. The common feature in all these approaches is a length scale in addition to \hbar . However it is only in the polymer approach that the quantization method itself comes with a scale due to the choice of Hilbert space, and in this sense the modifications are independent of the theory.

Increasing a collision's energy above the Planck scale, the extreme energy concentration in a small space will create a black hole with an event horizon behind which we cannot see. It is not unreasonable to suppose that this is not a lack of our experimental sophistication, but that nature possesses an absolute minimal length.

The formed black hole will evaporate through Hawking radiation. Moreover, the higher the energy of the collision, the more massive the created black hole, and the less energetic the Hawking radiation will be.

To express this more quantitatively (see e.g. the review [65]), we can imagine trying to probe the transplanckian distance d using energy of order $E \sim 1/d$. A

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black hole will form, with event horizon radius of $R_S \sim EG \sim G/d$ and temperature of $T \sim 1/RS \sim d/G$. As a result, the emitted thermal radiation will have a dominant wavelength increasing with the energy of the probing particle, and for transplanckian energies the scale probed is no longer the usual $d \sim 1/E$, but rather $d \sim EG$. It is natural that between the two regimes there will be a minimal observable length.

So far, attempts to incorporate gravity into relativistic quantum field theory run into problems, because taking into account smaller and smaller length-scales yields infinite results. A hypothetical minimal length could serve as a cutoff for a quantum gravity and remove the infinities.

In chapter 3, we continue our investigations for the different forms of the GUP and we study its implications in the subsequent chapters.

Chapter 3

The generalized uncertainty principle

In quantum mechanics, the Heisenberg uncertainty principle states a fundamental limit on the accuracy with which certain pairs of physical observables, such as the position and momentum of a particle, can be simultaneously known. In other words, the more precisely one observable is measured, the less precisely the other can be determined, or known.

Physical observables are described by operators acting on the Hilbert space of states. Given an observable A , we define an operator

$$\Delta A \equiv A - \langle A \rangle, \quad (3.0.1)$$

where the expectation value is to be taken for a certain physical state under consideration. The expectation value of $(\Delta A)^2$ is known as the dispersion of A . Because we have

$$\langle (\Delta A)^2 \rangle = \langle (A^2 - 2A\langle A \rangle + \langle A \rangle^2) \rangle = \langle A^2 \rangle - \langle A \rangle^2. \quad (3.0.2)$$

The last equation can also be considered as an definition of dispersion.

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Using the Schwartz inequality:

$$\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2, \quad (3.0.3)$$

which is valid for any ket state and bra state. Using this fact with

$$|\alpha\rangle = \Delta A|\alpha'\rangle, \quad (3.0.4)$$

$$|\beta\rangle = \Delta B|\beta'\rangle, \quad (3.0.5)$$

where the kets $|\alpha'\rangle$ or $|\beta'\rangle$ emphasize the fact that this consideration may be applied to any ket, we obtain:

$$(\Delta A)^2(\Delta B)^2 \geq |\langle\Delta A\Delta B\rangle|^2, \quad (3.0.6)$$

and this is possible only if the operators A and B are Hermitian operators i.e. $A = A^\dagger$ and $B = B^\dagger$.

To evaluate the right-hand side of Eq. (3.0.6), we note

$$\Delta A\Delta B = \frac{1}{2}[\Delta A, \Delta B] + \frac{1}{2}\{\Delta A, \Delta B\}, \quad (3.0.7)$$

where the commutator

$$([A, B])^\dagger = (AB - BA)^\dagger = BA - AB = -[A, B]. \quad (3.0.8)$$

In contrast, the anticommutator $\{\Delta A, \Delta B\}$ is obviously Hermitian, so by taking the expectation value Eq. (3.0.7), we get:

$$\Delta A\Delta B = \frac{1}{2}\langle[A, B]\rangle + \frac{1}{2}\langle\{\Delta A, \Delta B\}\rangle. \quad (3.0.9)$$

Since the expectation value of the Hermitian operator is purely real, and it is purely imaginary for the anti-Hermitian operator [6], then the first term of the right hand

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side in Eq. (3.0.9) will be purely imaginary and the second term will be purely real.

The right hand side of Eq. (3.0.6) now becomes

$$|\Delta A \Delta B|^2 = \frac{1}{4} |\langle [A, B] \rangle|^2 + \frac{1}{4} |\langle \{\Delta A, \Delta B\} \rangle|^2. \quad (3.0.10)$$

The omission of the second term in the right hand side leads to a strong inequality which is known as Cauchy-Schwartz inequality:

$$(\Delta A)^2 (\Delta B)^2 \geq \frac{1}{4} |\langle [A, B] \rangle|^2. \quad (3.0.11)$$

The most fundamental operators, the position operator \hat{x} and momentum operator \hat{p} , satisfy the canonical commutation relation

$$[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar. \quad (3.0.12)$$

As a consequence, for the position and momentum uncertainties Δx and Δp of a given state, the Heisenberg uncertainty relation holds:

$$\Delta x \Delta p \geq \frac{\hbar}{2}. \quad (3.0.13)$$

An important consequence is that in order to probe arbitrarily small length-scales, one has to use probes of sufficiently high energy, and thus momentum. This is the principle on which accelerators (such as LHC, FermiLab, etc) are based. There are reasons to believe that at high energies, when gravity becomes important, this is no longer true.

3.1 Minimal-length uncertainty relations

The existence of a minimal length is one of the most interesting predictions of some approaches of quantum gravity such as string theory as well as black hole physics.

3.1. *Minimal-length uncertainty relations*

This follows from String Theory since strings cannot interact at distances smaller than their size, so this yields *Generalized Uncertainty Principle* (GUP) [10]. From Black hole physics, the Heisenberg Uncertainty Principle (HUP), $\Delta x \sim \hbar/\Delta p$, breaks down for energies close to the Planck scale, when the corresponding Schwarzschild radius is comparable to the Compton wavelength ($\frac{\hbar}{mc}$) (both being approximately equal to the Planck length). Higher energies result in a further increase of the Schwarzschild radius, resulting in $\Delta x \approx \ell_{Pl}^2 \Delta p/\hbar$. The above observation, along with a combination of thought experiments and rigorous derivations suggest that the *Generalized Uncertainty Principle* (GUP) holds at all scales, and is represented by [10–20],

$$\Delta x_i \Delta p_i \geq \frac{\hbar}{2} [1 + \beta ((\Delta p)^2 + \langle p \rangle^2) + 2\beta (\Delta p_i^2 + \langle p_i \rangle^2)], \quad (3.1.1)$$

where $p^2 = \sum_j p_j p_j$, $\beta = \beta_0/(M_p c)^2 = \beta_0 \frac{\ell_p^2}{\hbar^2}$, $M_p =$ Planck mass, and $M_p c^2 =$ Planck energy.

It was shown in [18], that inequality (3.1.1) can follow from the following modified Heisenberg algebra

$$[x_i, p_j] = i\hbar(\delta_{ij} + \beta\delta_{ij}p^2 + 2\beta p_i p_j) . \quad (3.1.2)$$

This form ensures, via the Jacobi identity, that $[x_i, x_j] = 0 = [p_i, p_j]$ [19].

Defining [66, 67]:

$$x_i = x_{0i}, \quad (3.1.3)$$

$$p_i = p_{0i} (1 + \beta p_0^2), \quad (3.1.4)$$

where $p_0^2 = \sum_{j=1}^3 p_{0j} p_{0j}$ and with x_{0i}, p_{0j} satisfying the canonical commutation relations

$$[x_{0i}, p_{0j}] = i\hbar \delta_{ij}, \quad (3.1.5)$$

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it is easy to show that Eq. (3.1.2) is satisfied, to order β . Henceforth, the higher order terms of β could be neglected.

Note that p_{0i} could be defined as the momentum at low energy scale which is represented by $p_{0i} = -i\hbar d/dx_{0i}$, while p_i is considered as the momentum at the higher energy scales [66,67].

It is normally assumed that the dimensionless parameter β_0 is of the order of unity, in which case the β dependent terms are important only when energies (momenta) are comparable to the Planck energy (momentum), and lengths are comparable to the Planck length. However, if we do not impose this condition *a priori*, then this may signal the existence of a new physical length scale of the order of $\hbar\sqrt{\beta} = \sqrt{\beta_0}\ell_{Pl}$. Evidently, such an intermediate length scale cannot exceed the electroweak length scale $\sim 10^{17} \ell_{Pl}$ (as otherwise it would have been observed) and this implies that $\sqrt{\beta_0} \leq 10^{17}$. This tells us that β_0 cannot exceed about 10^{34} .

Using (3.1.4), any Hamiltonian of the form

$$H = \frac{p^2}{2m} + V(\vec{r}), \quad \vec{r} = (x_1, x_3, x_3) \quad (3.1.6)$$

can be written as [20]

$$H = \frac{p_0^2}{2m} + V(\vec{r}) + \frac{\beta}{m}p_0^4 + \mathcal{O}(\beta^2) \quad (3.1.7)$$

$$\equiv H_0 + H_1 + \mathcal{O}(\beta^2), \quad (3.1.8)$$

where

$$H_0 = \frac{p_0^2}{2m} + V(\vec{r}) \quad \text{and} \quad H_1 = \frac{\beta}{m}p_0^4 = \frac{\beta\hbar^4}{m}\nabla^4, \quad (3.1.9)$$

where in the last step, we used the position representation.

Thus, we see that *any* system with a well-defined quantum (or even classical) Hamiltonian H_0 is perturbed by H_1 , defined above, near the Planck scale. Such

corrections will continue to play a role irrespective of what other quantum gravity corrections one may consider. In other words, they are in some sense universal.

3.2 Other forms of generalized uncertainty principles

It should be noted that the uncertainty relations considered are not unique; several other forms have been considered in the existing literature.

3.2.1 Snyder's form

Some of the early articles to present a theory with quantized space-time are due to Snyder [68,69]. These papers investigated some ways to resolve the infinities problem in the early stages in the development of quantum field theory.

In [68,69], Snyder considers a de Sitter space, with real coordinates $\{\eta_0, \eta_1, \eta_2, \eta_3, \eta_4\}$. He defines the position and time operators by

$$X_i = ia\left(\eta_4 \frac{\partial}{\partial \eta_i} - \eta_i \frac{\partial}{\partial \eta_4}\right), \quad i = 1, 2, 3, \quad (3.2.1)$$

$$T = \frac{ia}{c}\left(\eta_4 \frac{\partial}{\partial \eta_i} + \eta_i \frac{\partial}{\partial \eta_4}\right), \quad (3.2.2)$$

acting on a functions of variables $\{\eta_0, \eta_1, \eta_2, \eta_3, \eta_4\}$, and where a is a natural unit of length, and c is the speed of light.

In addition, the energy and momentum operators are defined as

$$P_i = \frac{\hbar \eta_i}{a \eta_4}, \quad (3.2.3)$$

$$P_T = \frac{\hbar \eta_0}{a \eta_4}, \quad (3.2.4)$$

and thus the commutators between positions and momenta are given by

$$[X_i, P_j] = i\hbar(\delta_{ij} + \frac{a^2}{\hbar^2} P_i P_j). \quad (3.2.5)$$

The algebra described by Snyder is close to the generalized uncertainty commutation relation of Eq. (3.1.2).

3.2.2 Modified de Broglie relation

The modified de Broglie relation has been investigated by Hossenfelder et al. in [16]. They assume the wave number $k(p)$ to be an odd function and nearly linear for small values of p and approaching asymptotically some upper limit which is proportional to a minimal length $M_{Pl} \sim 1/L_{Pl}$. Such a function will have an expansion in p as follows,

$$k = p - \gamma \frac{p^3}{M_{Pl}^2}, \quad (3.2.6)$$

where γ is a constant which is not determined from the first principles. Considering the commutation relation between x and k will result in the generalized uncertainty principle in the following form

$$\Delta x \Delta p \geq \frac{\hbar}{2} \left| \left\langle \frac{\partial p}{\partial k} \right\rangle \right| = \frac{\hbar}{2} \left(1 + \gamma \frac{\langle p^2 \rangle}{M_{Pl}^2} \right), \quad (3.2.7)$$

and this gives the commutation relation by

$$[\hat{x}, \hat{p}] = i \frac{\partial p}{\partial k} = i\hbar \left(1 + \gamma \frac{p^2}{M_{Pl}^2}\right). \quad (3.2.8)$$

This agrees with the generalized uncertainty relation of Eq. (3.1.2).

3.2.3 Alternative approaches to the GUP

It should be mentioned that postulating non-standard commutation relations between the position and momentum operators is not the only way to define a theory with minimal length. Two approaches seem to be especially promising.

First, the doubly relativistic theory studied in [70, 71] found a group of transformations that have two invariants. In addition to the constant speed of light, it also assumes an invariant energy scale. This group is still a Lorentz group.

A nonlinear realization of Lorentz transformations in energy-momentum (E, p) space parametrized by an invariant length ℓ can be defined by the relations [71, 72]

$$\epsilon = Ef(\ell E, \ell^2 p^2), \quad (3.2.9)$$

$$\pi_i = p_i g(\ell E, \ell^2 p^2), \quad (3.2.10)$$

where (ϵ, π) are auxiliary linearly transforming variables which define the nonlinear Lorentz transformation of the physical energy-momentum (E, p) . Then we have two functions of two variables (f, g) which parametrize the more general nonlinear realization of Lorentz transformations, with rotations realized linearly, depending on a dimensional scale. The condition to recover the special relativistic theory in the low energy limit reduces to the condition $f(0, 0) = g(0, 0) = 1$. Each choice of the two functions f, g will lead to a generalization of the relativity principle with an invariant length scale. Lorentz transformation laws connecting the energy-momentum of a

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particle in different inertial frames differ from the standard special relativistic linear transformation laws which are recovered when $\ell E \ll 1$, $\ell^2 p^2 \ll 1$.

In order to have a quantum theory with such a deformed relativity principle, one should find the appropriate deformation of relativistic quantum field theory (QFT). Cortés and Gamboa succeeded to find a quantum form of the doubly special relativity in [72]. They found that the commutation relation between the canonical variables x and p should be modified in doubly special relativity as follows

$$[x_i, p_j] = i\hbar[(1 - \ell|p|)\delta_{ij} + \ell^2 p_i p_j]. \quad (3.2.11)$$

They conclude that there is a modification of the quantum mechanical commutators which becomes relevant when the energy approaches its maximum value and in the limit one gets a classical phase space.

There is another approach in which Padmanabhan and his collaborators in [73, 74] study a modified quantum theory by postulating that the path integral is invariant under a duality transformation of the form $x \rightarrow L_f^2/x$. This, again, yields a minimal length L_f .

3.3 Minimal length-maximal energy uncertainty relation

An intriguing prediction of various theories of quantum gravity (such as string theory) and black hole physics is the existence of a minimum measurable length. This has given rise to the GUP or equivalently, modified commutation relations between position coordinates and momenta as explained in the previous sections. The recently

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proposed *doubly special relativity* (DSR) theories on the other hand, also suggest a similar modification of commutators [72]. The commutators that are consistent with string theory, black holes physics, DSR, *and* which ensure $[x_i, x_j] = 0 = [p_i, p_j]$ (via the Jacobi identity) have the following form [21] (see Appendix A for the proof of the following equation.)¹

$$[x_i, p_j] = i\hbar \left(\delta_{ij} - \alpha \left(p\delta_{ij} + \frac{p_i p_j}{p} \right) + \alpha^2 (p^2 \delta_{ij} + 3p_i p_j) \right) \quad (3.3.1)$$

where $\alpha = \alpha_0 / M_{Pl} c = \alpha_0 \ell_{Pl} / \hbar$, M_{Pl} = Planck mass, $\ell_{Pl} \approx 10^{-35}$ m = Planck length, and $M_{Pl} c^2$ = Planck energy $\approx 10^{19}$ GeV.

In one dimension, Eq. (3.3.1) gives to $\mathcal{O}(\alpha^2)$

$$\begin{aligned} \Delta x \Delta p &\geq \frac{\hbar}{2} [1 - 2\alpha \langle p \rangle + 4\alpha^2 \langle p^2 \rangle] \\ &\geq \frac{\hbar}{2} \left[1 + \left(\frac{\alpha}{\sqrt{\langle p^2 \rangle}} + 4\alpha^2 \right) \Delta p^2 + 4\alpha^2 \langle p \rangle^2 - 2\alpha \sqrt{\langle p^2 \rangle} \right]. \end{aligned} \quad (3.3.2)$$

Commutators and inequalities similar to (3.3.1) and (3.3.2) were proposed and derived respectively in [10–20, 64, 66, 67, 72, 75, 76]. These in turn imply a minimum measurable length *and* a maximum measurable momentum, the latter following from the assumption that Δp characterizes the maximum momentum of a particle as well [77], and also from the fact that DSR predicts such a maximum (To the best of our knowledge, (3.3.1) and (3.3.2) are the only forms which imply both.)

$$\Delta x \geq (\Delta x)_{min} \approx \alpha_0 \ell_{Pl} \quad (3.3.3)$$

$$\Delta p \leq (\Delta p)_{max} \approx \frac{M_{Pl} c}{\alpha_0}. \quad (3.3.4)$$

¹The results do not depend on this particular form of GUP chosen, and continue to hold for a large class of variants, so long as an $\mathcal{O}(\alpha)$ term is present in the right-hand side of Eq. (3.3.1).

3.3. Minimal length-maximal energy uncertainty relation

Next, defining (see Appendix B)

$$x_i = x_{0i} , \quad p_i = p_{0i} (1 - \alpha p_0 + 2\alpha^2 p_0^2) , \quad (3.3.5)$$

with x_{0i}, p_{0j} satisfying the canonical commutation relations $[x_{0i}, p_{0j}] = i\hbar \delta_{ij}$, it can be shown that Eq. (3.3.1) is satisfied. Here, p_{0i} can be interpreted as the momentum at low energies (having the standard representation in position space, i.e. $p_{0i} = -i\hbar\partial/\partial x_{0i}$) and p_i as that at higher energies.

It is normally assumed that the dimensionless parameter α_0 is of the order of unity, in which case the α dependent terms are important only when energies (momenta) are comparable to the Planck energy (momentum), and lengths are comparable to the Planck length. However, if we do not impose this condition *a priori*, then this may signal the existence of a new physical length scale of the order of $\alpha\hbar = \alpha_0 \ell_{Pl}$. Evidently, such an intermediate length scale cannot exceed the electroweak length scale $\sim 10^{17} \ell_{Pl}$ (as otherwise it would have been observed) and this implies that $\alpha_0 \leq 10^{17}$. Using Eq. (3.3.5), a Hamiltonian of the form

$$H = \frac{p^2}{2m} + V(\vec{r}), \quad (3.3.6)$$

can be written as

$$H = H_0 + H_1 + \mathcal{O}(\alpha^3), \quad (3.3.7)$$

$$\text{where } H_0 = \frac{p_0^2}{2m} + V(\vec{r}), \quad (3.3.8)$$

$$\text{and } H_1 = -\frac{\alpha}{m} p_0^3 + \frac{5\alpha^2}{2m} p_0^4 . \quad (3.3.9)$$

Thus, we see that *any* system with a well-defined quantum (or even classical) Hamiltonian H_0 is perturbed by H_1 , defined above, near the Planck scale. Such corrections extend to relativistic systems as well [22], and given the robust nature of GUP, will

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continue to play a role irrespective of what other quantum gravity corrections one may consider. In other words, they are in some sense universal. The relativistic Dirac equation is modified in a similar way [22]. We will discuss this modification and its implications in chapter 5 of this thesis.

Chapter 4

Testing quantum gravity effects

In this chapter we investigate the effect of quantum gravity corrections on various quantum phenomena. We will see that *any* system with a well-defined quantum (or even classical) Hamiltonian H_0 is perturbed by H_1 , defined in Eqs. (3.3.8) and (3.3.9), near the Planck scale. Such corrections extend to relativistic systems as well [22], and given the robust nature of GUP, will continue to play a role irrespective of what other quantum gravity corrections one may consider. In other words, they are in this sense universal. Because this influences all the quantum Hamiltonians in a universal way, it predicts quantum gravity corrections to various quantum phenomena, and in the present chapter we compute these corrections to some quantum phenomena such as the Lamb shift, simple harmonic oscillator, Landau levels, and the tunneling current in a scanning tunneling microscope [23, 78].

4.1 The Landau levels

Consider a particle of mass m and charge e in a constant magnetic field $\vec{B} = B\hat{z}$, described by the vector potential $\vec{A} = B\hat{x}\hat{y}$ and the Hamiltonian

$$H_0 = \frac{1}{2m} (\vec{p}_0 - e\vec{A})^2 \quad (4.1.1)$$

$$= \frac{p_{0x}^2}{2m} + \frac{p_{0y}^2}{2m} - \frac{eB}{m} xp_{0y} + \frac{e^2 B^2}{2m} x^2. \quad (4.1.2)$$

Where p_0 is the conjugate momentum in case of the minimal electromagnetic coupling with \vec{A} .

Since p_{0y} commutes with H , replacing it with its eigenvalue $\hbar k$, we get

$$H_0 = \frac{p_{0x}^2}{2m} + \frac{1}{2} m \omega_c^2 \left(x - \frac{\hbar k}{m \omega_c} \right)^2 \quad (4.1.3)$$

where $\omega_c = eB/m$ is the cyclotron frequency. This is nothing but the Hamiltonian of a harmonic oscillator in the x direction, with its equilibrium position given by $x_0 \equiv \hbar k / m \omega_c$. Consequently, the eigenfunctions and eigenvalues are given, respectively, by

$$\psi_{k,n}(x, y) = e^{iky} \phi_n(x - x_0) \quad (4.1.4)$$

$$E_n = \hbar \omega_c \left(n + \frac{1}{2} \right), \quad n \in \mathbb{N} \quad (4.1.5)$$

where ϕ_n are the harmonic oscillator wavefunctions.

The GUP-corrected Hamiltonian assumes the form: [66, 67] [see Appendix C for details].

$$\begin{aligned} H &= \frac{1}{2m} (\vec{p}_0 - e\vec{A})^2 - \frac{\alpha}{m} (\vec{p}_0 - e\vec{A})^3 + \frac{5\alpha^2}{2m} (\vec{p}_0 - e\vec{A})^4 \\ &= H_0 - \sqrt{8m} \alpha H_0^{\frac{3}{2}} + 10 \alpha^2 m H_0^2, \end{aligned} \quad (4.1.6)$$

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where in the last step we have used Eqs. (4.1.1), (3.3.8) and (3.3.9). Evidently, the eigenfunctions remain unchanged. However, the eigenvalues are shifted by

$$\begin{aligned}\Delta E_{n(GUP)} &= \langle \phi_n | -\sqrt{8m} \alpha H_0^{\frac{3}{2}} + 10 \alpha^2 m H_0^2 | \phi_n \rangle \\ &= -\sqrt{8m} \alpha (\hbar\omega_c)^{\frac{3}{2}} \left(n + \frac{1}{2}\right)^{\frac{3}{2}} + 10 m \alpha^2 (\hbar\omega_c)^2 \left(n + \frac{1}{2}\right)^2, \quad (4.1.7)\end{aligned}$$

which can be written as

$$\frac{\Delta E_{n(GUP)}}{E_n^{(0)}} = -\sqrt{8m} \alpha (\hbar\omega_c)^{\frac{1}{2}} \left(n + \frac{1}{2}\right)^{\frac{1}{2}} + 10 m \alpha^2 (\hbar\omega_c) \left(n + \frac{1}{2}\right). \quad (4.1.8)$$

For $n=1$, we obtain the following relation

$$\frac{\Delta E_{1(GUP)}}{E_1^{(0)}} = -\frac{\sqrt{12m} (\hbar\omega_c)^{\frac{1}{2}}}{M_{Pl}c} \alpha_0 + \frac{15 m (\hbar\omega_c)}{M_{Pl}^2 c^2} \alpha_0^2. \quad (4.1.9)$$

For an electron in a magnetic field of $10T$, $\omega_c \approx 10^3$ GHz

$$\frac{\Delta E_{1(GUP)}}{E_1^{(0)}} \approx -10^{-26} \alpha_0 + 10^{-52} \alpha_0^2. \quad (4.1.10)$$

Thus, quantum gravity/GUP does affect the Landau levels. However, assuming $\alpha_0 \sim 1$ renders the correction too small to be measured. Without this assumption, due to an accuracy of one part in 10^3 in direct measurements of Landau levels using a scanning tunnel microscope (*STM*) (which is somewhat optimistic) [79], the upper bound on α_0 becomes

$$\alpha_0 < 10^{23}. \quad (4.1.11)$$

Note that this is more stringent than the one derived in previous works [66, 67].

4.2 Simple harmonic oscillator

We now consider a particle of mass m . The Hamiltonian of the simple harmonic oscillator with the GUP-corrected Hamiltonian assumes, using Eqs. (3.3.8) and (3.3.9), the following form

$$H = H_0 + H_1 = \frac{p_0^2}{2m} + \frac{1}{2}m\omega^2 x^2 - \frac{\alpha}{m}p_0^3 + \frac{5\alpha^2}{2}p_0^4. \quad (4.2.1)$$

Employing time-independent perturbation theory, the eigenvalues are shifted up to the first order of α by

$$\Delta E_{GUP} = \langle \psi_n | H_1 | \psi_n \rangle \quad (4.2.2)$$

where ψ_n are the eigenfunctions of the simple harmonic oscillator and are given by

$$\psi_n(x) = \left(\frac{1}{2^n n!} \right)^{\frac{1}{2}} \left(\frac{m\omega}{\pi \hbar} \right)^{\frac{1}{4}} e^{-\frac{m\omega x^2}{2\hbar}} H_n \left(\sqrt{\frac{m\omega}{\hbar}} x \right) \quad (4.2.3)$$

where

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \quad (4.2.4)$$

are the Hermite polynomials.

The p_0^3 term will not make any contribution to first order because it is an odd function and thus, it gives zero by integrating over a Gaussian integral. On the other hand, the p_0^4 term will make a nonzero contribution to first order. The contribution of the p_0^4 term to first order is given by

$$\Delta E_{0(GUP)}^{(1)} = \frac{5\alpha^2}{2m} \langle \psi_0 | \hbar^4 \frac{d^4}{dx^4} | \psi_0 \rangle \quad (4.2.5)$$

4.2. Simple harmonic oscillator

and thus we get

$$\Delta E_{0(GUP)}^{(1)} = \frac{5\alpha^2 \hbar^4}{2m} \left(\frac{\gamma}{\pi}\right)^{\frac{1}{2}} \gamma^2 \int dx e^{-\gamma x^2} (3 - 6\gamma x^2 + \gamma^2 x^4) \quad (4.2.6)$$

where γ is equal to $\frac{m\omega}{\hbar}$.

By integrating, we get the shift of the energy to first order of perturbation as follows

$$\Delta E_0^{(1)} = \frac{15}{8} \hbar^2 \omega^2 m \alpha^2 \quad (4.2.7)$$

or, equivalently,

$$\frac{\Delta E_0^{(1)}}{E_0^{(0)}} = \frac{15}{4} \hbar \omega m \alpha^2 . \quad (4.2.8)$$

We now compute the contribution of the p_0^3 term to second order of perturbation

$$\Delta E_n^{(2)} = \sum_{k \neq n} \frac{|\langle \psi_k | V_1 | \psi_n \rangle|^2}{E_n^{(0)} - E_k^{(0)}} \quad (4.2.9)$$

where

$$V_1 = i \frac{\alpha}{m} \hbar^3 \frac{d^3}{dx^3} . \quad (4.2.10)$$

In particular, we are interested in computing the shift in the ground state energy to second order

$$\Delta E_0^{(2)} = \sum_{k \neq 0} \frac{|\langle \psi_k | V_1 | \psi_0 \rangle|^2}{E_0^{(0)} - E_k^{(0)}} \quad (4.2.11)$$

and for this reason we employ the following properties of the harmonic oscillator eigenfunctions

$$\langle \psi_m | x | \psi_n \rangle = \begin{cases} 0, & m \neq n \pm 1 \\ \sqrt{\frac{n+1}{2\gamma}}, & m = n + 1 \\ \sqrt{\frac{n}{2\gamma}}, & m = n - 1 \end{cases} \quad (4.2.12)$$

and

$$\langle \psi_m | x^3 | \psi_0 \rangle = \sum_{k,l} \langle \psi_m | x | \psi_k \rangle \langle \psi_k | x | \psi_l \rangle \langle \psi_l | x | \psi_0 \rangle, \quad (4.2.13)$$

4.2. Simple harmonic oscillator

which is nonvanishing for the (l, k, m) triplets: $(1, 0, 1)$, $(1, 2, 1)$, and $(1, 2, 3)$.

Thus, the ground state energy is shifted by

$$\Delta E_0^{(2)} = \frac{\alpha^2 \hbar^6}{m^2} \sum_{m \neq 0} \frac{|\langle \psi_m | \frac{d^3}{dx^3} | \psi_0 \rangle|^2}{E_0^{(0)} - E_m^{(0)}}. \quad (4.2.14)$$

Since $|\psi_0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$, we have $\frac{d^3}{dx^3} |\psi_0\rangle = (3\gamma^2 x - \gamma^3 x^3) |\psi_0\rangle$. By employing these into Eq. (4.2.11), we get

$$\Delta E_0^{(2)} = \frac{\alpha^2 \hbar^6}{m^2} \gamma^4 \sum_{m \neq 0} \frac{|\langle \psi_m | (3x - \gamma x^3) | \psi_0 \rangle|^2}{E_0^{(0)} - E_m^{(0)}}. \quad (4.2.15)$$

Using Eqs. (4.2.12) and (4.2.13), the energy shift finally takes the form

$$\Delta E_0^{(2)} = -\frac{11}{2} \alpha^2 m \left(E_0^{(0)}\right)^2, \quad (4.2.16)$$

or, equivalently,

$$\frac{\Delta E_0^{(2)}}{E_0^{(0)}} = -\frac{11}{2} \alpha^2 m E_0^{(0)} = -\frac{11}{4} \alpha^2 m \hbar \omega. \quad (4.2.17)$$

It is noteworthy that there are some systems that can be represented by the harmonic oscillator such as heavy meson systems like charmonium [80]. The charm mass is $m_c \approx 1.3 \text{ GeV}/c^2$ and the binding energy ω of the system is roughly equal to the energy gap separating adjacent levels and is given by $\hbar\omega \approx 0.3 \text{ GeV}$. The correction due to GUP can be calculated at the second order of α . Using Eqs. (4.2.8) and (4.2.17), we found the shift in energy is given by

$$\frac{\Delta E_0^{(2)}}{E_0^{(0)}} = \alpha_0^2 \frac{m \hbar \omega}{M_{Pl}^2 c^2} \approx 2.7 \times 10^{-39} \alpha_0^2. \quad (4.2.18)$$

Once again, assuming $\alpha_0 \sim 1$ renders the correction too small to be measured. On the other hand, if such an assumption is not made, the current accuracy of precision

4.3. *The Lamb shift*

measurement in the case of J/ψ [81] is at the level of 10^{-5} . This sets the upper bound on α_0 to be

$$\alpha_0 < 10^{17}. \quad (4.2.19)$$

It should be stressed that this bound is in fact consistent with that set by the electroweak scale. Therefore, it could signal a new and intermediate length scale between the electroweak and the Planck scale.

4.3 The Lamb shift

For the hydrogen atom, $V(\vec{r}) = -k/r$ ($k = e^2/4\pi\epsilon_0 = \alpha\hbar c$, $e =$ electronic charge).

To first order, the perturbing Hamiltonian H_1 , shifts the wavefunctions to [82]

$$|\psi_{nlm}\rangle_1 = |\psi_{nlm}\rangle + \sum_{\{n'l'm'\} \neq \{nlm\}} \frac{e_{n'l'm'|nlm}}{E_n^{(0)} - E_{n'}^{(0)}} |\psi_{n'l'm'}\rangle, \quad (4.3.1)$$

where n, l, m have their usual significance, and $e_{n'l'm'|nlm} \equiv \langle \psi_{n'l'm'} | H_1 | \psi_{nlm} \rangle$. Using the expression $p_0^2 = 2m[H_0 + k/r]$ [20], the perturbing Hamiltonian reads

$$H_1 = -(\alpha\sqrt{8m}) \left[H_0 + \frac{k}{r} \right] \left[H_0 + \frac{k}{r} \right]^{\frac{1}{2}}. \quad (4.3.2)$$

So for GUP effect to α order, we have

$$e_{n'l'm'|nlm} = \langle \psi_{n'l'm'} | \left(-\frac{\alpha}{m} \right) p_0^2 p_0 | \psi_{nlm} \rangle. \quad (4.3.3)$$

It follows from the orthogonality of spherical harmonics that the above are non-vanishing if and only if $l' = l$ and $m' = m$

$$e_{200|100} = 2i\alpha\hbar \langle \psi_{200} | \left[H_0 + \frac{k}{r} \right] \left(\frac{\partial}{\partial r} \right) | \psi_{100} \rangle. \quad (4.3.4)$$

We utilize the following to calculate the shift in the energy:

4.3. The Lamb shift

1. the first term in the sum in Eq. (4.3.1) ($n' = 2$) dominates, since $E_n = -E_0/n^2$ ($E_0 = e^2/8\pi\epsilon_0 a_0 = k/2a_0 = 13.6 \text{ eV}$, $a_0 = 4\pi\epsilon_0\hbar^2/me^2 = 5.3 \times 10^{-11} \text{ metre}$, $m = \text{electron mass} = 0.5 \text{ MeV}/c^2$),
2. $\psi_{nlm}(\vec{r}) = R_{nl}(r)Y_{lm}(\theta, \phi)$,
3. $R_{10} = 2a_0^{-3/2}e^{-r/a_0}$, and $R_{20} = (2a_0)^{-3/2}(2 - r/a_0)e^{-r/2a_0}$,
4. $Y_{00}(\theta, \phi) = 1/(\sqrt{4\pi})$.

Thus, we get

$$e_{200|100} = -i\frac{2\alpha\hbar k}{a_0} \langle \psi_{200} | \frac{1}{r} | \psi_{100} \rangle = -i\frac{8\sqrt{2}\alpha\hbar k}{27a_0^2}. \quad (4.3.5)$$

Therefore, the first order shift in the ground state wavefunction is given by (in the position representation)

$$\begin{aligned} \Delta\psi_{100}(\vec{r}) &\equiv \psi_{100}^{(1)}(\vec{r}) - \psi_{100}^{(0)}(\vec{r}) = \frac{e_{200|100}}{E_1 - E_2} \psi_{200}(\vec{r}) \\ &= i\frac{32\sqrt{2}\alpha\hbar k}{81a_0^2 E_0} \psi_{200}(\vec{r}) = i\frac{64\sqrt{2}\alpha\hbar}{81a_0} \psi_{200}(\vec{r}). \end{aligned} \quad (4.3.6)$$

Next, we consider the Lamb shift for the n^{th} level of the hydrogen atom [83]

$$\Delta E_n^{(1)} = \frac{4\alpha^2}{3m^2} \left(\ln \frac{1}{\alpha} \right) |\psi_{nlm}(0)|^2. \quad (4.3.7)$$

Varying $\psi_{nlm}(0)$, the additional contribution due to GUP in proportion to its original value is given by

$$\frac{\Delta E_{n(GUP)}^{(1)}}{\Delta E_n^{(1)}} = 2 \frac{\Delta |\psi_{nlm}(0)|}{\psi_{nlm}(0)}. \quad (4.3.8)$$

Thus, for the ground state, we obtain

$$\frac{\Delta E_{1(GUP)}^{(1)}}{\Delta E_1^{(1)}} = \frac{64\hbar \alpha_0}{81a_0 M_{pl}c} \approx 1.2 \times 10^{-22} \alpha_0. \quad (4.3.9)$$

4.4. *Potential step*

The above result may be interpreted in two ways. First, if one assumes $\alpha_0 \sim 1$, then it predicts a nonzero, but virtually *unmeasurable* effect of GUP and thus of quantum gravity. On the other hand, if such an assumption is not made, the current accuracy of precision measurement of Lamb shift of about one part in 10^{12} [20, 84], sets the following upper bound on α_0 :

$$\alpha_0 < 10^{10}. \quad (4.3.10)$$

It should be stressed that this bound is more stringent than the ones derived in previous examples [66, 67], and is in fact consistent with that set by the electroweak scale. Therefore, it could signal a new and intermediate length scale between the electroweak and the Planck scale.

4.4 Potential step

Next, we study the one-dimensional potential step given by

$$V'(x) = V'_0 \theta(x) \quad (4.4.1)$$

where $\theta(x)$ is the usual step function. Assuming $E < V'_0$, the GUP-corrected Schrödinger equations to the left and right of the barrier are written, respectively, as

$$d^2\psi_{<} + k^2\psi_{<} + 2i\alpha\hbar d^3\psi_{<} = 0 \quad (4.4.2)$$

$$d^2\psi_{>} - k_1^2\psi_{>} + 2i\alpha\hbar d^3\psi_{>} = 0 \quad (4.4.3)$$

where $k = \sqrt{2mE/\hbar^2}$ and $k_1 = \sqrt{2m(V'_0 - E)/\hbar^2}$.

Considering solutions of the form $\psi_{<,>} = e^{mx}$, we get

$$m^2 + k^2 + 2i\alpha\hbar m^3 = 0 \quad (4.4.4)$$

$$m^2 - k_1^2 + 2i\alpha\hbar m^3 = 0 \quad (4.4.5)$$

4.4. *Potential step*

with the following solution sets to leading order in α , each consisting of three values of m

$$x < 0 : m = \left\{ ik', -ik'', \frac{i}{2\alpha\hbar} \right\} \quad (4.4.6)$$

$$x \geq 0 : m = \left\{ k'_1, -k''_1, \frac{i}{2\alpha\hbar} \right\} \quad (4.4.7)$$

where

$$k' = k(1 + k\alpha\hbar), \quad k'' = k(1 - k\alpha\hbar) \quad (4.4.8)$$

$$k'_1 = k_1(1 - i\alpha\hbar k_1), \quad k''_1 = k_1(1 + i\alpha\hbar k_1) . \quad (4.4.9)$$

Therefore, the wavefunctions take the form

$$\psi_{<} = Ae^{ik'x} + Be^{-ik''x} + Ce^{\frac{ix}{2\alpha\hbar}}, \quad x < 0 \quad (4.4.10)$$

$$\psi_{>} = De^{-k'_1x} + Ee^{\frac{ix}{2\alpha\hbar}}, \quad 0 \leq x \quad (4.4.11)$$

where we have omitted the left mover from $\psi_{>}$.

Now the boundary conditions at $x = 0$ consist of three equations (instead of the usual two)

$$d^n \psi_{<}|_0 = d^n \psi_{>}|_0, \quad n = 0, 1, 2 . \quad (4.4.12)$$

This leads to the following conditions:

$$A + B + C = D + E \quad (4.4.13)$$

$$i(k'A - k''B + \frac{C}{2\alpha\hbar}) = -k''_1 D + \frac{iE}{2\alpha\hbar} \quad (4.4.14)$$

$$k'^2 A + k''^2 B + \frac{C}{(2\alpha\hbar)^2} = \frac{E}{(2\alpha\hbar)^2} - k''^2_1 D . \quad (4.4.15)$$

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Assuming $C \sim E \sim \mathcal{O}(\alpha^2)$, we have the following solutions to leading order in α

$$\frac{B}{A} = \frac{ik' + k_1''}{ik'' - k_1''}, \quad (4.4.16)$$

$$\frac{D}{A} = \frac{2ik}{ik'' - k_1''}, \quad (4.4.17)$$

$$\frac{E - C}{(2\alpha\hbar)^2 A} = \frac{k'^2(ik'' - k_1'') + k''^2(ik' + k_1'') + k_1''^2(2ik)}{ik'' - k_1''}. \quad (4.4.18)$$

It can be easily shown that the GUP-corrected time-dependent Schrödinger equation admits the following modified conserved current density, charge density and conservation law, respectively [66, 67]:

$$J = \frac{\hbar}{2mi} \left(\psi^* \frac{d\psi}{dx} - \psi \frac{d\psi^*}{dx} \right) + \frac{\alpha\hbar^2}{m} \left(\frac{d^2|\psi|^2}{dx^2} - 3 \frac{d\psi}{dx} \frac{d\psi^*}{dx} \right), \quad (4.4.19)$$

$$\rho = |\psi|^2, \quad \frac{\partial J}{\partial x} + \frac{\partial \rho}{\partial t} = 0. \quad (4.4.20)$$

The conserved current is given as

$$J = J_0 + J_1 = \frac{\hbar k}{m} (|A|^2 - |B|^2) + \frac{2\alpha\hbar^2 k^2}{m} (|A|^2 + |B|^2) + \frac{|C|^2}{\alpha m}. \quad (4.4.21)$$

The reflection and transmission coefficients are given by

$$\begin{aligned}
 R &= \left| \frac{B}{A} \right|^2 \frac{1 - 2\alpha\hbar k}{1 + 2\alpha\hbar k} \\
 &= \left| \frac{ik' + k_1''}{ik'' - k_1''} \right|^2 \frac{1 - 2\alpha\hbar k}{1 + 2\alpha\hbar k} \\
 &= \frac{(k^2 + k_1^2)^2}{(k_1^2 + k^2)^2} \frac{1 - 2\alpha\hbar k}{1 + 2\alpha\hbar k} \\
 &\cong 1.
 \end{aligned} \tag{4.4.22}$$

$$T = \frac{-\frac{\alpha\hbar^2 k_1^2}{m} |D|^2 e^{-2k_1 x} + \frac{\alpha\hbar^2 k_1^2}{m} |D|^2 e^{-2k_1 x}}{\frac{\hbar k}{m} |A|^2 (1 + 2\alpha\hbar k)}, \tag{4.4.23}$$

$$= 0, \tag{4.4.24}$$

$$R + T = 1. \tag{4.4.25}$$

At this point we should note that GUP did not affect R and T up to $\mathcal{O}(\alpha)$.

4.5 Potential barrier

In this section we apply the above formalism to an STM and show that in an optimistic scenario, the effect of the GUP-induced term may be measurable. In an STM, free electrons of energy E (close to the Fermi energy) from a metal tip at $x = 0$, tunnel quantum mechanically to a sample surface a small distance away at $x = a$. This gap (across which a bias voltage may be applied) is associated with a potential barrier of height $V_0'' > E$ [85]. Thus

$$V''(x) = V_0'' [\theta(x) - \theta(x - a)] \tag{4.5.1}$$

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where $\theta(x)$ is the usual step function. The wave functions for the three regions, namely, $x \leq 0$, $0 \leq x \leq a$, and $x \geq a$, are ψ_1, ψ_2 , and ψ_3 , respectively, and satisfy the GUP-corrected time-independent Schrödinger equation

$$d^2\psi_{1,3} + k^2\psi_{1,3} + 2i\alpha\hbar d^3\psi_{1,3} = 0$$

$$d^2\psi_2 - k_1^2\psi_2 + 2i\alpha\hbar d^3\psi_2 = 0$$

where $k = \sqrt{2mE/\hbar^2}$ and $k_1 = \sqrt{2m(V_0'' - E)/\hbar^2}$.

The solutions to the aforementioned equations to leading order in α are

$$\psi_1 = Ae^{ik'x} + Be^{-ik''x} + Pe^{ix/2\alpha\hbar} \quad (4.5.2)$$

$$\psi_2 = Fe^{k_1'x} + Ge^{-k_1''x} + Qe^{ix/2\alpha\hbar} \quad (4.5.3)$$

$$\psi_3 = Ce^{ik'x} + Re^{ix/2\alpha\hbar} \quad (4.5.4)$$

where $k' = k(1 + \alpha\hbar k)$, $k'' = k(1 - \alpha\hbar k)$, $k_1' = k_1(1 - i\alpha\hbar k_1)$, $k_1'' = k_1(1 + i\alpha\hbar k_1)$ and A, B, C, F, G, P, Q, R are constants of integration. In the above, we have omitted the left mover from ψ_3 . Note the appearance of the new oscillatory terms with characteristic wavelengths $\sim \alpha\hbar$, due to the third order modification of the Schrödinger equation. The boundary conditions at $x = 0, a$ are given by

$$d^n\psi_1|_{x=0} = d^n\psi_2|_{x=0} \quad , \quad n = 0, 1, 2 \quad (4.5.5)$$

$$d^n\psi_2|_{x=a} = d^n\psi_3|_{x=a} \quad , \quad n = 0, 1, 2 \quad (4.5.6)$$

If we assume that $P \sim Q \sim R \sim \mathcal{O}(\alpha^2)$, we get the following solutions:

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$$\frac{C}{A} = \frac{i(k'k_1'' + k''k_1' + k'k_1' + k''k_1'')e^{-ik'a+k_1''a}}{e^{(k_1'+k_1'')a}(k' + ik_1')(k'' + ik_1'') - (k' - ik_1')(k'' - ik_1'')}, \quad (4.5.7)$$

$$\frac{B}{A} = \frac{k_1'' + ik'}{k_1'' - ik''} \left[e^{ik'a-k_1'a} \frac{C}{A} - 1 \right], \quad (4.5.8)$$

$$\frac{F}{A} = \frac{(1 + i\frac{k'}{k_1''})e^{ik'a-k_1'a} \frac{C}{A}}{1 + \frac{k_1'}{k_1''}}, \quad (4.5.9)$$

$$\frac{G}{A} = \frac{(1 - i\frac{k'}{k_1'})e^{ik'a+k_1''a} \frac{C}{A}}{1 + \frac{k_1''}{k_1'}}. \quad (4.5.10)$$

From Eq. (5.2.11), it follows that the transmission coefficient of the STM, given by the ratio of the right moving currents to the right and left of the barrier, namely, J_R and J_L , respectively, is to $\mathcal{O}(\alpha)$

$$T = \frac{J_R}{J_L} = \left| \frac{C}{A} \right|^2 - 2\alpha\hbar k \left| \frac{B}{A} \right|^2 \quad (4.5.11)$$

which gives using the solutions in Eqs. (4.5.7) and (4.5.8) the following final expression

$$T = T_0 [1 + 2\alpha\hbar k(1 - T_0^{-1})] \quad (4.5.12)$$

$$T_0 = \frac{16E(V_0'' - E)}{V_0''^2} e^{-2k_1a} \quad (4.5.13)$$

where T_0 is the standard STM transmission coefficient. The measured tunneling current is proportional to T (usually magnified by a factor \mathcal{G}), and using the following approximate (but realistic) values [85]

$$m = m_e = 0.5 \text{ MeV}/c^2, \quad E \approx V_0'' = 10 \text{ eV}$$

$$a = 10^{-10} \text{ m}, \quad I_0 = 10^{-9} \text{ A}, \quad \mathcal{G} = 10^9$$

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we get

$$\begin{aligned}\frac{\delta I}{I_0} &= \frac{\delta T}{T_0} = 10^{-26}, \\ \delta \mathcal{I} &\equiv \mathcal{G} \delta I = 10^{-26} A\end{aligned}\tag{4.5.14}$$

where we have chosen $\alpha_0 = 1$ and $T_0 = 10^{-3}$, also a fairly typical value. Thus, for the GUP-induced excess current $\delta \mathcal{I}$ to give the difference of the charge of just one electron, $e \simeq 10^{-19} C$, one would have to wait for a time

$$\tau = \frac{e}{\delta \mathcal{I}} = 10^7 s\tag{4.5.15}$$

or, equivalently, about 4 months, which can perhaps be argued to be not that long. We have chosen α_0 to be unity for simplicity, and demonstrated that the time τ for Planck scale effects to show up is not unreasonably long. We agree however, that it is still long compared to typical STM running times (a few hours). Nevertheless, for larger values of α_0 , τ may indeed be reduced to these time scales. In fact, higher values of α_0 and a more accurate estimate will likely reduce this time, and conversely, current studies may already be able to put an upper bound on α_0 .

What is perhaps more interesting is the following relation between the *apparent barrier height* $\Phi_A \equiv V_0'' - E$ and the (logarithmic) rate of increase of current with the gap, which follows from Eq. (4.5.12):

$$\sqrt{\Phi_A} = \frac{\hbar}{\sqrt{8m}} \left| \frac{d \ln I}{da} \right| - \frac{\alpha \hbar^2 (k^2 + k_1^2)^2}{8m(kk_1)} e^{2k_1 a} .\tag{4.5.16}$$

Note the GUP-induced deviation from the usual linear $\sqrt{\Phi_A}$ vs $|d \ln I/da|$ curve. The exponential factor makes this particularly sensitive to changes in the tip-sample distance a , and hence amenable to observations. Any such observed deviation may signal the existence of GUP and, thus, in turn an underlying theory of quantum gravity.

In this chapter we have investigated the consequences of quantum gravitational corrections to various quantum phenomena such as the Landau levels, simple harmonic oscillator, the Lamb shift, and the tunneling current in a scanning tunneling microscope and have found that the upper bounds on α_0 to be 10^{23} , 10^{17} , and 10^{10} from the first three respectively. The first one gives a length scale bigger than electroweak length that is not right experimentally. It should be stressed that the last three bounds are more stringent than the ones derived in the previous study [66,67], and might be consistent with that set by the electroweak scale. Therefore, it could signal a new and intermediate length scale between the electroweak and the Planck scale. On the other side, we have found that even if $\alpha_0 \sim 1$, we still might measure quantum gravitational corrections in a scanning tunneling microscopic case as was shown in Eq. (6.2.1). This is in fact an improvement over the general conclusion of [66,67], where it was shown that quantum gravitational effects are virtually negligible if the GUP parameter $\beta_0 \sim 1$, and appears to be a new and interesting result. It would also be interesting to apply our formalism to other areas including cosmology, black hole physics and Hawking radiation, selection rules in quantum mechanics, statistical mechanical systems etc. We will report some of these problems in the subsequent chapters.

Chapter 5

Is space fundamentally discrete?

We have proposed a GUP consistent with String Theory, Doubly Special Relativity and black hole physics, and have shown that this modifies all quantum mechanical Hamiltonians. In this chapter we will show that when applied to an elementary particle, it suggests that the space which confines it must be quantized. This suggests that space fundamentally is discrete, and that all measurable lengths are quantized in units of a fundamental length (which can be the Planck length) [21,22].

5.1 Discreteness of space in the non-relativistic case

In this section, we review our work in [21, 23]. We apply our proposed GUP to a single particle in a box of length L (with boundaries at $x = 0$ and $x = L$) in the nonrelativistic case to order of α and α^2 , and show that the box length must be quantized. Since this particle can be considered as a *test particle* to measure the dimension of the box, this suggests that space itself is quantized, as are all observable lengths.

5.1.1 Solution to order α

The wave function of the particle satisfies the following GUP-corrected Schrödinger equation inside the box of length L (with boundaries at $x = 0$ and $x = L$), where $V(\vec{r}) = 0$ (outside, $V = \infty$ and $\psi = 0$):

$$H\psi = E\psi, \quad (5.1.1)$$

is now written, to order α using Eqs. (3.3.8) and (3.3.9), as

$$d^2\psi + k^2\psi + 2i\alpha\hbar d^3\psi = 0, \quad (5.1.2)$$

where d^n stands for d^n/dx^n and $k = \sqrt{2mE/\hbar^2}$. A trial solution of the form $\psi = e^{mx}$ yields

$$m^2 + k^2 + 2i\alpha\hbar m^3 = 0, \quad (5.1.3)$$

with the following solution set to leading order in α : $m = \{ik', -ik'', i/2\alpha\hbar\}$, where $k' = k(1 + k\alpha\hbar)$ and $k'' = k(1 - k\alpha\hbar)$. Thus, the general wavefunction to leading order in ℓ_{Pl} and α is of the form

$$\psi = Ae^{ik'x} + Be^{-ik''x} + Ce^{ix/2\alpha\hbar}. \quad (5.1.4)$$

As is well known, the first two terms (with $k' = k'' = k$) and the boundary conditions $\psi = 0$ at $x = 0, L$ give rise to the standard quantization of energy for a particle in a box, namely $E_n = n^2\pi^2\hbar^2/2mL^2$. However, note the appearance of a new oscillatory term here, with characteristic wavelength $4\pi\alpha\hbar$ and momentum $1/4\alpha = M_{Pl}c/4\alpha_0$ (which corresponds to the Planck scale for $\alpha_0 = \mathcal{O}(1)$). This results in the new quantization mentioned above. Also, as this term should drop out in the $\alpha \rightarrow 0$ limit, one must have $\lim_{\alpha \rightarrow 0} |C| = 0$. We absorb any phase of A in ψ ,

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such that A is real. The boundary condition

$$\psi(0) = 0, \quad (5.1.5)$$

implies

$$A + B + C = 0. \quad (5.1.6)$$

Substituting for B in Eq. (5.1.4), we get

$$\begin{aligned} \psi &= 2iA \sin(kx) + C [-e^{-ikx} + e^{ix/2\alpha\hbar}] \\ &- \alpha\hbar k^2 x [i C e^{-ikx} + 2A \sin(kx)]. \end{aligned} \quad (5.1.7)$$

The remaining boundary condition

$$\psi(L) = 0, \quad (5.1.8)$$

yields

$$\begin{aligned} 2iA \sin(kL) &= |C| [e^{-i(kL+\theta_C)} - e^{i(L/2\alpha\hbar-\theta_C)}] \\ &+ \alpha\hbar k^2 L [i |C| e^{-i(kL+\theta_C)} + 2A \sin(kL)]. \end{aligned} \quad (5.1.9)$$

where $C = |C| \exp(-i\theta_C)$. Note that both sides of the above equation vanish in the limit $\alpha \rightarrow 0$, when $kL = n\pi$ ($n \in \mathbb{Z}$) and $C = 0$. Thus, when $\alpha \neq 0$, we must have $kL = n\pi + \epsilon$, where $\epsilon \in \mathbb{R}$ (such that energy eigenvalues E_n remain positive), and $\lim_{\alpha \rightarrow 0} \epsilon = 0$. This, along with the previously discussed smallness of $|C|$ ensures that the second line in Eq. (5.1.9) above falls off faster than $\mathcal{O}(\alpha)$, and hence can be dropped. Next, equating the real parts of the remaining terms of Eq. (5.1.9) (remembering that $A \in \mathbb{R}$), we get

$$\cos\left(\frac{L}{2\alpha\hbar} - \theta_C\right) = \cos(kL + \theta_C) = \cos(n\pi + \theta_C + \epsilon), \quad (5.1.10)$$

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which implies, to leading order, the following two series of solutions

$$\frac{L}{2\alpha\hbar} = \frac{L}{2\alpha_0\ell_{Pl}} = n\pi + 2q\pi + 2\theta_C \equiv p\pi + 2\theta_C, \quad (5.1.11)$$

$$\frac{L}{2\alpha\hbar} = \frac{L}{2\alpha_0\ell_{Pl}} = -n\pi + 2q\pi \equiv p\pi, \quad p \equiv 2q \pm n \in \mathbb{N}. \quad (5.1.12)$$

These show that there cannot even be a single particle in the box, unless its length is quantized as above. For other lengths, there is no way to probe or measure the box, even if it exists. Hence, effectively all measurable lengths are quantized in units of $\alpha_0\ell_{Pl}$. We interpret this as space essentially having a discrete nature. Note that the above conclusion holds for any unknown but *fixed* θ_C , which, however, determines the minimum measurable length, if any. It is hoped that additional physically motivated or consistency conditions will eventually allow one to either determine or at least put reasonable bounds on it.

The minimum length is $\approx \alpha_0\ell_{Pl}$ in each case. Once again, if $\alpha_0 \approx 1$, this fundamental unit is the Planck length. However, current experiments do not rule out discreteness smaller than about a thousandth of a Fermi, thus predicting the previously mentioned bound on α_0 . Note that similar quantization of length was shown in the context of loop quantum gravity in [44, 86, 87].

5.1.2 Solution to order α^2

We extend the previous solution to include the α^2 term in one dimension. Working to $\mathcal{O}(\alpha^2)$, the magnitude of the momentum at high energies as given by Eq. (3.3.5) reads

$$p = p_0(1 - \alpha p_0 + 2\alpha^2 p_0^2). \quad (5.1.13)$$

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The wavefunction satisfies the following GUP-corrected Schrödinger equation

$$d^2\psi + k^2\psi + 2i\hbar\alpha d^3\psi - 5\hbar^2\alpha^2 d^4\psi = 0, \quad (5.1.14)$$

where $k = \sqrt{2mE/\hbar^2}$ and $d^n \equiv d^n/dx^n$.

Substituting $\psi(x) = e^{mx}$, we obtain

$$m^2 + k^2 + 2i\alpha\hbar m^3 - 5(\alpha\hbar)^2 m^4 = 0, \quad (5.1.15)$$

with the following solution set to leading order in α^2 : $m = \{ik', -ik'', \frac{2+i}{5\alpha\hbar}, \frac{-2+i}{5\alpha\hbar}\}$, where $k' = k(1 + k\alpha\hbar)$ and $k'' = k(1 - k\alpha\hbar)$. Thus, the most general solution to leading order in ℓ_{Pl}^2 and α^2 is of the form

$$\psi(x) = Ae^{ik'x} + Be^{-ik''x} + Ce^{(2+i)x/5\alpha\hbar} + De^{(-2+i)x/5\alpha\hbar}. \quad (5.1.16)$$

Note again the appearance of new oscillatory terms, with characteristic wavelength $10\pi\alpha\hbar$, which as before, by virtue of C and D scaling as a power of α , disappear in the $\alpha \rightarrow 0$ limit. In addition, we absorb any phase of A in ψ that A is real. The boundary condition

$$\psi(0) = 0, \quad (5.1.17)$$

implies

$$A + B + C + D = 0, \quad (5.1.18)$$

and hence the general solution given in Eq. (5.1.16) becomes

$$\psi(x) = 2iA \sin(kx)e^{i\alpha k^2 \hbar x} - (C + D)e^{-ik''x} + e^{\frac{ix}{5\alpha\hbar}} [Ce^{\frac{2x}{5\alpha\hbar}} + De^{\frac{-2x}{5\alpha\hbar}}]. \quad (5.1.19)$$

If we now combine Eq. (5.1.19) and the remaining boundary condition

$$\psi(L) = 0, \quad (5.1.20)$$

5.2. Discreteness of space in the non-relativistic case

we get

$$2iA \sin(kL) = (C + D)e^{-i[\alpha k^2 \hbar L + k''L]} - \left[Ce^{\frac{2L}{5\alpha\hbar}} + De^{\frac{-2L}{5\alpha\hbar}} \right] e^{\frac{iL}{5\alpha\hbar}} e^{-i\alpha k^2 \hbar L}. \quad (5.1.21)$$

We can consider the exponentials $e^{-i\alpha k^2 \hbar L} \approx 1$, otherwise, since they are multiplied with C or D , terms of higher order in α will appear. Therefore, we have ($C = |C|e^{-i\theta_C}$, $D = |D|e^{-i\theta_D}$)

$$\begin{aligned} 2iA \sin(kL) &= \left[|C|e^{-i\theta_C} + |D|e^{-i\theta_D} \right] e^{-ikL} \\ &- \left[|C|e^{-i\theta_C} e^{\frac{2L}{5\alpha\hbar}} + |D|e^{-i\theta_D} e^{\frac{-2L}{5\alpha\hbar}} \right] e^{\frac{iL}{5\alpha\hbar}}. \end{aligned} \quad (5.1.22)$$

Now, equating the real parts of Eq. (5.1.22) (remembering that $A \in \mathbb{R}$), we have

$$\begin{aligned} 0 &= |C| \cos(\theta_C + kL) + |D| \cos(\theta_D + kL) \\ &- e^{\frac{2L}{5\alpha\hbar}} |C| \cos(\theta_C - \frac{L}{5\alpha\hbar}) - e^{\frac{-2L}{5\alpha\hbar}} |D| \cos(\theta_D - \frac{L}{5\alpha\hbar}). \end{aligned} \quad (5.1.23)$$

Note that the third term in the right hand side dominates over the other terms in the limit $\alpha \rightarrow 0$. Thus we arrive at the following equation to leading order

$$\cos(L/5\alpha\hbar - \theta_C) = 0. \quad (5.1.24)$$

This implies the quantization of space by the following equation

$$\frac{L}{5\alpha\hbar} = (2p + 1)\frac{\pi}{2} + \theta_C, \quad p \in \mathbb{N}. \quad (5.1.25)$$

Once again, even though the α^2 term has been included, the space quantization given in Eq. (5.1.25) suggests that the dimension of the box, and hence all measurable lengths are quantized in units of $\alpha_0 \ell_{Pl}$, and if $\alpha_0 \approx 1$, this fundamental unit is of the order of Planck length. And as before, the yet undetermined constant θ_C determines the minimum measurable length.

5.2 Discreteness of space in the relativistic case

In this section, we re-examine the aforementioned problem, but now assume that the particle is relativistic [22]. This we believe is important for several reasons, among which are that extreme high energy (ultra)-relativistic particles are natural candidates for probing the nature of spacetime near the Planck scale, and that most elementary particles in nature are fermions, obeying some form of the Dirac equation. Furthermore, as seen from below, attempts to extend our results to 2 and 3 dimensions seem to necessitate the use of matrices. However, we first start by examining the simpler Klein–Gordon equation.

5.2.1 Klein–Gordon equation in one dimension

The Klein–Gordon (KG) equation in 1-spatial dimension is

$$p^2\Phi(t, x) = \left(\frac{E^2}{c^2} - m^2c^2 \right) \Phi(t, x). \quad (5.2.1)$$

We see that this is identical to the Schrödinger equation, when one makes the identification: $2mE/\hbar^2 \equiv k^2 \rightarrow E^2/\hbar^2c^2 - m^2c^2/\hbar^2$. As a result, the quantization of length, which does not depend on k , continues to hold [21, 23].

However, in addition to fermions being the most fundamental entities, the 3-dimensional version of KG equation (5.2.1), when combined with Eq. (3.3.5), suffers from the drawback that the p^2 term translates to $p^2 = p_0^2 - 2\alpha p_0^3 + \mathcal{O}(\alpha^2) = -\hbar^2\nabla^2 + i2\alpha\hbar^3\nabla^{3/2} + \mathcal{O}(\alpha^2)$, of which the second term is evidently non-local. As we shall see in the next section, the Dirac equation can address both issues at once.

5.2.2 Dirac equation in one dimension

First we linearize $p_0 = \sqrt{p_{0x}^2 + p_{0y}^2 + p_{0z}^2}$ using the Dirac prescription, i.e. replace $p_0 \rightarrow \vec{\alpha} \cdot \vec{p}_0$, where α_i ($i = 1, 2, 3$) and β are the Dirac matrices, for which we use the following representation

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \quad (5.2.2)$$

The GUP-corrected Dirac equation can thus be written to $\mathcal{O}(\alpha)$ as¹

$$\begin{aligned} H\psi &= (c \vec{\alpha} \cdot \vec{p} + \beta mc^2) \psi(\vec{r}) \\ &= (c \vec{\alpha} \cdot \vec{p}_0 - c \alpha (\vec{\alpha} \cdot \vec{p}_0)(\vec{\alpha} \cdot \vec{p}_0) + \beta mc^2) \psi(\vec{r}) \\ &= E\psi(\vec{r}) \end{aligned} \quad (5.2.3)$$

which for 1-spatial dimension, say z , is in the position representation

$$\left(-i\hbar c \alpha_z \frac{d}{dz} + c \alpha \hbar^2 \frac{d^2}{dz^2} + \beta mc^2 \right) \psi(z) = E\psi(z). \quad (5.2.4)$$

Note that this is a second order differential equation instead of the usual first order Dirac equation (we have used $\alpha_z^2 = 1$). Thus, it has two linearly independent, positive energy solutions, which to $\mathcal{O}(\alpha)$ are

$$\psi_1 = N_1 e^{ikz} \begin{pmatrix} \chi \\ r\sigma_z\chi \end{pmatrix} \quad (5.2.5)$$

$$\psi_2 = N_2 e^{i\frac{z}{\alpha\hbar}} \begin{pmatrix} \chi \\ \sigma_z\chi \end{pmatrix} \quad (5.2.6)$$

¹In this subsection, we closely follow the formulation of [88].

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where m is the mass of the Dirac particle, $k = k_0 + \alpha \hbar k_0^2$, k_0 satisfies the usual dispersion relation $E^2 = (\hbar k_0 c)^2 + (mc^2)^2$, $r \equiv \frac{\hbar k_0 c}{E + mc^2}$ and $\chi^\dagger \chi = I$. Note that r runs from 0 (non-relativistic) to 1 (ultra-relativistic). k, k_0 could be positive (right moving) or negative (left moving). N_1, N_2 are suitable normalization constants. As in the case of the Schrödinger equation, here too a new *non-perturbative* solution ψ_2 appears, which should drop out in the $\alpha \rightarrow 0$ (i.e no GUP) limit. This has a characteristic wavelength $2\pi a \hbar$.

As noted in [88], to confine a relativistic particle in a box of length L in a consistent way avoiding the Klein paradox (in which an increasing number of negative energy particles are excited), one may take its mass to be z -dependent as was done in the MIT bag model of quark confinement:

$$\begin{aligned} m(z) &= M, \quad z < 0 \quad (\text{Region I}) \\ &= m, \quad 0 \leq z \leq L \quad (\text{Region II}) \\ &= M, \quad z > L \quad (\text{Region III}), \end{aligned} \tag{5.2.7}$$

where m and M are constants and we will eventually take the limit $M \rightarrow \infty$. Thus, we can write the general wavefunctions in the three regions

$$\psi_I = A e^{-iKz} \begin{pmatrix} \chi \\ -R\sigma_z \chi \end{pmatrix} + G e^{i\frac{z}{\alpha \hbar}} \begin{pmatrix} \chi \\ \sigma_z \chi \end{pmatrix} \tag{5.2.8}$$

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$$\begin{aligned} \psi_{II} = & B e^{ikz} \begin{pmatrix} \chi \\ r\sigma_z\chi \end{pmatrix} + C e^{-ikz} \begin{pmatrix} \chi \\ -r\sigma_z\chi \end{pmatrix} \\ & + F e^{i\frac{z}{\alpha\hbar}} \begin{pmatrix} \chi \\ \sigma_z\chi \end{pmatrix} \end{aligned} \quad (5.2.9)$$

$$\psi_{III} = D e^{iKz} \begin{pmatrix} \chi \\ R\sigma_z\chi \end{pmatrix} + H e^{i\frac{z}{\alpha\hbar}} \begin{pmatrix} \chi \\ \sigma_z\chi \end{pmatrix}, \quad (5.2.10)$$

where $E^2 = (\hbar K_0 c)^2 + (Mc^2)^2$, $K = K_0 + \alpha\hbar K_0^2$ and $R = \hbar K_0 c / (E + Mc^2)$. Thus, in the limit $M \rightarrow \infty$, $K \rightarrow +i\infty$, the terms associated with A and D go to zero. However, those with G and H do not. Moreover, it can be shown that the fluxes due to these terms do not vanish. Thus, we must set $G = 0 = H$. In addition, without loss of generality we choose $B = 1$ and $C = e^{i\delta}$ where δ is a real number. It can be shown that if one chooses $|C| \neq 1$ then the energy of the relativistic particle is complex. Finally, we must have $F \sim \alpha^s$, $s > 0$, such that this term goes to zero in the $\alpha \rightarrow 0$ limit. Now, boundary conditions akin to that for the Schrödinger equation, namely $\psi_{II} = 0$ at $z = 0$ and $z = L$ will require ψ_{II} to vanish identically. Thus, they are disallowed. Instead, we require the outward component of the Dirac current to be zero at the boundaries (the MIT bag model). This ensures that the particle is indeed confined within the box [89].

The conserved current corresponding to Eq. (5.2.4) can be shown to be

$$\begin{aligned} J_z &= \bar{\psi} \gamma^z \psi + i\hbar\alpha \left(\psi^\dagger \frac{d\psi}{dz} - \frac{d\psi^\dagger}{dz} \psi \right), \\ &\equiv J_{0z} + J_{1z}, \end{aligned} \quad (5.2.11)$$

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where $J_{0z} + J_{1z}$ are the usual and new GUP-induced currents, respectively. We will comment on J_{1z} shortly. First, the vanishing of the Dirac current $J^\mu = \bar{\psi}\gamma^\mu\psi$ at a boundary is equivalent to the condition $i\gamma \cdot n\psi = \psi$ there, where n is the outward normal to the boundary [89]. Applying this to J_{0z} for the wavefunction ψ_{II} at $z = 0$ and $z = L$ gives [88]

$$i\beta\alpha_z\psi_{II}|_{z=0} = \psi_{II}|_{z=0} \quad (5.2.12)$$

$$\text{and } -i\beta\alpha_z\psi_{II}|_{z=L} = \psi_{II}|_{z=L}, \quad (5.2.13)$$

respectively. Using the expression for ψ_{II} from (5.2.9), we get from (5.2.12) and (5.2.13), respectively,

$$\frac{B + C + F'e^{-i\pi/4}}{B - C} = ir \quad (5.2.14)$$

$$\frac{Be^{ikL} + Ce^{-ikL} + F'e^{i(L/\alpha\hbar+\pi/4)}}{Be^{ikL} - Ce^{-ikL}} = -ir, \quad (5.2.15)$$

(where $F' = \sqrt{2}F$), which in turn yield

$$(ir - 1) - F'e^{-i\pi/4} = (ir + 1)e^{i\delta} \quad (5.2.16)$$

$$(ir - 1) - F'e^{i(L/\alpha\hbar+\pi/4)}e^{ikL}e^{-i\delta} = (ir + 1)e^{i(2kL-\delta)}. \quad (5.2.17)$$

Note that conditions (5.2.16) and (5.2.17) imply

$$|B| = |C| + \mathcal{O}(\alpha), \quad (5.2.18)$$

which guarantees that

$$J_{1z} = -2c\alpha\hbar k(1 + r^2) [|B|^2 - |C|^2] = 0. \quad (5.2.19)$$

Furthermore, from (5.2.16) and (5.2.17) it follows that

$$kL = \delta = \arctan\left(-\frac{\hbar k}{mc}\right) + \mathcal{O}(\alpha) \quad (5.2.20)$$

$$\text{and } \frac{L}{\alpha\hbar} = \frac{L}{\alpha_0\ell_{Pl}} = 2p\pi - \frac{\pi}{2}, \quad p \in \mathbb{N}. \quad (5.2.21)$$

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The transcendental equation (5.2.20) gives the quantized energy levels for a relativistic particle in a box. Its $\alpha \rightarrow 0$ limit gives $k_0 L = \arctan\left(-\frac{\hbar k_0}{mc}\right)$ which is Eq. (17) of ref. [88], its non-relativistic limit gives $(k_0 + \alpha \hbar k_0^2)L = n\pi$, while its non-relativistic and $\alpha \rightarrow 0$ limit yields the Schrödinger equation result $k_0 L = n\pi$. Equation (5.2.21) on the other hand shows that such a particle cannot be confined in a box, unless the box length is quantized according to this condition. Note that this is *identical* to the quantization condition (7.1.4), which was derived using the Schrödinger equation (with the identification $4\theta_C \equiv -\pi/2$). This indicates the robustness of the result. As measuring spatial dimensions requires the existence and observation of at least one particle, the above result once again seems to indicate that effectively all measurable lengths are quantized in units of $\alpha_0 \ell_{Pl}$.

5.2.3 Dirac equation in two and three dimensions

We now generalize to a box in two or three dimensions defined by $0 \leq x_i \leq L_i, i = 1, \dots, d$ with $d = 1, 2, 3$. We start with the following ansatz for the wavefunction

$$\psi = e^{i\vec{t}\cdot\vec{r}} \begin{pmatrix} \chi \\ \vec{\rho} \cdot \vec{\sigma} \chi \end{pmatrix} \quad (5.2.22)$$

where \vec{t} and $\vec{\rho}$ are d -dimensional (spatial) vectors, and $\chi^\dagger \chi = I$ as before. In this case, Eq. (5.2.3) translates to

$$\begin{aligned} H\psi &= e^{i\vec{t}\cdot\vec{r}} \begin{pmatrix} ((mc^2 - \alpha \hbar^2 t^2) + c\hbar (\vec{t} \cdot \vec{\rho} + i\vec{\sigma} \cdot (\vec{t} \times \vec{\rho})))\chi \\ (c\hbar \vec{t} - (mc^2 + \alpha \hbar^2 t^2)\vec{\rho}) \cdot \vec{\sigma} \chi \end{pmatrix} \\ &= E\psi, \end{aligned} \quad (5.2.23)$$

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where we have used the identity $(\vec{t} \cdot \vec{\sigma})(\vec{\rho} \cdot \vec{\sigma}) = \vec{t} \cdot \vec{\rho} + i\vec{\sigma} \cdot (\vec{t} \times \vec{\rho})$. Eq. (5.2.23) implies $\vec{t} \times \vec{\rho} = 0$, i.e. $\vec{\rho}$ is parallel to \vec{t} , and two solutions for t , namely $t = k$ and $t = 1/\alpha\hbar$, and correspondingly $\rho = r$ and $\rho = 1$. The latter solutions for t and ρ are the (new) non-perturbative ones, which as we shall see, will give rise to quantization of space. Thus the vector \vec{t} for the two cases are $\vec{t} = \vec{k}$ and $\vec{t} = \frac{\hat{q}}{\alpha\hbar}$ and $\vec{\rho} = r\hat{k}$ and $\vec{\rho} = \hat{q}$ respectively, where \hat{q} is an arbitrary unit vector². Thus, putting in the normalizations, the two independent positive energy solutions are

$$\psi_1 = N_1 e^{i\vec{k} \cdot \vec{r}} \begin{pmatrix} \chi \\ r\hat{k} \cdot \vec{\sigma} \chi \end{pmatrix} \quad (5.2.24)$$

$$\psi_2 = N_2 e^{i\frac{\hat{q} \cdot \vec{r}}{\alpha\hbar}} \begin{pmatrix} \chi \\ \hat{q} \cdot \vec{\sigma} \chi \end{pmatrix} \quad (5.2.25)$$

with ψ_2 being the new GUP-induced eigenfunction.

Next, we consider the following wavefunction

$$\psi = \begin{pmatrix} \left[\prod_{i=1}^d (e^{ik_i x_i} + e^{-i(k_i x_i - \delta_i)}) + F e^{i\frac{\hat{q} \cdot \vec{r}}{\alpha\hbar}} \right] \chi \\ \sum_{j=1}^d \left[\prod_{i=1}^d (e^{ik_i x_i} + (-1)^{\delta_{ij}} e^{-i(k_i x_i - \delta_i)}) r\hat{k}_j \right. \\ \left. + F e^{i\frac{\hat{q} \cdot \vec{r}}{\alpha\hbar}} \hat{q}_j \right] \sigma_j \chi \end{pmatrix} \quad (5.2.26)$$

where $d = 1, 2, 3$, depending on the number of spatial dimensions and an overall normalization has been set to unity. The number of terms in row I and row II are $2^d + 1$ and $(2^d + 1) \times d$ respectively, i.e. (3, 3), (5, 10) and (9, 27) in 1, 2 and 3 dimensions, respectively. It can be easily shown that the above is a superposition of $F\psi_2$ and the

²Although one can choose $\hat{q} = \hat{k}$, *per se* our analysis does not require this to be the case. We will comment on this towards the end of this subsection.

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following 2^d eigenfunctions, for all possible combinations with ϵ_i ($i = 1, \dots, d$), with $\epsilon_i = \pm 1$

$$\Psi = e^{i(\sum_{i=1}^d \epsilon_i k_i x_i + \frac{(1-\epsilon_i)}{2} \delta_i)} \begin{pmatrix} \chi \\ r \sum_{i=1}^d \epsilon_i \hat{k}_i \sigma_i \chi \end{pmatrix} \quad (5.2.27)$$

where δ_i ($i = 1, \dots, d$) are phases to be determined shortly using boundary conditions.

Again, we impose the MIT bag boundary conditions $\pm i \beta \alpha_k \psi = \psi$, $k = 1, \dots, d$, with the + and - signs corresponding to $x_k = 0$ and $x_k = L_k$ respectively, ensuring vanishing flux through all six boundaries. First, we write the above boundary condition for any x_k , for the wavefunction given in Eq. (5.2.26). This yields the following 2-component equation

$$\pm \begin{pmatrix} i \sum_{j=1}^d \left[\prod_{i=1}^d (e^{ik_i x_i} + (-1)^{\delta_{ij}} e^{-i(k_i x_i - \delta_i)}) \sigma_k r_j \sigma_j \right. \\ \left. + F e^{i \frac{\hat{q} \cdot \vec{r}}{\alpha \hbar}} \sigma_k \hat{q}_j \sigma_j \right] \chi \\ -i \left[\prod_{i=1}^d (e^{ik_i x_i} + e^{-i(k_i x_i - \delta_i)}) + F e^{i \frac{\hat{q} \cdot \vec{r}}{\alpha \hbar}} \right] \sigma_k \chi \end{pmatrix} \\ = \psi . \quad (5.2.28)$$

Employing the MIT bag model boundary conditions and thus equating the rows I and II of Eq. (5.2.26) with the corresponding ones of Eq. (5.2.28) yields, respectively

$$\prod_{i=1}^d (e^{ik_i x_i} + e^{-i(k_i x_i - \delta_i)}) + F e^{i \frac{\hat{q} \cdot \vec{r}}{\alpha \hbar}} \\ = \pm \left[i \prod_{i=1}^d (e^{ik_i x_i} + (-1)^{\delta_{ik}} e^{-i(k_i x_i - \delta_i)}) r \hat{k}_k \right. \\ \left. + i F e^{i \frac{\hat{q} \cdot \vec{r}}{\alpha \hbar}} \hat{q}_k + i \sum_{j=1 \neq k}^d \left[\prod_{i=1}^d (e^{ik_i x_i} + (-1)^{\delta_{ij}} e^{-i(k_i x_i - \delta_i)}) r \hat{k}_j \sigma_k \sigma_j \right. \right. \\ \left. \left. + F e^{i \frac{\hat{q} \cdot \vec{r}}{\alpha \hbar}} \hat{q}_j \sigma_k \sigma_j \right] \right] \quad (5.2.29)$$

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and

$$\begin{aligned}
& \prod_{i=1}^d (e^{ik_i x_i} + e^{-i(k_i x_i - \delta_i)}) + F e^{i\frac{\hat{q}\cdot\vec{r}}{\alpha\hbar}} \\
& = \pm \left[i \prod_{i=1}^d (e^{ik_i x_i} + (-1)^{\delta_{ik}} e^{-i(k_i x_i - \delta_i)}) r \hat{k}_k \right. \\
& \quad + i F e^{i\frac{\hat{q}\cdot\vec{r}}{\alpha\hbar}} \hat{q}_k + i \sum_{j=1 \neq k}^d \left[\prod_{i=1}^d (e^{ik_i x_i} + (-1)^{\delta_{ij}} e^{-i(k_i x_i - \delta_i)}) r \hat{k}_j \sigma_j \sigma_k \right. \\
& \quad \left. \left. + F e^{i\frac{\hat{q}\cdot\vec{r}}{\alpha\hbar}} \hat{q}_j \sigma_j \sigma_k \right] \right]. \tag{5.2.30}
\end{aligned}$$

Note that the only difference between Eqs. (5.2.29) and (5.2.30) is in the order of σ_k and σ_j in the last two terms in the RHS. Thus, adding the two equations and using $\{\sigma_k, \sigma_j\} = 0$, these terms simply drop out. Next, dividing the rest by $f_{\bar{k}}(x_i, k_i, \delta_i) \equiv \prod_{i=1 \neq k}^d (e^{ik_i x_i} + e^{-i(k_i x_i - \delta_i)})$, where the subscript \bar{k} of $f_{\bar{k}}$ signifies the lack of dependence on (x_k, k_k, δ_k) , we get

$$\begin{aligned}
& e^{ik_k x_k} + e^{-i(k_k x_k - \delta_k)} + f_{\bar{k}}^{-1} F e^{i\frac{\hat{q}\cdot\vec{r}}{\alpha\hbar}} = \\
& \pm i (e^{ik_k x_k} - e^{-i(k_k x_k - \delta_k)}) r \hat{k}_k \pm i f_{\bar{k}}^{-1} F e^{i\frac{\hat{q}\cdot\vec{r}}{\alpha\hbar}} \hat{q}_k. \tag{5.2.31}
\end{aligned}$$

Note that for all practical purposes the boundary condition has factorized into its Cartesian components, at least in the α independent terms, which contain (x_k, k_k, δ_k) alone, i.e. no other index i . Eq. (5.2.31) yields, at $x_k = 0$ and $x_k = L_k$, respectively,

$$e^{i\delta_k} (1 + i r \hat{k}_k) = (i r \hat{k}_k - 1) + f_{\bar{k}}^{-1} F'_k e^{-i\theta_k} \tag{5.2.32}$$

and

$$e^{i(2k_k L_k - \delta_k)} (1 + i r \hat{k}_k) = (i r \hat{k}_k - 1) + f_{\bar{k}}^{-1} F'_k e^{i\theta_k} e^{i\frac{|q_k| L_k}{\alpha\hbar}} e^{i(k_k L_k - \delta_k)}, \tag{5.2.33}$$

where $F'_k \equiv \sqrt{1 + |\hat{q}_k|^2} F$, $\theta_k \equiv \arctan \hat{q}_k$ and we have assumed that $f_{\bar{k}}$ is evaluated at the same x_i ($i \neq k$) at both boundaries of x_k . Comparing Eqs. (5.2.32) and (5.2.33),

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which are the d -dimensional generalizations of Eqs. (5.2.16) and (5.2.17), we see that the following relations must hold

$$k_k L_k = \delta_k = \arctan\left(-\frac{\hbar k_k}{mc}\right) + \mathcal{O}(\alpha) \quad (5.2.34)$$

$$\frac{|\hat{q}_k| L_k}{\alpha \hbar} = \frac{|\hat{q}_k| L_k}{\alpha_0 \ell_{Pl}} = 2p_k \pi - 2\theta_k. \quad (5.2.35)$$

While Eq. (5.2.34) yields quantization of energy levels in d dimensions ($k_k L_k = n\pi$ in the non-relativistic limit), Eq. (5.2.35) shows that lengths in all directions are quantized. Further, one may choose the symmetric case $|\hat{q}_k| = 1/\sqrt{d}$ ³, in which case, it follows from Eq. (5.2.35) above

$$\frac{L_k}{\alpha_0 \ell_{Pl}} = (2p_k \pi - 2\theta_k) \sqrt{d}, \quad p_k \in \mathbb{N} \quad (5.2.36)$$

which reduces to Eq. (5.2.21) for $d = 1$. Note that the above also gives rise to quantization of measured areas ($N = 2$) and volumes ($N = 3$), as follows

$$A_N \equiv \prod_{k=1}^N \frac{L_k}{\alpha_0 \ell_{Pl}} = d^{N/2} \prod_{k=1}^N (2p_k \pi - 2\theta_k), \quad p_k \in \mathbb{N}. \quad (5.2.37)$$

5.2.4 Spherical cavity: Dirac equation in polar coordinates

Finally, we solve the Dirac equation with the GUP-induced terms in a spherical cavity, and show that only cavities of certain discrete dimensions can confine a relativistic particle. We follow the analysis of [90]. For related references, see [89,91]. A spherical

³Alternatively, assuming no direction is intrinsically preferred in space and the only *special* direction is provided by the particle momentum \vec{k} , one can make the identification $\hat{q} = \hat{k}$, in which case $|\hat{q}_k| = n_k / \sqrt{\sum_{i=1}^d n_i^2} \approx 1/\sqrt{d}$, assuming that the momentum quantum numbers $n_k \gg 1$ and approximately equal, when space is probed at the fundamental level with ultra high energy super-Planckian particles.

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cavity of radius R , defined by the potential

$$\begin{aligned} U(r) &= 0, \quad r \leq R, \\ &= U_0 \rightarrow \infty, \quad r > R \end{aligned} \quad (5.2.38)$$

yields the following Dirac equation in component form

$$c(\vec{\sigma} \cdot \vec{p}_0) \chi_2 + (mc^2 + U) \chi_1 - c\alpha p_0^2 \chi_1 = E \chi_1 \quad (5.2.39)$$

$$c(\vec{\sigma} \cdot \vec{p}_0) \chi_1 - (mc^2 + U) \chi_2 - c\alpha p_0^2 \chi_2 = E \chi_2, \quad (5.2.40)$$

where the Dirac spinor has the form $\psi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}$. It can be shown that the following operators commute with the GUP-corrected Hamiltonian: the total angular momentum operator (not to be confused with the Dirac current represented by the same letter) $\vec{J} = \vec{L} + \vec{\Sigma}/2$, $K = \beta(\vec{\Sigma} \cdot \vec{L} + I)$, where \vec{L} is the orbital angular momentum, $\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$, and $K^2 = J^2 + 1/4$. Thus, eigenvalues of J^2 and K , namely $j(j+1)$ and κ respectively, are related by $\kappa = \pm(j+1/2)$. Correspondingly, the Dirac spinor has the following form

$$\psi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} g_\kappa(r) \mathcal{Y}_{j\ell}^{j_3}(\hat{r}) \\ i f_\kappa(r) \mathcal{Y}_{j\ell'}^{j_3}(\hat{r}) \end{pmatrix}, \quad (5.2.41)$$

$$\begin{aligned} \mathcal{Y}_{j\ell}^{j_3} &= \begin{pmatrix} l & \frac{1}{2} & j_3 - \frac{1}{2} & \frac{1}{2} \\ & & | & \\ & & j & j_3 \end{pmatrix} Y_\ell^{j_3 - \frac{1}{2}}(\hat{r}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &+ \begin{pmatrix} l & \frac{1}{2} & j_3 + \frac{1}{2} & -\frac{1}{2} \\ & & | & \\ & & j & j_3 \end{pmatrix} Y_\ell^{j_3 + \frac{1}{2}}(\hat{r}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned} \quad (5.2.42)$$

where $Y_\ell^{j_\pm \frac{1}{2}}$ are spherical harmonics and $\begin{pmatrix} j_1 & j_2 & m_1 & m_2 \\ & & | & \\ & & J & M \end{pmatrix}$ are Clebsch-Gordon coefficients. χ_1 and χ_2 are eigenstates of L^2 with eigenvalues $\hbar^2 \ell(\ell+1)$ and $\hbar^2 \ell'(\ell'+1)$,

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respectively, such that the following hold

$$\begin{aligned} & \text{if } \kappa = j + \frac{1}{2} > 0, \\ & \text{then } \ell = \kappa = j + \frac{1}{2}, \quad \ell' = \kappa - 1 = j - \frac{1}{2}, \end{aligned} \quad (5.2.43)$$

$$\begin{aligned} & \text{and if } \kappa = -(j + \frac{1}{2}) < 0, \\ & \text{then } \ell = -(\kappa + 1) = j - \frac{1}{2}, \quad \ell' = -\kappa = j + \frac{1}{2}. \end{aligned} \quad (5.2.44)$$

Next, we use the following identities

$$\begin{aligned} \vec{\sigma} \cdot \vec{p}_0 &= \frac{\vec{\sigma} \cdot \vec{r}}{r^2} [(\vec{\sigma} \cdot \vec{r})(\vec{\sigma} \cdot \vec{p}_0)] \\ &= \frac{\vec{\sigma} \cdot \vec{r}}{r^2} [\vec{r} \cdot \vec{p}_0 + i\vec{\sigma} \cdot \vec{r} \times \vec{p}_0] \\ &= \frac{\vec{\sigma} \cdot \vec{r}}{r^2} \left[-i\hbar r \frac{d}{dr} + i\vec{\sigma} \cdot \vec{L} \right] \end{aligned} \quad (5.2.45)$$

$$(\vec{\sigma} \cdot \vec{L} + 1) \chi_{1,2} = \mp \kappa \chi_{1,2} \quad (5.2.46)$$

$$(\vec{\sigma} \cdot \hat{r}) \mathcal{Y}_{j\ell}^{j_3} = -\mathcal{Y}_{j\ell'}^{j_3}, \quad (\vec{\sigma} \cdot \hat{r}) \mathcal{Y}_{j\ell'}^{j_3} = -\mathcal{Y}_{j\ell}^{j_3} \quad (5.2.47)$$

where we have used $(\vec{\sigma} \cdot \vec{A})(\vec{\sigma} \cdot \vec{B}) = \vec{A} \cdot \vec{B} + i\vec{\sigma} \cdot (\vec{A} \times \vec{B})$, the related identity $(\vec{\sigma} \cdot \vec{r})(\vec{\sigma} \cdot \vec{r}) = r^2$, and the relation $p_0^2 F(r) Y_\ell^m = \hbar^2 \left[-\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} \right) + \frac{\ell(\ell+1)}{r^2} \right] F(r) Y_\ell^m$ for an arbitrary function $F(r)$, to obtain from Eqs. (5.2.39), (5.2.40)

$$\begin{aligned} & -c\hbar \frac{df_\kappa}{dr} + c \frac{(\kappa - 1)}{r} f_\kappa + (mc^2 + U) g_\kappa \\ & + c\alpha \hbar^2 \left[\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dg_\kappa}{dr} \right) - \frac{\ell(\ell+1)}{r^2} g_\kappa \right] = E g_\kappa \end{aligned} \quad (5.2.48)$$

$$\begin{aligned} & c\hbar \frac{dg_\kappa}{dr} + c \frac{(\kappa + 1)}{r} g_\kappa - (mc^2 + U) f_\kappa \\ & + c\alpha \hbar^2 \left[\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{df_\kappa}{dr} \right) - \frac{\ell'(\ell'+1)}{r^2} f_\kappa \right] = E f_\kappa. \end{aligned} \quad (5.2.49)$$

As in the case of rectangular cavities, Eqs. (5.2.48), (5.2.49) have the standard set of solutions, slightly perturbed by the GUP-induced term (represented by the $\mathcal{O}(\alpha)$

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terms below)

$$g_\kappa(r) = \tilde{N} j_\ell(p_0 r) + \mathcal{O}(\alpha), \quad (5.2.50)$$

$$\text{where } \ell = \begin{cases} \kappa, & \text{if } \kappa > 0 \\ -(\kappa + 1), & \text{if } \kappa < 0 \end{cases}$$

$$f_\kappa(r) = \tilde{N} \frac{\kappa}{|\kappa|} \sqrt{\frac{E - mc^2}{E + mc^2}} j_{\ell'}(p_0 r) + \mathcal{O}(\alpha), \quad (5.2.51)$$

$$\text{where } \ell' = \begin{cases} (\kappa - 1), & \text{if } \kappa > 0 \\ -\kappa, & \text{if } \kappa < 0 \end{cases}$$

where $j_l(x)$ are spherical Bessel functions. It can be shown that the MIT bag boundary condition (at $r = R$) is equivalent to [89, 90]

$$\bar{\psi}_\kappa \psi_\kappa = 0 \quad (5.2.52)$$

which in the massless (high energy) limit yields

$$[g_\kappa^2(r) - f_\kappa^2(r)] (\mathcal{Y}_{jl}^{j_3})^\dagger \mathcal{Y}_{jl}^{j_3} + \mathcal{O}(\alpha) = 0 \quad (5.2.53)$$

which in turn gives the quantization of energy (for energy eigenvalues obtained numerically from Eq. (5.2.53), see Table 2.1, Chapter 2, ref. [90]. These will also undergo tiny modifications $\mathcal{O}(\alpha)$).

But from the analysis of previous sections, we expect new non-perturbative solutions of the form $f_\kappa = \mathcal{F}_\kappa(r) e^{i\epsilon r/\alpha\hbar}$ and $g_\kappa = \mathcal{G}_\kappa(r) e^{i\epsilon r/\alpha\hbar}$ (where $\epsilon = \mathcal{O}(1)$) for which Eqs. (5.2.48), (5.2.49) simplify to

$$\alpha\hbar \frac{d^2 g_\kappa}{dr^2} = \frac{df_\kappa}{dr} \quad (5.2.54)$$

$$\alpha\hbar \frac{d^2 f_\kappa}{dr^2} = -\frac{dg_\kappa}{dr} \quad (5.2.55)$$

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where we have dropped terms which are ignorable for small a . These indeed have solutions

$$f_{\kappa}^{\mathcal{N}} = iN'e^{ir/\alpha\hbar} \quad (5.2.56)$$

$$g_{\kappa}^{\mathcal{N}} = N'e^{ir/\alpha\hbar}, \quad (5.2.57)$$

where similar to the constant C in [21], here one must have $\lim_{\alpha \rightarrow 0} N' = 0$, such that these new solutions drop out in the $\alpha \rightarrow 0$ limit. The boundary condition (5.2.52) now gives

$$|g_{\kappa}(r) + g_{\kappa}^{\mathcal{N}}(r)|^2 = |f_{\kappa}(r) + f_{\kappa}^{\mathcal{N}}(r)|^2, \quad (5.2.58)$$

which to $\mathcal{O}(\alpha)$ translates to

$$\begin{aligned} & [j_{\ell}^2(p_0R) - j_{\ell'}^2(p_0R)] \\ & + 2N' [j_{\ell}(p_0R) \cos(R/a\hbar) - j_{\ell'}(p_0R) \sin(R/\alpha\hbar)] = 0. \end{aligned} \quad (5.2.59)$$

This again implies the following conditions

$$j_{\ell}(p_0R) = j_{\ell'}(p_0R) \quad (5.2.60)$$

$$\tan(R/\alpha\hbar) = 1. \quad (5.2.61)$$

The first condition is identical to Eq. (5.2.53), and hence the energy quantization.

The second implies

$$\frac{R}{\alpha\hbar} = \frac{R}{\alpha_0\ell_{Pl}} = 2p\pi - \frac{\pi}{4}, \quad p \in \mathbb{N}. \quad (5.2.62)$$

This once again, the radius of the cavity, and hence the area and volume of spheres are seen to be quantized.

5.2. *Discreteness of space in the relativistic case*

In this section, we have studied a relativistic particle in a box in one, two and three dimensions (including a spherical cavity in three dimensions), using the Klein–Gordon and Dirac equations with corrections that follow from the Generalized Uncertainty Principle. We have shown that to confine the particle in the box, the dimensions of the latter would have to be quantized in multiples of a fundamental length, which can be the Planck length.

Chapter 6

GUP effects on equivalence and holographic principles

A possible discrepancy has been found between the results of a neutron interferometry experiment and Quantum Mechanics [24,25]. This experiment suggests that the weak equivalence principle is violated at small length scales, which quantum mechanics cannot explain. In this chapter, we investigate whether the GUP can explain the violation of the weak equivalence principle at small length scales. We find that the acceleration is no longer mass-independent because of the mass-dependence through the momentum p . Therefore, the equivalence principle is violated. We also tackle a naturally arising question of whether the number of states inside a volume of phase space does not change with time in the presence of the GUP. So, we calculate the consequences of the GUP on the Liouville theorem in statistical mechanics. We have found a new form of invariant phase space in the presence of the GUP. This result should modify the density states and affect the calculation of the entropy bound of local quantum field theory, the cosmological constant, black body radiation, etc. Furthermore, such modification may have observable consequences at length scales much larger than the Planck scale. This modification leads to a \sqrt{A} -type correction

to the bound of the maximal entropy of a bosonic field which might shed some light on the holographic theory.

6.1 The equivalence principle at short distance

Quantum mechanics does not violate the weak equivalence principle. This can be shown from studying Heisenberg equations of motion. For simplicity, consider 1-dimensional motion with the Hamiltonian given by

$$H = \frac{p^2}{2m} + V(x). \quad (6.1.1)$$

The Heisenberg equations of motion read,

$$\dot{x} = \frac{1}{i\hbar}[x, H] = \frac{p}{m}, \quad (6.1.2)$$

$$\dot{p} = \frac{1}{i\hbar}[p, H] = -\frac{\partial V}{\partial x}. \quad (6.1.3)$$

These equations ensure that the momentum at the quantum level is $p = m\dot{x}$ and the acceleration \ddot{x} is mass-independent as in classical physics. It is obvious that the equivalence principle is preserved at the quantum level, and it is clear that this result possibly contradicts experimental results [24, 25].

Let us study Eq(3.3.1) at the classical limit using the correspondence between the commutator in quantum mechanics and the Poisson bracket in classical mechanics,

$$\frac{1}{i\hbar}[\hat{P}, \hat{Q}] \implies \{P, Q\}, \quad (6.1.4)$$

so the classical limit of Eq(3.3.1) give

$$\{x_i, p_j\} = \delta_{ij} - \alpha(p\delta_{ij} + \frac{p_i p_j}{p}) + \alpha^2(p^2\delta_{ij} + 3p_i p_j). \quad (6.1.5)$$

6.1. *The equivalence principle at short distance*

The equations of motion are given by

$$\begin{aligned}\dot{x}_i &= \{x_i, H\} = \{x_i, p_j\} \frac{\partial H}{\partial p_j}, \\ \dot{p}_i &= \{p_i, H\} = -\{x_j, p_i\} \frac{\partial H}{\partial x_j}.\end{aligned}\tag{6.1.6}$$

Consider the effect of the GUP on 1—dimensional motion with the Hamiltonian given by,

$$H = \frac{p^2}{2m} + V(x).\tag{6.1.7}$$

The equations of motion will be modified as follows,

$$\dot{x} = \{x, H\} = (1 - 2\alpha p) \frac{p}{m},\tag{6.1.8}$$

$$\dot{p} = \{p, H\} = (1 - 2\alpha p) \left(-\frac{\partial V}{\partial x}\right),\tag{6.1.9}$$

where the momentum p is no longer equal to $m\dot{x}$.

Using (6.1.8,6.1.9), we can derive the acceleration given by,

$$m\ddot{x} = -(1 - 6\alpha p) \frac{\partial V}{\partial x}.\tag{6.1.10}$$

Notice that if the force $F = -\frac{\partial V}{\partial x}$ is gravitational and proportional to the mass m , the acceleration \ddot{x} is not mass-independent because of the mass-dependence through the momentum p . Therefore, the equivalence principle is dynamically violated because of the generalized uncertainty principle. It is a dynamical violation because the correction in Eq. (6.1.10) is a function of the momentum p . Note that we considered only the correction up to the first order of $\alpha = \alpha_0 \ell_{Pl}/\hbar$ which is sufficient to explain the violation of equivalence principle due to the existence of GUP.

Since the GUP is an aspect of various approaches to Quantum Gravity such as String Theory and Doubly Special Relativity (or DSR) Theories, as well as black hole

physics, it is important to predict the upper bounds on the quantum gravity parameter compatible with the experiment that was done in [24, 25]. This result agrees, too, with cosmological implications of the dark sector where a long-range force acting only between nonbaryonic particles would be associated with a large violation of the weak equivalence principle [92]. The violation of equivalence principle has been obtained, too, in the context of string theory [14, 93, 94] where the extended nature of strings are subject to tidal forces and do not follow geodesics.

6.2 The GUP and Liouville theorem

In this section, we continue our investigation of the consequences of our proposed commutation relation of Eq(3.3.1). What we are looking for is an analog of the Liouville theorem in the presence of the GUP. We should make sure that the number of states inside a volume of phase space does not change with time evolution in the presence of the GUP. If this is the case, this should modify the density states and affect the entropy bound of local quantum field theory, the Cosmological constant, black body radiation, etc. Furthermore, such a modification may have observable consequences at length scales much larger than the Planck scale. The Liouville theorem has been studied before with different versions of GUP, see e.g. [95].

Since we are seeking the number of states inside a volume of phase space that does not change with time, we assume the time evolutions of the position and momentum during δt are

$$\begin{aligned}x'_i &= x_i + \delta x_i , \\p'_i &= p_i + \delta p_i ,\end{aligned}\tag{6.2.1}$$

where

$$\begin{aligned}\delta x_i &= \{x_i, p_j\} \frac{\partial H}{\partial p_j} \delta t, \\ \delta p_i &= -\{x_j, p_i\} \frac{\partial H}{\partial x_j} \delta t.\end{aligned}\quad (6.2.2)$$

The infinitesimal phase space volume after this infinitesimal time evolution is

$$d^D \mathbf{x}' d^D \mathbf{p}' = \left| \frac{\partial(x'_1, \dots, x'_D, p'_1, \dots, p'_D)}{\partial(x_1, \dots, x_D, p_1, \dots, p_D)} \right| d^D \mathbf{x} d^D \mathbf{p}.\quad (6.2.3)$$

Using Eq(6.2.1), the Jacobian reads to the first order in δt

$$\left| \frac{\partial(x'_1, \dots, x'_D, p'_1, \dots, p'_D)}{\partial(x_1, \dots, x_D, p_1, \dots, p_D)} \right| = 1 + \left(\frac{\partial \delta x_i}{\partial x_i} + \frac{\partial \delta p_i}{\partial p_i} \right) + \dots.\quad (6.2.4)$$

Using Eq(6.2.2), we get

$$\begin{aligned}& \left(\frac{\partial \delta x_i}{\partial x_i} + \frac{\partial \delta p_i}{\partial p_i} \right) \frac{1}{\delta t} \\ &= \frac{\partial}{\partial x_i} \left[\{x_i, p_j\} \frac{\partial H}{\partial p_j} \right] - \frac{\partial}{\partial p_i} \left[\{x_j, p_i\} \frac{\partial H}{\partial x_j} \right] \\ &= \left[\frac{\partial}{\partial x_i} \{x_i, p_j\} \right] \frac{\partial H}{\partial p_j} + \{x_i, p_j\} \frac{\partial^2 H}{\partial x_i \partial p_j} \frac{\partial H}{\partial x_j} \\ &\quad - \left[\frac{\partial}{\partial p_i} \{x_j, p_i\} \right] \frac{\partial H}{\partial x_j} - \{x_j, p_i\} \frac{\partial^2 H}{\partial p_j \partial x_i} \\ &= - \left[\frac{\partial}{\partial p_i} \{x_j, p_i\} \right] \frac{\partial H}{\partial x_j} \\ &= - \frac{\partial}{\partial p_i} \left[\delta_{ij} - \alpha(p\delta_{ij} + \frac{p_i p_j}{p}) \right] \frac{\partial H}{\partial x_j} \\ &= \frac{\partial}{\partial p_i} \alpha(p\delta_{ij} + \frac{p_i p_j}{p}) \frac{\partial H}{\partial x_j} \\ &= \alpha(D+1) \frac{p_j}{p} \frac{\partial H}{\partial x_j}.\end{aligned}\quad (6.2.5)$$

The infinitesimal phase space volume after this infinitesimal evolution up to first order in α and δt is

$$d^D \mathbf{x}' d^D \mathbf{p}' = d^D \mathbf{x} d^D \mathbf{p} \left[1 + \alpha(D+1) \frac{p_i}{p} \frac{\partial H}{\partial x_i} \delta t \right].\quad (6.2.6)$$

6.2. The GUP and Liouville theorem

Now we are seeking the analog of the Liouville theorem in which the weighted phase space volume is invariant under time evolution. Let us check the infinitesimal evolution of $(1 - \alpha p')$ up to first order in α and δt

$$\begin{aligned}
(1 - \alpha p') &= 1 - \alpha \sqrt{p'_i p'_i} \\
&= 1 - \alpha \left[(p_i + \delta p_i)(p_i + \delta p_i) \right]^{\frac{1}{2}} \\
&\approx 1 - \alpha (p^2 + 2p_i \delta p_i)^{\frac{1}{2}} \\
&\approx 1 - \alpha \left[p^2 - 2p_i \{x_i, p_j\} \frac{\partial H}{\partial x_j} \delta t \right]^{\frac{1}{2}} \\
&\approx 1 - \alpha \left[p - \frac{1}{p} (p_j - 2\alpha p p_j) \frac{\partial H}{\partial x_j} \delta t \right] \\
&\approx (1 - \alpha p) + \alpha \frac{p_j}{p} (1 - 2\alpha p) \frac{\partial H}{\partial x_j} \delta t \\
&\approx (1 - \alpha p) \left[1 + \alpha \frac{p_j}{p} \frac{1 - 2\alpha p}{1 - \alpha p} \frac{\partial H}{\partial x_j} \delta t \right] \\
&\approx (1 - \alpha p) \left[1 + \alpha \frac{p_j}{p} \frac{\partial H}{\partial x_j} \delta t \right]. \tag{6.2.7}
\end{aligned}$$

Therefore, we get to first order in α and δt ,

$$(1 - \alpha p')^{-D-1} = (1 - \alpha p)^{-D-1} \left[1 - (D+1) \alpha \frac{p_j}{p} \frac{\partial H}{\partial x_j} \delta t \right] \tag{6.2.8}$$

This results in the following expression which is invariant under time evolution!

$$\frac{d^D \mathbf{x}' d^D \mathbf{p}'}{(1 - \alpha p')^{D+1}} = \frac{d^D \mathbf{x} d^D \mathbf{p}}{(1 - \alpha p)^{D+1}}. \tag{6.2.9}$$

If we integrate over the coordinates, the invariant phase space volume of Eq. (6.2.9) will be

$$\frac{V d^D \mathbf{p}}{(1 - \alpha p)^{D+1}}, \tag{6.2.10}$$

where V is the coordinate space volume. The number of quantum states per momentum space volume can be assumed to be

$$\frac{V}{(2\pi\hbar)^D} \frac{d^D \mathbf{p}}{(1 - \alpha p)^{D+1}} . \quad (6.2.11)$$

The modification in the number of quantum states per momentum space volume in (6.2.11) should have consequences on the calculation of the entropy bound of local quantum field theory, the cosmological constant, black body radiation, etc. In this chapter, we are investigating its consequences on the entropy bound of local quantum field theory. In the following two subsections we briefly introduce the holographic entropy bound proposed by 't Hooft [60] and the entropy bound of Local quantum field proposed by Yurtsever and Aste [96–98]. We treat the effects of the GUP on the entropy bound of a local quantum field.

6.2.1 The holographic entropy bound and local quantum field theory

The entropy of a closed spacelike surface containing a quantum bosonic field has been studied by 't Hooft [60] . For the field states to be observable for the outside world, 't Hooft assumed that their energy inside the surface should be less than 1/4 times its linear dimensions, otherwise the surface would lie within the Schwarzschild radius [60].

If the bosonic quantum fields are confined to a closed spacelike surface at a temperature T , the energy of the most probable state is

$$E = a_1 Z T^4 V, \quad (6.2.12)$$

where Z is the number of different fundamental particle types with mass less than T and a_1 a numerical constant of order one, all in natural units.

Now turning to the total entropy S , it is found that it is given by

$$S = a_2 Z V T^3, \quad (6.2.13)$$

where a_2 is another numeric constant of order one.

The Schwarzschild limit requires that

$$2E < \frac{V}{\frac{4}{3}\pi}. \quad (6.2.14)$$

Using Eq. (6.2.12), one finds

$$T < a_3 Z^{-\frac{1}{4}} V^{-\frac{1}{6}}, \quad (6.2.15)$$

so the entropy bound is given by

$$S < a_4 Z^{\frac{1}{4}} V^{\frac{1}{2}} = a_4 Z^{\frac{1}{4}} A^{\frac{3}{4}}, \quad (6.2.16)$$

where A is the boundary area of the system. At low temperatures, Z is limited by a dimensionless number, so that this entropy is small compared to that of a black hole, if the area A is sufficiently large. The black hole is the limit of maximum entropy

$$S_{max} = \frac{1}{4}A. \quad (6.2.17)$$

Therefore, for any closed surface without worrying about its geometry inside, all physics can be represented by degrees of freedom on this surface itself. This implies that quantum gravity can be described by a topological quantum field theory, for which all physical degrees of freedom can be projected onto the boundary [60]. This is known as the Holographic Principle.

6.2. *The GUP and Liouville theorem*

According to [96], the holographic entropy bound can be derived from elementary flat-spacetime quantum field theory when the total energy of Fock states is in a stable configuration against gravitational collapse by imposing a cutoff on the maximum energy of the field modes of the order of the Planck energy. This leads to an entropy bound of holographic type.

Considering a massless bosonic field confined to cubic box of size L , as has been done in [96–100], the total number of the quantized modes is given by

$$N = \sum_{\vec{k}} 1 \rightarrow \frac{L^3}{(2\pi)^3} \int d^3\vec{p} = \frac{L^3}{2\pi^2} \int_0^\Lambda p^2 dp = \frac{\Lambda^3 L^3}{6\pi^2}, \quad (6.2.18)$$

where Λ is the UV energy cutoff of the LQFT. The UV cutoff makes N finite. The Fock states can be constructed by assigning occupying number n_i to these N different modes:

$$|\Psi\rangle = |n(\vec{k}_1), n(\vec{k}_2), \dots, n(\vec{k}_N)\rangle \rightarrow |n_1, n_2, \dots, n_N\rangle. \quad (6.2.19)$$

The dimension of the Hilbert space is calculated by the number of occupancies $\{n_i\}$ which is finite if it is bounded. The non-gravitational collapse condition leads to finiteness of the Hilbert space:

$$E = \sum_{i=1}^N n_i \omega_i \leq E_{BH} = L. \quad (6.2.20)$$

It can be observed that N particle states with one particle occupying each mode ($n_i = 1$) corresponds to the lowest energy state with N modes simultaneously excited. In this case, it should satisfy the gravitational stability condition of Eq. (6.2.20). Hence, the energy bound is given by

$$E \rightarrow \frac{L^3}{2\pi^2} \int_0^\Lambda p^3 dp = \frac{\Lambda^4 L^3}{8\pi^2} \leq E_{BH}. \quad (6.2.21)$$

The last inequality implies

$$\Lambda^2 \leq \frac{1}{L}. \quad (6.2.22)$$

The maximum entropy is given by

$$S_{max} = - \sum_{j=1}^W \frac{1}{W} \ln \frac{1}{W} = \ln W, \quad (6.2.23)$$

where the bound of W is determined by

$$W = \dim \mathcal{H} < \sum_{m=0}^N \frac{z^m}{(m!)^2} \leq \sum_{m=0}^{\infty} \frac{z^m}{(m!)^2} = I_0(2\sqrt{z}) \sim \frac{e^{2\sqrt{z}}}{\sqrt{4\pi\sqrt{z}}}. \quad (6.2.24)$$

Here I_0 is the zeroth-order Bessel function of the second kind. Since z is given by

$$z = \sum_{i=1}^N L_i \rightarrow \frac{L^3}{2\pi^2} \int_0^\Lambda \left[\frac{E_{BH}}{p} \right] p^2 dp = \frac{\Lambda^2 L^4}{4\pi^2}. \quad (6.2.25)$$

Using UV-IR relation of Eq. (6.2.22), the bound can be given as follows

$$z \leq L^3. \quad (6.2.26)$$

Since the boundary area of the system is given by

$$A \sim L^2, \quad (6.2.27)$$

therefore, the bound for the maximum entropy of Eq(6.2.23) will be given by

$$S_{max} = \ln W \leq A^{3/4}. \quad (6.2.28)$$

This is just a brief summary of determining the entropy bound by using the Local Quantum Field Theory (LQFT).

6.2.2 The effect of GUP on the holographic entropy bound and LQFT

Consider a massless bosonic field confined to cubic box of size L , as has been done in the previous subsection, but now with including the GUP modification. Using Eq. (6.2.11), the total number of the quantized modes will be modified as follows:

$$N \rightarrow \frac{L^3}{2\pi^2} \int_0^\Lambda \frac{p^2 dp}{(1 - \alpha p)^4} \approx \frac{L^3}{2\pi^2} \left(\frac{\Lambda^3}{3} + \alpha \Lambda^4 \right). \quad (6.2.29)$$

We note the total number of states is increased due to the GUP correction. Note that, this result is valid subject to,

$$\frac{1}{\alpha} > \Lambda; \quad (6.2.30)$$

otherwise, the number of states will be infinite. This means that α gives a boundary on the cutoff Λ . This is consistent with our proposed GUP which predicts the existence of minimal measurable length as well as the maximal measurable momentum in Eq. (3.3.4) [See appendix D for an explanation of this result.]

Now turning to the modifications implied by GUP on the energy bound up to the first order of α , we find

$$E \rightarrow \frac{L^3}{2\pi^2} \int_0^\Lambda \frac{p^3 dp}{(1 - \alpha p)^4} \approx \frac{L^3}{2\pi^2} \left(\frac{\Lambda^4}{4} + \alpha \frac{4\Lambda^5}{5} \right) \leq E_{BH}. \quad (6.2.31)$$

Using Eqs. (6.2.20, 6.2.22) with the last inequality (6.2.31), we get the following UV-IR relation up to the first order of α :

$$\begin{aligned}\frac{L^3}{8\pi^2} \left(\Lambda^4 + \alpha \frac{16\Lambda^5}{5} \right) &\leq L, \\ \Lambda^4 \left(1 + \alpha \frac{16\Lambda}{5} \right) &\leq \frac{1}{L^2}, \\ \Lambda^2 &\leq \frac{1}{L} \left(1 - \frac{8\alpha}{5L^{\frac{1}{2}}} \right).\end{aligned}\tag{6.2.32}$$

On the other side, the modified maximum entropy has been calculated according to the following procedure

$$S_{max} = \ln W,\tag{6.2.33}$$

with $W \sim e^{2\sqrt{z}}$. Since z is given up to the first order of α by

$$z \rightarrow \frac{L^3}{2\pi^2} \int_0^\Lambda \left[\frac{E_{BH}}{p} \right] \frac{p^2 dp}{(1 - \alpha p)^4} \approx \frac{L^4}{2\pi^2} \left(\frac{\Lambda^2}{2} + \alpha \frac{4\Lambda^3}{3} \right),\tag{6.2.34}$$

one finds the bound when using UV-IR relation in Eq. (6.2.32):

$$\begin{aligned}z &\leq \frac{L^4}{4\pi^2} \left(\Lambda^2 + \alpha \frac{8\Lambda^3}{3} \right), \\ z &\leq \frac{L^4}{4\pi^2} \left(\frac{1}{L} \left(1 - \frac{8\alpha}{5L^{\frac{1}{2}}} \right) + \alpha \frac{8}{3L^{\frac{3}{2}}} \right), \\ z &\leq L^3 + \frac{16\alpha L^{\frac{5}{2}}}{15}.\end{aligned}\tag{6.2.35}$$

Using the boundary area of the system of Eq. (6.2.27), we find the bound for the maximum entropy will be modified as follows,

$$S_{max} = \ln W \leq A^{3/4} + \frac{16\alpha}{30} A^{1/2},\tag{6.2.36}$$

which clearly shows that the upper bound is increased due to the GUP. This means that the the maximum entropy that can be stored in a bounded region of space has been increased due to the presence of the GUP or, in other words, by considering the minimal length in Quantum Gravity. This is due to the increase in the number of states calculated in Eq. (6.2.29). This shows that the conjectured entropy of the truncated Fock space corrected by the GUP disagrees with 't Hooft's classical result which requires disagreement between the micro-canonical and canonical ensembles for a system with a large number of degrees of freedom due to the GUP-correction term. Then the holographic theory doesn't retain its good features. On the other side, since the GUP implies discreteness of space by itself as proposed in [21, 22, 78], therefore the discreteness of space will not leave the continuous symmetries such as rotation and Lorentz symmetry intact, which means in other words the holographic theory doesn't retain its good features [61]. The possibility of violating Holographic theory near the Planck scale has been discussed by many authors; see e.g [101, 102]. It seems that the holographic theory does not retain its good features by considering a minimal length in Quantum Gravity.

Chapter 7

Quantum gravity effects on the preheating phase of the universe

It is generally believed that there was a so-called *preheating* phase near the end of inflation, in which energy was rapidly transferred from the inflaton to matter fields. This was followed by *reheating*, in which thermalization took place, and most of the standard model particles in our universe were produced [103,104]. During preheating, coherent oscillations of the inflaton field $\varphi(t)$ around the minimum of its potential effectively contributed to a time-varying frequency term to the equations of motion of matter fields $\chi(t)$ coupled to it, thereby inducing instabilities by a well understood process known as *parametric resonance* (PR) [105–110]. These in turn resulted in an explosive particle production [111–113]¹(see also [103] for a recent review). Since the energy density of matter and radiation is exponentially small near the end of inflation, it has been argued that this transfer of energy from inflaton to matter fields must be fast. It was recently shown that (in)homogeneous noise, such as that arising out of quantum fluctuations, indeed increased the instability band and the rate of particle production [108–110]. Also it was shown in the past that these resonance effects were

¹It is also possible for PR to result in a particle production which will not be explosive [114,115]. We thank an anonymous referee of [27] for pointing this out to us.

sensitive to non-linearities in the equations of motion for χ , which in general were expected to increase the rate of particle production and result in an early termination of PR [106].

In this chapter, we completely analytically study the effect of one such important non-linearity, that predicted from the so-called *Generalized Uncertainty Principle* (GUP). This in turn has been predicted, from various approaches to Quantum Gravity, to replace the familiar Heisenberg's Uncertainty Principle near the Planck scale. We observe that, depending on the form of the GUP chosen, and initial conditions, an enhancement of particle production and an early termination of PR can indeed result.

7.1 Particle production: Parametric resonance

We review and closely follow the analysis of [105, 108–110], and consider an oscillating scalar field

$$\varphi(t) = h \cos(\theta t) \tag{7.1.1}$$

coupled to another scalar field χ representing the matter, via an interaction

$$\mathcal{L} \propto \frac{1}{2} \varphi \chi^2. \tag{7.1.2}$$

Then the evolution equation for χ is nothing but the Mathieu equation, of the form

$$\ddot{\chi} + \omega_0^2 \left(1 + h \cos \left[(2\omega_0 + \epsilon)t \right] \right) \chi = 0 \tag{7.1.3}$$

where the argument of the cosine, $\theta \equiv 2\omega_0 + \epsilon$, is so chosen to produce the strongest parametric resonance (PR) via the h -term. We shall assume $h, \epsilon/\omega_0 \ll 1$, and retain

7.2. Particle production: Parametric resonance

terms only to leading order in h, ϵ . Assuming a solution of the form

$$\chi(t) = a_0(t) \cos\left(\frac{\theta}{2}t\right) + b_0(t) \sin\left(\frac{\theta}{2}t\right) \quad (7.1.4)$$

with $a_0 \sim e^{s_0 t}$, $b_0 \sim e^{s_0 t}$, substituting in Eq. (7.1.3), using $\dot{a}_0 \sim \epsilon a_0$, $\dot{b}_0 \sim \epsilon b_0$ (thereby ignoring $\ddot{a}_0 \sim \epsilon^2 a_0$ and $\ddot{b}_0 \sim \epsilon^2 b_0$ terms), identities such as $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$ and $\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$, and ignoring weaker resonance terms of the form $\cos(n\frac{\theta}{2}t)$ and $\sin(n\frac{\theta}{2}t)$ ($n \in \mathbb{N} > 1$), we get

$$a_0 s_0 + \frac{b_0}{2} \left(\frac{h\omega_0}{2} + \epsilon \right) = 0 \quad (7.1.5)$$

$$b_0 s_0 + \frac{a_0}{2} \left(\frac{h\omega_0}{2} - \epsilon \right) = 0. \quad (7.1.6)$$

Solving the above two equations, we obtain

$$\frac{b_0}{a_0} = \sqrt{\frac{\frac{h\omega_0}{2} - \epsilon}{\frac{h\omega_0}{2} + \epsilon}} \equiv R \quad (7.1.7)$$

$$s_0 = \frac{1}{2} \sqrt{\left(\frac{1}{2}h\omega_0\right)^2 - \epsilon^2}. \quad (7.1.8)$$

Note that $0 \leq R < \infty$, with $R = 1$ corresponding to $a_0 = b_0$ and $R = 0$ (∞) corresponding to $b_0 = 0$ ($a_0 = 0$). Thus, when $s_0 \in \mathbb{R}$, the solution given in Eq. (7.1.4) grows exponentially in the so-called *Instability Region* of $h\omega_0$ given by the parameter range

$$-\frac{1}{2}h\omega_0 < \epsilon < \frac{1}{2}h\omega_0. \quad (7.1.9)$$

This is the phenomenon of Parametric Resonance [105].

7.2 Parametric resonance with non-linear terms

We introduce a generic non-linearity in the RHS of Eq. (7.1.3) of the following form

$$\ddot{\chi} + \omega_0^2 \left(1 + h \cos [(2\omega_0 + \epsilon)t] \right) \chi = \lambda f(\chi, \dot{\chi}) \quad (7.2.1)$$

where $f(\chi, \dot{\chi})$ is an arbitrary non-linear function. As we shall see later, λ is suppressed by powers of the Planck mass, and its effects may only show up at very high energies and very small length scales. We will thus treat this term perturbatively. Once again, we assume Eq. (7.2.1) has a solution of the form

$$\chi(t) = a(t) \cos \left(\frac{\theta}{2} t \right) + b(t) \sin \left(\frac{\theta}{2} t \right), \quad (7.2.2)$$

$$\text{where } a \sim e^{st}, b \sim e^{st} \quad (7.2.3)$$

with $a = a_0 + \lambda a_1, b = b_0 + \lambda b_1, s = s_0 + \lambda s_1$.

Thus we can substitute the ‘unperturbed’ solution given by Eq. (7.1.4) in the RHS of Eq. (7.2.1), simplify again using trigonometrical identities, and retain only the leading order resonance terms to write

$$\begin{aligned} \lambda f(\chi, \dot{\chi}) &= \lambda \omega_0 \sin \left(\frac{\theta}{2} t \right) f_1(a_0, b_0, s_0) \\ &+ \lambda \omega_0 \cos \left(\frac{\theta}{2} t \right) f_2(a_0, b_0, s_0). \end{aligned} \quad (7.2.4)$$

This modifies Eqs. (7.1.5, 7.1.6) to

$$as + \frac{b}{2} \left(\frac{h\omega_0}{2} + \epsilon \right) = -\frac{\lambda}{2} f_1 \quad (7.2.5)$$

$$bs + \frac{a}{2} \left(\frac{h\omega_0}{2} - \epsilon \right) = \frac{\lambda}{2} f_2. \quad (7.2.6)$$

Now we make a further assumption about the smallness of λ , namely that $\lambda \sim \epsilon^{1+p}$, $0 \leq p < 1$ and retain terms up to $\mathcal{O}(\epsilon^2)$ to obtain

$$s^2 = s_0^2 + \frac{\lambda}{4} \left[\frac{f_1}{b_0} \left(\frac{h\omega_0}{2} - \epsilon \right) - \frac{f_2}{a_0} \left(\frac{h\omega_0}{2} + \epsilon \right) \right]. \quad (7.2.7)$$

7.3. Parametric resonance with GUP

Comparing with the leading order expression $s^2 = s_0^2 + 2\lambda s_0 s_1$ we get

$$2s_0 s_1 = \frac{1}{4} \left[\frac{f_1}{b_0} \left(\frac{h\omega_0}{2} - \epsilon \right) - \frac{f_2}{a_0} \left(\frac{h\omega_0}{2} + \epsilon \right) \right] \quad (7.2.8)$$

$$= \frac{1}{4a_0} \left(\frac{h\omega_0}{2} + \epsilon \right) [Rf_1 - f_2] . \quad (7.2.9)$$

Thus to determine whether PR is enhanced ($s_1 > 0$) or diminished ($s_1 < 0$), one would need to find expressions for f_1, f_2 for specific models of non-linearity.

7.3 Parametric resonance with GUP

We start with a GUP-modified Hamiltonian in one dimension of the form

$$H = \frac{p^2}{2m} + V(\chi) = \frac{p_0^2}{2m} + V(\chi) + \frac{\kappa \alpha^{n-2}}{m} p_0^n , \quad (7.3.1)$$

where $\kappa = \pm 1$. This incorporates the two versions of GUP presented in chapter 3 , with $n = 4$ and $n = 3$, respectively, (cf. Eq. (3.1.7, 3.3.9)). The first of the equations of Hamilton, and its inverse, both to $\mathcal{O}(\alpha^{n-2})$ are given by

$$\dot{\chi} = \frac{\partial H}{\partial p_0} = \frac{p_0}{m} + \frac{\kappa n \alpha^{n-2}}{m} p_0^{n-1} \quad (7.3.2)$$

$$p_0 = m\dot{\chi} - n\kappa \alpha^{n-2} (m\dot{\chi})^{n-1} . \quad (7.3.3)$$

Then the second Hamilton equation $\dot{p}_0 = -\partial H/\partial \chi$ gives

$$m\ddot{\chi} [1 - \kappa n(n-1)\alpha^{n-2} m^{n-2} \dot{\chi}^{n-2}] = -\frac{\partial V}{\partial \chi} \quad (7.3.4)$$

$$\text{or, } m\ddot{\chi} + \frac{\partial V}{\partial \chi} + \kappa_1 \frac{\partial V}{\partial \chi} \dot{\chi}^{n-2} = 0 \quad (7.3.5)$$

where $\kappa_1 = \kappa n(n-1)\alpha^{n-2} m^{n-2}$. Thus for the time dependent harmonic oscillator, we get from Eq. (7.3.5) above

$$\ddot{\chi} + \omega_0^2 \left(1 + h \cos(\theta t) \right) \chi = -\kappa_1 \omega_0^2 \chi \dot{\chi}^{n-2} \quad (7.3.6)$$

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where we have ignored terms $\mathcal{O}(\kappa_1 h)$. Thus, comparing Eqs. (7.2.1) and (7.3.6), we obtain $\lambda f = -\kappa_1 \omega_0^2 \chi \dot{\chi}^{n-2}$. Further, we will make the identification $\lambda = \alpha^{n-2}$. Next we write the solution given in Eq. (7.2.2) as

$$\chi = ce^{i\frac{\theta}{2}t} + c^* e^{-i\frac{\theta}{2}t} \quad (7.3.7)$$

$$\dot{\chi} = Ae^{i\frac{\theta}{2}t} + A^* e^{-i\frac{\theta}{2}t} \quad (7.3.8)$$

$$c = \frac{1}{2} \left(a + \frac{b}{i} \right), \quad A = \frac{1}{2} \left[\left(\dot{a} + \frac{\theta}{2}b \right) + \frac{\dot{b} - \frac{\theta}{2}a}{i} \right] \quad (7.3.9)$$

from which the quantity $\chi \dot{\chi}^{n-2}$ that appears in the RHS of Eq. (7.3.6) reads

$$\chi \dot{\chi}^{n-2} = \left(ce^{i\frac{\theta}{2}t} + c^* e^{-i\frac{\theta}{2}t} \right) \left(Ae^{i\frac{\theta}{2}t} + A^* e^{-i\frac{\theta}{2}t} \right)^{n-2} \quad (7.3.10)$$

$$= \sum_{j=0}^{n-2} \binom{n-2}{j} A^j A^{*(n-2-j)} \left(ce^{i(2j-n+3)\frac{\theta}{2}t} + c^* e^{i(2j-n+1)\frac{\theta}{2}t} \right). \quad (7.3.11)$$

Setting $2j - n + 3 = \pm 1$ and $2j - n + 1 = \pm 1$ to extract the $e^{\pm i\frac{\theta}{2}t}$, i.e. the dominant resonance terms, we see $j = (n-2)/2, (n-4)/2$ and $j = n/2, (n-2)/2$, respectively. It is evident that n must be even and for the rest of this chapter, we assume that this is the case². Collecting these terms, simplifying, using $\dot{a} = sa$, $\dot{b} = sb$, and replacing $\{a, b, s\}$ by $\{a_0, b_0, s_0\}$ by noting that the above term is multiplied by the small parameter λ in the equation of motion, and finally comparing with Eq. (7.2.4)

²Higher order (weaker) resonance terms can of course arise for n odd.

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we get

$$f_1 = \ell a_0^3 \left[\frac{n}{n-2} (4s_0^2 + \theta^2) R + R(4s_0^2 - \theta^2) - 4s_0 \theta \right] \quad (7.3.12)$$

$$f_2 = \ell a_0^3 \left[\frac{n}{n-2} (4s_0^2 + \theta^2) + (4s_0^2 - \theta^2) + 4Rs_0 \theta \right] \quad (7.3.13)$$

$$\text{where } \ell = \frac{\kappa_2(1+R^2)}{16} \quad (7.3.14)$$

$$\text{and } \kappa_2 = -\kappa n(n-1)m^{n-2}\omega_0 \binom{n-2}{n/2} |A|^{n-4} \quad (7.3.15)$$

which using Eq. (7.2.9), implies in terms of the ratio R ,

$$2s_0 s_1 = \frac{\ell a_0^2}{4} \left(\frac{h\omega_0}{2} + \epsilon \right) \left[(R^2 - 1) \left(\frac{n}{n-2} (4s_0^2 + \theta^2) + (4s_0^2 - \theta^2) \right) - 8Rs_0 \theta \right]. \quad (7.3.16)$$

We use the last equation to summarize the sign of s_1 (Note that s_0 is positive)

	sign of s_1 ($\kappa > 0$)	sign of s_1 ($\kappa < 0$)
$R = 1$	+	-
$R = 0$	+	-
$R = \infty$	+	-
$R \neq 1$	\pm	\pm

Thus, we see that the initial conditions on the matter field (via R) and the GUP one is considering (via the sign of κ) determine whether there is an increase in the exponent of the matter field or not. The various auxiliary variables that were introduced (such as A , C , ℓ etc) do not play any role in it.

7.3.1 Instability region

Now setting $s = 0$ in Eqs. (7.2.5) and (7.2.6), we see that PR occurs when the modified instability region is given by

$$-\frac{1}{2}h\omega_0 - \frac{\lambda f_1}{b_0} < \epsilon < \frac{1}{2}h\omega_0 - \frac{\lambda f_2}{a_0} \quad (7.3.17)$$

whose width is thus

$$\begin{aligned} \Delta\epsilon &= h\omega_0 - \frac{\lambda}{a_0 R} (Rf_2 - f_1) \\ &= h\omega_0 - \lambda \left[\frac{4(1+R^2)}{R} s_0 \theta \ell \right] a_0^2 \\ &= h\omega_0 - 2\lambda h\omega_0 \theta \ell a_0^2 \\ &= h\omega_0 \left[1 - 2\lambda \theta \ell a_0^2 \right]. \end{aligned} \quad (7.3.18)$$

It follows that parametric resonance is maintained in the presence of GUP. However the region of instability increases when $\kappa > 0$ and vice-versa, i.e. it once again depends on the GUP under consideration, as well as the amplitude and oscillation frequency of the matter field, which may contribute to further enhancement of this region (but not on R). However, since such an increase or decrease would be proportional to inverse powers of the Planck mass, it may turn out to be too small to have an observable effect at present, although with more accurate experiments and improved observations, it may be detectable in the future. In the next section, we re-do the analysis for an expanding universe.

7.4 Parametric resonance in an expanding universe with nonlinear terms

We adopt the procedure outlined in [105, 107]. We first write the equation for the matter field in the presence of the inflaton field φ and in an expanding background [103], together with the non-linear terms described by Eq. (7.3.1) as

$$\ddot{\chi} + 3H\dot{\chi} + \omega_0^2 [1 + h \cos(2\omega_0 + \epsilon)t] \chi = \lambda f(\chi, \dot{\chi}), \quad (7.4.1)$$

where $H = \dot{a}/a$ is the Hubble parameter and $a(t)$ is the scale factor for FRW space-time. The re-definition

$$Y = a^{3/2} \chi \quad (7.4.2)$$

implying

$$\dot{\chi} = a^{-3/2} (\dot{Y} - \frac{3}{2} H Y) \quad (7.4.3)$$

$$\ddot{\chi} = a^{-3/2} (\ddot{Y} + \frac{9}{4} H^2 Y - 3H\dot{Y} - \frac{3}{2} \dot{H} Y) \quad (7.4.4)$$

substituted into (7.4.1) yields the following equation for the function Y

$$\ddot{Y} + \omega^2 Y = \lambda P[Y, \dot{Y}, a] \quad (7.4.5)$$

where

$$\omega^2(t) = \omega_0^2 (1 + h \cos \theta t) - \frac{9}{4} H^2 - \frac{3}{2} \dot{H}$$

with $\theta = 2\omega_0 + \epsilon$,

$$(7.4.6)$$

$$\text{and } P = a^{3/2} f[a^{-3/2} Y, a^{-3/2} (\dot{Y} - \frac{3}{2} H Y)]. \quad (7.4.7)$$

Next we consider the following cases.

7.4.1 Case when $H \neq 0$ and $\lambda = 0$

To gain some insight we first revisit the case where there is no nonlinearity. We assume a solution of the form

$$Y(t) = c_0 \zeta_0(a) \cos\left(\frac{\theta}{2}t\right) + d_0 \zeta_0(a) \sin\left(\frac{\theta}{2}t\right) \quad (7.4.8)$$

in which the effects of expansion are included in the scaling function $\zeta_0(a)$. Substituting into Eq. (7.4.5) with its RHS set to zero, one obtains

$$(A_1 + A_2) \cos\left(\frac{\theta}{2}t\right) + (B_1 + B_2) \sin\left(\frac{\theta}{2}t\right) + \mathcal{O}(\theta^2, h^2) = 0 \quad (7.4.9)$$

with

$$A_1 = (\dot{d}_0 \theta + \omega_0^2 c_0 + \frac{\omega_0^2 h}{2} c_0 - \frac{c_0 \theta^2}{4}) \zeta_0 \quad (7.4.10)$$

$$B_1 = (-\dot{c}_0 \theta + \omega_0^2 d_0 - \frac{\omega_0^2 h}{2} d_0 - \frac{d_0 \theta^2}{4}) \zeta_0 \quad (7.4.11)$$

$$A_2 = 2\dot{c}_0 \dot{\zeta}_0 + c_0 \ddot{\zeta}_0 + d_0 \dot{\zeta}_0 \theta - \frac{9}{4} H^2 c_0 \zeta_0 - \frac{3}{2} \dot{H} c_0 \zeta_0 \quad (7.4.12)$$

$$B_2 = 2\dot{d}_0 \dot{\zeta}_0 + d_0 \ddot{\zeta}_0 - c_0 \dot{\zeta}_0 \theta - \frac{9}{4} H^2 d_0 \zeta_0 - \frac{3}{2} \dot{H} d_0 \zeta_0 . \quad (7.4.13)$$

If Eq. (7.4.9) is to be justified, the coefficient of the sine and cosine must vanish. In addition, in order to ensure resonant behavior, we further set A_1 , A_2 , B_1 , and B_2 separately equal to zero. Thus, up to to order $\mathcal{O}(\epsilon, h)$, the coefficients A_1 and B_1 reduce to the Eqs. (7.1.5) and (7.1.6), i.e.,

$$A_1 = 2s_0 d_0 + c_0 \left(\frac{h\omega_0}{2} - \epsilon \right) = 0 \quad (7.4.14)$$

$$B_1 = -2s_0 c_0 - d_0 \left(\frac{h\omega_0}{2} + \epsilon \right) = 0 \quad (7.4.15)$$

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where s_0 is the characteristic ('unperturbed') exponent defined in Eq. (7.1.8). On the other hand, $A_2 = 0$ and $B_2 = 0$ yield

$$\begin{aligned} 2\dot{c}_0 \left(\frac{\dot{\zeta}_0}{\zeta_0} \right) + c_0 \left(\frac{\ddot{\zeta}_0}{\zeta_0} \right) + d_0 \left(\frac{\dot{\zeta}_0}{\zeta_0} \right) \theta - \left(\frac{9}{4} H^2 + \frac{3}{2} \dot{H} \right) c_0 &= 0 \\ 2\dot{d}_0 \left(\frac{\dot{\zeta}_0}{\zeta_0} \right) + d_0 \left(\frac{\ddot{\zeta}_0}{\zeta_0} \right) - c_0 \left(\frac{\dot{\zeta}_0}{\zeta_0} \right) \theta - \left(\frac{9}{4} H^2 + \frac{3}{2} \dot{H} \right) d_0 &= 0 \end{aligned}$$

which can be rewritten as

$$\frac{1}{\zeta_0 c_0} \frac{d}{dt} (c_0^2 \dot{\zeta}_0) + d_0 \left(\frac{\dot{\zeta}_0}{\zeta_0} \right) \theta - \left(\frac{9}{4} H^2 + \frac{3}{2} \dot{H} \right) c_0 = 0 \quad (7.4.16)$$

$$\frac{1}{\zeta_0 d_0} \frac{d}{dt} (d_0^2 \dot{\zeta}_0) - c_0 \left(\frac{\dot{\zeta}_0}{\zeta_0} \right) \theta - \left(\frac{9}{4} H^2 + \frac{3}{2} \dot{H} \right) d_0 = 0. \quad (7.4.17)$$

The two equations above may be combined to give (setting $c_0 = C_0 e^{s_0 t}$ and $d_0 = D_0 e^{s_0 t}$ in which C_0 and D_0 are constants)

$$\frac{d}{dt} (e^{2s_0 t} \dot{\zeta}_0) - \left(\frac{9}{4} H^2 + \frac{3}{2} \dot{H} \right) \zeta_0 e^{2s_0 t} = 0. \quad (7.4.18)$$

This equation may be solved following the procedure given in [107]. For a cosmic-time scale factor of the form $a(t) \propto t^q$, we have³

$$\dot{H} + H^2 = q(q-1)t^{-2} = \alpha(q)t^{-2}, \quad H = qt^{-1}, \quad (7.4.19)$$

then Eq. (7.4.18) becomes

$$\ddot{\zeta}_0 + 2s_0 \dot{\zeta}_0 - \beta(q) t^{-2} \zeta_0 = 0 \quad (7.4.20)$$

where we have introduced the coefficient $\beta(q) = \frac{3}{4}q^2 + \frac{3}{2}\alpha(q)$. If we use the fact that s_0 is of order ϵ and h , then we are allowed to approximate the above equation to

$$\ddot{\zeta}_0 - \beta(q) t^{-2} \zeta_0 \simeq 0. \quad (7.4.21)$$

³It is noteworthy that $\alpha(q) \geq 0$ for $q \geq 1$, and that for de Sitter expansion, $\alpha(\infty) = \infty$.

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The solution to this equation takes the form

$$\begin{aligned}\zeta_0(t) &= C_1 t^{\frac{1}{2}(1+\sqrt{1+4\beta})} + C_2 t^{\frac{1}{2}(1-\sqrt{1+4\beta})} \\ &= C_1 t^{\frac{3q}{2}} + C_2 t^{\frac{2-3q}{2}} .\end{aligned}\quad (7.4.22)$$

With this solution for $\zeta_0(t)$, the resonant solution reads

$$Y^\pm(t) = C_0 \zeta_0(t) e^{\pm s_0 t} \left[\cos\left(\frac{\theta}{2}t\right) \mp R \sin\left(\frac{\theta}{2}t\right) \right] .\quad (7.4.23)$$

where R stands for d_0/c_0 .

Following the normalization chosen in [114], C_0 has been determined to be $C_0 = \sqrt{\frac{1}{R\theta}}$. By comparing the solution given by Eq. (7.4.22) to the solution associated with the nonresonant case as given in [107], we infer that the second exponent in Eq. (7.4.22) must be neglected. Thus, the appropriate exponent for the resonant case is the first one. The requirement that $\zeta_0[a(t)] = 1$ when $a(t) = 1$ implies the normalization $\zeta_0 = a(t)^{3/2}$. Therefore, the full resonant solution in an expanding background behaves as

$$Y^\pm(t) = \sqrt{\frac{1}{R\theta}} e^{s_\pm t} \left[\cos\left(\frac{\theta}{2}t\right) \mp R \sin\left(\frac{\theta}{2}t\right) \right] \quad (7.4.24)$$

with the characteristic exponent $s_\pm(t)$ defined by

$$s_\pm(t) = \pm s_0 + \frac{3q}{2t} \ln t .\quad (7.4.25)$$

Clearly, the stability band width will decrease after taking the expanding background into account. In particular, following Eq. (7.1.9) the bounds of the Instability Region in a flat spacetime are

$$\epsilon_{min} = -\sqrt{\left(\frac{h\omega_0}{2}\right)^2} \quad \text{and} \quad \epsilon_{max} = +\sqrt{\left(\frac{h\omega_0}{2}\right)^2} \quad (7.4.26)$$

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while when the background is an expanding one as described above the bounds are modified as follows [105, 107, 111]

$$\epsilon_{min} = -\sqrt{\left(\frac{h\omega_0}{2}\right)^2 - 4\left(\frac{3q}{2t}\ln t\right)^2} \quad (7.4.27)$$

$$\epsilon_{max} = +\sqrt{\left(\frac{h\omega_0}{2}\right)^2 - 4\left(\frac{3q}{2t}\ln t\right)^2}. \quad (7.4.28)$$

Now we are in a position to generalize the above calculation to the nonlinear model governed by Eq. (7.4.5), and find the analogue of Eq. (7.4.18).

7.4.2 Case when $H \neq 0$ and $\lambda \neq 0$

We reconsider equation (7.4.5) with the nonlinear terms included

$$\ddot{Y} + \omega^2 Y = \lambda P[Y, \dot{Y}, a] \quad (7.4.29)$$

and thus, the solution will now be of the form

$$Y(t) = c(t)\zeta(a) \cos\left(\frac{\theta}{2}t\right) + d(t)\zeta(a) \sin\left(\frac{\theta}{2}t\right) \quad (7.4.30)$$

with $c = c_0 + \lambda c_1$, $d = d_0 + \lambda d_1$, $s = s_0 + \lambda s_1$, $\zeta = \zeta_0 + \lambda \zeta_1$.

Next we substitute the unperturbed solution given in Eq. (7.4.8) in the right-hand side of Eq. (7.4.29), and as in the previous sections perform the trigonometrical approximations and retain terms of order $\mathcal{O}(h)$. We thus write

$$\begin{aligned} \lambda P[Y, \dot{Y}, a] &= \lambda\omega_0 \sin\left(\frac{\theta}{2}t\right) P_1[c_0, d_0, s_0, \zeta_0, a] \\ &\quad + \lambda\omega_0 \cos\left(\frac{\theta}{2}t\right) P_2[c_0, d_0, s_0, \zeta_0, a]. \end{aligned} \quad (7.4.31)$$

Then we obtain up to order $\mathcal{O}(h, \lambda)$

$$(B - \lambda\omega_0 P_1) \sin\left(\frac{\theta}{2}t\right) + (A - \lambda\omega_0 P_2) \cos\left(\frac{\theta}{2}t\right) + \mathcal{O}(\theta^2, h^2, \lambda^2) = 0 \quad (7.4.32)$$

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with

$$B = B_1 + B_2, \quad A = A_1 + A_2, \quad (7.4.33)$$

and the modification of Eqs. (7.4.10), (7.4.13) to read

$$B_1 = \left(-c\theta + \omega_0^2 d - \frac{\omega_0^2 h}{2} d - \frac{d\theta^2}{4} \right) \zeta \quad (7.4.34)$$

$$A_1 = \left(\dot{d}\theta + \omega_0^2 c + \frac{\omega_0^2 h}{2} c - \frac{c\theta^2}{4} \right) \zeta \quad (7.4.35)$$

$$B_2 = 2\dot{d}\dot{\zeta} + d\ddot{\zeta} - c\dot{\zeta}\theta - \frac{9}{4}H^2 d\zeta - \frac{3}{2}\dot{H}d\zeta \quad (7.4.36)$$

$$A_2 = 2\dot{c}\dot{\zeta} + c\ddot{\zeta} + d\dot{\zeta}\theta - \frac{9}{4}H^2 c\zeta - \frac{3}{2}\dot{H}c\zeta. \quad (7.4.37)$$

Following the same methodology of the previous section, namely demanding resonant behavior, one can set

$$B_1 = \lambda\omega_0 P_1 \quad (7.4.38)$$

$$A_1 = \lambda\omega_0 P_2 \quad (7.4.39)$$

$$B_2 = 0 \quad (7.4.40)$$

$$A_2 = 0. \quad (7.4.41)$$

From the first two equations and up to order $\mathcal{O}(\epsilon, h, \lambda)$ it follows that

$$2sc + d\left(\frac{h\omega_0}{2} + \epsilon\right) = -\lambda\left(\frac{P_1}{\zeta_0}\right), \quad (7.4.42)$$

$$2sd + c\left(\frac{h\omega_0}{2} - \epsilon\right) = \lambda\left(\frac{P_2}{\zeta_0}\right). \quad (7.4.43)$$

We may now assume that $\lambda \sim \epsilon^{1+p}$ with $0 \leq p < 1$, and keep terms up to order $\mathcal{O}(\epsilon^2)$, we then find

$$2s_0s_1 = \frac{1}{4\zeta_0c_0}\left(\frac{h\omega_0}{2} + \epsilon\right)[RP_1 - P_2] \quad (7.4.44)$$

which reduces, for $\zeta_0 = 1$ when $a = 1$ to Eq. (7.2.9).

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Rewriting equations $A_2 = 0$ and $B_2 = 0$ as in the previous section yields

$$\frac{d}{dt}(e^{2st}\dot{\zeta}) - \left(\frac{9}{4}H^2 + \frac{3}{2}\dot{H}\right)\zeta e^{2st} = 0. \quad (7.4.45)$$

where s is defined by

$$s = s_0 + \lambda s_1 \quad (7.4.46)$$

and s_1 is given by Eq. (7.4.44). At this point, it should be noted that s_1 is time-dependent, so is s . Equation (7.4.45) may be written out as

$$\ddot{\zeta} + (2\lambda\dot{s}_1 t + 2s)\dot{\zeta} - \left(\frac{9}{4}H^2 + \frac{3}{2}\dot{H}\right)\zeta = 0 \quad (7.4.47)$$

giving at order $\mathcal{O}(\lambda^0)$ and $\mathcal{O}(\lambda)$, respectively,

$$\ddot{\zeta}_0 + 2s_0\dot{\zeta}_0 - \left(\frac{9}{4}H^2 + \frac{3}{2}\dot{H}\right)\zeta_0 = 0 \quad (7.4.48)$$

$$\ddot{\zeta}_1 + 2s_0\dot{\zeta}_1 - \left(\frac{9}{4}H^2 + \frac{3}{2}\dot{H}\right)\zeta_1 = -2(\dot{s}_1 t + s_1)\dot{\zeta}_0. \quad (7.4.49)$$

The solution to the first equation has been determined in the previous section to be

$$\zeta_0(t) = t^{\frac{3}{2}q}. \quad (7.4.50)$$

The task now is to substitute Eqs. (7.4.50) and (7.4.44) into Eq. (7.4.49), and ignore terms of order ϵ as they are small compared to the others. We employ Eq. (7.4.44) in order to evaluate s_1 but for this purpose we also need to specify the form of functions P_1 and P_2 . This would determine explicitly function ζ_1 and thus the complete resonant solutions in an expanding background where nonlinearities are present will be of the form

$$Y^\pm(t) \propto e^{\left[\pm s_0 + \frac{3q}{2i} \ln t + \lambda(s_1 + \frac{\zeta_1}{\zeta_0 t})\right]t} \left[\cos\left(\frac{\theta}{2}t\right) \mp R \sin\left(\frac{\theta}{2}t\right) \right] \quad (7.4.51)$$

where s_1 , ζ_0 , and ζ_1 are all known for a given nonlinearity expressed by P_1 and P_2 .

7.4.3 Parametric resonance in an expanding background with GUP

In this subsection we consider the nonlinearity to be the GUP term, namely,

$$\lambda P = -\kappa_1 \omega_0^2 \chi \dot{\chi}^{n-2} \quad (7.4.52)$$

with

$$\chi = a^{-3/2} Y, \quad \dot{\chi} = a^{-3/2} (\dot{Y} - \frac{3}{2} H Y). \quad (7.4.53)$$

In order to find the expressions of P_1 and P_2 defined in Eq. (7.4.31), one can follow the steps outlined in section V, and simply make the following replacements

$$\begin{aligned} a &\rightarrow a^{-3/2} c \zeta, & b &\rightarrow a^{-3/2} d \zeta, & c &\rightarrow \frac{1}{2} a^{-3/2} (c - id) \zeta, \\ A &\rightarrow B = a^{-3/2} \left(\bar{A} - \frac{3}{4} H (c - id) \zeta \right), \end{aligned} \quad (7.4.54)$$

with $\bar{A} = \frac{1}{2} [(\dot{c}\zeta + c\dot{\zeta} + d\zeta\frac{\theta}{2}) - i(\dot{d}\zeta + d\dot{\zeta} - c\zeta\frac{\theta}{2})]$. It should be stressed that the quantity “ a ” that appears on the right-hand side of the above replacements is the scale factor of the expanding universe. Thus the expressions of P_1 and P_2 read

$$\begin{aligned} P_1 &= \ell_0 a^{-9/2} c_0^3 \zeta_0^3 \\ &\times \left[\frac{n}{n-2} (4s_0^2 + \theta^2) R_0 + R_0 (4s_0^2 - \theta^2) - 4s_0 \theta \right] \end{aligned} \quad (7.4.55)$$

$$\begin{aligned} P_2 &= \ell_0 a^{-9/2} c_0^3 \zeta_0^3 \\ &\times \left[\frac{n}{n-2} (4s_0^2 + \theta^2) + (4s_0^2 - \theta^2) + 4R_0 s_0 \theta \right] \end{aligned} \quad (7.4.56)$$

$$\text{with } R_0 = \frac{d_0}{c_0}, \quad \ell_0 = \frac{\kappa_2^{(0)} (1 + R_0^2)}{16}, \quad (7.4.57)$$

$$\text{and } \kappa_2^{(0)} = -\kappa n (n-1) m^{n-2} \omega_0 \binom{n-2}{n/2} |B_0|^{n-4} \quad (7.4.58)$$

where B_0 is the quantity B as defined in Eq. (7.4.54) but in which c, d, ζ have been replaced with c_0, d_0, ζ_0 , respectively.

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Note that these two expressions reduce respectively to Eq. (7.3.12) and Eq. (7.3.13) when we take the limit $H = 0$, $a(t) = 1$, and $\zeta_0(a) = 1$. Furthermore, the expression of s_1 as given in Eq. (7.2.9) now takes the form

$$s_1 = \frac{\ell_0 a^{-9/2} c_0^2 \zeta_0^2}{4s_0} \left(\frac{h\omega_0}{2} + \epsilon \right) \left[(R_0^2 - 1) \left(\frac{n}{n-2} (4s_0^2 + \theta^2) + (4s_0^2 - \theta^2) \right) - 8R_0 s_0 \theta \right]. \quad (7.4.59)$$

where n can be equal to 4 or 3 depending on the version of GUP under consideration. In Section V was shown that when n is even one obtains the dominant resonance terms, while when n is odd one gets the higher order resonance terms which are weaker. Thus, employing $c_0 = C_0 e^{s_0 t}$, $a(t) = t^q$, $H = qt^{-1}$, $\zeta_0 = t^{\frac{3q}{2}}$, and keeping only terms of order $\mathcal{O}(\epsilon^0)$, Eq. (7.4.59) for $n = 4$ becomes

$$s_1 = -\frac{3\kappa m^2 \omega_0^3}{4s_0} (R_0^4 - 1) \left(\frac{h\omega_0}{2} + \epsilon \right) C_0^2 e^{2ts_0} t^{-\frac{3q}{2}}. \quad (7.4.60)$$

Accordingly, Eq. (7.4.49) is now written as

$$\begin{aligned} \ddot{\zeta}_1 - \beta(q)t^{-2}\zeta_1 &\simeq \frac{9\kappa m^2 \omega_0^3}{4s_0} q (R_0^4 - 1) \left(\frac{h\omega_0}{2} + \epsilon \right) \\ &\times C_0^2 e^{2ts_0} t^{-1}. \end{aligned} \quad (7.4.61)$$

The solution of Eq. (7.4.61) for ζ_1 together with the expression (7.4.60) of s_1 will determine the form of the solution (7.4.51).

Once again, from Eq. (7.4.60) one can infer about the sign of s_1 , which as in the case of a static background, depends only on the GUP considered (sign of κ) and the initial conditions on the matter field (via R).

	sign of s_1 ($\kappa > 0$)	sign of s_1 ($\kappa < 0$)
$R > 1$	–	+
$R < 1$	+	–

It is evident that $s_1 = 0$ for $R = 1$.

7.4.4 Instability region

In this case, the instability region is obtained by setting the exponent s of the complete resonant solutions in an expanding background with nonlinearities (Eq. (7.4.51)) to zero. The exponent is

$$s = \pm s_0 + \frac{3q}{2t} \ln t + \lambda \left(s_1 + \frac{\zeta_1}{\zeta_0 t} \right) \quad (7.4.62)$$

and setting $s = 0$ and using Eq. (7.1.8), one gets

$$\epsilon_{min} = -\sqrt{\left(\frac{h\omega_0}{2}\right)^2 - 4\left(\frac{3q}{2t} \ln t + \lambda\left(s_1 + \frac{\zeta_1}{\zeta_0 t}\right)\right)^2} \quad (7.4.63)$$

$$\epsilon_{max} = +\sqrt{\left(\frac{h\omega_0}{2}\right)^2 - 4\left(\frac{3q}{2t} \ln t + \lambda\left(s_1 + \frac{\zeta_1}{\zeta_0 t}\right)\right)^2}. \quad (7.4.64)$$

with the instability region given as usual by $\Delta\epsilon = \epsilon_{max} - \epsilon_{min}$. A number of comments are in order here. First, it is easily seen that switching off GUP by setting $\lambda = 0$ reduces Eqs. (7.4.63) and (7.4.64) to the ones relevant for an expanding universe without non-linearities, cf. Eqs. (7.4.27) and (7.4.28). Similarly, setting $\lambda = 0$ and $q = 0$ reduces the above to ordinary PR, and the corresponding range Eq. (7.1.9). Finally, although by setting $q = 0$ one should in principle recover the width given by Eq. (7.3.18), (by finding the solution for ζ_1 in Eq. (7.4.61) and using the expression for s_1 in Eq. (7.4.60)), this is seen much more easily by setting $s = 0$ in Eqs. (7.4.42) and (7.4.43), leading to the instability band

$$\Delta\epsilon = h\omega_0 - \frac{\lambda}{c_0 R \zeta_0} (RP_2 - P_1) \quad (7.4.65)$$

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which on using Eqs. (7.4.55) and (7.4.56) for P_1 and P_2 , respectively, reduces to

$$\begin{aligned}
 \Delta\epsilon &= h\omega_0 - \frac{\lambda\ell_0 a^{-9/2} c_0^2 \zeta_0^2}{R} \left[4(1 + R^2) s_0 \theta \right] \\
 &= h\omega_0 - 2\lambda h\omega_0 \theta \ell_0 c_0^2 \zeta_0^2 a^{-9/2} \\
 &= h\omega_0 \left[1 - 2\lambda \theta \ell_0 c_0^2 \zeta_0^2 a^{-9/2} \right].
 \end{aligned} \tag{7.4.66}$$

In this case too, the increase in the instability region depends on the GUP parameter, as well as parameters related to the expansion of the background, which may further magnify the GUP effect. Still the effect may remain small and unobservable due to powers of inverse Planck mass (via λ), although the situation can change with better experiments and observations in the future. It is evident that setting $a(t) = 1$ and $\zeta_0 = 1$, and also replacing $c_0 \rightarrow a_0$, reduces the above to Eq. (7.3.18).

Chapter 8

Are black holes created at LHC?

A possible discrepancy has been found between the results of recent experiments at the LHC and the predictions of large extra dimensions theories [28]. This experiment suggests that there are no signs of black holes at energies $3.5 - 4.5$ TeV, which large extra dimension theories cannot explain. In this chapter, we investigate whether the Generalized Uncertainty Principle (GUP), proposed by some approaches to quantum gravity such as String Theory and Doubly Special Relativity Theories (DSR), can explain the experimental results. This implies the necessity of existence of mechanisms such as GUP beside large extra dimensions theories to explain the experimental results while not ruling out extra dimensions theories.

The proposals for the existence of extra dimensions has opened up new doors of research in quantum gravity [116–119]. In particular, a host of interesting work is being done on different aspects of low-energy scale quantum gravity phenomenology. One of the most significant sub-fields is the study of black hole (BH) and brane production at the LHC [120].

In this chapter, we present a phenomenological study of the black holes in higher dimensions at the Large Hadron Collider (LHC) if GUP that follows from the Jacobi

identity is taken into consideration; see Eqs. (3.1.1, 3.3.1). If the black hole can be produced and detected, it would result in an additional mass threshold above the Planck scale at which new physics can be found. The scope of the present work is to investigate the effect of GUP on the Hawking temperature, entropy, and BH decay rate. We found that the BH thermodynamics dramatically changed if the GUP parameter is non-vanishing.

We also obtained an interesting result that black holes may not be detectable at the current LHC energy scales. This result possibly agrees with the very recent experiment that was done at LHC [28], which says that there are no signs of mini black holes at energies of 3.5 – 4.5 TeV.

The effect of the GUP on Black Holes has been studied before with different versions of GUP which does not follow from Jacobi Identity, see e.g. [121], however the previous studies predicted that BH's can be seen at the LHC energy scales in disagreement with the recent experimental results of LHC [28]. So our main result possibly agrees with the very recent experiment that was done at LHC [28]. At the same time it does not rule out string theory and large extra dimensions predictions if minimal length in quantum gravity is taken into consideration [29].

8.1 Hawking temperature–uncertainty relation connection

In this section, we review the connection between between standard Hawking temperature and uncertainty relation that has been proposed by Adler et al. in [77] and has been generalized in large extra dimensions by Cavaglia et al in [121]. A BH could

8.1. *Hawking temperature–uncertainty relation connection*

be modeled as a $(D - 1)$ -dimensional sphere of size equal to twice of Schwarzschild radius, r_s . Since the Hawking radiation is a quantum process, the emitted particle should obey the Heisenberg uncertainty relation. This leads to momentum-position uncertainty,

$$\Delta p_i \Delta x_j \geq \frac{\hbar}{2} \delta_{ij}, \quad (8.1.1)$$

where the uncertainty in position of emitted Hawking particle has its minimum value given by

$$\Delta x \approx 2r_s = 2\lambda_D \left[\frac{G_D M}{c^2} \right]^{\frac{1}{D-3}}, \quad (8.1.2)$$

where $\lambda_D = \left[\frac{16\pi}{(D-2)\Omega_{D-2}} \right]^{\frac{1}{D-3}}$, and $\Omega_D = \frac{2\pi^{\frac{D-1}{2}}}{\Gamma(\frac{D-1}{2})}$.

Using Eq. (8.1.1,8.1.2) with the argument used in [121] that $\Delta x_i \Delta p_i \approx \Delta x \Delta p$, the energy uncertainty of the emitted Hawking particle is given by

$$\Delta E \approx c \Delta p = c \frac{\hbar}{2 \Delta x} \approx c \frac{\hbar}{4 r_s} = \frac{M_p c^2}{4\lambda_D} \left(\frac{M}{M_p} \right)^{\frac{-1}{(D-3)}}. \quad (8.1.3)$$

From now on, we can assume $m = \frac{M}{M_p}$, where m is the mass in units of the Planck mass and the Planck mass M_p is given by $M_p = \left[\frac{\hbar^{D-3}}{c^{D-5} G_D} \right]^{\frac{1}{(D-2)}}$ in D-dimensions. As proposed by Adler et al. in [77], one can identify the energy uncertainty ΔE as the energy of the emitted photon from the black hole. Based on this argument, one can get the characteristic temperature of the emitted Hawking particle from the previous energy by just multiplying it with a calibration factor $\left(\frac{D-3}{\pi} \right)$ to give exactly the Hawking temperature [63] in D-dimensions of the spacetime as follows:

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$$T_H = \frac{D-3}{4\pi\lambda_D} M_p c^2 m^{\frac{-1}{(D-3)}}. \quad (8.1.4)$$

The thermodynamical properties of the BH can be computed via the usual thermodynamic relations. The entropy can be calculated using the first law of black hole thermodynamics,

$$dM = \frac{1}{c^2} T dS. \quad (8.1.5)$$

Using the mass in units of the Planck mass, m , one can rewrite Eq. (8.1.5) as follows:

$$dS = M_p c^2 \frac{1}{T} dm. \quad (8.1.6)$$

By integrating Eq. (8.1.6) using Eq. (8.1.4), one can obtain the the Bekenstein entropy [45] as follows

$$S = \frac{4\pi\lambda_D}{D-2} m^{(D-2)/(D-3)}. \quad (8.1.7)$$

The specific heat can be calculated using the thermodynamical relation

$$\mathcal{C} = T \frac{\partial S}{\partial T} = T \frac{\partial S}{\partial m} \frac{\partial m}{\partial T} = M_p c^2 \frac{\partial m}{\partial T}, \quad (8.1.8)$$

where we have used Eq. (8.1.6) in the last equation.

By differentiating Eq. (8.1.4) and substituting this into Eq. (8.1.8), the specific heat could be given by

$$\mathcal{C} = -4\pi\lambda_D m^{\frac{(D-2)}{(D-3)}}, \quad (8.1.9)$$

The Hawking temperature T_H can be used in the calculation of the emission rate. The emission rate might be calculated using Stefan-Boltzmann law if the energy loss

8.1. *Hawking temperature–uncertainty relation connection*

was dominated by photons. Assuming a \mathcal{D} -dimensional spacetime brane, the thermal emission in the bulk of the brane can be neglected and the black hole is supposed to radiate mainly on the brane [122–126], so the emission rate on the brane can be given by:

$$\frac{dM}{dt} \propto T^{\mathcal{D}}. \quad (8.1.10)$$

Because the black hole radiates mainly on the brane [122–126], i.e $\mathcal{D} = 4$, the emission rate can be found as following:

$$\frac{dm}{dt} = -\frac{\mu'}{t_p} m^{\frac{-2}{(\mathcal{D}-3)}}, \quad (8.1.11)$$

where $t_p = \left(\frac{\hbar G_{\mathcal{D}}}{c^{\mathcal{D}+1}}\right)^{\frac{1}{(\mathcal{D}-2)}}$ is the Planck time, and the form of μ can be found in [121].

The decay time of the black hole can be obtained by integrating Eq. (8.1.11) to give

$$\tau = \mu'^{-1} \left(\frac{D-3}{D-1}\right) m_i^{\frac{(D-1)}{(D-3)}} t_p. \quad (8.1.12)$$

Note that the calculated Hawking temperature T_H , Bekenstein entropy S , specific heat \mathcal{C} , emission rate $\frac{dm}{dt}$, and decay time τ lead to ***catastrophic evaporation*** as $m \rightarrow 0$. This can be explained as follows. Since $\mathcal{C} = 0$ only when $m = 0$, the black hole will continue to radiate until $m = 0$. But as the black hole approaches zero mass, its temperature approaches infinity with infinite radiation rate. This was just a brief summary for the Hawking radiation-Uncertainty principle connection, and the catastrophic implications of Hawking radiation as the black hole mass approaches zero. In the next two sections, we study BH thermodynamics if GUP is taken into consideration. The end-point of Hawking radiation is not catastrophic because GUP implies

the existence of BH remnants at which the specific heat vanishes and, therefore, the BH cannot exchange heat with the surrounding space. The GUP prevents BHs from evaporating completely, just like the standard uncertainty principle prevents the hydrogen atom from collapsing [77, 121].

8.2 GUP quadratic in Δp and BH thermodynamics

In this section, we analyze of BH thermodynamics if GUP proposed in [10–20] is taken into consideration.

The emitted particles as Hawking radiation are mostly photons and standard model (SM) particles. According to the ADD model of extra dimensions [116], photons and SM particles are localized to the brane. So the photons or SM particles have mainly 4-components momentum and the other components in the extra dimensions are equal to zero. For simplicity, we might assume from kinetic theory of gases which assumes a cloud of points in velocity space, equally spread in all directions (there is no reason particle would prefer to be moving in the x-direction, say, rather than the y-direction) and consider:

$$p_1 \approx p_2 \approx p_3 \tag{8.2.1}$$

This assumption leads to

$$\begin{aligned} p^2 &= \sum_{i=1}^3 p_i p_i \approx 3 p_i^2 \\ \langle p_i^2 \rangle &\approx \frac{1}{3} \langle p^2 \rangle. \end{aligned} \tag{8.2.2}$$

So Eq. (3.1.1) reads, using the argument used in [121],

$$\Delta x \Delta p \geq \frac{\hbar}{2} \left[1 + \frac{5}{3} \beta \langle p^2 \rangle \right], \quad (8.2.3)$$

Now, we want to find the relation between $\langle p^2 \rangle$ and Δp^2 . We can assume that we have a photon gas emitted from the BH like emission from a black body. Therefore, we might use Wien's Law which gives a temperature corresponds to a peak emission at energy given by

$$c \langle p \rangle = 2.821 T_H. \quad (8.2.4)$$

From the Hawking-Uncertainty connection proposed by Adler et al. in [77] and that was generalized in large extra dimensions by Cavaglia et al in [121], we have

$$T_H = \frac{D-3}{\pi} c \Delta p = \frac{1}{2.821} c \langle p \rangle. \quad (8.2.5)$$

We get the following relations using the relation $\langle p^2 \rangle = \Delta p^2 + \langle p \rangle^2$:

$$\begin{aligned} \langle p \rangle &= 2.821 \frac{D-3}{\pi} \Delta p = \sqrt{\mu} \Delta p, \\ \langle p^2 \rangle &= (1 + \mu) \Delta p^2, \quad \text{where} \quad \mu = \left(2.821 \frac{D-3}{\pi} \right)^2. \end{aligned} \quad (8.2.6)$$

Using Eqs. (8.2.2,8.2.6) in the inequality (8.2.3), we get

$$\Delta x \Delta p \geq \frac{\hbar}{2} \left[1 + \frac{5}{3} (1 + \mu) \beta_0 \ell_p^2 \frac{\Delta p^2}{\hbar^2} \right]. \quad (8.2.7)$$

By solving the inequality (8.2.7) as a quadratic equation in Δp , we obtain

8.2. GUP quadratic in Δp and BH thermodynamics

$$\frac{\Delta p}{\hbar} \geq \frac{\Delta x}{\frac{5}{3}(1+\mu)\beta_0\ell_p^2} \left[1 - \sqrt{1 - \frac{\frac{5}{3}(1+\mu)\beta_0\ell_p^2}{\Delta x^2}} \right]. \quad (8.2.8)$$

where we considered only the negative sign(-) solution which gives the standard uncertainty relation as $\frac{\ell_p}{\Delta x} \rightarrow 0$.

Using the same arguments that were used in the previous section, the modified Hawking temperature will be given by:

$$T'_H = \frac{D-3}{\pi\beta_0} \frac{M_p c^2}{\frac{5}{3}(1+\mu)} m^{\frac{1}{D-3}} \lambda_D \left[1 - \sqrt{1 - \frac{\frac{5}{3}(1+\mu)\beta_0}{4\lambda_D^2 m^{\frac{2}{D-3}}}} \right]. \quad (8.2.9)$$

$$= 2T_H \left[1 + \sqrt{1 - \frac{\frac{5}{3}(1+\mu)\beta_0}{4\lambda_D^2 m^{\frac{2}{D-3}}}} \right]^{-1}. \quad (8.2.10)$$

The modified Hawking temperature is physical so long as the black hole mass satisfies the following inequality:

$$4\lambda_D^2 m^{\frac{2}{D-3}} \geq \frac{5}{3}(1+\mu)\beta_0 \quad (8.2.11)$$

This tells us the black hole should have minimum mass M_{min} given by

$$M_{min} = M_p \left(\sqrt{\frac{\frac{5}{3}(1+\mu)}{4}} \right)^{D-3} \frac{D-2}{8\Gamma(\frac{D-1}{2})} (\sqrt{\beta_0}\sqrt{\pi})^{D-3}. \quad (8.2.12)$$

The endpoint of Hawking evaporation in the GUP-case is characterized by a Planck-size remnant with maximum temperature

$$T_{max} = 2T_H. \quad (8.2.13)$$

The emission rate can be calculated using the Stefan-Boltzmann Law, using Eq. (8.1.10,8.1.11). Since the BH is mostly emitting on the brane, we consider a 4-dimensional brane, so we get

8.3. GUP linear and quadratic in Δp and BH thermodynamics

$$\frac{dm}{dt} = -16 \frac{\mu'}{t_p} m^{\frac{-2}{D-3}} \left[1 + \sqrt{1 - \frac{\frac{5}{3} (1 + \mu) \beta_0}{4\lambda_D^2 m^{\frac{2}{D-3}}}} \right]^{-4} \quad (8.2.14)$$

The entropy can be calculated from the first law of BH-thermodynamics,

$$dS = \frac{2\pi}{D-3} \lambda_D m^{\frac{1}{D-3}} \left[1 + \sqrt{1 - \frac{\frac{5}{3} (1 + \mu) \beta_0}{4\lambda_D^2 m^{\frac{2}{D-3}}}} \right] dm. \quad (8.2.15)$$

The specific heat has been calculated in GUP-case to give

$$\begin{aligned} \mathcal{C} &\equiv T \frac{\partial S}{\partial T} = M_p c^2 \frac{\partial m}{\partial T} \\ &= -2\pi \lambda_d m^{\frac{(d-2)}{(d-3)}} \sqrt{1 - \frac{\frac{5}{3} (1 + \mu) \beta_0}{4\lambda_D^2 m^{\frac{2}{D-3}}}} \left(1 + \sqrt{1 - \frac{\frac{5}{3} (1 + \mu) \beta_0}{4\lambda_D^2 m^{\frac{2}{D-3}}}} \right). \end{aligned} \quad (8.2.16)$$

We note the BH specific heat vanishes at the minimum BH-mass. Therefore, the BH cannot exchange heat with the surrounding space. This may solve the problem of *catastrophic evaporation* of the BH that was discussed in the previous section.

8.3 GUP linear and quadratic in Δp and BH thermodynamics

In this section, we would like to find the inequality corresponding to Eq. (3.3.1) in $(D-1)$ -dimensions. Eq. (3.3.1) gives using the argument in [121],

$$\Delta x \Delta p \geq \frac{\hbar}{2} \left[1 - \alpha \langle p \rangle - \alpha \langle \frac{p_i^2}{p} \rangle + \alpha^2 \langle p^2 \rangle + 3\alpha^2 \langle p_i^2 \rangle \right]. \quad (8.3.1)$$

Using arguments in Eqs. (8.2.2, 8.2.6), in the inequality (8.3.1), we get

$$\Delta x \Delta p \geq \frac{\hbar}{2} \left[1 - \alpha_0 \ell_p \left(\frac{4}{3} \right) \sqrt{\mu} \frac{\Delta p}{\hbar} + 2 (1 + \mu) \alpha_0^2 \ell_p^2 \frac{\Delta p^2}{\hbar^2} \right]. \quad (8.3.2)$$

8.3. GUP linear and quadratic in Δp and BH thermodynamics

The last inequality is (and as far as we know the only one) following from Eq. (3.3.1).

By solving the inequality (8.3.2) as quadratic equation in Δp , we obtain

$$\frac{\Delta p}{\hbar} \geq \frac{2\Delta x + \alpha_0 \ell_p \left(\frac{4}{3} \sqrt{\mu}\right)}{4(1+\mu) \alpha_0^2 \ell_p^2} \left[1 - \sqrt{1 - \frac{8(1+\mu) \alpha_0^2 \ell_p^2}{(2\Delta x + \alpha_0 \ell_p \left(\frac{4}{3} \sqrt{\mu}\right))^2}} \right]. \quad (8.3.3)$$

where we considered only the negative sign(-) solution which gives the standard uncertainty relation as $\frac{\ell_p}{\Delta x} \rightarrow 0$.

Using the same arguments that were used in the previous section, the modified Hawking temperature will be given by:

$$T'_H = \frac{D-3}{\pi \alpha_0^2} \frac{M_p c^2}{(1+\mu)} \left(m^{\frac{1}{D-3}} \lambda_D + \frac{\alpha_0 \sqrt{\mu}}{3} \right) \left[1 - \sqrt{1 - \frac{(1+\mu) \alpha_0^2}{2 \left(\lambda_D m^{\frac{1}{D-3}} + \frac{\alpha_0 \sqrt{\mu}}{3} \right)^2}} \right] \quad (8.3.4)$$

$$= 2T_H \left(1 + \frac{\alpha_0 \sqrt{\mu}}{3 \lambda_D m^{\frac{1}{D-3}}} \right)^{-1} \left[1 + \sqrt{1 - \frac{(1+\mu) \alpha_0^2}{2 \left(\lambda_D m^{\frac{1}{D-3}} + \frac{\alpha_0 \sqrt{\mu}}{3} \right)^2}} \right]^{-1} \quad (8.3.5)$$

The modified Hawking temperature is physical so long as the black hole mass satisfies the following inequality:

$$(1+\mu) \alpha_0^2 \leq 2 \left(\lambda_D m^{\frac{1}{D-3}} + \frac{\alpha_0 \sqrt{\mu}}{3} \right)^2 \quad (8.3.6)$$

This tells us the black hole should have minimum mass M_{min} given by

$$M_{min} = M_p \left(\sqrt{\frac{(1+\mu)}{2}} - \sqrt{\frac{\mu}{9}} \right)^{D-3} \frac{D-2}{8\Gamma(\frac{D-1}{2})} (\alpha_0 \sqrt{\pi})^{D-3}. \quad (8.3.7)$$

The endpoint of Hawking evaporation in the GUP-case is characterized by a Planck-size remnant with maximum temperature

$$T_{max} \approx 2 \left[\frac{\frac{3(1+\mu)}{2} + \sqrt{\frac{\mu(\mu+1)}{2}}}{\frac{3}{2} + \frac{7}{6}\mu} \right] T_H. \quad (8.3.8)$$

8.4. *NO black holes at LHC current energy scales due to GUP*

The emission rate can be calculated using the Stefan–Boltzmann Law. Using Eqs. (8.1.10, 8.1.11), we get for a 4–dimensional brane:

$$\frac{dm}{dt} = -16 \frac{\mu'}{t_p} m^{\frac{-2}{D-3}} \left(1 + \frac{\alpha_0 \sqrt{\mu}}{3\lambda_D m^{\frac{1}{D-3}}} \right)^{-4} \left[1 + \sqrt{1 - \frac{(1+\mu)\alpha_0^2}{2\left(\lambda_D m^{\frac{1}{D-3}} + \frac{\alpha_0 \sqrt{\mu}}{3}\right)^2}} \right]^{-4} \quad (8.3.9)$$

The entropy can be calculated from the first law of BH-thermodynamics,

$$dS = \frac{2\pi}{D-3} \lambda_D m^{\frac{1}{D-3}} \left(1 + \frac{\alpha_0 \sqrt{\mu}}{3\lambda_D m^{\frac{1}{D-3}}} \right) \left[1 + \sqrt{1 - \frac{(1+\mu)\alpha_0^2}{2\left(\lambda_D m^{\frac{1}{D-3}} + \frac{\alpha_0 \sqrt{\mu}}{3}\right)^2}} \right] dm. \quad (8.3.10)$$

The specific heat has been calculated in GUP-case to give

$$\begin{aligned} \mathcal{C} = & -\frac{2\pi}{\lambda_D} m^{\frac{D-4}{D-3}} \left(\lambda_D m^{\frac{1}{D-3}} + \frac{\alpha_0 \sqrt{\mu}}{3} \right)^2 \sqrt{1 - \frac{(1+\mu)\alpha_0^2}{2\left(\lambda_D m^{\frac{1}{D-3}} + \frac{\alpha_0 \sqrt{\mu}}{3}\right)^2}} \\ & \left[1 + \sqrt{1 - \frac{(1+\mu)\alpha_0^2}{2\left(\lambda_D m^{\frac{1}{D-3}} + \frac{\alpha_0 \sqrt{\mu}}{3}\right)^2}} \right]. \end{aligned} \quad (8.3.11)$$

We note the BH specific heat vanishes at the minimum BH-mass. Therefore, the BH cannot exchange heat with the surrounding space.

8.4 NO black holes at LHC current energy scales due to GUP

In this section, we use the calculations in the last two sections to investigate whether black holes could be formed at LHC energy scales. From Eqs. (8.2.12, 8.3.7), we note

8.4. *NO black holes at LHC current energy scales due to GUP*

that black holes can be formed with masses larger than M_p in D -dimensions. The model of GUP- black holes in higher dimensions has three unknown parameters: D , M_P , and $\beta_0(\alpha_0)$. If we fix the GUP-parameters to be $\beta_0 = 1(\alpha_0 = 1)$, the values for the minimum black hole masses in the extra dimensions, using Eqs. (8.2.12, 8.3.7), are shown in the following Table.

Table 8.1: BH minimal mass for different dimensions with assuming $M_p \approx 1$ TeV.

	GUP-Quadratic: $\beta_0 = 1$	GUP-Linear&Quadratic: $\alpha_0 = 1$	GUP-Quadratic of [121]
D	M_{\min}	M_{\min}	M_{\min}
6	> 13.365 TeV	> 3.0526 TeV	> 2.0944 TeV
7	> 103.47 TeV	> 13.232 TeV	> 3.0843 TeV
8	> 910.92 TeV	> 65.900 TeV	> 3.9479 TeV
9	> 8857.0 TeV	> 364.90 TeV	> 4.5216 TeV
10	> 93340 TeV	> 2200.1 TeV	> 4.7247 TeV
11	> 1.0538×10^6 TeV	> 14250 TeV	> 4.5661 TeV

Table 8.2: The Schwarzschild radius $R_s = \frac{1}{\sqrt{\pi}M_p} \left[\frac{M_{BH}}{M_p} \frac{8\Gamma(\frac{D-1}{2})}{D-2} \right]^{\frac{1}{D-3}}$ with $M_p \approx 1$ TeV.

	GUP-Quadratic: $\beta_0 = 1$	GUP-Linear&Quadratic: $\alpha_0 = 1$	GUP-Quadratic of [121]
D	R_s	R_s	R_s
6	> 1.86	> 1.13	> 1
7	> 2.4	> 1.43	> 0.99960
8	> 2.97	> 1.75	> 0.99758
9	> 3.5	> 2.1	> 1.0003
10	> 4.1	> 2.4	> 0.99988
11	> 4.65	> 2.7	> 0.99982

In table 8.1, a BH in D -dimensions at fixed $\beta_0 = 1$ can form only for energies equal to or larger than its minimum mass. We considered the Planck scale (M_p) ≈ 1 TeV [81, 127–131].

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This means BH's (if a GUP-quadratic in Δp is only considered) in $D = 6$ can form only at energies not less than 13.3 TeV, and for $D = 8$, they can form only for energies not less than 910 TeV, and for BH's in $D = 10$, they can only form for energies not less than 93340 TeV.

Turning to GUP-linear and quadratic in Δp case, we found that the black hole can be formed at energies less than the ones predicted by GUP-quadratic case, but they are still larger than the current energy scales of LHC. The BH's in $D = 6$ can form only at energies not less than 3.05 TeV, and BH's can form in $D = 7$ for energies not less than 13.23 TeV. For $D = 11$, BH's can form only for energies not less than 14250 TeV.

In table 8.1, we compare our results with the results proposed in [121]. The previous studies in [121] predicted that BH's might be seen at the energy scales of LHC in disagreement with the recent experimental results of LHC [28]. Our results agree with the results of the experiment [28] and at the same time do not rule out the string theory and extra dimensions predictions if minimal length in quantum gravity is taken into consideration. We found that black holes can be formed at energies much higher than the current energy scales of LHC. Predictions of mini black holes forming at collision energies of a few TeV's were based on theories that consider the gravitational effects of extra dimensions of space like string theory and large extra dimensions theories [116–120]. But scientists at the Compact Muon Solenoid (CMS) detector in LHC now say they have found no signs of mini black holes at energies of 3.5–4.5 TeV. Our proposed model of GUP can justify why higher energies larger than the current scale of LHC is needed to form String Theory (Large extra dimensions)-Proving Black Holes.

Chapter 9

Summary

In this thesis, we presented certain phenomenological aspects of quantum gravity. In the introduction we reviewed the fundamental ideas of the general theory of relativity as well as quantum mechanics. Approaches to quantum gravity have been discussed in chapter 2. The canonical approach has been reviewed in which one makes use of a Hamiltonian formalism and identifies appropriate canonical variables and conjugate momenta. Examples include quantum geometrodynamics (where gravity is described in metric form) and loop quantum gravity (where gravity is described by a connection integrated around a closed loop). They are characterized by a constraint equation of the form

$$H\psi = 0, \tag{9.0.1}$$

where H denotes the full Hamilton operator for the gravitational field as well as all nongravitational fields; ψ is the full wave functional for these degrees of freedom. In the geometrodynamical approach, this equation is called the Wheeler–DeWitt equation, in honour of the physicists Wheeler and DeWitt, who first discussed this equation in detail. The loop approach goes mainly back to work by Ashtekar, Smolin, Rovelli

and many others.

As can be recognized from the stationary form of equation (9.0.1), these theories are explicitly timeless, that is, devoid of any classical time parameter. They thus solve the problem of time by getting rid of time at the fundamental level. This should happen in the other approaches, but the situation there is much less clear. The attempt to do quantum gravity in string theory has been briefly discussed. This is the main approach to construct a unifying quantum framework of all interactions. The quantum aspect of the gravitational field only emerges in a certain limit in which the different interactions can be distinguished from each other. All particles have their origin in excitations of fundamental strings. The fundamental scale is given by the string length; it is supposed to be of the order of the Planck length, although the Planck length is here a derived quantity. Quantum general relativity as well as string theory have found applications for quantum black holes and for quantum cosmology. Both approaches have, for restricted situations, proposed a microscopic explanation for the black-hole entropy. The corresponding microscopic states are either those of spin networks (in loop quantum gravity) or D-branes (in string theory).

We reviewed the basic development of the generalized uncertainty principle in chapter 3. Evidence from string theory, quantum geometry and black hole physics suggests that the usual Heisenberg uncertainty principle needs certain modification(s). This evidence has an origin in the quantum fluctuations of the background metric. The generalized uncertainty principle provides the existence of a minimal length scale in nature which is of the order of the Planck length. We reviewed some different forms of GUP which predict the existence of a minimal measurable length as well as the form derived in the context of Doubly Special Relativity which predicts the existence

of maximum measurable momentum. The last section in chapter 3 introduces our proposed form of generalized uncertainty principle which is consistent with string theory, black hole physics, doubly special relativity and ensures that the Jacobi identity is satisfied. Our proposed form of GUP predicts the existence of a minimal measurable length as well as the maximum measurable momentum.

We investigated some phenomenological aspects of our proposed form of GUP in chapter 4. The new form of GUP adds a correction term to the quantum or even the classical Hamiltonian as computed in Eqs. (3.3.8) and (3.3.9), near the Planck scale. We computed these corrections in various quantum phenomena such as Landau levels, simple harmonic oscillator, Lamb shift, and scanning tunneling microscope and have found that the upper bound on the GUP parameter α_0 would be 10^{23} , 10^{17} , 10^{10} , and 1. The correction in Landau levels is excluded experimentally because it corresponds to a length scale bigger than the electroweak length scale. The last three bounds are stringent and correspond to length scales which are smaller than the electroweak scale. This might imply the existence of an intermediate length scale between electroweak scale and Planck scale. In an optimistic scenario, we hope some of the effects may be measurable, that the current approach might open up a phenomenological window to Quantum Gravity, and that this would strengthen the synergy among experimentalists and theorists.

In chapter 5, we present another important result of our proposed form of GUP, the fundamental discreteness of space. We applied our proposed GUP to an elementary particle inside a box of length L . When we computed the corrections up to the first order and second order of the GUP parameter α , we noted the appearance of new oscillatory terms with a characteristic wavelength $\sim \pi\alpha\hbar$ in the general solutions

(5.1.4), (5.1.16). By imposing the appropriate boundary conditions, i.e. $\psi = 0$ at $x = 0, L$, the new oscillatory terms show that there cannot be a particle confined inside a box unless its length is quantized in terms of $\alpha = \alpha_0 \ell_{Pl}$ units. Since the box can be considered anywhere, we interpret the result as space essentially having a discrete nature. We learned from the results in chapter 4 that α_0 may take higher values, so the experiments do not rule out discreteness of space smaller than about a thousandth of a Fermi. The discreteness of space result has been extended to the relativistic case in one, two and three dimensions using the GUP-corrected Klein-Gordon equation and GUP-corrected Dirac equation. As measurements of lengths, areas and volumes require the existence and use of such particles, we interpret this as effective quantization of these quantities. Similar quantization has been obtained in the context of loop quantum gravity. Note that although existence of a fundamental length is apparently inconsistent with special relativity and Lorentz transformations (fundamental length in *whose frame?*), it is indeed consistent, and in perfect agreement, with Doubly Special Relativity Theories. It is hoped that the essence of these results will continue to hold in curved spacetimes, and even if possible fluctuations of the metric can be taken into account in a consistent way.

The discreteness of space raises a fundamental question: how do we re-state general covariance if not all physical systems can be described exactly using differentiable functions? One possible answer could be that the observable physical quantities are discrete, but transformations between allowable representations of those quantities could be continuous.

We investigated in chapter 6 the success of our proposed form of GUP in explaining the experimental results of violation of the weak equivalence principle at short

distance which quantum mechanics can not explain. We have shown that, by studying Heisenberg equations of motions in the presence of GUP, the acceleration is no longer mass-independent because of the mass-dependence through the momentum p . Therefore, the equivalence principle is dynamically violated. We also computed the consequences of our proposed form of GUP on the Liouville theorem. We found a new form of an invariant phase space in the presence of the GUP. We applied our approach to the calculation of the entropy bound of local quantum field theory. This led to a $\sqrt{\text{Area}}$ -type correction to the bound of the maximal entropy of a bosonic field. This showed that the conjectured entropy of the truncated Fock space corrected by GUP disagrees with 't Hooft's classical result. This agreed with the discreteness of space implications which does not leave the continuous symmetries such as translation, rotation and full Lorentz symmetry intact, and hence the holographic theory doesn't retain its good features due to discreteness of space.

In Chapter 7, we investigated the consequences of GUP on the parametric resonance in the post-inflation preheating in a static as well as expanding background. We showed that depending on the exact form of the GUP and initial conditions, the phenomenon of parametric resonance and the corresponding instability band can increase, potentially resulting in higher rates of particle production and an early termination of the above. We believe the inclusion of GUP takes into account (at least partially) remnant Planck scale effects on the reheating of the universe.

It would be interesting to study the effects of back-reaction of the produced particles in our set-up. We also believe that taking up the GUP term the way it is

considered in this work is tacitly including remnant Planck scale effects on the reheating theory of the universe. Identifying those effects requires further investigations. It would also be interesting to apply our approach to string-inspired models, for which nonlinearities enter the matter field equation via the Born-Infeld term, *viz*, $\mathcal{L} \sim 1 - \sqrt{1 - \dot{\chi}^2}$. This would perhaps shed some light on the inflationary reheating theory in the context of extended objects, e.g, D-branes. Last, since our matter field Hamiltonian (Eq. (3.3.9)) is non-relativistic, it might be worthwhile studying its relativistic generalization (e.g. via a GUP modified Klein-Gordon equation), to have a better understanding of the massless limit among other things. Note that our general formalism, including Eqs. (7.2.1) and (7.4.1) remain well suited for such generalizations, as well as for a large class to non-linear corrections to parametric resonance and particle production. We hope to study these issues in the future.

Another phenomenological aspect of our proposed GUP has been investigated in chapter 8. In this chapter, we studied the possible discrepancy that has been found between the results of the very recent experiment that was done at LHC [28] and the predictions of large extra dimensions theories. Some large extra dimensions theories predicted the existence of BH's at LHC [120, 121] energy scales which disagree with the recent experiment that was done at LHC [28]. We investigated whether the GUP can explain the formation of black holes at energies higher than the energy scales of LHC. We have shown that, by studying the Hawking-Uncertainty connection, that we reviewed in the same chapter, the black holes can be formed in the range between 3.05 – 14250 TeV for GUP-linear and quadratic in Δp , and they can be formed in the range between 13.36 – 1.05×10^6 TeV for GUP-quadratic in Δp . Both cases say Black Holes can be formed at energies higher than the current energy scales of LHC. We

Summary

conclude that mechanisms such as GUP are necessary beside large extra dimensions theories to explain the experimental results. We think that this is another success of GUP in explaining the experimental results after its success in explaining the violation of the weak equivalence principle at short distance that was presented in chapter 6. In the future, it would be appropriate to apply our approach to the calculations of the cosmological constant, black body radiation, etc. We hope to report on these in the future.

We hope that our work will make some useful contributions to quantum gravity phenomenology.

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Appendix

A Proof for Eq. 3.3.1

Since black hole physics and string theory suggest a modified Heisenberg algebra (which is consistent with GUP) quadratic in the momenta (see e.g. [10–14, 16]) while DSR theories suggest one that is linear in the momenta (see e.g. [72, 75]), we try to incorporate both of the above, and start with the most general algebra with linear and quadratic terms

$$\begin{aligned}
 [x_i, p_j] &= i\hbar(\delta_{ij} + \delta_{ij}\alpha_1 p + \alpha_2 \frac{p_i p_j}{p} + \beta_1 \delta_{ij} p^2 \\
 &+ \beta_2 p_i p_j) .
 \end{aligned} \tag{A1}$$

Assuming that the coordinates commute among themselves, as do the momenta, it follows from the Jacobi identity that

$$-[[x_i, x_j], p_k] = [[x_j, p_k], x_i] + [[p_k, x_i], x_j] = 0 . \tag{A2}$$

Employing Eq. (A1) and the commutator identities, and expanding the right hand side, we get (summation convention assumed)

$$\begin{aligned}
 0 &= [[x_j, p_k], x_i] + [[p_k, x_i], x_j] \\
 &= i\hbar(-\alpha_1 \delta_{jk} [x_i, p] - \alpha_2 [x_i, p_j p_k p^{-1}] - \beta_1 \delta_{jk} [x_i, p_l p_l] \\
 &\quad - \beta_2 [x_i, p_j p_k]) - (i \leftrightarrow j) \\
 &= i\hbar(-\alpha_1 \delta_{jk} [x_i, p] - \alpha_2 ([x_i, p_j] p_k p^{-1} + p_j [x_i, p_k] p^{-1} \\
 &\quad + p_j p_k [x_i, p^{-1}]) - \beta_1 \delta_{jk} ([x_i, p_l] p_l + p_l [x_i, p_l]) \\
 &\quad - \beta_2 ([x_i, p_j] p_k + p_j [x_i, p_k])) - (i \leftrightarrow j) .
 \end{aligned} \tag{A3}$$

Proof for Eq. 3.3.1

To simplify the right hand side of Eq. (A3), we now evaluate the following commutators:

(i) $[x_i, p]$ to $\mathcal{O}(p)$

Note that

$$[x_i, p^2] = [x_i, p \cdot p] = [x_i, p]p + p[x_i, p] \quad (\text{A4})$$

$$\begin{aligned} &= [x_i, p_k p_k] = [x_i, p_k]p_k + p_k[x_i, p_k] \\ &= i\hbar (\delta_{ik} + \alpha_1 p \delta_{ik} + \alpha_2 p_i p_k p^{-1}) p_k + i\hbar p_k (\delta_{ik} \\ &\quad + \alpha_1 p \delta_{ik} + \alpha_2 p_i p_k p^{-1}) \quad (\text{to } \mathcal{O}(p) \text{ using (A1)}) \\ &= 2i\hbar p_i [1 + (\alpha_1 + \alpha_2)p] . \end{aligned} \quad (\text{A5})$$

Comparing (A4) and (A5), we get

$$[x_i, p] = i\hbar (p_i p^{-1} + (\alpha_1 + \alpha_2)p_i) . \quad (\text{A6})$$

(ii) $[x_i, p^{-1}]$ to $\mathcal{O}(p)$

Using

$$0 = [x_i, I] = [x_i, p \cdot p^{-1}] = [x_i, p]p^{-1} + p[x_i, p^{-1}] \quad (\text{A7})$$

it follows that

$$\begin{aligned} [x_i, p^{-1}] &= -p_i^{-1}[x_i, p]p^{-1} \\ &= -i\hbar p^{-1} (p_i p^{-1} + (\alpha_1 + \alpha_2)p_i) p^{-1} \\ &= -i\hbar p_i p^{-3} (1 + (\alpha_1 + \alpha_2)p) . \end{aligned} \quad (\text{A8})$$

Proof for Eq. (3.3.5)

Substituting (A6) and (A8) in (A3) and simplifying, we get

$$\begin{aligned} 0 &= [[x_j, p_k], x_i] + [[p_k, x_i], x_j] \\ &= ((\alpha_1 - \alpha_2)p^{-1} + (\alpha_1^2 + 2\beta_1 - \beta_2)) \Delta_{jki} \end{aligned} \quad (\text{A9})$$

where $\Delta_{jki} = p_i \delta_{jk} - p_j \delta_{ik}$. Thus one must have $\alpha_1 = \alpha_2 \equiv -\alpha$ (with $\alpha > 0$; The negative sign follows from Ref. [72]), and $\beta_2 = 2\beta_1 + \alpha_1^2$. Since from dimensional grounds it follows that $\beta \sim \alpha^2$, for simplicity, we assume $\beta_1 = \alpha^2$. Hence $\beta_2 = 3\alpha^2$, and we get Eq. (3.3.1) of this thesis, namely,

$$[x_i, p_j] = i\hbar \left(\delta_{ij} - \alpha \left(p \delta_{ij} + \frac{p_i p_j}{p} \right) + \alpha^2 (p^2 \delta_{ij} + 3p_i p_j) \right). \quad (\text{A10})$$

B Proof for Eq. (3.3.5)

We would like to express the momentum p_j in terms of the *low energy momentum* p_{0j} (such that $[x_i, p_{0j}] = i\hbar \delta_{ij}$). Since Eq. (A10) is quadratic in p_j , the latter can at most be a cubic function of the p_{0i} . We start with the most general form consistent with the index structure

$$p_j = p_{0j} + a p_0 p_{0j} + b p_0^2 p_{0j}, \quad (\text{B1})$$

where $a \sim \alpha$ and $b \sim \alpha^2$. From Eq. (B1) it follows that

$$\begin{aligned} [x_i, p_j] &= [x_i, p_{0j} + a p_0 p_{0j} + b p_0^2 p_{0j}] \\ &= i\hbar \delta_{ij} + a ([x_i, p_0] p_{0j} + p_0 [x_i, p_{0j}]) \\ &\quad + b ([x_i, p_0] p_0 p_{0j} + p_0 [x_i, p_0] p_{0j} + p_0^2 [x_i, p_{0j}]). \end{aligned} \quad (\text{B2})$$

Next, we use the following four results to $\mathcal{O}(a)$ and $[x_i, p_{0j}] = i\hbar$ in Eq. (B2):

(i) $[x_i, p_0] = i\hbar p_{0i} p_0^{-1}$, which follows from Eq. (A6) when $\alpha_i = 0$, as well from the corresponding Poisson bracket.

(ii) $p_j = p_{0j}(1 + ap_0) + \mathcal{O}(a^2) \simeq p_{0j}(1 + ap)$ [from Eq. (B1)]. Therefore, $p_{0j} \simeq \frac{p_j}{1+ap} \simeq (1 - ap)p_j$.

(iii) $p_0 = (p_{0j} p_{0j})^{\frac{1}{2}} = ((1 - ap)^2 p_j p_j)^{1/2} = (1 - ap)p$.

(iv) $p_{0i} p_0^{-1} p_{0j} = (1 - ap)p_i (1 - ap)^{-1} p^{-1} (1 - ap)p_j = (1 - ap)p_i p_j p^{-1}$.

Thus, Eq. (A2) yields

$$\begin{aligned} [x_i, p_j] &= i\hbar \delta_{ij} + ia\hbar (p\delta_{ij} + p_i p_j p^{-1}) \\ &+ i\hbar(2b - a^2)p_i p_j + i\hbar(b - a^2)p^2 \delta_{ij}. \end{aligned} \quad (\text{B3})$$

Comparing with Eq. (A10), it follows that $a = -\alpha$ and $b = 2\alpha^2$. In other words

$$p_j = p_{0j} - \alpha p_0 p_{0j} + 2\alpha^2 p_0^2 p_{0j} = p_{0j} (1 - \alpha p_0 + 2\alpha^2 p_0^2) \quad (\text{B4})$$

which is Eq. (3.3.5) in this thesis.

C Minimal coupling to electromagnetism

The minimal coupling prescription that we used in [23] and that is shown in Eq. (4.1.6) in this thesis, is the standard procedure for any Lagrangian/Hamiltonian describing charged particles in an electromagnetic field, including the ones that we used therein. It has very little to do with the space-time dimension under consideration.

We present below an outline of the proof, following [132] (see article 16 therein). We start with the Lagrangian for a free particle of mass m , with coordinates x_i and

velocity v_i , $v = \sqrt{v_i v_i}$ (i runs over the spatial dimensions, and summation convention is assumed)

$$L = f(v) . \quad (\text{C1})$$

The conjugate momenta are given by

$$p_i = \frac{\partial f(v)}{\partial v_i} , \quad (\text{C2})$$

with its inverse

$$v_i = v_i(p_j) \quad \text{and} \quad v = v(p_j) . \quad (\text{C3})$$

A Legendre transformation yields the Hamiltonian

$$H(x_i, p_j) = -L + p_i v_i \quad (\text{C4})$$

$$= -f(v(p_j)) + p_i v_i(p_j) . \quad (\text{C5})$$

Next, we assume that the particle also has a charge e , and is in an electromagnetic field, given by the vector and scalar potentials A_i and ϕ respectively. The Lagrangian now has the extra potential dependent terms

$$L = f(v) + e(A_i v_i - \phi) . \quad (\text{C6})$$

Now the conjugate momentum is

$$p_{0i} = \frac{\partial L}{\partial v_i} = \frac{\partial f}{\partial v_i} + eA_i \quad (\text{C7})$$

$$= p_i + eA_i , \quad (\text{C8})$$

and the Hamiltonian

$$H(x_i, p_{0j}) = -L + p_{0i} v_i \quad (\text{C9})$$

$$= -f(v) - e(A_i v_i - \phi) + (p_i + eA_i) v_i \quad (\text{C10})$$

$$= -f(v(p_j)) + p_i v_i(p_j) + e\phi \quad (\text{C11})$$

$$= -f(v(p_{0j} - eA_j)) + (p_{0i} - eA_i) v_i(p_{0j} - eA_j) + e\phi. \quad (\text{C12})$$

Note that A_i drops out of Eq. (C11), which (apart from the $e\phi$ term), is identical to Eq. (C5). This is subsequently restored in Eq. (C12) by the replacement $p_i \rightarrow p_{0i} - eA_i$. This constitutes proof of the minimal coupling prescription, which as seen is valid for *any* $f(v)$. The number of space-time dimensions or the exact form of the vector potential plays no role in the above. One can of course incorporate a non-electromagnetic potential V by simply adding it to $e\phi$. Well known examples of the above prescription include, e.g.

- a non-relativistic charged particle of mass m , for which $f(v) = mv^2/2$ and $H = p^2/2m + e\phi = (\vec{p}_0 - e\vec{A})^2/2m + e\phi$, and
- a relativistic charged particle of mass m , for which $f(v) = -mc^2\sqrt{1 - v^2/c^2}$ and $H = \sqrt{p^2c^2 + m^2c^4} + e\phi = \sqrt{(\vec{p}_0 - e\vec{A})^2c^2 + m^2c^4} + e\phi$.

For the GUP corrected Hamiltonian in Section I of [23], the corresponding Lagrangian can be easily written and the electromagnetic potentials added (these and the following expressions are written up to second order in α)

$$L = \frac{1}{2}mv^2 + \alpha m^2 v^3 + 2\alpha^2 m^3 v^4 + e(A_i v_i - \phi) . \quad (\text{C13})$$

The expressions for momenta and conversely for velocity are given by

$$p_{0i} = mv_i + 3\alpha m^2 v v_i + 8\alpha^2 m^3 v^2 v_i + eA_i \quad (\text{C14})$$

$$\equiv p_i + eA_i \quad (\text{C15})$$

$$v_i = \frac{p_i}{m} - \frac{3\alpha}{m} p p_i + \frac{10\alpha^2}{m} p^2 p_i, \quad (\text{C16})$$

which satisfy the canonical relation $\{x_i, p_{0j}\} = 1$. It should be stressed that in [23], p signified the momentum at *high energies*, whose Poisson/commutator bracket with position was different from the canonical one. Employing Eq. (C15) one gets the Hamiltonian

$$H = \frac{1}{2m} p^2 - \frac{\alpha}{m} p^3 + \frac{5\alpha^2}{2m} p^4 \quad (\text{C17})$$

$$= \frac{1}{2m} (\vec{p}_0 - e\vec{A})^2 - \frac{\alpha}{m} (\vec{p}_0 - e\vec{A})^3 + \frac{5\alpha^2}{2m} (\vec{p}_0 - e\vec{A})^4, \quad (\text{C18})$$

which is the Hamiltonian in Section II of [23] and which is Eq. (4.1.6) in this thesis. This is precisely what we used to compute GUP corrections to Landau levels, and would also be relevant for any system involving electromagnetic interactions in which the GUP can have some effect.

D Number of states and maximum energy

To explain the result obtained in Sec. 6.2.2, we found an increase, in the number of states inside the considered volume because of the following reasons. The number of states is proportional to the considered volume divided by $(\Delta p \Delta x)^s$, where s is the number of degrees of freedom. Now, $\Delta x \Delta p$ decreases by considering the GUP form in Eq. (3.3.2), note the negative term which is of the first order of α in Eq. (3.3.2). The

positive term in Eq. (3.3.2), which is a second order of α , has a smaller contribution than the negative term, so the net result is a decreasing of $\Delta p \Delta x$ which results in an increasing in the number of states. To make the point more clear, the second order term ($\approx \alpha^2 \Lambda^5$) contributes to Eq. (6.2.29) as follows :

$$N \approx \frac{L^3}{2\pi^2} \left(\frac{\Lambda^3}{3} + \alpha \Lambda^4 - \alpha^2 \Lambda^5 \right). \quad (D1)$$

If we assume that the second order term contribution is bigger than the first order term in Eq. (D1), then we will have:

$$\begin{aligned} \alpha \Lambda^4 &< \alpha^2 \Lambda^5, \\ 1 &< \alpha \Lambda, \\ \frac{1}{\alpha} &< \Lambda. \end{aligned} \quad (D2)$$

This of course is not consistent with the condition in Eq. (6.2.30) which assumes the existence of maximum measurable momentum $\approx \frac{1}{\alpha}$, and also is not consistent with the predicted maximum measurable momentum in Eq. (3.3.4). But, if we consider that the first order term has a bigger contribution than the second order term in Eq. (D1), we will have:

$$\begin{aligned} \alpha \Lambda^4 &> \alpha^2 \Lambda^5, \\ 1 &> \alpha \Lambda, \\ \frac{1}{\alpha} &> \Lambda. \end{aligned} \quad (D3)$$

This is completely consistent with the condition in Eq. (6.2.30) which assumes the existence of maximum measurable momentum $\approx \frac{1}{\alpha}$ and hence it is consistent with the predicted maximum measurable momentum in Eq. (3.3.4).

To summarize, the contribution of the second order term of α^2 should be smaller than the contribution of the first order term of α , otherwise, this will violate the condition of the existence of maximum measurable momentum that we obtained in our proposed model of GUP in Eq. (3.3.4). This was the reason why we ignored the second order term contribution due to its smaller contribution than the first order term ($\alpha\Lambda^4$) and its contribution will not change the main result of increasing the number of states.